

N° 32 – August 2014

Italian Journal of Pure and Applied Mathematics

ISSN 2239-0227

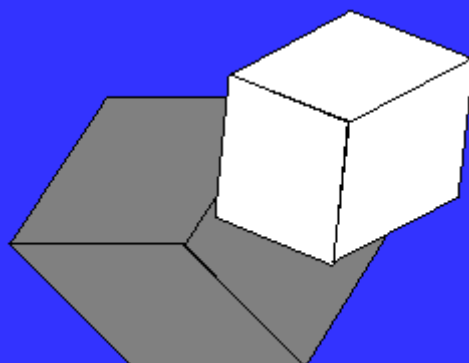
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Italian Journal of Pure and Applied Mathematics – N. 32–2014

On OD-characterizability of a certain alternating and symmetric group A. Mahmiani, M.R. Darafsheh	pp.	7-14
Some uniqueness results on meromorphic functions sharing two sets Abhijit Banerjee, Pranab Bhattacharjee	pp.	15-32
A note on P-regular semirings M.K. Dubey	pp.	33-36
Some classes of generalized Minmax polynomials H.M. Srivastava, Gospava B. Djordjević	pp.	37-48
On γ-s-Urysohn closed and γ-s-regular closed spaces Sabir Hussain, Ennis Rosas	pp.	49-56
(0,2; 0)-Interpolation on the unit circle Swarnima Bahadur	pp.	57-66
On laplace transforms of generalized whittaker function of multi-variables $M_{i, \delta_1 - \delta_k} (x_1, \dots, x_k)$ M. Kamarujjama, Waseem A. Khan	pp.	67-72
Fekete-Szegő problem for concave univalent functions defined by Sălăgean operator Alawiah Ibrahim, Maslina Darus, Sever S. Dragomir	pp.	73-86
r-Weak cb spaces D. Bhattacharya, L. Dey	pp.	87-102
More properties on flexible graded modules Fida Moh'd, Mashhoor Refai	pp.	103-114
On $(N(k), \xi)$-semi-Riemannian 3-manifolds D.G. Prakasha, H.G. Nagaraja, G. Somashekhar	pp.	115-124
A new signing algorithm based on elliptic curve discrete logarithms and quadratic residue problems Nedal Tahat, Emad E. Abdallah	pp.	125-132
Eccentricity, space bending, dimension M. Nitu, Fl. Smarandache, M.E. Selariu	pp.	133-142
Strongly duo and duo right S-Acts Mohammad Roueentan, Majid Ershad	pp.	143-154
On automatic boundedness of linear operators on convex bornological spaces Abdelaziz Tajmouati	pp.	155-164
Stability solutions of nonlinear nabla fractional difference equations using fixed point theorems J. Jagan Mohan, N. Shobanadevi, G.V.S.R. Deekshitulu	pp.	165-184
Exp-function method using modified Riemann-Liouville derivative for singularly perturbed Boussinesq equations of fractional-order Qazi Mahmood Ul Hassan, Syed Tauseef Mohyud-Din	pp.	185-192
Common fixed point for self and nonself-maps through an implicit relation T. Phaneendra, D. Surekha	pp.	193-202
A characterization of projective special linear group $L_3(5)$ by nse Shitian Liu	pp.	203-212
New inequalities of Hermite-Hadamard type for functions whose first derivatives absolute values are s-convex Feixiang Chen, Yuming Feng	pp.	213-222
On Thompson's conjecture for $\text{Aut}(J_2)$ and $\text{Aut}(McL)$ Yanheng Chen, Yuming Feng, Guiyun Chen	pp.	223-234
On some growth properties of differential polynomials in the light of relative order Sanjib Kumar Datta, Tanmay Biswas, M.D. Azizul Hoque	pp.	235-246
Jensen type weighted inequalities for functions of selfadjoint and unitary operators S.S. Dragomir	pp.	247-264
ϕJ-multipliers and ϕJ-multipliers quadratic on Jordan Banach algebras Abdelaziz Tajmouati	pp.	265-276
A note on Boolean subsets of orthomodular posets Dietmar Dorninger, Helmut Länger	pp.	277-282
Some derivations on the bounds for the zeros of entire functions based on slowly changing functions Sanjib Kumar Datta, Dilip Chandra Pramanik	pp.	283-300
Soft intersection h-ideals of hemirings and its applications Xueling Ma, Jianming Zhan	pp.	301-308
Characterizations of regular Abel-Grassmann's groupoids by the properties of their $(\in, \in \vee_q k)$-fuzzy ideals Xueling Ma, Jianming Zhan, Madad Khan, Tariq Aziz	pp.	309-324
Secret key distribution technique using theory of numbers S. Srinivasan, P. Muralikrishna, N. Chandramowliswaran	pp.	325-328
Some characterizations of intra-regular Abel Grassmann's groupoids Xueling Ma, Jianming Zhan, Madad Khan, Tariq Aziz	pp.	329-346
Some double lacunary sequence spaces Kuldip Raj, Sunil K. Sharma, Seema Jamwal	pp.	347-358
A new approach to a certain generalized integral transform for certain space of Boehmians S.K.Q. Al-Omari	pp.	359-368
On solitary wave solutions of nonlinear time-fractional Fornberg-Whitham equation Qazi Mahmood Ul Hassan, Syed Tauseef Mohyud-Din	pp.	369-378
Global exponential stability of impulsive hybrid dynamical systems with any time delay Xingjie Wu, Yang Liu, Zongmin Qiao	pp.	379-392

From Newton to Kepler. One simple derivation of Kepler's laws from Newton's ones. František Mošna	pp.	393-400
Hopf Modules in the weak Yetter-Drinfeld categories Yin Yanmin	pp.	401-414
Right alternative rings with $x(yz)-y(xz)$ in the center K. Madhusudhan Reddy, K. Suvarna	pp.	415-418
Hyers-Ulam stability of linear differential equations of second order with constant coefficient Jianming Xue	pp.	419-424
A theorem about finite groups with special conjugacy classes Xianglin Du, Yuming Feng, Jinkui Liu	pp.	425-430
Energy of an intuitionistic fuzzy graph B. Praba, V.M. Chandrasekaran, G. Deepa	pp.	431-444
On upper and lower almost contra-ω-continuous multifunctions C. Carpintero, N. Rajesh, E. Rosas, S. Saranyasri	pp.	445-460
Irreducible ideals in rings B. Venkateswarlu, R. Vasu Babu, E. Yohannes, T. Embiale	pp.	461-466
Inclusion results on subclasses of starlike and convex functions associated with struve functions T. Janani, G. Murugusundaramoorthy	pp.	467-476
The modified ($w=g$)-expansion method and its applications for solving the modified generalized Vakhnenko equation Elsayed M.E. Zayed, Ahmed H. Arnous	pp.	477-492
Rough neutrosophic sets Said Broumi, Florentin Smarandache, Mamoni Dhar	pp.	493-502
Neutrosophic parametrized soft set theory and its decision making Said Broumi, Irfan Deli, Florentin Smarandache	pp.	503-514
Young type inequalities for matrices Yang Peng	pp.	515-518
Bounds for the eigenvalues of matrices Limin Zou, Youyi Jiang	pp.	519-524
On thompson's conjecture for alternating group Shitian Liu, Yanhua Huang	pp.	525-532
Application of bipolar fuzzy soft sets in K-algebras M. Akram, N.O. Alsherei, K.P. Shum, A. Farooq	pp.	533-546
Semigroup distances of finite groupoids Barbora Batíková, Šárka Dvořáková, Milan Trch	pp.	547-560
On how to construct left semimodules from the right ones Barbora Batíková, TomášKepka, Petr Němec	pp.	561-578
Toward a new algorithm for systems of fractional differential-algebraic equations H.M. Jaradat, M. Zurigat, Safwan Al-Shara', Qutaibeh Katatbeh	pp.	579-594
A characterization of higher derivations on Banach algebras T.L. Shatery, S. Hejazian	pp.	595-602

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Up to December 2013 this journal is exchanged with the following periodicals:

1.	Acta Cybernetica - Szeged	H
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65.	Demonstratio Mathematica - Warszawa	PL
66.	Discusiones Mathematicae - Zielona Góra	PL
67.	Divulgaciones Matemáticas - Universidad del Zulia	YV
68.	Doctoral Thesis - Department of Mathematics Umea University	SW
69.	Extracta Mathematicae - Badajoz	E
70.	Fasciculi Mathematici - Poznan	PL
71.	Filomat - University of Nis	SRB
72.	Forum Mathematicum - Mathematisches Institut der Universität Erlangen	D
73.	Functiones et Approximatio <i>Commentarii Mathematici</i> - Adam Mickiewicz University	L
74.	Funkcialaj Ekvacioj - Kobe University	J
75.	Fuzzy Systems & A.I. Reports and Letters - Iași University	RO
76.	General Mathematics - Sibiu	RO
77.	Geometria - Fasciculi Mathematici - Poznan	PL
78.	Glasnik Matematički - Zagreb	CRO
79.	Grazer Mathematische Berichte – Graz	A
80.	Hiroshima Mathematical Journal - Hiroshima	J
81.	Hokkaido Mathematical Journal - Sapporo	J

82.	Houston Journal of Mathematics - Houston - Texas	USA
83.	IJMSI - Iranian Journal of Mathematical Sciences & Informatics, Tarbiat Modares University, Tehran	IR
84.	Illinois Journal of Mathematics - University of Illinois Library - Urbana	USA
85.	Informatica - The Slovene Society Informatika - Ljubljana	SLO
86.	Internal Reports - University of Natal - Durban	SA
87.	International Journal of Computational and Applied Mathematics – University of Qiongzhou, Hainan	PRC
88.	International Journal of Science of Kashan University - University of Kashan	IR
89.	Iranian Journal of Science and Technology - Shiraz University	IR
90.	Irish Mathematical Society Bulletin - Department of Mathematics - Dublin	IRL
91.	IRMAR - Inst. of Math. de Rennes - Rennes	F
92.	Israel Mathematical Conference Proceedings - Bar-Ilan University - Ramat -Gan	IL
93.	Izvestiya: Mathematics - Russian Academy of Sciences and London Mathematical Society	RU
94.	Journal of Applied Mathematics and Computing – Dankook University, Cheonan – Chungnam	ROK
95.	Journal of Basic Science - University of Mazandaran – Babolsar	IR
96.	Journal of Beijing Normal University (Natural Science) - Beijing	PRC
97.	Journal of Dynamical Systems and Geometric Theory - New Delhi	IND
98.	Journal Egyptian Mathematical Society – Cairo	ET
99.	Journal of Mathematical Analysis and Applications - San Diego California	USA
100.	Journal of Mathematics of Kyoto University - Kyoto	J
101.	Journal of Science - Ferdowsi University of Mashhad	IR
102.	Journal of the Bihar Mathematical Society - Bhagalpur	IND
103.	Journal of the Faculty of Science – Tokyo	J
104.	Journal of the Korean Mathematical Society - Seoul	ROK
105.	Journal of the Ramanujan Mathematical Society - Mysore University	IND
106.	Journal of the RMS - Madras	IND
107.	Kumamoto Journal of Mathematics - Kumamoto	J
108.	Kyungpook Mathematical Journal - Taegu	ROK
109.	L'Enseignement Mathématique - Genève	CH
110.	La Gazette des Sciences Mathématiques du Québec - Université de Montréal	CAN
111.	Le Matematiche - Università di Catania	I
112.	Lecturas Matematicas, Soc. Colombiana de Matematica - Bogotá	C
113.	Lectures and Proceedings International Centre for Theoretical Physics - Trieste	I
114.	Lucrările Seminarului Matematic – Iași	RO
115.	m-M Calculus - Matematički Institut Beograd	SRB
116.	Matematicna Knjiznica - Ljubljana	SLO
117.	Mathematica Balcanica – Sofia	BG
118.	Mathematica Bohemica - Academy of Sciences of the Czech Republic Praha	CZ
119.	Mathematica Macedonica, St. Cyril and Methodius University, Faculty of Natural Sciences and Mathematics - Skopje	MK
120.	Mathematica Montisnigri - University of Montenegro - Podgorica	MNE
121.	Mathematica Moravica - Cacak	SRB
122.	Mathematica Pannonica - Miskolc - Egyetemvaros	H
123.	Mathematica Scandinavica - Aarhus - Copenhagen	DK
124.	Mathematica Slovaca - Bratislava	CS
125.	Mathematicae Notae - Universidad Nacional de Rosario	AR
126.	Mathematical Chronicle - Auckland	NZ
127.	Mathematical Journal - Academy of Sciences - Uzbekistan	CSI
128.	Mathematical Journal of Okayama University - Okayama	J
129.	Mathematical Preprint - Dep. of Math., Computer Science, Physics – University of Amsterdam	NL
130.	Mathematical Reports - Kyushu University - Fukuoka	J
131.	Mathematics Applied in Science and Technology – Sangyo University, Kyoto	J
132.	Mathematics Reports Toyama University - Gofuku	J
133.	Mathematics for Applications - Institute of Mathematics of Brno University of Technology, Brno	CZ
134.	MAT - Prepublicacions - Universidad Austral	AR
135.	Mediterranean Journal of Mathematics – Università di Bari	I
136.	Memoirs of the Faculty of Science - Kochi University - Kochi	J
137.	Memorias de Mathematica da UFRJ - Instituto de Matematica - Rio de Janeiro	BR
138.	Memorie lincee - Matematica e applicazioni - Accademia Nazionale dei Lincei	I
139.	Mitteilungen der Naturforschenden Gesellschaften beider Basel	CH
140.	Monografii Matematiche - Universitatea din Timisoara	RO
141.	Monthly Bulletin of the Mathematical Sciences Library – Abuja	WAN
142.	Nagoya Mathematical Journal - Nagoya University, Tokyo	J
143.	Neujahrsblatt der Naturforschenden Gesellschaft - Zürich	CH
144.	New Zealand Journal of Mathematics - University of Auckland	NZ
145.	Nieuw Archief voor Wiskunde - Stichting Mathematicae Centrum – Amsterdam	NL
146.	Nihonkai Mathematical Journal - Niigata	J
147.	Notas de Algebra y Analisis - Bahia Blanca	AR
148.	Notas de Logica Matematica - Bahia Blanca	AR
149.	Notas de Matematica Discreta - Bahia Blanca	AR
150.	Notas de Matematica - Universidad de los Andes, Merida	YV
151.	Notas de Matematicas - Murcia	E
152.	Note di Matematica - Lecce	I
153.	Novi Sad Journal of Mathematics - University of Novi Sad	SRB
154.	Obzornik za Matematiku in Fiziku - Ljubljana	SLO
155.	Octagon Mathematical Magazine - Braşov	RO
156.	Osaka Journal of Mathematics - Osaka	J
157.	Periodica Matematica Hungarica - Budapest	H
158.	Periodico di Matematiche - Roma	I
159.	Pliska - Sofia	BG
160.	Portugaliae Mathematica - Lisboa	P
161.	Posebna Izdanja Matematičkog Instituta Beograd	SRB
162.	Pre-Publicações de Matematica - Univ. de Lisboa	P
163.	Preprint - Department of Mathematics - University of Auckland	NZ
164.	Preprint - Institute of Mathematics, University of Lodz	PL
165.	Proceeding of the Indian Academy of Sciences - Bangalore	IND
166.	Proceeding of the School of Science of Tokai University - Tokai University	J
167.	Proceedings - Institut Teknologi Bandung - Bandung	RI
168.	Proceedings of the Academy of Sciences Tasked – Uzbekistan	CSI
169.	Proceedings of the Mathematical and Physical Society of Egypt – University of Cairo	ET
170.	Publicaciones del Seminario Matematico Garcia de Galdeano - Zaragoza	E
171.	Publicaciones - Departamento de Matemática Universidad de Los Andes Merida	YV
172.	Publicaciones Matematicas del Uruguay - Montevideo	U
173.	Publicaciones Mathematicae - Debrecen	H

174.	Publicacions matemàtiques - Universitat Autònoma, Barcelona	E
175.	Publications de l'Institut Mathématique - Beograd	SRB
176.	Publications des Séminaires de Mathématiques et Informatiques de Rennes	F
177.	Publications du Département de Mathématiques, Université Claude Bernard - Lyon	F
178.	Publications Mathématiques - Besançon	F
179.	Publications of Serbian Scientific Society - Beograd	SRB
180.	Publikacije Elektrotehničkog Fakulteta - Beograd	SRB
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184.	Quasigroups and Related Systems - Academy of Science - Kishinev Moldova	CSI
185.	Ratio Mathematica - Università di Pescara	I
186.	Recherche de Mathématique - Institut de Mathématique Pure et Appliquée Louvain-la-Neuve	B
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188.	Rendiconti del Seminario Matematico - Università di Padova	I
189.	Rendiconti dell'Istituto Matematico - Università di Trieste	I
190.	Rendiconti di Matematica e delle sue Applicazioni - Roma	I
191.	Rendiconti lincei - Matematica e applicazioni - Accademia Nazionale dei Lincei	I
192.	Rendiconti Sem. - Università di Cagliari	I
193.	Report series - Auckland	NZ
194.	Reports Math. University of Stockholm - Stockholm	SW
195.	Reports - University Amsterdam	NL
196.	Reports of Science Academy of Tajikistan - Dushanbe	TAJ
197.	Research Reports - Cape Town	SA
198.	Research Reports - University of Umea - Umea	SW
199.	Research Report Collection (RGMIA) Melbourne	AUS
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201.	Review of Research, Faculty of Science, Mathematics Series - Institute of Mathematics University of Novi Sad	SRB
202.	Review of Research Math. Series - Novi Sad	YN
203.	Revista Ciencias Matem. - Universidad de la Habana	C
204.	Revista Colombiana de Matemáticas - Bogotá	C
205.	Revista de Matemáticas Aplicadas - Santiago	CH
206.	Revue Roumaine de Mathématiques Pures et Appliquées - Bucaresti	RO
207.	Ricerca Operativa AIRO - Genova	I
208.	Ricerche di Matematica - Napoli	I
209.	Rivista di Matematica - Università di Parma	I
210.	Sains Malaysiana - Selangor	MAL
211.	Saitama Mathematical Journal - Saitama University	J
212.	Sankhya - Calcutta	IND
213.	Sarajevo Journal of Mathematics	BIH
214.	Sciences Bulletin, DPRK, Pyongyang	KR
215.	Scientific Review - Beograd	SRB
216.	Scientific Studies and Research, Vasile Alecsandri University Bacau	RO
217.	Semesterbericht Funktionalanalysis - Tübingen	D
218.	Séminaire de Mathématique - Université Catholique, Louvain la Neuve	B
219.	Seminario di Analisi Matematica - Università di Bologna	I
220.	Serdica Bulgaricae Publicationes Mathematicae - Sofia	BG
221.	Serdica Mathematical Journal - Bulgarian Academy of Sciences, University of Sofia	BG
222.	Set-Valued Mathematics and Applications - New Delhi	IND
223.	Sitzungsberichte der Mathematisch Naturwissenschaftlichen Klasse Abteilung II - Wien	A
224.	Southeast Asian Bulletin of Mathematics - Southeast Asian Mathematical Society	PRC
225.	Studia Scientiarum Mathematica Hungarica - Budapest	H
226.	Studia Universitatis Babes Bolyai - Cluj Napoca	RO
227.	Studii și Cercetări Matematice - București	RO
228.	Studii și Cercetări Științifice, ser. Matematică - Universitatea din Bacău	RO
229.	Sui Hak - Pyongyang DPR of Korea	KR
230.	Tamkang Journal of Mathematics - Tamsui - Taipei	TW
231.	Thai Journal of Mathematics - Chiang Mai	TH
232.	Task Quarterly	PL
233.	The Journal of the Academy of Mathematics Indore	IND
234.	The Journal of the Indian Academy of Mathematics - Indore	IND
235.	The Journal of the Nigerian Mathematical Society (JNMS) - Abuja	WAN
236.	Theoretical and Applied Mathematics - Kongju National University	ROK
237.	Thesis Reprints - Cape Town	SA
238.	Tohoku Mathematical Journal - Sendai	J
239.	Trabalhos do Departamento de Matemática Univ. - San Paulo	BR
240.	Travaux de Mathématiques - Bruxelles	B
241.	Tsukuba Journal of Mathematics - University of Tsukuba	J
242.	UCNW Math. Preprints Prifysgol Cymru - University of Wales - Bangor	GB
243.	Ukrainii Matematiskii Journal - Kiev	RU
244.	Uniwersytetu Jagiellońskiego Acta Mathematica - Krakow	PL
245.	Verhandlungen der Naturforschenden Gesellschaft - Basel	CH
246.	Vierteljahrsschrift der Naturforschenden Gesellschaft - Zürich	CH
247.	Volumenes de Homenaje - Universidad Nacional del Sur Bahía Blanca	AR
248.	Yokohama Mathematical Journal - Yokohama	J
249.	Yugoslav Journal of Operations Research - Beograd	SRB
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ON OD-CHARACTERIZABILITY OF A CERTAIN ALTERNATING AND SYMMETRIC GROUP

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Abstract. In this paper we will show that if G is a finite group with the same order and degree pattern as the alternating group on 27 letters, then G is isomorphic to A_{27} . Furthermore we will show that there are three non-isomorphic finite groups with the same order and degree pattern as the symmetric group on 27 letters.

Keywords: prime graph, characterization, alternating groups, symmetric groups.

2000 Mathematics Subject Classification: 20D05.

1. Introduction and preliminaries

Let G be a finite group. The set of all the prime divisors of G is denoted by $\pi(G)$ and the set of elements order in G is denoted by $\pi_e(G)$. The prime graph or the Gruenberg–Kegel graph of G is a simple graph denoted by $\Gamma(G)$ or $GK(G)$ and is defines as follows. Vertices of $\Gamma(G)$ are elements of $\pi(G)$ and two distinct vertices p and q are joined by an edge if G possesses an element of order pq . In this case, we will write $p \sim q$, otherwise $p \not\sim q$ means that G does not have an element of order pq . The number of connected components of $\Gamma(G)$ is denoted by $t(G)$ and we write $\pi_i = \pi_i(G)$, $1 \leq i \leq t(G)$ for these connected components. In the case that $2 \in \pi(G)$ we choose $\pi_1(G)$ for the component containing 2. For a natural number n we write $\pi(n)$ for the set of all primes dividing n . With this notation $|G|$ can be expressed as a product of the numbers $m_1, m_2, \dots, m_{t(G)}$ where m_i s are positive integers with $\pi(m_i) = \pi_i$. The numbers m_i , $1 \leq i \leq t(G)$, are called the order components of G . We write $OC(G) = \{m_1, m_2, \dots, m_{t(G)}\}$ and call it the set of order components of G . The set of prime graph components of G is denoted by $T(G) = \{\pi_1(G), \pi_2(G), \dots, \pi_{t(G)}(G)\}$. Throughout this paper we use standard notation, in particular $\mathbb{P}$ denotes the set of all the prime numbers. All groups considered in this paper are finite.

Definition 1. Let G be a finite group with $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where $p_i \in P$ and $\alpha_i \in \mathbb{N}$. Let $\Gamma(G)$ be the prime graph of G . For $p \in \pi(G)$, the degree of the vertex p in $\Gamma(G)$ is the number of vertices adjacent to p and it is denoted by $\deg(p)$, i.e., $\deg(p) = |\{q \in \pi(G) \mid p \sim q\}|$. We choose an ordering such as $p_1 < p_2 < \dots < p_k$ in $\pi(G)$ and define $D(G) = (\deg(p_1), \deg(p_2), \dots, \deg(p_k))$. We call $D(G)$ the degree pattern of G .

Definition 2. Let k be a natural number. A group M is called k -fold OD-characterizable if there are exactly k non-isomorphic groups G such that $|G| = |M|$ and $D(G) = D(M)$. A 1-fold OD-characterizable group is called an OD-characterizable group. In particular, an OD-characterizable group is uniquely determined by its order and the degrees of its vertices.

Definition 3. Let $n \in \mathbb{N}$. A finite non-abelian simple group G is called a simple K_n -group if $|\pi(G)| = n$.

The Gruenberg–Kegel graph (prime graph) of a finite group G for the first time was defined in [5] and its significance can be found in many recent researches. Characterization of finite groups by their orders and degree pattern will help us to know the properties of almost simple groups. In [1] it is shown that the alternating group A_p , where p and $p - 2$ are primes, and all sporadic simple groups are OD-characterizable. It is also shown that certain groups of Lie type are OD-characterizable, but the projective symplectic group $SP_6(3)$ is 2-fold OD-characterizable. In [4] it is proved that all the simple K_4 -groups except A_{10} are OD-characterizable whereas A_{10} is 2-fold OD-characterizable. In [2] it is proved that all simple groups of order less than 10^8 except A_{10} and $U_4(2)$ are OD-characterizable while A_{10} and $U_4(2)$ are 2-fold OD-characterizable. In [3] some almost simple groups related to $L_2(49)$ are considered and it is proved that if $G = L_2(49) : V$, where V is isomorphic to the Klein fourgroup, is the full automorphism group of $L_2(49)$, then G is 9-fold OD-characterizable, while if W is a proper subgroup of V , then $H = L_2(49) : W$ is OD-characterizable. In [6] the authors proved that the alternating group of degree 16 is OD-characterizable. In this article our aim is to consider the OD-characterizability of the alternating and symmetric groups A_{27} and S_{27} . The importance of these groups is that both of them have connected prime graph. Figure 1 shows the prime graph of A_{27} .

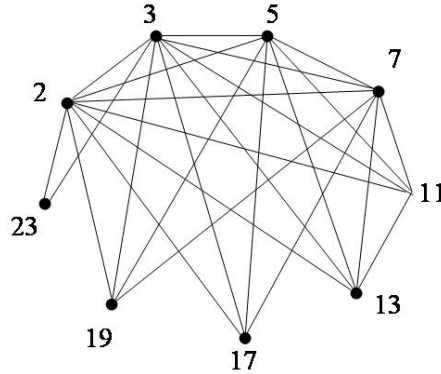


Figure 1. The prime graph of A_{27}

For the proof we need to know about finite non-abelian simple groups of order at most 19. In general for a given prime p , we denote by $\mathfrak{S}_p$ the set of all non-abelian simple groups G such that $p \in \pi(G) \subseteq \{1, 2, \dots, p\}$. Therefore Table 1 lists all $\mathfrak{S}_{19}$ groups. These are given in [7] and we reproduce them in Table 1 below.

Table 1: Non-abelian simple groups G with $\pi(G) \subseteq \{1, 2, 3, \dots, 19\}$

G	$ G $	$ Out(G) $
$\mathfrak{S}_5$	$\pi(G) \subseteq \{2, 3, 5\}$	
$\mathbb{A}_5 \cong L_2(4) \cong L_2(5)$	$2^2.3.5$	2
$\mathbb{A}_6 \cong L_2(9)$	$2^3.3^2.5$	4
$S_4(3) \cong U_4(2)$	$2^6.3^4.5$	2
$\mathfrak{S}_7$	$7 \in \pi(G) \subseteq \{2, 3, 5, 7\}$	
$L_2(7) \cong L_3(2)$	$2^3.3.7$	2
$L_2(8)$	$2^3.3^2.7$	3
$U_3(3)$	$2^5.3^3.7$	2
$\mathbb{A}_7$	$2^3.3^2.5.7$	2
$L_2(49)$	$2^4.3.5.7^2$	4
$U_3(5)$	$2^4.3^2.5^3.7$	6
$L_3(4)$	$2^6.3^2.5.7$	12
$\mathbb{A}_8 \cong L_4(2)$	$2^6.3^2.5.7$	2
$\mathbb{A}_9$	$2^6.3^4.5.7$	2
J_2	$2^7.3^3.5^2.7$	2
$\mathbb{A}_{10}$	$2^7.3^4.5^2.7$	2
$U_4(3)$	$2^7.3^6.5.7$	8
$S_4(7)$	$2^8.3^2.5^2.7^4$	2
$S_6(2)$	$2^9.3^4.5.7$	1
$O_8^+(2)$	$2^{12}.3^5.5^2.7$	6
$\mathfrak{S}_{11}$	$11 \in \pi(G) \subseteq \{2, 3, 5, 7, 11\}$	$ Out(G) $
$L_2(11)$	$2^2.3.5.11$	2
M_{11}	$2^4.3^2.5.11$	1
M_{12}	$2^6.3^3.5.11$	2
$U_5(2)$	$2^{10}.3^5.5.11$	2
M_{22}	$2^7.3^2.5.7.11$	2
$\mathbb{A}_{11}$	$2^7.3^4.5^2.7.11$	2
McL	$2^7.3^6.5^3.7.11$	2
HS	$2^9.3^2.5^3.7.11$	2
$\mathbb{A}_{12}$	$2^9.3^5.5^2.7.11$	2
$U_6(2)$	$2^{15}.3^6.5.7.11$	6
$\mathfrak{S}_{13}$	$13 \in \pi(G) \subseteq \{2, 3, 5, \dots, 13\}$	$ Out(G) $
$L_3(3)$	$2^4.3^3.13$	2
$L_2(25)$	$2^3.3.5^2.13$	4
$U_3(4)$	$2^6.3.5^2.13$	4
$S_4(5)$	$2^6.3^2.5^4.13$	2
$L_4(3)$	$2^7.3^6.5.13$	4

${}^2F_4(2)'$	$2^{11}.3^3.5^2.13$	2
$L_2(13)$	$2^2.3.7.13$	2
$L_2(27)$	$2^2.3^3.7.13$	6
$G_2(3)$	$2^6.3^6.7.13$	2
${}^3D_4(2)$	$2^{12}.3^4.7^2.13$	3
$Sz(8)$	$2^6.5.7.13$	3
$L_2(64)$	$2^6.3^2.5.7.13$	6
$U_4(5)$	$2^7.3^4.5^6.7.13$	4
$L_3(9)$	$2^7.3^6.5.7.13$	4
$S_6(3)$	$2^9.3^9.5.7.13$	2
$O_7(3)$	$2^9.3^9.5.7.13$	2
$G_2(4)$	$2^{12}.3^3.5^2.7.13$	2
$S_4(8)$	$2^{12}.3^4.5.7^2.13$	6
$O_8^+(3)$	$2^{12}.3^{12}.5^2.7.13$	24
$L_5(3)$	$2^9.3^{10}.5.11^2.13$	2
A_{13}	$2^9.3^5.5^2.7.11.13$	2
A_{14}	$2^{10}.3^5.5^2.7^2.11.13$	2
A_{15}	$2^{10}.3^6.5^3.7^2.11.13$	2
$L_6(3)$	$2^{11}.3^{15}.5.7.11^2.13^2$	4
Suz	$2^{13}.3^7.5^2.7.11.13$	2
A_{16}	$2^{14}.3^6.5^3.7^2.11.13$	2
Fi_{22}	$2^{17}.3^9.5^2.7.11.13$	2
$\mathfrak{S}_{17}$	$17 \in \pi(G) \subseteq \{2, 3, 5, \dots, 17\}$	$ Out(G) $
$L_2(17)$	$2^4.3^2.17$	2
$L_2(16)$	$2^4.3.5.17$	4
$S_4(4)$	$2^8.3^2.5^2.17$	4
He	$2^{10}.3^3.5^2.7^3.17$	2
$O_8^-(2)$	$2^{12}.3^4.5.7.17$	2
$L_4(4)$	$2^{12}.3^4.5^2.7.17$	4
$S_8(2)$	$2^{16}.3^5.5^2.7.17$	1
$U_4(4)$	$2^{12}.3^2.5^3.13.17$	4
$U_3(17)$	$2^6.3^4.7.13.17^3$	6
$O_{10}(2)$	$2^{20}.3^6.5^2.7.11.17$	2
$L_2(13^2)$	$2^3.3.5.7.13^2.17$	4
$S_4(13)$	$2^6.3^2.5.7^2.13^4.17$	2
$L_3(16)$	$2^{12}.3^2.5^2.7.13.17$	24
$S_6(4)$	$2^{18}.3^4.5^3.7.13.17$	2
$O_8^+(4)$	$2^{24}.3^5.5^4.7.13.17^2$	12
$F_4(2)$	$2^{24}.3^6.5^2.7^2.13.17$	2
$\mathbb{A}_{17}$	$2^{14}.3^6.5^3.7^2.11.13.17$	2
$\mathbb{A}_{18}$	$2^{15}.3^8.5^3.7^2.11.13.17$	2
$\mathfrak{S}_{19}$	$19 \in \pi(G) \subseteq \{2, 3, 5, \dots, 19\}$	$Out(G)$

$L_2(19)$	$2^2.3^2.5.19$	2
$L_3(7)$	$2^5.3^2.5^3.19$	6
$U_3(8)$	$2^9.3^4.7.19$	18
$U_3(19)$	$2^5.3^2.5^2.7^3.19^3$	2
$L_4(7)$	$2^9.3^4.5^2.7.19$	4
J_3	$2^7.3^5.5.17.19$	2
J_4	$2^3.3.5.7.11.19$	1
$L_3(11)$	$2^4.3.5^2.7.11^3.19$	2
HN	$2^{14}.3^6.5^6.7.11.19$	2
$U_4(8)$	$2^{18}.3^7.5.7^2.13.19$	6
$\mathbb{A}_{19}$	$2^{15}.3^8.5^3.7^2.11.13.17.19$	2
$\mathbb{A}_{20}$	$2^{17}.3^8.5^4.7^2.11.13.17.19$	2
$\mathbb{A}_{21}$	$2^{17}.3^9.5^4.7^3.11.13.17.19$	2
$\mathbb{A}_{22}$	$2^{18}.3^9.5^4.7^3.11^2.13.17.19$	2
${}^2E_6(2)$	$2^{36}.3^9.5^2.7^2.11.13.17.19$	6

2. Main result

At first, we consider the alternating group on 27 letters, $\mathbb{A}_{27}$, and will show it is OD-characterizable. It is easy to verify that $|\mathbb{A}_{27}| = 2^{22}.3^{13}.5^6.7^3.11^2.13^2.17.19.23$ and $D(\mathbb{A}_{27}) = (8, 8, 7, 7, 5, 5, 4, 4, 2)$. Let G be a finite group such that $|G| = |\mathbb{A}_{27}|$, $\pi(G) = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$, $D(G) = (8, 8, 7, 7, 5, 5, 4, 4, 2)$, where $D(G)$ is the degree pattern of G in its prime graph. Since $\deg(2) = 8$ we deduce that the prime 2 is joined to all vertices in $\pi(G) - \{2\}$, hence $\Gamma(G)$ is a connected graph. Now it is easy to see that $\Gamma(G) = \Gamma(\mathbb{A}_{27})$ and that

$$\pi_e(G) \supseteq \left\{ \begin{array}{l} 2, 3, 5, 7, 11, 13, 17, 19, 23, 6, 10, 14, 22, 26, 34, 38, 46, 15, 27, \\ 33, 39, 51, 57, 69, 35, 55, 65, 85, 95, 77, 91, 119, 133, 143 \end{array} \right\}.$$

Lemma 1. *Let K be the maximal normal solvable subgroup of G . Then K is a $\{2, 3\}$ -group. In particular, G is a non-solvable group.*

Proof. Let p be a prime divisor of the order of K and S_p be a Sylow p -subgroup of K . Since $K \trianglelefteq G$, by the Frattini argument we obtain $G = KN_G(S_p)$. Next we distinguish several cases:

Case (1) $p = 23$. In this case, S_{23} is a cyclic group of order 23, hence $\overline{N} = \frac{N_G(S_{23})}{C_G(S_{23})}$ is isomorphic to a subgroup of $\mathbb{Z}_{22}$. Therefore, $|\overline{N}|$ is a divisor of 22. But $\deg(23)$ in $\Gamma(G)$ is 2 and 23 is joined to 2 and 3 only, hence $C_G(S_{23})$ is a $\{2, 3, 23\}$ -group. Thus, $|N_G(S_{23})| = 2^\alpha.3^\beta.11^\gamma.23$, where $0 \leq \alpha \leq 22$, $0 \leq \beta \leq 13$, $0 \leq \gamma \leq 2$ and from $G = KN_G(S_{23})$ we deduce that $19 \mid |K|$. Since K is assumed to be solvable, we can consider a $\{19 - 23\}$ -Hall subgroup of K which has order 19.23

and should be cyclic. This implies $19 \sim 23$ which is a contradiction because $\mathbb{A}_{27}$ has no element of order $19 \cdot 23$. We conclude that $23 \nmid |K|$.

Case (2) $p = 19$ or 17 . This case is treated with the same technique as in case (1) and a contradiction is obtained.

Case (3) $p = 13$ or 11 . First, we consider $p = 13$, the case $p = 11$ is treated similarly. If a Sylow 13-subgroup of K has order 13, then with the same techniques as in case (1) we obtain a contradiction, therefore we may assume that a Sylow 13-subgroup of K has order $|S_{13}| = 13^2$. We have $\bar{N} = \frac{N_G(S_{23})}{C_G(S_{23})} \leq \text{Aut}(S_{13})$. If S_{13} is a cyclic group, then $|\text{Aut}(S_{13})| = 13 \cdot 12$ and if S_{13} is elementary abelian, then $|\text{Aut}(S_{13})| = |GL_2(13)| = 2^5 \cdot 3^2 \cdot 7 \cdot 13$. Therefore, in any case we deduce that $\bar{N}$ is a $\{2, 3, 7, 13\}$ -group. Since $\deg(13) = 5$ and $13 \sim 2$, $13 \sim 3$, $13 \sim 5$, $13 \sim 7$, $13 \sim 11$, we obtain that $N_G(S_{23})$ is a $\{2, 3, 5, 7, 11, 13\}$ -group, hence $19 \nmid |K|$. Again considering a $\{19, 13\}$ -Hall subgroup in K results in a contradiction. Therefore, $|K|$ cannot be divisible by 13 or 11.

Case (4) $p = 7$. In this case, a Sylow 7-subgroup of K has order 7, 7^2 or 7^3 . If $|S_7| = 7$ or 7^2 , then using the same techniques as in case (1) and case (3) we will obtain a contradiction. Hence, suppose $|S_7| = 7^3$. From $G = KN_G(S_7)$ and the fact that $23 \nmid |K|$, we deduce that $23 \mid |N_G(S_7)|$, so S_7 is normalized by an element σ of G with order 23. Since G has no element of order $23 \cdot 7$, $\langle \sigma \rangle$ should act fixed-point-freely on S_7 , implying $23 \mid 7^3 - 1$, a contradiction.

Case (5) $p = 5$. In this case, $|S_5| = 5^\alpha$, $1 \leq \alpha \leq 6$. By case (1), we have $23 \nmid |K|$, and from $G = KN_G(S_5)$ we deduce that S_5 is normalized by an element of order 23, hence $23 \mid 5^\alpha - 1$. But considering all the numbers $1 \leq \alpha \leq 6$ impossibility of the last divisibility is proved.

Therefore, we have proved that $|K|$ can only be divisible by 2 and 3 and part of the lemma is proved. If G is a solvable group, then it has a $\{23, 19\}$ -Hall subgroup of order $23 \cdot 19$ implying that $23 \sim 19$, a contradiction. ■

In the following lemma, we keep fixed the notation used in Lemma 1.

Lemma 2. $\frac{G}{K}$ is an almost simple group, i.e. $S \leq \frac{G}{K} \leq \text{Aut}(S)$ where S is a simple group isomorphic to $\mathbb{A}_{27}$.

Proof. Let us put $\bar{G} = \frac{G}{K}$ and $\bar{S} = \text{Soc}(\bar{G})$ where $\text{Soc}(\bar{G})$ denotes the socle of the group $\bar{G}$, i.e., the subgroup of $\bar{G}$ generated by the set of all the minimal normal subgroups of $\bar{G}$. Then, $\bar{S} \cong S_1 \times S_2 \times \dots \times S_n$ where S_i 's are finite non-abelian simple groups. We assume $n \geq 2$ and derive a contradiction.

If $23 \mid |\bar{S}|$, then the order of only one S_i is divisible by 23. We assume $23 \mid |S_1|$. Because only $23 \sim 2$ and $23 \sim 3$ hold in the prime graph of G , S_i , $i \geq 2$, must be $\{2, 3\}$ -groups contradicting simplicity of S_i . Hence, $23 \nmid |\bar{S}|$.

Since K is a $\{2, 3\}$ -group, we have $23 \mid |\bar{G}|$. From $\frac{N_{\bar{G}}(\bar{S})}{C_{\bar{G}}(\bar{S})} = \frac{\bar{G}}{C_{\bar{G}}(\bar{S})} \leq \text{Aut}(\bar{S})$, we deduce that $23 \in \pi(\bar{G}) \subseteq \pi(\text{Aut}(\bar{S}))$. But $|\text{Aut}(\bar{S})| = |\bar{S}| \cdot |\text{Out}(\bar{S})|$, hence $23 \mid |\text{Out}(\bar{S})|$. Let $P_1, P_2, \dots, P_k$ be non-isomorphic simple groups among $S_1, \dots, S_n$

such that $\bar{S} \cong P_1^{t_1} \times P_2^{t_2} \times \dots \times P_k^{t_k}$, where $t_1 + t_2 + \dots + t_k = n$. We have $\text{Out}(\bar{S}) = \text{Out}(P_1^{t_1}) \times \text{Out}(P_2^{t_2}) \times \dots \times \text{Out}(P_k^{t_k})$, and from $23 \nmid |\bar{S}|$ we have $23 \nmid |P_i|$, for all $1 \leq i \leq k$, so P_i s are simple $\{2, 3, 5, 7, 11, 13, 17, 19\}$ -group and by Table 1 we obtain $23 \nmid |\text{Out}(P_i)|$, $1 \leq i \leq k$. But $\text{Aut}(P_i^{t_i}) \cong P_i \text{Wr} S_{t_i}$, and from $23 \mid |\text{Aut}(P_1^{t_1})| = |P_1|^{t_1} \cdot t_1!$ we obtain $23 \mid t_1!$ which implies $t_1 \geq 23$. Therefore, $(23!)_2 \times 2^{23} = 2^{42}$ must divide the order of G , a contradiction. Therefore, $n = 1$ and $\bar{S}$ is a simple group. ■

Next, we consider the fact that $|\bar{G}| \mid |\text{Aut}(\bar{S})|$, $|\bar{G}| = 2^{22} \cdot 3^{13} \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23$, and $\bar{S}$ an S_{19} -group. Now, by Table 1, we obtain $\bar{S} \cong \mathbb{A}_{27}$ and $\bar{S} \leq \frac{G}{K} \leq \text{Aut}(\bar{S})$.

Theorem 1. *G is isomorphic to $\mathbb{A}_{27}$.*

Proof. By Lemma 2, $\mathbb{A}_{27} \leq \frac{G}{K} \leq \text{Aut}(\mathbb{A}_{27}) = \mathbb{S}_{27}$, therefore $\frac{G}{K} \cong \mathbb{A}_{27}$ or $\mathbb{S}_{27}$. If $\frac{G}{K} \cong \mathbb{A}_{27}$, then $|K| = 1$, because $|G| = |\mathbb{A}_{27}|$, so $G \cong \mathbb{A}_{27}$. If $\frac{G}{K} \cong \mathbb{S}_{27}$, then $2 \mid |K| = 1$, which is impossible. ■

Next, we consider OD-characterizability of the symmetric group $\mathbb{S}_{27}$. The degree pattern of $\mathbb{S}_{27}$ is the same as the degree pattern of $\mathbb{A}_{27}$, i.e., $D(\mathbb{S}_{27}) = (8, 8, 7, 7, 5, 5, 4, 4, 2)$, but $|\mathbb{S}_{27}| = 2^{23} \cdot 3^{13} \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23$. We assume that G is a finite group with $|G| = |\mathbb{S}_{27}|$ and $D(G) = D(\mathbb{S}_{27})$. As in the case of $\mathbb{A}_{27}$, if K is the maximal normal solvable subgroup of G , then we can prove that K is a $\{2, 3\}$ -group. Also, if we set $\bar{G} = \frac{G}{K}$ and $\bar{S} = \text{Soc}(\bar{G})$, then similar techniques yields $\bar{S} \cong \mathbb{A}_{27}$ and $\bar{S} \leq \bar{G} \leq \text{Aut}(\bar{S})$.

Theorem 2. *The symmetric group $\mathbb{S}_{27}$ is 3-fold OD-characterizable.*

Proof. Since $\mathbb{A}_{27} \leq \frac{G}{K} \leq \text{Aut}(\mathbb{A}_{27}) = \mathbb{S}_{27}$ we will obtain $\frac{G}{K} = \mathbb{A}_{27}$ or $\mathbb{S}_{27}$. If $\frac{G}{K} = \mathbb{S}_{27}$, then we obtain $|K| = 1$ and $G = \mathbb{S}_{27}$. If $\frac{G}{K} = \mathbb{A}_{27}$, then $|K| = 2$, hence $K \leq Z(G)$. Since K is the maximal normal solvable subgroup of G we obtain $K = Z(G)$ and G is the central extension of $\mathbb{Z}_2$ by $\mathbb{A}_{27}$. Therefore, we have two possibilities for G , i.e., $G = \mathbb{Z}_2 \times \mathbb{A}_{27}$ or $G = \mathbb{Z}_2 \cdot \mathbb{A}_{27}$ the covering group of $\mathbb{A}_{27}$. Both groups have the same degree pattern and orders as $\mathbb{S}_{27}$ and theorem is proved. ■

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Accepted: 19.04.2010

SOME UNIQUENESS RESULTS ON MEROMORPHIC FUNCTIONS SHARING TWO SETS

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Abstract. Using the notion of weighted sharing, we prove some uniqueness theorems of meromorphic functions that share two sets. The results in this paper improve and supplement some recent ones of the first author and consequently provide a better answer to the famous question of Gross than that was given previously.

Keywords: meromorphic functions, uniqueness, weighted sharing, shared set.

2000 Mathematics Subject Classification: 30D35.

1. Introduction, definitions and results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane $\mathbb{C}$. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any non-constant meromorphic function $h(z)$ we denote by $S(r, h)$ any quantity satisfying

$$S(r, h) = o(T(r, h)) \quad (r \rightarrow \infty, r \notin E).$$

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f - a$ and $g - a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f - a$ and $g - a$ have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and $E_f(S) = \bigcup_{a \in S} \{z : f(z) - a = 0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity, the set $\bigcup_{a \in S} \{z : f(z) - a = 0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$, we say that f and g share the set S CM. On the other hand, if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM. Evidently, if S contains only one element, then it coincides with the usual definition of CM (respectively, IM) shared values.

Let m be a positive integer or infinity and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $E_m(a; f)$ the set of all a -points of f with multiplicities not exceeding m , where an a -point is counted according to its multiplicity. If, for some $a \in \mathbb{C} \cup \{\infty\}$, $E_\infty(a; f) = E_\infty(a; g)$, we say that f, g share the value a CM. For a set S of distinct elements of $\mathbb{C}$, we define $E_m(S, f) = \bigcup_{a \in S} E_m(a, f)$. The condition

$E_m(S, f) = E_m(S, g)$ obviously implies $E_j(S, f) = E_j(S, g)$ for all $1 \leq j \leq m$.

Inspired by the Nevanlinna's three and four values theorems, in 1970s F. Gross and C.C. Yang started to study the similar but more general questions of two functions that share sets of distinct elements instead of values. For instance, they proved that if f and g are two non-constant entire functions and S_1, S_2 and S_3 are three distinct finite sets such that $f^{-1}(S_i) = g^{-1}(S_i)$ for $i = 1, 2, 3$, then $f \equiv g$. In 1976, F. Gross proposed the following question in [8]:

Question A *Can one find two finite sets S_j ($j = 1, 2$) such that any two non-constant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical ?*

In [8], Gross wrote: *If the answer of Question A is affirmative it would be interesting to know how large both sets would have to be ?*

Yi [21] and independently Fang and Xu [7] gave the same answer in this direction.

In 2003, Lin and Yi posed the following question.

Question B ([19]) *Can one find two finite sets S_j ($j = 1, 2$) such that any two non-constant meromorphic functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical ?*

Gradually, the research on *Question B* gained pace and today it has become one of the most prominent branches of the uniqueness theory. For the last two decades several attempts have been made by different authors to consider the shared value problems relative to a meromorphic function sharing two sets and at the same time give affirmative answers to *Question B* under weaker hypothesis {see [1]-[7], [10], [14]-[21], [28]-[28], [29]-[30]}.

In 1994, Yi [20] gave an affirmative answer to *Question B* and proved that there exist two finite sets S_1 (with 2 elements) and S_2 (with 9 elements) such that any two non-constant meromorphic functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical.

Inspired by *Question B*, we will consider the uniqueness of two non-constant meromorphic functions satisfying $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$. According to what we know by now, perhaps this type of results were first obtained by Li and Yang in [17], where they proved that there exists one finite set S with 15 elements such that any two non-constant meromorphic functions satisfying $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$ must be identical. In 1995, Yi [21] and independently Li and Yang [17] proved that there exists a set S of 11 elements such that any two non-constant meromorphic functions with $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$ must be identical. In 1997 Fang and Guo in [6] exhibited a set S of nine elements with this property.

In 2002, Yi [25] proved the following result in which he not only reduced the cardinalities of the set S but also relaxed the sharing of the poles from CM to IM.

Theorem A. ([25]; see also [29]) *Let n be a positive integer such that $n \geq 8$, and let a, b be two nonzero complex numbers satisfying $ab^{n-2} \neq 2$. Then the polynomial*

$$(1.1) \quad P(w) = aw^n - n(n-1)w^2 + 2n(n-2)bw - (n-1)(n-2)b^2$$

has only simple zeros. Let $S = \{w \mid P(w) = 0\}$. If f and g are two non-constant meromorphic functions satisfying $E_f(S) = E_g(S)$ and $\overline{E}_f(\{\infty\}) = \overline{E}_g(\{\infty\})$ then $f \equiv g$.

Dealing with the question of Gross in [6], Fang and Lahiri obtained a unique range set S with smaller cardinalities than that obtained previously imposing some restrictions on the poles of f and g .

Theorem B. ([6]) *Let $S = \{z : z^n + az^{n-1} + b = 0\}$ where $n (\geq 7)$ is an integer and a and b are two nonzero constants such that $z^n + az^{n-1} + b = 0$ has no multiple root. If f and g are two non-constant meromorphic functions having no simple poles such that $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$ then $f \equiv g$.*

Let $S = \{z : z^7 - z^6 - 1 = 0\}$ and

$$f = \frac{e^z + e^{2z} + \dots + e^{6z}}{1 + e^z + \dots + e^{6z}}, \quad g = \frac{1 + e^z + \dots + e^{5z}}{1 + e^z + \dots + e^{6z}}$$

Obviously, $f = e^z g$, $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$ but $f \not\equiv g$. So, for the validity of *Theorem B*, f and g must not have any simple pole.

In 2001, an idea of gradation of sharing known as weighted sharing has been introduced by I. Lahiri in [12], [13] which measures how close a shared value is to being shared CM or to being shared IM. In the following definition we explain the notion.

Definition 1.1 [12], [13] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a -points of f , where an a -point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k .

We write f, g share (a, k) to mean that f, g share the value a with weight k . Clearly if f, g share (a, k) then f, g share (a, p) for any integer p , $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2 [12] Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a nonnegative integer or ∞ . We denote by $E_f(S, k)$ the set $\bigcup_{a \in S} E_k(a; f)$.

Clearly, $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.

Using the notion of weighted sharing of sets, Lahiri [14] proved the following theorem, which improved *Theorem B*.

Theorem C. ([14]) *Let S be defined as in Theorem B and $n (\geq 7)$ be an integer. If for two non-constant meromorphic functions f and g , $\Theta(\infty; f) + \Theta(\infty; g) > 1$, $E_f(S, 2) = E_g(S, 2)$ and $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$ then $f \equiv g$.*

Recently, the first author [3] has generalized *Theorem C* by investigating the problem of further relaxation of the nature of sharing the set $\{\infty\}$ and obtained the following result.

Theorem D. ([3]) *Let S be defined as in Theorem B and $n (\geq 7)$ be an integer. If, for two non-constant meromorphic functions f and g , $\Theta(\infty; f) + \Theta(\infty; g) > 1 + \frac{29}{6nk + 6n - 5}$, $E_f(S, 2) = E_g(S, 2)$ and $E_f(\{\infty\}, k) = E_g(\{\infty\}, k)$, where $0 \leq k < \infty$, then $f \equiv g$.*

In the mean time, the first author [1] has improved *Theorem B* by relaxing the nature of sharing the set S and proved the following result.

Theorem E. *Let S be defined as in Theorem B. If for two non-constant meromorphic functions f and g , $\Theta(\infty; f) > \frac{1}{2}$, $\Theta(\infty; g) > \frac{1}{2}$ and $E_3(S, f) = E_3(S, g)$, $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$, then $f \equiv g$.*

In the paper, we consider a new range set different from those mentioned earlier and with the help of the same we will improve and supplement *Theorems D* and *E*.

The following theorems are the main results of the paper.

Theorem 1.1 *Let*

$$S = \left\{ z : \frac{(n-1)(n-2)}{4} z^n - \frac{n(n-2)}{2} z^{n-1} + \frac{n(n-1)}{4} z^{n-2} - 1 = 0 \right\},$$

where $n (\geq 7)$ is an integer. If, for two non-constant meromorphic functions f and g , $E_f(S, 2) = E_g(S, 2)$ and $E_f(\{\infty\}, k) = E_g(\{\infty\}, k)$, where $0 \leq k < \infty$ and $\min\{\Theta_f, \Theta_g\} > 4 - \frac{n}{2} + \frac{3}{nk+n-1}$, then $f \equiv g$, where $\Theta_f = 2 \Theta(0; f) + 2 \Theta(1; f) + \Theta(\infty; f)$ and Θ_g can be similarly defined.

Corollary 1.1 *Let S be given as in Theorem 1.1. If, for two non-constant meromorphic functions f and g , $E_f(S, 2) = E_g(S, 2)$, $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$ and $\min\{\Theta_f, \Theta_g\} > 4 - \frac{n}{2}$, where Θ_f and Θ_g have the same meaning as in Theorem 1.1, then $f \equiv g$.*

Theorem 1.2 *Let*

$$S = \left\{ z : \frac{(n-1)(n-2)}{4} z^n - \frac{n(n-2)}{2} z^{n-1} + \frac{n(n-1)}{4} z^{n-2} - 1 = 0 \right\},$$

where $n (\geq 8)$ is an integer. If for two non-constant meromorphic functions f and g $E_f(S, 2) = E_g(S, 2)$ and $E_f(\{\infty\}, 2) = E_g(\{\infty\}, 2)$, then $f \equiv g$.

Theorem 1.3 *Let S be defined as in Theorem 1.1. If for two non-constant meromorphic functions f and g , $E_3(S, f) = E_3(S, g)$, $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$ and $\min\{\Theta_f, \Theta_g\} > 4 - \frac{n}{2}$, where Θ_f and Θ_g have the same meaning as in Theorem 1.1, then $f \equiv g$.*

Theorem 1.4 *Let S be defined as in Theorem 1.1. If for two non-constant meromorphic functions f and g , $E_m(S, f) = E_m(S, g)$, $E_f(\{\infty\}, k) = E_g(\{\infty\}, k)$, where $0 \leq k < \infty$, $m \geq 4$ and $\min\{\Theta_f, \Theta_g\} > 4 - \frac{n}{2} + \frac{3}{nk+n-1}$, where Θ_f and Θ_g have the same meaning as in Theorem 1.1 then $f \equiv g$.*

It is assumed that the readers are familiar with the standard definitions and notations of the value distribution theory as those are available in [9]. We are still going to explain some notations as these are used in the paper.

Definition 1.3 [11] For a value a in the extended complex plane, we denote by $N(r, a; f | = 1)$ the reduced counting function of simple a points of f in $|z| < r$. Denote by $N(r, a; f | \leq m)$ ($N(r, a; f | \geq m)$, respectively) the counting function of those a -points of f in $|z| < r$, where the multiplicity of each point is not greater (not less, respectively) than m , m is a positive integer, and each point is counted according to its multiplicity. Denote by $N(r, a; f | < m)$ ($N(r, a; f | > m)$, respectively) the counting function of those a -points of f in $|z| < r$, where the multiplicity of each point is less (greater, respectively) than m , and each point is counted according to its multiplicity. Denote by $\overline{N}(r, a; f | \leq m)$, $\overline{N}(r, a; f | \geq m)$, $\overline{N}(r, a; f | < m)$, $\overline{N}(r, a; f | > m)$ the reduced forms of $N(r, a; f | \leq m)$, $N(r, a; f | \geq m)$, $N(r, a; f | < m)$, $N(r, a; f | > m)$ respectively.

Definition 1.4 Let f and g be two non-constant meromorphic functions such that f and g share a value a IM where $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a -point of f with multiplicity p , an a -point of g with multiplicity q . We denote by $\overline{N}_L(r, a; f)$ ($\overline{N}_L(r, a; g)$) the counting function of those a -points of f and g where $p > q$ ($q > p$), each a -point is counted only once.

Definition 1.5 Let f and g be two non-constant meromorphic functions and m be a positive integer such that $E_m(a; f) = E_m(a; g)$ where $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a -point of f with multiplicity $p > 0$, an a -point of g with multiplicity $q > 0$. We denote by $\overline{N}_L^{(m)}(r, a; f)$ ($\overline{N}_L^{(m)}(r, a; g)$) the counting function of those common a -points of f and g , where $p > q$ ($q > p$), each a -point is counted only once.

Definition 1.6 [13] We denote by $N_2(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f | \geq 2)$.

Definition 1.7 Let m be a positive integer. Also let z_0 be a zero of $f(z) - a$ of multiplicity p and a zero of $g(z) - a$ of multiplicity q . We denote by $\overline{N}_{f \geq m+1}(r, a; f | g \neq a)$ ($\overline{N}_{g \geq m+1}(r, a; g | f \neq a)$) the reduced counting functions of those a -points of f and g , where $p \geq m+1$ and $q = 0$ ($q \geq m+1$ and $p = 0$).

Definition 1.8 [8], [9] Let f, g share $(a, 0)$. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a -points of f whose multiplicities differ from the multiplicities of the corresponding a -points of g .

Clearly, $\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$. For $E_m(a; f) = E_m(a; g)$ we can define $\overline{N}_*(r, a; f, g)$ in a similar manner and we note that here $\overline{N}_*(r, a; f, g) = \overline{N}_L^{(m)}(r, a; f) + \overline{N}_L^{(m)}(r, a; g) + \overline{N}_{f \geq m+1}(r, a; f | g \neq a) + \overline{N}_{g \geq m+1}(r, a; g | f \neq a)$.

2. Lemmas

In this section, we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. Henceforth we shall denote by H and V the following two functions

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1} \right)$$

and

$$V = \left(\frac{F'}{F-1} - \frac{F'}{F} \right) - \left(\frac{G'}{G-1} - \frac{G'}{G} \right) = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}.$$

Lemma 2.1 [13] *If F, G be two non-constant meromorphic functions such that they share $(1, 1)$ and $H \not\equiv 0$, then*

$$N(r, 1; F | = 1) = N(r, 1; G | = 1) \leq N(r, H) + S(r, F) + S(r, G).$$

Lemma 2.2 [19] *If F, G be two non-constant meromorphic functions such that $E_1(1; F) = E_1(1; G)$ and $H \not\equiv 0$ then*

$$N(r, 1; F | = 1) \leq N(r, H) + S(r, F) + S(r, G).$$

Lemma 2.3 *Let f and g be two non-constant meromorphic functions sharing $(1, m)$, where $1 \leq m < \infty$. Then*

$$\begin{aligned} \overline{N}(r, 1; f) + \overline{N}(r, 1; g) - N(r, 1; f | = 1) + \left(m - \frac{1}{2}\right) \overline{N}_*(r, 1; f, g) \\ \leq \frac{1}{2} [N(r, 1; f) + N(r, 1; g)] \end{aligned}$$

Proof. Let z_0 be a 1- point of f of multiplicity p and a 1-point of g of multiplicity q . Since f, g share $(1, m)$, we note that the 1-points of f and g up to multiplicity m are same. When $p = q = 1$, z_0 is counted once, both in left and right hand side of the above inequality but when $2 \leq p = q \leq m$, z_0 is counted 2 times in the left hand side of the above inequality whereas it is counted p times in the right hand side of the above inequality. If $p = m + 1$, then the possible values of q are as follows. (i) $q = m + 1$, (ii) $q \geq m + 2$. When $p = m + 2$, then q can take the following possible values (i) $q = m + 1$, (ii) $q = m + 2$, (iii) $q \geq m + 3$. Similar explanations hold if we interchange p and q . Clearly, when $p = q \geq m + 1$, z_0 is counted 2 times in the left hand side and $p \geq m + 1$ times in the right hand side of the above inequality. If $p > q \geq m + 1$, in view of *Definition 1.8* we know z_0 is counted $m + \frac{3}{2}$ times in the left hand side of the inequality and $\frac{p+q}{2} \geq m + \frac{3}{2}$ times in the right hand side of the above inequality. If $q > p$, we can explain similarly. Hence the lemma follows. ■

Lemma 2.4 *Let f and g be two non-constant meromorphic functions such that $E_m(1; f) = E_m(1; g)$, where $1 \leq m < \infty$. Then*

$$\begin{aligned} \overline{N}(r, 1; f) + \overline{N}(r, 1; g) - N(r, 1; f | = 1) + \left(\frac{m}{2} - \frac{1}{2}\right) \{ \overline{N}_{f \geq m+1}(r, 1; f | g \neq 1) \\ + \overline{N}_{g \geq m+1}(r, 1; g | f \neq 1) \} + \left(m - \frac{1}{2}\right) \{ \overline{N}_L^{(m)}(r, 1; f) + \overline{N}_L^{(m)}(r, 1; g) \} \\ \leq \frac{1}{2} [N(r, 1; f) + N(r, 1; g)] \end{aligned}$$

Proof. By the condition that $E_m(1; f) = E_m(1; g)$, we note that every common zero of $f - 1$ and $g - 1$ up to multiplicity m has the same multiplicities related to f and g . Let z_0 be a 1-point of f with multiplicity p and a 1-point of g with multiplicity q . If $p = m + 1$, then the possible values of q are as follows: (i) $q = m + 1$; (ii) $q \geq m + 2$; (iii) $q = 0$. Similarly, if $p = m + 2$, the possible values of q are as follows: (i) $q = m + 1$; (ii) $q = m + 2$; (iii) $q \geq m + 3$; (iv) $q = 0$. If $p \geq m + 3$, we can find the possible values of q similarly. Now, Lemma 2.4 follows from the above explanation. ■

Lemma 2.5 Suppose that F, G share $(1, 0), (\infty, 0)$ and a is a complex number satisfying $a \neq 0, 1$. If $H \not\equiv 0$ then

$$N(r, H) \leq \overline{N}(r, 0; F | \geq 2) + \overline{N}(r, 0; G | \geq 2) + \overline{N}(r, a; F | \geq 2) + \overline{N}(r, a; G | \geq 2) \\ + \overline{N}_*(r, 1; F, G) + \overline{N}_*(r, \infty; F, G) + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G'),$$

where $\overline{N}_0(r, 0; F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)(F-a)$ and $\overline{N}_0(r, 0; G')$ is similarly defined.

Proof. By the definition of H we verify that the possible poles of H result from the following six cases: (i) The multiple zeros of F and G . (ii) The multiple a -points of F and G . (iii) Those common poles of F and G , where each such pole of F and G has different multiplicities related to F and G . (iv) Those common 1-points of F and G , where each such point has different multiplicities related to F and G . (v) The zeros of F' which are not zeros of $F(F-1)(F-a)$. (vi) The zeros of G' which are not zeros of $G(G-1)(G-a)$. Now proceeding as in the proof of Lemma 2.4, we can get the result of the lemma. ■

Lemma 2.6 Let $E_m(1; F) = E_m(1; G)$, let F, G share $(\infty, 0)$ and let a be a complex number satisfying $a \neq 0, 1$. If $H \not\equiv 0$, then

$$N(r, H) \leq \overline{N}(r, 0; F | \geq 2) + \overline{N}(r, 0; G | \geq 2) + \overline{N}(r, a; F | \geq 2) + \overline{N}(r, a; G | \geq 2) \\ + \overline{N}_L^{(m)}(r, 1; F) + \overline{N}_L^{(m)}(r, 1; G) + \overline{N}_{F \geq m+1}(r, 1; F | G \neq 1) \\ + \overline{N}_{G \geq m+1}(r, 1; G | F \neq 1) + \overline{N}_*(r, \infty; F, G) + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G'),$$

where $\overline{N}_0(r, 0; F')$ and $\overline{N}_0(r, 0; G')$ has the same meaning as in Lemma 2.5.

Proof. The proof is obvious. ■

Lemma 2.7 [18] Let f be a non-constant meromorphic function and $P(f) = a_0 + a_1f + a_2f^2 + \dots + a_nf^n$, where $a_0, a_1, a_2, \dots, a_n$ are constants and $a_n \neq 0$. Then $T(r, P(f)) = nT(r, f) + O(1)$.

Next, we set

$$(2.1) \quad F = \frac{(n-1)(n-2)}{4}f^n - \frac{n(n-2)}{2}f^{n-1} + \frac{n(n-1)}{4}f^{n-2},$$

$$(2.2) \quad G = \frac{(n-1)(n-2)}{4}g^n - \frac{n(n-2)}{2}g^{n-1} + \frac{n(n-1)}{4}g^{n-2},$$

where f, g are two non-constant meromorphic functions and $n \geq 3$ is an integer.

Lemma 2.8 Let F, G be given by (2.1) and (2.2), where $n \geq 7$ is an integer and $H \not\equiv 0$. Suppose α_1 and α_2 are the roots of the equation

$$\frac{(n-1)(n-2)}{4}z^2 - \frac{n(n-2)}{2}z + \frac{n(n-1)}{4} = 0.$$

If F, G share $(1, m)$ and f, g share (∞, k) , where $2 \leq m < \infty$. Then, for a complex number $a (\neq 0, 1)$,

$$\begin{aligned} \frac{3n}{2} \{T(r, f) + T(r, g)\} &\leq 2\{\overline{N}(r, 0; f) + \overline{N}(r, 0; g)\} + N_2(r, \alpha_1; f) \\ &\quad + N_2(r, \alpha_1; g) + N_2(r, \alpha_2; f) + N_2(r, \alpha_2; g) \\ &\quad + N_2(r, a; F) + N_2(r, a; G) + \overline{N}(r, \infty; f) \\ &\quad + \overline{N}(r, \infty; g) + \overline{N}_*(r, \infty; f, g) - \left(m - \frac{3}{2}\right) \\ &\quad \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g). \end{aligned}$$

Proof. By the second fundamental theorem we get

$$\begin{aligned} (2.3) \quad 2\{T(r, F) + T(r, G)\} &\leq \overline{N}(r, 1; F) + \overline{N}(r, 0; F) + \overline{N}(r, a; F) \\ &\quad + \overline{N}(r, \infty; F) + \overline{N}(r, 1; G) + \overline{N}(r, 0; G) + \overline{N}(r, a; G) + \overline{N}(r, \infty; G) \\ &\quad - N_0(r, 0; F') - N_0(r, 0; G') + S(r, F) + S(r, G). \end{aligned}$$

Using Lemmas 2.1, 2.3, 2.5 and 2.7 we see that

$$\begin{aligned} (2.4) \quad \overline{N}(r, 1; F) + \overline{N}(r, 1; G) &\leq \frac{1}{2} [N(r, 1; F) + N(r, 1; G)] \\ &\quad + N(r, 1; F | = 1) - \left(m - \frac{1}{2}\right) \overline{N}_*(r, 1; F, G) \\ &\leq \frac{n}{2} \{T(r, f) + T(r, g)\} + \overline{N}(r, 0; f) + \overline{N}(r, 0; g) \\ &\quad + \overline{N}(r, \alpha_1; f | \geq 2) + \overline{N}(r, \alpha_2; f | \geq 2) \\ &\quad + \overline{N}(r, \alpha_1; g | \geq 2) + \overline{N}(r, \alpha_2; g | \geq 2) \\ &\quad + \overline{N}(r, a; F | \geq 2) + \overline{N}(r, a; G | \geq 2) \\ &\quad + \overline{N}_*(r, \infty; f, g) - \left(m - \frac{3}{2}\right) \overline{N}_*(r, 1; F, G) \\ &\quad + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G') + S(r, f) + S(r, g). \end{aligned}$$

Using (2.4) in (2.3) the lemma follows in view of Definition 1.6. ■

Lemma 2.9 Let F, G be given by (2.1) and (2.2), where $n \geq 7$ is an integer and $H \neq 0$. Suppose $\alpha_i, i = 1, 2$ has the same meaning as given in Lemma 2.8. If $E_m(1; F) = E_m(1; G)$ and f, g share (∞, k) , where m, k are integers such that $1 \leq m < \infty$ and $k \geq 0$. Then for a complex number $a (\neq 0, 1)$

$$\begin{aligned} \frac{3n}{2} \{T(r, f) + T(r, g)\} &\leq 2\{\overline{N}(r, 0; f) + \overline{N}(r, 0; g)\} + N_2(r, \alpha_1; f) \\ &\quad + N_2(r, \alpha_1; g) + N_2(r, \alpha_2; f) + N_2(r, \alpha_2; g) \\ &\quad + N_2(r, a; F) + N_2(r, a; G) + \overline{N}(r, \infty; f) \\ &\quad + \overline{N}(r, \infty; g) + \overline{N}_*(r, \infty; f, g) \end{aligned}$$

$$\begin{aligned}
& - \left(\frac{m}{2} - \frac{3}{2} \right) \left\{ \overline{N}_{F \geq m+1}(r, 1; F \mid G \neq 1) + \overline{N}_{G \geq m+1}(r, 1; G \mid F \neq 1) \right\} \\
& - \left(m - \frac{3}{2} \right) \left\{ \overline{N}_L^m(r, 1; F) + \overline{N}_L^m(r, 1; G) \right\} + S(r, f) + S(r, g).
\end{aligned}$$

Proof. Using Lemmas 2.2, 2.4, 2.6 and 2.7 we obtain

$$\begin{aligned}
(2.5) \quad & \overline{N}(r, 1; F) + \overline{N}(r, 1; G) \\
& \leq \frac{1}{2} [N(r, 1; F) + N(r, 1; G)] + N(r, 1; F \mid = 1) \\
& - \left(\frac{m}{2} - \frac{1}{2} \right) \left\{ \overline{N}_{f \geq m+1}(r, 1; f \mid g \neq 1) + \overline{N}_{g \geq m+1}(r, 1; g \mid f \neq 1) \right\} \\
& - \left(m - \frac{1}{2} \right) \left\{ \overline{N}_L^m(r, 1; f) + \overline{N}_L^m(r, 1; g) \right\} \\
& \leq \frac{n}{2} \{T(r, f) + T(r, g)\} + \overline{N}(r, 0; f) + \overline{N}(r, 0; g) \\
& + \overline{N}(r, \alpha_1; f \mid \geq 2) + \overline{N}(r, \alpha_2; f \mid \geq 2) \\
& + \overline{N}(r, \alpha_1; g \mid \geq 2) + \overline{N}(r, \alpha_2; g \mid \geq 2) + \overline{N}(r, a; F \mid \geq 2) \\
& + \overline{N}(r, a; G \mid \geq 2) + \overline{N}_*(r, \infty; f, g) \\
& - \left(\frac{m}{2} - \frac{3}{2} \right) \left\{ \overline{N}_{F \geq m+1}(r, 1; F \mid G \neq 1) + \overline{N}_{G \geq m+1}(r, 1; G \mid F \neq 1) \right\} \\
& - \left(m - \frac{3}{2} \right) \left\{ \overline{N}_L^m(r, 1; F) + \overline{N}_L^m(r, 1; G) \right\} \\
& + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G') + S(r, f) + S(r, g).
\end{aligned}$$

Using (2.5) in (2.3), the lemma follows in view of Definition 1.6. ■

Lemma 2.10 *Let f, g be two non-constant meromorphic functions and suppose $\alpha_i, i = 1, 2$ has the same meaning as given in Lemma 2.8. Then*

$$(n-1)^2(n-2)^2 f^{n-2}(f-\alpha_1)(f-\alpha_2) g^{n-2}(g-\alpha_1)(g-\alpha_2) \not\equiv b,$$

where b is a non-zero constant and $n \geq 5$ is an integer.

Proof. On the contrary, suppose that

$$(2.6) \quad (n-1)^2(n-2)^2 f^{n-2}(f-\alpha_1)(f-\alpha_2) g^{n-2}(g-\alpha_1)(g-\alpha_2) \equiv b.$$

Let z_0 be a zero of f with multiplicity p . Then z_0 is a pole of g with multiplicity q such that

$$(2.7) \quad (n-2)p = (n-2)q + 2q = nq.$$

From (2.7) we see that $2q = (n-2)(p-q) \geq n-2$ and so $p = \frac{n}{n-2} q \geq \frac{n}{2}$.

Let z_0 be a zero of $f - \alpha_i$ $i = 1, 2$ with multiplicity p . Then z_0 is a pole of g with multiplicity q such that $p = (n - 2)q + 2q = nq \geq n$.

Since the poles of f are the zeros of g and $g - \alpha_i$ $i = 1, 2$, we get

$$\begin{aligned} \overline{N}(r, \infty; f) &\leq \overline{N}(r, 0; g) + \overline{N}(r, \alpha_1; g) + \overline{N}(r, \alpha_2; g) \\ &\leq \frac{2}{n} N(r, 0; g) + \frac{1}{n} N(r, \alpha_1; g) + \frac{1}{n} N(r, \alpha_2; g) \\ &\leq \frac{4}{n} T(r, g). \end{aligned}$$

By the second fundamental theorem we get

$$\begin{aligned} 2T(r, f) &\leq \overline{N}(r, 0; f) + \overline{N}(r, \alpha_1; f) + \overline{N}(r, \alpha_2; f) + \overline{N}(r, \infty; f) + S(r, f) \\ &\leq \frac{2}{n} N(r, 0; f) + \frac{1}{n} N(r, \alpha_1; f) + \frac{1}{n} N(r, \alpha_2; f) + \frac{4}{n} T(r, g) + S(r, f) \\ &\leq \frac{4}{n} T(r, f) + \frac{4}{n} T(r, g) + S(r, f). \end{aligned}$$

i.e.,

$$(2.8) \quad \left(2 - \frac{4}{n}\right) T(r, f) \leq \frac{4}{n} T(r, g) + S(r, f).$$

Similarly,

$$(2.9) \quad \left(2 - \frac{4}{n}\right) T(r, g) \leq \frac{4}{n} T(r, f) + S(r, g)$$

Adding (2.8) and (2.9) we get

$$\left(2 - \frac{8}{n}\right) \{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g),$$

a contradiction for $n \geq 5$. This proves the lemma. ■

Lemma 2.11 [5] *Let f, g be two non-constant meromorphic functions and suppose $n (\geq 6)$ is an integer. If*

$$(2.10) \quad \begin{aligned} &\frac{(n-1)(n-2)}{2} f^n - n(n-2) f^{n-1} + \frac{n(n-1)}{2} f^{n-2} \\ &\equiv \frac{(n-1)(n-2)}{2} g^n - n(n-2) g^{n-1} + \frac{n(n-1)}{2} g^{n-2}, \end{aligned}$$

then $f \equiv g$.

Lemma 2.12 [22] *If F, G share $(\infty, 0)$ and $V \equiv 0$ then $F \equiv G$.*

Lemma 2.13 *Let F, G be given by (2.1) and (2.2), where $n \geq 7$ is an integer and $V \not\equiv 0$. If F, G share $(1, 2)$, f, g share (∞, k) , where $0 \leq k < \infty$, then the poles of F and G are zeros of V and*

$$\begin{aligned} (nk + n - 1) \overline{N}(r, \infty; f | \geq k + 1) &= (nk + n - 1) \overline{N}(r, \infty; g | \geq k + 1) \\ &\leq \overline{N}(r, 0; f) + \overline{N}(r, \alpha_1; f) + \overline{N}(r, \alpha_2; f) \\ &\quad + \overline{N}(r, 0; g) + \overline{N}(r, \alpha_1; g) + \overline{N}(r, \alpha_2; g) \\ &\quad + \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g), \end{aligned}$$

where α_i $i = 1, 2$ has the same meaning as in Lemma 2.8.

Proof. Since f, g share $(\infty; k)$, it follows that F, G share $(\infty; nk)$ and so a pole of F with multiplicity $p(\geq nk + 1)$ is a pole of G with multiplicity $r(\geq nk + 1)$ and vice versa. We note that F and G have no pole of multiplicity q where $nk < q < nk + n$. Now using the *Milloux theorem* [9, p. 55] and Lemma 2.7 we get from the definition of V

$$m(r, V) = S(r, f) + S(r, g).$$

Hence

$$\begin{aligned} (nk + n - 1) \overline{N}(r, \infty; f | \geq k + 1) &= (nk + n - 1) \overline{N}(r, \infty; g | \geq k + 1) \\ &= (nk + n - 1) \overline{N}(r, \infty; F | \geq nk + n) \\ &\leq N(r, 0; V) \\ &\leq T(r, V) + O(1) \\ &\leq N(r, \infty; V) + m(r, V) + O(1) \\ &\leq N(r, \infty; V) + S(r, f) + S(r, g) \\ &\leq \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}_*(r, 1; F, G) \\ &\quad + S(r, f) + S(r, g) \\ &\leq \overline{N}(r, 0; f) + \overline{N}(r, \alpha_1; f) + \overline{N}(r, \alpha_2; f) \\ &\quad + \overline{N}(r, 0; g) + \overline{N}(r, \alpha_1; g) + \overline{N}(r, \alpha_2; g) \\ &\quad + \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g). \end{aligned}$$

This proves the lemma. ■

Lemma 2.14 *Let F, G be given by (2.1), where $n \geq 7$ is an integer. Also let S be given as in Theorem 1.1. If $E_f(S, 0) = E_g(S, 0)$, then $S(r, f) = S(r, g)$.*

Proof. Since $E_f(S, 0) = E_g(S, 0)$, it follows that F and G share $(1, 0)$. We first note that the polynomial

$$p(z) = \frac{(n-1)(n-2)}{4} z^n - \frac{n(n-2)}{2} z^{n-1} + \frac{n(n-1)}{4} z^{n-2} - 1$$

has only simple zeros. In fact,

$$p'(z) = \frac{n(n-1)(n-2)}{4} z^{n-3} (z-1)^2.$$

Also we note that $p(0), p(1) \neq 0$. Thus all the zeros of $p(z)$ are simple and we denote them by w_j , $j = 1, 2, \dots, n$. Since F, G share $(1, 0)$ from the second fundamental theorem we have

$$\begin{aligned} (n-2)T(r, g) &\leq \sum_{j=1}^n \overline{N}(r, w_j; g) + S(r, g) \\ &= \sum_{j=1}^n \overline{N}(r, w_j; f) + S(r, g) \\ &\leq nT(r, f) + S(r, g). \end{aligned}$$

Similarly, we can deduce

$$(n-2)T(r, f) \leq nT(r, g) + S(r, f).$$

The last inequalities imply $T(r, f) = O(T(r, g))$ and $T(r, g) = O(T(r, f))$ and so we have $S(r, f) = S(r, g)$. $\blacksquare$

3. Proofs of the theorems

Proof of Theorem 1.1. Let F, G be given by (2.1) and (2.2). Since $E_f(S, 2) = E_g(S, 2)$ and $E_f(\{\infty\}, k) = E_g(\{\infty\}, k)$ it follows that F, G share $(1, 2)$ and $(\infty, nk + n - 1)$. So $\overline{N}_*(r, \infty; f, g) = \overline{N}_*(r, \infty; F, G) \leq \overline{N}(r, \infty; F | \geq nk + n) = \overline{N}(r, \infty; f | \geq k + 1)$. By a simple computation it can be easily seen that 1 is a root with multiplicity 3 of $F - \frac{1}{2}$ and hence $F - \frac{1}{2} = (f - 1)^3 Q_{n-3}(f)$, where $Q_{n-3}(f)$ is a polynomial in f of degree $n - 3$ and thus

$$\begin{aligned} N_2\left(r, \frac{1}{2}; F\right) &\leq 2\overline{N}(r, 1; f) + N(r, 0; Q_{n-3}(f)) \\ &\leq 2\overline{N}(r, 1; f) + (n-3)T(r, f) + S(r, f). \end{aligned}$$

Suppose that $H \neq 0$. Then $F \neq G$. So, it follows from Lemma 2.12 that $V \neq 0$. Hence, from Lemma 2.8 with $a = \frac{1}{2}$, $m = 2$, and Lemma 2.13, we obtain for $\varepsilon(> 0)$

$$\begin{aligned} &\left(\frac{n}{2} + 1\right) \{T(r, f) + T(r, g)\} \\ &\leq 2 \{ \overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \overline{N}(r, 1; f) + \overline{N}(r, 1; g) \} + \overline{N}(r, \infty; f) \\ &\quad + \overline{N}(r, \infty; g) + \overline{N}_*(r, \infty; f, g) - \frac{1}{2} \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g) \\ &\leq (5 - 2\Theta(0; f) - 2\Theta(1; f) - \Theta(\infty; f) + \varepsilon) T(r, f) \\ &\quad + (5 - 2\Theta(0; g) - 2\Theta(1; g) - \Theta(\infty; g) + \varepsilon) T(r, g) \\ &\quad + \frac{1}{nk + n - 1} [3T(r, f) + 3T(r, g)] + S(r, f) + S(r, g). \end{aligned}$$

That is

$$\begin{aligned}
 (3.1) \quad & \left(\frac{n}{2} - 4 + \Theta_f - \frac{3}{nk+n-1} - \varepsilon \right) T(r, f) \\
 & + \left(\frac{n}{2} - 4 + \Theta_g - \frac{3}{nk+n-1} - \varepsilon \right) T(r, g) \\
 & \leq S(r, f) + S(r, g).
 \end{aligned}$$

Without loss of generality, we may suppose that there exists a set I with infinite linear measure such that

$$T(r, g) \leq T(r, f), \quad r \in I.$$

From (3.1) and Lemma 2.14, we have

$$\left[\Theta_f + \Theta_g - 8 + n - \frac{6}{nk+n-1} - 2\varepsilon \right] T(r, g) \leq S(r, g), \quad r \in I \setminus E,$$

which leads to a contradiction for sufficiently small $\varepsilon > 0$. Hence $H \equiv 0$. Then

$$(3.2) \quad F \equiv \frac{aG + b}{cG + d},$$

where a, b, c, d are constants such that $ad - bc \neq 0$. Also

$$(3.3) \quad T(r, F) = T(r, G) + O(1).$$

We now consider the following cases.

Case I. Let $ac \neq 0$. From (3.2) we get

$$(3.4) \quad \overline{N}(r, \infty; G) = \overline{N}\left(r, \frac{a}{c}; F\right).$$

So, in view of (3.3), by the second fundamental theorem we get

$$\begin{aligned}
 T(r, F) & \leq \overline{N}(r, 0; F) + \overline{N}(r, \infty; F) + \overline{N}\left(r, \frac{a}{c}; F\right) + S(r, F) \\
 & = \overline{N}(r, 0; f) + 2T(r, f) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + S(r, f) \\
 & \leq 5T(r, f) + S(r, f),
 \end{aligned}$$

as $r \in I$ and $r \rightarrow \infty$. This, together with Lemma 2.7, gives $n \leq 5$, which contradicts the condition $n \geq 7$.

Case II. Let $a \neq 0$ and $c = 0$. Then $F = \alpha G + \beta$, where $\alpha = \frac{a}{d}$ and $\beta = \frac{b}{d}$.

If F has no 1-point, by the second fundamental theorem we get

$$\begin{aligned}
 T(r, F) & \leq \overline{N}(r, 0; F) + \overline{N}(r, \infty; f) + S(r, F) \\
 & \leq 3T(r, f) + \overline{N}(r, \infty; f) + S(r, f),
 \end{aligned}$$

as $r \in I$ and $r \rightarrow \infty$. This, together with Lemma 2.7, gives $n \leq 4$, which contradicts the condition $n \geq 7$.

If F and G have a common 1-point, then $\alpha + \beta = 1$, and so

$$(3.5) \quad F \equiv \alpha G + 1 - \alpha.$$

Suppose $\alpha \neq 1$. If $1 - \alpha \neq \frac{1}{2}$, then in view of (3.3) and the second fundamental theorem we get

$$\begin{aligned} 2T(r, F) &\leq \overline{N}(r, 0; F) + \overline{N}(r, 1 - \alpha; F) + \overline{N}\left(r, \frac{1}{2}; F\right) + \overline{N}(r, \infty; F) + S(r, F) \\ &\leq 3T(r, f) + \overline{N}(r, 0; G) + (n - 2)T(r, f) + \overline{N}(r, \infty; f) + S(r, f) \\ &\leq (n + 5)T(r, f) + S(r, f), \end{aligned}$$

as $r \in I$ and $r \rightarrow \infty$. This, together with Lemma 2.7, gives $n \leq 5$, which contradicts the condition $n \geq 7$. If $\alpha = \frac{1}{2}$, then we have from (3.5)

$$F \equiv \frac{1}{2}(G + 1).$$

So, by the second fundamental theorem we can obtain, using (3.3), that

$$\begin{aligned} 2T(r, G) &\leq \overline{N}(r, 0; G) + \overline{N}\left(r, \frac{1}{2}; G\right) + \overline{N}(r, -1; G) + \overline{N}(r, \infty; G) + S(r, G) \\ &\leq 3T(r, g) + (n - 2)T(r, g) + \overline{N}(r, 0; F) + \overline{N}(r, \infty; g) + S(r, g) \\ &\leq (n + 5)T(r, g) + S(r, g), \end{aligned}$$

as $r \in I$ and $r \rightarrow \infty$. This, together with Lemma 2.7, gives $n \leq 5$, which contradicts the condition $n \geq 7$.

So $\alpha = 1$ and hence $F \equiv G$. So, by Lemma 2.11, we get $f \equiv g$.

Case III. Let $a = 0$ and $c \neq 0$. Then $F \equiv \frac{1}{\gamma G + \delta}$, where $\gamma = \frac{c}{b}$ and $\delta = \frac{d}{b}$. If F has no 1-points, then as in Case II we can deduce a contradiction. If F and G have a common 1-point, then $\gamma + \delta = 1$ and so

$$(3.6) \quad F \equiv \frac{1}{\gamma G + 1 - \gamma}.$$

Suppose $\gamma \neq 1$. If $\gamma \neq -1$, then by the second fundamental theorem we get

$$\begin{aligned} 2T(r, F) &\leq \overline{N}(r, 0; F) + \overline{N}\left(r, \frac{1}{1 - \gamma}; F\right) + \overline{N}\left(r, \frac{1}{2}; F\right) + \overline{N}(r, \infty; f) + S(r, f) \\ &\leq 3T(r, f) + \overline{N}(r, 0; G) + (n - 2)T(r, f) + \overline{N}(r, \infty; f) + S(r, f) \\ &\leq (n + 5)T(r, f) + S(r, f), \end{aligned}$$

as $r \in I$ and $r \rightarrow \infty$. This, together with Lemma 2.7, gives $n \leq 5$, which contradicts the condition $n \geq 7$. If $\gamma = -1$ from (3.6) we have

$$F \equiv \frac{1}{-G + 2}.$$

Now, the second fundamental theorem with the help of (3.3) yields

$$\begin{aligned}
2T(r, G) &\leq \overline{N}(r, 0; G) + \overline{N}\left(r, \frac{1}{2}; G\right) + \overline{N}(r, 2; G) + \overline{N}(r, \infty; G) + S(r, G) \\
&\leq 3T(r, g) + (n-2)T(r, g) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + S(r, g) \\
&\leq (n+3)T(r, g) + S(r, g),
\end{aligned}$$

as $r \in I$ and $r \rightarrow \infty$. This, together with Lemma 2.7, gives $n \leq 3$, which contradicts the condition $n \geq 7$. So, we must have $\gamma = 1$ which implies $FG \equiv 1$, which is impossible by Lemma 2.10. This completes the proof of the theorem. ■

Proof of Corollary 1.1. Let F, G be given by (2.1) and (2.2). Since $E_f(S, 2) = E_g(S, 2)$ and $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$, it follows that f, g share (∞, k) for all large k . Also since $\min\{\Theta_f, \Theta_g\} > 4 - \frac{n}{2}$, for sufficiently large k we can have $\min\{\Theta_f, \Theta_g\} > 4 - \frac{n}{2} + \frac{3}{nk+n-1}$ and hence, by Theorem 1.1, we get the conclusion of Corollary 1.1. So, Corollary 1.1 can be treated as a special case of Theorem 1.1. ■

Proof of Theorem 1.2. Let F, G be given by (2.1) and (2.2). Since $E_f(S, 2) = E_g(S, 2)$ and $E_f(\{\infty\}, 2) = E_g(\{\infty\}, 2)$ it follows that F, G share $(1, 2)$ and $(\infty, 3n-1)$. Suppose that $H \not\equiv 0$. Now proceeding in the same way as done in the proof of Theorem 1.1, using Lemma 2.8 with $a = \frac{1}{2}$, $m = 2$ and Lemma 2.13 for $k = 0$ and $k = 2$ we obtain

$$\begin{aligned}
&\left(\frac{n}{2} + 1\right) \{T(r, f) + T(r, g)\} \\
&\leq 2 \{\overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \overline{N}(r, 1; f) + \overline{N}(r, 1; g)\} + 2\overline{N}(r, \infty; f) \\
&\quad + \overline{N}(r, \infty; f | \geq 3) - \frac{1}{2}\overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g) \\
&\leq 4 \{T(r, f) + T(r, g)\} + \frac{6}{n-1} \{T(r, f) + T(r, g)\} \\
&\quad + \frac{3}{3n-1} \{T(r, f) + T(r, g)\} + S(r, f) + S(r, g),
\end{aligned}$$

that is

$$(3.7) \quad \left(\frac{n}{2} - 3 - \frac{6}{n-1} - \frac{3}{3n-1}\right) \{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g).$$

Clearly (3.7) implies a contradiction for $n \geq 8$ and hence $H \equiv 0$ and the rest of the theorem can be proved in the line of proof of Theorem 1.1. ■

Proof of Theorem 1.3. Let F, G be given by (2.1) and (2.2). $E_3(S, f) = E_3(S, g)$, $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$ it follows that $E_3(1, F) = E_3(1; G)$, and F, G share (∞, ∞) . We omit the detail proof since using Lemmas 2.9, 2.10 and 2.11 the proof of the theorem can be carried out along the line of the proof of Theorem 1.1. ■

Proof of Theorem 1.4. Let F, G be given by (2.1) and (2.2). $E_m(S, f) = E_m(S, g)$, $E_f(\{\infty\}, k) = E_g(\{\infty\}, k)$ it follows that $E_m(1, F) = E_m(1; G)$, and F, G share $(\infty, nk + n - 1)$. We omit the detail proof since the same can be done in the line of proof of Theorem 1.1. ■

Acknowledgement. The authors wish to thank the referees for their valuable comments and suggestions.

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Accepted: 1.08.2010

A NOTE ON P -REGULAR SEMIRINGS

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Abstract. The notion of P -regular semiring is introduced and characterization of the same has been given. We also study the representation of the elements in terms of quasi-ideals of weak P -regular semirings relative to the right k -ideal P .

2000 Mathematical Subject Classification: 16Y30.

1. Introduction and preliminaries

Throughout this paper S will denote a semiring. A semiring is a commutative monoid $(S, +, 0_S)$ having additive identity zero 0_S and a semigroup $(S, \cdot)$ which are connected by ring like distributivity and $0_S \cdot x = x \cdot 0_S = 0_S$ for all $x \in S$. A left (right) ideal of a semiring S is a non-empty subset I of S such that $a+b \in I$ and $ra(ar) \in I$ for all $a, b \in I$ and $r \in S$. An ideal of a semiring S is a non-empty subset I of S such that I is both left and right ideal of S . A left (right) ideal I of S is called a left (right) k -ideal if $x+y \in I$, $x \in S$, $y \in I$ imply that $x \in I$. An ideal I of a semiring S is called a k -ideal if it is both left k -ideal and right k -ideal. If A is an ideal (resp. left, right) of a semiring S then

$$\overline{A} = \{a \in S : a + x \in A \text{ for some } x \in A\}$$

is called k -closure of A . It can easily be verified that an ideal (resp. left, right) A of S is a k -ideal if and only if $A = \overline{A}$. An additive subsemigroup Q of a semiring S is called a quasi-ideal of S if $QS \cap SQ \subseteq Q$. Clearly, every quasi-ideal of a semiring S is a subsemiring of S .

2. P -regular semirings

In this section, we study the concept of P -regular semiring and some properties related to the same.

Definition 2.1. Let S be a semiring and P be a right k -ideal of S . Then semiring S is said to be a P -regular if for each $a \in S$, there exists $x \in S$ such that $a + p_1 = axa + p_2$ for some $p_1, p_2 \in P$ and $axP \subseteq P$ and a semiring S is said to be weak P -regular if for each $a \in S$, there exists $x \in S$ such that $a = axa + p$ for some $p \in P$ and $axP \subseteq P$.

Definition 2.2. Let S be a semiring and P be a right k -ideal of S . An element $e \in S$ is called idempotent relative to P if $e + p' = e^2 + p''$ for some $p', p'' \in P$ and $eP \subseteq P$.

Clearly, it is easy to see that if $P = \{0\}$ then P -regular semiring is a regular and idempotent element is same as in semiring theory. From definition it is also clear that $ax = e$ is an idempotent element relative to P . Consider the semiring $Z_4 = \{0, 1, 2, 3\}$ with respect to addition and multiplication modulo 4 and $I = \{0, 2\}$ is a right as well as k -ideal of Z_4 . Clearly Z_4 is not regular because $2 \neq 2 \odot x \odot 2$ for $x \in Z_4$ but it is P -regular.

Theorem 2.3. *The semiring S with unity is P -regular if and only if every right ideal of S be of the form $aS + P$ where $a \in S$ has the form $aS + P = eS + P$, where e is an idempotent relative to P .*

Proof. Suppose S is P -regular. Therefore for each $a \in S$, there exists $x \in S$ such that $a + p_1 = axa + p_2$ for some $p_1, p_2 \in P$ and $axP \subseteq P$. Now $aS + P = axaS + P \subseteq axS + P = eS + P$. Therefore $aS + P \subseteq eS + P$. Again, $eS + P = axS + P \subseteq aS + P$. From above conclusion, we have $eS + P = aS + P$. Conversely, suppose that $aS + P = eS + P$ and $e + p' = e^2 + p''$ for some $p', p'' \in P$ and $eP \subseteq P$. To show S is P -regular. From above, we can write $a + p_1 = ey + p_2$ and $e + p_3 = ax + p_4$. Now

$$(1) \quad ea + p_3a = axa + p_4a$$

Again, $ea + ep_1 = e^2y + ep_2$. Adding $p''y + p_2$ on both sides, we get $ea + ep_1 + p_2 + p''y = e^2y + ep_2 + p_2 + p''y = ey + ep_2 + p'y + p_2 = a + p_1 + p'y + ep_2$, which implies

$$(2) \quad ea + z_1 = a + z_2$$

where $z_1 = ep_1 + p_2 + p''y$, $z_2 = p'y + ep_2 + p_1 \in P$. Adding z_1 in (1) and p_3a in (2), we get $a + z_2 + p_3a = axa + p_4a + z_1$. Thus $a + p_5 = axa + p_6$. Also, $e + p_3 = ax + p_4$ implies $ep + p_3p = axp + p_4p \in P$ (because $ep \in eP \subseteq P$ and $p_3p \subseteq P$) implies $axP \subseteq P$ (since P is a right k -ideal of S). Therefore S is a P -regular. ■

Proposition 2.4. *Suppose P is a right k -ideal of S and I is an ideal of S such that $P \subseteq I$. If S is P -regular then I is P -regular.*

Proof. Suppose S is P -regular and I is an ideal of S such that $P \subseteq I$. Then for each $a \in I$ there exists $x \in S$ and $p_1, p_2 \in P$ such that $a + p_1 = axa + p_2$ and $axP \subseteq P$. Let $y = xax \in I$. Then $axa + p_1xa = axaxa + p_2xa$ implies $p_2 + axa + p_1xa = aya + p_2xa + p_2$ implies $a + p_1 + p_1xa = aya + p_2xa + p_2$ implies $a + p' = aya + p''$ for some $p', p'' \in P$. Also, $ayP = a(xax)p = ax(axp) \in axP \subseteq P$. Thus I is a P -regular. ■

Theorem 2.5. *Let a semiring S be a P -regular. Then every right k -ideal R and left k -ideal L of S has the form $(P + R) \cap \overline{(P + L)} \subseteq \overline{P + RL}$.*

Proof. Let S be a P -regular and let $a \in (P + R) \cap \overline{(P + L)}$. Then $a \in S$ can be written as $a = p + r$ and $a + p_1 + l_1 = p_2 + l_2$ for some $p, p_1, p_2 \in P$, and $r \in R$ and $l_1, l_2 \in L$. Since S is a P -regular therefore for each $a \in S$ there exists $x \in S$ such that $a + p_3 = axa + p_4$ for some $p_3, p_4 \in P$ and $axP \subseteq P$. Now $(a)x(a + p_1 + l_1) = (p + r)x(p_2 + l_2)$ which implies that $axa + axp_1 + axl_1 = pxp_2 + pxl_2 + rxp_2 + rxl_2$ implies $axa + p_4 + axp_1 + axl_1 = p_4 + pxp_2 + pxl_2 + rxp_2 + rxl_2$ which gives $a + p_3 + axp_1 + axl_1 = pxp_2 + pxl_2 + rxp_2 + rxl_2 + p_4$. Since $axp_2 = pxp_2 + rxp_2 \in P$ and P is a right k -ideal of S . Therefore $rxp_2 \in P$ and also $axl_1 = pxl_1 + rxl_1 \in P + RL$. Therefore $a \in \overline{P + RL}$. Thus $(P + R) \cap \overline{(P + L)} \subseteq \overline{P + RL}$. ■

Note. The equality holds if $P + R$ is a right k -ideal of S because suppose $a \in \overline{(P + RL)}$. This implies $a \in \overline{(P + R)}$ and $a \in \overline{(P + L)}$. Therefore, $a \in (P + R) \cap \overline{(P + L)}$ (as $P + R$ is a right k -ideal of S). Hence $(P + R) \cap \overline{(P + L)} = \overline{P + RL}$.

Theorem 2.6. *The semiring S is a weak P -regular if every right ideal R and left ideal L of S has the form $(P + R) \cap (P + L) = P + RL$.*

Proof. The proof is same as in Theorem 2.6. ■

Proposition 2.7. *Let S be a weak P -regular semiring and R be any right ideal of S . Then the following holds:*

- (i) $R + P = R^2 + P$
- (ii) if $R^2 \subseteq P$ then $R \subseteq P$.

Proof. (i). Since S is a weak P -regular therefore for each $a \in S$ there exists $x \in S$ and some $p \in P$ such that $a = axa + p$ and $axP \subseteq P$. Suppose $a \in R$. Then $a = (ax)a + p \in R^2 + P$. This implies $R \subseteq R^2 + P$. This implies $R + P \subseteq R^2 + P$. But $R^2 + P \subseteq R + P$ (always). Therefore $R + P = R^2 + P$.

(ii). Since $R^2 \subseteq P$ therefore $R^2 + P \subseteq P$. This gives $R + P \subseteq P$. Therefore $R \subseteq P$ (as P is a right k -ideal of S). ■

Theorem 2.8. *If a semiring S is weak P -regular, then every element of a quasi-ideal Q can be represented as the sum of two elements of P and Q .*

Proof. Let S be a weak P -regular semiring and Q be a quasi-ideal of S . Then any $q \in Q$ can be written as $q = qxq + p$ and $qxP \subseteq P$ for some $p \in P$ and $x \in S$. Since Q is a quasi-ideal of S therefore $qxq \in QSQ \subseteq QS \cap SQ \subseteq Q$ and therefore we have $q = p + qxq \in P + Q$. ■

Theorem 2.9. *Let S be a weak P -regular semiring and Q_1 and Q_2 are quasi-ideals of S . If $q \in Q_1 \cap Q_2$, then this element q can be expressed as $q = p + q_1 x q_2$ for some $p \in P, x \in S, q_1 \in Q_1$ and $q_2 \in Q_2$ and also $q_1 x q_2 x P \subseteq P$.*

Proof. Let $q \in Q_1 \cap Q_2$. Since S is a weak P -regular then there exists $x \in S$ such that $q = q x q + p$ for some $p \in P$ and $q x P \subseteq P$. Since $Q_1 \cap Q_2$ is a quasi-ideal of S therefore the element $q \in Q_1 \cap Q_2$ can be written as both $q = p_1 + q_1 \in Q_1$ and $q = p_2 + q_2 \in Q_2$ for some $p_1, p_2 \in P, q_1 \in Q_1, q_2 \in Q_2$. Since S is a weak P -regular therefore the element $q \in S$ can be written as $q = p_3 + q x q$ for some $p_3 \in P$ and $q x P \subseteq P$. Now $q = p_3 + q x q = p_3 + (p_1 + q_1)x(p_2 + q_2) = p_3 + q_1 x p_2 + q_1 x q_2 + p_1 x p_2 + p_1 x q_2 = p_6 + q_1 x q_2$ where $p_6 \in P$ because $q x p_2 = p_1 x p_2 + q_1 x p_2 \in P$ and P is a right k -ideal of S , therefore $q_1 x p_2 \in P$. Again, $q x P = p_6 x P + q_1 x q_2 x P \subseteq P$. Since P is a right k -ideal of S therefore $q_1 x q_2 x P \subseteq P$. ■

Theorem 2.10. *Let S be a weak P -regular. Then every quasi-ideal Q of S can be written as $P + Q = P + Q S Q$.*

Proof. Let S be a weak P -regular and Q be a quasi-ideal of S . Then $Q S Q \subseteq (Q S) \cap (S Q) \subseteq Q$ holds. Therefore $P + Q S Q \subseteq P + Q$.

Conversely, let $a \in P + Q$. Then $a = p + q$ for some $p \in P$ and $q \in Q$. By weak P -regularity of S , we have $a = a x a + p_1$, for some $p_1 \in P, x \in S$ and $a x P \subseteq P$. This gives $a = (p + q)x(p + q) + p_1 = p x p + p x q + q x p + q x q + p_1$. Now $a x p = (p + q)x p = p x p + q x p \in P$ and P is a right k -ideal of S . Therefore $q x p \in P$. Thus we have $a = (p x p + p x q + q x p + p_1) + q x q \in P + Q S Q$. Thus $P + Q \subseteq P + Q S Q$. Hence $P + Q = P + Q S Q$. ■

Theorem 2.11. *Let S be weak P -regular. If Q_1 and Q_2 are quasi-ideals of S , then $P + (Q_1 \cap Q_2) = P + (Q_1 S Q_2) \cap (Q_2 S Q_1)$.*

Proof. Let S be a weak P -regular and Q_1 and Q_2 be quasi-ideals of S . If $q \in P + Q_1 \cap Q_2$. Then q can be written as $q = p + q'$ for some $p \in P$ and $q' \in Q_1 \cap Q_2$. Also, by weak P -regularity of S , there exists $x \in S$ such that $q' = p' + q' x q'$, for some $p' \in P$. Hence we have $q = p + p' + q' x q' \in P + (Q_1 S Q_2) \cap (Q_2 S Q_1)$. Therefore, $P + Q_1 \cap Q_2 \subseteq P + (Q_1 S Q_2) \cap (Q_2 S Q_1)$. Conversely, since $(Q_1 S Q_2) \cap (Q_2 S Q_1) \subseteq (Q_1 S \cap S Q_2) \cap (Q_2 S \cap S Q_1) \subseteq Q_1 \cap Q_2$, then $P + (Q_1 S Q_2 \cap Q_2 S Q_1) \subseteq P + (Q_1 \cap Q_2)$. Hence $P + (Q_1 S Q_2 \cap Q_2 S Q_1) = P + (Q_1 \cap Q_2)$. ■

Acknowledgment. The author wishes to express his thanks to the referee for giving fruitful suggestion and valuable comments.

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Accepted: 10.09.2010

SOME CLASSES OF GENERALIZED MINMAX POLYNOMIALS

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Abstract. The so-called MinMax numbers $\{M_n\}$ and their subsidiary numbers $\{N_n\}$ for the Pell numbers $\{P_n\}$ were studied by (for example) A.F. Horadam [3]. These MinMax numbers $\{M_n\}$ are positive integers which are the minimal and maximal representations by means of the Pell numbers. Analogous results for the MinMax numbers $\{D_n\}$ and their subsidiary numbers $\{R_n\}$, and for the modified Pell numbers $\{q_n\}$, are obtained in [3], $Q_n := 2q_n$ being the Pell-Lucas numbers. A.F. Horadam [4], on the other hand, expanded this MinMax number system to the algebraic polynomials $\{M_n(x)\}$, $\{N_n(x)\}$, $\{D_n(x)\}$ and $\{R_n(x)\}$. Our aim in this paper is to investigate results, which are similar to those in [4], but which hold true instead for the following generalized sequences of polynomials:

$$\{P_{n,m}(x)\} \quad \text{and} \quad \{Q_{n,m}(x)\} \quad (m \in \mathbb{N}; n \in \mathbb{N} \cup \{0\}),$$

$\mathbb{N}$ being the set of positive integers.

Keywords and phrases: Pell numbers; Modified Pell numbers; MinMax numbers; Pell-Lucas numbers; Recurrence relations; Explicit representations; Algebraic polynomials; MinMax polynomials; Fibonacci polynomials; Lucas polynomials.

2010 Mathematics Subject Classification: Primary 11B83; Secondary 11B37, 11B39.

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1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ the set of *positive* integers. We also write

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}.$$

The generalized Pell polynomials $\{P_{n,m}(x)\}$, and the generalized Pell-Lucas polynomials $\{Q_{n,m}(x)\}$ and $\{q_{n,m}^*(x)\}$, are defined by (see [1])

$$(1.1) \quad P_{n,m}(x) = 2xP_{n-1,m}(x) + P_{n-m,m}(x) \quad (n \geq m; m, n \in \mathbb{N}_0)$$

with $P_{0,m}(x) = 0$ and $P_{n,m}(x) = (2x)^{n-1}$ ($n = 1, \dots, m-1$);

$$(1.2) \quad Q_{n,m}(x) = 2xQ_{n-1,m}(x) + Q_{n-m,m}(x) \quad (n \geq m; m, n \in \mathbb{N}_0)$$

with $Q_{0,m}(x) = 2$ and $Q_{n,m}(x) = (2x)^n$ ($n = 1, \dots, m-1$; $m \in \mathbb{N} \setminus \{1\}$; $n \in \mathbb{N}$) and

$$(1.3) \quad q_{n,m}^*(x) = 2xq_{n-1,m}^*(x) + q_{n-2m,m}^*(x) \quad (n \geq 2m; m, n \in \mathbb{N}_0)$$

with $q_{0,m}^*(x) = 1$ and $q_{n,m}^*(x) = (2x)^{n-1}$ ($n = 1, \dots, 2m-1$; $m, n \in \mathbb{N}$).

In their special case when $m = 2$, the polynomials

$$P_{n,2}\left(\frac{x}{2}\right) \quad \text{and} \quad Q_{n,2}\left(\frac{x}{2}\right)$$

are the same as the familiar Fibonacci and Lucas polynomials, respectively (see [2]; see also [6]).

By using the recurrence relations (1.1), (1.2) and (1.3), and their corresponding initial values, we can compute the first few members of the polynomials

$$\{P_{n,2}(x) \equiv P_n(x)\}, \quad \{Q_{n,2}(x) \equiv Q_n(x)\} \quad \text{and} \quad \{q_{n,1}^*(x) \equiv q_n^*(x)\},$$

which are given in Table 1, Table 2 and Table 3 below.

Table 1

$$\begin{aligned} P_0(x) &= 0 \\ P_1(x) &= 1 \\ P_2(x) &= 2x \\ P_3(x) &= 4x^2 + 1 \\ P_4(x) &= 8x^3 + 4x \\ P_5(x) &= 16x^4 + 12x^2 + 1 \\ P_6(x) &= 32x^5 + 32x^3 + 6x \\ P_7(x) &= 64x^6 + 80x^4 + 24x^2 + 1 \\ P_8(x) &= 126x^7 + 192x^5 + 80x^3 + 8x \\ P_9(x) &= 256x^8 + 448x^6 + 240x^4 + 40x^2 + 1 \\ P_{10}(x) &= 512x^9 + 1024x^7 + 672x^5 + 160x^3 + 10x. \end{aligned}$$

Table 2

$$\begin{aligned}
Q_0(x) &= 2 \\
Q_1(x) &= 2x \\
Q_2(x) &= 4x^2 + 2 \\
Q_3(x) &= 8x^3 + 6x \\
Q_4(x) &= 16x^4 + 16x^2 + 2 \\
Q_5(x) &= 32x^5 + 40x^3 + 10x \\
Q_6(x) &= 64x^6 + 96x^4 + 36x^2 + 2 \\
Q_7(x) &= 128x^7 + 224x^5 + 112x^3 + 14x \\
Q_8(x) &= 256x^8 + 512x^6 + 320x^4 + 64x^2 + 2 \\
Q_9(x) &= 512x^9 + 1152x^7 + 864x^5 + 240x^3 + 18x \\
Q_{10}(x) &= 1024x^{10} + 2560x^8 + 2240x^6 + 800x^4 + 100x^2 + 2.
\end{aligned}$$

Table 3

$$\begin{aligned}
q_0^*(x) &= 1 \\
q_1^*(x) &= 1 \\
q_2^*(x) &= 2x + 1 \\
q_3^*(x) &= 4x^2 + 2x + 1 \\
q_4^*(x) &= 8x^3 + 4x^2 + 4x + 1 \\
q_5^*(x) &= 16x^4 + 8x^3 + 12x^2 + 4x + 1 \\
q_6^*(x) &= 32x^5 + 16x^4 + 32x^3 + 12x^2 + 6x + 1 \\
q_7^*(x) &= 64x^6 + 32x^5 + 80x^4 + 32x^3 + 24x^2 + 6x + 1 \\
q_8^*(x) &= 128x^7 + 64x^6 + 192x^5 + 80x^4 + 80x^3 + 24x^2 + 8x + 1 \\
q_9^*(x) &= 256x^8 + 128x^7 + 448x^6 + 192x^5 + 240x^4 + 80x^3 + 40x^2 + 8x + 1.
\end{aligned}$$

Our present investigation is motivated essentially by the earlier works of Horadam (see, for details, [3]) dealing with the so-called MinMax numbers $\{M_n\}$ and their subsidiary numbers $\{N_n\}$ for Pell numbers $\{P_n\}$, and also with the MinMax numbers $\{D_n\}$ and their subsidiary numbers $\{R_n\}$ and the modified Pell numbers $\{q_n\}$, $Q_n := 2q_n$ being the Pell-Lucas numbers. These MinMax numbers $\{M_n\}$ are positive integers which are the minimal and maximal representations by means of the Pell numbers. Analogous results for the MinMax numbers $\{D_n\}$ and their subsidiary numbers $\{R_n\}$, and also for the modified Pell numbers $\{q_n\}$. Subsequently, Horadam [4] expanded this MinMax number system to the algebraic polynomials $\{M_n(x)\}$, $\{N_n(x)\}$, $\{D_n(x)\}$ and $\{R_n(x)\}$. The main object of this paper is to present results, which are similar to those in [4], but which hold true instead for the following generalized sequences of polynomials:

$$\{P_{n,m}(x)\} \quad \text{and} \quad \{Q_{n,m}(x)\} \quad (m \in \mathbb{N}; n \in \mathbb{N} \cup \{0\}),$$

which are defined here by (1.1) and (1.2), respectively.

2. The MinMax polynomials $\{M_{n,m}(x)\}$

The polynomials $\{M_{n,m}(x)\}$ ($m \in \mathbb{N}$; $n \in \mathbb{N}_0$) are defined by means of the following recurrence relation:

$$(2.1) \quad M_{n,m}(x) = 2xM_{n-1,m}(x) + M_{n-2m,m}(x) + l \quad (n \geq 2m; m, n \in \mathbb{N})$$

with $M_{0,m}(x) = 0$ and $M_{n,m}(x) = (2x)^{n-1}$ ($n = 1, \dots, m$) and $M_{m+1,m}(x) = (2x)^m + 1$ and $M_{n,m}(x) = 2xM_{n-1,m}(x)$ ($n = m+2, \dots, 2m-1$), where

$$l = \begin{cases} 0 & (n \neq mk + 1; k \in \mathbb{N}_0) \\ 1 & (n = mk + 1). \end{cases}$$

One of our main results is asserted by Theorem 1 below.

Theorem 1. *For $m \in \mathbb{N}$ and $n \geq m$, each of the following equalities holds true:*

$$(2.2) \quad M_{n,m}(x) = \sum_{i=0}^{[n/m]} P_{n-mi,2m}(x)$$

and

$$(2.3) \quad M_{n,m}(x) = \frac{P_{n+1,2m}(x) + P_{n+1-m,2m}(x) - l}{2x},$$

where

$$l = \begin{cases} 0 & (n \neq mk; k \in \mathbb{N}_0) \\ 1 & (n = mk). \end{cases}$$

Proof. We make use of the principle of mathematical induction on n in order to prove equalities (2.2) and (2.3). First of all, it is easy to prove equality (2.2) for

$$n = 0, 1, \dots, 2m - 1.$$

Suppose now that (2.2) holds true for $n = mn$ ($n \in \mathbb{N}$). Then, for $n = mn + 1$,

we apply the recurrence relation (2.1) to get

$$\begin{aligned}
M_{mn+1,m}(x) &= 2xM_{mn,m}(x) + M_{mn+1-2m,m}(x) + 1 \\
&= 2x \sum_{i=0}^n P_{mn-mi,2m}(x) + \sum_{i=0}^{n-2} P_{mn+1-2m-mi,2m}(x) + 1 \\
&= \sum_{i=0}^{n-2} (2xP_{mn-mi,2m} + P_{mn+1-2m-mi,2m}) \\
&\quad + 2x(P_{m,2m}(x) + P_{0,2m}(x)) + 1 \\
&= \sum_{i=0}^{n-2} P_{mn+1-mi,2m}(x) + P_{m+1,2m}(x) + P_{1,2m}(x) \\
&= \sum_{i=0}^n P_{mn+1-mi,2m}(x),
\end{aligned}$$

which proves the the first assertion (2.2) of Theorem 1.

Next, clearly, equality (2.3) holds true for $n = 0, 1, \dots, 2m - 1$, by means of the recurrence relations (1.1) and (2.1). Suppose that (2.3) holds true for $n = mn$. Then, for $n = mn + 1$, we find that

$$\begin{aligned}
M_{mn+1,m}(x) &= 2xM_{mn,m}(x) + M_{mn+1-2m,m}(x) + 1 \\
&= 2x \frac{P_{mn+1,2m}(x) + P_{mn+1-m,2m}(x) - 1}{2x} \\
&\quad + \frac{P_{mn+2-2m,2m}(x) + P_{mn+2-3m,2m}(x)}{2x} + 1 \\
&= \frac{P_{mn+2,2m}(x) + P_{mn+2-m,2m}(x)}{2x},
\end{aligned}$$

which, by the principle of mathematical induction on n , proves the second assertion (2.3) of Theorem 1. Our proof of Theorem 1 is thus completed. $\blacksquare$

3. The polynomials $\{N_{n,m}(x)\}$

The polynomials $\{N_{n,m}(x)\}$ ($m \in \mathbb{N}$; $n \in \mathbb{N}_0$) are defined by

$$(3.1) \quad N_{n,m}(x) = M_{n+1,m}(x) + M_{n+1-2m,m}(x) \quad (n \geq 2m),$$

which, in conjunction with (2.1), shows that the polynomials $\{N_{n,m}(x)\}$ satisfy the following recurrence relation:

$$(3.2) \quad N_{n,m}(x) = 2xN_{n-1,m}(x) + N_{n-2m,m}(x) + l$$

with $N_{0,m}(x) = 1$ and $N_{n,m}(x) = (2x)^n$ ($n = 1, \dots, m-1$) and $N_{m,m}(x) = (2x)^m + 1$ and $N_{n,m}(x) = 2xN_{n-1,m}(x)$ ($n = m+1, \dots, 2m-1$), where

$$l = \begin{cases} 0 & (n \neq mk; k \in \mathbb{N}_0) \\ 2 & (n = mk). \end{cases}$$

Theorem 2. *The polynomials $\{N_{n,m}(x)\}$ satisfy the following relationship:*

$$(3.3) \quad N_{n,m}(x) = \frac{Q_{n+1,2m} + Q_{n+1-m,2m}(x) - l}{2x},$$

where

$$l = \begin{cases} 0 & (n \neq mk + m - 1; k \in \mathbb{N}_0) \\ 2 & (n = mk + m - 1). \end{cases}$$

Proof. By using (1.2) and (3.2), we readily obtain

$$N_{m-1,m}(x) = \frac{Q_{m,2m}(x) + Q_{0,2m}(x) - 2}{2x},$$

which shows that relationship (3.3) holds true for $n = m - 1$. If we assume that (3.3) holds true for $n \neq mk$, then, for $n = mk$, we find by using the recurrence relation (3.2) that

$$\begin{aligned} N_{mk,m}(x) &= 2xN_{mk-1,m}(x) + N_{mk-2m,m}(x) + 2 \\ &= 2x \left(\frac{Q_{mk,2m}(x) + Q_{mk-m,2m}(x) - 2}{2x} \right) \\ &\quad + \frac{Q_{mk+1-2m,2m}(x) + Q_{mk+1-3m,2m}(x)}{2x} + 2 \\ &= \frac{Q_{mk+1,2m}(x) + Q_{mk+1-m,2m}(x)}{2x}, \end{aligned}$$

which evidently completes our proof of Theorem 2. ■

By some suitable algebraic manipulations based upon (3.3), we get the following result.

Theorem 3. *For $m \in \mathbb{N}$ and $n \geq m$, the following relations hold true:*

$$(3.4) \quad N_{n,m}(x) - N_{n-m,m}(x) = Q_{n,2m}(x)$$

and

$$(3.5) \quad N_{n,m}(x) - N_{n-2m,m}(x) = Q_{n,2m}(x) + Q_{n-m,2m}(x).$$

Proof. Upon setting $n = m$ in (3.4), we get

$$N_{m,m}(x) - N_{0,m}(x) = Q_{m,2m}(x),$$

which, by virtue of (3.2) and Table 2, coincides with the identity:

$$(2x)^m + 1 - 1 = (2x)^m.$$

If (3.4) holds true for some fixed positive integer n ($n \geq m$), then

$$\begin{aligned}
 & N_{n+1,m}(x) - N_{n+1-m,m}(x) \\
 &= 2xN_{n,m}(x) + N_{n+1-2m,m}(x) \\
 &\quad - 2xN_{n-m,m}(x) - N_{n+1-3m,m}(x) \\
 &= 2x[N_{n,m}(x) - N_{n-m,m}(x)] \\
 &\quad + [N_{n+1-2m,m}(x) - N_{n+1-3m,m}(x)] \\
 &= 2xQ_{n,2m}(x) + Q_{n+1-2m,2m}(x) \\
 (3.6) \quad &= Q_{n+1,2m}(x),
 \end{aligned}$$

which implies that relation (3.4) is holds true when n is replaced by $n + 1$.

The second assertion (3.5) of Theorem 3 is a *direct* consequence of the first assertion (3.4). ■

4. The subsidiary MinMax polynomials $\{D_{n,m}(x)\}$

Instead of the MinMax polynomials for the generalized Pell polynomials $\{P_{n,m}(x)\}$ defined by (1.1), we now consider the analogous polynomials for the polynomials $\{Q_{n,m}(x)\}$ and $\{q_{n,m}^*(x)\}$, which are defined by means of the recurrence relations (1.2) and (1.3), respectively. We thus define the MinMax polynomials $\{D_{n,m}(x)\}_{n \in \mathbb{N}_0}$ by means of the following recurrence relation:

$$(4.1) \quad D_{n,m}(x) = 2xD_{n-1,m}(x) + D_{n-2m,m}(x) + l \quad (n \geq 2m; m, n \in \mathbb{N})$$

with $D_{0,m}(x) = 0$ and $D_{n,m}(x) = (2x)^{n-1}$ ($n = 1, \dots, m$) and $D_{m+1,m}(x) = (2x)^m + 2$ and $D_{n,m}(x) = 2xD_{n-1,m}(x)$ ($n = m+2, \dots, 2m-1$), where

$$l = \begin{cases} 0 & (n \neq mk + 1; k \in \mathbb{N}_0) \\ 2 & (n = mk + 1). \end{cases}$$

Theorem 4 below provides the connection between the classes of polynomials $\{M_{n,m}(x)\}$ and $\{D_{n,m}(x)\}$.

Theorem 4. *For $m \in \mathbb{N}$ and $n \geq m$ the following relationships hold true:*

$$(4.2) \quad M_{n,m}(x) + M_{n-m,m}(x) = D_{n,m}(x) \quad (n > m);$$

$$(4.3) \quad D_{n,m}(x) = \frac{q_{n+1,m}^*(x) + q_{n+1-m,m}^*(x) - l}{2x} \quad (n \geq m),$$

where

$$l = \begin{cases} 0 & (n \neq mk; k \in \mathbb{N}_0) \\ 2 & (n = mk); \end{cases}$$

$$(4.4) \quad D_{n,m}(x) - D_{n-m,m}(x) = q_{n,m}^*(x) \quad (n \geq m)$$

and

$$(4.5) \quad D_{n,m}(x) - D_{n-2m,m}(x) = q_{n,m}^*(x) + q_{n-m,m}^*(x) \quad (n \geq 2m).$$

Proof. We first assume that (4.2) holds true for some positive integer n . Then, for $n \mapsto n+1$, we get

$$\begin{aligned} M_{n+1,m}(x) + M_{n+1-m,m}(x) &= 2xM_{n,m}(x) + M_{n+1-2m,m}(x) + l \\ &\quad + 2xM_{n-m,m}(x) + M_{n+1-3m,m}(x) + l \\ &= 2x[M_{n,m}(x) + M_{n-m,m}(x)] + M_{n+1-2m,m}(x) \\ &\quad + M_{n+1-3m,m}(x) + 2l \\ &= 2xD_{n,m}(x) + D_{n+1-2m,m}(x) + 2l \\ &= D_{n+1,m}(x) \quad (2l = 2 \text{ or } 2l = 0). \end{aligned}$$

Next, we suppose that the relationship (4.3) holds true for $n = mk$. Then, for $n = mk+1$, we get

$$\begin{aligned} D_{mk+1,m}(x) &= 2xD_{mk,m}(x) + D_{mk+1-2m,m}(x) + 2 \\ &= 2x \frac{q_{mk+1,m}^*(x) + q_{mk+1-m,m}^*(x) - 2}{2x} \\ &\quad + \frac{q_{mk+1-2m,m}^*(x) + q_{mk+1-3m,m}^*(x)}{2x} + 2 \\ &= \frac{1}{2x} \left[2xq_{mk+1,m}^*(x) + q_{mk+1-2m,m}^*(x) \right. \\ &\quad \left. + 2xq_{mk+1-m,m}^*(x) + q_{mk+1-3m,m}^*(x) \right] \\ &= \frac{q_{mk+2,m}^*(x) + q_{mk+2-m,m}^*(x)}{2x}. \end{aligned}$$

For $n = m$ in (4.4), we are led at once to the following obvious identity:

$$D_{m,m}(x) - D_{0,m}(x) = q_{m,m}^*(x),$$

so (4.4) holds true for $n = m$. Suppose that (4.4) holds true for some positive integer n ($n \geq m$). Then, for $n \mapsto n+1$, we get

$$\begin{aligned} D_{n+1,m}(x) - D_{n+1-m,m}(x) &= 2xD_{n,m}(x) + D_{n+1-2m,m}(x) + l \\ &\quad - 2xD_{n-m,m}(x) - D_{n+1-3m,m}(x) - l \\ &= 2x(D_{n,m}(x) - D_{n-m,m}(x)) + D_{n+1-2m,m}(x) \\ &\quad - D_{n+1-3m,m}(x) \\ &= 2xq_{n,m}^*(x) + q_{n+1-2m,m}^*(x) \\ &= q_{n+1,m}^*(x). \end{aligned}$$

The last assertion (4.5) of Theorem 4 is an immediate consequence of assertion (4.4). The proof of Theorem 4 is thus completed. ■

We now present another interesting result.

Theorem 5. *The polynomials $\{D_{n,m}(x)\}$ possess the following representation:*

$$(4.6) \quad D_{n,m}(x) = \sum_{i=0}^{\lfloor (n-1)/m \rfloor} q_{n-mi,m}^*(x) \quad (m, n \in \mathbb{N}).$$

Proof. For $n = 1$ in (4.6), we have the following obvious special case:

$$D_{1,m}(x) = \sum_{i=0}^0 q_{1-mi,m}^*(x),$$

which shows that that formula (4.6) holds true for $n = 1$.

Suppose now that (4.6) holds true for $n = mk$. Then, for $n = mk + 1$, we get

$$\begin{aligned} D_{mk+1,m}(x) &= 2xD_{mk,m}(x) + D_{mk+1-2m,m}(x) + 2 \\ &= 2x \sum_{i=0}^{\lfloor (mk-1)/m \rfloor} q_{mk-mi,m}^*(x) + \sum_{i=0}^{\lfloor (mk-2m)/m \rfloor} q_{mk+1-2m,m}^*(x) + 2 \\ &= 2x \sum_{i=0}^{k-1} q_{mk-mi,m}^*(x) + \sum_{i=0}^{k-2} q_{mk+1-2m,m}^*(x) + 2 \\ &= \sum_{i=0}^{k-2} (2xq_{mk-mi,m}^*(x) + q_{mk+1-2m-mi,m}^*(x)) \\ &\quad + 2xq_{mk-m(k-1),m}^*(x) + 2 \\ &= \sum_{i=0}^{k-2} q_{mk+1-mi,m}^*(x) + 2xq_{m,m}^*(x) + 1 + 1 \\ &= \sum_{i=0}^{k-2} q_{mk+1-mi,m}^*(x) + q_{m+1,m}^*(x) + 1 \\ &= \sum_{i=0}^k q_{mk+1-mi,m}^*(x) \quad \left(k = \left\lceil \frac{n}{m} \right\rceil\right). \end{aligned}$$

For $n = mk + 2$, we find from the recurrence relation (4.1) that

$$\begin{aligned}
 D_{mk+2,m}(x) &= 2xD_{mk+1,m}(x) + D_{mk+2-2m,m}(x) \\
 &= 2x \sum_{i=0}^k q_{mk+1-mi,m}^*(x) + \sum_{i=0}^{k-2} q_{mk+2-2m-mi,m}^*(x) \\
 &= \sum_{i=0}^{k-2} q_{mk+2-mi,m}^*(x) + 2xq_{m+1,m}^*(x) + 2xq_{1,m}^*(x) \\
 &= \sum_{i=0}^{k-2} q_{mk+2-mi,m}^*(x) + q_{m+2,m}^*(x) + q_{2,m}^*(x) \\
 &= \sum_{i=0}^k q_{mk+2-mi,m}^*(x).
 \end{aligned}$$

This, evidently, completes our proof of Theorem 5. ■

5. The subsidiary MinMax polynomials $\{R_{n,m}(x)\}$

In this section, we introduce and investigate the sequence of polynomials $\{R_{n,m}(x)\}_{m \in \mathbb{N}}$, which are the subsidiary MinMax polynomials of $\{D_{n,m}(x)\}$ for $\{q_{n,m}(x)\}$, where

$$2q_{n,2}(x) = Q_{n,2}(x).$$

Thus, by definition, we have

$$(5.1) \quad R_{n,m}(x) = 2xR_{n-1,m}(x) + R_{n-2m,m}(x) + l \quad (n \geq 2m; m \in \mathbb{N})$$

with $R_{0,m}(x) = 0$ and $R_{n,m}(x) = (2x)^n$ ($n = 1, \dots, m-1$) and $R_{m,m}(x) = (2x)^m + 2$ and $R_{n,m}(x) = 2xR_{n-1,m}(x)$ ($n = m+1, \dots, 2m-1$), where

$$l = \begin{cases} 0 & (n \neq mk; k \in \mathbb{N}_0) \\ 2 & (n = mk). \end{cases}$$

The connection between the polynomials $\{R_{n,m}(x)\}$, $\{N_{n,m}(x)\}$ and $\{D_{n,m}(x)\}$ is given by Theorem 6 below.

Theorem 6. *Each of the following relationships holds true:*

$$(5.2) \quad R_{n,m}(x) = D_{n+1,m}(x) + D_{n+1-2m,m}(x) \quad (n \geq 2m),$$

$$(5.3) \quad R_{n,m}(x) = N_{n,m}(x) + N_{n-m,m}(x) \quad (n \geq m),$$

$$(5.4) \quad R_{n,m}(x) - R_{n-m,m}(x) = q_{n+1,m}^*(x) + q_{n+1-2m,m}^*(x) \quad (n \geq 2m)$$

and

$$(5.5) \quad R_{n,m}(x) - R_{n-m,m}(x) = N_{n,m}(x) - N_{n-2m,m}(x) \quad (n \geq 2m).$$

Proof. Suppose that (5.2) is correct for $n = mk$. Then, for $n = mk + 1$, the recurrence relation (5.1) yields

$$\begin{aligned} R_{mk+1,m}(x) &= 2xR_{mk,m}(x) + R_{mk+1-2m,m}(x) \\ &= 2x(D_{mk+1,m}(x) + D_{mk+1-2m,m}(x)) + D_{mk+2-2m,m}(x) + D_{mk+2-4m,m}(x) \\ &= D_{mk+2,m}(x) + D_{mk+2-2m,m}(x), \end{aligned}$$

which proves relationship (5.2).

By using the initial values for $R_{n,m}(x)$ and $N_{n,m}(x)$, we can easily prove relationship (5.3). Let relationship (5.3) hold true for $n \neq mk$. Then

$$\begin{aligned} R_{mk,m}(x) &= 2xR_{mk-1,m}(x) + R_{mk-2m,m}(x) + 4 \\ &= 2x(N_{mk-1,m}(x) + N_{mk-1-m,m}(x)) + N_{mk-2m,m}(x) + N_{mk-3m,m}(x) + 4 \\ &= 2xN_{mk-1,m}(x) + N_{mk-2m,m}(x) + 2 + 2xN_{mk-1-m,m}(x) + N_{mk-3m,m}(x) + 2 \\ &= N_{mk,m}(x) + N_{mk-m,m}(x), \end{aligned}$$

which proves relationship (5.3).

Suppose now that the relationship (5.4) holds true for some positive integer n . Then, for $n \mapsto n + 1$, we find from the recurrence relation (5.1) in conjunction with (3.1) that

$$\begin{aligned} &R_{n+1,m}(x) - R_{n+1-m,m}(x) \\ &= 2xR_{n,m}(x) + R_{n+1-2m,m}(x) + l - 2xR_{n-m,m}(x) - R_{n+1-3m,m}(x) - l \\ (5.6) \quad &= 2x[R_{n,m}(x) - R_{n-m,m}(x)] + R_{n+1-2m,m}(x) - R_{n+1-3m,m}(x) \\ &= 2xq_{n+1,m}^*(x) + 2xq_{n+1-2m,m}^*(x) + q_{n+2-2m,m}^*(x) + q_{n+2-4m,m}(x) \\ &= q_{n+2,m}^*(x) + q_{n+2-2m,m}^*(x), \end{aligned}$$

which proves relationship (5.4).

It is easily observed that the last assertion (5.5) of Theorem 6 is a *direct* consequence of relationship (5.3). ■

6. A set of sequences of numbers

In their special cases when $x = 0$, the above-investigated MInMax polynomials $\{M_{n,m}(x)\}$, $\{N_{n,m}(x)\}$, $\{D_{n,m}(x)\}$ and $\{R_{n,m}(x)\}$ would lead us to some interesting sequences of MinMax numbers, which we denote by $\{M_{n,m}\}$, $\{N_{n,m}\}$, $\{D_{n,m}\}$ and $\{R_{n,m}\}$, respectively. Some of these sequences of MinMax numbers are given below:

$$\begin{aligned}
M_{n,2} &: \{0, 1, 0, 1, 0, 2, 0, 2, 0, 3, 0, 3, 0, 4, 0, 4, \dots\}; \\
M_{n,3} &: \{0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 2, 0, 0, 3, 0, 0, 3, \dots\}; \\
M_{m,m} &: \{0, 1, \underbrace{0, \dots, 0}_{m-1}, 1, \underbrace{0, \dots, 0}_{m-1}, 2, \underbrace{0, \dots, 0}_{m-1}, \dots\}. \\
N_{n,2} &: \{1, 0, 1, 0, 3, 0, 3, 0, 5, 0, 5, \dots\}; \\
N_{n,3} &: \{1, 0, 0, 1, 0, 0, 3, 0, 0, 3, 0, 0, 5, 0, 0, 5, \dots\}; \\
N_{m,m} &: \{1, \underbrace{0, \dots, 0}_{m-1}, 1, \underbrace{0, \dots, 0}_{m-1}, 3, \underbrace{0, \dots, 0}_{m-1}, 3, \underbrace{0, \dots, 0}_{m-1}, 5, \dots\}. \\
D_{n,2} &: \{0, 1, 0, 2, 0, 3, 0, 4, 0, 5, 0, \dots\}; \\
D_{n,3} &: \{0, 1, 0, 0, 2, 0, 0, 3, 0, 0, 4, 0, 0, 5, \dots\}; \\
D_{m,m} &: \{0, 1, \underbrace{0, \dots, 0}_{m-1}, 2, \underbrace{0, \dots, 0}_{m-1}, \dots\}. \\
R_{n,2} &: \{0, 0, 2, 0, 4, 0, 6, 0, 8, 0, 10, \dots\}; \\
R_{n,3} &: \{0, 0, 0, 2, 0, 0, 4, 0, 0, 6, 0, 0, 8, 0, 0, 10, \dots\}; \\
R_{m,m} &: \{\underbrace{0, \dots, 0}_{m-1}, 2, \underbrace{0, \dots, 0}_{m-1}, \dots\}.
\end{aligned}$$

Acknowledgements. The second-named author was supported, in part, by the *Ministry of Science of the Republic of Serbia* under Grant 144003.

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Accepted: 20.04.2011

ON γ -s-URYSOHN CLOSED AND γ -s-REGULAR CLOSED SPACES**Sabir Hussain**

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Abstract. In this paper, we introduced and studied γ -s-Urysohn spaces and γ -s-regular closed spaces. Several characterizations and properties of these classes of spaces have been obtained.

Keywords. γ -closed (open), γ -interior (closure), γ -semi-open(closed), γ -s-Urysohn, γ -s-regular, γ -s-adherent, γ -irresolute.

AMS Subject Classification: 54A05, 54A10, 54D10, 54D99.

1. Introduction

S. Kasahara [14] introduced and discussed an operation γ of a topology τ into the power set $P(X)$ of a space X . H. Ogata [16] introduced the concept of γ -open sets and investigated the related topological properties of the associated topology τ_γ and τ by using operation γ .

S. Hussain and B. Ahmad [1]-[6] and [10]-[13] continued studying the properties of γ -operations on topological spaces and investigated many interesting results. Recently, B. Ahmad and S. Hussain [2], [13] defined and discussed γ -semi-open sets in topological spaces. They explored many interesting properties of γ -semi-open sets. It is interesting to mention that γ -semi-open sets generalized γ -open sets introduced by H. Ogata [17].

In 1982, S.P. Arya and M.P. Bhamini introduced the concept of s-Urysohn. A space X is said to be s-Urysohn, if for any two distinct points x and y of X , there exist semi-open sets U and V containing x and y respectively such that $cl(U) \cap cl(V) = \emptyset$. The concept of s-regular space was introduced by Mahaeshwari and Prasad in 1975. In 1984, Arya and Bhamini introduced and discussed s-regular-closed sets [8].

In 2009, S. Hussain and B. Ahmad defined a new axiom called γ -s-regularity. It is interesting to mention that this axiom is a generalization of the axiom of s-regularity [15] as well as semi-regularity [9].

In this paper, we introduced and studied γ -s-Urysohn spaces and γ -s-regular closed spaces. A γ -s-Urysohn (resp., γ -s-regular [12]) space X is said to be γ -s-Urysohn-closed (resp., γ -s-regular-closed), if it is γ -closed in every γ -s-Urysohn (respt. γ -s-regular [12]) space in which it can be embedded. Several characterizations and properties of these classes of spaces have been obtained.

First, we recall some definitions and results used in this paper. Here after, we shall write a space in place of a topological space.

Preliminaries

Definition 2.1. [14] Let X be a space. An operation $\gamma : \tau \rightarrow P(X)$ is a function from τ to the power set of X such that $V \subseteq V^\gamma$, for each $V \in \tau$, where V^γ denotes the value of γ at V . The operations defined by $\gamma(G) = G$, $\gamma(G) = cl(G)$ and $\gamma(G) = intcl(G)$ are examples of operation γ .

Definition 2.2. [14] Let $A \subseteq X$. A point $x \in A$ is said to be γ -interior point of A if there exists an open nbd N of x such that $N^\gamma \subseteq A$ and we denote the set of all such points by $int_\gamma(A)$. Thus

$$int_\gamma(A) = \{x \in A : x \in N \in \tau \text{ and } N^\gamma \subseteq A\} \subseteq A.$$

Note that A is γ -open [14] iff $A = int_\gamma(A)$. A set A is called γ -closed [1] iff $X - A$ is γ -open.

Definition 2.3. [16] A point $x \in X$ is called a γ -closure point of $A \subseteq X$, if $U^\gamma \cap A \neq \emptyset$, for each open nbd U of x . The set of all γ -closure points of A is called γ -closure of A and is denoted by $cl_\gamma(A)$. A subset A of X is called γ -closed, if $cl_\gamma(A) \subseteq A$. Note that $cl_\gamma(A)$ is contained in every γ -closed superset of A .

Definition 2.4. [16] An operation γ on τ is said to be regular, if for any open nbds U, V of $x \in X$, there exists an open nbd W of x such that $U^\gamma \cap V^\gamma \supseteq W^\gamma$.

Definition 2.5. [16] An operation γ on τ is said to be open, if for any open nbd U of each $x \in X$, there exists γ -open set B such that $x \in B$ and $U^\gamma \supseteq B$.

Definition 2.6. [13] A subset A of a space X is said to be a γ -semi-open set, if there exists a γ -open set O such that $O \subseteq A \subseteq cl_\gamma(O)$. The set of all γ -semi-open

sets is denoted by $SO_\gamma(X)$. A is γ -semi-closed if and only if $X - A$ is γ -semi-open in X . Note that A is γ -semi-closed if and only if $\text{int}_\gamma \text{cl}_\gamma(A) \subseteq A$ [2].

Definition 2.7. [2] Let A be a subset of a space X . The intersection of all γ -semi-closed sets containing A is called γ -semi-closure of A and is denoted by $\text{scl}_\gamma(A)$. Note that A is γ -semi-closed if and only if $\text{scl}_\gamma(A) = A$. The set of all γ -semi-closed subsets of A is denoted by $SC_\gamma(A)$.

Definition 2.8. [2] Let A be a subset of a space X . The union of γ -semi-open subsets contained in A is called γ -semi-interior of A and is denoted by $\text{shint}_\gamma(A)$.

Definition 2.9. [13] An operation γ on τ is said to be semi-regular, if for any semi-open sets U, V of $x \in X$, there exists a semi-open W of x such that $U^\gamma \cap V^\gamma \supseteq W^\gamma$.

Note that, if γ is semi-regular operation, then intersection of two γ -semi-open sets is γ -semi-open [13].

Definition 2.10. [2] A subset A of a space X is said to be γ -semi-regular, if it is both γ -semi-open and γ -semi-closed. The class of all γ -semi-regular sets of X is denoted by $SR_\gamma(A)$. Note that, if γ is a regular operation, then the union of γ -semi-regular sets is γ -semi-regular.

Definition 2.11. [16] A space X is said to be γ - T_2 space, if for each distinct points $x, y \in X$ there exists open sets U, V such that $x \in U, y \in V$ and $U^\gamma \cap V^\gamma = \phi$.

Definition 2.12. [3] Let X be a space and $A \subseteq X$. A point $x \in X$ is said to be a γ -adherent point of A , if $U^\gamma \cap A \neq \phi$, for every γ -open set U such that $x \in U$. A point x is a γ -adherent point for A if and only if $x \in \text{cl}_\gamma(A)$.

3. γ -S-Urysohn closed spaces

Definition 3.1. A space X is said to be γ -s-Urysohn, if for any two distinct points x and y there exist γ -semi-open sets U and V such that $x \in U, y \in V$ and $\text{cl}_\gamma(U) \cap \text{cl}_\gamma(V) = \phi$.

Definition 3.2. A γ -s-Urysohn space X is said to be γ -s-Urysohn-closed, if it is γ -closed in every γ -s-Urysohn space in which it can be embedded.

Definition 3.3. A filter base ξ is said to be a γ -s-Urysohn filter base, if whenever x is not γ -adherent point of ξ , there exists a γ -semi-open set U containing x such that $\text{cl}_\gamma(U) \cap \text{cl}_\gamma(F) = \phi$, for some $F \in \xi$. A γ -open cover δ of X is said to be a γ -s-Urysohn cover, if there exists a γ -semi-open cover ϱ of X such that for each $V \in \varrho$, there is a $U \in \delta$ such that $\text{cl}_\gamma(V) \subseteq U$.

We are interesting in characterize the γ -s-Urysohn-closed space when it is a γ - T_2 and γ -s-Urysohn space.

Theorem 3.4. *Let X be a γ -s-Urysohn and γ - T_2 space and γ be a regular and open operation. Then the following are equivalent:*

- (1) X is γ -s-Urysohn-closed.
- (2) Every γ -s-Urysohn cover δ of X has a finite subfamily δ^* such that the γ -closures of members of δ^* cover X .
- (3) Every γ -open γ -s-Urysohn filter base has nonempty γ -adherence.

Proof. (1) $\Rightarrow$ (2). Let X be γ -s-Urysohn-closed and let δ be a γ -s-Urysohn cover of X . Suppose on the contrary that for no finite subfamily of δ , the γ -closures of the members of the subfamily cover X . Let $p \notin X$ and let $Y = \{p\} \cup X$. Let $\mathfrak{S} = \tau \cup \{\{p\} \cup \{X - \bigcup_{i=1}^n cl_\gamma(U_i), U_i \in \delta\}\}$. Then $\mathfrak{S}$ is a topology on Y . Define an operation $\gamma\mathfrak{S}$ on $\mathfrak{S}$ as follows:

$$\begin{aligned} \gamma\mathfrak{S}(U) &= \gamma(U) & \text{if } U \in \tau \text{ and} \\ \gamma\mathfrak{S}(U) &= U & \text{otherwise.} \end{aligned}$$

We shall prove that $(Y, \mathfrak{S})$ is $\gamma\mathfrak{S}$ -s-Urysohn. Let x and y be two distinct points in Y . If $x \neq p \neq y$, there exist disjoint γ -semi-open sets H and K containing x and y respectively such that $cl_{\gamma\mathfrak{S}}(H) \cap cl_{\gamma\mathfrak{S}}(K) = \emptyset$. Suppose now, that one of x and y , say is p . Since δ is a γ -s-Urysohn cover of X , there exists a γ -semi-open cover ϱ of X such that for each $V \in \varrho$, there exists a $U_V \in \delta$ where $cl_\gamma(V) \subseteq U_V$. Let $x \in V \in \varrho$. Now, $\{p\} \cup (X - cl_\gamma(U_V))$ is $\gamma\mathfrak{S}$ -open (and hence $\gamma\mathfrak{S}$ -semi-open) set containing $y = p$. Since $cl_\gamma(V) \subseteq U_V$, $cl_{\gamma\mathfrak{S}}(V) \cap (\{p\} \cup (X - U_V)) = \emptyset$. That is, $cl_{\gamma\mathfrak{S}}(V) \cap cl_{\gamma\mathfrak{S}}(\{p\} \cup (X - cl_\gamma(U_V))) = \emptyset$, since $\gamma\mathfrak{S}$ is regular. Hence $(Y, \mathfrak{S})$ is $\gamma\mathfrak{S}$ -s-Urysohn. But X is not a γ -closed subset of $(Y, \mathfrak{S})$ since $p \in cl_{\gamma\mathfrak{S}}(X)$. This is a contradiction to the fact that X is γ -s-Urysohn-closed. Hence (2) is true.

(2) $\Rightarrow$ (3). We suppose contrarily that ξ be a γ -s-Urysohn filter base without any γ -adherent point. Then $\delta = \{X - cl_\gamma(F) : F \in \xi\}$ is a γ -open cover of X . We shall prove that δ is a γ -s-Urysohn cover. Let $x \in X$. Since x is not a γ -adherent point of ξ , there exists a γ -semi-open set V_x such that $cl_\gamma(V_x) \cap cl_\gamma(F_x) = \emptyset$ for some $F_x \in \xi$. Therefore, there exist $X - cl_\gamma(F_x) \in \delta$ such that $cl_\gamma(V_x) \subseteq X - cl_\gamma(F_x)$. Hence, $\varrho = \{V_x : x \in X\}$ and $cl_\gamma(V_x) \subseteq X - cl_\gamma(F_x)$ is a γ -semi-open over of X . Thus δ is a γ -s-Urysohn cover. Therefore, $X = \bigcup_{i=1}^n cl_\gamma(X - cl_\gamma(F_{x_i})) \subseteq \bigcup_{i=1}^n cl_\gamma(X - F_{x_i}) = \bigcup_{i=1}^n (X - (F_{x_i}))$, since each F_{x_i} is γ -open. Thus $X = X - \bigcap_{i=1}^n (F_{x_i})$ which means that $\bigcap_{i=1}^n (F_{x_i}) = \phi$, a contradiction. Thus (3) is proved.

(3) $\Rightarrow$ (1). Contrarily, suppose that X be a γ -s-Urysohn space which is not γ -s-Urysohn-closed. Let Y be a γ -s-Urysohn space in which X is embedded. If possible, suppose that X is not a γ -closed subset of Y . Let $p \in cl_\gamma(X) - X$ and $\delta = \{U \cap X \text{ where } U \text{ is a } \gamma\text{-open subset of } Y \text{ containing } p\}$. Then δ is a γ -s-Urysohn filter base in X . Since X is a γ -s-Urysohn and $\gamma - T_2$, it can be easily verified that δ has no γ -adherent point in X . This is a contradiction. This completes the proof.

Theorem 3.5. *Every γ -clopen subset of a γ -s-Urysohn-closed space is γ -s-Urysohn-closed.*

Proof. Let X be a γ -s-Urysohn-closed and let $Y \subseteq X$ be γ -open. Let ξ be a γ -s-Urysohn filter base in Y with empty γ -adherence. Y being γ -open is γ -s-Urysohn. Also ξ is a γ -open filter base in X . We shall claim that ξ is a γ -s-Urysohn filter base in X . If every point of x is a γ -adherent point of ξ in X , then ξ is of course γ -s-Urysohn in X . Let x be a point in X which is not a γ -adherent point of ξ in X . Since every γ -semi-open subset V of X containing x have nonempty intersection with every $F \in \xi$. Then $V \cap Y$ is a γ -semi-open subset of X [2] and hence of Y . Since Y is a γ -semi-closed subset of X , $x \in Y$. Thus $V \cap Y$ is a γ -semi-open subset of Y containing x having nonempty intersection with every member of ξ . Also, Y being γ -open, every γ -semi-open subset of Y is of the form $V \cap Y$, where V is a γ -semi-open subset of X . Hence every γ -semi-open subset of Y containing x intersects every member of ξ , which is a contradiction to the fact that ξ is a γ -s-Urysohn filter base in Y . Y being γ -closed. ξ can not have a γ -adherent point in X , since it has empty γ -adherent in Y . This is a contradiction. Hence the proof.

Definition 3.6. A point x is said to be a γ -s-adherent point of a filter base ξ , if $x \in scl_\gamma(F)$, for every $F \in \xi$.

Definition 3.7. [17] Let (X, τ) and (Y, δ) be spaces. Let $\gamma : \rightarrow P(X)$ and $\beta : \delta \rightarrow P(Y)$ be operations. Let $(X \times Y, \tau \times \delta)$ be the product space and let $\rho : \tau \times \delta \rightarrow P(X \times Y)$ be an operations on $\tau \times \delta$. Then ρ is called associative with (γ, β) , if $(U \times V)^\rho = U^\gamma \times V^\beta$ holds for each nonvoid $U \in \tau$ and nonvoid $V \in \delta$.

It is known in [17] that, if $A \subseteq X$ and $B \subseteq Y$. Then $cl_\rho(A \times B) = cl_\gamma(A) \times cl_\beta(B)$.

Lemma 3.8. *Let X be a space and γ be a regular operation such that every γ -open filter base in X has nonempty γ -s-adherent and let Y be an arbitrary space. If ξ is a ρ -open ρ -s-Urysohn filter base in $X \times Y$, then $P_y(\xi)$ is a ρ -open ρ -s-Urysohn filter base in Y , where P_y is the projection function from $X \times Y$ onto Y . Where γ , β and ρ are operations on X , Y and $X \times Y$ respectively.*

Proof. ξ is a ρ -open filter base in $X \times Y$. Then $P_x(\xi)$ is a γ -open filter base in X and hence has a γ -s-adherent point, say $x \in X$. We shall prove that $P_x(\xi)$ is a β -s-Urysohn filter base in Y . $P_y(\xi)$ is a β -open filter base in Y . Suppose that y is not a β -adherent point of $P_y(\xi)$. Then (x, y) is not a ρ -adherent point of ξ in $X \times Y$. It is given that ξ is a ρ -s-Urysohn filter base in $X \times Y$. Hence there exists a ρ -semi-open subset $U \times V$ of $X \times Y$ containing (x, y) such that $cl_\rho(U \times V) \cap cl_\rho(F) = \emptyset$, for some $F \in \xi$. Since $U \times V$ is a ρ -semi-open subset of $X \times Y$ containing (x, y) , U is a γ -semi-open subset of X containing x and V is a β -semi-open subset of Y containing y [2]. Also, we have $cl_\rho(U \times V) \cap cl_\rho(F) = \emptyset$. That is, $(cl_\gamma(U) \times cl_\beta(V)) \cap cl_\rho(F) = \emptyset$ for some $F \in \xi$. Hence $cl_\beta(V) \cap cl_\beta(P_y(F)) = \emptyset$ for some $F \in \xi$, since x is a γ -s-adherent point of $P_x(\xi)$.

4. γ -s-Regular spaces

Definition 4.1. A γ -s-regular space is said to be γ -s-regular-closed, if it is γ -closed in every γ -s-regular space in which it can be embedded.

Definition 4.2. A cover δ is said to be a γ -s-regular cover, if there exists a γ -semi-open cover ϱ such that the γ -semi-closures of members of ϱ refine δ .

Definition 4.3. A filter base is said to be a γ -s-regular filter base, if it is equivalent to a γ -semi-closed filter base.

Theorem 4.4. Let X be a γ - T_2 space and γ be a regular operation. Then the following are equivalent for a γ -s-regular space X :

- (1) X is γ -s-regular-closed.
- (2) Every γ -open γ -s-regular cover has a finite subcover.
- (3) Every γ -s-regular filter base has nonempty γ -adherence.

Proof. (1) $\Rightarrow$ (2). Let δ be a γ -open γ -s-regular cover of X and suppose that δ does not have a finite subcover and $p \notin X$. Let $Y = \{p\} \cup X$. Let $\mathfrak{S} = \tau \cup \{\{p\} \cup \{X - \bigcup_{i=1}^n U_i : U_i \in \delta\}\}$. Then $\mathfrak{S}$ is a topology on Y . Define an operation $\gamma\mathfrak{S}$ on $\mathfrak{S}$ as follows: $\gamma\mathfrak{S}(U) = \gamma(U)$ if $U \in \tau$ and $\gamma\mathfrak{S}(U) = U$ otherwise. We shall prove that $(Y, \mathfrak{S})$ is $\gamma\mathfrak{S}$ -s-regular.

Case (1). Let $x \in Y$, where $x \neq p$ and B be a $\mathfrak{S}$ -closed set containing neither x nor p . Then $Y - B$ is a $\mathfrak{S}$ -open set containing both x and p . Since δ is a γ -s-regular cover, there exists a γ -semi-open cover ϱ of X such that $\{scl_\gamma(V) : V \in \varrho\}$ is a refinement of δ . Since $x \in X$, $x \in V$ for some $V \in \varrho$. Now, $x \in Y - B$ where $(Y - B)$ is a $\gamma\mathfrak{S}$ -open set containing p . Hence $Y - B = \{p\} \cup \{X - \bigcup_{i=1}^n (U_i) : U_i \in \delta\}$. Therefore, $x \in V \cap (\{p\} \cup (X - \bigcup_{i=1}^n (U_i)))$. The set $V \cap (\{p\} \cup (X - \bigcup_{i=1}^n (U_i)))$ is a $\mathfrak{S}$ -open set, being an intersection of $\mathfrak{S}$ -open (and hence $\mathfrak{S}$ -semi-open) set. Thus $V \cap (\{p\} \cup (X - \bigcup_{i=1}^n (U_i)))$ is a $\mathfrak{S}$ -semi-open set containing x . Also $Y - B = \{p\} \cup (X - \bigcup_{i=1}^n (U_i)) = Y - \bigcup_{i=1}^n (U_i)$. Hence $B = \bigcup_{i=1}^n (U_i)$. Thus $V \cap (\{p\} \cup (X - \bigcup_{i=1}^n (U_i)))$ and $\bigcup_{i=1}^n (U_i)$ are disjoint $\mathfrak{S}$ -semi-open sets containing x and B respectively.

Case 2. Now, let us suppose that $x = p$ and B is $\mathfrak{S}$ -closed set not containing p . Then $Y - B = \{p\} \cup \{X - \bigcup_{i=1}^n (U_i) : U_i \in \delta\}$. Therefore, $\{p\} \cup \left(X - \bigcup_{i=1}^n (U_i)\right)$ is $\mathfrak{S}$ -open set and hence a $\gamma\mathfrak{S}$ -semi-open set containing p and $\bigcup_{i=1}^n U_i$ is a $\mathfrak{S}$ -open set containing B .

Case 3. Suppose that $x \neq p$ and let B be a $\gamma\mathfrak{S}$ -closed set containing p . Then $Y - B$ is a $\mathfrak{S}$ -open set containing x and not containing p . Hence $Y - B$ is a τ -open set containing x . Therefore there exists a γ -semi-open subset V of X such that $x \in V \subseteq scl_\gamma(V) \subseteq Y - B$. Thus $(Y, \mathfrak{S})$ is $\gamma\mathfrak{S}$ -s-regular. But X is not a γ -closed subset of Y since every $\gamma\mathfrak{S}$ -open set containing p intersect X . This is a contradiction. Hence δ has a finite subcover.

(2) $\Rightarrow$ (3). Let ξ be a γ -s-regular filter base without a γ -adherent point in X . Then $\{X - cl_\gamma(F) : F \in \xi\}$ is a γ -open cover of X . We shall prove that this is a γ -s-regular cover. Let $x \in X$. Then x is not a γ -adherent point of ξ , so that $x \notin cl_\gamma(F_x)$, for some $F_x \in \xi$. Since X is γ -s-regular, there exists a γ -semi-open set V_x containing x such that $x \in V_x \subseteq scl_\gamma(V_x) \subseteq X - cl_\gamma(F_x)$. Thus $\{V_x : x \in X \text{ and } scl_\gamma(V_x) \subseteq X - cl_\gamma(F_x)\} = \delta$ is a γ -semi-open cover of X such that $\{scl_\gamma(V_x) : V_x \in \delta\}$ is a refinement of $\{X - cl_\gamma(F_x) : F_x \in \delta\}$. Hence $\{X - cl_\gamma(F) : F \in \delta\}$ is a γ -s-regular cover of X . Therefore there exist finite many members $F_1, F_2, \dots, F_n$ of δ such that $X = \bigcup_{i=1}^n (X - cl_\gamma(F_i)) \subseteq \bigcup_{i=1}^n (X - F_i) = X - \bigcap_{i=1}^n F_i$. That is

$\bigcap_{i=1}^n F_i = \phi$, a contradiction to the fact that δ is a filter base.

(3) $\Rightarrow$ (1). If possible, suppose that X is not γ -s-regular-closed. Let X be embedded in a γ -s-regular space Y and let $cl_{\gamma_y}(X) - X \neq \phi$. Let $p \in cl_{\gamma_y}(X) - X$. Let $\delta = \{X \cap U : U \text{ is a nbd of } p \text{ in } Y\}$ and let $\varrho = \{X \cap V : V \text{ is the } \gamma\text{-semi-closure of a } \gamma\text{-semi-open subset of } Y \text{ containing } p\}$. Since Y is γ -s-regular, δ and ϱ are equivalent and they are filter bases in X . It can be verified that each $V \in \varrho$ is a γ -semi-closed subset of X . Hence δ is a γ -s-regular filter base in X . δ does not have a γ -adherent point in X , since X is γ - T_2 . This is a contradiction. Hence the proof.

Definition 4.5. [6] A function $f : X \rightarrow Y$ is said to be γ -irresolute, if the inverse image of every γ -semi-open set is γ -semi-open.

Theorem 4.6. *If $X = \Pi_{i \in I} X_i$ is γ -s-regular-closed, then each X_i is γ -s-regular-closed provided each X_i is γ -s-regular.*

Proof. Let ξ_i be a γ -s-regular filter base in X_i . Let δ_i be the γ -semi-closed filter base in X_i which is equivalent to ξ_i . Then $\{P_i^{-1}(F_i) : F_i \in \xi_i\}$ is a filter base in X and $\{P_i^{-1}(U_i) : U_i \in \xi_i\}$ is a γ -semi-closed filter base in X , since the projection mapping is γ -irresolute[6]. Also $\{P_i^{-1}(U_i) : U_i \in \xi_i\}$ is equivalent to $\{P_i^{-1}(F_i) : F_i \in \xi_i\}$. Thus $\{P_i^{-1}(F_i) : F_i \in \xi_i\}$ is a γ -s-regular filter base in X . If $x = (x_i)$ is a γ -adherent of $\{P_i^{-1}(F_i) : F_i \in \xi_i\}$ in X . Then x_i is a γ -adherent point of ξ_i . Thus X_i is γ -s-regular-closed.

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Accepted: 15.06.2011

(0, 2; 0)-INTERPOLATION ON THE UNIT CIRCLE**Swarnima Bahadur**

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Abstract. In this paper, we study the explicit representation and convergence of Pál-type weighted $(0, 2; 0)$ -interpolation on two pairwise disjoint sets of nodes on the unit circle, which are obtained by projecting vertically the zeros of $(1 - x^2) P_n(x)$ and $P'_n(x)$ respectively on the unit circle, where $P_n(x)$ stands for n^{th} Legendre polynomial.

Keywords: Legendre polynomial, weighted $(0, 2)$ -interpolation, Pál-type interpolation, regularity, explicit forms, convergence.

2000 Mathematics Subject Classification: 41A05, 30E10.

1. Introduction

P. Turán initiated the study of $(0, 2)$ -interpolation in order to get approximate solution of the differential equation $y'' + f \cdot y = 0$. The first result was published by J. Surányi and P. Turán [19] in 1955. J. Balázs [2] introduced a generalization of this problem and considered the weighted $(0, 2)$ -interpolation on the zeros of ultraspherical polynomial $P_n^{(\lambda)}(x)$, $\lambda > -1$. Then several mathematicians [5], [8], [9], [13], [15], [16], [22], [23], [24] considered the weighted $(0, 2)$ -interpolation on various set of nodes. O. Kiš [11] was first to consider the interpolation problem on the unit circle and then a lot of mathematicians [17], [18], [20] studied different interpolation problems on the unit circle. In paper [1], author proved the convergence theorem of weighted $(0, 2)^*$ -interpolation on the projected nodes on the unit circle.

In case of Pál-type interpolation, P. Mathur and S. Datta [14] considered the weighted Pál-type $(0, 2; 0)$ -interpolation on $(-\infty, \infty)$. Recently M. Lenard [12] has considered the $(0, 2; 0)$ and $(0; 0, 2)$ -type interpolation problem on the zeros of Legendre polynomial $P_n(x)$ and give the explicit formulae. H.P. Dikshit [7] considered the existence of Pál-type interpolation on the non-uniformly distributed nodes on the unit circle. Later on, in [2], [3], author considered the convergence of $(0, 1; 0)$ and $(0; 0, 1)$ -interpolation on the sets obtained by projecting vertically

the zeros of $(1 - x^2) P_n(x)$ and $P'_n(x)$ respectively on the unit circle, where $P_n(x)$ stands for n^{th} Legendre polynomial. In paper [4], author considered the $(0; 0, 2)$ -interpolation on the projected nodes of $(1 - x^2) P_n(x)$ and $P'_n(x)$ respectively on the unit circle and proved a convergence theorem. The aim of this paper is to consider explicit representation and convergence of weighted Pál-type $(0, 2; 0)$ -interpolation on the unit circle.

Let $Z_n = \{z_k = k = 0, 1, \dots, 2n + 1\}$ satisfying

$$(1.1) \quad \begin{cases} z_0 = 1, z_{2n+1} = -1 \\ z_k = \cos \theta_k + i \sin \theta_k, \\ z_{n+k} = -z_k, \quad k = 1(1)n \end{cases}$$

and $T_n = \{\omega_k = k = 1, \dots, 2n - 2\}$ such that

$$(1.2) \quad \begin{cases} \omega_k = \cos \phi_k + i \sin \phi_k, \\ \omega_{n+k} = -\omega_k, \quad k = 1(1)n - 1 \end{cases}$$

be two set of nodes such that the weighted $(0, 2)$ -interpolation is prescribed on the one set of nodes, whereas Lagrange interpolation on the points of other one

In Section 2 we give some preliminaries, in Section 3 we describe the problem, in Section 4 we represent the explicit forms of interpolatory polynomials, and in Sections 5 and 6 we give the estimates and convergence of interpolatory polynomials.

2. Preliminaries

In this section, we shall give some well known results, which we shall use in our present paper.

The differential equation satisfied by $P_n(x)$ is

$$(2.1) \quad (1 - x^2) P_n''(x) - 2x P_n'(x) + n(n + 1) P_n(x) = 0$$

$$(2.2) \quad W(z) = \prod_{k=1}^{2n} (z - z_k) = K_n P_n \left(\frac{1 + z^2}{2z} \right) z^n$$

$$(2.3) \quad H(z) = \prod_{k=1}^{2n-2} (z - \omega_k) = K_n^* P'_n \left(\frac{1 + z^2}{2z} \right) z^{n-1}$$

We shall require the fundamental polynomials of Lagrange interpolation based on Z_n and T_n

$$(2.4) \quad L_k(z) = \frac{W(z)}{W'(z_k)(z - z_k)}$$

$$(2.5) \quad l_k(z) = \frac{H(z)}{H'(\omega_k)(z - \omega_k)}$$

$$(2.6) \quad J_k(z) = \int_0^z z^{2n+2} L_k(z) dz$$

$$(2.7) \quad J_{1j}(z) = \int_0^z z^{2n+j} W(z) dz, \quad j = 0, 1,$$

which satisfies

$$(2.8) \quad J_{1j}(-1) = (-1)^{j+1} J_{1j}(1).$$

For $-1 \leq x \leq 1$,

$$(2.9) \quad |P_n(x)| \leq 1$$

$$(2.10) \quad (1 - x^2)^{\frac{1}{4}} |P_n(x)| \leq \left(\frac{2}{n\pi} \right)^{\frac{1}{2}}$$

$$(2.11) \quad (1 - x^2)^{\frac{3}{4}} |P'_n(x)| \leq \sqrt{2n}$$

$$(2.12) \quad |P'_n(x)| \leq \frac{n(n+1)}{2}.$$

If u_k be the zeros of $P'_n(x)$, then by [6]

$$(2.13) \quad P_n(u_k) > \frac{1}{\sqrt{8\pi k}}.$$

Let $x_k = \cos \theta_k$, ($k = 1, 2, \dots, n$) be the zeros of the n^{th} Legendre polynomial $P_n(x)$, with $1 > x_1 > x_2 > \dots > -1$, then

$$(2.14) \quad \begin{cases} (1 - x_k^2)^2 \geq k^2 n^{-2}, & k = 1, \dots, \left[\frac{n}{2} \right] \\ (1 - x_k^2)^2 \geq (n - k + 1)^2 n^{-2}, & k = \left[\frac{n}{2} \right] + 1, \dots, n \end{cases}$$

$$(2.15) \quad \begin{cases} |P'_n(x_k)| \geq k^{-\frac{3}{2}} n^2, & k = 1, \dots, \left[\frac{n}{2} \right] \\ |P'_n(x_k)| \geq (n - k + 1)^{-\frac{3}{2}} n^2, & k = \left[\frac{n}{2} \right] + 1, \dots, n \end{cases}$$

For more details, see [20], [21].

3. The Problem

Let $\{z_k\}_{k=0}^{2n+1}$ and $\{\omega_k\}_{k=1}^{2n-2}$ be two disjoint set of nodes obtained by projecting vertically the zeros of $(1-x^2)P_n(x)$ and $P'_n(x)$ respectively on the unit circle, where $P_n(x)$ stands for n^{th} Legendre polynomial and $w(z) \in \mathbb{C}$ be a given function, called weight function. Here we are interested to determine the convergence of interpolatory polynomials satisfying the conditions:

$$(3.1) \quad \begin{cases} R_n(z_k) = \alpha_k, & k = 0, 1, 2, \dots, 2n+1 \\ \{w(z)R_n(z)\}''_{z=z_k} = \beta_k, & k = 1, 2, \dots, 2n \\ R_n(\omega_k) = \gamma_k, & k = 1, 2, \dots, 2n-2 \end{cases}$$

where $w(z) = (1-z^2)^{\frac{5}{2}}$ and α_k, β_k and γ_k are complex constants.

4. Explicit representation of interpolatory polynomials

We shall write $R_n(z)$ satisfying (3.1) as

$$(4.1) \quad R_n(z) = \sum_{k=0}^{2n+1} \alpha_k A_k(z) + \sum_{k=1}^{2n} \beta_k B_k(z) + \sum_{k=1}^{2n-2} \gamma_k C_k(z),$$

where $A_k(z)$, $B_k(z)$ and $C_k(z)$ are unique polynomials each of degree $\leq 6n-1$ determined by the following conditions:

For $k = 0(1)2n+1$

$$(4.2) \quad \begin{cases} A_k(z_j) = \delta_{kj}, & j = 0(1)2n+1 \\ \left\{ (1-z^2)^{\frac{5}{2}} A_k(z) \right\}''_{z=z_j} = 0, & j = 1(1)2n \\ A_k(\omega_j) = 0, & j = 1(1)2n-2 \end{cases}$$

For $k = 1(1)2n$

$$(4.3) \quad \begin{cases} B_k(z_j) = 0, & j = 0(1)2n+1 \\ \left\{ (1-z^2)^{\frac{5}{2}} B_k(z) \right\}''_{z=z_j} = \delta_{kj}, & j = 1(1)2n \\ B_k(\omega_j) = 0, & j = 1(1)2n-2 \end{cases}$$

For $k = 1(1)2n-2$

$$(4.4) \quad \begin{cases} C_k(z_j) = 0, & j = 0(1)2n+1 \\ \left\{ (1-z^2)^{\frac{5}{2}} C_k(z) \right\}''_{z=z_j} = 0, & j = 1(1)2n \\ C_k(\omega_j) = \delta_{kj}, & j = 1(1)2n-2 \end{cases}$$

Theorem 4.1. For $k = 1(1)2n - 2$

$$(4.5) \quad C_k(z) = \frac{z^{-2n}(z^2 - 1)W^2(z)L_k(z)}{\omega_k^{-2n}(\omega_k^2 - 1)W^2(\omega_k)L_k(\omega_k)} l_k(z) \\ + \frac{z^{-2n-1}W(z)H(z)}{\omega_k^{-2n}(\omega_k^2 - 1)W^2(\omega_k)H'(\omega_k)L_k(\omega_k)} \{G_k(z) + c_1 J_{10}(z) + c_2 J_{11}(z)\}$$

where

$$(4.6) \quad G_k(z) = \int_0^z (1 - z^2) \frac{zW'(z) + c_k W(z)}{z - \omega_k} L_k(z) dz$$

with

$$(4.7) \quad c_k = -\omega_k \frac{W'(\omega_k)}{W(\omega_k)} \\ c_1 = -\frac{G_k(1) - G_k(-1)}{2J_{10}(1)} \quad \text{and} \\ c_2 = -\frac{G_k(1) + G_k(-1)}{2J_{11}(1)}.$$

Proof. Consider (4.5).

Clearly, $C_k(z_j) = 0$ for $j = 1(1)2n$ and $C_k(\omega_j) = \delta_{kj}$ for $j = 1(1)2n - 2$.

Also, for $z = \pm 1$, we get (4.7).

Further, from $\left\{ (1 - z^2)^{\frac{5}{2}} C_k(z) \right\}_{z=z_j}'' = 0$ for $j = 1(1)2n$, we get $G'_k(z)$, which on integration gives (4.6).

Theorem 4.2. For $k = 1(1)2n$

$$(4.8) \quad B_k(z) = z^{-2n-1}W(z)H'(z) \{b_k J_k(z) + b_1 J_{10}(z) + b_2 J_{11}(z)\}$$

where

$$(4.9) \quad b_k = \frac{1}{2z_k(z_k^2 - 1)^{\frac{5}{2}} W'(z_k)H'(z_k)}$$

$$(4.10) \quad b_1 = -b_k \frac{J_k(1) - J_k(-1)}{2J_{10}(1)} \quad \text{and} \quad b_2 = -b_k \frac{J_k(1) + J_k(-1)}{2J_{11}(1)}.$$

Proof. Consider (4.8).

Clearly, $B_k(z_j) = 0$ for $j = 1(1)2n$ and $B_k(\omega_j) = \delta_{kj}$ for $j = 1(1)2n - 2$.

Again, for $z = \pm 1$, we get (4.10).

Further, from $\left\{ (1 - z^2)^{\frac{5}{2}} B_k(z) \right\}_{z=z_j}'' = \delta_{kj}$ for $j = 1(1)2n$, we get (4.9) for $j = k$, owing to (2.7) and (2.14)-(2.15).

For $j \neq k$, one can verify the result.

Theorem 4.3. For $k = 1(1)2n$,

$$(4.11) \quad A_k(z) = \frac{(z^2 - 1) H(z) L_k^2(z)}{(z_k^2 - 1) H(z_k)} + \frac{z^{-2n-1} W(z) H(z)}{(z_k^2 - 1) W'(\omega_k) H(z_k)} \{T_k(z) + a_1 J_{10}(z) + a_2 J_{11}(z)\} + a_k B_k(z)$$

where

$$(4.12) \quad a_k = - (z_k^2 - 1)^{\frac{5}{2}} \left\{ \frac{H''(z_k)}{H'(z_k)} + \frac{2z_k}{(z_k^2 - 1)} \frac{H'(z_k)}{H(z_k)} + \frac{35z_k}{(z_k^2 - 1)^2} + \frac{7}{(z_k^2 - 1)} + \frac{28z_k}{(z_k^2 - 1)} L'_k(z_k) + 4 \frac{H'(z_k)}{H(z_k)} L'_k(z_k) \right\}$$

$$(4.13) \quad T_k(z) = \int_0^z z^{2n+1} (1 - z^2) \frac{L_k(z) - L'_k(z_k) L_k(z)}{z - z_k} dz$$

$$(4.14) \quad a_1 = -\frac{T_k(1) - T_k(-1)}{2J_{10}(1)} \quad \text{and} \quad a_2 = -\frac{T_k(1) + T_k(-1)}{2J_{11}(1)}.$$

Further, for $k = 0, 2n + 1$

$$(4.15) \quad A_k(z) = z^{-2n-1} W(z) H(z) \{a_1 J_{10}(z) + a_2 J_{11}(z)\}$$

where

$$(4.16) \quad a_1 = \frac{1}{2W(1)H(1)J_{10}(1)}$$

$$(4.17) \quad a_2 = \frac{1}{2W(1)H(1)J_{11}(z)}.$$

Proof. One can check that in this theorem (4.11) and (4.15) are polynomials of required degree satisfying (4.2), so we omit the details of proof.

5. Estimation of interpolatory polynomials

For convergence of interpolatory polynomial, we need the estimates of fundamental polynomials:

Lemma 1. *Let $A_k(z)$ be defined in (4.11), then for $|z| \leq 1$, we have*

$$(5.1) \quad \sum_{k=1}^{2n} |A_k(z)| \leq cn^{\frac{3}{2}} \log n$$

$$(5.2) \quad |A_0(z)| \leq c, \quad |A_{2n+1}(z)| \leq c,$$

where c is a constant independent of n and z .

Lemma 2. *Let $B_k(z)$ be defined in (4.8), then for $|z| \leq 1$, we have*

$$(5.3) \quad \sum_{k=1}^{2n-2} |B_k(z)| \leq c \frac{\log n}{n^{\frac{1}{2}}}$$

where c is a constant independent of n and z .

Lemma 3. *Let $C_k(z)$ be defined in (4.5), then for $|z| \leq 1$, we have*

$$(5.4) \quad \sum_{k=1}^{2n-2} |C_k(z)| \leq cn^{\frac{3}{2}} \log n$$

where c is a constant independent of n and z .

Proof of Lemma 3. Consider (4.5) and using (2.10)-(2.15), we get (5.4), owing to results from [20], [21]. So we can omit the details of the proof.

Proof of Lemma 2. Consider (4.8), we have

$$(5.5) \quad |B_k(z)| \leq |P_n(x) P'_n(x)| \{ |b_k| |J_k(z)| + |b_1| |J_{10}(z)| + |b_2| |J_{11}(z)| \},$$

where

$$(5.6) \quad |b_k| \leq \frac{1}{(1 - x_k^2)^{\frac{7}{4}} |P'_n(x_k)|^2}$$

$$(5.7) \quad |J_k(z)| \leq \max_{|z| \leq 1} |L_k(z)| \int_0^z t^{2n+2} dt,$$

by substituting $z = e^{i\theta}$ ($0 \leq \theta < 2\pi$).

Combining (5.5)-(5.8) and using (2.10)-(2.15), we get (5.3).

Similarly one can prove Lemma 1.

6. Convergence

In this section we prove the following:

Theorem 6.1. *Let $f(z)$ be continuous in $|z| \leq 1$ and analytic in $|z| < 1$. Let the arbitrary numbers β_k 's be such that*

$$(6.1) \quad |\beta_k| = o\left(n^{\frac{3}{2}}\omega_2(f, n^{-1})\right), \quad k = 1(1)2n.$$

Then the sequence $\{R_n\}$ be defined by

$$(6.2) \quad R_n(z) = \sum_{k=0}^{2n+1} f(z_k)A_k(z) + \sum_{k=1}^{2n} \beta_k B_k(z) + \sum_{k=1}^{2n-2} \gamma_k C_k(z)$$

satisfies the relation

$$(6.3) \quad |R_n(z) - f(z)| = o\left(n^{\frac{3}{2}} \log n \cdot \omega_2(f, n^{-1})\right)$$

where $\omega_2(f, n^{-1})$ is the modulus of smoothness of $f(z)$.

Remark. Let $f(z)$ be continuous in $|z| \leq 1$ and $f' \in Lip \alpha$, $\alpha > \frac{1}{2}$, then the sequence $\{R_n\}$ converges uniformly to $f(z)$ in $|z| \leq 1$, follows from (6.3) provided

$$(6.4) \quad \omega_2(f, n^{-1}) = o(n^{-1-\alpha}).$$

To prove Theorem 6.1, we shall need the following:

Let $f(z)$ be continuous in $|z| \leq 1$ and analytic in $|z| < 1$. Then there exists a polynomial $F_n(z)$ of degree $2n - 2$ satisfying Jackson's inequality

$$(6.5) \quad |f(z) - F_n(z)| \leq c\omega_2(f, n^{-1}), \quad z = e^{i\theta} (0 \leq \theta < 2\pi)$$

and also an inequality due to O. Kiš [11] viz. :

$$(6.6) \quad F_n^{(m)}(z) \leq cn^m \omega_2\left(f, \frac{1}{n}\right),$$

for positive integer m .

Proof. Let $z = e^{i\theta}$ ($0 \leq \theta < 2\pi$), using (6.1), (6.2), (6.4)–(6.6) and Lemmas 1, 2 and 3, we get the result.

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Accepted: 10.12.2011

ON LAPLACE TRANSFORMS OF GENERALIZED WHITTAKER FUNCTION OF MULTI-VARIABLES $M_{\lambda, \mu_1 \dots \mu_k}(x_1, \dots, x_k)$

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Abstract. In this paper, a Laplace transform of generalized Whittaker function is derived which is used to obtain further some partly bilateral and partly unilateral generating function and series expansion. Some special cases are also discussed.

Keywords: Generalized Whittaker function, Lauricella's function, Appell's function and Laplace transform.

2000 AMS Subject Classification: 33C15, 33C65, 33C70.

1. Introduction and definition

A Whittaker function $M_{\lambda, \mu}$ was introduced by Whittaker [2] (see also Whittaker and Watson [3]) in terms of the confluent hypergeometric function ${}_1F_1$ (or Kummer's function)

$$(1.1) \quad M_{\lambda, \mu}(x) = x^{\mu+1/2} e^{(-1/2)x} {}_1F_1\left(\frac{1}{2} + \mu - \lambda, 2\mu + 1; x\right).$$

Further generalization of the Whittaker function $M_{\lambda, \mu}$ was introduced by Humbert [6; p.63 (15)] in the following form

$$(1.2) \quad M_{\lambda, \mu_1 \dots \mu_k}(x_1, \dots, x_k) = x_1^{\mu_1+1/2} \dots x_k^{\mu_k+1/2} \exp\left[\frac{-1}{2}(x_1 + \dots + x_k)\right] \\ \cdot \Psi_2^{(k)}[\mu_1 + \dots + \mu_k - \lambda + k/2; 2\mu_1 + 1, \dots, 2\mu_k + 1; x_1, \dots, x_k],$$

where $\Psi_2^{(k)}$ denotes Humbert's confluent hypergeometric function of n -variables [6; p. 62(11)]

$$\begin{aligned}
& \Psi_2^{(k)}[a; c_1, \dots, c_k; x_1, \dots, x_k] \\
(1.3) \quad &= \sum_{m_1 \dots m_k=0}^{\infty} \frac{(a)_{m_1+\dots+m_k}}{(c_1)_{m_1} \dots (c_k)_{m_k}} \frac{x_1^{m_1} \dots x_k^{m_k}}{m_1! \dots m_k!} (\max\{|x_1|, |x_2|, \dots, |x_k|\} < \infty).
\end{aligned}$$

A Lauricella function $F_C^{(k)}$ [6; p. 60] generalized the Appell function F_4 to a function of k variables which is defined as

$$\begin{aligned}
(1.4) \quad F_C^{(k)}[a, b; c_1, \dots, c_k; x_1, \dots, x_k] &= F_{0:1;\dots;1}^{2:0;\dots;0} \left[\begin{array}{ccc} a, b & : \_ ; \dots ; & - & ; \\ & & & x_1, \dots, x_k \\ \_ & : c_1 ; \dots ; & c_k & ; \end{array} \right] \\
&= \sum_{m_1, m_2, \dots, m_k=0}^{\infty} \frac{(a)_{m_1+\dots+m_k} (b)_{m_1+\dots+m_k}}{(c_1)_{m_1} \dots (c_k)_{m_k}} \frac{x_1^{m_1} \dots x_k^{m_k}}{m_1! \dots m_k!} \\
&\quad (\sqrt{|x_1|} + \dots + \sqrt{|x_k|} < 1).
\end{aligned}$$

2. Integral transforms

We first establish the following integral

$$\begin{aligned}
(2.1) \quad I &= \int_0^{\infty} u^{\nu-1} e^{-pu} M_{\lambda, \mu_1 \dots \mu_k}(x_1 u, \dots, x_k u) du \\
&= \frac{x_1^{\mu_1+1/2} \dots x_k^{\mu_k+1/2} \Gamma(b)}{(p+X)^b} F_C^{(k)} \left[a, b; 2\mu_1+1, \dots, 2\mu_k+1; \frac{x_1}{p+X}, \dots, \frac{x_k}{p+X} \right],
\end{aligned}$$

where $a = \mu_1 + \dots + \mu_k + \frac{k}{2} - \lambda$, $b = \mu_1 + \dots + \mu_k + \frac{k}{2} + \nu$, $X = \frac{x_1 + \dots + x_k}{2}$ and $Re(p+X) > 0$.

For $k = 2$, equation (2.1) reduces to

$$\begin{aligned}
(2.2) \quad I &= \int_0^{\infty} u^{\nu-1} e^{-pu} M_{\lambda, \mu_1, \mu_2}(x_1 u, x_2 u) du \\
&= \frac{x_1^{\mu_1+1/2} x_2^{\mu_2+1/2} \Gamma(\mu_1 + \mu_2 + 1 + \nu)}{(p+X)^{\mu_1+\mu_2+1+\nu}} \\
&\quad \cdot F_4 \left[\mu_1+\mu_2+1-\lambda, \mu_1+\mu_2+1+\nu; 2\mu_1+1, 2\mu_2+1; \frac{x_1}{p+X}, \frac{x_2}{p+X} \right] \\
&\quad (Re(p+X) > 0)
\end{aligned}$$

where as for $k = 1$, equation (2.1) reduces to a known result [1; p. 215(11)].

Proof of result (2.1)

By using definitions (1.2) and (1.3), the L.H.S. of integral (2.1) is given by

$$\begin{aligned}
I &= \int_0^\infty u^{\nu-1} e^{-pu} (x_1 u)^{\mu_1+1/2} \cdots (x_k u)^{\mu_k+1/2} \exp \left[\frac{-1}{2} (x_1 u + \cdots + x_k u) \right] \\
&\quad \cdot \Psi_2^{(k)} [\mu_1 + \cdots + \mu_k - \lambda + k/2; 2\mu_1 + 1, \dots, 2\mu_k + 1; x_1 u, \dots, x_k u] du \\
&= x_1^{\mu_1+1/2} \cdots x_k^{\mu_k+1/2} \sum_{m_1 \cdots m_k=0}^\infty \frac{(a)_{m_1+\cdots+m_k}}{(2\mu_1+1)_{m_1} \cdots (2\mu_k+1)_{m_k}} \frac{(x_1)^{m_1} \cdots (x_k)^{m_k}}{m_1! \cdots m_k!} \\
&\quad \cdot \int_0^\infty u^{\mu_1+\cdots+\mu_k+k/2+m_1+\cdots+m_k+\nu-1} e^{-(p+\frac{x_1+\cdots+x_k}{2})u} du,
\end{aligned}$$

and then, using the integral transform [1, p.137(1)], we get the main result (2.1).

In view of integral (2.1), we can also establish the following integral

$$\begin{aligned}
(2.3) \quad I &= \int_0^\infty u^{\nu-1} e^{-pu-zu^2} M_{\lambda, \mu_1 \cdots \mu_k} (x_1 u, \dots, x_k u) du \\
&= x_1^{\mu_1+1/2} \cdots x_k^{\mu_k+1/2} \sum_{q=0}^\infty \frac{(-1)^q z^q \Gamma(b+2q)}{q! (p+X)^{b+2q}} \\
&\quad \cdot F_C^{(k)} \left[a, b+2q; 2\mu_1+1, \dots, 2\mu_k+1; \frac{x_1}{p+X}, \dots, \frac{x_k}{p+X} \right].
\end{aligned}$$

where $a = \mu_1 + \cdots + \mu_k + \frac{k}{2} - \lambda$, $b = \mu_1 + \cdots + \mu_k + \frac{k}{2} + \nu$, $X = \frac{x_1 + \cdots + x_k}{2}$ and $Re(p+X) > 0$.

Clearly, if $z = 0$ in (2.3), the above result reduces to the result (2.1).

3. Generating relations

A well known modified generating relation of Exton is given by [5; p. 147(3)]

$$(3.1) \quad \exp \left(s + t - \frac{xt}{s} \right) = \sum_{m=-\infty}^\infty \sum_{n=m^*}^\infty \frac{s^m t^n}{m! n!} {}_1F_1(-n; m+1; x)$$

where ${}_1F_1(-n; m+1; x)/m!n! = L_n^{(m)}(x)/(m+n)!$, ${}_1F_1$ and $L_n^{(m)}$ are confluent hypergeometric function and Laguerre polynomials, respectively (see [4; p. 200(1)]). Pathan and Yasmeen [7] modified the above result (3.1) of Exton, by defining $m^* = \max[0, -m]$ and

$$\begin{aligned}
\frac{L_n^{(n)}(x)}{(m+n)!} &= \frac{1}{n!} \sum_{r=m^*}^n \frac{(-n)_r x^r}{(m+r)! r!} \quad \text{if } n \geq m^* \\
&= 0 \quad \text{if } 0 \leq n < m^*.
\end{aligned}$$

A set of expansions

$$x^r = 2^r \sum_{m=-\infty}^\infty \sum_{n=m^*}^\infty \frac{(-n)_r (x/2)^{m+n}}{m! n!} {}_1F_1(-n; m+1; x) \quad (3.2)$$

for $r = 0, 1, 2, \dots$, has been obtained by Exton [5, p. 148(8)] from (3.1) by taking successive partial derivatives with respect to t and letting $s = t = x/2$.

On replacing s, t and x by su, tu and xu , respectively, in (3.1), multiplying both sides by

$$u^{\nu-1} e^{-pu} M_{\lambda, \mu_1 \dots \mu_k}(x_1 u, \dots, x_k u)$$

integrating the multiple series with respect to u between the limits zero and infinity. Using integral (2.1) and definition (1.2) and adjusting the parameters, we get

$$\begin{aligned} & F_C^{(k)}[a, b; 2\mu_1+1, \dots, 2\mu_k+1; \frac{x_1}{p-s-t+\frac{xt}{s}+X}, \dots, \frac{x_k}{p-s-t+\frac{xt}{s}+X}] \\ &= \frac{(p-s-t+\frac{xt}{s}+X)^b}{(p+X)^b} \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{s^m t^n (b)_{m+n}}{m! n! (p+X)^{m+n}} \\ (3.3) \quad & \sum_{l=0}^n \frac{(-n)_l (b+m+n)_l}{(m+1)_l l!} \left(\frac{x}{p+X} \right)^l \\ & F_C^{(k)} \left[a, b+m+n+l; 2\mu_1+1, \dots, 2\mu_k+1; \frac{x_1}{p+X}, \dots, \frac{x_k}{p+X} \right] \end{aligned}$$

where $a = \mu_1 + \dots + \mu_k + \frac{k}{2} - \lambda$, $b = \mu_1 + \dots + \mu_k + \frac{k}{2} + \nu$, $X = \frac{x_1 + \dots + x_k}{2}$ and $Re(p) > 0$, $Re(p+X) > 0$.

Similarly, using the result (3.2) in place of (3.1), we get

$$\begin{aligned} & F_C^{(k)}[a, b+r; 2\mu_1+1, \dots, 2\mu_k+1; \frac{x_1}{p+X}, \dots, \frac{x_k}{p+X}] \\ &= \frac{2^r}{(b)_r x^r} \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{(-n)_r (b)_{m+n} (x/2)^{m+n}}{m! n! (p+X)^{m+n-r}} \\ (3.4) \quad & \cdot \sum_{l=0}^n \frac{(-n)_l (b+m+n)_l}{(m+1)_l l!} \left(\frac{x}{p+X} \right)^l \\ & \cdot F_C^{(k)} \left[a, b+m+n+l; 2\mu_1+1, \dots, 2\mu_k+1; \frac{x_1}{p+X}, \dots, \frac{x_k}{p+X} \right] \end{aligned}$$

where $a = \mu_1 + \dots + \mu_k + \frac{k}{2} - \lambda$, $b = \mu_1 + \dots + \mu_k + \frac{k}{2} + \nu$, $X = \frac{x_1 + \dots + x_k}{2}$ and $Re(p) > 0$, $Re(p+X) > 0$.

By setting $s = t = x/2$ in (3.3), the result reduces to

$$\begin{aligned} & F_C^{(k)}[a, b; 2\mu_1+1, \dots, 2\mu_k+1; \frac{x_1}{p+X}, \dots, \frac{x_k}{p+X}] \\ (3.5) \quad &= \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{(x/2)^{m+n} (b)_{m+n}}{m! n! (p+X)^{m+n}} \sum_{l=0}^n \frac{(-n)_l (b+m+n)_l}{(m+1)_l l!} \left(\frac{x}{p+X} \right)^l \\ & \cdot F_C^{(k)} \left[a, b+m+n+l; 2\mu_1+1, \dots, 2\mu_k+1; \frac{x_1}{p+X}, \dots, \frac{x_k}{p+X} \right] \end{aligned}$$

where $a = \mu_1 + \cdots + \mu_k + \frac{k}{2} - \lambda$, $b = \mu_1 + \cdots + \mu_k + \frac{k}{2} + \nu$, $X = \frac{x_1 + \cdots + x_k}{2}$ and $Re(p) > 0$, $Re(p + X) > 0$. which can also be obtained from (3.4) by taking $r = 0$.

4. Special cases

On setting $x = 0$ in (3.3), the result reduces to

$$(4.1) \quad \begin{aligned} & F_C^{(k)} \left[a, b; 2\mu_1 + 1, \dots, 2\mu_k + 1; \frac{x_1}{p - s - t + X}, \dots, \frac{x_k}{p - s - t + X} \right] \\ &= \frac{(p - s - t + X)^b}{(p + X)^b} \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{s^m t^n (b)_{m+n}}{m! n! (p + X)^{m+n}} \\ & \cdot F_C^{(k)} \left[a, b + m + n; 2\mu_1 + 1, \dots, 2\mu_k + 1; \frac{x_1}{p + X}, \dots, \frac{x_k}{p + X} \right] \end{aligned}$$

where $a = \mu_1 + \cdots + \mu_k + \frac{k}{2} - \lambda$, $b = \mu_1 + \cdots + \mu_k + \frac{k}{2} + \nu$, $X = \frac{x_1 + \cdots + x_k}{2}$ and $Re(p) > 0$, $Re(p + X) > 0$.

For $k = 2$, equation (4.1) reduces to

$$(4.2) \quad \begin{aligned} & F_4 \left[a, b; 2\mu_1 + 1, 2\mu_2 + 1; \frac{x_1}{p - s - t + X}, \frac{x_2}{p - s - t + X} \right] \\ &= \frac{(p - s - t + X)^b}{(p + X)^b} \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{s^m t^n (b)_{m+n}}{m! n! (p + X)^{m+n}} \\ & \cdot F_4 \left[a, b + m + n; 2\mu_1 + 1, 2\mu_2 + 1; \frac{x_1}{p + X}, \frac{x_2}{p + X} \right] \end{aligned}$$

where $a = \mu_1 + \mu_2 + 1 - \lambda$, $b = \mu_1 + \mu_2 + 1 + \nu$, $X = \frac{x_1 + x_2}{2}$ and $Re(p) > 0$, $Re(p + X) > 0$.

For $k = 2$, equation (3.5) reduces to

$$(4.3) \quad \begin{aligned} & F_4 \left[a, b; 2\mu_1 + 1, 2\mu_2 + 1; \frac{x_1}{p + X}, \frac{x_2}{p + X} \right] \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{(x/2)^{m+n} (b)_{m+n}}{m! n! (p + X)^{m+n}} \sum_{l=0}^n \frac{(-n)_l (b + m + n)_l}{(m + 1)_l l!} \left(\frac{x}{p + X} \right)^l \\ & \cdot F_4 \left[a, b + m + n + l; 2\mu_1 + 1, 2\mu_2 + 1; \frac{x_1}{p + X}, \frac{x_2}{p + X} \right] \end{aligned}$$

where $a = \mu_1 + \mu_2 + 1 - \lambda$, $b = \mu_1 + \mu_2 + 1 + \nu$, $X = \frac{x_1 + x_2}{2}$ and $Re(p) > 0$, $Re(p + X) > 0$.

For $k = 1$, equation (3.5) reduces to

$$\begin{aligned}
 & {}_2F_1\left[a, b; 2\mu_1 + 1; \frac{x_1}{p + X}\right] \\
 (4.4) \quad &= \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{(x/2)^{m+n} (b)_{m+n}}{m! n! (p + X)^{m+n}} \sum_{l=0}^n \frac{(-n)_l (b + m + n)_l}{(m + 1)_l l!} \left(\frac{x}{p + X}\right)^l \\
 & \quad \cdot {}_2F_1\left[a, b + m + n + l; 2\mu_1 + 1; \frac{x_1}{p + X}\right]
 \end{aligned}$$

where $a = \mu_1 + \frac{1}{2} - \lambda$, $b = \mu_1 + \frac{1}{2} + \nu$, $X = \frac{x_1}{2}$ and $Re(p) > 0$, $Re(p + X) > 0$.

By putting $x_1 = 0$ and $x_2 = 0$ in (4.3), the above result reduces to a known result of Pathan and Yasmeen [7; p. 242(2.3)]

$$(4.5) \quad 1 = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{(x/2p)^{m+n} (c)_{m+n}}{(m + n)!} P_n^{(m, c-1)} \left(\frac{p - 2x}{p}\right)$$

where the Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$ is defined by [4; p. 254(1)]

$$P_n^{(\alpha, \beta)}(x) = \frac{(1 + \alpha)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, 1 + \alpha + \beta + n & ; & \frac{1 - x}{2} \\ 1 + \alpha & ; & \end{matrix} \right].$$

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Accepted: 14.07.2012

FEKETE-SZEGÖ PROBLEM FOR CONCAVE UNIVALENT FUNCTIONS DEFINED BY SĂLĂGEAN OPERATOR

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Abstract. Let $C_0(\alpha)$ denote the class of concave univalent functions defined in the open unit disk U . In this paper, we investigate the sharp upper bounds of Fekete-Szegő functional with real and complex parameter λ for the class of concave univalent functions defined by Sălăgean differential operator.

Keywords: Concave, Univalent function, Sălăgean differential operator.

2000 Mathematics Subject Classification: 30C45, 30C10.

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1. Introduction

Let S denote the class of all analytic and univalent functions

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

defined on the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

Denote by $S^*(\beta)$, $C(\beta)$ and $K(\alpha, \beta)$, the classes of starlike functions of order β , convex functions of order β and close-to-convex functions of order α type β respectively, which are analytically defined as follows:

$$\begin{aligned} \text{(i)} \quad S^*(\beta) &= \left\{ f \in A : \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \beta, z \in U, 0 \leq \beta < 1 \right\}, \\ \text{(ii)} \quad C(\beta) &= \left\{ f \in A : \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \beta, z \in U, 0 \leq \beta < 1 \right\}, \\ \text{(iii)} \quad K(\alpha, \beta) &= \left\{ f \in A : \operatorname{Re} \left(\frac{f'(z)}{g'(z)} \right) > \alpha, g(z) \in C(\beta), \right. \\ &\quad \left. z \in U, 0 \leq \alpha < 1, 0 \leq \beta < 1 \right\}. \end{aligned}$$

In 1933, Fekete and Szegő [19] obtained the maximum value of $|a_3 - \lambda a_2^2|$ as a function of the real parameter λ , namely

$$|a_3 - \lambda a_2^2| \leq 1 + 2 \exp \left(\frac{-2\lambda}{1-\lambda} \right),$$

for the class S of analytic and univalent functions given by (1.1). This inequality is sharp for each $\lambda \in [0, 1]$. In the literature, there exists a large number of results of the Fekete-Szegő functional $|a_3 - \lambda a_2^2|$ for various subclasses of S , such as the class of $S^*(\beta)$, $C(\beta)$ and $K(\alpha, \beta)$. For instance, Keogh and Merkers [10], Kaplan [26], Koepf [27] solved the Fekete-Szegő problem for close-to-convex functions. Nasr and Gawad [20], Gawad and Thomas [12], Darus and Thomas [18], Ibrahim and Darus [4] and others generalized this result for the class of functions that are close-to-convex functions of order α and type β . Later, Avkhadiev et al. [8], [9] and Bhowmik et al. [5], [6], they gave another treatment of Fekete-Szegő problem by considering the class of concave univalent functions given by (1.1).

Also, there are several authors that proved this type of result for the Fekete-Szegő functional for the class of function defined by differential operator, see [16], [3], for example, by using the Sălăgean differential operator D^k [11], for $f \in S$ which is defined by

$$\begin{aligned} \text{(i)} \quad D^0 f(z) &= f(z), \\ \text{(ii)} \quad D^1 f(z) &= Df(z) = z + \sum_{n=2}^{\infty} n a_n z^n, \end{aligned}$$

$$(iii) \quad D^k f(z) = D(D^{k-1} f(z)) = z + \sum_{n=2}^{\infty} n^k a_n z^n; \quad k = 1, 2, \dots$$

Denote by S_k^* , the class of k -starlike functions which is analytically defined as follows:

$$(1.2) \quad S_k^* = \left\{ f(z) \in S : \operatorname{Re} \left(\frac{D^{k+1} f(z)}{D^k f(z)} \right) > 0, \quad k = 0, 1, 2, \dots, \quad z \in U \right\}.$$

In this paper, we investigated the sharp upper bounds of Fekete-Szegő functional $|a_3 - \lambda a_2^2|$ for the class of concave univalent functions with real and complex parameter λ , where the function of f is defined by Sălăgean differential operator (1.2).

2. Preliminary results

A function $f : U \rightarrow \mathbb{C}$ is said to belong to the family $C_0(\alpha)$ if f satisfies the following conditions:

- (a) f is analytic in U with the standard normalization $f(0) = f'(0) - 1 = 0$. In addition it satisfies $f(1) = \infty$.
- (b) f maps conformally onto a set whose complement with respect to $\mathbb{C}$ is convex.
- (c) The opening angle of $f(U)$ at ∞ is less than or equal to $\pi\alpha$, $\alpha \in (1, 2]$.

The class $C_0(\alpha)$ is referred to concave univalent functions and for a detailed discussion about concave functions we refer to [8], [9], [17] and the references therein. Recently, the class $C_0(\alpha)$ of concave function was considered by Bhowmik et al. [5], [6].

We recall the analytic characterization for the functions in $C_0(\alpha)$, $\alpha \in (1, 2]$: $f \in C_0(\alpha)$ if and only if $\operatorname{Re} P_f(z) > 0$, $z \in U$, where

$$P_f(z) = \frac{2}{\alpha - 1} \left[\frac{(\alpha + 1)}{2} \frac{1 + z}{1 - z} - 1 - z \frac{f''(z)}{f'(z)} \right].$$

In [5], [6] they used this characterization and proved the following theorem.

Theorem 1 *Let $\alpha \in (1, 2]$. A function $f \in C_0(\alpha)$ if, and only if, there exist a starlike function $\phi \in S^*$ such that $f(z) = \Lambda_\phi(z)$ where*

$$\Lambda_\phi(z) = \int_0^z \frac{1}{(1-t)^{\alpha+1}} \left(\frac{t}{\phi(t)} \right)^{(\alpha-1)/2} dt$$

and S^* denote the family of starlike functions g defined by $g \in S^*$ if and only if $\operatorname{Re} \left(\frac{zg'(z)}{g(z)} \right) > 0$.

The objective of the present paper is to give some generalizations of the result of Fekete-Szegő problem given by Bhowmik et al. [5] for the starlike function defined by Sălăgean differential operator $D^k f$, $k = 0, 1, 2, \dots$, which is $f \in S_k^*$ is characterized by the condition (1.2).

In order to prove our main results, we need to recall the following lemma.

Lemma 1 [27] Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S^*$. Then $|b_3 - \lambda b_2^2| \leq \max \{1, |3 - 4\lambda|\}$, which is sharp for the Koebe function k if $|\lambda - 3/4| \geq 1/4$ and for $(k(z))^{1/2} = \frac{z}{1-z^2}$ if $|\lambda - 3/4| \leq 1/4$.

3. Main result and its proof

We consider the Fekete-Szegő functional $|a_3 - \lambda a_2^2|$ for real and complex parameter λ . Our results are contained in the following theorems.

Theorem 1 Let $f \in C_0(\alpha)$ have the expansion given by (1.1), $\alpha \in (1, 2]$, $k = 0, 1, 2, \dots$. If λ is real, then we have

$$12|a_3 - \lambda a_2^2| \leq \begin{cases} (3 + 2^{2k})(2 - 3\lambda)\alpha^2 + 3(1 - 2^{2k})(1 - 2\alpha)\lambda + 6(1 - 3^k)\alpha + 2(3^{k+1} - 2^{2k}), & \text{if } \lambda \leq \lambda_0; \\ 4[(2 - 3\lambda)\alpha^2 + 1], & \text{if } \lambda_0 \leq \lambda \leq \frac{2(\alpha - 1)}{3\alpha}; \\ \frac{4[(10 - 9\lambda)\alpha + (2 - 3\lambda)]}{3(2 - \lambda) - (2 - 3\lambda)\alpha}, & \text{if } \frac{2(\alpha - 1)}{3\alpha} \leq \lambda \leq \frac{2}{3}; \\ 12(1 - \lambda)\alpha \sqrt{\frac{12(1 - \lambda)}{(4 - 3\lambda)^2 - (3\lambda - 2)^2\alpha^2}}, & \text{if } \frac{2}{3} \leq \lambda \leq \lambda_2; \\ 4[(3\lambda - 2)\alpha^2 - 1], & \text{if } \lambda_2 \leq \lambda \leq \frac{2(\alpha + 2)}{3(\alpha + 1)}; \\ (3 + 2^{2k})(3\lambda - 2)\alpha^2 + 3(2^{2k} - 1)(1 - 2\alpha)\lambda + 6(3^k - 1)\alpha + 2(2^{2k} - 3^{k+1}), & \text{if } \lambda \geq \frac{2(\alpha + 2)}{3(\alpha + 1)}; \end{cases}$$

where

$$\lambda_0 = \frac{2^{2k-2}(\alpha + 1) - 3^k}{3(2^{2k-3})(\alpha - 1)} \text{ and } \lambda_2 = \frac{2}{3} + \frac{1}{6\alpha^2} \left(\sqrt{8\alpha^2 + 1} - 1 \right).$$

The inequalities are sharp.

Theorem 2 Let $f \in C_0(\alpha)$ have the expansion given by (1.1), $\alpha \in (1, 2]$, $k = 0, 1, 2, \dots$. If λ are complex numbers, then we have

$$|a_3 - \lambda a_2^2| \leq \max \left\{ 1, \frac{1}{12} (\alpha + 1) \nu(\alpha, \lambda) \right\},$$

where

$$\begin{aligned} \nu(\alpha, \lambda) = & |(2 - 3\lambda)(\alpha + 1) + 2| + 2(\alpha - 1)|3\lambda - 2| \\ & + \left(\frac{\alpha - 1}{\alpha + 1} \right) |6 + [2 - 3(\alpha - 1)\lambda]|. \end{aligned}$$

Proof. We recall from Theorem 1 that for $f \in C_0(\alpha)$ if and only if there exist a function $\phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n \in S_k^*$, $k = 0, 1, 2, \dots$ such that

$$(3.1) \quad f'(z) = \frac{1}{(1-z)^{\alpha+1}} \left(\frac{z}{D^n \phi(z)} \right)^{(\alpha-1)/2},$$

where f has the form given by (1.1) and D^n is the Sălăgean operator. Comparing the coefficients of z and z^2 on the both sides of the series expansion (3.1), we obtain that

$$a_2 = \frac{(\alpha + 1)}{2} - 2^{k-2}(\alpha - 1)\phi_2$$

and

$$\begin{aligned} a_3 = & \frac{1}{6}(\alpha + 1)(\alpha + 2) - \frac{2^{k-1}}{3}(\alpha^2 - 1)\phi_2 \\ & - \frac{3^{k-1}}{2}(\alpha - 1)\phi_3 + \frac{2^{2k-3}}{3}(\alpha - 1)\phi_2^2, \end{aligned}$$

respectively.

A computation yields that

$$\begin{aligned} (3.2) \quad a_3 - \lambda a_2^2 = & \frac{(\alpha + 1)^2}{4} \left[\frac{2(\alpha + 2)}{3(\alpha + 1)} - \lambda \right] \\ & + 2^{k-2}(\alpha^2 - 1) \left(\lambda - \frac{2}{3} \right) \phi_2 - \frac{3^{k-1}}{2}(\alpha - 1) \\ & \times \left[\phi_3 - \left(\frac{2^{2k-2}(\alpha + 1) - 3\lambda(2^{2k-3})(\alpha - 1)}{3^k} \right) \phi_2^2 \right] \end{aligned}$$

Now, we need to investigate the maximum values of the function $|a_3 - \lambda a_2^2|$ by considering several cases of λ .

Case 1: Consider the first case for all $\lambda \leq \frac{2^{2k-2}(\alpha + 1) - 3^k}{3(2^{2k-3})(\alpha - 1)}$.

We observe that the assumption on λ is seen to be equivalent to

$$\frac{1}{3^k} [2^{2k-2}(\alpha + 1) - 3\lambda(2^{2k-3})(\alpha - 1)] \geq 1$$

and the first term in equation (3.2) is nonnegative. Hence, using the Lemma 1 for the last term in (3.2), we have

$$\begin{aligned} & \left| \phi_3 - \left(\frac{2^{2k-2}(\alpha+1) - 3\lambda(2^{2k-3})(\alpha-1)}{3^k} \right) \phi_2^2 \right| \\ & \leq \frac{2^{2k}(\alpha+1) - 3\lambda(2^{2k-1})(\alpha-1)}{3^k} - 3 \end{aligned}$$

and noticing that for $\phi \in S_k^*$, $|\phi_n| \leq n^{1-k}$, $k = 2, 3, \dots$, we have from the equality (3.2) that

$$\begin{aligned} |a_3 - \lambda a_2^2| & \leq \frac{(\alpha+1)^2}{4} \left[\frac{2(\alpha+2)}{3(\alpha+1)} - \lambda \right] + 2^{k-2}(\alpha^2-1) \left(\frac{2}{3} - \lambda \right) |\phi_2| \\ & \quad + \frac{3^{k-1}}{2}(\alpha-1) \left| \phi_3 - \left(\frac{2^{2k-2}(\alpha+1) - 3\lambda(2^{2k-3})(\alpha-1)}{3^k} \right) \phi_2^2 \right| \\ & = \frac{(\alpha+1)(\alpha+2)}{6} - \frac{\lambda}{4}(\alpha+1)^2 + \frac{(\alpha^2-1)}{2} \left(\frac{2}{3} - \lambda \right) \\ & \quad + \frac{3^{k-1}}{2}(\alpha-1) \left(\frac{2^{2k}(\alpha+1) - 3\lambda(2^{2k-1})(\alpha-1)}{3^k} - 3 \right). \end{aligned}$$

It can be simplified to

$$\begin{aligned} |a_3 - \lambda a_2^2| & \leq \frac{1}{12} \left[(3 + 2^{2k})(2 - 3\lambda)\alpha^2 + 3(1 - 2^{2k})(1 - 2\alpha)\lambda \right. \\ & \quad \left. + 6(1 - 3^k)\alpha + 2(3^{k+1} - 2^{2k}) \right], \end{aligned}$$

$$\text{for } \lambda \in \left(\infty, \frac{2^{2k-2}(\alpha+1) - 3^k}{3(2^{2k-3})(\alpha-1)} \right).$$

Case 2: Let $\lambda \geq \frac{2(\alpha+2)}{3(\alpha+1)}$.

For this case, the first term in (3.2) is nonnegative. The condition on λ in particular gives $\lambda \geq \frac{2}{3}$ and therefore our assumption on λ implies that

$$\frac{2^{2k-2}(\alpha+1) - 3\lambda(2^{2k-3})(\alpha-1)}{3^k} \leq \frac{2^{2k}}{3^k} \left(\frac{1}{2} \right).$$

Again, it follows from Lemma 1, that

$$\left| \phi_3 - \frac{2^{2k-2}(\alpha+1) - 3\lambda(2^{2k-3})(\alpha-1)}{3^k} \phi_2^2 \right| \leq 3 - \frac{2^{2k}(\alpha+1) - 3\lambda(2^{2k-1})(\alpha-1)}{3^k}.$$

In view of these observation and an use of the inequality that $|\phi_2| \leq 2^{1-k}$, equality (3.2) gives

$$(3.3) \quad \begin{aligned} |a_3 - \lambda a_2^2| &\leq \frac{(\alpha+1)^2}{4} \left[\lambda - \frac{2(\alpha+2)}{3(\alpha+1)} \right] + 2^{k-2} (\alpha^2 - 1) \left(\lambda - \frac{2}{3} \right) (2^{1-k}) \\ &\quad + \frac{3^{k-1}}{2} (\alpha - 1) \left(3 - \frac{2^{2k}(\alpha+1) - 3\lambda(2^{2k-1})(\alpha-1)}{3^k} \right). \end{aligned}$$

Thus, simplifying the right hand side expression (3.3), we obtain that

$$\begin{aligned} |a_3 - \lambda a_2^2| &\leq \frac{1}{12} \left[(3 + 2^{2k}) (3\lambda - 2) \alpha^2 + 3 (2^{2k} - 1) (1 - 2\alpha) \lambda \right. \\ &\quad \left. + 6 (3^k - 1) \alpha - 2 (3^{k+1} - 2^{2k}) \right], \end{aligned}$$

for $\lambda \in \left[\frac{2(\alpha+2)}{3(\alpha+1)}, \infty \right)$.

Case 3: Consider λ , where

$$\lambda \in \left(\frac{2^{2k-2}(\alpha+1) - 3^k}{3(2^{2k-3})(\alpha-1)}, \frac{2(\alpha+2)}{3(\alpha+1)} \right).$$

Now we deal with the case by using the formulas (3.1) and (3.2) together with the representation formula for $\phi(z) \in S_k^*$. Let us define $w(z)$ by

$$(3.4) \quad \frac{D^{k+1}\phi(z)}{D^k\phi(z)} = \frac{1 + zw(z)}{1 - zw(z)}; \quad (w(z) \neq 1)$$

where $w : U \rightarrow \bar{U}$ is a function analytic in U with the Taylor series

$$w(z) = \sum_{n=0}^{\infty} c_n z^n.$$

Comparing the coefficients of z and z^2 in (3.4), we get that

$$(3.5) \quad \phi_2 = 2^{1-k} c_0 \quad \text{and} \quad \phi_3 = \frac{1}{3^k} (c_1 + 3c_0^2).$$

Inserting these resulting formulas (3.5) into (3.2) yields

$$(3.6) \quad \begin{aligned} a_3 - \lambda a_2^2 &\leq \frac{(\alpha+1)^2}{4} \left[\frac{2(\alpha+2)}{3(\alpha+1)} - \lambda \right] \\ &\quad + 2^{k-2} (\alpha^2 - 1) \left(\lambda - \frac{2}{3} \right) (2^{1-k} c_0) \\ &\quad + \frac{3^{k-1}}{2} (\alpha - 1) \left[\frac{1}{3^k} (c_1 + 3c_0^2) \right. \\ &\quad \left. - \left(\frac{2^{2k-2}(\alpha+1) - 3\lambda(2^{2k-3})(\alpha-1)}{3^k} \right) (2^{2-2k}) c_0^2 \right] \\ &= A + Bc_0 + Cc_0^2 + Dc_1, \end{aligned}$$

where

$$\begin{aligned} A &= \frac{1}{6}(\alpha + 2)(\alpha + 1) - \frac{\lambda}{4}(\alpha + 1)^2, \\ B &= \frac{1}{6}(\alpha^2 - 1)(3\lambda - 2), \\ C &= -\frac{1}{12}(\alpha - 1)[4 - 2\alpha + 3\lambda(\alpha - 1)], \\ D &= -\frac{1}{6}(\alpha - 1). \end{aligned}$$

Hence, by using the well known inequalities that $|c_0| \leq 1$ and $|c_1| \leq 1 - |c_0|^2$, from (3.6) we obtain that

$$(3.7) \quad |a_3 - \lambda a_2^2| \leq |A + Bc_0 + Cc_0^2| + \frac{1}{6}(\alpha - 1)(1 - |c_0|^2).$$

Now, in order to determine the maximum value of (3.7), let $c_0 = re^{i\theta}$, then we consider the quadratic expression

$$(3.8) \quad \begin{aligned} f(r, \theta) &= |A + Bc_0 + Cc_0^2|^2 \\ &= (A - Cr^2)^2 + B^2r^2 + 2Br(A + Cr^2)\cos\theta + 4ACr^2\cos^2\theta, \end{aligned}$$

where $\cos\theta \in [-1, 1]$, $r \in (0, 1]$. For getting the upper bounds of $|a_3 - \lambda a_2^2|$, we have to find the biggest value of (3.8) for r in the interval $(0, 1]$. So, let $x = \cos\theta$, then from (3.8) we have

$$(3.9) \quad h(x) = (A - Cr^2)^2 + B^2r^2 + 2Br(A + Cr^2)x + 4ACr^2x^2.$$

We have to determine the maximum value of (3.9) for $x \in [-1, 1]$. So, for this, we need to consider the several subclasses of λ , where

$$\lambda \in \left(\frac{2^{2k-2}(\alpha + 1) - 3^k}{3(2^{2k-3})(\alpha - 1)}, \frac{2(\alpha + 2)}{3(\alpha + 1)} \right).$$

Case 3A: First, consider

$$\lambda \in \left(\frac{2^{2k-2}(\alpha + 1) - 3^k}{3(2^{2k-3})(\alpha - 1)}, \frac{2(\alpha - 2)}{3(\alpha - 1)} \right).$$

We observe that for λ in this interval, we have $A > 0$, $B < 0$, $C > 0$ and $A + Cr^2 > 0$ for $r \in (0, 1]$, and (3.9) attains its maximum value at $x = -1$. Therefore, it gives that

$$(3.10) \quad |a_3 - \lambda a_2^2| \leq g(r) = A - Br + Cr^2 + \frac{1}{3}(\alpha - 1)(1 - r^2).$$

By a simple calculation, we show that the maximum value of (3.10) attains at the boundary of r , i.e. $r = 1$. Therefore

$$g(r) \leq g(1) = A - B + C = \frac{1}{3}[(2 - 3\lambda)\alpha^2 + 1].$$

Case 3B: Let $\lambda = \frac{2(\alpha - 2)}{3(\alpha - 1)}$.

In this case, we get $C = 0$, therefore $h(x)$ becomes a linear function,

$$(3.11) \quad h(x) = A^2 + B^2 r^2 + 2BrAx.$$

It is easy to show that the maximum value of (3.11) occurs at $x = -1$ and $r = 1$. Again we get the maximum value of $|a_3 - \lambda a_2^2|$ as the previous case.

Case 3C: Let $\lambda \in \left(\frac{2(\alpha - 2)}{3(\alpha - 1)}, \frac{2(\alpha - 1)}{3\alpha} \right)$.

In this interval, the quadratic function (3.9) has maximum value at

$$x(r) = -\frac{B}{4} \left(\frac{1}{Cr} + \frac{r}{A} \right),$$

where $x(r)$ is monotonic increasing in $r \in (0, 1]$ and $x(1) < -1$. Hence we get the upper bound as in Cases 3A and 3B. As conclusion, from the Cases 3A, 3B and 3C give us that

$$|a_3 - \lambda a_2^2| \leq \frac{1}{3} [(2 - 3\lambda)\alpha^2 + 1]$$

for

$$\lambda \in \left(\frac{2^{2k-2}(\alpha + 1) - 3^k}{3(2^{2k-3})(\alpha - 1)}, \frac{2(\alpha - 1)}{3\alpha} \right).$$

Case 3D: Let $\lambda \in \left[\frac{2(\alpha - 1)}{3\alpha}, \frac{2}{3} \right)$.

From the Case 3C, the inequality $x(1) < -1$ gives that

$$\frac{2(3\lambda + 4\alpha^2 - 12\alpha^2\lambda + 9\alpha^2\lambda^2 - 4)}{[(3\lambda - 4) + \alpha(3\lambda - 2)][\alpha(3\lambda - 2) - (3\lambda - 4)]} < 0,$$

hence it shows that

$$p(\lambda) = 9\alpha^2\lambda^2 + (3 - 12\alpha^2)\lambda + 4(\alpha^2 - 1) < 0$$

where $\lambda < \frac{2}{3}$. Factorizing $p(\lambda)$, we have

$$(3.12) \quad \lambda_1 = \frac{2}{3} - \frac{1}{6\alpha^2} \left(1 + \sqrt{8\alpha^2 + 1} \right)$$

and

$$(3.13) \quad \lambda_2 = \frac{2}{3} - \frac{1}{6\alpha^2} \left(1 - \sqrt{8\alpha^2 + 1} \right).$$

It is clear that $\lambda_1 < \lambda_2$. Therefore, for $\lambda \in \left[\frac{2(\alpha - 1)}{3\alpha}, \lambda_1 \right)$, functions (3.9) and (3.10) have their maximum value at

$$x = -1 \text{ and } r_m = \frac{-3B}{-6C + \alpha - 1} \in (0, 1]$$

respectively. Hence the upper bound of Fekete-Szegő functional is given by

$$(3.14) \quad \begin{aligned} |a_3 - \lambda a_2^2| &\leq g(r_m) = A - Br_m + Cr_m^2 + \frac{1}{3}(\alpha - 1)(1 - r_m^2) \\ &= \frac{4[(10 - 9\lambda)\alpha + (2 - 3\lambda)]}{3(2 - \lambda) - (2 - 3\lambda)\alpha}. \end{aligned}$$

Next, we consider for $\lambda \in \left[\lambda_1, \frac{2}{3}\right)$. In this interval, the quadratic equation (3.9) attains its maximum value at

$$x(r) = \frac{-B(A + Cr^2)}{4ACr}$$

with

$$h(x(r)) = -\frac{1}{4AC}(B^2 - 4AC)(A - Cr)^2.$$

Hence, the Fekete-Szegő functional satisfies the following inequality

$$(3.15) \quad \begin{aligned} |a_3 - \lambda a_2^2| &\leq \sqrt{h(x(r))} + \frac{(\alpha - 1)}{6}(1 - r^2) \\ &= (A - Cr)\sqrt{1 - \frac{B^2}{4AC}} + \frac{(\alpha - 1)}{6}(1 - r^2) = k(r). \end{aligned}$$

The maximum value of $g(r)$,

$$g(r) = A - Br + Cr^2 + \frac{(\alpha - 1)}{6}(1 - r^2)$$

and (3.15) occurs at

$$r_m = \frac{-B}{-2C + \frac{(\alpha-1)}{3}} \text{ and } r_0 = \frac{B}{2C + \sqrt{1 - \frac{B^2}{4AC}}}$$

respectively. It is easy to show that (3.15) is monotonic decreasing for $r \geq r_0$. Hence, the maximum value of $|a_3 - \lambda a_2^2|$ is given by (3.14).

For $\lambda = \frac{2}{3}$, we get $B = 0$ and $C = \frac{1}{6}(1 - \alpha)$. Thus, the maximum value

$$(3.16) \quad |a_3 - \lambda a_2^2| = \frac{\alpha}{3},$$

occurs at $x = \cos \theta = 0$ and $r \in (0, 1]$.

From (3.14), (3.15) and (3.16) we concluded that

$$|a_3 - \lambda a_2^2| \leq \frac{4[(10 - 9\lambda)\alpha + (2 - 3\lambda)]}{3(2 - \lambda) - (2 - 3\lambda)\alpha}$$

for $\lambda \in \left[\frac{2(\alpha - 1)}{3\alpha}, \frac{2}{3}\right]$.

Case 3E: Let $\lambda \in \left(\frac{2}{3}, \lambda_2\right]$, where λ_2 is given by (3.13).

In this interval, we have $B > 0$. So that (3.9) attains its maximum value at $x = 1$. Then, we consider the function

$$l(r) = h(1) = A + Br + Cr^2 + \frac{(\alpha - 1)}{6} (1 - r^2).$$

Again, by a simple calculation shows that the maximum value of $l(r)$ to be occurred at

$$r_n = \frac{B}{-2C + \frac{(\alpha-1)}{3}},$$

hence the maximum of the function (3.15) to be attained at

$$r_1 = \frac{B}{-2C \left(1 + \sqrt{1 - \frac{B^2}{4AC}}\right)} \in (0, 1].$$

It is easily to prove that $r_1 < r_n \leq 1$. Since $k(r)$ is monotonic increasing function, then

$$k(r) \leq k(1) = (A - C) \sqrt{1 - \frac{B^2}{4AC}},$$

which gives that

$$|a_3 - \lambda a_2^2| \leq k(1) = (1 - \lambda) \alpha \sqrt{\frac{12(1 - \lambda)}{(4 - 3\lambda)^2 - (3\lambda - 2)^2 \alpha^2}}$$

for $\lambda \in \left(\frac{2}{3}, \lambda_2\right]$.

Case 3F: Finally, we consider the case for $\lambda \in \left(\lambda_2, \frac{2(\alpha + 2)}{3(\alpha + 1)}\right)$.

For these λ , we see that $A < 0$, $B > 0$, $C < 0$, $A + Cr^2 < 0$ and the maximum value of function (3.7) is attained for $x = -1$, i.e.

$$\eta(x) = -A + Br - Cr^2 + \frac{(\alpha - 1)}{6} (1 - r^2).$$

We get $\eta(r) \leq \eta(1)$ for all λ in these interval and hence

$$|a_3 - \lambda a_2^2| \leq -A + B - C = \frac{1}{3} [(3\lambda - 2) \alpha^2 - 1].$$

Thus, the proof of Theorem 1 is complete.

Further, substitute (3.5) into (3.2) yields

$$\begin{aligned} 12(a_3 - \lambda a_2^2) &= (\alpha + 1) [(2 - 3\lambda)(\alpha + 1) + 2] + 2(\alpha^2 - 1)(3\lambda - 2)c_0 \\ &\quad + (\alpha - 1)(6 + [2 - 3(\alpha - 1)\lambda])c_0^2 + 2(1 - \alpha)c_1. \end{aligned}$$

Hence for λ complex numbers, we have

$$(3.17) \quad \begin{aligned} 12 |a_3 - \lambda a_2^2| &\leq (\alpha + 1) |(2 - 3\lambda)(\alpha + 1) + 2| \\ &\quad + 2(1 - \alpha) |c_1| + 2(\alpha^2 - 1) |3\lambda - 2| |c_0| \\ &\quad + (\alpha - 1) |6 + [2 - 3(\alpha - 1)\lambda]| |c_0|^2. \end{aligned}$$

Using the well known inequality that $|c_0| \leq 1$ and $|c_1| \leq 1 - |c_0|^2$, then from (3.17) we get

$$12 |a_3 - \lambda a_2^2| \leq \frac{1}{12} (\alpha + 1) \nu(\alpha, \lambda)$$

for $\operatorname{Re} \{\nu(\alpha, \lambda)\} > 0$, where

$$\begin{aligned} \nu(\alpha, \lambda) &= |(2 - 3\lambda)(\alpha + 1) + 2| + 2(1 - \alpha) |3\lambda - 2| \\ &\quad + \frac{(\alpha - 1)}{\alpha + 1} |6 + [2 - 3(\alpha - 1)\lambda]|. \end{aligned}$$

Thus, the proof of Theorem 2 is complete. ■

Remark 1 Taking $k = 0$ and λ real numbers, we deduce a result of Bhowmik et al. [5].

Other problems related to Fekete-Szegő functional for further reading can be found in ([1], [2], [7], [13], [14], [15], [21], [22], [23], [24], [25]).

Acknowledgements: The second author is fully supported by LRGS/TD/2011/UKM/ICT/03/02.

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Accepted: 16.09.2012

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Abstract. The purpose of the paper is to introduce the notion of r -weak cb spaces as a generalization of weak cb spaces. The r -weak cb property of a space is defined with the help of a stronger form of normal upper semi continuous functions viz. strongly normal upper semi continuous (s-nusc) function. A stronger form of regular closed subsets called strongly regular (s-regular) closed subsets turns out to be the natural tool for defining the new function. The r -weak cb spaces are characterized by a increasing cover of s-regular open (i.e. complement of s-regular closed) subsets and a decreasing sequence of s-regular closed subsets. Some of the properties of the newly introduced spaces and its interrelationship with other spaces are investigated.

Keywords and phrases: locally finite, σ -locally finite, cozero refinement, s -regular closed, s-nusc.

2000 Mathematics Subject Classification: 54D60.

1. Introduction

Throughout by a space we mean a completely regular (Hausdorff) space. For basic definitions of zero-sets, cozero-sets, F_σ -sets, G_δ -sets, regular open and regular closed sets, refinement, cover etc. we refer [9] and [13].

The concept of regular G_δ -subsets was introduced by J. Mack [14]. A subset H of a topological space X is called a regular G_δ -subset if H is an intersection of a sequence of closed sets whose interiors contain H .

Equivalently, if $H = \bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty} Cl_X G_n$, where each G_n is open subset of X , then H is a regular G_δ -subset. The complement of a regular G_δ -subset is called regular F_σ , i.e., a subset V of a space X is said to be a regular F_σ if $V = \bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} Int_X F_n$, where each F_n is closed subset of X . Properties of regular G_δ and regular F_σ -subsets have been studied in [3]. The intersection (resp. union) of two regular F_σ (resp. regular G_δ)-subsets is a regular F_σ (resp. regular G_δ)-subset. The countable union (resp. intersection) of regular F_σ (resp. regular G_δ)-subsets is a regular F_σ (resp. regular G_δ)-subset. Also every zero-set is regular G_δ and every cozero-set is regular F_σ .

A real valued function on a topological space X is said to be locally bounded if each point has a neighbourhood (nbd) on which the function is bounded. Let $C(X)$ denotes the set of all real valued continuous functions on X . A collection of subsets $\{A_\alpha\}$ of a topological space is said to be locally finite (resp. σ -locally finite) if each point $x \in X$ has a nbd which has non-empty intersection only with a finite (resp. countable) number of members of the collection. Also for two families of sets $\mathcal{U}$ and $\mathcal{V}$, $\mathcal{V}$ is said to be a refinement of $\mathcal{U}$, if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$. A family of continuous functions $\mathcal{F}$ is locally finite (resp. subordinate to a cover $\mathcal{U}$) if the collection of cozero-sets associated with $\mathcal{F}$ is locally finite collection of sets (resp. is a refinement of $\mathcal{U}$). A family $\mathcal{F} \subseteq C(X)$ is a partition of unity of a space X if each function in $\mathcal{F}$ maps X into $[0, 1]$ and $\sum f(x) = 1$ at each point $x \in X$, where the summation is taken over all $f \in \mathcal{F}$. The constant function, on any set, whose constant value is the real number r is denoted by $\mathbf{r}$, e.g. $\mathbf{1}$ is used to represent the constant function whose constant value is 1, i.e., $\mathbf{1}(x) = 1$ for all x . For all real valued function f on a set, the symbols f^+ and f^- are respectively defined as follows:

$$f^+(x) = \begin{cases} f(x) & \text{for } f(x) > 0 \\ 0 & \text{for } f(x) \leq 0 \end{cases}$$

and

$$f^-(x) = \begin{cases} f(x) & \text{for } f(x) < 0 \\ 0 & \text{for } f(x) \geq 0 \end{cases}$$

Thus with these notations the function $h_n = [(n+1)f - 1]^+ \times [(n-1)f - 1]^-$ is explained as $h_n(x) = [(n+1)f(x) - 1]^+ \times [(n-1)f(x) - 1]^-$. For real valued functions f and g , the functions $f \vee g$ and $f \wedge g$ are defined by the formulae $(f \vee g)(x) = \max\{f(x), g(x)\}$ and $(f \wedge g)(x) = \min\{f(x), g(x)\}$ respectively. If $f, g \in C(X)$ then the functions $f \vee g$, $f \wedge g$, f^+ and f^- are also the members of $C(X)$.

Real valued semi continuous functions play an important role in topology. Lower (resp. upper) semi continuous functions abbreviated as lsc (resp. usc) functions and their stronger forms are found in the literature. A function $f : X \rightarrow R$ is lsc (resp. usc) at a point $x_0 \in X$, if for each $\epsilon > 0$, there exists an nbd of x_0 , say $N(x_0)$, such that for all $x \in N(x_0)$, $f(x) > f(x_0) - \epsilon$ (resp.

$f(x) < f(x_0) - \epsilon$). Equivalently, a function $f : X \rightarrow R$ is lsc (resp. usc) iff for all $r \in R$, the set $\{x : f(x) > r\}$ (resp. $\{x : f(x) < r\}$) is an open set [13]. A lsc (resp. usc) function $f : X \rightarrow R$ is normal lower (resp. upper) semi continuous or nlsc (resp. nusc) at a point $x_0 \in X$, if $f(x_0) < r$ (resp. $f(x_0) > r$) and any open set U containing x_0 , there exists a non-empty open set V , such that $Cl_X V \subseteq U$ and $f(y) < r$ (resp. $f(y) > r$) for all $y \in V$. Equivalently, a lsc (resp. usc) function $f : X \rightarrow R$ is nlsc (resp. nusc) iff for all $r \in R$, the set $\{x : f(x) < r\}$ (resp. $\{x : f(x) > r\}$) is a union of regular closed sets [8].

A Tychonoff space X is Oz iff every regular closed set of X is a zero-set [5]. Equivalently, a Tychonoff space X is Oz iff every regular closed set is the intersection of a countable collection of regular closed nbds [6]. A topological space X is said to be a cb space if for each locally bounded function h , there exists $f \in C(X)$ such that $f \geq |h|$ [12]. The cb property is stronger than countable paracompactness, but equivalent to it for normal spaces. A space X is said to be weak cb if each locally bounded lsc function is bounded above by a continuous function. The weak cb and countable paracompactness are independent properties in Tychonoff spaces [15]. Two important characterizations of cb (resp. weak cb) spaces are: a space X is cb (resp. weak cb) iff (i) given an usc (resp. nusc) function h on X there exists $f \in C(X)$ such that $f \geq h$, (ii) for each decreasing sequence $\{F_n\}$ of closed (resp. regular closed) subsets of X with empty intersection, there exists a sequence of zero-sets $\{Z_n\}$ with $Z_n \supseteq F_n$ and $\bigcap_n Z_n = \phi$.

It would be interesting to study whether a further generalization of weak cb spaces could be introduced characterized by equivalent conditions like (i) and (ii) of weak cb spaces, where the nusc function in (i) and the regular closed subsets in (ii) are respectively replaced by their stronger forms. Further it is known that the characteristic function of regular closed (resp. regular open) subset of a space is nusc (resp. nlsc). This motivates us to search for a class of subsets which is stronger than that of regular closed subsets and whose characteristic function is at the same time stronger than the nusc function. In this direction it is found that the desired characteristics are present in the class of s -regular closed subsets which is defined below.

2. s -Regular closed subset and s -nusc function

Definition 2.1 A subset F (resp. G) of X is said to be strongly regular closed or s -regular closed (resp. strongly regular open or s -regular open) if $F = Cl_X B$ (resp. $G = Int_X A$), where B (resp. A) is a regular F_σ (resp. regular G_δ)-subset of X .

Theorem 2.2 *The complement of an s -regular open subset is s -regular closed and vice versa.*

Proof. Let G be an s -regular open subset of X . Then $G = Int_X A$, where A is a regular G_δ -subset. Now $X - G = X - Int_X A = Cl_X(X - A) = Cl_X B$,

where $B = X - A$ is a regular F_σ -subset of X . Conversely, the complement of an s-regular closed set is s-regular open. ■

From the definition it follows that every s-regular open (resp. s-regular closed) subset is regular open (resp. regular closed). Now we cite an example to show that a regular open (resp. regular closed) subset may not be s-regular open (resp. s-regular closed).

Example 2.3 Let us consider the spaces of ordinals, $\mathbf{W}=[0, \omega_1)$ and $\mathbf{W}^*=[0, \omega_1]$, the one point compactification of $\mathbf{W}$, where ω_1 denotes the first uncountable ordinal. Again let us consider a subset A of $\mathbf{W}$ which is cofinal in $\mathbf{W}$, but not a tail. Then A is open in $\mathbf{W}$ and hence in $\mathbf{W}^*$ and $B = Cl_{\mathbf{W}^*}A$ is a regular closed subset of $\mathbf{W}^*$, which contains ω_1 but not a tail of $\mathbf{W}^*$. Next we show that B is not a regular G_δ . Now any open set in $\mathbf{W}^*$ containing ω_1 must contain a tail $\mathbf{W} - W(\alpha) = \{\sigma : \sigma > \alpha\}$, α is an ordinal. If possible, let B is a regular G_δ -subset of $\mathbf{W}^*$. Then $B = \bigcap_n G_n = \bigcap_n Cl_{\mathbf{W}^*}G_n$, where each G_n is open in $\mathbf{W}^*$ and contains ω_1 . Then for each $n \in \mathbb{N}$, there exists an ordinal α_n such that $\{\sigma : \sigma > \alpha_n\} \subseteq G_n$. Let $\alpha^* > \sup_n \alpha_n$, then $\alpha^* \in \mathbf{W}$ and $\{\sigma : \sigma > \alpha^*\} \subseteq \bigcap_n G_n = B$, i.e., B contains a tail $\mathbf{W} - W(\alpha^*)$, which is a contradiction. Thus B is not even a G_δ and hence not a regular G_δ . Therefore any regular G_δ -subset containing ω_1 must contain a tail $\{\sigma : \sigma > \alpha\}$, α is an ordinal. Let C be a regular G_δ -subset of $\mathbf{W}^*$ containing ω_1 . Then $B \neq C$ and also $Int_{\mathbf{W}^*}B = A$ and $IntC$ contains a tail, which in turn implies that $IntB \neq IntC$ and hence the regular open subset $IntB$ is not equal to the s-regular open subset $IntC$.

This example also shows that the regular closed subset $X - A = X - Int_X B = Cl_X(X - B)$, where $X = \mathbf{W}^*$, is not s-regular closed.

With the help of s-regular closed subsets next we define functions to be called strongly nlsc (in short s-nlsc) and strongly nusc (in short s-nusc) functions.

Definition 2.4 A lsc (resp. usc) function $f : X \rightarrow R$ is said to be s-nlsc (resp. s-nusc) if for each real number $r \in R$, the set $\{x : f(x) < r\}$ (resp. $\{x : f(x) > r\}$) is a countable union of s-regular closed subsets of X .

The rationale of taking countable unions of the above definition stems from the fact that closures of countable unions of s-regular closed subsets are s-regular closed (Lemma 3.4), unlike regular closed subsets where closures of arbitrary unions of regular closed subsets are again regular closed. The following theorem provides us with examples of such functions.

Theorem 2.5 *The characteristic function of an s-regular open subset is s-nlsc.*

Proof. Let G be an s-regular open subset in a space X . The characteristic function f of G is defined by

$$\begin{aligned} f(x) &= 1, & \text{for all } x \in G \\ &= 0, & \text{for all } x \in X - G \end{aligned}$$

It can be easily seen that f is lsc.

We prove that f is s-nlsc. For this we show that $\{x : f(x) < r\}$ is a countable union of s-regular closed subsets of X . We consider the following cases.

Case (i). $r < 0$, $\{x : f(x) < r\} = \emptyset = Cl_X \emptyset$, and $\emptyset$ is regular F_σ .

Case (ii). $0 \leq r < 1$, $\{x : f(x) < r\} = X - G$, which is s-regular closed and can be thought of as a countable union of s-regular closed subsets.

Case (iii). $r \geq 1$, $\{x : f(x) > r\} = \emptyset$, which is s-regular closed.

Hence f is s-nlsc. ■

Dually we can prove that the characteristic function of an s-regular closed subset is s-nusc.

Remark 2.6 From Hardy and Woods [11] we quote here a similar result for nusc (resp. nlsc) function. The characteristic function of a regular closed (resp. open) subset is nusc (resp. nlsc).

Next, we find the relationship between nlsc and s-nlsc functions. That every s-nlsc function is nlsc follows directly from Definition 2.4 and the characterization of nlsc functions [8]. But the converse is not true which follows from the following example.

Example 2.7 In Example 2.3 we have seen that the subset A of $X = \mathbf{W}^*$ is regular open, but not s-regular open and $X - A = F$ is regular closed, but not s-regular closed. Thus, in view of Remark 2.6, it follows that the characteristic function χ_A of A is nlsc. We want to show that χ_A is not s-nlsc. Let us suppose the contrary. Then, by definition, $\{x : \chi_A(x) < r\}$ is a countable union of s-regular closed subsets of X . Then, for $r = \frac{1}{2}$, $\{x : \chi_A(x) < r\} = X - A$ and thus $X - A$ is a countable union of s-regular closed subsets. Hence, $X - A$ is an F_σ -subset, since each s-regular closed subset is closed. In Example 2.3, it has been shown that $Cl_X A$ is regular closed, but not a regular G_δ (not even a G_δ). Thus $X - Cl_X A = Int_X(X - A) = X - A$ (since $X - A$ is open in X) is not regular F_σ (not even a F_σ) and hence cannot be expressed as a countable union of closed subsets. Thus, we arrive at a contradiction. Therefore, χ_A is not s-nlsc.

Similarly, the characteristic function χ_F of F is nusc, but not s-nusc.

3. r -Weak cb spaces and its properties

In this section, we introduce r -weak cb spaces as a generalization of weak cb spaces and study its properties.

Definition 3.1 A space X is called r -weak cb space if for each locally bounded s-nusc function h in X , there exists $f \in C(X)$ such that $|h| \leq f$.

As remarked earlier, our aim is to characterize the r -weak cb space in terms of an increasing cover of s -regular open and a decreasing sequence of s -regular closed subsets. To achieve our goal, we first prove some results in the form of lemmas which will be utilized in the desired characterization.

Lemma 3.2 *If f is strictly positive s -nlsc function on X , then $\frac{1}{f}$ is s -nusc.*

Proof. Let f be an s -nlsc function on X which is strictly positive. Now, we show that $g = \frac{1}{f}$ is s -nusc. Clearly, $\{x : g(x) > 0\} = X$ is a countable union of s -regular closed subsets. Let $r > 0$. Now,

$$\{x : g(x) > r\} = \left\{x : \frac{1}{f(x)} > r\right\} = \left\{x : f(x) < \frac{1}{r}\right\},$$

which is a countable union of s -regular closed subsets, since f is s -nlsc. Thus, for each $r \in R$, the set $\{x : g(x) > r\}$ is a countable union of s -regular closed subsets. Hence $g = \frac{1}{f}$ is s -nusc. $\blacksquare$

Lemma 3.3 *Let $\{B_n\}$ be a countable increasing cover of X consisting of s -regular open subsets and g be a function defined by $g(x) = 1$ if $x \in B_1$ and $g(x) = \frac{1}{n}$ for $x \in B_n - B_{n-1}$, $n \geq 2$, then g is s -nlsc function.*

Proof. Let us consider a countable increasing s -regular open cover $\{B_n\}$ of X . Now, let us define a function g by $g(x) = 1$ if $x \in B_1$ and $g(x) = \frac{1}{n}$ for $x \in B_n - B_{n-1}$, $n \geq 2$. Clearly, g is lsc. Now, we see that the function g is actually s -nlsc. For this, we need to consider the following cases only as $g > 0$ for all x .

Case (i). Let $0 < r \leq 1$, we can always find $n \in N$ such that $\frac{1}{n+1} \leq r < \frac{1}{n}$.

Now, $\{x \in X : g(x) < r\} = \left\{x \in X : g(x) < \frac{1}{n}\right\} = X - B_n$, which is s -regular closed and can be thought of a countable union of s -regular closed subsets.

Case (ii). Let $r > 1$, then $\{x : g(x) < r\} = X$, which is a countable union of s -regular closed subsets. Thus, for each real number r , the set $\{x : g(x) < r\}$ is a countable union of s -regular closed subsets of X . Hence g is an s -nlsc function. $\blacksquare$

Lemma 3.4 *If $A = \bigcup_n Cl_X G_n$, where each G_n is regular F_σ in X , $n \in N$, then $Cl_X A$ is s -regular closed.*

Proof. Let $V = \bigcup_n G_n$, which is regular F_σ . Now, we show that $Cl_X A = Cl_X V$. Clearly, $V \subseteq A$ and so $Cl_X V \subseteq Cl_X A$.

Again, $Cl_X V = Cl_X \left(\bigcup_n G_n \right) \supseteq \bigcup_n Cl_X G_n = A$, which implies $Cl_X Cl_X V \supseteq Cl_X A$, i.e., $Cl_X V \supseteq Cl_X A$. Therefore, $Cl_X A = Cl_X V$, which is an s-regular closed. ■

We are now in a position to characterize the r-weak cb spaces in terms of an increasing cover of s-regular open subsets and a decreasing sequence of s-regular closed subsets having empty intersection.

Theorem 3.5 *Let X be any topological space. Then the following statements are equivalent.*

- a) X is r-weak cb.
- b) For a given strictly positive s-nlsc function g on X , there exists $f \in C(X)$ such that $0 < f(x) \leq g(x)$ for all $x \in X$.
- c) For each countable increasing s-regular open cover of X , there exists a locally finite partition of unity subordinate that cover.
- d) Each countable increasing s-regular open cover of X has a locally finite cozero refinement.
- e) Each countable increasing s-regular open cover of X has a σ -locally finite cozero refinement.
- f) Each countable increasing s-regular open cover of X has a countable cozero refinement.
- g) Given a decreasing sequence $\{P_n : n \in N\}$ of s-regular closed subsets of X with $\bigcap_{n \in N} P_n = \phi$, there exists a decreasing sequence $\{Z_n : n \in N\}$ of zero-sets with $\bigcap_{n \in N} Z_n = \phi$, such that $P_n \subseteq Z_n$ for every n .

Proof. (a) $\Rightarrow$ (b). Let g be a strictly positive s-nlsc function on X . Then $\frac{1}{g}$ is s-nusc (Lemma 3.2). Again, $\frac{1}{g}$ being strictly positive and usc, it is easy to see that $\frac{1}{g}$ is locally bounded. Now, by (a), there exists $f_1 \in C(X)$ such that $0 < \frac{1}{g(x)} \leq f_1(x)$ for all $x \in X$. Then $f(x) = \frac{1}{f_1(x)}$ is such that $f \in C(X)$ and $0 < f(x) \leq g(x)$ for all $x \in X$.

(b) $\Rightarrow$ (c). Let us consider a countable increasing s-regular open cover $\{B_n\}$ of X . Now, let us define a function g by $g(x) = 1$ if $x \in B_1$ and $g(x) = \frac{1}{n}$

for $x \in B_n - B_{n-1}$, $n \geq 2$. Then, g is an s-nlsc function (Lemma 3.3), which is also strictly positive. So, there exists $f \in C(X)$ such that $0 < f(x) \leq g(x)$ for all $x \in X$. Now, let us define a sequence of continuous functions h_n on X as $h_n = [(n+1)f - 1]^+[(n-1)f - 1]^-$, $n \in N$, where $\mathbf{1}(x) = 1$ for all $x \in X$. We see that $h_n(x) = 0$ for $f(x) \leq \frac{1}{n+1}$ or $f(x) \geq \frac{1}{n-1}$ for $n > 1$. Thus

$$Z(h_n) = \left\{ x : f(x) \leq \frac{1}{n+1} \right\} \cup \left\{ x : f(x) \geq \frac{1}{n-1} \right\},$$

i.e.,

$$\begin{aligned} X - Z(h_n) &= \left\{ x : \frac{1}{n+1} < f(x) < \frac{1}{n-1} \right\}, \quad \text{for all } n > 1, \text{ and} \\ X - Z(h_1) &= \left\{ x : \frac{1}{2} < f(x) \right\}, \quad \text{for } n = 1. \end{aligned}$$

Now, we show that the sequence of cozero-sets $\{X - Z(h_n)\}$ is locally finite. Let us consider a point $y \in X$ such that $f(y) = r$, then $0 < r \leq 1$. Since f is continuous, the set $\{x : r - \delta < f(x) < r + \delta\}$, $\delta > 0$ is open and clearly $y \in \{x : r - \delta < f(x) < r + \delta\}$. Now, by the Archimedean property of real number, we can choose $m \in N$ such that $r - \delta > \frac{1}{m-1}$. Then $\{x : r - \delta < f(x) < r + \delta\}$ is a nbd of y which has empty intersection with $X - Z(h_n)$ for $n \geq m$. So the cozero-sets $\{X - Z(h_n)\}$ forms a locally finite family, i.e., $\{h_n\}$ is locally finite. Now, $\{h_n\}$ being a locally finite family of continuous functions, $h(x) = \sum h_n(x)$ is continuous. Next, we show that h is non-vanishing. For this let $p \in X$ and $f(p) = a$, $0 < a \leq 1$. If $a = 1$, then $p \in \left\{ x : \frac{1}{2} < f(x) \right\} = X - Z(h_1)$, so $h_1(p) \neq 0$. On the other hand, if $0 < a < 1$, we can find $n \in N$ such that $p \in \left\{ x : \frac{1}{n+1} < f(x) < \frac{1}{n-1} \right\} = X - Z(h_n)$. In this case, $h_n(p) \neq 0$. Thus $h(p) = \sum h_n(p) \neq 0$. Since h is non-vanishing, $\frac{1}{h}$ is defined and belongs to $C(X)$ and $\left\{ \frac{h_n}{h} \right\}$ is a locally finite partition of unity. Since

$$\begin{aligned} X - Z\left(\frac{h_n}{h}\right) &= \left\{ x : \frac{1}{n+1} < f(x) < \frac{1}{n-1} \right\}, \quad \text{for all } n > 1 \text{ and} \\ X - Z\left(\frac{h_n}{h}\right) &= \left\{ x : \frac{1}{2} < f(x) \right\}, \quad \text{for } n = 1, \end{aligned}$$

we have

$$X - Z\left(\frac{h_n}{h}\right) \subseteq \left\{ x : \frac{1}{n+1} < f(x) \right\} = B_n \quad \text{and} \quad \bigcup_n \left\{ X - Z\left(\frac{h_n}{h}\right) \right\} = X.$$

(c) $\Rightarrow$ (d). Let us consider a countable increasing s-regular open cover $\{B_n\}$ of X , then there exists a locally finite partition of unity $\{F_\alpha\}$, say $\alpha \in \Lambda$, an index set, subordinate to the cover $\{B_n\}$. So by definition $\{X - Z(F_\alpha)\}$ forms a locally finite cozero refinement of the cover $\{B_n\}$.

(d) $\Rightarrow$ (e). Since every locally finite collection is σ -locally finite, the result is obvious.

(e) $\Rightarrow$ (f). Let $\mathcal{A}$ be a σ -locally finite cozero refinement of the countable increasing s-regular open cover $\{B_n\}$ of X . Then $\mathcal{A} = \cup \mathcal{A}_m$, where each $\mathcal{A}_m$ is locally finite collection of cozero-sets. Let $A_{m,n}$ be the union of the sets $\{A_{m,\alpha}\}$ in $\mathcal{A}_m$ such that $A_{m,\alpha} \subseteq B_n$. Since $A_{m,n}$ is the union of a locally finite family of cozero-sets, it is a cozero-set [1]. Thus $\{A_{m,n}\}$ is a countable cozero refinement of $\{B_n\}$.

(f) $\Rightarrow$ (g). Let us consider a decreasing sequence $\{F_n : n \in N\}$ of s-regular closed subsets of X with $\bigcap_n F_n = \phi$. Then $\{X - F_n\}$ is an increasing s-regular open cover of X . Let $\mathcal{M}$ be a countable subset of $C(X)$ such that $\{X - Z(f)\}$, $f \in \mathcal{M}$, is a refinement of the cover $\{X - F_n\}$. Let us denote by Z_n , the intersection of all those $Z(f)$, $f \in \mathcal{M}$ for which $Z(f) \supseteq F_n$. Each Z_n , being a countable intersection of zero sets, is a zero set [9]. Now $\bigcap_n Z_n \subseteq \bigcap_{f \in \mathcal{M}} Z(f) = \phi$, since $\{X - Z(f)\}$ is a cover of X . Thus $\{Z_n\}$ is the desired sequence of zero-sets.

(g) $\Rightarrow$ (a). Let h be a locally bounded s-nusc function on X and define $P_n = \{x : h(x) > n\}$, $n \in N$. The set P_n need not be countable but each P_n can be expressed as a countable union of s-regular closed subsets of X (Definition 2.4). Hence $F_n = Cl_X P_n (\subseteq \{x : h(x) \geq n\})$ is an s-regular closed subset (Lemma 3.4). Then, clearly, $\{F_n\}$ is a decreasing sequence of s-regular closed subsets of X with $\bigcap_n F_n = \phi$. So, by the given condition, there exists a decreasing sequence $\{Z_n = Z(g_n) : n \in N\}$ of zero sets with $\bigcap_{n \in N} Z(g_n) = \phi$ such that $F_n \subseteq Z(g_n)$

for every n , $g_n \in C(X)$. Let us define $f_n = 1 - \left(\bigvee_{i \leq n} n|g_i| \right) \wedge 1$, $n \in N$. Since $\bigcap_{n \in N} Z(g_n) = \phi$ for each $x \in X$, there exists $i \in N$ such that $g_i(x) \neq 0$. So,

we can find $j \in N$ such that $|g_i(y)| > \frac{1}{j}$ for all y in an nbd $N(x)$ of x by the continuity of the function g_i and the Archimedean property of real numbers. Thus $j|g_i(y)| > 1$ for all $y \in N(x)$. If $n \geq i$ and $n \geq j$, then $n|g_i(y)| > j|g_i(y)| > 1$ for all $y \in N(x)$. Choosing $n \geq \max\{i, j\}$, $n|g_i(y)| > 1$ and thus $(\bigvee_{i \leq n} n|g_i|)(y) > 1$ and, consequently, $\left(\bigvee_{i \leq n} n|g_i| \right) \wedge 1 = 1$. Thus, by the definition of f_n , $f_n(y) = 0$ for all $y \in N(x)$, provided $n \geq \max\{i, j\}$, i.e., $y \in X - Z(f_n)$, $y \in N(x)$ only for finite number of values of n . These show that $\{f_n\}$ is locally finite. Therefore, by the locally finite property of $\{f_n\}$, $\sum_n f_n \in C(X)$, it follows that $f = 1 + \sum_n f_n \in C(X)$.

On F_n , we have $g_i(x) = 0$ for $i \leq n$ and hence $f_i(x) = 1$ for $i \leq n$. Thus, $f(x) \geq 1 + n > h(x)$ on $F_n - F_{n+1}$, for each $n \in N$. Also, for $\{X - F_1\}$, $h(x) < 1$, but $f(x) \geq 1$. So, $f(x) \geq h(x)$, for all $x \in X$. This proves the theorem. ■

Next, we study some properties of r -weak cb spaces. In the following theorems it will be shown that, like weak cb spaces, this generalized space also possesses the properties that a cozero subspace of an r -weak cb space and its product with a locally compact paracompact space are again r -weak cb spaces. To prove the first result we need the following lemma.

Lemma 3.6 *If h is s -nusc on X , then h^+ is also s -nusc.*

Proof. Let us consider the set $\{x : h^+(x) < \lambda\}$, $\lambda \in R$. Since $h^+ \geq 0$, we need only to consider the case $\lambda \geq 0$. Now, $\{x : h^+(x) < \lambda\} = \{x : h(x) < \lambda\}$, which is open. Thus h^+ is usc.

Now, consider the set $\{x : h^+(x) > \lambda\}$, $\lambda \in R$.

Case (i). $\lambda < 0$, $\{x : h^+(x) > \lambda\} = X$, which is s -regular closed and can be thought of a countable union of s -regular closed subsets.

Case (ii). $\lambda \geq 0$, $\{x : h^+(x) > \lambda\} = \{x : h(x) > \lambda\}$, which is a countable union of s -regular closed subsets, h being s -nusc.

Hence h^+ is also s -nusc. ■

Theorem 3.7 *Each cozero subspace of an r -weak cb space is r -weak cb.*

Proof. Let X be an r -weak cb space and Y be a cozero subspace of X . Then $Y = X - Z(g)$ for some $g \in C(X)$, $0 \leq g \leq 1$. Let $F_n = \left\{x \in X : \frac{1}{n} \leq g(x)\right\}$. For a given locally bounded s -nusc function h on Y , let us define a sequence of functions $\{h_n\}$, $n \in N$ on X as follows:

$$h_n(x) = h^+(x) \text{ on } F_n \subseteq Y \text{ and } h_n(x) = 0 \text{ on } X - F_n.$$

It can be easily seen that h_n is a locally bounded s -nusc on X for each $n \in N$. Since X is r -weak cb, there exists $f_n \in C(X)$ such that $h_n \leq f_n$ for each n .

Next, we define the continuous functions $g_n : X \rightarrow [0, 1]$ for $n > 2$, as $g_n = 1 - [(n-1)(n-2)g - (n-2)]^+ \wedge 1$, where $1(x) = 1$ etc. for all $x \in X$. Now

$$\begin{aligned} F_{n-2} &= \left\{x \in X : \frac{1}{n-2} \leq g(x)\right\} \\ &= \{x \in X : (n-1)(n-2)g(x) - (n-1) \geq 0\} \\ &= \{x \in X : (n-1)(n-2)g(x) - (n-2) \geq 1\}. \end{aligned}$$

So, $g_n(x) = 1 - 1 = 0$ for all $x \in F_{n-2}$, which implies that $Z(g_n) \supseteq F_{n-2}$. Similarly, it can be checked that $g_n(x) = 1$ for all $x \in X - F_{n-1}$. Assuming $g_1 = g_2 = 1$, we set $f = \vee_n (f_n g_n)|_Y$. Next, we show that $\{g_n|_Y\}$ is a locally finite family in $C(Y)$.

If $y \in Y$, then $g(y) \neq 0$. So, we can find an integer $i \in N$ such that $g(y) > \frac{1}{i}$, i.e., $ig(y) - 1 > 0$. As $g \in C(X)$, the set $\{x : ig(x) - 1 > 0\} = M$, say, is open and $y \in M$. Then, M is a nbd of y which does not meet $X - Z(g_{n+2})$ for $n \geq i$, since $X - Z(g_{n+2}) \subseteq X - F_n = \{x : ng(x) - 1 < 0\}$. Thus, $\{g_n|Y\}$ is a locally finite family in $C(Y)$, which implies that $\{(f_n g_n)|Y\}$ is also locally finite. So $f \in C(Y)$.

Again, since $g_n(x) = 1$ for $x \in X - F_{n-1}$, $F_{n-1} \subseteq F_n$ and hence $g_n(x) = 1$ for $x \in F_n - F_{n-1}$. Therefore, on $F_n - F_{n-1} \subseteq F_n \subseteq Y$, $f = \bigvee_n f_n g_n = \bigvee_n f_n \geq f_n$ for each $n \in N$. Finally, on $F_n - F_{n-1}$, $h^+ = h_n \leq f_n \leq f$ for all $n \in N$, which implies that $h(x) \leq f(x)$ on Y . Hence Y is r-weak cb. ■

Since every clopen subset is cozero, we have the following:

Corollary 3.8 *Every clopen subspace of r-weak cb space is r-weak cb.*

To prove that the product of an r-weak cb space with a locally compact paracompact space is again r-weak cb we first prove the result considering a special case when the other space is compact. To that end, the following lemma is needed.

Lemma 3.9 *Let $p : X \times Y \rightarrow X$ be the projection map and Y be compact (T_1 -space). Then for each s-regular closed subset A of $X \times Y$, $p(A)$ is an s-regular closed subset of X .*

Proof. Let $A = Cl_{X \times Y} G$, where G is a regular F_σ -subset of $X \times Y$, be an s-regular closed subset of $X \times Y$. We are to show that $p(A)$ is an s-regular closed subset of X . It is known that the projection map is open, continuous and surjective. Further since Y is compact, the projection of $X \times Y$ onto X is a closed mapping [14]. Therefore, $p(A) = p(Cl_{X \times Y} G) = Cl_X(p(G))$. It remains to show that $p(G)$ is a regular F_σ -subset of X . Let $G = \bigcup_n F_n = \bigcup_n Int_{X \times Y} F_n$, where each F_n is closed in $X \times Y$. Let $K_n = p(F_n)$, then each K_n is closed in X . Also $p(G) = p\left(\bigcup_n F_n\right) = \bigcup_n p(F_n) = \bigcup_n K_n$. Finally we show that $\bigcup_n K_n = \bigcup_n Int_X K_n$. It suffices to show that $\bigcup_n K_n \subseteq \bigcup_n Int_X K_n$. Now, p being an open mapping, $p(Int_{X \times Y} F_n) \subseteq Int_X p(F_n) = Int_X K_n$, for each $n \in N$. Hence $\bigcup_n p(Int_{X \times Y} F_n) \subseteq \bigcup_n Int_X K_n$, i.e., $p\left(\bigcup_n Int_{X \times Y} F_n\right) \subseteq \bigcup_n Int_X K_n$, i.e., $p\left(\bigcup_n F_n\right) \subseteq \bigcup_n Int_X K_n$, i.e., $\bigcup_n p(F_n) \subseteq \bigcup_n Int_X K_n$, i.e., $\bigcup_n K_n \subseteq \bigcup_n Int_X K_n$. Therefore, $p(G) = \bigcup_n K_n = \bigcup_n Int_X K_n$ is a regular F_σ -subset of X . ■

Theorem 3.10 *The product of an r-weak cb space and a compact (Hausdorff) space is r-weak cb.*

Proof. Let X be an r-weak cb space and Y be a compact (Hausdorff) space. Let $\{K_n\}$, $n \in N$ be a decreasing sequence of s-regular closed subsets of $X \times Y$ with $\bigcap_n K_n = \phi$. Let $p : X \times Y \rightarrow X$ be the projection map. By Lemma 3.9,

$p(K_n) = A_n$, say, is an s-regular closed subset of X for each $n \in N$ and $\{A_n\}$ is also decreasing. Next, we prove that $\bigcap_n A_n = \phi$. For any $x \in X$, $\{x\} \times Y$ is compact. Thus, the family $\{K_n \cap (\{x\} \times Y)\}$ of closed sets does not have the finite intersection property, in other words $\{x\} \times Y$ fails to meet K_n for some n . Therefore $\{A_n\}$ has empty intersection. Now, X being r-weak cb, there exists a decreasing sequence $\{Z_n\}$ of zero-sets having empty intersection such that $Z_n \supseteq A_n$. Lastly, $\{Z_n \times Y\}$ is a sequence of zero-sets in $X \times Y$ with empty intersection such that $Z_n \times Y \supseteq K_n$ for each $n \in N$. Hence, in view of Theorem 3.5 (g), $X \times Y$ is r-weak cb. ■

The conditions of the above theorem could be weakened by assuming locally compact paracompact (Hausdorff) property of the space Y . This will be shown in Theorem 3.17. For this purpose, we require some basic results regarding embedding properties of s-regular closed subsets, which are discussed first. In this direction, we consider an example to illustrate the difficulty in embedding an s-regular closed subset of a space to a subspace of it.

Example 3.11 Let $X = [0, 1] \cup [2, 3]$ and $Y = \{0\} \cup [2, 3] \subseteq X$. Now $G = \left(0, \frac{1}{2}\right) \cup (2, 2.5)$ is regular F_σ in X and hence $F = Cl_X G = \left[0, \frac{1}{2}\right] \cup [2, 2.5]$ is s-regular closed in X . Then, $V = G \cap Y = (2, 2.5)$ is also a regular F_σ in Y . But $0 \notin Cl_Y V$, whereas $0 \in Cl_X G$. Thus $F \cap Y = \{0\} \cup [2, 2.5]$ and $Cl_Y V = [2, 2.5]$, which shows that $F \cap Y \neq Cl_Y V$.

Thus, we see that if $F = Cl_X G$, where G is regular F_σ in X , be an s-regular closed subset of a space X and $Y \subseteq X$, then for $V = G \cap Y$ the equality $F \cap Y = Cl_Y V$ may not be true always.

Now, we investigate the condition under which $F \cap Y = Cl_Y V$ holds.

Theorem 3.12 *If Y be a subset of X with the property that $Cl_X G \cap Y \subseteq Cl_Y V$, where G is regular F_σ in X and $V = G \cap Y$, then, for each s-regular closed subset $F = Cl_X G$ of X , $F \cap Y = Cl_Y V$ is s-regular closed in Y .*

Proof. Let $F = Cl_X G$, where G is regular F_σ -subset of X . Then, $G \cap Y$ is a regular F_σ -subset of Y [4]. Let $V = G \cap Y$. Then $Cl_Y V$ is s-regular closed subset of Y . We are to show that $F \cap Y = Cl_Y V$. Now, $Cl_Y V \subseteq Cl_X V \subseteq Cl_X G = F$, i.e., $Cl_Y V \cap Y \subseteq F \cap Y$, i.e., $Cl_Y V \subseteq F \cap Y$ as $Cl_Y V \subseteq Y$.

Conversely, let $p \in F \cap Y = Cl_X G \cap Y$. Then, by our assumption, $p \in Cl_Y V$. Hence, $F \cap Y \subseteq Cl_Y V$. Therefore, $F \cap Y = Cl_Y V$ is s-regular closed subset of Y . ■

The assumption made in the above theorem is fulfilled by wide category of subspaces, e.g. open subspaces, dense subspaces [9, Art. 0.12] etc.

Next, we investigate the relationship between the properties of preservation of trace of an s-regular closed subset and the restriction of an s-nlsc (or s-nusc) function of a space to a subspace of it. In this direction first we see that the preservation of trace of an s-regular closed subset implies that the restriction of an s-nlsc (resp. s-nusc) function to a subspace is s-nlsc (resp. s-nusc).

Theorem 3.13 *The restriction of an s-nusc (resp. s-nlsc) function of a space X to a subset Y of X is also s-nusc (resp. s-nlsc), provided that the trace of each s-regular closed subset of X to Y is an s-regular closed subset of Y .*

Proof. First we prove the theorem for the s-nusc function. The proof for s-nlsc case is similar. Let f be an s-nusc function on X and Y is a subset of X . Let $f|_Y = g$. For each real number r , $S_r(f) = \{x \in X : f(x) > r\}$ is a countable union of s-regular closed subsets of X , since f is s-nusc on X . Now, $S_r(f) \cap Y$ is a countable union of s-regular closed subsets of Y , by our assumption. Hence, $S_r(f) \cap Y = \{y \in Y : g(y) > r\}$. Hence, g is s-nusc function on Y , since $\{y \in Y : g(y) > r\} = S_r(f) \cap Y$. ■

But it is interesting to see that the restriction of an s-nlsc (resp. s-nusc) function of a space to a closed subspace is again s-nlsc (resp. s-nusc).

Theorem 3.14 *The restriction of an s-nlsc (resp. s-nusc) function of a space X to a closed subset C having Lindelöf frontier is also s-nlsc (resp. s-nusc).*

Proof. Let us prove the theorem for the s-nlsc function. The s-nusc case may be dealt dually. Let f be an s-nlsc function on X and let C be a closed subset of X . Clearly, $g = f|_C$ is lsc. Now consider the set $F_r = \{x : f(x) < r\}$, $r \in R$. By definition, F_r is a countable union of s-regular closed subsets of X , i.e. $F_r = \bigcup_n Cl_X U_n$, where U_n is a regular F_σ -subset of X for each $n \in N$. Now, $\{t \in C : g(t) < r\} = F_r \cap C = \bigcup_n (Cl_X U_n \cap C)$. We propose to show that the trace $Cl_X U_n \cap C$ of s-regular closed subsets $Cl_X U_n$ generated by the s-nlsc function f on X is again s-regular closed in C . Let $V_n = U_n \cap C$, then V_n is a regular F_σ -subset of C [4] and there exists a regular F_σ -subset W_n , $n \in N$, $W_n \supseteq V_n$ such that $\bigcup_n Cl_X U_n \cap C = \bigcup_n Cl_C W_n$. Since $Cl_C V_n \subseteq Cl_X U_n \cap C$ for each n , so we have $\bigcup_n Cl_C V_n \subseteq \bigcup_n (Cl_X U_n \cap C)$. Now let $p \in \bigcup_n (Cl_X U_n \cap C)$. Then, $p \in Cl_X U_n \cap C$ for some n , and every nbd of p in X meets U_n . Now, we claim that either every nbd of p in C meets V_n , i.e., $p \in Cl_C V_n$ or else p is an isolated point in C . Clearly, $f(p) = g(p) < r$. Now, f being lsc, the sets $\{x \in X : f(x) > f(p) - \epsilon\}$, $\epsilon > 0$ form basic nbds of p in X . Then, each such nbd meets $V_n = U_n \cap C$ and hence $p \in Cl_X V_n = Cl_C V_n$ provided $\{x \in C : r > f(x) > f(p) - \epsilon\} - \{p\} \neq \phi$. Otherwise, p would be an isolated point of C . As C is closed, all such points p satisfying $p \in Cl_X U_n \cap C$, but $p \notin Cl_C V_n$ for some n , must be a boundary point of C . Let G be the collection of all such p 's. Now, we see that G is indeed a regular F_σ -subset of C . By our assumption, $Fr_X C$ is Lindelöf and $G \subseteq Fr_X C$ consists of isolated points. Hence G must be a countable set, which in turn implies that G is a regular F_σ subset. Now, it is plain to see that $\left(\bigcup_n Cl_C V_n\right) \cup G = \bigcup_n (Cl_X U_n \cap C)$ and $\{t \in C : g(t) < r\} = F_r \cap C = \left(\bigcup_n Cl_C V_n\right) \cup G = \bigcup_n (Cl_X U_n \cap C)$ is an s-regular closed of C and hence g is s-nlsc. ■

Corollary 3.15 *Let $X \times C \subseteq X \times Y$, where C is compact in Y . Then for each s -nlsc (resp. s -nusc) function f of $X \times Y$, $f|(X \times C)$ is s -nlsc (resp. s -nusc).*

Proof. Since compact subsets are closed in Hausdorff spaces and $Fr_{X \times Y}(X \times C)$ is compact and hence Lindelöf, the corollary at once follows from Theorem 3.14. ■

Finally, before going to the proof of the main theorem we quote a result from [2].

Result 3.16 *Let $S \subseteq X$ be open in X and f and g are respectively members of $C(S)$ and $C(X)$. Then, the function h on X defined by $h = f + g$ on S and $h = g$ on $X - S$ is continuous.*

Theorem 3.17 *The product of a r -weak cb space and a locally compact paracompact (Hausdorff) space is r -weak cb.*

Proof. Let X be a r -weak cb space and Y be a locally compact paracompact (Hausdorff) space. Since Y is locally compact, each point of Y has a compact nbd. Consider a compact nbd K_y containing $y \in Y$. Then there exists an open set O_y in Y such that $y \in O_y \subseteq K_y$. Again Y being paracompact, the cover of Y by the open sets $\{O_y : y \in Y\}$ has a locally finite open refinement $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$, say. Then, clearly, the members of the family $\bar{\mathcal{U}} = \{ClU_\alpha : \alpha \in \Lambda\}$ are compact sets. Since Y is normal (as it is paracompact Hausdorff) and the open cover $\mathcal{U}$ of Y is point finite (as it is locally finite), there is an open cover $\mathcal{V} = \{V_\alpha : \alpha \in \Lambda\}$ such that $ClV_\alpha \subseteq U_\alpha$ for each $\alpha \in \Lambda$ [7].

Now, let h be a locally bounded s -nusc function on $X \times Y$. By Theorem 3.10, $X \times ClU_\alpha$ is r -weak cb. The function $h|X \times ClU_\alpha$ being s -nusc on $X \times ClU_\alpha$ (Corollary 3.15) and $X \times ClU_\alpha$ is r -weak cb, the function $h|X \times ClU_\alpha$ is bounded above by a continuous function f in $X \times ClU_\alpha$. The disjoint sets ClV_α and $Y - U_\alpha$ are such that ClV_α is compact and $Y - U_\alpha$ is a closed set of the completely regular space Y . Hence ClV_α and $Y - U_\alpha$ are completely separated [9] and this implies that $X \times V_\alpha \subseteq X \times ClV_\alpha$ and $X \times (Y - U_\alpha)$ are completely separated. Thus, there exists a function e_α in $C(X \times Y)$ such that $e_\alpha = 1$ on $X \times V_\alpha$ and $e_\alpha = 0$ on $X \times (Y - U_\alpha)$. Finally, define a function g_α on $X \times Y$ as $g_\alpha = e_\alpha + f$ on $X \times V_\alpha$ and $g_\alpha = e_\alpha$ elsewhere. Since $X \times V_\alpha$ is open in $X \times Y$, it can be seen that $g_\alpha \in C(X \times Y)$ (Result 3.16) and $g_\alpha(X \times Y) \geq |h(x, y)|$ on $X \times V_\alpha$ and $g_\alpha = 0$ on $X \times (Y - U_\alpha)$. Again, since $\{U_\alpha : \alpha \in \Lambda\}$ is locally finite, $g = \bigvee_{\alpha \in \Lambda} g_\alpha$ exists in $C(X \times Y)$ and clearly $g \geq h$. Hence the theorem. ■

Next we study the relationship of r -weak cb space with other spaces.

Theorem 3.18 *Every weak cb space is r -weak cb.*

Proof. Let X be a weak cb space and $\{P_n : n \in N\}$ be a decreasing sequence of s -regular closed subsets of X with $\bigcap_{n \in N} P_n = \phi$. Now every s -regular closed subset is regular closed. Hence $\{P_n\}$ is a decreasing sequence of regular closed subsets with $\bigcap_{n \in N} P_n = \phi$. By weak cb property there exists a decreasing sequence

$\{Z_n : n \in N\}$ of zero-sets with $\bigcap_{n \in N} Z_n = \phi$, such that $P_n \subseteq Z_n$ for every n . Therefore X is r -weak cb . ■

Corollary 3.19 *Every completely regular (Hausdorff) pseudocompact space is r -weak cb .*

Proof. Since every completely regular (Hausdorff) pseudocompact space is weak cb [15], hence the result follows from Theorem 3.18. ■

Theorem 3.20 *Every Oz space is r -weak cb .*

Proof. Since every Oz space is weak cb , the result follows immediately from Theorem 3.18. ■

That an r -weak cb space may fail to be countably paracompact, can be shown by the following example.

Example 3.21 The Tychonoff plank T , being pseudocompact [9], is an r -weak cb space by Corollary 3.19. But T is not countably paracompact [15].

Conclusion

In this paper, a generalized weak cb space called r -weak cb space has been introduced with the help of s -nlsc function and its various properties have been studied. This space includes the well known cb , weak cb spaces, but fails to imply countably paracompactness. The results obtained are interesting in the sense that various properties possessed by the cb and weak cb spaces are also possessed by the generalized r -weak cb spaces. This study opens up the future scope of investigation for the properties of the newly introduced s -nusc (s -nlsc) functions, and interrelationship between r -weak cb and nd -spaces [10], where it is known that every nd -space is countably paracompact and together with weak cb it implies cb property.

Acknowledgement. The authors would like to thank the referees for their valuable suggestions.

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Accepted: 13.09.2012

MORE PROPERTIES ON FLEXIBLE GRADED MODULES

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Abstract. In this paper, we study the structure of flexible graded modules over various types of graded rings such as first strong and augmented graded rings. Also, we introduce the notions of flexibly simple and flexibly Noetherian modules, and investigate various properties of such modules.

Keywords: graded rings, graded modules, flexible graded modules, augmented graded modules.

2010 Mathematics Subject Classification: 13A02.

1. Introduction

Strongly graded modules were a reasonable generalization of strongly graded rings, which play an important role in producing interesting theorems and results in Graded Ring Theory and Graded Module Theory. Of course, this is due to the nice structure of a ring or a module when it is equipped with a strong graduation. A natural question arises about whether we can extend the category of strongly graded rings and modules in order to increase the application area of these nice results. As a step on this path, first and second strongly graded rings were presented by Refai [5] to generalize strongly graded rings. Another step was presented by Refai and Moh'D [10], where first strongly graded rings were produced to generalize strongly graded modules. A second generalization of strongly graded modules was the flexible graded modules, exhibited in [12].

This paper continues the work done in [12], where we further investigate about flexible modules. In particular, we study the structure of these modules over various types of graded rings, and observe the behavior of the submodules of such modules. In addition, several characterizations related to flexible modules are listed in different places in the article. The main result in this paper demonstrates that over augmented graded rings, flexible and augmented modules are the same.

In Section 2, we list some important facts about flexible modules. These facts will be useful in the sequel sections.

In Section 3, we study flexible graded modules when the ring is first strong. We study various properties in this stage. In particular, we give a partial answer to the validity of the statement “A module is flexible iff every submodule is flexible”. Also, a partial answer of the question “whether every module can be graduated to be flexible?” is presented. On the other hand, we generalize some results of [3]. The main result of this section asserts that for flexible modules over commutative graded rings, every submodule can be graduated to become flexible. This assertion can be interpreted as “In flexible modules over commutative rings, every discussion about a property concerning submodules can be carried out regarding them as flexible submodules”. This fact allows us to use the nice structure and the smoothness of flexible modules.

In Section 4, we further study the structure of flexible modules over augmented graded rings. The main result in this section shows that flexible modules and augmented graded modules are the same over augmented graded rings. This result allows us to build augmented graded modules through flexible modules, and avoid checking out the rather complicated conditions of augmented graded modules.

In Section 5, we introduce the notions of flexibly simple and flexibly Noetherian modules. These concepts generalize the concepts of flexible simple and flexible Noetherian modules [14], which were defined only for flexible modules. We show that in flexible modules over commutative graded rings, “flexibly simple” (resp. “flexibly Noetherian”) property implies “simple” (resp. “Noetherian”) property.

2. Preliminaries

This section presents some necessary background of graded rings and graded modules considered in this paper. More details can be found in [5, 6, 7, 12]. For a general background of graded rings and graded modules, we advise the reader to look in [1, 2, 3]. Throughout this paper, unless otherwise stated, G is a group with identity e , $R = \bigoplus_{g \in G} R_g$ is a G -graded ring with unity 1, and $M = \bigoplus_{g \in G} M_g$ is a G -graded left R -module. The set $\text{supp}(R, G) = \{g \in G : R_g \neq 0\}$ is called the support of R . The support of M , $\text{supp}(M, G)$, is defined similarly. To avoid repetition, we assume $R \neq 0$, and $M \neq 0$. One more thing, all modules considered in this article are left modules.

Definition 2.1 [12] Let R be a G -graded ring, and M a G -graded R -module. Then M is said to be flexible if $M_g = R_g M_e$, for every $g \in G$.

Proposition 2.2 [12] Let R be a G -graded ring, and M a G -graded R -module. Then M is flexible iff $M = R M_e$.

Remark 2.3 [12] Let M be a flexible G -graded R -module. Then

1. $M \neq 0$ iff $M_e \neq 0$ iff $e \in \text{supp}(M, G)$.
2. $\text{supp}(M, G) \subseteq \text{supp}(R, G)$.

Definition 2.4 [12] Let R be a G -graded ring, and M be a flexible R -module. A G -graded R -submodule N of M is said to be flexible, if N is a flexible R -module.

Proposition 2.5 [12] Let R be a G -graded ring, and M be a G -graded R -module. If X is an R_e -submodule of M_e , then RX is a flexible R -submodule of M .

3. Flexible modules over first strongly graded rings

This section is devoted to study the structure of flexible modules over first strongly graded rings. In addition, some different properties are considered. For instance, flexibility (i.e., to become flexible) results such as Theorem 3.8 and Proposition 3.10.

Definition 3.1 [5] A G -graded ring R is said to be first strong iff $R_g R_h = R_{gh}$, for all $g, h \in \text{supp}(R, G)$ iff $R_g R_{g^{-1}} = R_e$, for all $g \in \text{supp}(R, G)$ iff $1 \in R_g R_{g^{-1}}$, for all $g \in \text{supp}(R, G)$.

Proposition 3.2 [5] If R is a first strongly G -graded ring, then $\text{supp}(R, G)$ is a subgroup of G .

Lemma 3.3 [10] Let R be a first strongly G -graded ring, and M be an R -module. If N and L are R_e -submodules of M , then $R_g(N \cap L) = R_g N \cap R_g L$, for every $g \in G$.

The following proposition tells that on first strongly graded rings, flexibility of a graded module is completely determined by the behavior of the supports of the ring and the module.

Proposition 3.4 [10] Let R be a first strongly G -graded ring, and M be a G -graded R -module. Then M is flexible iff $\text{supp}(M, G) = \text{supp}(R, G)$.

Lemma 3.5 [10] Suppose R is a first strongly G -graded ring, and M is an R -module. If X is an R -submodule of M , then $R_g X = X$, for every $g \in \text{supp}(R, G)$.

Proposition 3.6 [14] *Let R be a first strongly G -graded ring, and M be a flexible R -module. Then every G -graded R -submodule of M is also flexible.*

Proof. We have $R_{g^{-1}}M_g = M_e$, for every $g \in \text{supp}(R, G)$. Let $X = \bigoplus_{g \in G} X_g$ be a G -graded R -submodule of M . Fix $g \in \text{supp}(R, G)$. By Lemma 3.5, $R_g X = X$. So, Lemma 3.3 implies $X_g = R_g R_{g^{-1}}(X \cap M_g) = R_g(X \cap M_e) = R_g X_e$. If $g \notin \text{supp}(R, G)$, Proposition 3.4 yields $X_g = R_g X_e$. ■

Proposition 3.6 asserts that for flexible modules over first strongly graded rings, there is no graded submodule of M , which is not flexible. In the next two results, if R is commutative, we prove two facts. The first fact states that we can drop the “first strong” condition from Proposition 3.6. The second fact states that not only are graded submodules in this proposition flexible, but also that every submodule is graded and flexible.

Lemma 3.7 *Let R be a commutative G -graded ring, and M be a G -graded R -module. If N is an R -submodule of $R M_e$, then there exists an R_e -submodule X of M_e such that $N = R X$. Further, N is a flexible G -graded R -submodule of M .*

Proof. Define the set X by $X = \{x \in M_e : r x \in N, \text{ for some } r \in R\}$. Then X is an R_e -submodule of M_e . Moreover, $N = R X$, and it is a G -graded flexible R -submodule of M , with $N_g = R_g X$, for each $g \in G$. ■

Theorem 3.8 *Let R be a commutative G -graded ring, and M be a flexible G -graded R -module. Then every R -submodule of M is G -graded and flexible R -submodule of M .*

Proof. Apply Lemma 3.7 to M . ■

Remark 3.9 Theorem 3.8 shows how strong the structure of flexible modules is over commutative rings. In such modules, any property of submodules can be considered as a property of graded or even flexible submodules. This fact gives us the opportunity to use the nice properties of flexible modules. Hence, a question arises “Given an R -module M and a group G , Does a graduation of M by G exist such that M turns into a flexible R -module?” We give a partial answer to this question. Hopefully, we will continue searching for more answers in the near future.

Although the following proposition is not related to modules over first strongly graded rings, we prefer to put it in this section to keep the flow of the article homogeneous, especially since it was, in the first place, an indirect outcome of Proposition 3.6.

Proposition 3.10 *Let R be a G -graded ring, and M be a free R -module. Then there exists a graduation of M by G such that M is a flexible G -graded R -module.*

Proof. Let T be a basis of M . We have $X = R_e T$ is an R_e -submodule of M . For each $g \in G$, set $M_g = R_g X$. It is not difficult to see that $M_g = R_g M_e$, for each $g \in G$, and $M = \sum_{g \in G} M_g$. Let $z \in M_h \cap (\sum_{g \in G - \{h\}} M_g)$. Then there exist $r_g \in R_g$, and $x^g \in X$, where $g \in G$, such that $z = r_h x^h = \sum_{g \in G - \{h\}} r_g x^g$. So, $r_h x^h - \sum_{g \in G - \{h\}} r_g x^g = 0$. Since $x^g \in X$, there exist $t^g \in T$, and $r^g \in R_e$ such that $x^g = r^g t^g$, where $g \in G$. Thus, $r_h r^h t^h - \sum_{g \in G - \{h\}} r_g r^g t^g = 0$. Since T is linearly independent, we have $r_h r^h = 0$, which implies $z = 0$. Therefore, $M = \bigoplus_{g \in G} M_g$. Consequently, M is a flexible G -graded R -module. ■

Recall that if M is an R -module, $S \subseteq R$, and $0 \neq x \in M$. Then x is called S -torsion free, if $rx \neq 0$, whenever $0 \neq r \in S$. Otherwise, x is said to be S -torsion. In addition, A subset N of M is S -torsion free, if every element of N is S -torsion free. In the upcoming work, we give a partial converse of Proposition 3.6, as shown in Corollary 3.14.

Lemma 3.11 *Let R be a G -graded ring, and M be a flexible R -module. If M_e contains an R -torsion free element, then $\text{supp}(M, G) = \text{supp}(R, G)$.*

Proof. Let $g \in G$. Since M_e has a torsion free element, say x , we have $M_g = 0$ iff $R_g = 0$. Actually, if $M_g = R_g M_e = 0$, then $R_g x = 0$, and hence $R_g = 0$. Therefore, $\text{supp}(M, G) = \text{supp}(R, G)$. ■

Proposition 3.12 *Let R be a G -graded ring, and M be a flexible R -module such that M_e contains an R -torsion free element. If $g \in \text{supp}(R, G)$, then R_g contains a regular element (an element which is not a zero divisor) iff M_g contains an R -torsion free element.*

Proof. Suppose R_g contains an element r_g , which is not a zero divisor. By assumption, let $x_e \in M_e$ be an R -torsion free element. We have $0 \neq r_g x_e \in M_g$. Moreover, if $\alpha \in R$, and $\alpha(r_g x_e) = 0$, then $(\alpha r_g) x_e = 0$, which implies in turn $\alpha r_g = 0$. Thus, $\alpha = 0$. Hence, the element $r_g x_e$ is an R -torsion free in M_g .

For the converse, assume M_g contains an R -torsion free element. By Lemma 3.11, $g \in \text{supp}(M, G)$. Let $x_g = r_g x_e \in M_g$ be an R -torsion free element, where $r_g \in R_g$ and $x_e \in M_e$. Let $h \in \text{supp}(R, G)$, and $\alpha \in R_h$ such that $\alpha r_g = 0$. Then $(\alpha r_g) x_e = 0$ or $\alpha(r_g x_e) = 0$. Hence, $\alpha x_g = 0$. Since x_g is R -torsion free, we obtain $\alpha = 0$. Consequently, R_g has no zero divisors in R . ■

In fact, Proposition 3.12 allows us to switch the discussion between torsion free elements in M , and regular elements in R .

Proposition 3.13 *Let R be a G -graded ring, and M be a flexible R -module such that M_g has a torsion free element, for every $g \in \text{supp}(M, G)$. Then R is first strong iff every G -graded cyclic submodule of M , with a homogeneous generator, is flexible.*

Proof. Assume every G -graded cyclic submodule of M , with a homogeneous generator, is flexible. By Lemma 3.11, $\text{supp}(M, G) = \text{supp}(R, G)$. Let $g \in \text{supp}(R, G)$, and x_g be an R -torsion free element in M_g . By assumption, Rx_g is flexible. Fix $h \in \text{supp}(R, G)$. We have $R_{hg^{-1}}x_g = (Rx_g)_h = R_h(Rx_g)_e = R_hR_{g^{-1}}x_g$. Since x_g is R -torsion free, we get $R_{hg^{-1}} = R_hR_{g^{-1}}$. Now, put $h = g$ in the last equation to obtain $R_e = R_gR_{g^{-1}}$. Therefore, R is first strong. The converse follows by Proposition 3.6. ■

Corollary 3.14 *Let R be a G -graded ring, and M be a flexible R -module such that M_g has a torsion free element, for every $g \in \text{supp}(M, G)$. Then R is first strong iff every G -graded R -submodule of M is flexible.*

As final applications of flexible modules over first strongly graded rings, we curve away to homological algebra to generalize some results of [3]. The proof of Proposition 3.15 resembles the proof of the similar result of [3]. So, we will not put it in the body of the article. On the other hand, the proof of Proposition 3.16 is inferred by combining Proposition 2.1 of [11], and the same result of [3], which is valid for strongly graded rings.

Proposition 3.15 [14] *Let R be a first strongly G -graded ring such that $\text{supp}(R, G)$ is finite, M be an R -module, and $N = \bigoplus_{g \in G} N_g$ be a flexible R -module. Then*

1. *For each $g \in \text{supp}(R, G)$, the function $\varphi : \text{Hom}_R(M, N) \longrightarrow \text{Hom}_{R_e}(M, N_g)$ defined by $\varphi(f) = \pi_g \circ f$ is a group isomorphism, where $\pi_g : N \longrightarrow N_g$ is the canonical projection on the g -th component of N .*
2. *For each $g \in \text{supp}(R, G)$, the function $\psi : \text{Hom}_R(N, M) \longrightarrow \text{Hom}_{R_e}(N_g, M)$ defined by $\psi(f) = f \circ i_g$ is a group isomorphism, where $i_g : N_g \longrightarrow N$ is the canonical inclusion of the g -th component of N .*

Proposition 3.16 [14] *Let R be a first strongly G -graded ring such that $\text{supp}(R, G)$ is finite, M be an R -module, and $N = \bigoplus_{g \in G} N_g$ be a flexible R -module. Then*

1. *For each $g \in \text{supp}(R, G)$, $\text{Ext}_R^n(M, N) \cong \text{Ext}_{R_e}^n(M, N_g)$.*
2. *For each $g \in \text{supp}(R, G)$, $\text{Ext}_R^n(N, M) \cong \text{Ext}_{R_e}^n(N_g, M)$.*

4. Flexible modules over augmented graded rings

In this section, we further study the structure of flexible modules over augmented G -graded rings. We show in particular that over augmented graded rings, flexible modules and augmented modules are the same.

Definition 4.1 [7] *A G -graded ring R is said to be augmented if the following conditions hold:*

1. $R_e = \bigoplus_{g \in G} R_{e-g}$ is a G -graded ring.
2. For each $g \in G$, there exists $r_g \in R_g$ such that $R_g = R_e r_g$. We assume $r_e = 1$.
3. If $r_g \neq 0$ and $r_h \neq 0$ are as in (2), then $r_g r_h = r_{gh}$.
4. If $r_g \neq 0$ and $r_h \neq 0$ are as in (2), and $x, y \in R_e$, then $(xr_g)(yr_h) = xy r_{gh}$.

An r_g that appears in condition (2) of Definition 4.1 is called a g -representative. The set of all selected nonzero representatives is denoted by $\Lambda(R, G)$. Thus, the set $\Lambda(R, G)$ may vary as the representatives vary. However, once an augmented graded ring is under consideration, we fix $\Lambda(R, G)$.

Proposition 4.4 below modifies Proposition 1.12 of [9], where it shows that augmented graded rings are a subcategory of first strongly graded rings. We begin with the following lemma, which we omit its proof, because it is trivial.

Lemma 4.2 *Let R be an augmented G -graded ring. Then $g \in \text{supp}(R, G)$ iff $r_g \in \Lambda(R, G)$.*

Remark 4.3 [7] If R is an augmented G -graded ring, then

1. For every $g \in G$, R_g is a G -graded R_e -module, with the graduation $R_{g-h} = R_{e-h} r_g$.
2. For every $g, h, g', h' \in G$, $R_{g-h} R_{g'-h'} \subseteq R_{gg'-hh'}$.
3. Condition (4) of Definition 4.1 is equivalent to the condition

$$(x_{e-h} r_g)(x_{e-h'} r_{g'}) = x_{e-h} x_{e-h'} r_g r_{g'},$$

for all $h, h' \in G$, and $r_g, r_{g'} \in \Lambda(R, G)$.

Proposition 4.4 *Every augmented G -graded ring is first strong.*

Proof. Assume R is an augmented G -graded ring. Let $g, h \in \text{supp}(R, G)$. By Lemma 4.2, $R_g R_h = R_e r_g R_e r_h = R_e R_e r_g r_h = R_e r_{gh} = R_{gh}$. Therefore, R is first strong. ■

Corollary 4.5 *Let R be an augmented G -graded ring. Then*

1. $\text{supp}(R, G)$ is a subgroup of G .
2. $\Lambda(R, G)$ is a multiplicative group.
3. $\text{supp}(R, G)$ is group-isomorphic to $\Lambda(R, G)$.
4. $R_g = r_g R_e = R_e r_g$, for every $g \in \text{supp}(R, G)$.

Proof. 1. Follows directly from Propositions 3.2 and 4.4.

2. Apply Lemma 2.3 of [9] along with condition (1).

3. Apply Lemma 2.3 and conditions (1) and (2).

4. Apply (1) and condition (4) of Definition 4.1. ■

Remark 4.6 Corollary 4.5 gives us the right to cancel the condition “ $\text{supp}(R, G)$ is a subgroup of G ” from all results in [9].

Definition 4.7 [6] Let R be an augmented G -graded ring. A G -graded R -module M is said to be augmented if the following conditions hold:

1. $M_g = \bigoplus_{h \in G} M_{g-h}$ is a G -graded R_e -module.
2. $R_{g-h} M_{g'-h'} \subseteq M_{gg'-hh'}$, for every $g, h, g', h' \in G$.

Given an augmented G -graded ring, we let λ be the sum of elements of $\Lambda(R, G)$. If we set $r_g = 0$, for every $g \notin \text{supp}(R, G)$, and $r_g \in \Lambda(R, G)$, otherwise, we can write $\lambda = \sum_{g \in G} r_g$. The next proposition shows that a flexible module over an augmented ring is completely determined by M_e and λ .

Proposition 4.8 *Let R be an augmented G -graded ring, and M be a G -graded R -module. Then M is flexible iff $M = \lambda M_e$.*

Proof. Suppose M is flexible R -module. It follows from Propositions 3.4 and 4.4 that $\text{supp}(M, G) = \text{supp}(R, G)$. Let $g \in G$. We have $M_g = R_g M_e = r_g M_e$. Hence, $M = \bigoplus_{g \in G} (r_g M_e) = \lambda M_e$.

Conversely, Suppose $M = \lambda M_e$. Then $M = (\sum_{g \in G} r_g) M_e = \bigoplus_{g \in G} R_g M_e$. Since $R_g M_e \subseteq M_g$, we obtain $M_g = R_g M_e$, for every $g \in G$. Therefore, M is flexible. ■

Corollary 4.9 *Let R be an augmented G -graded ring, and M be an augmented G -graded R -module. If $\text{supp}(M, G) = \text{supp}(R, G)$, then M is flexible, and $M_{g-h} = r_g M_{e-h}$, for every $g, h \in G$.*

Proof. Assume $\text{supp}(M, G) = \text{supp}(R, G)$. By Propositions 3.4 and 4.4, M is flexible. Proposition 4.8 implies $M_g = r_g M_e$, for all $g \in G$, where $r_g \in \Lambda(R, G)$, if $g \in \text{supp}(R, G)$, and $r_g = 0$, otherwise. Thus, $M_{g-h} = r_g M_{e-h}$, for every $g, h \in G$. ■

The following proposition gives the converse of Proposition 4.8.

Proposition 4.10 *Let R be an augmented G -graded ring, and M be a flexible G -graded R -module such that M_e is a G -graded R_e -module. Then M is an augmented G -graded R -module.*

Proof. First, we prove that M_g is a G -graded R_e -module, with $M_{g-h} = r_g M_{e-h}$, for all $g, h \in G$. Propositions 3.4 and 4.4 yield $\text{supp}(M, G) = \text{supp}(R, G)$. Let $g \in \text{supp}(R, G)$ and $h \in G$. Then $\sum_{h \in G} r_g M_{e-h} \subseteq M_g$. On the other hand, let $x \in M_g$. Then $x = \sum_{h \in G} r_g m_{e-h} \in \sum_{h \in G} r_g M_{e-h}$ and hence $M_g = \sum_{h \in G} r_g M_{e-h}$. Let $\sigma \in G$, and $x \in r_g M_{e-\sigma} \cap \sum_{h \in G-\{\sigma\}} r_g M_{e-h}$. Then $x = r_g m_{e-\sigma} = \sum_{h \in G-\{\sigma\}} r_g m_{e-h}$, where $m_{e-h} \in M_{e-h}$, for every $h \in G$. By Corollary 4.5, r_g is a unit. So we get $m_{e-\sigma} - \sum_{h \in G-\{\sigma\}} m_{e-h} = 0$. Thus, we obtain $m_{e-\sigma} = 0$, and then $x = 0$. Therefore, $r_g M_{e-\sigma} \cap \sum_{h \in G-\{\sigma\}} r_g M_{e-h} = 0$. Consequently, $M_g = \bigoplus_{h \in G} r_g M_{e-h}$. The same result obviously holds if $g \notin \text{supp}(R, G)$. For each $g, h \in G$, set $M_{g-h} = r_g M_{e-h}$. Let $\sigma \in G$. Then $R_{e-\sigma} M_{g-h} = r_g (R_{e-\sigma} M_{e-h}) \subseteq M_{g-\sigma h}$. As a result, we obtain M_g is a G -graded R_e -module.

Now, we show that $R_{g-h} M_{\sigma-\tau} \subseteq M_{g\sigma-h\tau}$, for every $g, h, \sigma, \tau \in G$. In fact, if $g, \sigma \in \text{supp}(R, G)$, we have $R_{g-h} M_{\sigma-\tau} = r_g R_{e-h} r_\sigma M_{e-\tau} = r_g r_\sigma R_{e-h} M_{e-\tau} \subseteq r_g r_\sigma M_{e-h\tau} = M_{g\sigma-h\tau}$. The case where either $g \notin \text{supp}(R, G)$ or $\sigma \notin \text{supp}(R, G)$ is easy.

As a conclusion, M is an augmented G -graded R -module. ■

The following result is the main result in this section. It sums up both Corollary 4.9 and Proposition 4.10 in the statement “Over augmented graded rings, flexible modules and augmented modules are the same”. The proof is a quick application of Corollary 4.9 and Proposition 4.10.

Theorem 4.11 *Let R be an augmented G -graded ring, and M be a G -graded R -module such that M_e is a G -graded R_e -module. Then M is flexible R -module iff M is an augmented G -graded R -module, and $\text{supp}(R, G) = \text{supp}(M, G)$.*

5. Flexibly simple and flexibly Noetherian modules

In this section, We define flexibly simple and flexibly Noetherian modules, and give an analogous study to that exhibited in [8]. We start by noticing that the combination of Propositions 2.2 and 2.5 yields the fact which says “given a G -graded ring R , and a G -graded R -module M , then a G -graded submodule N of M is flexible iff there exists an R_e -submodule N_e of M_e such that $N = RN_e$ ”. So, regardless M is flexible or not, we deduce that RM_e is the largest flexible R -submodule of M , and 0 is the smallest flexible R -submodule of M .

Now, we move to our definition of flexibly simple modules. To avoid any confusion, it is important to draw the reader’s attention to the fact that Definition 5.1 is a generalization of the definition of flexible simple modules [14], which is only defined for flexible modules.

Definition 5.1 Let R be a G -graded ring, and M be a G -graded R -module. Then M is called flexibly simple if 0 and RM_e are the only flexible R -submodules of M .

The structure of flexible modules indicates a strong relationship between flexibly simple R -modules and simple R_e -modules, as the following proposition illustrates.

Proposition 5.2 *Let R be a G -graded ring, and M be a G -graded R -module. Then M is flexibly simple iff M_e is a simple R_e -module.*

Proof. Suppose M is a flexibly simple R -module. Let $X \neq 0$ be an R_e -submodule of M_e . By Proposition 2.5, RX is a nonzero flexible R -submodule of M . Therefore, $RX = RM_e$, and hence $X = M_e$. Consequently, M_e is a simple R_e -module.

Conversely, suppose M_e is a simple R_e -module. Let $N \neq 0$ be a flexible R -submodule of M . We have $N_e \neq 0$ is an R_e -submodule of M_e . Thus, $N_e = M_e$, which implies that $N = RM_e$. ■

In the following corollary, the definition of gr -simple modules (i.e., graded simple modules) can be found in [2].

Corollary 5.3 *Let R be a first strongly G -graded ring, and M be a G -graded R -module such that $e \in \text{supp}(M, G)$. Then the following statements are equivalent:*

1. RM_e is gr -simple R -module.
2. M_e is simple R_e -module.
3. M is flexibly simple R -module.

Proof. The equivalence of (1) and (3) follows from Proposition 5.2. The implication “(1) $\Rightarrow$ (2)” is a well known result of Graded Module Theory [2]. For “(2) $\Rightarrow$ (1)”, apply Propositions 3.6 and 5.2. ■

In the following result, we show that nontrivial flexibly simple R -modules are only valid over non-strongly graded rings. In fact, this is one reason that urges mathematicians to seek well-structured graded rings, rather than strongly graded rings, over which modules obtain nice properties. Examples of such structures are first and second strongly graded rings, as well as augmented graded rings.

Proposition 5.4 *Suppose R is a strongly G -graded ring, and M is non-cyclic G -graded R -module. Then M is not flexibly simple. In particular, if M is flexibly simple R -module, then R is not strong.*

Proof. Since R is strong, M is flexible and $M_e \neq 0$. Pick $0 \neq x \in M_e$. Set $N = Rx$. We have $0 \neq N \neq M$ is a G -graded R -submodule of M . Moreover, N is flexible because if $g \in G$, $N_g = R_g x = (R_g R_e) x = R_g (R_e x) = R_g N_e$. ■

Next we study flexibly Noetherian R -modules. Again, Definition 5.5 generalizes the definition of flexible Noetherian modules [14], which is defined only for flexible modules.

Definition 5.5 Let R be a G -graded ring, and M be a G -graded R -module. Then M is said to be flexibly Noetherian if every ascending chain of flexible R -submodules of M terminates.

An immediate consequence of Definition 5.5 is that every flexibly simple R -module is flexibly Noetherian.

Proposition 5.6 Let R be a G -graded ring, and M be a G -graded R -module. Then M is a flexibly Noetherian R -module iff M_e is a Noetherian R_e -module.

Proof. Suppose M is a flexibly Noetherian R -module. Let $X^{(1)} \subseteq X^{(2)} \subseteq \dots$ be an ascending chain of R_e -submodules of M_e . Then by Proposition 2.5, $RX^{(1)} \subseteq RX^{(2)} \subseteq \dots$ is the corresponding ascending chain of flexible R -submodules of M . By assumption, there exist $n \in \mathbb{N}$ such that $RX^{(n)} = RX^{(n+1)} = \dots$. Hence, we obtain $X^{(n)} = X^{(n+1)} = \dots$. Therefore, M_e is a Noetherian R_e -module.

For the converse, assume M_e is a Noetherian R_e -module.

Let $X^{(1)} \subseteq X^{(2)} \subseteq \dots$ be an ascending chain of flexible R -submodules of M . We have $X_e^{(1)} \subseteq X_e^{(2)} \subseteq \dots$ is an ascending chain of R_e -submodules of M_e . By assumption, there exists $n \in \mathbb{N}$ such that $X_e^{(n)} = X_e^{(n+1)} = \dots$. Hence,

$$RX_e^{(n)} = RX_e^{(n+1)} = \dots, \quad \text{or} \quad X^{(n)} = X^{(n+1)} = \dots$$

That is, M is flexibly Noetherian. ■

The following corollaries generalize Corollary 4.2.6 and Remark 4.2.10 of [14], respectively. The proof of the first one follows obviously from Corollary 5.3.

Corollary 5.7 Let R be a first strongly G -graded ring, and M be a flexible G -graded R -module. Then the following statements are equivalent:

1. M is gr -simple (resp. gr -Noetherian) R -module.
2. M_e is simple (resp. Noetherian) R_e -module.
3. M is flexibly simple (resp. flexibly Noetherian) R -module. □

Corollary 5.8 Let R be a commutative G -graded ring, and M be a flexible G -graded R -module. Then the following statements are equivalent:

1. M is simple (resp. Noetherian) R -module.
2. M is gr -simple (resp. gr -Noetherian) R -module.
3. M_e is simple (resp. Noetherian) R_e -module.
4. M is flexibly simple (resp. flexibly Noetherian) R -module.

Proof. Apply Theorem 3.8 (resp. Proposition 5.6) and Proposition 5.2. ■

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Accepted: 4.11.2012

ON $(N(k), \xi)$ -SEMI-RIEMANNIAN 3-MANIFOLDS**D.G. Prakash**

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Abstract. The object of the present paper is to study 3-dimensional $(N(k), \xi)$ -semi-Riemannian manifolds. We study $(N(k), \xi)$ -semi-Riemannian 3-manifolds which are Ricci-semi-symmetric, locally ϕ -symmetric and have η -parallel Ricci tensor.

Key words and phrases: $(N(k), \xi)$ -semi-Riemannian 3-manifold, Ricci-semi-symmetric, locally ϕ -symmetric, η -parallel Ricci tensor, η -Einstein manifold.

MSC(2000): 53C25, 53C50.

1. Introduction

Let (M, g) be an n -dimensional semi-Riemannian manifold [12] equipped with a semi-Riemannian metric g . If $\text{index}(g)=1$ then g is a Lorentzian metric and (M, g) a Lorentzian manifold [4]. If g is positive definite then g is an usual Riemannian metric and (M, g) a Riemannian manifold. The notion of $(N(k), \xi)$ -semi-Riemannian structure was introduced and studied by Tripathi and Gupta [21] to unify $N(k)$ -contact metric [3], Sasakian [5], [14], (ϵ) -Sasakian [17], [22], Kenmotsu [10], para-Sasakian [15], (ϵ) -para-Sasakian structures [20].

In this paper we study 3-dimensional $(N(k), \xi)$ -semi-Riemannian manifolds. The paper is organized as follows. Section 2 is devoted to some basic definitions and properties of almost contact metric, almost para contact metric and $(N(k), \xi)$ -semi-Riemannian manifolds. Further, we prove that an $(N(k), \xi)$ -semi-Riemannian 3-manifold is a space form if and only if the scalar curvature r of the manifold is equal to $6k$. In Section 3, we show that a Ricci-semi-symmetric $(N(k), \xi)$ -semi-Riemannian 3-manifold is a space-form. In Section 4, a necessary and sufficient condition for an $(N(k), \xi)$ -semi-Riemannian 3-manifold to be locally ϕ -symmetric is obtained. Section 5 contains some results on $(N(k), \xi)$ -semi-Riemannian 3-manifold with η -parallel Ricci tensor.

2. Preliminaries

Let M be an n -dimensional differentiable manifold endowed with an almost contact structure (ϕ, ξ, η) , where ϕ is a $(1, 1)$ -tensor field, ξ is a vector field and η is a 1-form on M satisfying

$$(2.1) \quad \eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi,$$

where I denotes the identity transformation. It follows from (2.1) that

$$(2.2) \quad \eta \cdot \phi = 0, \quad \phi(\xi) = 0.$$

If there exists a semi-Riemannian metric g satisfying

$$(2.3) \quad g(\phi X, \phi Y) = g(X, Y) - \epsilon \eta(X) \eta(Y), \quad \forall X, Y \in \chi(M),$$

where $\epsilon = \pm 1$, then the structure (ϕ, ξ, η, g) is called an (ϵ) -almost contact metric structure and M is called an (ϵ) -almost contact metric manifold. For an (ϵ) -almost contact metric manifold, we have

$$(2.4) \quad \eta(X) = \epsilon g(X, \xi) \text{ and } \epsilon = g(\xi, \xi) \quad \forall X \in \chi(M).$$

When $\epsilon = 1$ and index of g is 0 then M is the usual Sasakian manifold and M is a Lorentz-Sasakian manifold for $\epsilon = -1$ and index of g is 1.

If $d\eta(X, Y) = g(\phi X, Y)$, then M is said to have (ϵ) -contact metric structure (ϕ, ξ, η, g) . For $\epsilon = 1$ and g Riemannian, M is the usual contact metric manifold [5]. A contact metric manifold with $\xi \in N(k)$, is called a $N(k)$ -contact metric manifold [1, 6]. If moreover, this structure is normal, that is, if

$$(2.5) \quad [\phi X, \phi Y] + \phi^2[X, Y] - \phi[X, \phi Y] - \phi[\phi X, Y] = -2d\eta(X, Y)\xi,$$

then the (ϵ) -contact metric structure is called an (ϵ) -Sasakian structure and the manifold endowed with this structure is called (ϵ) -Sasakian manifold. The physical importance of indefinite Sasakian manifolds have been pointed out by Duggal in [9].

An (ϵ) -almost contact metric structure (ϕ, ξ, η, g) is (ϵ) -Sasakian if and only if

$$(2.6) \quad (\nabla_X \phi)Y = g(X, Y)\xi - \epsilon\eta(Y)X, \quad \forall X, Y \in \chi(M),$$

where ∇ is the Levi-Civita connection with respect to g . Also we have

$$(2.7) \quad \nabla_X \xi = -\epsilon\phi X \quad \forall X \in \chi(M).$$

An almost contact metric manifold is a Kenmotsu manifold [10] if

$$(2.8) \quad (\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X.$$

By (2.8), we have

$$\nabla_X \xi = X - \eta(X)\xi.$$

If in (2.1), the condition $\phi^2 = -I + \eta \otimes \xi$ is replaced by

$$(2.9) \quad \phi^2 = I - \eta \otimes \xi$$

then (M, g) is called an (ϵ) -almost paracontact metric manifold equipped with an (ϵ) -almost paracontact metric structure (ϕ, ξ, η, g) .

An (ϵ) -almost paracontact metric structure is called (ϵ) -para-Sasakian structure [20] if

$$(2.10) \quad (\nabla_X \phi)Y = -g(\phi X, \phi Y)\xi - \epsilon\eta(Y)\phi^2 X,$$

where ∇ is Levi-Civita connection with respect to the metric g . A manifold endowed with an (ϵ) -para-sasakian structure is called (ϵ) -para-Sasakian manifold [20]. For $\epsilon = 1$ and g Riemannian, M is the usual para-Sasakian manifold [15].

$(N(k), \xi)$ -semi-Riemannian manifold

The k -nullity distribution [18] of (M, g) is the distribution

$$(2.11) \quad N(k) : p \rightarrow N_p(k) = \{Z \in T_p M : R(X, Y)Z = k(g(Y, Z)X - g(X, Z)Y)\},$$

where k is a real number.

An $(N(k), \xi)$ -semi-Riemannian manifold consists of a semi-Riemannian manifold (M, g) , a k -nullity distribution $N(k)$ on (M, g) and a non-null unit vector field ξ in (M, g) belonging to $N(k)$. Throught the paper we assume that $X, Y, Z, U, V, W \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields in M , unless specifically stated otherwise. Let ξ be a non null unit vector field in (M, g) and η its associated 1-form. Thus

$$g(\xi, \xi) = \epsilon,$$

where $\epsilon = 1$ or -1 according as ξ is spacelike or timelike, and

$$(2.12) \quad a) g(X, \xi) = \epsilon\eta(X), \quad b) \eta(\xi) = 1.$$

In an n -dimensional $(N(k), \xi)$ -semi-Riemannian manifold (M, g) , the following relations hold [21]:

$$(2.13) \quad R(X, Y)\xi = \epsilon k\{\eta(Y)X - \eta(X)Y\},$$

$$(2.14) \quad R(\xi, X)Y = \epsilon k\{\epsilon g(X, Y)\xi - \eta(Y)X\},$$

$$(2.15) \quad \eta(R(X, Y)Z) = k\{\eta(X)g(Y, Z) - \eta(Y)g(X, Z)\},$$

$$(2.16) \quad S(X, \xi) = \epsilon k(n - 1)\eta(X),$$

In a 3-dimensional Riemannian manifold we have

$$(2.17) \quad R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y \\ - \frac{r}{2} [g(Y, Z)X - g(X, Z)Y],$$

where Q is the Ricci operator, i.e., $g(QX, Y) = S(X, Y)$ and r is the scalar curvature of the manifold. Putting $Z = \xi$ in (2.17) and using (2.13) and (2.16), we have

$$(2.18) \quad \epsilon(\eta(Y)QX - \eta(X)QY) = \left(-\epsilon k + \frac{r}{2}\epsilon\right)(\eta(Y)X - \eta(X)Y).$$

Putting $Y = \xi$ in (2.18) and then using (2.12(b)) and (2.16) (for $n=3$), we get

$$(2.19) \quad QX = \frac{1}{2}\{(r - 2k)X - (r - 6k)\eta(X)\xi\},$$

that is,

$$(2.20) \quad S(X, Y) = \frac{1}{2}\{(r - 2k)g(X, Y) - \epsilon(r - 6k)\eta(X)\eta(Y)\}.$$

An $(N(k), \xi)$ -semi-Riemannian manifold M is said to be η -Einstein if its Ricci tensor S is of the form

$$(2.21) \quad S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$$

for any vector fields X, Y where a, b are functions on M . Hence from (2.20) we can state the following:

Lemma 1 *A 3-dimensional $(N(k), \xi)$ -semi-Riemannian manifold is an η -Einstein manifold.*

By using (2.19) and (2.20) in (2.17), we obtain

$$(2.22) \quad R(X, Y)Z = \left(\frac{r}{2} - 2k\right)\{g(Y, Z)X - g(X, Z)Y\} \\ - \left(\frac{r}{2} - 3k\right)\{g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi \\ + \epsilon\eta(Y)\eta(Z)X - \epsilon\eta(X)\eta(Z)Y\}.$$

An $(N(k), \xi)$ -semi-Riemannian 3-manifold is a space of constant curvature then it is an indefinite space form.

Remark. Relations (2.19), (2.20) and (2.22) are true for

1. An $N(k)$ -contact metric 3-manifold [8] if $\epsilon = 1$,
2. A Sasakian 3-manifold if $k = 1$ and $\epsilon = 1$,
3. A Kenmotsu 3-manifold [7] if $k = -1$ and $\epsilon = 1$,
4. An (ϵ) -Sasakian 3-manifold if $k = 1$ and $\epsilon k = 1$,
5. A para-Sasakian 3-manifold [2] if $k = -1$ and $\epsilon = 1$,
6. An (ϵ) -para-Sasakian 3-manifold [19] if $k = -\epsilon$ and $\epsilon k = -1$.

Lemma 2 *A 3-dimensional $(N(k), \xi)$ -semi-Riemannian manifold is a space form if and only if the scalar curvature $r = 6k$.*

Consequently, for a 3-dimensional $(N(k), \xi)$ -semi-Riemannian manifold, we have the following table:

M	S =	r =
<i>$N(k)$-contact metric</i>	$\frac{1}{2}\{(r - 2k)g - (r - 6k)\eta \otimes \eta\}$	$6k$
<i>Sasakian</i>	$\frac{1}{2}\{(r - 2)g - (r - 6)\eta \otimes \eta\}$	6
<i>Kenmotsu</i>	$\frac{1}{2}\{(r + 2)g - (r + 6)\eta \otimes \eta\}$	-6
<i>(ϵ)-Sasakian</i>	$\frac{1}{2}\{(r - 2\epsilon)g - \epsilon(r - 6\epsilon)\eta \otimes \eta\}$	6ϵ
<i>para-Sasakian</i>	$\frac{1}{2}\{(r + 2)g - (r + 6)\eta \otimes \eta\}$	-6
<i>(ϵ)-para Sasakian</i>	$\frac{1}{2}\{(r + 2\epsilon)g - \epsilon(r + 6\epsilon)\eta \otimes \eta\}$	-6ϵ

Proof. Let a 3-dimensional $(N(k), \xi)$ -semi-Riemannian manifold be an indefinite space form. Then

$$(2.23) \quad R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\}, \quad X, Y, Z \in \chi(M),$$

where c is the constant curvature of the manifold. By using the definition of Ricci curvature and (2.23) we have

$$(2.24) \quad S(X, Y) = 2cg(X, Y).$$

If we use (2.24) in the definition of the scalar curvature we get

$$(2.25) \quad r = 6c.$$

From (2.24) and (2.25) one can easily see that

$$(2.26) \quad S(X, Y) = \frac{r}{3}g(X, Y).$$

By plugging $X = Y = \xi$ in (2.20) and using (2.26) we obtain

$$(2.27) \quad r = 6k.$$

Conversely, if $r = 6k$, then from the equation (2.22) we can easily see that the manifold is a space form. This completes the proof. ■

3. Ricci-semi-symmetric $(N(k), \xi)$ -semi-Riemannian 3-manifolds

A semi-Riemannian manifold M is said to be Ricci semi-symmetric [13] if its Ricci tensor S satisfies the condition

$$(3.28) \quad R(X, Y) \cdot S = 0, \quad X, Y \in \chi(M),$$

where $R(X, Y)$ acts as a derivation on S . Ricci-semisymmetric manifold is a generalization of manifold of constant curvature, Einstein manifold, Ricci symmetric manifold, symmetric manifold and semisymmetric manifold. Ricci-semisymmetric condition for Kenmotsu 3-manifolds, (ϵ) -para-Sasakian 3-manifolds and LP-Sasakian 3-manifolds are studied in [7], [19] and [16] respectively.

Let M be a Ricci-semi-symmetric $(N(k), \xi)$ -semi-Riemannian 3-manifold. From (3.28) we have

$$(3.29) \quad S(R(X, Y)U, V) + S(U, R(X, Y)V) = 0.$$

If we put $X = \xi$ in (3.29) and use (2.14), then we get

$$(3.30) \quad kg(Y, U)S(\xi, V) - \epsilon K\eta(U)S(Y, V) + kg(Y, V)S(U, \xi) - \epsilon K\eta(V)S(U, Y) = 0.$$

By using (2.16) in (3.30) we obtain

$$(3.31) \quad \epsilon K\{2kg(Y, U)\eta(V) - \eta(U)S(Y, V) - 2kg(Y, V)\eta(U) - \eta(V)S(U, Y)\} = 0.$$

Consider that $\{e_1, e_2, e_3\}$ be an orthonormal basis of the $T_p M$, $p \in M$. Then, by putting $X = U = e_i$ in (2.2) and taking the summation for $1 \leq i \leq 3$, we have

$$(3.32) \quad \epsilon k\{8k\eta(V) - \epsilon S(V, \xi) - r\eta(V)\} = 0.$$

Again, by using (2.16) in (3.32), we get

$$(3.33) \quad \epsilon k(6k - r)\eta(V) = 0,$$

which gives $r = 6k$. This implies, in view of Lemma 2, that the manifold is a space form.

Therefore, we have the following:

Theorem 1 *A Ricci-semi-symmetric $(N(k), \xi)$ -semi-Riemannian 3-manifold is a space form.*

From Theorem 1 and the above table, we can state the following corollaries:

Corollary 1 *A Ricci-semi-symmetric $N(k)$ -contact metric 3-manifold is a manifold of constant scalar curvature $6k$.*

Corollary 2 *A Ricci-semi-symmetric Sasakian 3-manifold is a manifold of constant positive scalar curvature 6.*

Corollary 3 [7] *A Ricci-semi-symmetric Kenmotsu 3-manifold is a manifold of constant negative scalar curvature -6 .*

Corollary 4 *A Ricci-semi-symmetric (ϵ) -Sasakian 3-manifold is an indefinite space form.*

Corollary 5 [2] *A Ricci-semi-symmetric para-Sasakian 3-manifold is a manifold of constant negative scalar curvature -6 .*

Corollary 6 [19] *A Ricci-semi-symmetric (ϵ) -para-Sasakian 3-manifold is an indefinite space form.*

4. Locally ϕ -symmetric $(N(k), \xi)$ -semi-Riemannian 3-manifolds

Definition 1 An $(N(k), \xi)$ -semi-Riemannian manifold is said to be locally ϕ -symmetric if

$$\phi^2(\nabla_W R)(X, Y, Z) = 0,$$

for all vector fields W, X, Y, Z orthogonal to ξ . This notion was introduced for Sasakian manifolds by Takahashi [17].

Now, differentiating (2.22) covariantly with respect to W , we get

$$\begin{aligned} (\nabla_W R)(X, Y, Z) = & \frac{1}{2}(\nabla_W r)\{g(Y, Z)X - g(X, Z)Y - g(Y, Z)\eta(X)\xi \\ & + g(X, Z)\eta(Y)\xi - \epsilon\eta(Y)\eta(Z)X + \epsilon\eta(X)\eta(Z)Y\} \\ & - \frac{(r - 6k)}{2}\{g(Y, Z)((\nabla_W \eta)(X)\xi + \eta(X)\nabla_W \xi) \\ & - g(X, Z)((\nabla_W \eta)(Y)\xi + \eta(Y)\nabla_W \xi) \\ & + \epsilon((\nabla_W \eta)(Y)\eta(Z)X + (\nabla_W \eta)(Z)\eta(Y)X) \\ & - \epsilon((\nabla_W \eta)(X)\eta(Z)Y + (\nabla_W \eta)(Z)\eta(X)Y)\}. \end{aligned}$$

Taking W, X, Y, Z orthogonal to ξ , we have

$$\begin{aligned} (\nabla_W R)(X, Y, Z) = & \frac{1}{2}(\nabla_W r)\{g(Y, Z)X - g(X, Z)Y\} \\ (4.34) \quad & - \frac{(r - 6k)}{2}\{g(Y, Z)(\nabla_W \eta)(X)\xi - g(X, Z)(\nabla_W \eta)(Y)\xi\}. \end{aligned}$$

Applying ϕ^2 on both sides of the above equation and using $\phi \cdot \xi = 0$, we have

$$(4.35) \quad \phi^2((\nabla_W R)(X, Y, Z)) = \frac{1}{2}(\nabla_W r)\{g(Y, Z)\phi^2 X - g(X, Z)\phi^2 Y\}.$$

Now taking X, Y are orthogonal to ξ , we obtain

$$(4.36) \quad \phi^2((\nabla_W R)(X, Y, Z)) = -\frac{1}{2}(\nabla_W r)\{g(Y, Z)X - g(X, Z)Y\}$$

Hence from (4.36), we can state the following:

Theorem 2 *An $(N(k), \xi)$ -semi-Riemannian 3-manifold is locally ϕ -symmetric if and only if the scalar curvature r is constant.*

If an $(N(k), \xi)$ -semi-Riemannian 3-manifold is Ricci semi-symmetric, then we have showed that $r = 6k$, that is r is constant.

Therefore, from Theorem (2), we have

Theorem 3 *A Ricci-semi-symmetric $(N(k), \xi)$ -semi-Riemannian 3-manifold is locally ϕ -symmetric.*

5. $(N(k), \xi)$ -semi-Riemannian 3-manifold with η -parallel Ricci tensor

Definition 2 The Ricci tensor S of an $(N(k), \xi)$ -semi-Riemannian manifold M is called η -parallel if it satisfies

$$(5.37) \quad (\nabla_Z S)(\phi X, \phi Y) = 0$$

for all vector fields X, Y and Z . The notion of Ricci- η -parallelity for Sasakian manifolds was introduced by Kon in [11].

Now, let us consider a 3-dimensional $(N(k), \xi)$ -semi-Riemannian manifold with η -parallel Ricci tensor. Then, from (2.20), we get

$$(5.38) \quad S(\phi X, \phi Y) = \frac{1}{2}(r - 2k)[g(\phi X, \phi Y)].$$

Differentiating (5.38) covariantly along Z , we have

$$(5.39) \quad (\nabla_Z S)(\phi X, \phi Y) = \frac{1}{2}dr(Z)g(\phi X, \phi Y).$$

If the Ricci tensor is η -parallel, then from (5.37) and (5.39) one can get

$$\frac{1}{2}dr(Z)g(\phi X, \phi Y) = 0.$$

From which, it follows that

$$dr(Z) = 0,$$

for all Z . This leads us to the following:

Theorem 4 *Let M be an $(N(k), \xi)$ -semi-Riemannian 3-manifold with η -parallel Ricci tensor. The the scalar curvature r is constant.*

In view of Theorem (2) and Theorem (4), we have the following:

Theorem 5 *An $(N(k), \xi)$ -semi-Riemannian 3-manifold with η -parallel Ricci tensor is locally ϕ -symmetric.*

Acknowledgement. The first author (DGP) is thankful to University Grants Commission, New Delhi, India for financial support in the form of Major Research Project [F.No. 39-30/2010 (SR), dated: 23-12-2012].

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Accepted: 12.02.2013

A NEW SIGNING ALGORITHM BASED ON ELLIPTIC CURVE DISCRETE LOGARITHMS AND QUADRATIC RESIDUE PROBLEMS

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Abstract. In this paper we propose a new digital signature algorithm for authenticity and integrity of a digital message. The core idea behind our approach is concentrated on using two hard problems in the signing process. The elliptic curve discrete logarithm and quadratic residue are engaged in a sophisticated manner to do the signing. The new proposed scheme provides higher level of security than other techniques that use a single hard problem. Clearly, Cybercriminals have to solve the two underlying hard problems simultaneously to destroy embedded signature. Extensive experimental results on several signed documents are performed to demonstrate the robustness of the proposed scheme against the most common attacks on digital signatures. Moreover, the computational complexity of the new scheme requires reasonable number of operations in both signing and verifying algorithms. **Keywords:** Cryptography, Digital Signature, Quadratic Residue, Elliptic Curve Discrete Logarithms, heuristically secure.

1. Introduction

In modern cryptography, the security and robustness of the digital signature algorithms are based on the difficulty of solving some hard theoretical problems such as factoring and discrete logarithm [2], [3], [14]. In the literature, various digital signature algorithms that use two theoretical hard problems [4], [5], [15]

have been proposed. One common feature of these algorithms is that they are depending on a number-theoretical problem and thus their implementations depend heavily on modular exponentiation which is known to be time consuming with high computational complexity.

Elliptic curves are used in cryptography by Koblitz [7] and Miller [10] to overcome the costly modular exponentiation. The cryptographic schemes with security lie on the so-called elliptic curve discrete logarithm (ECDLP) [7], [10] are proved to provide longer security and better efficiency than both integer factorization system and discrete logarithm systems. The ECDLP has become a turning point of the rigorous development of cryptographic schemes [1], [9], [6], [13], [11], [12].

Motivated by the need for secure digital signature scheme, we propose a robust signature approach using the ECDLP and quadratic residue problem (QRP) hard problems. The new scheme offers better security than all other schemes based on either ECDLP or QRP. This is because the probability of solving two hard problems simultaneously by adversaries is believed to be negligible. Moreover, the proposed scheme does not involve any modular exponentiation operation in all algorithms. The remainder of this paper is organized as follows. In Section 2, we briefly review some background material and describe the ECDLP and QRP hard problems. In Section 3 we introduce the proposed approach and describe in detail the key generation, the signing and the verification algorithms. In Sections 4 and 5, we present some security analysis and the performance evaluation of the proposed approach. Finally, we conclude and point out future directions in Section 5.

1.1. Elliptic curve

In this section we describe some elementary tools on elliptic curves and define the two underlying hard problems ECDLP and QRP.

Definition 1.1 Let K be a field of characteristic neither of 2 nor 3, then an elliptic curve can be expressed as:

$$(1.1) \quad y^2 = x^3 + ax + b$$

where $a, b \in K$ and $4a^3 + 27b^2 \neq 0$. The set $E(K)$ consists of all point (x, y) , $x, y \in K$ which satisfy the defining equation (1.1) together with a special point $\mathcal{O}$ called the point at infinity. Let G be a point on the elliptic curve defined in equation (1.1) and if n is the smallest positive integer satisfying the equation $nG = \mathcal{O}$, then we say that G has an order n and is called the base point. See [6], [7], [8], [10] for complete discussion on how to add and to multiply elliptic curves points.

- ECDLP: Let G and C be two elliptic curve points on equation (1.1). Then find a positive integer k such that $kG = C$.
- QRP: Let p, q are two strong primes of large size and γ is an integer. Then compute an integer γ such that $\gamma \equiv \beta^2 \pmod{pq}$

2. The new signature scheme

The proposed signature scheme consists of four main phases: initialization, algorithm for key generation, algorithm for signing messages, and algorithm for signature verification.

2.1. Initialization

- The field $K = F_p$ of order p , where p is be a large prime number and $p - 1$ have two prime factors $\bar{p}$ and $\bar{q}$.
- Two coefficient $a, b \in F_p$ that define the equation $y^2 = x^3 + ax + b(\text{mod } p)$ over F_p .
- $n = \bar{p}\bar{q}$, so that $n/(p - 1)$ is a root points of elliptic curve construct a circulating subgroup. G is a generating element for subgroup and its rank equals n .
- $h(.)$ is a secure hash function.

2.2. Algorithm for generating keys

Step 1 Pick randomly two integers α and β from $\mathbb{Z}_n$

Step 2 Compute an integer k such that $k\alpha^{4\beta} \equiv -1(\text{mod } n)$

Step 3 Calculate $T = kG$

The signer publishes his public keys as (p, G, n, T) and his corresponding private keys as (α, β, k) .

2.3. Algorithm for signing message

Suppose the verifier wants the signer's signature on his message m . The signer then:

Step 1 Select an integer $r \in \mathbb{Z}_n$ such that $\gcd(r, n) = 1$

Step 2 Compute $R = r\beta^2G = (x_1, x_2)$, where $u \equiv x_1(\text{mod } n)$

Step 3 Calculate $s \equiv \frac{1}{2}(r + h(m)u^2\beta^2)(\text{mod } n)$

Step 4 Calculate $v \equiv \frac{1}{2}\alpha^{2\beta}(r - h(m)u^2\beta^2)(\text{mod } n)$

The original signer then produces (R, s, v) as a signature of message M .

2.4. Algorithm for verifying signature

Verifier tests the validity of the signature (R, s, v) by checking the following steps

Step 1 Compute $W_1 = s^2G + V^2T \pmod{n}$

Step 2 Compute $W_2 = h(m)u^2R \pmod{n}$

Accept the signature as valid if and only if $W_1 = W_2$

Theorem 2.1 *If the algorithms for keys and signing message are run smoothly then the validation of signature is correct.*

Proof. Squaring s and v

$$\begin{aligned} s^2 &\equiv \frac{1}{4}(r^2 + h(m)^2u^4\beta^4 + 2rh(m)u^2\beta^2) \pmod{n} \\ v^2 &\equiv \frac{1}{4}\alpha^{4\beta}(r^2 + h(m)^2u^4\beta^4 - 2rh(m)u^2\beta^2) \pmod{n} \end{aligned}$$

we have

$$\begin{aligned} s^2G &= \frac{1}{4}(r^2 + h(m)^2u^4\beta^4 + 2rh(m)u^2\beta^2)G \pmod{n} \\ v^2T &= \frac{1}{4}\alpha^{4\beta}(r^2 + h(m)^2u^4\beta^4 - 2rh(m)u^2\beta^2)kG \pmod{n} \\ &= \frac{1}{4}k\alpha^{4\beta}(r^2 + h(m)^2u^4\beta^4 - 2rh(m)u^2\beta^2)G \pmod{n} \\ &= -\frac{1}{4}(r^2 + h(m)^2u^4\beta^4 - 2rh(m)u^2\beta^2)G \pmod{n} \end{aligned}$$

Thus

$$\begin{aligned} W_1 &= s^2G + v^2T \\ &= \frac{1}{4}(r^2 + h(m)^2u^4\beta^4 + 2rh(m)u^2\beta^2)G - \frac{1}{4}(r^2 + h(m)^2u^4\beta^4 - 2rh(m)u^2\beta^2)G \\ &= h(m)u^2r\beta^2G \\ &= h(m)u^2R \\ &= W_2 \end{aligned}$$

We accept the signature if $W_1 = W_2$. With the knowledge of the signer's public key (n, T) and the signature (R, s, v) of m , the verifier can authenticate the message m because the verifier can be convinced that the message was really signed by the signer. Else, the signature (R, s, v) is invalid.

3. Security analysis

One of the known models of security in cryptography is heuristic or ad-hoc model. We show that our scheme is heuristically secure by considering several possible attacks by an adversary (Adv). For every attack, we give valid reasons of why this attack fails.

Attack 1: Adv wishes to obtain secret keys (α, β, k) using all information available in the system. In this case, Adv needs to solve $T = kG \pmod{n}$ and $k\alpha^{4\beta} \equiv -1 \pmod{n}$ which are clearly infeasible due to the difficulty of solving the ECDLP and factoring problem.

Attack 2: Adv tries to derive the signature (R, s, v) for a given message m by fixing the value of two integers in order to find the remaining one. In this case, Adv randomly fixes either (R, s) or (R, v) or (s, v) to find s, v or R respectively to satisfy $W_1 = W_2$. Obviously, this is as difficult as solving QRP and ECDLP simultaneously. For example, say Adv fixes the value (R, s) and tries to figure out the value of v . Adv then needs to solve the following equations that can be reduced from

$$(2.1) \quad s^2G + v^2T = h(m)u^2R$$

Adv start by computing

$$\gamma = v^2T$$

where γ is known and can be calculated easily. Note that, solving the above equation is as hard as solving ECDLP. But, even if ECDLP is solvable then the above equation is reducible to the next equation as below:

$$(2.2) \quad \lambda \equiv kv^2 \pmod{n}$$

where λ is known but solving this equation is hard as solving the QRP.

Attack 3: Adv may also try collecting t valid signatures (R_j, s_j, v_j) on message M_j where $j = 1, 2, \dots, t$ and attempts to find secret keys of the signature scheme. In this case, Adv has s equations as follows:

$$\begin{aligned} s_1^2 + v_1^2k &= r_1h(M_1)u_1^2\beta^2 \\ s_2^2 + v_2^2k &= r_2h(M_2)u_2^2\beta^2 \\ &\vdots \\ s_t^2 + v_t^2k &= r_th(M_t)u_t^2\beta^2 \end{aligned}$$

In the above s equations, there are $(t + 2)$ variables that is k, β and r_j , where $j = 1, 2, \dots, t$ which are unknown by the Adv. Hence, k and β remain hard to detect because Adv can generate infinite solutions of the above system of equations but cannot figure out which one is correct.

Attack 4: It is assumed that Adv is able to solve ECDLP problem. In this case, Adv knows k and $r\beta^2 \equiv \xi \pmod{n}$ but cannot figure out the values of r and β because breaking of QRP is difficult. Now from $s^2G + v^2T = h(m)u^2R$, Adv will have

$$(2.3) \quad s^2 + v^2k \equiv rh(m)u^2\beta^2 \pmod{n}$$

Adv can launch Attack 2 but will not be successful due to the hardness of breaking QRP. Since Adv knows k , Adv may try to obtain α and β from equation

$$(2.4) \quad k\alpha^{4\beta} \equiv -1 \pmod{n}$$

but still fail although Adv does know the factorization of n .

Attack 5: It is assumed that Adv is able to solve FAC problem. That means, he knows the prime factorization $\bar{p}$ and $\bar{q}$. In this case, Adv will learn nothing about k, α and β from the equation (2.4). Adv also has no information about k and (r, β) from equation (2.3) because there are three unknowns in the equation.

4. Performance evaluation

In this section we investigate the efficiency and the performance of our proposed scheme in terms of number of keys, computational complexity and communication costs. The complete lists of the notations that we used to analyze the performance are shown in Table 1.

Table 1: Lists of the notations that we used to analyze the performance of the proposed scheme

Notation	Description
SK	Number of secret keys
PK	Number of public keys
T_{mul}	The time complexity for executing the modular multiplication
T_{add}	The time complexity for executing the modular addition
T_{exp}	The time complexity for executing the modular exponentiation
T_{ec-add}	The time complexity for executing the addition of two elliptic curve points
T_{ec-mul}	The time complexity for executing the multiplication on elliptic curve points
T_{sqr}	The time for modular square computation
T_r	The time complexity for selecting a random integer
T_h	The time complexity for performing a one-way hash function h

The performance of our new signature scheme is summarized as follows: The number of keys in this scheme is given by $PK = 4$, and $SK = 3$. The signer needs $T_{ec-mul} + 6T_{mul} + 6T_{sqr} + T_{exp} + T_r + T_h$ time complexity to create a signature on any message, m . The signature validation verifier needs $3T_{ec-mul} + 3T_{sqr} + T_{mul} + T_h$. Finally the communication costs and the parameters for signing message and verifying signature are respectively given by $3|n|$ and $2|n|$.

To describe the efficiency performance in terms of T_{mul} , we use the conversion proposed in [8]. It converts various operations units to the time complexity for executing the modular multiplication.

$$T_{exp} \approx 240T_{mul}; T_{ec-mul} \approx 29T_{mul}; T_{ec-add} \approx 0.12T_{mul}$$

Assuming the time complexity for T_h and T_r are negligible, we found that, our signing message requires $275T_{mul} + 6T_{sqr}$ time complexity and our verifying signature requires $88T_{mul} + 3T_{sqr}$ time complexity. The widely used RSA signature scheme needs $240T_{mul}$ in both signature message and in its verifying signature. clearly our new proposed scheme is more efficient than RSA and yet our scheme is based on two hard problems which offer a longer security than RSA.

5. Conclusion

In this paper, we presented a new computationally inexpensive digital signature scheme based on the ECDLP and QRP problems. The new scheme offers higher level of security than other algorithms that based on ECDLP or QRP. The performance of the proposed method was evaluated through extensive experiments that clearly showed a perfect resiliency against a wide range of attacks including the key-only attack, ECDLP and QRP attacks, the chosen-message and feed attacks. Furthermore, it requires only minimal and acceptable number of operations in both signing and verifying algorithm.

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Accepted: 09.03.2013

Motto:

*The science wouldn't be so good today,
if yesterday we hadn't thought about today.*

Grigore C. Moisil

ECCENTRICITY, SPACE BENDING, DIMENSION

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Abstract. The main goal of this paper is to present new transformations, previously non-existent in traditional mathematics, that we call centric mathematics (CM) but that became possible due to the new born eccentric mathematics, and, implicitly, to the supermathematics (SM).

As shown in this work, the new geometric transformations, namely conversion or transfiguration, wipe the boundaries between discrete and continuous geometric forms, showing that the first ones are also continuous, being just apparently discontinuous.

Abbreviations and annotations

C ▶ Circular and Centric,	E ▶ Eccentric and Eccentrics,
F ▶ Function,	M ▶ Mathematics,
CE ▶ Circular Eccentric,	F CE ▶ FCE,
CM ▶ Centric M,	EM ▶ Eccentric M,
SM ▶ Super M,	F CM ▶ FCM,
F EM ▶ FEM,	F SM ▶ FSM

Introduction: conversion or transfiguration

In *linguistics* a **word** is the fundamental unit to communicate a meaning. It can be composed by one or more *morphemes*. Usually, a word is composed by a basic part, named *root*, where one can attach *affixes*. To define some *concepts* and to express the domain where they are available, sometimes more words are needed: two, in our case. In this paper several new concepts are introduced and they are related to *SuperMathematics* (SM).

The principal new idea in this paper is that it introduces a new mathematical transformation with a large significance in the fields of Physics, previously non-existent in the original classical mathematics, named herein as centric mathematics (CM). They became possible thanks to this new mathematics called Eccentric Mathematics (EM) and to the Super Mathematics (SM), which are puts together with (CM) with (EM). The (CM) is now a particular case of a linear numeric eccentricity for $s = 1$ in (SM).

Supermathematical conversion

The concept is the easiest and methodical *idea* which reflects a finite of one or more series of attributes, where these attributes are *essentials*.

The concept is a minimal coherent and usable information, relative to an object, action, property or a defined event.

According to the Explicatory Dictionary, **the conversion** is, among many other definitions/meanings, defined as “changing the nature of an object”.

Next, we will talk about this thing, about transforming/changing/converting, previously impossible in the ordinary classic mathematics, now named also CENTRIC (CM), of some forms in others, and that became possible due to the new born mathematics, named ECCENTRIC (EM) and to the new built-in mathematical complements, named temporarily also SUPERMATHEMATICS (SM).

We talk about the *conversion* of a circle into a square, of a sphere into a cube, of a circle into a triangle, of a cone into a pyramid, of a cylinder into a prism, of a circular torus in section and shape into a square torus in section and/or form, etc. (Fig.1).

Supermathematical Conversion (SMC) is an internal pry for the mathematical dictionary enrichment, which consists in building-up of a new denomination, with one or more new terms (two in our case), by assimilating some words from the current language in a specialized domain, as Mathematics, with the intention to name and adequate the new operations that became possible only due to the new born *eccentric mathematics*, and implicitly to *supermathematics*. Because previously mentioned conversions could not be made until today in MC, but only in SM, we need to call them as SUPERMATHEMATICAL conversion (SMC).

In [14] the continuous transformation of a circle into a square was named also *eccentric transformation*, because, in that case, the linear numeric eccentricity s varies/grows from 0 to 1, being a slide from centric mathematics domain $MC \rightarrow s = 0$ to the eccentric mathematics, $ME(s \neq 0) \rightarrow s \in (0, 1]$, where the circular form draws away more and more from the circular form until reaching a perfect square ($s = \pm 1$).

Eccentric transformation

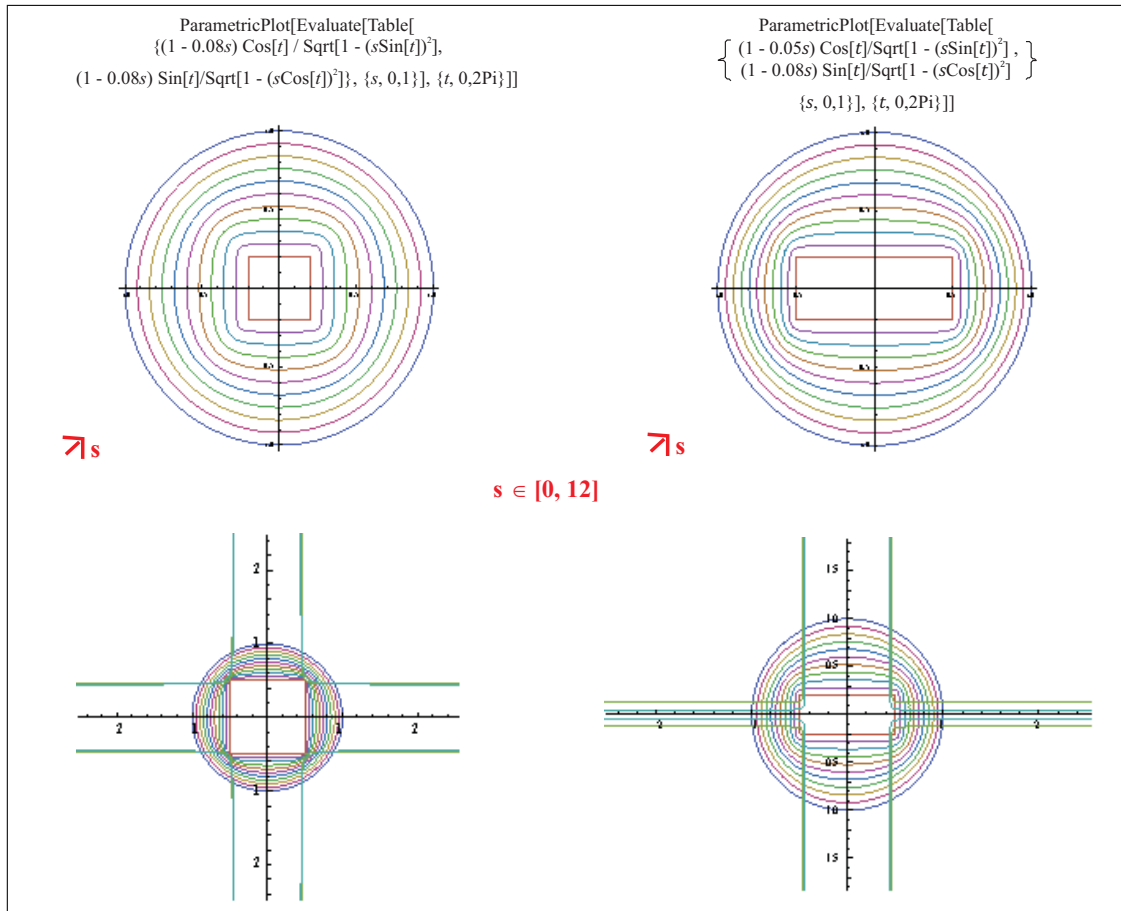


Figure 1: Conversion or transfiguration in 2D of a circle into a square and/or into a rectangle ►ECCENTRIC TRANSFORMATION

In the same work, the reverse transformation, of a square into a circle, was named as *centering transformation*. Same remarks are valid also for transforming a circle into a rectangle and a rectangle into a circle (Fig. 2). Most modern physicists and mathematicians consider that the numbers represent the reality's language. The truth is that the forms are those which generate all physical laws.

Centering transformation

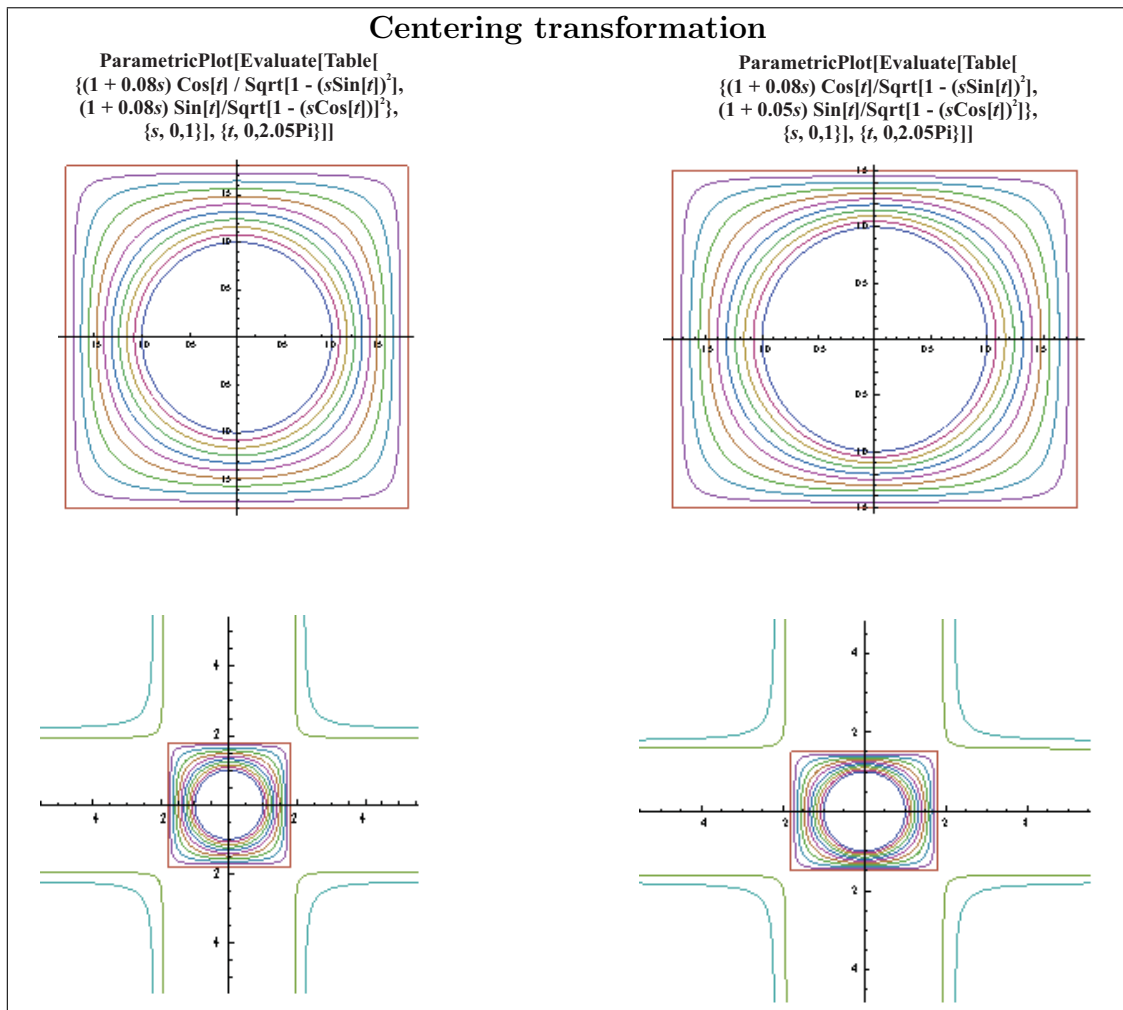


Figure 2: Conversion or transfiguration in 2D of a square and/or a rectangle into a circle ►CENTERING TRANSFORMATION

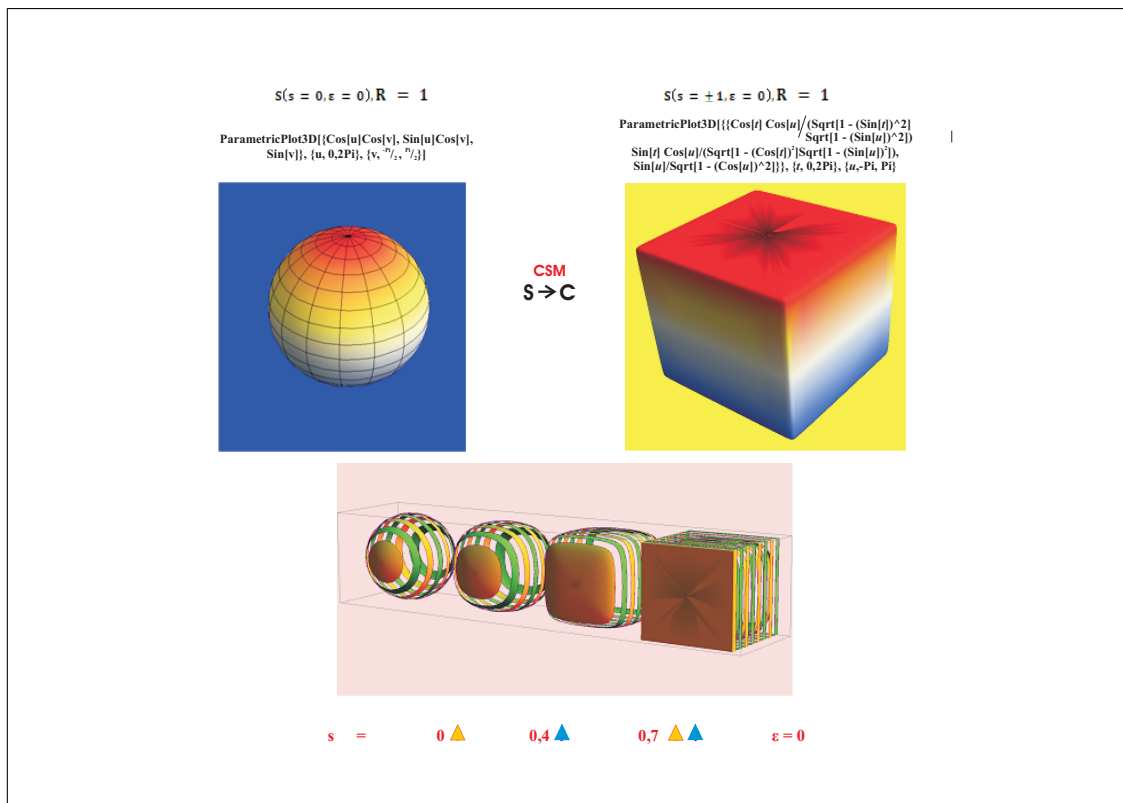


Figure 3: The conversion of a sphere into a cube

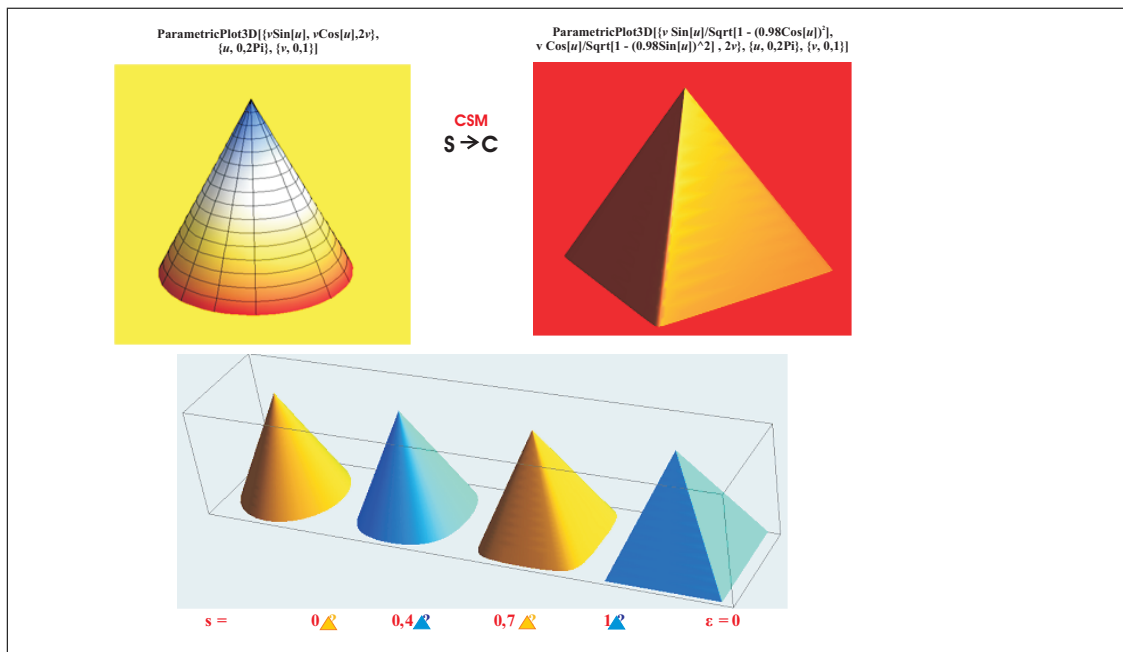


Figure 4: The Conversion of a cone into a pyramid

Look what the famous Romanian physicist Prof. Dr. Liviu Sofonea in *Representative Geometries and Physical Theories*, Ed. Dacia, Cluj-Napoca, p. 24, in 1984, in the chapter named *Mathematical geometry and physical geometry* wrote: In the *centric mathematical geometry* one does what can be done, how can be done, with what can be done, and in supermathematical geometry we can do what must be done, with what must be done, as we will proceed. In the *supermathematical geometry*, between the elements of the 'CM scaffold', one can introduce as many other constructive elements as we want, which will give an infinitely denser scaffold structure, much more durable and, consequently, higher, able to offer an unseen high level and an extremely deep description and gravity.

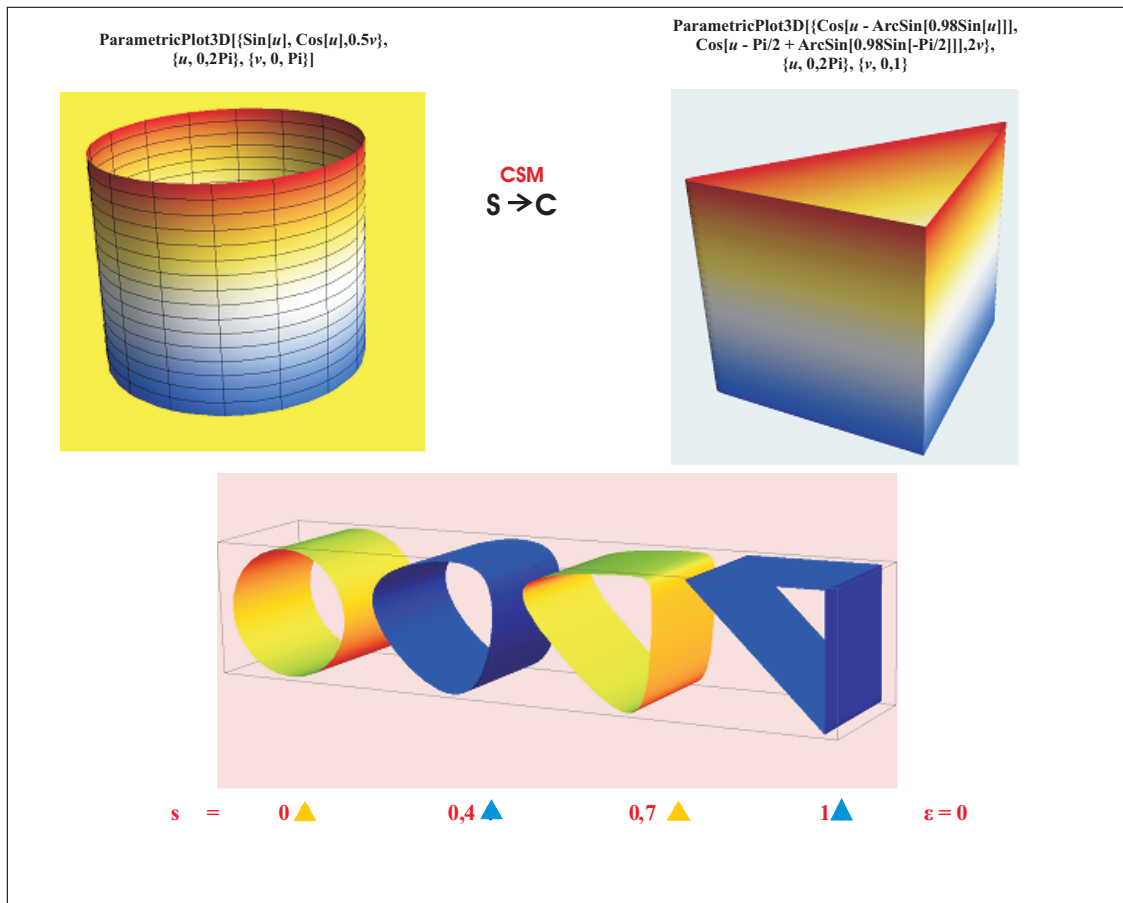


Figure 5: The Conversion or transfiguration of a cylinder into a prism



Figure 6: The conversion or transfiguration of the circular thorus into a square thorus, both in form and in section

The *fundamental principles* of the geometry are, according to their topological dimensions: the *corps* (3), the *line* (2), and the *point* (0).

The *elementary principles* of geometry are the point, the line, the space, the curve, the plane, the geometrical figures (such as the segment, triangle, square, rectangle, rhombus, the polygons, the polyhedrons, etc. the arcs, circle, ellipse, hyperbola, the scroll, the helix, etc.) both in 2D and in 3D spaces.

With the fundamental geometrical elements are defined and built all the forms and geometrical structures of the objects:

- *Discrete (discontinuous) statically forms*, or directly starting from a finite (discrete) set of points statically bonded with lines and planes.
- *Continuous (or dynamical, mechanical) forms*, starting from a single point and considering its motion, therefore the time, and obtaining in this way continuous forms of curves, as trajectories of points or curves, in the plane (2D) or in the space (3D)

Consequently, one has considered, and still is considering, the existence of two geometries: the *geometry of discontinuous*, or discrete geometry, and the *geometry of the continuum*.

As both objects limited by plane surfaces (cube, pyramid, prism), *apparently discontinuous*, as those limited by different kinds of *continuous surfaces* (sphere, cone, cylinder) can be described with the same parametric equations, the first ones for numerical eccentricity $s = \pm 1$ and the last ones for $s = 0$, it results that in SM there exists only one geometry, *the geometry of the continuum*.

In other words, the SM erases the boundaries between continuous and discontinuous, as SM erased the boundaries between linear and nonlinear, between centric and eccentric, between ideal/perfection and real, between circular and hyperbolic, between circular and elliptic, etc.

Between the values of numerical eccentricity of $s = 0$ and $s = \pm 1$, there exist an infinity of values, and for each value, an infinity of geometrical objects, which, each of them has the right to a geometrical existence.

If the geometrical mathematical objects for $s \in [0 \vee \pm 1]$ belong to the centric ordinary mathematics (CM) (circle $\rightarrow$ square, sphere $\rightarrow$ cube, cylinder $\rightarrow$ prism, etc.), those for $s \in (0, \pm 1)$ have forms, equations and denominations unknown in this centric mathematics (CM). They belong to the new mathematics, the eccentric mathematics (EM) and, implicitly, to the supermathematics (SM), which is a reunion of the two mathematics: centric and eccentric, that means $SM = MC \cup ME$.

Concluding remarks

The principal new idea in this paper is that it introduces a new mathematical transformation with a large significance in the fields of Physics, previously inexistent in the original classical mathematics named here as centric mathematics (CM); and now they became possible thanks to this new mathematics, called Eccentric Mathematics (EM), and to the Super Mathematics (SM), which are put together: (CM) with (EM). The (CM) is now a particular case of a linear numeric eccentricity for $s = 1$ in (SM).

In this paper the authors prove that these new geometric transformations, named Conversions or Transfigures, eliminate the borders between the discrete and continuous forms, showing that the first ones are also continuous but only apparently continuous. They mean: the conversion of a circle in a square, of a sphere in a cube, of a circle in a triangle, of a cone in a pyramid, of a cylinder in a prism, of a torus with circular section in a torus with a square section, etc. Also, they consider this eccentricity in the formation and deformation of the space. The authors claim that all of these transformations are possible because of the eccentricity considered as 4-th up to n-th dimension of the space to complete the usual accepted (x, y, z) dimension. This is the reason why they consider the eccentricity as a dimension of the formation or deformation of the space.

The extension of a three dimensional space to a n -dimensional space became possible if the linear constant eccentricity e and the angle eccentricity *epsilon* which are the polar coordinates of the eccentricity E (e , *epsilon*) became of one or multiple variables considered eccentricities too.

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Accepted: 30.03.2013

STRONGLY DUO AND DUO RIGHT S -ACTS

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Abstract. New kinds of acts, namely strongly duo and duo acts over a monoid are introduced and investigated. This leads to the study of the relation between these kinds of acts and other classes of acts, such as injective, projective and multiplication. It is shown that a projective act is duo if and only if it is a multiplication act. In addition if S is commutative, then a cyclic S -act is strongly duo if and only if it is cyclic quasi-injective.

Keywords: duo, strongly duo, right S -act.

2000 Mathematics Subject Classification: 20M30.

1. Introduction and preliminaries

In this paper, S is a monoid and an S -act A_S (or A) is a unitary right S -act. Let A be a right S -act and let B be a subact of A . We say that B is *fully invariant* if $f(B) \subseteq B$ for every endomorphism f of A and A is called *duo* if every subact of A is fully invariant. This notion generalizes the concept of right duo semigroups (semigroups for which every right ideal is two sided) see [1]. Also a right S -act A is called *strongly duo* if for every subact B of A the trace of B in A is equal to B , i.e., $tr(B, A) = \bigcup_{f \in \text{Hom}(B, A)} f(B) = B$ (see the properties of trace in [4]

page 146). It is clear that every strongly duo act is duo. In this paper several equivalent conditions to being strongly duo are given. For instance, it is shown that a right S -act A is strongly duo if and only if it is duo and is quasi-injective relative to all inclusions from its cyclic subacts. Also we prove that a projective right S -act A is duo if and only if it is multiplication (i.e., every subact of A is of the form AI for some right ideal I of S). It is proved that if S_S is strongly

duo, then all right S -acts are torsion free (divisible) and S is left reversible. To obtain a characterization for strongly duo acts, we need the following definition (see Definition 1.4.20 of [4]).

Definition 1.1 Suppose S is a monoid and A is a right S -act. For an element $a \in A$ we define the *annihilator* of a as $\text{ann}(a) := \{(s, t) \in S \times S : as = at\}$.

Theorem 1.2 Let S be a monoid and let A be a right S -act. Then the following are equivalent:

- (i) A is strongly duo.
- (ii) Every subact of A is strongly duo.
- (iii) Every finitely generated subact of A is strongly duo.
- (iv) If B, C are subacts of A and B is a homomorphic image of C , then $B \subseteq C$.
- (v) If $\text{ann}(a) \subseteq \text{ann}(b)$ for some $a, b \in A$, then $b \in aS$.

Proof. (i) $\longrightarrow$ (ii) and (ii) $\longrightarrow$ (iii) are clear.

(iii) $\longrightarrow$ (i) Suppose B is a subact of A and $f : B \longrightarrow A$ is a homomorphism. Let b be an element of B and let $C = bS \cup f(b)S$. If $g = f|_{bS}$, then, clearly, $f(b) = g(b) \in \text{tr}(bS, C)$. By assumption, C is strongly duo and so $f(b) \in \text{tr}(bS, C) = bS$. It follows that $\text{tr}(B, A) = B$.

(i) $\longrightarrow$ (iv) If B, C are two subacts of A and $f : C \longrightarrow B$ is an epimorphism, then $B = \text{Im}(f) \subseteq \text{tr}(C, A) = C$.

(iv) $\longrightarrow$ (v) Suppose $\text{ann}(a) \subseteq \text{ann}(b)$ for some $a, b \in A$. Define $f : aS \longrightarrow bS$ by $f(as) = bs$ for every $s \in S$. Clearly, f is a well-defined epimorphism and so $bS \subseteq aS$ by assumption.

(v) $\longrightarrow$ (i) Suppose B is a subact of A and $f \in \text{Hom}(B, A)$. If $b \in B$, then $\text{ann}(b) \subseteq \text{ann}(f(b))$ and hence $f(b) \in bS \subseteq B$ by assumption. Consequently, $\text{tr}(B, A) = B$. ■

The following corollary shows that, there are acts which are not strongly duo.

Corollary 1.3 Suppose S is a monoid and A is a strongly duo right S -act. If S_S is a subact of A , then $A_S = S_S$.

Proof. By part (ii) of the previous theorem, S_S is strongly duo. Also every cyclic right S -act is a homomorphic image of S_S . Thus the result holds by part (iv) of Theorem 1.2. ■

Definition 1.4 A right S -act A is called *completely ordered* if for any two subacts B, C of A , either $B \subseteq C$ or $C \subseteq B$.

Proposition 1.5 *Suppose S is a monoid and A is a completely ordered right S -act. If A satisfies the descending chain condition on cyclic subacts, then A is strongly duo. In particular if S is a finite principal right ideal monoid, then S_S is strongly duo.*

Proof. For elements $a, b \in A$ with condition $\text{ann}(a) \subseteq \text{ann}(b)$, we show that $b \in aS$. If $b \notin aS$, then by assumption $aS \subseteq bS$ and so $a = bs$ for some $s \in S$. Since $bsS \supseteq bs^2S \supseteq \dots$, by the hypothesis there exists $n \in \mathbb{N}$ such that $(bs^n)S = (bs^{n+1})S$ and so $bs^n = bs^{n+1}t$ for some $t \in S$. Thus $as^{n-1} = as^nt$. Since $\text{ann}(a) \subseteq \text{ann}(b)$, $bs^{n-1} = bs^nt$. Repeat this argument to obtain $a = ast$. Thus $b = bst = (bs)t = at \in aS$, which is a contradiction. Therefore, $bS \subseteq aS$ and A_S is strongly duo by Theorem 1.2. In case that S is a principal right ideal monoid, S_S is completely ordered. Hence the finite condition on S implies that S_S is strongly duo. ■

Corollary 1.6 *Suppose S is a monoid which satisfies the descending chain condition on principal right ideals. If A is a completely ordered right S -act, then A is strongly duo.*

Proof. We show that A satisfies the descending chain condition on cyclic subacts. If $a, b \in A$ and $bS \subseteq aS$, then there exists $s \in S$ such that $bS = asS$. Thus every descending chain of cyclic subacts of A has the form, $aS \supseteq as_1S \supseteq as_1s_2S \supseteq \dots$. Now consider the descending chain, $S \supseteq s_1S \supseteq s_1s_2S \supseteq \dots$. By assumption there exists $n \in \mathbb{N}$ such that $s_1\dots s_nS = s_1\dots s_{n+1}S$. It follows that A satisfies the descending chain condition on cyclic subacts and by the previous proposition is strongly duo. ■

The following proposition reveals some similarities between strongly duo acts and cohopfian modules.

Proposition 1.7 *If S is a monoid and A is a strongly duo right S -act, then every monomorphism $f : A \longrightarrow A$ is an epimorphism.*

Proof. Suppose $f : A \longrightarrow A$ is a monomorphism and define $g : f(A) \longrightarrow A$ by $g(f(a)) = a$ for every $a \in A$. Since f is a monomorphism, g is a well-defined homomorphism and clearly $g(f(A)) = A$. Since A is strongly duo, $g(f(A)) \subseteq \text{tr}(f(A), A) = f(A)$ and so $f(A) = A$, that is f is an epimorphism. ■

Now, we study the properties of duo acts. As the following lemma shows a good source of duo acts is provided by multiplication acts. By [3] an S -act A is called *multiplication* if every subact of A is of the form AI , for some right ideal I of S . Recall that a monoid S is said to be *left reversible* if every two right ideals of S have a nonempty intersection. Also for a right S -act A and subact B of A , the factor act A/B is defined as A/ρ_B , where ρ_B is the Rees congruence (see [4]).

Lemma 1.8 *Over a monoid S the following statements hold:*

- (i) A right S -act A is duo if and only if for each endomorphism f of A and each element a of A , $f(a) = at$ for some $t \in S$. In particular, if S is commutative and A is a duo right S -act, then $\text{End}(A)$ is a commutative monoid.
- (ii) Let $B \subseteq C$ be subacts of a right S -act A . If B and C/B are fully invariant subacts of A and A/B respectively, then C is fully invariant in A .
- (iii) If S_S is duo, then for any two elements $s, t \in S$, $st = tx$ for some $x \in S$.
- (iv) S_S is duo if and only if every right ideal of S is a two sided ideal.
- (v) If S_S is duo, then S is left reversible.
- (vi) Every multiplication right S -act is duo.

Proof. (i) Note that if A is duo, then $f(aS) \subseteq aS$ for all $f \in \text{End}(A)$ and $a \in A$. In particular, if S is commutative and A is a duo right S -act, then the first part of (i) implies that $\text{End}(A)$ is commutative.

(ii) Let $f \in \text{End}(A)$ and define $\bar{f} : A/B \rightarrow A/B$ by $\bar{f}(\bar{a}) = \overline{f(a)}$. As B is fully invariant in A , $\bar{f}$ is a well-defined homomorphism and $\bar{f}(C) = \overline{f(C)} = \overline{f(C/B)}$ and so $f(C) \subseteq C$.

(iii) For every $s \in S$ define $\lambda_s : S \rightarrow S$ by $\lambda_s(t) = st$ for every $t \in S$. Thus by part (i), $\lambda_s(t) \in tS$ for every $t \in S$ and the result follows. (iv) and (v) are obvious by part (i). (vi) If B is a subact of a multiplication right S -act A , then $B = AI$ for some right ideal I of S and so for every endomorphism f of A , $f(B) = f(AI) = f(A)I \subseteq AI = B$. ■

Corollary 1.9 Suppose T is a proper submonoid of a monoid S . Then S_T the right T -act S is not duo.

Proof. Suppose $s \in S \setminus T$ and define $f : S_T \rightarrow S_T$ by $f(x) = sx$ for every $x \in S$. If S_T is duo, then by the previous lemma, $s = f(1) = 1t$ for some $t \in T$ and so $s \in T$, which is a contradiction. ■

Proposition 1.10 Suppose S is a monoid and A is a completely ordered right S -act. If S_S is duo and A satisfies the ascending chain condition on cyclic subacts, then A is duo.

Proof. Let $f : A \rightarrow A$ be an endomorphism of A and let $a_1 \in A$. Suppose for some elements $a_2, a_3, \dots$ of A , $f(a_1) = a_2$, $f(a_2) = a_3, \dots$. Since A is completely ordered, for every $n \in \mathbb{N}$, either $a_n \in a_{n+1}S$ or $a_{n+1} \in a_nS$. If for every $n \in \mathbb{N}$, $a_nS \subset a_{n+1}S$, then we have the ascending chain $a_1S \subset a_2S \subset a_3S \dots$, which is a contradiction. Thus there exists the smallest $n \in \mathbb{N}$ such that $a_{n+1}S \subseteq a_nS$. Hence $a_{n-1} \in a_nS, a_{n-2} \in a_{n-1}S, \dots, a_1 \in a_2S$ and so $a_1 \in a_nS$. If $a_1 = a_ns$ for some $s \in S$, then $f(a_1) = f(a_n)s = a_{n+1}s$. Since $a_{n+1} \in a_nS$, $a_{n+1} = a_nt$ for some $t \in S$ and so $f(a_1) = a_n ts$. By Lemma 1.8.(iii), $ts = sx$ for some $x \in S$ and hence $f(a_1) = a_n sx = a_1 x$. Now the result follows by Lemma 1.8.(i). ■

Now, the behavior of duo acts under taking subacts and homomorphic images is considered.

Definition 1.11 Let S be a monoid. Then an S -act A is called (*cyclic*) *quasi-injective* if for every (cyclic) subact B of A and for every homomorphism $f \in \text{Hom}(B, A)$, there exists a homomorphism $g \in \text{Hom}(A, A)$ which extends f , i.e., $g|_B = f$. Also A is called *quasi-projective* if for a given epimorphism $g : A \rightarrow C$ and homomorphism $f : A \rightarrow C$, there exists a homomorphism $h : A \rightarrow A$ such that $gh = f$.

Proposition 1.12 Suppose S is a monoid and A is a duo right S -act. Then the following statements hold:

- (i) If A is quasi-injective, then every subact of A is duo and quasi-injective.
- (ii) If A is quasi-projective, then every homomorphic image of A is duo and every Rees factor of A is quasi-projective.

Proof. (i) Suppose B is a subact of A and C is a subact of B . Let f be an endomorphism of B . Since A is quasi-injective, f can be lifted to an endomorphism $\bar{f}$ of A . Thus $\bar{f}(C) = f(C)$, which is contained in C because A is duo. Also it is easy to see that B is quasi-injective.

(ii) Let B be a homomorphic image of A and let g be an endomorphism of B . Suppose $B = \frac{A}{\rho}$ for some right congruence ρ on A and $\frac{C}{\rho}$ is a subact of B . Since A is quasi-projective, there exists an endomorphism g^* of A such that $\pi \circ g^* = g \circ \pi$, where π is the natural epimorphism. Since A is duo, $g^*(C) \subseteq C$ and so $g(\frac{C}{\rho}) \subseteq \frac{C}{\rho}$. Thus B is duo. By a similar proof as part (ii) of Lemma 1.8, we can conclude that every Rees factor of A is quasi-projective. ■

It is easy to see that every homomorphic image of any multiplication right S -act is multiplication. Also by Proposition 1.12.(ii), if S_S is duo, then every cyclic right S -act is duo. Thus, by Lemma 1.8, we have the following result.

Corollary 1.13 Over a monoid S the following statements are equivalent:

- (i) S_S is duo.
- (ii) S_S is multiplication.
- (iii) Every cyclic right S -act is multiplication.
- (iv) Every cyclic right S -act is duo.

Proposition 1.14 Suppose S is a monoid and A is a right S -act. Then the following hold:

- (i) If every countably generated subact of A is duo, then A is duo.
- (ii) If A is duo and cyclic quasi-injective, then A is strongly duo.

Proof. (i) Suppose $a \in A$ and $f \in \text{End}(A)$. Let $B = aS \cup f(a)S \cup f(f(a))S \cup \dots$. Clearly, B is a countably generated subact of A and $g := f|_B$ is an element of $\text{End}(B)$. Thus, by Lemma 1.8.(i) and by assumption, $g(a) = as$ for some $s \in S$. Hence $f(a) = g(a) = as$ for some $s \in S$ and by Lemma 1.8.(i) A is duo.

(ii) Let B be a subact of A . Since A is cyclic quasi-injective, for every $b \in B$ and every homomorphism $f : B \rightarrow A$, there exists $\bar{f} : A \rightarrow A$ such that $\bar{f}(bS) = f(bS)$. Hence the duo condition on A implies that $f(b) = \bar{f}(b) \in bS \subseteq B$, proving that A is strongly duo. ■

Note that, by Propositions 1.12.(i), 1.14.(ii) and Corollary 1.3, over every monoid S there exist right S -acts which are not strongly duo (duo). To investigate the relation between projective acts and duo acts we need the concept of the "dual basis" for acts. In [3], Khaksari et al generalized the concept of dual basis from modules to acts over commutative monoids. But it is easy to see that the condition "commutativity" is not necessary for this result. Regarding this observation we have the following theorem.

Theorem 1.15 *Let S be a monoid and let A be a projective right S -act. Then A is duo if and only if A is multiplication.*

Proof. If A is multiplication, then by Lemma 1.8.(vi), A is duo. Conversely, suppose A is duo and B is a subact of A . Let $A^* = \text{Hom}(A, S)$. Since A is projective, by Theorem 1 of [3], there exists a subset $T = \{(x_\alpha : f_\alpha) : \alpha \in \Lambda\}$ of $A \times A^*$ such that for every $x \in A$, $x = x_\alpha f_\alpha(x)$, where $(x_\alpha, f_\alpha) \in T$. Let I be the right ideal generated by the elements of the form $f_\alpha(x)$ for $x \in B$ and $\alpha \in \Lambda$. We claim that $B = AI$. If $x \in B \subseteq A$, then $x = x_\alpha f_\alpha(x)$ for some $\alpha \in \Lambda$ and $(x_\alpha, f_\alpha) \in T$ and hence $x \in AI$. Now suppose $x \in B$, $a \in A$ and $\alpha \in \Lambda$. Thus $\lambda_a \circ f_\alpha \in \text{End}(A)$, where $\lambda_a : S \rightarrow A$ is defined by $\lambda_a(s) = as$ for every $s \in S$. Consequently, $af_\alpha(x) = \lambda_a(f_\alpha(x)) \in \lambda_a(f_\alpha(xS)) \subseteq xS$ because A is duo. Hence $AI \subseteq B$ and so $B = AI$. ■

Corollary 1.16 *Suppose S is an idempotent monoid (i.e., $I^2 = I$ for every right ideal I of S) and A is a projective right S -act. Then the following statements are equivalent:*

- (i) A is multiplication.
- (ii) A is strongly duo.
- (iii) A is duo.

Proof. (i) $\longleftrightarrow$ (iii) holds by the previous theorem.

(i) $\longrightarrow$ (ii) Suppose for some right ideal I of S , $B = AI$ is a subact of A . Then for every $f \in \text{Hom}(B, A)$, $f(B) = f(AI) = f(AI^2) = f(AI)I \subseteq AI = B$ and so A is strongly duo.

(ii) $\longrightarrow$ (iii) is clear. ■

Lemma 1.17 *Suppose S is a monoid and S_S is strongly duo. If I is a projective right ideal of S , then $I^2 = I$.*

Proof. By Theorem 1 of [3], $Itr(I, S) = I$ for every projective right ideal I of S . Since S_S is strongly duo, $tr(I, S) = I$ and so $I^2 = I$. ■

A monoid S is called *right hereditary* if all right ideals of S are projective.

Theorem 1.18 *Suppose S is a commutative monoid. Then the following hold:*

- (i) *A right S -act A is strongly duo if and only if it is duo and cyclic quasi-injective. In particular, a cyclic right S -act A is strongly duo if and only if it is cyclic quasi-injective.*
- (ii) *If $E(S)$ is the injective envelope of the right S -act S_S , then S_S is injective if and only if $E(S)$ is a duo right S -act.*
- (iii) *If S is right hereditary, then S_S is strongly duo if and only if $I^2 = I$ for every right ideal I of S .*

Proof. (i) By Proposition 1.14.(ii), every duo cyclic quasi-injective right S -act is strongly duo. Now let A be a strongly duo right S -act. Clearly, A is duo. We show that A is cyclic quasi-injective. For this, let $f : aS \rightarrow A$ be a homomorphism where $a \in A$. Then $f(a) \in aS$ because A is strongly duo. Thus $f(a) = as$ for some $s \in S$. Define $\bar{f} : A \rightarrow A$ by $\bar{f}(x) = xs$ for every $x \in A$. Since S is commutative, $\bar{f}$ is a well-defined homomorphism which is an extension of f . Now the result follows by the definition of cyclic quasi-injectivity. Also over a commutative monoid S , every cyclic right S -act is duo. Thus a cyclic right S -act A is strongly duo if and only if it is cyclic quasi-injective.

(ii) If $E(S)$ is a duo right S -act, then by Proposition 1.14.(ii), $E(S)$ is strongly duo and by Corollary 1.3, $S_S = E(S)$ and so S_S is injective. The converse is obvious because S is commutative and so it is duo.

(iii) The necessity follows by Lemma 1.17. Conversely, by Theorem 2 of [3], every right ideal I of S is multiplication and so it is strongly duo by Corollary 1.16. ■

Remark. Note that, in the previous theorem, the condition "cyclic quasi-injectivity" is necessary. Indeed, if $S = (\mathbb{Z}, .)$, then S_S is clearly duo, but by Theorem 1.2, S_S is not strongly duo. In fact S_S is not cyclic quasi-injective.

A right S -act A is called *torsion free* if for any $a, b \in A$ and for any right cancelable element $c \in S$ the equality $ac = bc$ implies $a = b$.

Proposition 1.19 *Suppose S is a monoid and S_S is strongly duo. Then the following hold:*

- (i) *Every left cancelable element of S is invertible.*

- (ii) All right S -acts are divisible (torsion free).
- (iii) Every right cancelable element of S is invertible.
- (iv) If S is left (right) cancelative, then S is a group.

Proof. (i) If c is a left cancelable element of S , then $\text{ann}(c) \subseteq \text{ann}(s)$ for every $s \in S$. Thus by Theorem 1.2, $s \in cS$ for every $s \in S$ and therefore $S = cS$. Consequently $cx = 1$ for some $x \in S$. Also since S_S is duo, by Lemma 1.8.(iii), $1 = cx = xy$ for some $y \in S$, which implies c is invertible.

(ii) By (i) all right S -acts are torsion free. Also by (i) S_S is divisible and hence all right S -acts are divisible by Proposition 3.2.2 of [4].

(iii) is clear by part (ii) and Theorem 4.6.1 of [4] and Lemma 1.8.(iii).

(iv) holds by parts (i) and (iii). ■

Proposition 1.20 Suppose S is a monoid and $\{A_i : i \in I\}$ is a family of right S -acts. Then the following hold:

- (i) If $\prod_{i \in I} A_i \left(\prod_{i \in I} A_i \right)$ is duo, then for every $i \in I$, A_i is duo.
- (ii) If $\prod_{i \in I} A_i \left(\prod_{i \in I} A_i \right)$ is strongly duo, then for every $i \in I$, A_i is strongly duo.

Proof. (i) Suppose $A = \prod_{i \in I} A_i$ is duo. For a fixed element $j \in I$, let $f_j \in \text{End}(A_j)$.

Define $f : A \longrightarrow A$, by $f(\{a_i\}_{i \in I}) = \{b_i\}_{i \in I}$, where

$$(1.1) \quad b_i = \begin{cases} a_i, & i \neq j; \\ f_j(a_j), & i = j, \end{cases}$$

for every element $\{a_i\}_{i \in I}$ of $A = \prod_{i \in I} A_i$. Clearly $f \in \text{End}(A)$ and by Lemma

1.8.(i), $f(\{a_i\}_{i \in I}) = \{a_i\}_{i \in I} s = \{a_i s\}_{i \in I}$ for some $s \in S$. Thus, for every, $a_j \in A_j$, $f_j(a_j) = a_j s$, for some $s \in S$, and so A_j is duo by Lemma 1.8.(i). The proof for $A = \prod_{i \in I} A_i$ is straightforward and will be omitted.

(ii) is obvious by Theorem 1.2. ■

In general, if $\{A_i : i \in I\}$ is a family of duo (strongly duo) right S -acts, then $\prod_{i \in I} A_i$ and $\prod_{i \in I} A_i$ are not duo (strongly duo). See the following example.

Example 1.21

- (i) If S is a monoid and θ_S is the one element right S -act, then clearly θ_S is duo (strongly duo). But we can easily see that $\theta_S \sqcup \theta_S$ is not duo (strongly duo).
- (ii) Suppose S is a commutative monoid which is not a group and S_S is principally weakly injective. By Theorem 1.18.(i), S_S is strongly duo and so it is duo, but $(S \times S)_S$ is not duo (strongly duo). To see this, define $f : (S \times S)_S \longrightarrow (S \times S)_S$ by $f(x, y) = (y, x)$, for every $(x, y) \in S \times S$. It is clear that f is a homomorphism. Thus if $(S \times S)_S$ is duo, then by Lemma 1.8.(i), for every $(x, y) \in S \times S$, $f(x, y) = (x, y)s$ for some $s \in S$. It implies $S = xS = Sx$ for every $x \in S$, and hence S is a group, a contradiction.

2. Strongly duo and duo acts over monoids with a zero element

In this section, we study the behavior of duo and strongly duo acts under the formation of 0-decomposition of an act. We assume that S is a monoid with a zero. If A is a right S -act, then A is called *centered* if A has a unique zero element. Let A be a centered right S -act with a zero element θ_A . By a 0-decomposition of A we mean an expression $A = \bigcup_{i \in I} A_i$ of A such that for every i , A_i is a centered subact of A and for all distinct $i, j \in I$, $A_i \cap A_j = \theta_A$. In what follows, we will use the notation $A = \coprod_{i \in I}^0 A_i$ for this concept. By [2], over monoids with a zero element, every centered right S -act has a 0-decomposition.

Proposition 2.1 *Suppose S is a monoid and A is a right S -act. If $A = \coprod_{i \in I}^0 A_i$ is a 0-decomposition of A , then the following hold:*

- (i) *For every $i \in I$, A_i is a fully invariant subact of A if and only if $\text{Hom}(A_i, A_k) = 0$ for all distinct $i, k \in I$.*
- (ii) *If A is duo, then for every $i \in I$, A_i is duo.*

Proof. (i) Suppose $i \in I$ and A_i is a fully invariant subact of A . For $k \neq i$ in I , let $f \in \text{Hom}(A_i, A_k)$ and suppose $j_k : A_k \longrightarrow A$ is the inclusion map and $\pi_i : A \longrightarrow A_i$ is the canonical projection. Clearly $\bar{f} = j_k \circ f \circ \pi_i$ is an endomorphism of A and so by assumption $\bar{f}(A_i) \subseteq A_i$. Since $\bar{f}(A_i) = f(A_i) \subseteq A_k$, $f(A_i) \subseteq A_i \cap A_k = 0$ and hence $f = 0$. Conversely, suppose $i \in I$ and for every $k \neq i$ in I , $\text{Hom}(A_i, A_k) = 0$. We show that A_i is a fully invariant subact of A . For this let $f \in \text{End}(A)$ and suppose $j_i : A_i \longrightarrow A$ is the inclusion map and $\pi_i : A \longrightarrow A_i$ is the canonical projection. It is clear that for every $k \neq i$ in I , $\pi_k \circ f \circ j_i \in \text{Hom}(A_i, A_k) = 0$, where $\pi_k : A \longrightarrow A_k$ is the canonical projection. Thus $f(A_i) \subseteq \bigcup_{k \in I} (\pi_k f j_i)(A_i) \subseteq \pi_i f j_i(A_i) \subseteq A_i$.

(ii) Suppose $i \in I$ and $f_i \in \text{End}(A_i)$. If $\pi_i : A \rightarrow A_i$ is the canonical projection and $j_i : A_i \rightarrow A$ is the inclusion map, then $f = j_i \circ f_i \circ \pi_i \in \text{End}(A)$. Thus for every subact B of A_i , $f_i(B) = f(B) \subseteq B$ because A is duo. ■

Theorem 2.2 Suppose S is a monoid and A is a right S -act. If $A = \coprod_{i \in I}^0 A_i$ is a 0-decomposition of A , then A is duo if and only if for every $i \in I$, A_i is duo and $\text{Hom}(A_i, A_k) = 0$ for all distinct $i, k \in I$.

Proof. The necessity follows by the previous proposition. Conversely, suppose B is a subact of A and $f \in \text{End}(A)$. Let $\pi_k : A \rightarrow A_k$ denote the canonical projection and $j_k : A_k \rightarrow A$ denote the inclusion map for every $k \in I$. Thus $\pi_k \circ f \circ j_k(A_k \cap B) \subseteq A_k \cap B$ because $\pi_k \circ f \circ j_k \in \text{End}(A_k)$ and A_k is duo for every $k \in I$. Also by assumption, $\pi_k \circ f \circ j_i(A_k \cap B) = 0$ for all distinct $k, i \in I$. Since $B = \bigcup_{k \in I} (A_k \cap B)$, $f(B) \subseteq \bigcup_{k \in I} f(A_k \cap B) \subseteq \bigcup_{k \in I} \pi_k f j_k(A_k \cap B) \subseteq \bigcup_{k \in I} (A_k \cap B) \subseteq B$, i.e., B is fully invariant, which implies A is duo. ■

Theorem 2.3 Suppose S is a commutative monoid and A is a right S -act. If $A = \coprod_{i \in I}^0 A_i$ is a 0-decomposition of A , then A is strongly duo if and only if $A_i \sqcup^0 A_j$ is strongly duo for all distinct $i, j \in I$.

Proof. The necessity follows by Theorem 1.2. Conversely, suppose for every $i \neq j \in I$, $A_i \sqcup^0 A_j$ is strongly duo. Since $A_i \sqcup^0 A_j$ is duo, by Proposition 2.1.(i), $\text{Hom}(A_i, A_j) = 0$ for every $i \neq j \in I$. Also by Theorem 1.2, for every $i \in I$, A_i is strongly duo and so is duo. Hence by Theorem 2.2, A is duo. Now we show that A is cyclic quasi-injective. Suppose aS is a cyclic subact of A and $f \in \text{Hom}(aS, A)$. If $a \in A_i$ and $f(a) \in A_j$ for some $i \neq j \in I$, then $a, f(a) \in A_i \sqcup^0 A_j$. Since $\text{ann}(a) \subseteq \text{ann}(f(a))$, by Theorem 1.2, $f(a) \in aS \subseteq A_i$ and so $f(a) \in A_i \cap A_j = 0$. Define $\bar{f} : A \rightarrow A$ by $\bar{f}(x) = \theta_A$ for every $x \in A$. Clearly $\bar{f}$ is an extension of f and so in this case, A is cyclic quasi-injective. Now suppose $a, f(a) \in A_i$ for some $i \in I$. Since A_i is strongly duo, A_i is cyclic quasi-injective by Theorem 1.18.(i). Hence there exists $f : A_i \rightarrow A_i$ such that $\bar{f}(a) = f(a)$. Define $g : A \rightarrow A$ by

$$(2.1) \quad g(x) = \begin{cases} \bar{f}(x), & x \in A_i; \\ \theta_A, & x \notin A_i. \end{cases}$$

It is easy to see that g is a well-defined homomorphism and is an extension of f . It follows that A is cyclic quasi-injective and the proof is complete by Theorem 1.18.(i). ■

Definition 2.4 Suppose S is a monoid and A is a right S -act. We say that A is *weakly duo* if every non-zero subact of A contains a non-zero subact which is fully invariant in A .

Theorem 2.5 *Let S be a monoid and A be a quasi-projective right S -act. The following are equivalent:*

- (i) *A is a duo right S -act.*
- (ii) *Every factor of A is weakly duo.*

Proof. (i) $\longrightarrow$ (ii) By Proposition 1.12.(ii), every factor of A is duo and so is weakly duo.

(ii) $\longrightarrow$ (i) Suppose B is a non-zero subact of A such that B is not fully invariant. By the hypothesis, there exists a non-zero subact $C \subset B$ which is fully invariant in A . Let $D = \bigcup \{L : L \text{ is a fully invariant subact of } A \text{ and } L \subset B\}$. Clearly D is a fully invariant subact of A and $D \neq B$. Thus B/D is a non-zero subact of A/D and so by assumption there exists a subact $E \subseteq B$ such that E/D is a non-zero fully invariant subact of A/D . Thus $D \subset E \subseteq B$. By Lemma 1.8.(ii), E is fully invariant in A and by the choice of D we must have $B = E$, a contradiction. ■

By the previous theorem we give the following result.

Corollary 2.6 *If S is a monoid, then S_S is duo if and only if every cyclic right S -act is weakly duo.*

Theorem 2.7 *If S is a monoid and A is a projective right S -act, then the following are equivalent:*

- (i) *A is duo.*
- (ii) *A is multiplication.*
- (iii) *Every factor of A is weakly duo.*

Proof. The result follows by Theorems 1.15 and 2.5. ■

Recall that a subact B of an S -act A is called *essential* in A , if $B \cap C \neq 0$ for each $0 \neq C \leq A$ (see [5]).

Proposition 2.8 *If S is a monoid and A is a duo right S -act, then for every monomorphism $f : A \longrightarrow A$, $f(A)$ is an essential subact of A .*

Proof. Suppose $f : A \longrightarrow A$ is a monomorphism and $0 \neq B$ is a subact of A such that $B \cap f(A) = 0$. Since A is duo, $f(B) \subseteq B$, and so $f(B) \subseteq B \cap f(A)$. Thus $f(B) = 0$ and so $B = 0$, a contradiction. ■

Acknowledgment. The authors would like to thank the referee for providing valuable comments and suggestions.

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Accepted: 03.05.2013

ON AUTOMATIC BOUNDEDNESS OF LINEAR OPERATORS ON CONVEX BORNOLOGICAL SPACES

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Abstract. In this paper, using the notion of separating space of a linear operator defined on a bornological vector space introduced in [5], we give some useful criteria to study the automatic boundedness of operators. In particular, we give necessary and sufficient conditions in order that operators should be bounded (Theorem 3.1 and Theorem 4.1).

Keywords and phrases: bornological vector space, separating space, Mackey convergence, linear operator, automatic boundedness.

1. Introduction

In [4], Sinclair studied the necessary conditions for continuity of homomorphisms, derivations and pair of operators acting on a Banach space.

The aim of the present paper is to extend some of this results in case of bornological vector space (bvs) and consequently obtains some techniques to answer the boundedness problem for linear operators.

We extend naturally the notion of separating space of some linear operator S between (bvs) X and (bvs) Y (see Definition 3.1 below). The notion of separating space characterizes the continuity of linear operators

The nice properties of the notion of separating space is for an linear operator T acting on (bvs) X then, T is bounded if, and only if, its separating space is reduce a zero (see Theorem 3.1 below).

In the following we introduce some techniques which are necessary to study the boundedness of homomorphisms, derivations and pair of linear operators. Particularly, we give the characterization of bounded operators acting in bornological quotient (see Theorem 4.1 below).

2. Preliminaries

Recall that a bornology on a set X is a family $\mathcal{B}$ of subset X such that $\mathcal{B}$ is a covering of X , hereditary under inclusion and stable under finite union. The pair $(X, \mathcal{B})$ is called bornological set.

A subfamily $\mathcal{B}'$ of $\mathcal{B}$ is said to be base of bornology $\mathcal{B}$, if every element of $\mathcal{B}$ is contained in an element of $\mathcal{B}'$.

Let X and Y be two bornological set, a map X of Y is called bounded if the image of every bounded subset of X is bounded in Y .

A bornology $\mathcal{B}$ on $\mathbb{K}$ -vector space E is said to be vector bornology on E if the maps $(x, y) \mapsto x + y$ and $(\lambda, x) \mapsto \lambda.x$ are bounded.

We called a bornological vector space(b.v.s) any pair $(X, \mathcal{B})$ consisting of a vector space E and a vector bornology $\mathcal{B}$ on E .

A vector bornology on a vector space is called a convex vector bornology if it is stable under the formation of convex hull.

A bornological vector space is said a convex bornological vector space (cbvs) if it bornology is convex.

A (b.v.s) space is called of type M_1 if, it satisfies the countability condition of Mackey:

For every sequence of bounded $(B_k)_k$ in E , there exists a sequence of scalars $(\lambda_k)_{k \geq 0}$ such that $\bigcup_{k=0}^{\infty} \lambda_k B_k$ is bounded in E .

Observe that every Banach algebra is a multiplicative convex bornological complete algebra of type M_1 . Also, if E is unital topological algebra with continuous inverse such that it is F -space, then E is a multiplicative convex bornological complete algebra of type M_1 .

A sequence $(x_n)_{n \geq 0}$ in bornological vector space (b.v.s) E is said Mackey-convergent to 0 (or converge bornological to 0) if there exists a bounded set $B \subset E$ such that

$$\forall \varepsilon > 0 \exists n_0 \in \mathbb{N} \ n \geq n_0 \text{ implies } x_n \in \varepsilon B.$$

If E is (cbvs), then $(x_n)_{n \geq 0}$ is Mackey-convergent to 0 if there exists a bounded disk $B \subset E$ such that $(x_n)_{n \geq 0} \subset E_B$ and $(x_n)_{n \geq 0}$ converges to 0 in E_B , where (E_B, p_B) is the vector space spanned by B and endowed with the semi-norm p_B gauge of B .

Let E be a (bvs), E is said separated if there is not a non-zero bounded line in E , equivalently, every sequence Mackey-convergent its limit is unique.

A (bvs) E is separated if, and only if, for every bounded disk B the space (E_B, p_B) is a normed space.

A set B in a (bvs) E is said M -closed (or b-closed) if every sequence $(x_n)_{n \geq 0} \subseteq B$ Mackey convergent in E its limit belongs in B .

Let E a separated (bvs) and let F be a subspace of E , the bornological quotient space E/F is separated, if and only if, F is b-closed in E .

Let E be a (cbvs) and A is a disk in E . A is called a completant disk if the space (E_A, p_A) spanned by A and semi-normed by the gauge of A is a Banach space.

A (cbvs) E is called a complete convex bornological vector space if its bornology has a base consisting of completant disks.

Let E be a (bvs) and $A \subset E$, the bornological closure (briefly b-closure or M-closure) of A denoted $\overline{A}$ is the intersection of all bornological closed subsets of E containing A . A (cbvs) E satisfies M -closed properties if for any subset A of E we have $\overline{A} = A^{(1)}$ where $A^{(1)}$ is the set of limits in Mackey-sense of the sequences belonging in A .

For bornology with a nets see [1], [2]. Recall that every bornology of a (cbvs) having a countable base has a net.

Theorem 2.1 (bornological closed graph theorem) *Let $(E, \mathcal{B})$ be a complete (cbvs) and $(E', \mathcal{B}')$ be a (cbvs) such that $\mathcal{B}'$ has a net. Then, every linear map $u : E \rightarrow E'$ with a bornological closed graph in $E \times E'$ is bounded.*

Theorem 2.2 (bornological isomorphism) *Let E be a complete (cbvs) and let F be a (cbvs) where its bornology has a net. Then every bijective bounded linear map $u : F \rightarrow E$ is a bornological isomorphism.*

For details, see [1], [2], [3], or [6].

3. Separating space

Definition 3.1 [5] Let X and Y two (bvs), let T a linear map between X and Y . We called separating space of T the subset of Y denotes by $\sigma(T)$

$$\sigma(T) = \{y \in Y / \exists (x_n)_n \subset X : x_n \xrightarrow{M} 0 \text{ and } T(x_n) \xrightarrow{M} y\}.$$

Proposition 3.1 *Let X and Y two (cbvs) of type M_1 . Then, every separating space of linear map $T : X \rightarrow Y$ is a b-closed subspace in Y .*

Proof. Evidently, $\sigma(T)$ is a subspace vector of Y . $\sigma(T)$ is b-closed, indeed.

Let $(y_k)_{k \geq 0}$ a sequence in $\sigma(T)$ converging to y in Y . We prove that $y \in \sigma(T)$. For every $k \in \mathbb{N}^*$, $y_k \in \sigma(T)$. Then, there is a sequence $(x_{n,k})_n \subset X$ such that

$$x_{n,k} \xrightarrow{M} 0 \text{ and } T(x_{n,k}) \xrightarrow{M} y_k.$$

Since $x_{n,k} \xrightarrow{M} 0$, then there is a bounded disc B_k in X :

$$(x_{n,k})_n \subset B_k \text{ and } \lim_{n \rightarrow +\infty} p_{B_k}(x_{n,k}) = 0.$$

$(B_k)_{k \geq 0}$ is a sequence of bounded disc in X which is of type M_1 , hence there is $(\lambda_k)_k \subset \mathbb{R}^{*+}$ and there is circled bounded B such that $B_k \subset \lambda_k B$, therefore $p_B \leq p_{B_k}$, for any $k \in \mathbb{N}$. (B can be disked, if not we take the disked hull of B).

Consequently, there exists a bounded disked B in X such that

$$(x_{n,k})_{n,k} \subset X_B,$$

we have

$$\forall k \geq 1 \exists N_k \in \mathbb{N}, n \geq N_k \implies p_B(x_{n,k}) \leq \frac{1}{k}.$$

On other hand, $(T(x_{n,k}))_n$ converges bornological to y_k .

Again, by the same argument, there exists a bounded disked B' in Y such that $[T(x_{n,k}) - y_k]_n \subset Y_{B'}$, we have

$$\forall k \geq 1 \exists N'_k \in \mathbb{N}, n \geq N'_k \implies p_{B'}(T(x_{n,k}) - y_k) \leq \frac{1}{k}.$$

Consequently, there exists a sequence $(z_k)_k \subset X_B$ such that $(T(z_k) - y_k)_k \subset Y_{B'}$ and, for every $k \in \mathbb{N}^*$, we have

$$p_B(z_k) < \frac{1}{k} \text{ and } p_{B'}(T(z_k) - y_k) < \frac{1}{k}.$$

Since $y_k \rightarrow^M y$, then there is a bounded disked C in Y such that

$$(y_k - y)_{k \geq 1} \subset Y_C \text{ and } p_C(y_k - y) < \frac{1}{k}.$$

Let D be the disked hull of $B' \cup C$. We have $p_D \leq p_{B'}$, $p_D \leq p_C$, $Y_{B'} \subset Y_D$ and $Y_C \subset Y_D$.

On other hand, for every $k \in \mathbb{N}^*$, we have

$$T(z_k) - y = (T(z_k) - y_k) + (y_k - y).$$

This gives

$$(T(z_k) - y)_{k \geq 1} \subset Y_D \text{ and } \lim_{k \rightarrow +\infty} p_D(T(z_k) - y) = 0.$$

We conclude that, $y \in \sigma(T)$. ■

Proposition 3.2 *Let X and Y two (cbvs) and let $T : X \rightarrow Y$ be a linear map. Then, we have*

- i) $\sigma(T) = \{0\}$ if, and only if, the graph of T is b -closed.
- ii) Let R and S be two linear operators of X into Y , if $TR = ST$, then

$$S(\sigma(T)) \subset \sigma(T).$$

Proof. Let $G(T)$ the graph of T .

- i) Suppose that $\sigma(T) = \{0\}$.

Let $(x_n, T(x_n))_{n \geq 0} \subset G(T)$ such that $(x_n, T(x_n)) \xrightarrow{M} (x, y) \in X \times Y$. Then, $x_n \xrightarrow{M} x$ and $T(x_n) \xrightarrow{M} y$. Since, $T(x_n - x) = T(x_n) - T(x)$, then $T(x_n - x) \xrightarrow{M} y - T(x)$.

Consequently, $(y - T(x)) \in \sigma(T)$. Hence, $y = T(x)$.

Conversely, suppose that $G(T)$ is b-closed, and let $y \in \sigma(T)$. There exists $(x_n)_{n \geq 0} \subset X$ such that:

$$x_n \xrightarrow{M} 0 \text{ and } T(x_n) \xrightarrow{M} y.$$

Since, $(x_n, T(x_n))_{n \geq 0} \subset G(T)$ and $(x_n, T(x_n))_n \xrightarrow{M} (0, y)$. Then, $(0, y) \in G(T)$. Therefore, $y = 0$.

ii) Let $y \in \sigma(T)$. There exists $(x_n)_{n \geq 0} \subset X$ such that $x_n \xrightarrow{M} 0$ and $T(x_n) \xrightarrow{M} y$. R being bounded, then $R(x_n) \xrightarrow{M} 0$.

On other hand, $TR(x_n) = ST(x_n)$ and S is bounded then $TR(x_n) \xrightarrow{M} S(y)$.

Consequently, $S(y) \in \sigma(T)$, i.e., $S(\sigma(T)) \subset \sigma(T)$. ■

Theorem 3.1 *Let X be a complete (cbvs) and Y a (cbvs) such that its bornology has a net. Let $T : X \rightarrow Y$ be a linear map. Then T is bounded if, and only if, $\sigma(T) = \{0\}$.*

Proof. Applique the b-closed graph theorem's and Proposition 3.2. ■

4. Characterization of bounded operators

Proposition 4.1 *Let X and Y two (cbvs) of type M_1 and Z a separated (bvs). Suppose that X is complete and the bornology of Y has a net. Let $S : X \rightarrow Y$ be linear and $R : Y \rightarrow Z$ be bounded linear map. Then*

i) RS is bounded if, and only if, $R(\sigma(S)) = \{0\}$.

ii) $[R\sigma(S)]^{(1)} = \sigma(RS)$.

For the proof. we shall need the following lemma.

Lemma 4.1 *Let X be a (bvs) and F be a vector subspace of E . Let $\varphi : E \rightarrow E/F$ the canonical surjection. Then, For every sequence $(x_n)_{n \geq 0} \subset E$ such that $(\varphi(x_n))_{n \geq 0}$ bornological converges to 0 in E/F , there exists a sequence $(y_n)_{n \geq 0} \subset F$ such that $(x_n - y_n)_{n \geq 0}$ bornological converges to 0 in E .*

Proof. Let $(x_n)_{n \geq 0} \subset E$ such that $(\varphi(x_n))_n$ bornological converges to 0 in E/F . There exists increasing sequence $(\epsilon_n)_n \subset \mathbb{R}^{*+}$ converging to 0 and a bounded disked B in E such that

$$\varphi(x_n) \in \epsilon_n \varphi(B), \forall n \in \mathbb{N}.$$

Let $(z_n)_n \subset B$ such that $\varphi(x_n) = \epsilon_n \varphi(z_n)$ for every n . Then, $(x_n - \epsilon_n z_n)_n \subset F$. We set $y_n = x_n - \epsilon_n z_n, \forall n \in \mathbb{N}$. Then

$$x_n - y_n = \epsilon_n z_n \in \epsilon_n B, \forall n \in \mathbb{N}.$$

Therefore, $(x_n - y_n)_n$ bornological converges to 0. ■

Proof of Proposition 4.1

i) Suppose that RS is bounded.

Let $y \in \sigma(S)$. Then there exists $(x_n)_{n \geq 0} \subset X$ such that $x_n \xrightarrow{M} 0$ and $S(x_n) \xrightarrow{M} y$.

R being bounded, then $RS(x_n) \xrightarrow{M} R(y)$.

On other hand, by hypothesis RS is bounded then, $RS(x_n) \xrightarrow{M} 0$.

Z is separated, thus $R(y) = 0$, i.e: $R\sigma(S) = \{0\}$.

Conversely, suppose that $R(\sigma(S)) = \{0\}$.

Let $Q : Y \rightarrow Y/\sigma(S)$ the canonical quotient map.

Consider $R_0 : Y/\sigma(S) \rightarrow Z$ defined by: $R_0(y + \sigma(S)) = R(y)$.

Clearly, R_0 is well defining and we have $R = R_0Q$. Therefore, $R_0QS = RS$.

Since R_0 is bounded, it suffice to show that QS is bounded and for this we prove that $\sigma(QS) = \{0\}$ (Theorem 3.1).

Let $y + \sigma(S) \in \sigma(QS)$. Then there exists $(x_n)_{n \geq 0} \subset X$ such that

$$x_n \xrightarrow{M} 0 \text{ and } QS(x_n) \xrightarrow{M} Q(y) = y + \sigma(S).$$

Then, $Q(S(x_n) - y) \xrightarrow{M} 0$ in $Y/\sigma(S)$.

By Lemma 4.1, there exists a sequence $(y_n)_{n \geq 0} \subset \sigma(S)$ such that:

$$S(x_n) - y - y_n \xrightarrow{M} 0 \text{ in } Y.$$

Thus, for every $k \in \mathbb{N}^*$, there exists $(x_{n,k})_{n \geq 0} \subset X$ such that:

$$x_{n,k} \xrightarrow{M} 0 \text{ and } S(x_{n,k}) \xrightarrow{M} y_k.$$

Now, as already showed in proposition 3.2, we conclude that there exists two bounded disked B and B' in X and Y respectively and a sequence $(z_n)_{n \geq 0} \subset X_B$ such that

$$(S(x_n) - y_n)_{n \geq 0} \subset Y_{B'}$$

and, for every $k \in \mathbb{N}^*$, we have

$$p_B(z_k) < \frac{1}{k} \text{ and } p_{B'}(S(x_z) - y_k) < \frac{1}{k}.$$

Since

$$S(x_n - z_n) - y = (S(x_n) - y - y_n) + (y - S(z_n)),$$

the sequence $(S(x_n - z_n))_{n \geq 0}$ bornological converges to y . Since $(x_n - z_n)_{n \geq 0}$ bornological converges to 0. Then, $y \in \sigma(S)$.

Consequently, $\sigma(QS) = \{0\}$.

ii) We shows that $[R\sigma(S)]^{(1)} = \sigma(RS)$.

We have $R(\sigma(S)) \subset \sigma(RS)$, indeed.

Let $y \in \sigma(S)$. There exists $(x_n)_{n \geq 0} \subset X$ such that

$$x_n \xrightarrow{M} y \text{ and } S(x_n) \xrightarrow{M} y.$$

R being bounded, then $R(S(x_n)) \xrightarrow{M} R(y)$. This gives, $R(y) \in \sigma(RS)$. Hence, $\sigma(RS)$ is b-closed (Proposition 3.2). Then

$$[R\sigma(S)]^{(1)} \subset \sigma(RS).$$

Next, to show the converse inclusion, consider the canonical quotient map:

$$Q_0 : Z \longrightarrow Z/[R\sigma(S)]^{(1)}$$

such that

$$Q_0(z) = z + [R\sigma(S)]^{(1)}.$$

Then, Q_0 is bounded. Thus Q_0R is bounded.

On other hand, $Q_0[R\sigma(S)] = \{\bar{0}\}$ where $\bar{0}$ is the class of 0. Then, by Proposition 3.2, Q_0RS is bounded. Therefore,

$$Q_0\sigma(RS) = \{\bar{0}\}.$$

Thus

$$\sigma(RS) \subset [R\sigma(S)]^{(1)}.$$

The proof is complete. ■

Remark 1 In the conditions of Proposition 4.1, the subspace $S^{-1}[\sigma(S)]$ is b-closed in X .

Proof. $S^{-1}[\sigma(S)] = \text{Ker}(QS) = (QS)^{-1}(\{\bar{0}\})$. Since $\sigma(S)$ is b-closed and $Y/\sigma(S)$ is separated, then, $S^{-1}[\sigma(S)]$ is b-closed in X . ■

Theorem 4.1 *Let X and Y two (cbvs) of type M_1 and $S : X \rightarrow Y$ be linear map. Suppose that X is complete and the bornology of Y has a net.*

Let X_0 and Y_0 two subspaces b-closed of X and Y respectively such that $S(X_0) \subset Y_0$.

Let $S_0 : X/X_0 \rightarrow Y/Y_0$ defined by:

$$S_0(x + X_0) = S(x) + Y_0.$$

Then, S_0 is bounded if, and only if, $\sigma(S) \subset Y_0$.

Proof. Suppose that S_0 is bounded. Let $y \in \sigma(S)$. There exists $(x_n)_{n \geq 0} \subset X$ such that $x_n \xrightarrow{M} 0$ and $S(x_n) \xrightarrow{M} y$. Then

$$S_0(x_n + X_0) = S(x_n) + Y_0 \xrightarrow{M} y + Y_0 \text{ and } S_0(x_n + X_0) \xrightarrow{M} S_0(X_0).$$

Y_0 being b-closed, then Y/Y_0 is separated. Consequently, $y + Y_0 = S_0(X_0) \subset Y_0$. Thus, $y \in Y_0$.

Conversely, suppose that $\sigma(S) \subset Y_0$. Consider the canonical quotient map $Q : Y \rightarrow Y/Y_0$. Q is bounded by the definition of the quotient bornology, and we have

$$Q(\sigma(S)) \subset Q(Y_0) = \{\bar{0}\}.$$

Then, QS is bounded, (by Proposition 3.1). Since $S(X_0) \subset Y_0$, we have:

$$QS(X_0) = \{\bar{0}\}.$$

On other hand,

$$S_0(x + X_0) = S(x) + Y_0 = QS(x).$$

Then, S_0 is bounded. ■

Remark 2 In the conditions of Proposition 4.2, we have

$$\sigma(S|X_0) \subset Y_0 \cap \sigma(S),$$

where S/X_0 is the restriction of S in X_0 .

Proposition 4.2 *Let X and Y two (cbvs) of type M_1 such that X is complete and the bornology of Y has a net. Let $Z_1, Z_2, \dots, Z_n$ the complete (cbvs) and $T_1, T_2, \dots, T_n$ the bounded linear maps of Z_j into X such that $X = \sum_{j=1}^n T_j Z_j$. Let $S : X \longrightarrow Y$ be bounded linear map. Then, we have*

$$[\sigma(ST_1) + \dots + \sigma(ST_n)]^{(1)} = \sigma(S).$$

Proof. The proof is establish in two parts.

1°) Suppose that, for every $j \in \{1, 2, \dots, n\}$, ST_j is bounded.

Let $T : Z \longrightarrow X$ defined by

$$T(z_1, \dots, z_n) = T_1(z_1) + \dots + T_n(z_n)$$

such that

$$Z = Z_1 \oplus Z_2 \oplus \dots \oplus Z_n.$$

Since for $j = 1, \dots, n$, T_j is bounded, by definition the bornology of Z , T is bounded.

On other hand we have

$$ST(z) = S(T_1(z_1) + \dots + T_n(z_n)) = ST_1(z_1) + \dots + ST_n(z_n).$$

Then, ST is bounded.

Now, we prove again that S is bounded:

Let $x \in X$. Then $x = T(z)$, where $z \in Z$. Thus, $S(x) = ST(z)$.

Let B a bounded in $X = \sum_{j=1}^n T_j Z_j$. Then, there exists $B_1, B_2, \dots, B_n$ bounded in Z_j such that

$$B \subset T_1 B_1 + T_2 B_2 + \dots + T_n B_n$$

Therefore,

$$S(B) \subset ST_1(B_1) + ST_2(B_2) + \dots + ST_n(B_n).$$

Since ST_j is bounded, for every $j \in \{1, 2, \dots, n\}$, $\sum_{j=1}^n ST_j(B_j)$ is bounded in Y .

Thus, $S(B)$ is bounded in Y . Consequently, S is bounded. Then, $\sigma(S) = \{0\}$.

Since, $\sigma(ST_j) = \{0\}$, $\forall j = 1, 2, \dots, n$. Thus :

$$\left[\sum_{j=1}^n \sigma(ST_j) \right]^{(1)} = \sigma(S)$$

2°) General Case:

We have:

$$\sigma(ST_j) \subset \sigma(S), \quad \forall j = 1, 2, \dots, n.$$

Indeed, let $y \in \sigma(ST_j)$. There is a sequence $(z_j^n)_{n \geq 0} \subset Z_j$ such that

$$z_j^n \xrightarrow{M} 0 \quad \text{et} \quad ST_j(z_j^n) \xrightarrow{M} y.$$

Since, by hypothesis, T_j are bounded, then

$$T_j(z_j^n) \xrightarrow{M} 0 \quad \text{and} \quad S[T_j(z_j^n)] \xrightarrow{M} y.$$

Then, $y \in \sigma(S)$, i.e., $\sigma(ST_j) \subset \sigma(S)$. This gives

$$\sum_{j=1}^n \sigma(ST_j) \subset \sigma(S).$$

Since $\sigma(S)$ is b-closed, we have

$$\left[\sum_{j=1}^n \sigma(ST_j) \right]^{(1)} \subseteq \sigma(S).$$

Next, we shows the converse inclusion.

Let $Q : Y \longrightarrow Y/W$, where $W = \left[\sum_{j=1}^n \sigma(ST_j) \right]^{(1)}$. Then, Q is bounded.

On other hand, we have

$$Q[\sigma(ST_j)] \subset Q(W) = \{\bar{0}\}.$$

Then,

$$Q[\sigma(ST_j)] = \{\bar{0}\}.$$

By Proposition 3.2, we conclude that QST_j are bounded. Therefore, by the first case it results that QS is bounded. Then, $Q[\sigma(S)] = \{\bar{0}\}$, i.e: $\sigma(S) \subset W$. ■

Acknowledgement. The author would like to thank the referee for helpful suggestions and remarks.

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Accepted: 3.05.2013

STABILITY OF NONLINEAR NABLA FRACTIONAL DIFFERENCE EQUATIONS USING FIXED POINT THEOREMS

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Abstract. Difference equations are often used to analyze sampled data systems, in which stability problems are considered to be important. This is evident from a large number of research papers dedicated to it. However, stability results for nonlinear nabla fractional difference equations are not yet reported. The present article discusses stability of nonlinear nabla fractional difference equations of Riemann–Liouville and Caputo type, using fixed point theory.

Keywords: fractional order, difference equation, zero solution, fixed point, stability.

AMS Subject Classification: 39A10, 39A99.

1. Introduction

Fractional calculus deals with the study of fractional order integrals and derivatives and their diverse applications [11]. The analogous theory for discrete fractional calculus was initiated and a series of papers continuing this research has appeared [3], [4], [5], [6], [7], [8], [9], [10], [12], [13].

Qualitative properties of fractional difference equations assume importance in the absence of closed form solutions. One such qualitative property, which has wide applications, is the stability of solutions. Motivated by the application of fixed point theory in the study of stability of differential equations, we use Krasnoselskii fixed point theorem, Schauder fixed point theorem and discrete Arzela–Ascoli theorem to discuss the stability of nonlinear nabla fractional difference equations.

The present article is organized as follows. Section 2 contains basic definitions and results concerning nabla discrete fractional calculus. In Sections 3 and 4, we establish sufficient conditions for stability of nonlinear nabla fractional difference equations of Riemann–Liouville and Caputo type. We illustrate our main results through few examples in Section 5.

2. Nabla discrete fractional calculus

Throughout the article, we consider the discrete time scale [2]

$$\mathbb{T} = \mathbb{N}_a = \{a, a+1, a+2, \dots\},$$

where $a \in \mathbb{R}$ is fixed. For any function $f : \mathbb{N}_a \rightarrow \mathbb{R}$, the backward difference or nabla operator is defined as $\nabla f(t) = f(t) - f(t-1)$ for $t \in \mathbb{N}_{a+1}$.

Definition 2.1. For any real numbers α and t , the α rising function is defined by

$$(2.1) \quad t^{\bar{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \quad t \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}, \quad 0^{\bar{\alpha}} = 0.$$

Lemma 2.1. For any real numbers a and b , the quotient expansion of two gamma functions at infinity is given by

$$\frac{\Gamma(t+a)}{\Gamma(t+b)} = t^{a-b} \left[1 + O\left(\frac{1}{t}\right) \right], \quad |t| \rightarrow \infty.$$

Regarding the rising factorial function we observe the following properties.

Lemma 2.2. Assume the following factorial functions are well defined. For any positive real numbers t , α and β ,

1. $\nabla t^{\bar{\alpha}} = \alpha t^{\overline{\alpha-1}}$.
2. $t^{\bar{\alpha}}(t+\alpha)^{\bar{\beta}} = t^{\overline{\alpha+\beta}}$.
3. $t^{\overline{\alpha+\beta}} = \sum_{j=0}^t C(t, j)(t-j)^{\bar{\alpha}} j^{\bar{\beta}}$.
4. If $\alpha < t \leq r$, then $t^{\overline{-\alpha}} \geq r^{\overline{-\alpha}}$.
5. If $\alpha \geq \beta$, then $t^{\overline{-\alpha}} \leq t^{\overline{-\beta}}$.

6. If $0 < \alpha < 1$, then $(t - \alpha\beta)^{\overline{\alpha\beta}} \geq [(t - \beta)^{\overline{\beta}}]^\alpha$.

7. If $\alpha > 1$, then $[(t + \beta)^{\overline{-\beta}}]^\alpha < \frac{\Gamma(1 + \alpha\beta)}{[\Gamma(1 + \beta)]^\alpha} (t + \alpha\beta)^{\overline{-\alpha\beta}}$.

Proof. The proofs of (1), (2), (3) are straightforward. The proof of (4) follows from Eulers infinite product

$$\Gamma(t) = \frac{1}{t} \prod_{n=1}^{\infty} \frac{(1 + \frac{1}{n})^t}{(1 + \frac{t}{n})}.$$

Consider

$$\begin{aligned} t^{\overline{-\alpha}} &= \frac{\Gamma(t - \alpha)}{\Gamma(t)} = \frac{t}{t - \alpha} \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-\alpha} \left(\frac{n + t}{n + t - \alpha}\right) \\ &\geq \frac{r}{r - \alpha} \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-\alpha} \left(\frac{n + r}{n + r - \alpha}\right) \\ &= \frac{\Gamma(r - \alpha)}{\Gamma(r)} = r^{\overline{-\alpha}}. \end{aligned}$$

The proof of (5) follows from Lemma 2.1. Consider

$$\frac{t^{\overline{-\alpha}}}{t^{\overline{-\beta}}} = \frac{\Gamma(t - \alpha)}{\Gamma(t - \beta)} = t^{\beta - \alpha} \left[1 + O\left(\frac{1}{t}\right)\right] = \frac{1}{t^{\alpha - \beta}} \left[1 + O\left(\frac{1}{t}\right)\right] \leq 1.$$

For the proofs of (6) and (7), we refer [4], [5]. ■

Definition 2.2. (Nabla Fractional Sum [8], [13]) Let $f : \mathbb{N}_a \rightarrow \mathbb{R}$ and $\alpha > 0$ be given. Then the α^{th} -order nabla fractional sum of f is given by

$$(2.2) \quad \nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha-1}} \text{ for } t \in \mathbb{N}_a$$

where $\rho(s) = s - 1$. Also, we define the trivial sum by $\nabla_a^{-0} f(t) = f(t)$ for $t \in \mathbb{N}_a$.

Definition 2.3. (R-L Nabla Fractional Difference [8], [13]) Let $f : \mathbb{N}_a \rightarrow \mathbb{R}$, $\alpha > 0$ be given and let $N \in \mathbb{N}$ be chosen such that $N - 1 < \alpha \leq N$. Then, the α^{th} -order Riemann–Liouville nabla fractional difference of f is given by

$$(2.3) \quad \nabla_a^\alpha f(t) = \nabla^N \nabla_a^{-(N-\alpha)} f(t) \text{ for } t \in \mathbb{N}_{a+N}.$$

For $\alpha = 0$, we get $\nabla_a^0 f(t) = f(t)$ for $t \in \mathbb{N}_a$.

Definition 2.4. (Caputo Nabla Fractional Difference [12]) Let $f : \mathbb{N}_a \rightarrow \mathbb{R}$, $\alpha > 0$ be given and let $N \in \mathbb{N}$ be chosen such that $N - 1 < \alpha \leq N$. Then, the α^{th} -order Caputo nabla fractional difference of f is given by

$$(2.4) \quad \nabla_{a*}^\alpha f(t) = \nabla_a^{-(N-\alpha)} \left[\nabla^N f(t) \right] \text{ for } t \in \mathbb{N}_{a+N}.$$

For $\alpha = 0$, we set $\nabla_{a*}^0 f(t) = f(t)$ for $t \in \mathbb{N}_a$.

Theorem 2.3. [7], [13] (Power Rule) *Let $\alpha > 0$ and $\mu > -1$. Then, for $t \in \mathbb{N}_a$, we have*

$$1. \nabla_a^{-\alpha}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)}(t-a)^{\overline{\alpha+\mu}}.$$

$$2. \nabla_a^{\alpha}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(t-a)^{\overline{\mu-\alpha}}.$$

Let $f(t, r) : \mathbb{N}_a \times \mathbb{R} \rightarrow \mathbb{R}$, $u(t) : \mathbb{N}_a \rightarrow \mathbb{R}$ and $0 < \alpha < 1$. Consider a nonlinear nabla fractional difference equation of Riemann–Liouville type together with an initial condition of the form

$$(2.5) \quad \nabla_{a-1}^{\alpha} u(t) = f(t, u(t)), \quad t \in \mathbb{N}_{a+1},$$

$$(2.6) \quad \nabla_{a-1}^{-(1-\alpha)} u(t) \Big|_{t=a} = u(a) = u(0).$$

From [13], $u(t)$ is a solution of the initial value problem (2.5)–(2.6) if and only if $u(t)$ has the following representation

$$(2.7) \quad u(t) = \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} u_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^t (t-\rho(s))^{\overline{\alpha-1}} f(s, u(s)), \quad t \in \mathbb{N}_a.$$

If we consider a nonlinear nabla fractional difference equation of Caputo type together with an initial condition of the form

$$(2.8) \quad \nabla_{a*}^{\alpha} u(t) = f(t, u(t)), \quad t \in \mathbb{N}_{a+1},$$

$$(2.9) \quad u(a) = u_0.$$

Then from [13], $u(t)$ is a solution of the initial value problem (2.8)–(2.9) if and only if $u(t)$ has the following representation

$$(2.10) \quad u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^t (t-\rho(s))^{\overline{\alpha-1}} f(s, u(s)), \quad t \in \mathbb{N}_a.$$

Definition 2.5. The solution $u = \varphi(t)$ of the initial value problem (2.5)–(2.6) or (2.8)–(2.9) is said to be

1. stable, if given $\epsilon > 0$ and $t_0 \geq 0$, there exists $\delta = \delta(\epsilon, t_0)$ such that $|u_0 - \varphi(t_0)| < \delta \Rightarrow |u(t, x_0, t_0) - \varphi(t)| < \epsilon$ for all $t \geq t_0$.
2. attractive, if there exists $\eta = \eta(t_0)$ such that $|u_0 - \varphi(t_0)| < \eta$ implies $u(t, x_0, t_0) \rightarrow \varphi(t)$ as $t \rightarrow \infty$.
3. asymptotically stable if it is stable and attractive.

Definition 2.6. The space l_a^{∞} is the set of real sequences defined on the set of positive integers where any individual sequence is bounded with respect to the usual supremum norm. Clearly, l_a^{∞} is a Banach space under the supremum norm.

Definition 2.7. A set Ω of sequences in l_a^∞ is uniformly Cauchy (or equi-Cauchy), if for every $\epsilon > 0$, there exists a positive integer N such that $|u(i) - u(j)| < \epsilon$, whenever $i, j > N$ for any $u = u(t)$ in Ω .

Theorem 2.4. (Discrete Arzela-Ascoli's theorem) *A bounded, uniformly Cauchy subset Ω of l_a^∞ is relatively compact.*

Theorem 2.5. (Krasnoselskii's fixed point theorem) *Let S be a nonempty, closed, convex and bounded subset of the Banach space X and let $A : X \rightarrow X$ and $B : S \rightarrow X$ be two operators such that*

1. A is a contraction with constant $L < 1$,
2. B is continuous, BS resides in a compact subset of X ,
3. $[x = Ax + By, y \in S] \implies x \in S$.

Then the operator equation $Ax + Bx = x$ has a solution in S .

Theorem 2.6. (Schauder fixed point theorem) *Let Ω be a closed, convex and nonempty subset of a Banach space X . Let $T : \Omega \rightarrow \Omega$ be a continuous mapping such that $T\Omega$ is a relatively compact subset of X . Then T has at least one fixed point in Ω . That is, there exists an $x \in \Omega$ such that $Tx = x$.*

3. Riemann–Liouville type fractional difference equation

Let l_a^∞ be the set of all real sequences $u = \{u(t)\}_{t=a}^\infty$ with norm $\|u\| = \sup_{t \in \mathbb{N}_a} |u(t)|$, then l_a^∞ is a Banach space. Define the operators

$$(3.1) \quad Pu(t) = \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} u_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^t (t-\rho(s))^{\overline{\alpha-1}} f(s, u(s)).$$

$$(3.2) \quad Au(t) = \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} u_0.$$

$$(3.3) \quad Bu(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^t (t-\rho(s))^{\overline{\alpha-1}} f(s, u(s)).$$

Lemma 3.1. *Let the following condition be satisfied:*

- (I) *There exist constants $\beta_1 \in (\alpha, 1)$ and $L_1 \geq 0$ such that*

$$(3.4) \quad |f(t, u(t))| \leq L_1 (t-a)^{\overline{-\beta_1}}, \text{ for } t \in \mathbb{N}_{a+1}.$$

Define

$$(3.5) \quad S_1 = \{u(t) : |u(t)| \leq (t-a)^{\overline{-\gamma_1}} \text{ for } t \in \mathbb{N}_{a+n_1+1}\}$$

where $\gamma_1 = (-1/2)(\alpha - \beta_1)$ and $n_1 \in \mathbb{N}$ such that

$$(3.6) \quad \frac{|u_0|}{\Gamma(\alpha)}(n_1 - \gamma_1)^{\overline{(1/2)(\alpha + \beta_1) - 1}} + \frac{L_1 \Gamma(1 - \beta_1)}{\Gamma(1 + \alpha - \beta_1)}(n_1 - \gamma_1)^{-\overline{\gamma_1}} \leq 1.$$

Then the operator B is continuous and BS_1 is a compact subset of $\mathbb{R}$ for $t \in \mathbb{N}_{a+n_1+1}$.

Proof. Clearly, $\gamma_1 > 0$. Using Lemma 2.1, for $t \in \mathbb{N}_{a+1}$,

$$(t - a)^{-\overline{\gamma_1}} = \frac{\Gamma(t - a - \gamma_1)}{\Gamma(t - a)} = (t - a)^{-\gamma_1} \left[1 + O\left(\frac{1}{t - a}\right) \right].$$

Further, $(t - a)^{-\overline{\gamma_1}} \rightarrow 0$ as $t \rightarrow \infty$. Then, there exists $n_1 \in \mathbb{N}$ such that $(t - a)^{-\overline{\gamma_1}} \rightarrow 0$ for $(t - a) > n_1$. Clearly $\gamma_1 < 1 - (1/2)(\alpha + \beta_1)$ and $n_1 - \gamma_1 < t - a$. Using Lemma 2.2, we get $(n_1 - \gamma_1)^{\overline{(1/2)(\alpha + \beta_1) - 1}} \leq (n_1 - \gamma_1)^{-\overline{\gamma_1}} \leq (t - a)^{-\overline{\gamma_1}} \rightarrow 0$ and $(n_1 - \gamma_1)^{-\overline{\gamma_1}} \leq (t - a)^{-\overline{\gamma_1}} \rightarrow 0$. Thus, we have the inequality (3.6) which implies that the set S_1 exists. Now, we show that B maps S_1 in S_1 . Clearly, S_1 is a closed, bounded, and convex subset of $\mathbb{R}$. Applying condition (I), Theorem 2.3, Lemma 2.2 and (3.4), we have

$$\begin{aligned} |Bu(t)| &\leq \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha-1}} |f(s, u(s))| \\ &\leq \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha-1}} L_1 (s - a)^{-\overline{\beta_1}} \\ &= L_1 \nabla_a^{-\alpha} (t - a)^{-\overline{\beta_1}} \\ &= \frac{L_1 \Gamma(1 - \beta_1)}{\Gamma(1 + \alpha - \beta_1)} (t - a)^{\overline{\alpha - \beta_1}} \\ &= \frac{L_1 \Gamma(1 - \beta_1)}{\Gamma(1 + \alpha - \beta_1)} (t - a - \gamma_1)^{-\overline{\gamma_1}} (t - a)^{-\overline{\gamma_1}} \\ &\leq \frac{L_1 \Gamma(1 - \beta_1)}{\Gamma(1 + \alpha - \beta_1)} (n_1 - \gamma_1)^{-\overline{\gamma_1}} (t - a)^{-\overline{\gamma_1}} \\ &\leq (t - a)^{-\overline{\gamma_1}}, \end{aligned}$$

implies $BS_1 \subset S_1$ for $t \in \mathbb{N}_{a+n_1+1}$. Next, we show that B is continuous on S_1 . Let $\epsilon > 0$ be given. Then, there exists $T_1 \in \mathbb{N}$ and $T_1 \geq n_1$ such that, for $t \in \mathbb{N}_{a+T_1+1}$,

$$(3.7) \quad \frac{L_1 \Gamma(1 - \beta_1)}{\Gamma(1 + \alpha - \beta_1)} (t - a)^{\overline{\alpha - \beta_1}} < \frac{\epsilon}{2}.$$

Since $(t - a) > T_1 \geq n_1$ and $0 < \gamma_1 < \beta_1 - \alpha < 1$, we have

$$(t - a)^{\overline{\alpha - \beta_1}} \leq (t - a)^{-\overline{\gamma_1}} \rightarrow 0,$$

which implies the validity of (3.7). Let $\{u_n\}$ be a sequence such that $u_n \rightarrow u$. For $t \in \{a + n_1 + 1, a + n_1 + 2, \dots, a + T_1\}$, applying the continuity of f , we have

$$\begin{aligned} & |Bu_n(t) - Bu(t)| \\ & \leq \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha-1}} |f(s, u_n(s)) - f(s, u(s))| \\ & \leq \left[\frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha-1}} \right] \left[\max_{s \in \{a+1, a+2, \dots, a+T_1\}} |f(s, u_n(s)) - f(s, u(s))| \right] \\ & = \frac{(t-a)^{\overline{\alpha}}}{\Gamma(\alpha+1)} \left[\max_{s \in \{a+1, a+2, \dots, a+T_1\}} |f(s, u_n(s)) - f(s, u(s))| \right] \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

For $t \in \mathbb{N}_{a+T_1+1}$,

$$\begin{aligned} |Bu_n(t) - Bu(t)| & \leq \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha-1}} [|f(s, u_n(s))| + |f(s, u(s))|] \\ & \leq \frac{2L_1}{\Gamma(\alpha)} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha-1}} (s-a)^{-\overline{\beta_1}} \\ & = 2L_1 \nabla_a^{-\alpha} (t-a)^{\overline{-\beta_1}} = \frac{2L_1 \Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)} (t-a)^{\overline{\alpha-\beta_1}} < \epsilon. \end{aligned}$$

Thus, for all $t \in \mathbb{N}_{a+n_1+1}$, $|Bu_n(t) - Bu(t)| \rightarrow 0$ as $n \rightarrow \infty$ implies B is continuous. Finally, we show that BS_1 is relatively compact. Let $t_1, t_2 \in \mathbb{N}_{a+T_1+1}$ such that $t_2 > t_1$. Then, we have

$$\begin{aligned} & |Bu(t_1) - Bu(t_2)| \\ & = \left| \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t_1} (t_1 - \rho(s))^{\overline{\alpha-1}} f(s, u(s)) - \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t_2} (t_2 - \rho(s))^{\overline{\alpha-1}} f(s, u(s)) \right| \\ & \leq \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t_1} (t_1 - \rho(s))^{\overline{\alpha-1}} |f(s, u(s))| + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t_2} (t_2 - \rho(s))^{\overline{\alpha-1}} |f(s, u(s))| \\ & \leq \frac{L_1 \Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)} (t_1 - a)^{\overline{\alpha-\beta_1}} + \frac{L_1 \Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)} (t_2 - a)^{\overline{\alpha-\beta_1}} < \epsilon. \end{aligned}$$

Thus, $\{Bu : u \in S_1\}$ is a bounded and uniformly Cauchy subset implies BS_1 is relatively compact. Hence the proof. $\blacksquare$

Lemma 3.2. Assume that condition (I) holds. Then a solution of (2.5) is in S_1 for $t \in \mathbb{N}_{a+n_1+1}$.

Proof. Clearly, A is a contraction mapping with the constant 0, implies condition (1) of Theorem 2.5 holds. Using Lemma 3.1, the operator B is continuous and BS_1 is a compact subset of $\mathbb{R}$, implies condition (2) of Theorem 2.5 holds. Also

$Pu = Au + Bu$ and u is the solution of (2.5)–(2.6) if it is a fixed point of the operator P . Now we prove condition (3) of Theorem 2.5. Let us suppose, for a fixed $v \in S_1$, $u = Au + Bv$. Applying condition (I) and (3.6), we have

$$\begin{aligned}
 |u(t)| &\leq |Au(t)| + |Bv(t)| \\
 &\leq \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} |u_0| + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^t (t-\rho(s))^{\overline{\alpha-1}} |f(s, v(s))| \\
 &\leq \frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)} |u_0| + \frac{L_1 \Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)} (t_1-a)^{\overline{\alpha-\beta_1}} \\
 (3.8) \quad &= \frac{|u_0|}{\Gamma(\alpha)} (t-a)^{\overline{-\gamma_1}} (t-a-\gamma_1)^{\overline{1/2(\alpha+\beta_1)-1}} \\
 &+ \frac{L_1 \Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)} (t-a)^{\overline{-\gamma_1}} (t-a-\gamma_1)^{\overline{-\gamma_1}} \\
 &= \left[\frac{|u_0|}{\Gamma(\alpha)} (n_1-\gamma_1)^{\overline{1/2(\alpha+\beta_1)-1}} \right. \\
 &\quad \left. + \frac{L_1 \Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)} (n_1-\gamma_1)^{\overline{-\gamma_1}} \right] (t-a)^{\overline{-\gamma_1}} \\
 &\leq (t-a)^{\overline{-\gamma_1}},
 \end{aligned}$$

for $t \in \mathbb{N}_{a+n_1+1}$. Thus, $u \in S_1$. According to Theorem 2.5, u is a fixed point of P implies u is the solution of (2.5)–(2.6). Hence the proof. ■

Theorem 3.3. *Assume that condition (I) holds, then the zero solution of (2.5) is attractive.*

Proof. Using Lemma 3.1, the solutions of (2.5) exist in S_1 and tend to 0 as $t \rightarrow \infty$. Hence the proof. ■

Theorem 3.4. *Assume that the following condition is satisfied:*

(II) *There exist constants $\beta_2 \in (\alpha, 1)$ and $L_2 \geq 0$ such that*

$$(3.9) \quad |f(t, u(t)) - f(t, v(t))| \leq L_2 (t-a)^{\overline{-\beta_2}} |u-v|, \quad \text{for } t \in \mathbb{N}_{a+1}.$$

Then, the solutions of (2.5) are stable provided that

$$(3.10) \quad c = L_2 \Gamma(1-\beta_2) < 1.$$

Proof. Let $u(t)$ be the solution of (2.5)–(2.6), and let $\widetilde{u}(t)$ be the solution of (2.5) with the initial condition $\widetilde{u}(0) = \widetilde{u}_0$. Applying condition (II), we have

$$\begin{aligned}
|u(t) - \tilde{u}(t)| &\leq \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} |u_0 - \tilde{u}_0| \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^t (t-\rho(s))^{\overline{\alpha-1}} |f(s, u(s)) - f(s, \tilde{u}(s))| \\
&\leq \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} |u_0 - \tilde{u}_0| + \frac{L_2}{\Gamma(\alpha)} \sum_{s=a+1}^t (t-\rho(s))^{\overline{\alpha-1}} (s-a)^{\overline{-\beta_2}} |u - \tilde{u}| \\
&= \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} |u_0 - \tilde{u}_0| + L_2 \nabla_a^{-\alpha} (t-a)^{\overline{-\beta_2}} |u - \tilde{u}| \\
&= \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} |u_0 - \tilde{u}_0| + \frac{L_2 \Gamma(1-\beta_2)}{\Gamma(1+\alpha-\beta_2)} (t-a)^{\overline{\alpha-\beta_2}} |u - \tilde{u}| \\
&\leq \frac{(2)^{\overline{\alpha-1}}}{\Gamma(\alpha)} |u_0 - \tilde{u}_0| + \frac{L_2 \Gamma(1-\beta_2)}{\Gamma(1+\alpha-\beta_2)} (1)^{\overline{\alpha-\beta_2}} |u - \tilde{u}| \\
&= \alpha |u_0 - \tilde{u}_0| + L_2 \Gamma(1-\beta_2) |u - \tilde{u}| \\
&< \alpha |u_0 - \tilde{u}_0| + c |u - \tilde{u}|,
\end{aligned}$$

which implies that

$$(3.11) \quad |u - \tilde{u}| < \frac{\alpha}{1-c} |u - \tilde{u}_0|.$$

For any given $\varepsilon > 0$, let $\delta = \frac{(1-c)}{\alpha} \varepsilon$. Then, $|u - \tilde{u}_0| < \delta$ implies $|u - \tilde{u}| < \varepsilon$, which shows that the solutions of (2.5) are stable. This completes the proof. ■

Theorem 3.5. Assume that conditions (I) and (II) hold, then the zero solution of (2.5) is asymptotically stable provided that (3.10) holds.

Theorem 3.6. Assume that the following condition is satisfied:

(III) There exist constants $\beta_3 \in (\alpha, \frac{1+\alpha}{2})$, $\gamma_2 = \frac{1}{2}(1-\alpha)$ and $L_3 \geq 0$ such that

$$(3.12) \quad |f(t, u(t))| \leq L_3 (t-a+\gamma_2)^{\overline{-\beta_3}} |u(t)| \quad \text{for } t \in \mathbb{N}_{a+1}.$$

Then, the zero solution of (2.5) is attractive.

Proof. Set $S_2 = \{u(t) : |u(t)| \leq (t-a)^{\overline{-\gamma_2}} \text{ for } t \in \mathbb{N}_{a+n_2+1}\}$, where $n_2 \in \mathbb{N}$ such that

$$(3.13) \quad \frac{|u_0|}{\Gamma(\alpha)} (n_2 - \gamma_2)^{\overline{-\gamma_2}} + \frac{L_3 \Gamma(1-\beta_3-\gamma_2)}{\Gamma(1+\alpha-\beta_3-\gamma_2)} (n_2 - \gamma_2)^{\overline{\alpha-\beta_3}} \leq 1.$$

Clearly $\beta_3 - \alpha > 0$. Using Lemma 2.1, for $t \in \mathbb{N}_{a+1}$,

$$(t-a)^{\overline{\alpha-\beta_3}} = \frac{\Gamma(t-a+\alpha-\beta_3)}{\Gamma(t-a)} = (t-a)^{-(\beta_3-\alpha)} \left[1 + O\left(\frac{1}{t-a}\right) \right].$$

Further, $(t-a)^{\overline{-\gamma_2}} \rightarrow 0$ as $t \rightarrow \infty$. Then, there exists $n_2 \in \mathbb{N}$ such that $(t-a)^{\overline{\alpha-\beta_3}} \rightarrow 0$ for $(t-a) > n_2$. Clearly, $\beta_3 - \alpha < \gamma_2$ and $n_2 - \gamma_2 < t - a$.

Using Lemma 2.2, we get $(n_2 - \gamma_2)^{-\overline{\gamma_2}} \leq (n_2 - \gamma_2)^{\overline{\alpha - \beta_3}} \leq (t - a)^{\overline{\alpha - \beta_3}} \rightarrow 0$ and $(n_2 - \gamma_2)^{\overline{\alpha - \beta_3}} \leq (t - a)^{\overline{\alpha - \beta_3}} \rightarrow 0$. Thus, we have the inequality (3.13) which implies that the set S_2 exists. Now, we prove condition (3) of Theorem 2.5. Suppose, for a fixed $v \in S_2$ and for all $u \in \mathbb{R}$, $u = Au + Bv$. Applying condition (III), we have

$$\begin{aligned}
|u(t)| &\leq |Au(t)| + |Bv(t)| \\
&\leq \frac{|u_0|}{\Gamma(\alpha)}(t - a + 1)^{\overline{\alpha - 1}} + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha - 1}} |f(s, v(s))| \\
&\leq \frac{|u_0|}{\Gamma(\alpha)}(t - a + 1)^{\overline{\alpha - 1}} + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha - 1}} L_3(s - a + \gamma_2)^{\overline{-\beta_3}} |v(s)| \\
&\leq \frac{|u_0|}{\Gamma(\alpha)}(t - a + 1)^{\overline{\alpha - 1}} + \frac{L_3}{\Gamma(\alpha)} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha - 1}} (s - a + \gamma_2)^{\overline{-\beta_3}} (s - a)^{\overline{-\gamma_2}} \\
&= \frac{|u_0|}{\Gamma(\alpha)}(t - a + 1)^{\overline{\alpha - 1}} + \frac{L_3}{\Gamma(\alpha)} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha - 1}} (s - a)^{\overline{-\beta_3 - \gamma_2}} \\
&\leq \frac{|u_0|}{\Gamma(\alpha)}(t - a)^{\overline{\alpha - 1}} + \frac{L_3 \Gamma(1 - \beta_3 - \gamma_2)}{\Gamma(1 + \alpha - \beta_3 - \gamma_2)} (t - a)^{\overline{\alpha - \beta_3 - \gamma_2}} \\
&= \frac{|u_0|}{\Gamma(\alpha)}(t - a)^{\overline{-\gamma_2 - \gamma_2}} + \frac{L_3 \Gamma(1 - \beta_3 - \gamma_2)}{\Gamma(1 + \alpha - \beta_3 - \gamma_2)} (t - a)^{\overline{\alpha - \beta_3 - \gamma_2}} \\
&= \frac{|u_0|}{\Gamma(\alpha)}(t - a)^{\overline{-\gamma_2}} (t - a - \gamma_2)^{\overline{-\gamma_2}} \\
&+ \frac{L_3 \Gamma(1 - \beta_3 - \gamma_2)}{\Gamma(1 + \alpha - \beta_3 - \gamma_2)} (t - a)^{\overline{-\gamma_2}} (t - a - \gamma_2)^{\overline{\alpha - \beta_3}} \\
&= \left[\frac{|u_0|}{\Gamma(\alpha)} (t - a - \gamma_2)^{\overline{-\gamma_2}} + \frac{L_3 \Gamma(1 - \beta_3 - \gamma_2)}{\Gamma(1 + \alpha - \beta_3 - \gamma_2)} (t - a - \gamma_2)^{\overline{\alpha - \beta_3}} \right] (t - a)^{\overline{-\gamma_2}} \\
&\leq \left[\frac{|u_0|}{\Gamma(\alpha)} (n_2 - \gamma_2)^{\overline{-\gamma_2}} + \frac{L_3 \Gamma(1 - \beta_3 - \gamma_2)}{\Gamma(1 + \alpha - \beta_3 - \gamma_2)} (n_2 - \gamma_2)^{\overline{\alpha - \beta_3}} \right] (t - a)^{\overline{-\gamma_2}} \\
&\leq (t - a)^{\overline{-\gamma_2}}
\end{aligned}$$

for $t \in \mathbb{N}_{a+n_2+1}$. Thus, condition (3) of Theorem 2.5 holds. The proof of condition (2) of Theorem 2.5 is similar to that of Lemma 3.1, and we omit it. Therefore, by Theorem 2.5, P has a fixed point $u \in S_2$ implies u is the solution of (2.5)–(2.6). Moreover, all functions in S_2 tend to 0 as $t \rightarrow \infty$, then the zero solution of (2.5) tends to zero as $t \rightarrow \infty$, which shows that the zero solution of (2.5) is attractive. This completes the proof. $\blacksquare$

Theorem 3.7. Assume that conditions (II) and (III) hold, then the zero solution of (2.5) is asymptotically stable provided that (3.10) holds.

Theorem 3.8. Assume that the following condition is satisfied:

(IV) There exist constants $\eta \in (0, 1)$, $\beta_4 \in (\alpha, \frac{2+\alpha\eta}{2+\eta})$ and $L_4 \geq 0$ such that

$$(3.14) \quad |f(t, u(t))| \leq L_4(t - a + 1)^{\overline{-\beta_4}} \left| u \left(t + \frac{\beta_4 - \alpha}{2} \right) \right|^\eta \text{ for } t \in \mathbb{N}_{a+1}.$$

Then, the zero solution of (2.5) is attractive.

Proof. Set

$$(3.15) \quad S_3 = \{u(t) : |u(t)| \leq (t-a)^{-\overline{\gamma_3}} \text{ for } t \in \mathbb{N}_{a+n_3+1}\}$$

where $\gamma_3 = (1/2)(\beta_4 - \alpha)$ and $n_3 \in \mathbb{N}$ satisfies that

$$(3.16) \quad \frac{|u_0|}{\Gamma(\alpha)}(n_3 - \gamma_3)^{\overline{\beta_4 - \gamma_3 - 1}} + \frac{L_4 \Gamma(1 - \beta_4 - \gamma_3 \eta)}{\Gamma(1 + \alpha - \beta_4 - \gamma_3 \eta)}(n_3 - \gamma_3)^{-\overline{\gamma_3}} \leq 1$$

Clearly $\gamma_3 > 0$. Using Lemma 2.1, for $t \in \mathbb{N}_{a+1}$,

$$(t-a)^{-\overline{\gamma_3}} = \frac{\Gamma(t-a-\gamma_3)}{\Gamma(t-a)} = (t-a)^{-\gamma_3} \left[1 + O\left(\frac{1}{t-a}\right) \right].$$

Further, $(t-a)^{-\overline{\gamma_3}} \rightarrow 0$ as $t \rightarrow \infty$. Then, there exists $n_3 \in \mathbb{N}$ such that $(t-a)^{-\overline{\gamma_3}} \rightarrow 0$ for $(t-a) > n_3$. Clearly $\gamma_3 < 1 + \gamma_3 - \beta_4$ and $n_3 - \gamma_3 < t-a$. Using Lemma 2.2, we get $(n_3 - \gamma_3)^{\overline{\beta_4 - \gamma_3 - 1}} \leq (n_3 - \gamma_3)^{-\overline{\gamma_3}} \leq (t-a)^{-\overline{\gamma_3}} \rightarrow 0$ and $(n_3 - \gamma_3)^{-\overline{\gamma_3}} \leq (t-a)^{-\overline{\gamma_3}} \rightarrow 0$. Thus, we have the inequality (3.16) which implies that the set S_3 exists. Now, we prove condition (3) of Theorem 2.5 only, and the remaining part of the proof is similar to that of Theorem 3.6. Since $\eta \in (0, 1)$, $\beta_4 \in \left(\alpha, \frac{2+\alpha\eta}{2+\eta}\right)$ and $\gamma_3 = (1/2)(\beta_4 - \alpha)$, then $\gamma_3, \gamma_3\eta, \alpha + \gamma_3 \in (0, 1)$ and $\beta_4 + \gamma_3\eta \in (\alpha, 1)$. Suppose, for a fixed $v \in S_3$ and for all $u \in \mathbb{R}$, $u = Au + Bv$. Applying condition (IV), Lemma 2.2 and (3.16), we have

$$\begin{aligned} |u(t)| &\leq |Au(t)| + |Bv(t)| \\ &\leq \frac{|u_0|}{\Gamma(\alpha)}(t-a+1)^{\overline{\alpha-1}} + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^t (t-\rho(s))^{\overline{\alpha-1}} L_4(s-a+1)^{-\overline{\beta_4}} \left| v\left(s + \frac{\beta_4 - \alpha}{2}\right) \right|^\eta \\ &\leq \frac{|u_0|}{\Gamma(\alpha)}(t-a+1)^{\overline{\alpha-1}} + \frac{L_4}{\Gamma(\alpha)} \sum_{s=a+1}^t (t-\rho(s))^{\overline{\alpha-1}} (s-a+\gamma_3\eta)^{-\overline{\beta_4}} ((s-a+\gamma_3)^{-\overline{\gamma_3}})^\eta \\ &\leq \frac{|u_0|}{\Gamma(\alpha)}(t-a+1)^{\overline{\alpha-1}} + \frac{L_4}{\Gamma(\alpha)} \sum_{s=a+1}^t (t-\rho(s))^{\overline{\alpha-1}} (s-a+\gamma_3\eta)^{-\overline{\beta_4}} (s-a+\gamma_3\eta)^{-\overline{\gamma_3\eta}} \\ &\leq \frac{|u_0|}{\Gamma(\alpha)}(t-a)^{\overline{\alpha-1}} + \frac{L_4}{\Gamma(\alpha)} \sum_{s=a+1}^t (t-\rho(s))^{\overline{\alpha-1}} (s-a+\gamma_3\eta)^{-\overline{\beta_4 - \gamma_3\eta}} \\ &\leq \frac{|u_0|}{\Gamma(\alpha)}(t-a)^{\overline{\alpha-1}} + \frac{L_4}{\Gamma(\alpha)} \sum_{s=a+1}^t (t-\rho(s))^{\overline{\alpha-1}} (s-a)^{-\overline{\beta_4 - \gamma_3\eta}} \\ &= \frac{|u_0|}{\Gamma(\alpha)}(t-a)^{\overline{\beta_4 - \gamma_3 - \gamma_3 - 1}} + \frac{L_4 \Gamma(1 - \beta_4 - \gamma_3 \eta)}{\Gamma(1 + \alpha - \beta_4 - \gamma_3 \eta)}(t-a)^{\overline{\alpha - \beta_4 - \gamma_3 \eta}} \\ &\leq \frac{|u_0|}{\Gamma(\alpha)}(t-a)^{\overline{\beta_4 - \gamma_3 - \gamma_3 - 1}} + \frac{L_4 \Gamma(1 - \beta_4 - \gamma_3 \eta)}{\Gamma(1 + \alpha - \beta_4 - \gamma_3 \eta)}(t-a)^{\overline{\alpha - \beta_4}} \end{aligned}$$

$$\begin{aligned}
&= \frac{|u_0|}{\Gamma(\alpha)}(t-a)^{\overline{\beta_4-\gamma_3-\gamma_3-1}} + \frac{L_4\Gamma(1-\beta_4-\gamma_3\eta)}{\Gamma(1+\alpha-\beta_4-\gamma_3\eta)}(t-a)^{\overline{-\gamma_3-\gamma_3}} \\
&= \frac{|u_0|}{\Gamma(\alpha)}(t-a)^{\overline{-\gamma_3}}(t-a-\gamma_3)^{\overline{\beta_4-\gamma_3-1}} + \frac{L_4\Gamma(1-\beta_4-\gamma_3\eta)}{\Gamma(1+\alpha-\beta_4-\gamma_3\eta)}(t-a)^{\overline{-\gamma_3}}(t-a-\gamma_3)^{\overline{-\gamma_3}} \\
&= \left[\frac{|u_0|}{\Gamma(\alpha)}(t-a-\gamma_3)^{\overline{\beta_4-\gamma_3-1}} + \frac{L_4\Gamma(1-\beta_4-\gamma_3\eta)}{\Gamma(1+\alpha-\beta_4-\gamma_3\eta)}(t-a-\gamma_3)^{\overline{-\gamma_3}} \right] (t-a)^{\overline{-\gamma_3}} \\
&\leq \left[\frac{|u_0|}{\Gamma(\alpha)}(n_3-\gamma_3)^{\overline{\beta_4-\gamma_3-1}} + \frac{L_4\Gamma(1-\beta_4-\gamma_3\eta)}{\Gamma(1+\alpha-\beta_4-\gamma_3\eta)}(n_3-\gamma_3)^{\overline{-\gamma_3}} \right] (t-a)^{\overline{-\gamma_3}} \\
&\leq (t-a)^{\overline{-\gamma_3}}
\end{aligned}$$

for $t \in \mathbb{N}_{a+n_3+1}$. Thus, condition (3) of Theorem 2.5 holds. Hence the proof. ■

4. Caputo type fractional difference equation

Let l_a^∞ be the set of all real sequences $u = \{u(t)\}_{t=a}^\infty$ with norm $\|u\| = \sup_{t \in \mathbb{N}_a} |u(t)|$, then l_a^∞ is a Banach space. Define the operator

$$(4.1) \quad Tu(t) = u_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^t (t-\rho(s))^{\overline{\alpha-1}} f(s, u(s)),$$

Obviously, $u(t)$ is a solution of (2.8)–(2.9) if it is a fixed point of the operator T .

Lemma 4.1. *Assume that the following condition is satisfied:*

(V) *there exist constants γ_4 , $L_5 > 0$ such that*

$$(4.2) \quad \left| u_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^t (t-\rho(s))^{\overline{\alpha-1}} f(s, u(s)) \right| \leq L_5(t-a)^{\overline{-\gamma_4}} \text{ for } t \in \mathbb{N}_{a+1}.$$

Then there exists a solution for (2.8).

Proof. Define the set

$$(4.3) \quad S_4 = \{u(t) : |u(t)| \leq L_5(t-a)^{\overline{-\gamma_4}} \text{ for } t \in \mathbb{N}_{a+1}\}.$$

Clearly, S_4 is a closed, bounded and convex subset of $\mathbb{R}$. Using Lemma 2.1, we have

$$(t-a)^{\overline{-\gamma_4}} = \frac{\Gamma(t-a-\gamma_4)}{\Gamma(t-a)} = (t-a)^{-\gamma_4} \left[1 + O\left(\frac{1}{t-a}\right) \right]$$

for $t \in \mathbb{N}_{a+1}$. Then, $(t-a)^{\overline{-\gamma_4}} \rightarrow 0$ as $t \rightarrow \infty$. To prove that T has a fixed point, first we show that T maps S_4 in S_4 . For $t \in \mathbb{N}_{a+1}$, from condition (V), we have $|Tu(t)| \leq L_5(t-a)^{\overline{-\gamma_4}}$ implies $TS_4 \subset S_4$. Next, we show that T is continuous on S_4 . Let $\epsilon > 0$ be given. Then there exists a $N_1 \in \mathbb{N}_{a+1}$ such that $t > N_1$ and $L_5(t-a)^{\overline{-\gamma_4}} < \frac{\epsilon}{2}$. Let $\{u_n\}$ be a sequence such that $u_n \rightarrow u$. For $t \in \{a+1, a+2, a+3, \dots, N_1\}$, applying the continuity of f , we have

$$\begin{aligned}
 |Tu_n(t) - Tu(t)| &\leq \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha-1}} |f(s, u_n(s)) - f(s, u(s))| \\
 &\leq \left[\frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha-1}} \right] \max_{s \in \{a+1, a+2, a+3, \dots, N_1\}} |f(s, u_n(s)) - f(s, u(s))| \\
 &= \frac{\Gamma(t - a + \alpha)}{\Gamma(\alpha + 1)\Gamma(t - a)} \max_{s \in \{a+1, a+2, a+3, \dots, N_1\}} |f(s, u_n(s)) - f(s, u(s))| \rightarrow 0 \\
 &\quad \text{as } n \rightarrow \infty.
 \end{aligned}$$

For $t \in \{N_1 + 1, N_1 + 2, \dots\}$, we have

$$\begin{aligned}
 |Tu_n(t) - Tu(t)| &\leq \left| \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha-1}} (f(s, u_n(s)) - f(s, u(s))) \right| \\
 &\leq 2L_5(t - a)^{\overline{-\gamma_4}} < \epsilon.
 \end{aligned}$$

Thus, for all $t \in \mathbb{N}_{a+1}$, $|Tu_n(t) - Tu(t)| \rightarrow 0$ as $n \rightarrow \infty$, which means that T is continuous. Finally, we show that TS_4 is relatively compact. Let $t_1, t_2 \in \mathbb{N}_{a+1}$ and $t_2 > t_1 \geq N_1$, we have

$$\begin{aligned}
 |Tu(t_2) - Tu(t_1)| &\leq \left| \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t_2} (t_2 - \rho(s))^{\overline{\alpha-1}} f(s, u(s)) \right| \\
 (4.4) \quad &+ \left| \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t_1} (t_1 - \rho(s))^{\overline{\alpha-1}} f(s, u(s)) \right| \\
 &\leq L_5(t_2 - a)^{\overline{-\gamma_1}} + L_5(t_1 - a)^{\overline{-\gamma_1}} < \epsilon.
 \end{aligned}$$

Therefore, $\{Tu : u \in S_4\}$ is a bounded and uniformly Cauchy subset. Hence, by Theorem 2.4, TS_4 is relatively compact. According to Theorem 2.6, T has a fixed point in S_4 which is the solution of (2.8) - (2.9). Hence the proof. ■

Theorem 4.2. *Assume that condition (V) holds. Then the zero solution of (2.8) is attractive.*

Proof. By Lemma 4.1, the solutions of (2.8) exist and are in S_4 . But all functions in S_4 tend to 0 as $t \rightarrow \infty$. Then, the zero solution of (2.8) tend to zero as $t \rightarrow \infty$. This completes the proof. ■

Theorem 4.3. *Assume that the following condition is satisfied:*

(VI) *there exist constants $\gamma_5 \in (\alpha, 1)$ and $L_6 > 0$ such that*

$$(4.5) \quad |f(t, u(t)) - f(t, v(t))| \leq L_6(t - a)^{\overline{-\gamma_5}} |u - v| \text{ for } t \in \mathbb{N}_{a+1}.$$

Then, the solutions of (2.8) are stable provided that

$$(4.6) \quad c = L_6\Gamma(1 - \gamma_5) < 1.$$

Proof. Let $u(t)$ be the solution of (2.8)–(2.9), and let $\widetilde{u}(t)$ be the solution of (2.8) satisfying the initial condition $\widetilde{u}(0) = \widetilde{u}_0$. Applying condition (VI), we have

$$\begin{aligned}
|u(t) - \widetilde{u}(t)| &\leq |u_0 - \widetilde{u}_0| + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha-1}} |f(s, u(s)) - f(s, \widetilde{u}(s))| \\
&\leq |u_0 - \widetilde{u}_0| + \frac{L_6}{\Gamma(\alpha)} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha-1}} (s - a)^{\overline{-\gamma_5}} |u - \widetilde{u}| \\
&= |u_0 - \widetilde{u}_0| + L_6 \nabla_a^{-\alpha} (t - a)^{\overline{-\gamma_5}} |u - \widetilde{u}| \\
&= |u_0 - \widetilde{u}_0| + \frac{L_6 \Gamma(1 - \gamma_5)}{\Gamma(1 + \alpha - \gamma_5)} (t - a)^{\overline{\alpha - \gamma_5}} |u - \widetilde{u}| \\
&\leq |u_0 - \widetilde{u}_0| + L_6 \Gamma(1 - \gamma_5) |u - \widetilde{u}| \\
&< |u_0 - \widetilde{u}_0| + c |u - \widetilde{u}|
\end{aligned}$$

which implies that

$$(4.7) \quad |u - \widetilde{u}| < \frac{\alpha}{1 - c} |u - \widetilde{u}_0|.$$

For any given $\varepsilon > 0$, let $\delta = \frac{(1-c)}{\alpha} \varepsilon$. Then, $|u - \widetilde{u}_0| < \delta$ implies $|u - \widetilde{u}| < \varepsilon$, which shows that the solutions of (2.8) are stable. This completes the proof. ■

Theorem 4.4. *Assume that conditions (V) and (VI) hold. Then, the zero solution of (2.8) is asymptotically stable provided that (4.6) holds.*

Lemma 4.5. *Assume that the following condition is satisfied:*

(VII) *there exist constants $\gamma_6 \in (\alpha, 1)$ and $L_7 > 0$ such that*

$$(4.8) \quad \left| \frac{(t - a)^{\overline{\alpha}}}{\Gamma(1 - \alpha)} u_0 + f(t, u(t)) \right| \leq L_7 (t - a)^{\overline{-\gamma_6}} \text{ for } t \in \mathbb{N}_{a+1}.$$

Then, there exists a solution for (2.8).

Proof. Define the set

$$(4.9) \quad S_5 = \left\{ u(t) : |u(t)| \leq \frac{L_7 \Gamma(1 - \gamma_6)}{\Gamma(1 + \alpha - \gamma_6)} (t - a)^{\overline{\alpha - \gamma_6}} \right\} \text{ for } t \in \mathbb{N}_{a+1}.$$

From the definition, clearly, S_5 is a closed, bounded and convex subset of $\mathbb{R}$. First, we show that T maps S_5 in S_5 . From condition (VII), we have

$$\begin{aligned}
 |Tu(t)| &= \left| u_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha-1}} f(s, u(s)) \right| \\
 &= \left| \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha-1}} \frac{(s-a)^{\overline{\alpha}}}{\Gamma(1-\alpha)} u_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha-1}} f(s, u(s)) \right| \\
 &= \left| \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha-1}} \left[\frac{(s-a)^{\overline{\alpha}}}{\Gamma(1-\alpha)} u_0 + f(s, u(s)) \right] \right| \\
 &\leq \frac{L_7}{\Gamma(\alpha)} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha-1}} (s-a)^{\overline{-\gamma_6}} \\
 &= L_7 \nabla_a^{-\alpha} (t-a)^{\overline{-\gamma_6}} \\
 &= \frac{L_7 \Gamma(1-\gamma_6)}{\Gamma(1+\alpha-\gamma_6)} (t-a)^{\overline{\alpha-\gamma_6}}
 \end{aligned}$$

for $t \in \mathbb{N}_{a+1}$ implies $TS_5 \subset S_5$. The remaining proof of TS_5 to be relatively compact is similar to that of Lemma 4.1, and we omit it. By Theorem 2.6, T has a fixed point in S_5 which is the solution of (2.8)–(2.9). Hence the proof. ■

Theorem 4.6. *Assume that condition (VII) holds. Then the zero solution of (2.8) is attractive.*

Theorem 4.7. *Assume that conditions (VI) and (VII) hold. Then the zero solution of (2.8) is asymptotically stable provided that (4.6) holds.*

Lemma 4.8. *Assume that the following condition is satisfied:*

(VIII) *there exist constants $\beta_5 > \frac{1}{1-\alpha}$, $\frac{\alpha}{\beta_5-1} < \gamma_7 < \frac{1}{\beta_5}$ and $L_8 > 0$, such that*

$$(4.10) \quad \left| \frac{(t-a)^{\overline{\alpha}}}{\Gamma(1-\alpha)} u_0 + f(t, u(t)) \right| \leq L_8 |u(t+\gamma_7)|^{\beta_5} \text{ for } t \in \mathbb{N}_{a+1}.$$

Then, there exists a solution for (2.8) provided that

$$(4.11) \quad \frac{L_8 \Gamma(1-\beta_5 \gamma_7)}{\Gamma(1+\alpha-\beta_5 \gamma_7)} \frac{\Gamma(1+\beta_5 \gamma_7)}{[\Gamma(1+\gamma_7)]^{\beta_5}} \leq 1.$$

Proof. From $\beta_5 > \frac{1}{1-\alpha}$, we have that $\frac{\alpha}{\beta_5-1} < \frac{1}{\beta_5}$ which implies that γ_7 exists. In addition, $\gamma_7 < \frac{1}{\beta_5}$ means that $\Gamma(1-\beta_5 \gamma_7) > 0$ and $\Gamma(1+\alpha-\beta_5 \gamma_7) > 0$, $\frac{\alpha}{\beta_5-1} < \gamma_7$ implies that $\alpha - \beta_5 \gamma_7 < -\gamma_7$. Define the set

$$(4.12) \quad S_6 = \left\{ u(t) : |u(t)| \leq (t-a)^{\overline{-\gamma_7}} \text{ for } t \in \mathbb{N}_{a+1} \right\}.$$

We show that T maps S_6 in S_6 . Applying condition (VIII), Lemma 2.2 and (4.11), we have

$$\begin{aligned}
|Tu(t)| &= \left| u_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha-1}} f(s, u(s)) \right| \\
&= \left| \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha-1}} \left[\frac{(s-a)^{\overline{\alpha}}}{\Gamma(1-\alpha)} u_0 + f(s, u(s)) \right] \right| \\
&\leq \frac{L_8}{\Gamma(\alpha)} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha-1}} |u(s + \gamma_7)|^{\beta_5} \\
&\leq \frac{L_8}{\Gamma(\alpha)} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha-1}} [(t - a + \gamma_7)^{-\overline{\gamma_7}}]^{\beta_5} \\
&\leq \frac{L_8}{\Gamma(\alpha)} \frac{\Gamma(1 + \beta_5 \gamma_7)}{[\Gamma(1 + \gamma_7)]^{\beta_5}} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha-1}} (t - a + \beta_5 \gamma_7)^{-\overline{\beta_5 \gamma_7}} \\
&\leq \frac{L_8}{\Gamma(\alpha)} \frac{\Gamma(1 + \beta_5 \gamma_7)}{[\Gamma(1 + \gamma_7)]^{\beta_5}} \sum_{s=a+1}^t (t - \rho(s))^{\overline{\alpha-1}} (t - a)^{-\overline{\beta_5 \gamma_7}} \\
&= L_8 \frac{\Gamma(1 + \beta_5 \gamma_7)}{[\Gamma(1 + \gamma_7)]^{\beta_5}} \nabla_a^{-\alpha} (t - a)^{-\overline{\beta_5 \gamma_7}} \\
&= \frac{L_8 \Gamma(1 - \beta_5 \gamma_7)}{\Gamma(1 + \alpha - \beta_5 \gamma_7)} \frac{\Gamma(1 + \beta_5 \gamma_7)}{[\Gamma(1 + \gamma_7)]^{\beta_5}} (t - a)^{\overline{\alpha - \beta_5 \gamma_7}} \\
&\leq \frac{L_8 \Gamma(1 - \beta_5 \gamma_7)}{\Gamma(1 + \alpha - \beta_5 \gamma_7)} \frac{\Gamma(1 + \beta_5 \gamma_7)}{[\Gamma(1 + \gamma_7)]^{\beta_5}} (t - a)^{-\overline{\gamma_7}} \leq (t - a)^{-\overline{\gamma_7}}
\end{aligned}$$

for $t \in \mathbb{N}_{a+1}$ implies $TS_6 \subset S_6$. The remaining part of the proof is similar to that of Lemma 4.1, so we omit it. $\blacksquare$

Theorem 4.9. *Assume that the condition (VIII) and (4.11) hold, then the zero solution of (2.8) is attractive.*

Theorem 4.10. *Assume that the condition (VI) and (VIII) hold, then the zero solution of (2.8) is asymptotically stable provided that (4.6) and (4.11) hold.*

5. Examples

Example 5.1.

$$(5.1) \quad \nabla_{-1}^{0.5} u(t) = (0.3)t^{-\overline{0.25}} \sin(u(t)), \quad \nabla_{-1}^{-0.5} u(t) \Big|_{t=0} = u(0), \quad t \in \mathbb{N}_1.$$

Here, $f(t, u(t)) = (0.3)t^{\overline{-0.25}} \sin(u(t))$, $t \in \mathbb{N}_1$. Clearly, $|f(t, u(t))| \leq (0.3)t^{\overline{-0.25}}$ implies condition (I) holds. Further, $|f(t, u(t)) - f(t, v(t))| \leq (0.3)t^{\overline{-0.25}}|u - v|$ implies condition (II) is satisfied. Here $L_2 = 0.3$ and $\beta_2 = 0.25$ and we have $c = L_2\Gamma(1 - \beta_2) = (0.3)\Gamma(0.75) \approx (0.3)(1.2254) < 1$, implies that the inequality (3.10) holds. Hence, the solution of (5.1) is asymptotically stable by Theorem 3.5.

Example 5.2.

$$(5.2) \quad \nabla_{-1}^{0.5}u(t) = (0.1)(t+1)t^{\overline{-0.6}}u(t), \quad \nabla_{-1}^{-0.5}u(t)\Big|_{t=0} = u(0), \quad t \in \mathbb{N}_1.$$

Here $f(t, u(t)) = (0.1)(t+1)t^{\overline{-0.6}}u(t)$, $t \in \mathbb{N}_1$, where $\beta_3 = 0.6$ and $\alpha = 0.5$. Clearly, $\beta_3 \in \left(\alpha, \frac{1+\alpha}{2}\right)$, $\gamma_2 = \frac{1}{2}(1 - \alpha) = 0.25$ and $L_3 = 0.1$ such that

$$|f(t, u(t))| = |(0.1)(t+1)t^{\overline{-0.6}}u(t)| \leq (0.1)(t+0.25)t^{\overline{-0.6}}|u(t)|$$

which implies that condition (III) is satisfied. Further,

$$|f(t, u(t)) - f(t, v(t))| \leq (0.1)(t+1)t^{\overline{-0.6}}|u - v| \leq (0.1)(t)t^{\overline{-0.6}}|u - v|,$$

implies that condition (II) is satisfied. Here, $L_2 = 0.1$, $\beta_2 = 0.6$ and $\alpha = 0.5$. Then, we have $c = L_2\Gamma(1 - \beta_2) = (0.1)\Gamma(0.4) \approx (0.1)(2.2181) < 1$, implies that the inequality (3.10) holds. Hence, the solution of (5.2) is asymptotically stable by Theorem 3.7.

Example 5.3.

$$(5.3) \quad \nabla_{-1}^{0.5}u(t) = (0.2)(t+1.5)t^{\overline{-0.6}}[u(t+0.05)]^{0.3}, \quad \nabla_{-1}^{-0.5}u(t)\Big|_{t=0} = u(0), \quad t \in \mathbb{N}_1.$$

Here, $f(t, u(t)) = (0.2)(t+1.5)t^{\overline{-0.6}}[u(t+0.05)]^{0.3}$, where $\beta_4 = 0.6$, $\eta = 0.3$, $L_4 = 0.2$ and $\alpha = 0.5$. Clearly, $\eta \in (0, 1)$, $\beta_4 \in \left(\alpha, \frac{2+\alpha\eta}{2+\eta}\right)$ such that

$$|f(t, u(t))| = |(0.2)(t+1.5)t^{\overline{-0.6}}[u(t+0.05)]^{0.3}| \leq (0.2)(t+1)t^{\overline{-0.6}}|u(t+0.05)|^{0.3}$$

which implies condition (IV) is satisfied. Hence, the zero solution of (5.3) is attractive by Theorem 3.8.

Example 5.4.

$$(5.4) \quad \nabla_{0*}^{0.5}u(t) = (0.2)t^{\overline{-0.75}}\sin(u(t)), \quad u(0) = 0, \quad t \in \mathbb{N}_1.$$

Using (2.10), we get

$$u(t) = \frac{(0.2)}{\Gamma(0.5)} \sum_{s=1}^t (t - \rho(s))^{\overline{-0.5}} s^{\overline{-0.75}} \sin(u(s)).$$

Now, consider

$$\left| \frac{(0.2)}{\Gamma(0.5)} \sum_{s=1}^t (t - \rho(s))^{-0.5} s^{-0.75} \sin(u(s)) \right| \leq \frac{(0.2)}{\Gamma(0.5)} \sum_{s=1}^t (t - \rho(s))^{-0.5} s^{-0.75} \leq t^{-0.25}$$

implies that (V) holds. Further, $|f(t, u(t)) - f(t, v(t))| \leq (0.2)t^{-0.75}|u - v|$ implies condition (VI) is satisfied. Here, $L_6 = 0.2$, $\gamma_5 = 0.75$ and we have $c = L_6\Gamma(1 - \gamma_5) = 0.2\Gamma(0.25) \approx (0.2)(3.6256) < 1$, implies (4.6) holds. Hence, the zero solution of (5.4) is asymptotically stable by Theorem 4.4.

Example 5.5.

$$(5.5) \quad \nabla_{0*}^{0.5} u(t) = (0.2)t^{-0.75} \sin(u(t)) - u_0 \frac{t^{0.5}}{\Gamma(0.5)}, \quad u(0) = u_0, \quad t \in \mathbb{N}_1.$$

Using (2.10), we get

$$u(t) = u_0 + \frac{1}{\Gamma(0.5)} \sum_{s=1}^t (t - \rho(s))^{-0.5} \left[(0.2)s^{-0.75} \sin(u(s)) - u_0 \frac{s^{0.5}}{\Gamma(0.5)} \right].$$

Now, consider

$$\left| u_0 \frac{t^{0.5}}{\Gamma(0.5)} + f(t, u(t)) \right| \leq (0.2)t^{-0.75} |\sin(u(s))| \leq (0.2)t^{-0.75} \leq t^{-0.75}$$

implies that (VII) holds. Further, $|f(t, u(t)) - f(t, v(t))| \leq 0.2t^{-0.75}|u - v|$ implies condition (VI) is satisfied. Here, $L_6 = 0.2$, $\gamma_5 = 0.75$ and we have $c = L_6\Gamma(1 - \gamma_5) = 0.2\Gamma(0.25) \approx (0.2)(3.6256) < 1$, implies (4.6) holds. Hence, the zero solution of (5.5) is asymptotically stable by Theorem 4.7.

Example 5.6.

$$(5.6) \quad \nabla_{0*}^{0.1} u(t) = (0.5)[u(t + 0.2)]^2 - u_0 \frac{t^{0.5}}{\Gamma(0.5)}, \quad u(0) = u_0, \quad t \in \mathbb{N}_1.$$

Using (2.10), we get

$$u(t) = \frac{1}{\Gamma(0.5)} \sum_{s=1}^t (t - \rho(s))^{-0.5} \left[(0.5)[u(s + 0.2)]^2 - u_0 \frac{s^{0.5}}{\Gamma(0.5)} \right].$$

Now, consider

$$\left| u_0 \frac{t^{0.5}}{\Gamma(0.5)} + f(t, u(t)) \right| = \left| (0.5)[u(t + 0.2)]^2 \right| \leq (0.5)|u(t + 0.2)|^2$$

implies that (VIII) holds. Here, $L_8 = 0.5$, $\beta_5 = 2$ and $\gamma_7 = 0.2$ and we have

$$\frac{L_8\Gamma(1 - \beta_5\gamma_7)}{\Gamma(1 + \alpha - \beta_5\gamma_7)} \frac{\Gamma(1 + \beta_5\gamma_7)}{[\Gamma(1 + \gamma_7)]^{\beta_5}} = \frac{(0.5)\Gamma(0.6)}{\Gamma(0.7)} \frac{\Gamma(1.4)}{[\Gamma(1.2)]^2} \approx 0.6039 < 1,$$

implies (4.11) is satisfied. Hence, the zero solution of (5.6) is attractive by Theorem 4.9.

Acknowledgements. For this work, the author J. Jagan Mohan was supported by the University Grants Commission Start Up Grant, Government of India.

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Accepted: 6.05.2013

EXP-FUNCTION METHOD USING MODIFIED RIEMANN– LIOUVILLE DERIVATIVE FOR SINGULARLY PERTURBED BOUSSINESQ EQUATIONS OF FRACTIONAL-ORDER

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Abstract. This paper witnesses the combination of an efficient transformation and Exp-function method to construct generalized solitary wave solutions of the nonlinear singularly perturbed sixth-order Boussinesq equations of fractional-order. Computational work and subsequent numerical results re-confirm the efficiency of proposed algorithm. It is observed that suggested scheme is highly reliable and may be extended to other nonlinear differential equations of fractional order.

Keywords: perturbed sixth-order Boussinesq equations, fractional calculus, exp-function method, modified Riemann-Liouville derivative.

1. Introduction

The subject of fractional calculus [1], [2] is a rapidly growing field of research, at the interface between chaos, probability, differential equations, and mathematical physics. In recent years, nonlinear fractional differential equations (NFDEs) have gained much interest due to exact description of nonlinear phenomena of many real-time problems. The fractional calculus is also considered as a novel topic [3], [4]; has gained considerable popularity and importance during the recent past. It has been the subject of specialized conferences, workshops and treatises or so, mainly due to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. Some of the areas of present-day applications of fractional models [5]–[8] include fluid flow, solute transport or dynamical processes in self-similar and porous structures, diffusive transport akin to diffusion, material viscoelastic theory, electromagnetic theory, dynamics of earthquakes, control theory of dynamical systems, optics and signal processing, bio-sciences, economics, geology, astrophysics, probability and statistics, chemical

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physics, and so on. As a consequence, there has been an intensive development of the theory of fractional differential equations, see [1]–[8] and the references therein. Recently, He and Wu [9] developed a very efficient technique which is called exp-function method for solving various nonlinear physical problems. The through study of literature reveals that Exp-function method has been applied on a wide range of differential equations and is highly reliable. The exp-function method has been extremely useful for diversified nonlinear problems of physical nature and has the potential to cope with the versatility of the complex nonlinearities of the problems. The subsequent works have shown the complete reliability and efficiency of this algorithm. He et.al. [10]–[11] used this scheme to find periodic solutions of evolution equations; Mohyud-Din [12–13] extended the same for nonlinear physical problems including higher-order BVPs; Oziz [14] tried this novel approach for Fisher's equation; Wu et.al. [15], [16] for the extension of solitary, periodic and compacton-like solutions; Yusufoglu [17] for MBBN equations, Zhang [18] for high-dimensional nonlinear evolutions; Zhu [19], [20] for the Hybrid-Lattice system and discrete m KdV lattice; Kudryashov [21] for exact soliton solutions of the generalized evolution equation of wave dynamics; Momani [22] for an explicit and numerical solutions of the fractional KdV equation; The basic motivation of this paper is the development of an efficient combination comprising an efficient transformation, exp-function method using Jumarie's derivative approach [23]–[26] and its subsequent application to construct generalized solitary wave solutions of the nonlinear singularly perturbed sixth-order Boussinesq equations of fractional-order [27]–[28]. It is to be highlighted that Ebaid [29] proved that $c = d$ and $p = q$ are the only relations that can be obtained by applying exp-function method to any nonlinear ordinary differential equation.

Theorem 1 [29] Suppose that $u^{(r)}$ and $(u^{(\gamma)})^\lambda$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r and γ are both positive integers. Then the balancing procedure using the Exp-function ansatz; $U(\eta) = \frac{\sum_{n=-c}^d a_n \exp(n\eta)}{\sum_{m=-p}^q b_m \exp(m\eta)}$, leads to $c = d$ and $p = q, \forall r, s, \lambda \geq 1$.

Theorem 2 [29] Suppose that $u^{(r)}$ and $u^{(s)}u^k$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r, s and Ω are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c = d$ and $p = q, \forall r, s, k \geq 1$.

Theorem 3 [29] Suppose that $u^{(r)}$ and $(u^{(s)})^\Omega$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r, s and Ω are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c = d$ and $p = q, \forall r, s \geq 1, \forall \Omega \geq 2$.

Theorem 4 [29] Suppose that $u^{(r)}$ and $(u^{(s)})^\Omega u^\lambda$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r, s, Ω and λ are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c = d$ and $p = q, \forall r, s, \Omega, \lambda \geq 1$.

2. Jumarie's fractional derivative

Jumarie's fractional derivative is a modified Riemann-Liouville derivative defined as [27]–[30]

$$(1) \quad D_t^\alpha f(x) = \left\{ \begin{array}{l} \frac{1}{\Gamma(-\alpha)} \int_0^x (x-t)^{-\alpha-1} (f(t) - f(0)) dt, \alpha \leq 0, \\ \frac{1}{\Gamma(-\alpha)} \frac{d}{dx} \int_0^x (x-t)^{-\alpha} (f(t) - f(0)) dt, 0 \leq \alpha \leq 1 \\ [f^{\alpha-n} (x)^n]^n, n \leq \alpha \leq n+1, n \geq 1 \end{array} \right\}$$

where $f: \mathbb{R} \rightarrow \mathbb{R}$, $x \rightarrow f(x)$ denotes a continuous (but not necessarily differentiable) function. Some useful formulas and results of Jumarie's modified Riemann-Liouville derivative were summarized in [27]–[28].

$$(2) \quad D_x^\alpha c = 0, \quad \alpha \geq 0, \quad c = \text{constant}$$

$$(3) \quad D_x^\alpha [cf(x)] = cD_x^\alpha f(x) \quad \alpha \geq 0, \quad c = \text{constant}$$

$$(4) \quad D_x^\alpha x^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}, \quad \beta \geq \alpha \geq 0,$$

$$(5) \quad D_x^\alpha [f(x)g(x)] = [D_x^\alpha f(x)g(x) + f(x)[D_x^\alpha g(x)].$$

$$(6) \quad D_x^\alpha f(x(t)) = f'_x(x) \cdot x^\alpha(t).$$

3. Exp-function method [28]–[31]

We consider the general nonlinear FPDE of the type

$$(7) \quad P(u, u_t, u_x, u_{xx}, u_{xxx}, \dots, D_t^\alpha u, D_x^\alpha u, D_{xx}^\alpha u, \dots) = 0, \quad 0 < \alpha \leq 1,$$

where $D_t^\alpha u, D_x^\alpha u, D_{xx}^\alpha u$ are the modified Riemann-Liouville derivative of u with respect to t, x, xx , respectively.

Using a transformation [32]

$$(8) \quad \eta = kx + \frac{\omega t^\alpha}{\Gamma(1+\alpha)} + \eta_0, \quad k, \omega, \eta_0 \text{ are all constants with } k, \omega \neq 0$$

we can rewrite equation (7) in the following nonlinear ODE;

$$(9) \quad Q(u, u', u'', u''', u^{iv}) = 0,$$

where the prime denotes derivative with respect to η .

According to the Exp-function method, we assume that the wave solution can be expressed in the following form

$$(10) \quad u(\eta) = \frac{\sum_{n=-c}^d a_n \exp[n\eta]}{\sum_{m=-p}^q b_m \exp[m\eta]}$$

where p, q, c and d are positive integers which are known to be further determined, a_n and b_m are unknown constants. We can rewrite equation (4) in the following equivalent form

$$(11) \quad u(\eta) = \frac{a_c \exp(c\eta) + \cdots + a_{-d} \exp(-d\eta)}{b_p \exp(p\eta) + \cdots + b_{-q} \exp(-q\eta)}.$$

This equivalent formulation plays an important and fundamental part for finding the analytic solution of problems. To determine the value of q and p by using [25],

$$(12) \quad p = c, \quad q = d$$

4. Solution procedure

In this section, we apply the exp-function method to construct generalized solitary solutions for Burger's Equations of fractional-order. Numerical results are very encouraging. In this section, we apply exp-function method to construct generalized solitary of the singularly perturbed sixth-order Boussinesq equations. Numerical results are very encouraging.

Example 4.1 Consider the singularly perturbed sixth-order Boussinesq equation where a and k are arbitrary constants.

$$u_{tt} = u_{xx} + (p(u))_{xx} + \beta u_{xxx} + \delta u_{xxxxx}$$

Taking $\beta = 1$, $\delta = 1$ and $p(u) = 3u^2$, the model equation is given as

$$(13) \quad u_{tt} = u_{xx} + 3(u^2)_{xx} + u_{xxxx}$$

with the initial conditions

$$u(x, 0) = \frac{2ak^2 e^{kx}}{(1 + ae^{kx})}, \quad u_t(x, 0) = \frac{2ak^3 \sqrt{1 + k^2} (1 - ae^{kx}) e^{kx}}{(1 + ae^{kx})^3}$$

Using (8), equation (13) can be converted to an ordinary differential equation

$$(14) \quad (\omega^2 + k^2) u'' + 6k^2 (u'^2 + u''u) u u''' + k^4 u^{(iv)} = 0,$$

where the prime denotes the derivative with respect to η . The solution of the equation (13) can be expressed in the form, equation (11). To determine the value of c and p , by using [25],

$$(15) \quad p = c, \quad q = d$$

Case 4.1.1. We can freely choose the values of c and d , but we will illustrate that the final solution does not strongly depend upon the choice of values of c and d . For simplicity, we set $p = c = 1$ and $q = d = 1$ equation (11) reduces to

$$(16) \quad u(\eta) = \frac{a_1 \exp[\eta] + a_0 + a_{-1} \exp[-\eta]}{b_1 \exp[\eta] + a_0 + b_{-1} \exp[-\eta]}.$$

Substituting equation (16) into equation (14), we have

$$(17) \quad \frac{1}{A} \left[\begin{array}{l} c_5 \exp(5\eta) + c_4 \exp(4\eta) + c_3 \exp(3\eta) + c_2 \exp(2\eta) + \\ c_1 \exp(\eta) + c_0 + c_{-1} \exp(-\eta) + c_{-2} \exp(-2\eta) + \\ c_{-3} \exp(-3\eta) + c_{-4} \exp(-4\eta) + c_{-5} \exp(-5\eta) \end{array} \right] = 0$$

$A = 5(b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta))^6$ where c_i ($i = -5, -4, \dots, 4, 5$) are constants obtained by Maple 16.

Equating the coefficients of $\exp(n\eta)$ to be zero, we obtain

$$(18) \quad \left(\begin{array}{l} c_{-5} = 0, c_{-4} = 0, c_{-3} = 0, c_{-2} = 0, c_{-1} = 0, c_0 = 0, \\ c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0, c_5 = 0 \end{array} \right)$$

By solving (18), we get different solutions of (13).

1st Solution set:

$$(19) \quad \begin{aligned} a_{-1} &= \frac{-1}{24} \frac{b_0^2 (K^4 - \omega^2 + K^2)}{K^2 b_1}, a_0 = \frac{1}{6} \frac{b_0 (5K^4 + \omega^2 - K^2)}{K^2}, \\ a_1 &= -\frac{1}{6} \frac{b_1 (K^4 - \omega^2 + K^2)}{K^2}, b_{-1} = \frac{1}{4} \frac{b_0^2}{b_1}, b_0 = b_0, b_1 = b_1 \end{aligned}$$

Therefore, we obtained the following generalized solitary solution $u(x, t)$ of equation (13)

$$(20) \quad u(x, t) = \frac{\left(\frac{-1}{24} \frac{b_0^2 (K^4 - \omega^2 + K^2)}{K^2 b_1} e^{-\frac{xK + \omega t^\alpha}{\Gamma(1+\alpha)}} + \frac{1}{6} \frac{b_0 (5K^4 + \omega^2 - K^2)}{K^2} - \frac{1}{6} \frac{b_1 (K^4 - \omega^2 + K^2)}{K^2} e^{\frac{xK + \omega t^\alpha}{\Gamma(1+\alpha)}} \right)}{\frac{1}{4} \frac{b_0^2}{b_1} e^{-\frac{xK + \omega t^\alpha}{\Gamma(1+\alpha)}} + b_0 + b_1 e^{\frac{xK + \omega t^\alpha}{\Gamma(1+\alpha)}}}$$

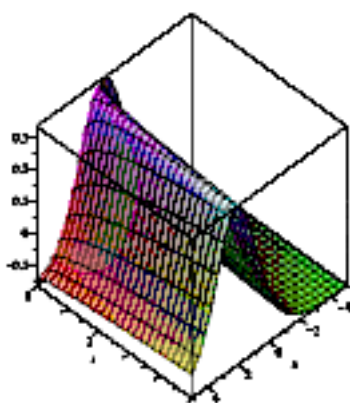


Figure 4.1(a) $\alpha = 0.25$

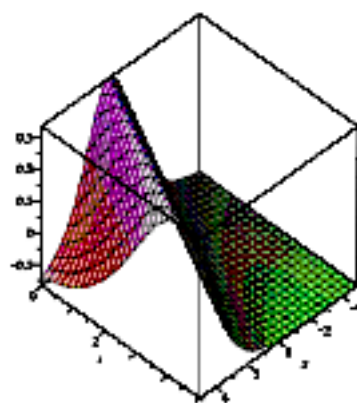


Figure 4.1(b) $\alpha = 1$

Case 4.1.2. If $p = c = 2$ and $q = d = 1$, then equation (11) reduces to

$$(21) \quad u(\eta) = \frac{a_2 \exp[2\eta] + a_1 \exp[\eta] + a_0 + a_{-1} \exp[-\eta]}{b_2 \exp[2\eta] + b_1 \exp[\eta] + a_0 + b_{-1} \exp[-\eta]}.$$

Proceeding as before, we obtain

1st Solution set:

$$(22) \quad \left\{ \begin{array}{l} a_{-1} = \frac{-1}{24} \frac{b_0^2 (K^4 - \omega^2 + K^2)}{K^2 b_1}, a_0 = \frac{1}{6} \frac{b_0 (5K^4 + \omega^2 - K^2)}{K^2}, \\ a_1 = -\frac{1}{6} \frac{b_1 (K^4 - \omega^2 + K^2)}{K^2}, b_{-1} = \frac{1}{4} \frac{b_0^2}{b_1}, b_0 = b_0, b_1 = b_1 \end{array} \right\},$$

Hence we get the generalized solitary wave solution of equation (13) as follows
(23)

$$u(x, t) = \frac{\left(\frac{-1}{24} \frac{b_0^2 (K^4 - \omega^2 + K^2)}{K^2 b_1} e^{-(xK + \omega t^\alpha)} + \frac{1}{6} \frac{b_0 (5K^4 + \omega^2 - K^2)}{K^2} - \frac{1}{6} \frac{b_1 (K^4 - \omega^2 + K^2)}{K^2} e^{xK + \omega t^\alpha} \right)}{\frac{1}{4} \frac{b_0^2 e^{-(xK + \omega t^\alpha)}}{b_1} + b_0 + b_1 e^{xK + \omega t^\alpha}}$$

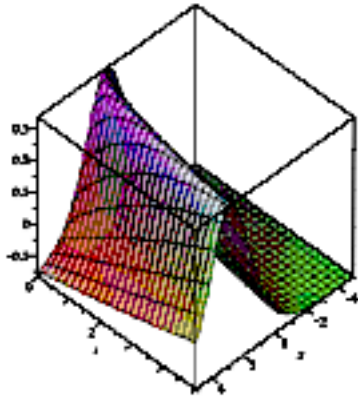


Figure 4.1(c) $\alpha = 0.25$

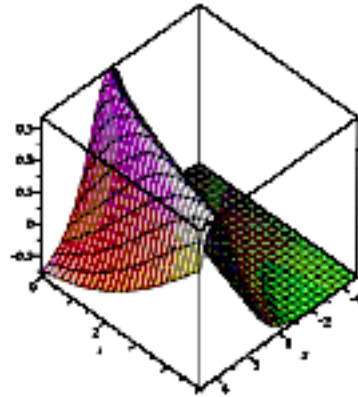


Figure 4.1(d) $\alpha = 1$

We get the same soliton solutions which clearly illustrate that final solution does not strongly depends upon these parameters.

5. Conclusion

In this paper, we applied exp-function method to construct generalized solitary solutions of the nonlinear fractional order singularly perturbed sixth-order Boussinesq equations. It is observed that the Exp-function method is very convenient to apply and is very useful for finding solutions of a wide class of nonlinear problems.

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COMMON FIXED POINT FOR SELF AND NONSELF-MAPS THROUGH AN IMPLICIT RELATION

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Abstract. Common fixed point theorems for three self and nonself-maps have been proved through the notions of property E.A., orbital completeness and weak compatibility under an implicit relation. The results of Singh and Mishra (1997), Singh and Asish Kumar (2006), Khan and Dolmo (2007) and Imdad and Ali (2008) are then particular cases.

Keywords: property E.A. weakly compatible maps, implicit relation, orbitally complete metric space.

2010 Mathematics Subject Classification: 54H25.

1. Introduction and preliminaries

Let (X, d) be a metric space. Self-maps S and A on X are *compatible* [2] if

$$(1.1) \quad \lim_{n \rightarrow \infty} d(SAx_n, ASx_n) = 0$$

whenever $\langle x_n \rangle_{n=1}^{\infty} \subset X$ is such that

$$(1.2) \quad \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Ax_n = p \quad \text{for some } p \in X.$$

A point $x \in X$ is a *coincidence point* of self-maps S and A on X if $Sx = Ax$. Self-maps which commute at their coincidence points are called *weakly compatible* [3]. Thus S and A are weakly compatible if $SAx = ASx$ whenever $x \in X$ is such that $Ax = Sx$. It is obvious that every compatible pair is weakly compatible. One can find examples from [3] for weakly compatible maps which are not compatible. Let $\mathbb{R}_+$ be the space of nonnegative real numbers. A *contractive modulus* $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with the choice $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$ is *upper semicontinuous* (abbreviated as *usc*) if for each $t_0 \geq 0$, $\limsup_{n \rightarrow \infty} \phi(t_n) \leq \phi(t_0)$ whenever $\langle t_n \rangle_{n=1}^\infty \subset \mathbb{R}_+$ is such that $\lim_{n \rightarrow \infty} t_n = 0$.

With these notations, Singh and Mishra, [11] proved the following result:

Theorem 1.1 *Let S , T and A be self-maps on X satisfying the inclusions*

$$(1.3) \quad S(X) \subset A(X) \quad \text{and} \quad T(X) \subset A(X)$$

and the contractive-type condition

$$(1.4) \quad \begin{aligned} & d(Sx, Ty) \\ & \leq \phi \left(\max \left\{ d(Ax, Ay), d(Sx, Ax), d(Ty, Ay), \frac{d(Ty, Ax) + d(Sx, Ay)}{2} \right\} \right) \end{aligned}$$

for all $x, y \in X$,

where ϕ is a nondecreasing, usc contractive modulus. Suppose that one of $S(X)$, $T(X)$ and $A(X)$ is a complete subspace of X . If

(a) *(A, S) and (A, T) are weakly compatible,*

then the three maps S , T and A will have a unique common fixed point.

From the definition of compatibility, it is clear that self-maps S and A are *noncompatible* on X if (1.2) holds good but $\lim_{n \rightarrow \infty} d(Sx_n, Ax_n)$ is either $\neq 0$ or $+\infty$ for some $\langle x_n \rangle_{n=1}^\infty \subset X$. It may be noted from [4] that both compatible and noncompatible maps are included in the class of maps with (1.2). Self-maps S and A on X are said to satisfy the *property E.A.* [1] if (1.2) holds good for some $\langle x_n \rangle_{n=1}^\infty \subset X$, where the common limit p is called a *tangent point*. It is known that weak compatibility and property E.A. are independent.

With these ideas, the following theorem was proved in [10]:

Theorem 1.2 *Let S , T and $A : Y \rightarrow X$ satisfy the inequality*

$$(1.5) \quad \begin{aligned} & d(Sx, Ty) \\ & < \max \left\{ d(Ax, Ay), \alpha d(Ax, Sx), \alpha d(Ay, Ty), \frac{d(Ax, Ty) + d(Sx, Ay)}{2} \right\}, \end{aligned}$$

for all $x, y \in X$ with $0 < \alpha < 1$

and both conditions

$$(b) \quad \overline{S(Y)} \subset A(Y),$$

$$(c) \quad \overline{T(Y)} \subset A(Y),$$

where Y an arbitrary nonempty subset of X . Suppose one of the pairs (S, A) and (T, A) satisfies the property E.A. on Y . Then there is a coincidence point common to S , T and A in Y . Further, if $Y=X$ and (a) of Theorem 1.1 holds good, then S , T and A will have a unique common fixed point in X .

In this paper, we prove the generalizations of Theorem 1.1 and Theorem 1.2 using implicit relations.

2. Main results

Given $x_0 \in X$ and S, T and A , self-maps on X , if there exist points $x_1, x_2, x_3, \dots$ in X such that

$$(2.1) \quad Sx_{2n-2} = Ax_{2n-1}, Tx_{2n-1} = Ax_{2n}, \quad \text{for } n = 1, 2, 3, \dots$$

then the sequence $\langle Ax_n \rangle_{n=1}^\infty \subset X$ is an (S, T, A) -orbit or simply an orbit at x_0 . The metric space X is (S, T, A) -orbitally complete or orbitally complete [7], [8] at x_0 if every Cauchy sequence in some orbit at $x_0 \in X$ converges in X . It easily follows that every complete metric space is orbitally complete at each of its points. However the converse is not true (cf. [7], [8]).

Let $\mathbb{R}_+^6$ denote the space of 6-tuples of nonnegative real numbers in this paper, and $\psi : \mathbb{R}_+^6 \rightarrow \mathbb{R}$, lower semicontinuous in each coordinate variable such that

(C₁) ψ is nondecreasing in the fifth and sixth coordinate variables,

(C₂) For every $l \geq 0, m \geq 0$, there is a constant $0 \leq \omega < 1$ such that

$$(2.2) \quad \min\{\psi(l, m, m, l, l+m, 0), \psi(l, m, l, m, 0, l+m)\} \leq 0 \Rightarrow l \leq \omega m,$$

(C₃) $\psi(l, l, 0, 0, l, l) > 0$ for all $l > 0$.

Such implicit-type relations were first introduced by Popa [9], which covers several contractive conditions in proving fixed point theorems.

Theorem 2.1 *Let S , T and A be self-maps on X satisfying the implicit-type inequality*

$$(2.3) \quad \psi(d(Sx, Ty), d(Ax, Ay), d(Sx, Ax), d(Ty, Ay), d(Ty, Ax), d(Sx, Ay)) \leq 0$$

for all $x, y \in X$.

Suppose that either (S, A) or (T, A) satisfies the property E.A. and that one of the following statements holds good:

1. $A(X)$ is orbitally complete at some $x_0 \in X$,
2. $\overline{S(X)} \subset A(X)$,
3. $\overline{T(X)} \subset A(X)$.

Then there is a coincidence point u common to S , T and A . Further, if

(a°) (S, A) or (T, A) is weakly compatible,

then the point of coincidence of S , T and A with respect to u will be their unique common fixed point.

Proof. Suppose that (S, A) satisfies the property E.A. Then (1.2) holds good for some $\langle x_n \rangle_{n=1}^\infty \subset X$. We claim that $q = \lim_{n \rightarrow \infty} Tx_n = p$. Writing $x = y = x_n$ in (2.3), we find that

$$\psi(d(Sx_n, Tx_n), d(Ax_n, Ax_n), d(Ax_n, Sx_n), d(Ax_n, Tx_n), d(Ax_n, Tx_n), d(Ax_n, Sx_n)) < 0.$$

Applying the limit as $n \rightarrow \infty$ in this, using (1.2) and the lower semicontinuity of ψ , we get

$$\psi(d(p, q), 0, 0, d(p, q), d(p, q), 0) \leq 0,$$

which gives (2.2) with $l = d(p, q)$ and $m = 0$. Therefore, by (C_2) , we get $d(p, q) \leq 0$ or $p = q$. Thus

$$(2.4) \quad \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Ax_n = p \quad \text{for some } p \in X.$$

While if (T, A) satisfies the property E.A., there is some $\langle y_n \rangle_{n=1}^\infty \subset X$ such that

$$(2.5) \quad \lim_{n \rightarrow \infty} Ty_n = \lim_{n \rightarrow \infty} Ay_n = r \quad \text{for some } r \in X.$$

Then (2.3) with $x = y = y_n$ gives

$$\psi(d(Sy_n, Ty_n), d(Ay_n, Ay_n), d(Ay_n, Sy_n), d(Ay_n, Ty_n), d(Ay_n, Ty_n), d(Ay_n, Sy_n)) < 0.$$

As $n \rightarrow \infty$, this along with (2.5), $s = \lim_{n \rightarrow \infty} Sx_n$ and the lower semicontinuity of ψ imply that

$$\psi(d(s, r), 0, d(r, s), 0, 0, d(r, s)) \leq 0.$$

So that (C_2) with $l = d(s, r)$ and $m = 0$ gives $d(s, r) \leq 0$ or $s = r$, proving (2.4).

Case 2.1: Suppose that $A(X)$ orbitally complete at $x_0 \in X$. Then (2.4) implies that $p \in A(X)$ so that $p = Au$ for some $u \in X$. From (2.3) with $x = u$ and $y = p_n$, we would get

$$\psi(d(Su, Tp_n), d(Au, Ap_n), d(Au, Su), d(Ap_n, Tp_n), d(Au, Tp_n), d(Ap_n, Su)) < 0.$$

Employing the limit as $n \rightarrow \infty$ and using (2.4), this gives

$$\psi(d(Su, Au), 0, d(Au, Su), 0, 0, d(Au, Su)) \leq 0,$$

which in view of (2.2) with $l = d(Au, Su)$ and $m = 0$ together with (C_2) gives $d(Au, Su) = 0$ or

$$(2.6) \quad Au = Su = p.$$

Again from (2.3) with $x = u = y$ and (2.6), it follows that

$$\psi(d(Su, Tu), d(Au, Au), d(Au, Su), d(Au, Tu), d(Au, Su), d(Au, Tu)) < 0$$

or $\psi(d(p, Tu), 0, 0, d(p, Tu), 0, d(p, Tu)) < 0$ so that from (C_2) with $l = d(p, Tu)$ and $m = 0$, we get $d(Ap, Tu) = 0$ or $Tu = p$, that is

$$(2.7) \quad Au = Su = Tu = p.$$

Case 2.1: Suppose that (2.1) holds good. Then $\langle Sx_n \rangle_{n=1}^\infty \subset S(X)$ and hence lies in $\overline{S(X)}$ so that $p \in \overline{S(X)} \subset A(X)$. It follows that u is a common coincidence from Case (2.1).

Case 2.1: Suppose that (2.1) holds good. Then as above, we get that $\langle Tx_n \rangle_{n=1}^\infty$ converges in $\overline{T(X)}$ so that $p \in \overline{S(X)} \subset A(X)$ and (2.7) follows from Case (2.1).

To prove the second part of Theorem 2.1, we suppose that (S, A) is weakly compatible. From (2.6), we find that $SAu = ASu$ or $Ap = Sp$. But then from (2.3) with $x = y = p$, we get

$$\begin{aligned} \psi(d(Sp, Tp), d(Ap, Ap), d(Ap, Sp), d(Ap, Tp), d(Ap, Tp), d(Ap, Sp)) < 0 \text{ or} \\ \psi(d(Sp, Tp), 0, 0, d(Sp, Tp), d(Sp, Tp), 0) < 0, \end{aligned}$$

which, in view of (C_2) with $l = d(Sp, Tp)$ and $m = 0$, gives $d(Sp, Tu) = 0$ or $Sp = Tp$, that is

$$(2.8) \quad Ap = Sp = Tp.$$

If (T, A) is weakly compatible, from (2.7), it follows that $Tp = Ap$. Then taking $x = y = p$ in (2.3) and using this and (C_2) with $l = d(Sp, Tp)$ and $m = 0$ and simplifying, we get (2.8).

Now, p is a fixed point of S . In fact, from (2.3) with $x = u$ and $y = p$, we get

$$\begin{aligned} \psi(d(Su, Tp), d(Au, Ap), d(Au, Su), d(Ap, Tp), d(Au, Tp), d(Ap, Su)) < 0 \text{ or} \\ \psi(d(p, Sp), d(p, Sp), 0, 0, d(p, Sp), d(Sp, p)) < 0. \end{aligned}$$

Again using (C_2) in this, we get $d(p, Sp) = 0$. Hence p is a common fixed point of S, T and A , in view of (2.8).

Finally, suppose that q is also a common fixed point of S, T and A with $p \neq q$ so that $d(p, q) > 0$, then writing $x = p$ and $y = q$ in (2.3), we obtain

$$\psi(d(p, q), d(p, q), d(p, p), d(q, q), d(p, q), d(q, p)) < 0,$$

which is a contradiction to the choice (C_3) . This shows that $p = q$, that is the common fixed point of S, T and A is unique. ■

The following example emphasizes that none of the conditions (2.1), (2.1) and (2.1) can be dropped in Theorem 2.1 to ensure a common coincidence point for the three self-maps:

Example 2.1 Let

$$\psi(l_1, l_2, l_3, l_4, l_5, l_6) = l_1 - q \max \left\{ l_2, l_3, l_4, \frac{l_5 + l_6}{2} \right\},$$

where $0 \leq q < 1$. Then ψ satisfies (C_1) and

$$\psi(l, m, m, l, l + m, 0) = l - q \max\{l, m\} = \psi(l, m, l, m, 0, l + m)$$

so that from (2.2) we get

$$(2.9) \quad l \leq q \max\{l, m\}.$$

It may be noted that (2.9) is trivial if $l = 0$. Therefore let $l > 0$.

Again, if $\max\{l, m\} = l$, (2.9) would give a contradiction that $l \leq ql < l$, since $q < 1$. Thus $\max\{l, m\} = m$ so that $l \leq qm$, proving (C_2) .

Finally, $\psi(l, l, 0, 0, l, l) = (1 - q)l \leq 0$, which is a contradiction to the choice of ψ if $l > 0$, and (C_3) follows.

Let $X = \mathbb{R}_+$ with $d(x, y) = |x - y|$ for all $x, y \in X$. Define

$$Sx = \begin{cases} \frac{1}{2} & (x = 0) \\ \frac{x}{2} & (x > 0), \end{cases} \quad Tx = \begin{cases} \frac{1}{2} & (x = 0) \\ \frac{2x}{3} & (x > 0) \end{cases} \quad \text{and} \quad Ax = \begin{cases} 1 & (x = 0) \\ \frac{3x}{4} & (x > 0). \end{cases}$$

Note that $S(X) = T(X) = A(X) = (0, \infty)$. But $\overline{S(X)} = \overline{T(X)} = [0, \infty)$ so that (2.1) and (2.1) are not satisfied. For $x_0 = 0$, we choose

$$x_1 = \frac{2}{3}, \quad x_{2n} = 3 \left(\frac{16}{27} \right)^n, \quad x_{2n+1} = 2 \left(\frac{16}{27} \right)^n \quad \text{for } n \geq 1,$$

the orbit is

$$\left\{ \frac{1}{2}, \frac{4}{3}, \frac{8}{9}, \dots, \frac{9}{4} \left(\frac{16}{27} \right)^n, \frac{3}{2} \left(\frac{16}{27} \right)^n, \dots \right\}.$$

While, for $x_0 > 0$, choose

$$x_1 = \frac{2}{3}x_0, x_{2n} = \left(\frac{16}{27}\right)^n x_0, x_{2n+1} = \frac{2}{3} \left(\frac{16}{27}\right)^n x_0, n = 1, 2, 3, \dots,$$

so that its orbit is

$$\left\{ \frac{x_0}{2}, \left(\frac{2}{3}\right)^2 x_0, \dots, \frac{3}{4} \left(\frac{16}{27}\right)^n x_0, \frac{1}{2} \left(\frac{16}{27}\right)^n x_0, \dots \right\}.$$

In each case, the orbit converges to 0; (S, A) and (T, A) satisfy the property E.A. However, since the limit 0 does not belong to $A(X)$, the subspace $A(X)$ cannot not be orbitally complete at each x_0 . Thus, (2.1) fails and the three maps do not have a common coincidence point, though X is complete. In other words, at least one of (2.1), (2.1) and (2.1) is necessary in Theorem 2.1 to obtain a common coincidence point.

Remark 2.1 Set

$$\psi(l_1, l_2, l_3, l_4, l_5, l_6) = l_1 - \phi \left(\max \left\{ l_2, l_3, l_4, \frac{l_5 + l_6}{2} \right\} \right),$$

where ϕ is a nondecreasing usc contractive modulus. Then (C_1) holds good. Ignoring the triviality $l = 0$, we assume that $l > 0$, and $m \geq 0$ so that

$$\psi(l, m, m, l, l + m, 0) = l - \phi(\max\{l, m\}) = \psi(l, m, l, m, 0, l + m)$$

and from (2.2) we get

$$l - \phi(\max\{l, m\}) \leq 0 \quad \text{or} \quad l \leq \phi(\max\{l, m\}),$$

which would give a contradiction that $l \leq \phi(l) < l$ if $\max\{l, m\} = l$. Thus $\max\{l, m\} = m$ so that $l \leq \phi(m)$ and hence $l \leq qm$ for some $0 < q < 1$ proving (C_2) . Finally the choice of ϕ gives

$$\psi(l, l, 0, 0, l, l) = l - \phi \left(\max \left\{ l, 0, 0, \frac{l+l}{2} \right\} \right) = l - \phi(l) > 0,$$

and hence (C_3) .

The choice of ψ reveals that (1.4) is a special case of (2.3). Given $x_0 \in X$, using (1.3), one can define an orbit $\langle Ax_n \rangle_{n=1}^\infty$ at x_0 with the choice (2.1). But, from [11], $\langle Ax_n \rangle_{n=1}^\infty$ is a Cauchy sequence in $A(X)$. Condition (1) holds good whenever if $A(X)$ is a complete subspace of X and hence $Ax_n \rightarrow p$ as $n \rightarrow \infty$ for some $p \in A(X)$ so that $p = Au$ for some $u \in X$. Then $\langle Sx_{2n-2} \rangle_{n=1}^\infty$ and $\langle Tx_{2n-1} \rangle_{n=1}^\infty$ being its subsequences also converge to $p = Au$. That is

$$(2.10) \quad \lim_{n \rightarrow \infty} Sx_{2n-2} = \lim_{n \rightarrow \infty} Tx_{2n-1} = \lim_{n \rightarrow \infty} Ax_{2n} = p = Au.$$

Write $x = y = x_{2n}$ in (1.4). Then applying the limit as $n \rightarrow \infty$, $s = \lim_{n \rightarrow \infty} Tx_{2n}$ and using (2.10), we find that

$$d(p, s) \leq \phi \left(\max \left\{ 0, 0, d(s, p), \frac{d(s, p) + 0}{2} \right\} \right),$$

so that $d(p, s) = 0$ or $p = s$. This proves that (T, A) satisfies the property E.A. While taking $x = y = x_{2n-1}$ in (1.4), applying the limit as $n \rightarrow \infty$, using (2.10), $t = \lim_{n \rightarrow \infty} Sx_{2n-1}$ and proceeding as above, it follows that (S, A) satisfies the property E.A. If $S(X)$ is orbitally complete, then $p \in S(X)$ and (1.3) gives, $p \in A(X)$ and the conclusion follows from the earlier case. Similarly the case that $T(X)$ is orbitally complete can be handled. The conclusion follows from Theorem 2.1. Thus Theorem 1.1 is a particular case of Theorem 2.1.

Remark 2.2 When $T = S$ in (2.1), we get an (S, A) -orbit [6] at $x_0 \in X$ given by

$$(2.11) \quad Sx_{n-1} = Ax_n \quad \text{for } n = 1, 2, 3, \dots$$

and hence the (S, A) -orbital completeness of X at x_0 . Therefore, taking $T = S$ in Theorem 2.1, we get

Corollary 2.1 (Theorem 3.1, [4]) *Let S and A be self-maps on X satisfying the property E.A. and the inequality*

$$(2.12) \quad \psi(d(Sx, Sy), d(Ax, Ay), d(Sx, Ax), d(Sy, Ay), d(Sx, Ay), d(Sy, Ax)) \leq 0$$

for all $x, y \in X$,

If $A(X)$ is complete and (S, A) is a weakly compatible pair, then S and A have a unique common fixed point.

Let Y an arbitrary nonempty subset of X . Imitating the proof of Theorem 2.1, we can establish the following result for nonself-maps:

Theorem 2.2 *Let S, T and $A : Y \rightarrow X$ satisfy (2.3) for all $x, y \in Y$. Suppose that either (S, A) or (T, A) satisfies the property E.A. on Y , and any one of the following conditions holds good:*

(1°) $A(Y)$ is closed subspace of Y ,

(2°) $\overline{S(Y)} \subset A(Y)$,

(3°) $\overline{T(Y)} \subset A(Y)$.

Then there is a coincidence point u common to S, T and A in Y . Further if the point of common coincidence of S, T and A with respect to u lies in Y , it will be their unique common fixed point, provided either (S, A) or (T, A) is a weakly compatible pair.

Remark 2.3 Write

$$\psi(l_1, l_2, l_3, l_4, l_5, l_6) = l_1 - \max \left\{ l_2, \alpha l_3, \alpha l_4, \frac{l_5 + l_6}{2} \right\}$$

in Theorem 2.2, where $0 < \alpha < 1$. Then (C_1) – (C_3) hold good and (1.5) is a particular case of (2.3). Interestingly, Theorem 1.2 required both the containments (2°) and (3°) and weak compatibility of both the pairs (S, A) and (T, A) . Also a common fixed point was obtained under the condition that $Y = X$. However, the proof of our second results suggests that weak compatibility and property E.A. of either pair is sufficient to obtain a common fixed point under a generalized inequality, even without the condition that $Y = X$.

Finally taking

$$\psi(l_1, l_2, l_3, l_4, l_5, l_6) = l_1 - \max \left\{ l_2, \beta l_3 + \alpha l_4, \frac{l_5 + l_6}{2} \right\},$$

where $\beta \geq 0$ and $0 < \alpha < 1$ and $T = S$ in Theorem 2.2, we get

Corollary 2.2 (Theorem 3.1, [5]) *Let $S, A : Y \rightarrow X$ satisfying*

$$(2.13) \quad \begin{aligned} & d(Sx, Sy) \\ & < \max \left\{ d(Ax, Ay), \beta d(Sx, Ax) + \alpha d(Sy, Ay), \frac{d(Sy, Ax) + d(Sx, Ay)}{2} \right\} \\ & \text{for all } x, y \in Y \text{ with } x \neq y, \end{aligned}$$

Suppose that either $A(Y)$ is a complete subspace of Y or $S(Y)$ is a complete subspace of Y with $S(Y) \subset A(Y)$. Then (S, A) have a coincidence point u in Y . Further if the point of coincidence of S and A with respect to u is in Y , then S and A will have a unique common fixed point in Y , provided S and A are weakly compatible.

Acknowledgements. The author wishes to express sincere thanks to the referee for his/her valuable suggestions in improving the paper.

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Accepted: 21.06.2013

A CHARACTERIZATION OF PROJECTIVE SPECIAL LINEAR GROUP $L_3(5)$ BY nse

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Abstract. Let G be a group and $\omega(G)$ be the set of element orders of G . Let $k \in \omega(G)$ and s_k be the number of elements of order k in G . Let $\text{nse}(G) = \{s_k | k \in \omega(G)\}$. In Khatami et al. and Liu's works, the authors proved that the groups $L_3(2)$ and $L_3(4)$ are unique determined by nse. In this paper, we prove that if G is a group such that $\text{nse}(G) = \text{nse}(L_3(5))$, then $G \cong L_3(5)$.

Keywords and phrases: Element order, Projective special linear group, Thompson's problem, Number of elements of the same order.

AMS Subject Classification: 20D05, 20D06, 20D20.

1. Introduction

In 1987, Thompson posed a very interesting problem related to algebraic number fields as follows (see [17]).

Thompson's Problem. Let $T(G) = \{(n, s_n) | n \in \omega(G) \text{ and } s_n \in \text{nse}(G)\}$, where s_n is the number of elements with order n . Suppose that $T(G) = T(H)$. If G is a finite solvable group, is it true that H is also necessarily solvable?

It was proved that: Let G be a group and M some simple K_i -group, $i = 3, 4$, then $G \cong M$ if and only if $|G| = |M|$ and $\text{nse}(G) = \text{nse}(M)$ (see [13], [14]). The group A_{12} is characterized by nse and order (see [8]). Recently, all sporadic simple groups are characterizable by nse and these orders (see [1]).

Comparing the sizes of elements of same order but disregarding the actual orders of elements in $T(G)$ of the Thompson's Problem, in other words, it remains only $\text{nse}(G)$, whether can it characterize finite simple groups? Up to now, some groups especial for $L_2(q)$, where $q = 7, 8, 9, 11, 13$, can be characterized by only the set $\text{nse}(G)$ (see [7], [16]). The author has proved that the groups $L_3(4)$, $L_5(2)$

and $U_3(5)$ are characterizable by nse (see [9], [10] and [11], respectively). In this paper, it is shown that the group $L_3(5)$ also can be characterized by $\text{nse}(L_3(5))$.

We introduce some unfamiliar notations which will be used in this note. Let $a \cdot b$ denote the products of an integer a by an integer b . If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n . Let G be a group and r a prime. Then we denote the number of Sylow r -subgroups P_r of G by n_r or $n_r(G)$. The set of element orders of G is denoted by $\omega(G)$. Let $k \in \omega(G)$ and s_k be the number of elements of order k in G . Let $\text{nse}(G) = \{s_k | k \in \omega(G)\}$. Let $\pi(G)$ denote the set of prime p such that G contains an element of order p . The other notations are standard (see [2]).

2. Some lemmas

Lemma 2.1 [4] *Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G | g^m = 1\}$, then $m \mid |L_m(G)|$.*

Lemma 2.2 [12] *Let G be a finite group and $p \in \pi(G)$ be odd. Suppose that P is a Sylow p -subgroup of G and $n = p^s m$ with $(p, m) = 1$. If P is not cyclic and $s > 1$, then the number of elements of order n is always a multiple of p^s .*

Lemma 2.3 [16] *Let G be a group containing more than two elements. If the maximal number s of elements of the same order in G is finite, then G is finite and $|G| \leq s(s^2 - 1)$.*

Lemma 2.4 [5, Theorem 9.3.1] *Let G be a finite solvable group and $|G| = mn$, where $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $(m, n) = 1$. Let $\pi = \{p_1, \dots, p_r\}$ and h_m be the number of Hall π -subgroups of G . Then $h_m = q_1^{\beta_1} \cdots q_s^{\beta_s}$ satisfies the following conditions for all $i \in \{1, 2, \dots, s\}$:*

- (1) $q_i^{\beta_i} \equiv 1 \pmod{p_j}$ for some p_j .
- (2) The order of some chief factor of G is divided by $q_i^{\beta_i}$.

Definition 2.5 A finite group G is called a simple K_n -group, if G is a simple group with $|\pi(G)| = n$.

Remark 2.6 If G is a simple K_1 -group, then G is a cyclic group of order p .

Remark 2.7 If G is a simple K_2 -group, then by Thompson' $p^a q^b$ Theorem, G is soluble. Therefore there is no simple K_2 -group.

Lemma 2.8 [18] *Let G be a simple K_4 -group. Then G is isomorphic to one of the following groups:*

- (1) A_7, A_8, A_9 or A_{10} .
- (2) M_{11}, M_{12} or J_2 .

(3) One of the following:

- (a) $L_2(r)$, where r is a prime and $r^2 - 1 = 2^a \cdot 3^b \cdot v^c$ with $a \geq 1$, $b \geq 1$, $c \geq 1$, and v is a prime greater than 3.
- (b) $L_2(2^m)$, where $2^m - 1 = u$, $2^m + 1 = 3t^b$ with $m \geq 2$, u , t are primes, $t > 3$, $b \geq 1$.
- (c) $L_2(3^m)$, where $3^m + 1 = 4t$, $3^m - 1 = 2u^c$ or $3^m + 1 = 4t^b$, $3^m - 1 = 2u$, with $m \geq 2$, u , t are odd primes, $b \geq 1$, $c \geq 1$.

(4) One of the following 28 simple groups: $L_2(16)$, $L_2(25)$, $L_2(49)$, $L_2(81)$, $L_3(4)$, $L_3(5)$, $L_3(7)$, $L_3(8)$, $L_3(17)$, $L_4(3)$, $S_4(4)$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_6(2)$, $O_8^+(2)$, $G_2(3)$, $U_3(4)$, $U_3(5)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_4(3)$, $U_5(2)$, $Sz(8)$, $Sz(32)$, ${}^2D_4(2)$ or ${}^2F_4(2)'$.

Lemma 2.9 Let G be a simple K_4 -group and $31 \mid |G| \mid 2^5 \cdot 3 \cdot 5^3 \cdot 31$. Then $G \cong L_3(5)$ or $L_2(31)$.

Proof. Since G is a K_4 -group, then by Lemma 2.8(1)(2), order consideration rules out these cases.

So, by Lemma 2.8(3), the following cases are considered.

Case 1. $G \cong L_2(r)$, where $r \in \{3, 5, 31\}$.

- Let $r = 3$, then $|\pi(r^2 - 1)| = 1$, which contradicts $|\pi(r^2 - 1)| = 3$.
- Let $r = 5$, then $|\pi(r^2 - 1)| = 2$, which contradicts $|\pi(r^2 - 1)| = 3$.
- Let $r = 31$, then $|\pi(31^2 - 1)| = 3$. So $G \cong L_2(31)$ since $|G| = 2^5 \cdot 3 \cdot 5 \cdot 31$.

Case 2. $G \cong L_2(2^m)$, where $u \in \{3, 5, 31\}$.

- Let $u = 3$, then $m = 2$ and so $5 = 3t^b$. But the equation has no solution in $\mathbb{N}$, a contradiction.
- Let $u = 5$, then the equation $2^m - 1 = 5$ has no solution in $\mathbb{N}$, a contradiction..
- Let $u = 31$, then $32 = 2^m$ and so $m = 5$. So $33 = 3t^b$, and so $t = 11$, which is a contradiction since $11 \nmid |G|$.

Case 3. $G \cong L_2(3^m)$.

The following two cases need to be considered.

- Subcase 3.1. $3^m + 1 = 4t$ and $3^m - 1 = 2u^c$.
Let $t \in \{3, 5, 31\}$.
If $t = 3, 5, 31$, the equation $3^m + 1 = 4t$ has no solution. So the case can be ruled out.

- Subcase 3.2. $3^m + 1 = 4t^b$ and $3^m - 1 = 2u$.

Let $u \in \{3, 5, 31\}$.

If $u = 3, 5, 31$, then the equation $3^m - 1 = 2u$ has no solution in $\mathbb{N}$, a contradiction.

By Lemma 2.8(4), order consideration rules out the groups except for $L_3(5)$. Therefore, G is isomorphic to $L_2(31)$ or $L_3(5)$.

This completes the proof. $\blacksquare$

3. Main theorem and its proof

Let G be a group such that $\text{nse}(G) = \text{nse}(L_3(5))$, and s_n be the number of elements of order n . By Lemma 2.3 we have that G is finite. We note that $s_n = k\phi(n)$, where k is the number of cyclic subgroups of order n . Also, we note that if $n > 2$, then $\phi(n)$ is even. If $m \in \omega(G)$, then by Lemma 2.1 and the above discussion, we have

$$(3.1) \quad \begin{cases} \phi(m) \mid s_m \\ m \mid \sum_{d \mid m} s_d \end{cases}$$

Theorem 3.1 *Let G be a group with $\text{nse}(G) = \text{nse}(L_3(5)) = \{1, 775, 15500, 15624, 18600, 24800, 31000, 37200, 62000, 120000\}$. Then $G \cong L_3(5)$.*

Proof. We first prove that $\pi(G) \subseteq \{2, 3, 5, 31\}$, then show that $|G| = |L_3(5)|$ and so by [14], $G \cong L_3(5)$.

By (3.1), $\pi(G) \subseteq \{2, 3, 5, 7, 11, 31, 37201\}$.

If $k > 2$, then $\phi(k)$ is even, $s_2 = 775$, and so $2 \in \pi(G)$.

If $5, 31, 3 \in \pi(G)$, then $s_5 = 15624$, $s_{31} = 120000$, $s_3 = 15500$, 24800 , 62000 .

In the following, we prove $37201 \notin \pi(G)$.

If $37201 \in \pi(G)$, then by (3.1), $s_{37201} = 37200$.

If $2 \cdot 37201 \in \omega(G)$, then $s_{74402} \notin \text{nse}(G)$.

Therefore $2 \cdot 37201 \notin \omega(G)$.

It follows that the Sylow 37201-subgroup of G acts fixed point freely on the set of elements of order 2 and so $|P_{37201}| \mid s_2$, a contradiction. Thus, $37201 \notin \pi(G)$.

If $2 \cdot 7 \in \omega(G)$, then by Lemma 2.3 of [15], $s_{2 \cdot 7} = s_7 \cdot t$ for some integer t . Then $s_{2 \cdot 7} = s_7$. But by Lemma 2.1, $2 \cdot 7 \mid 1 + s_2 + s_7 + s_{2 \cdot 7}(240776)$, a contradiction. It follows that the Sylow 7-subgroup of G acts fixed point freely on the set of elements of order 2 and $7 \mid s_2$, a contradiction. Hence $7 \notin \pi(G)$.

If $2^a \in \omega(G)$, then $\phi(2^a) = 2^{a-1} \mid s_{2^a}$ and so $0 \leq a \leq 7$.

If $3^a \in \omega(G)$, then $1 \leq a \leq 3$.

If $5^a \in \omega(G)$, then $1 \leq a \leq 5$. If $5^5 \in \omega(G)$, then $s_{3125} \notin \text{nse}(G)$ and so $1 \leq a \leq 4$.

If $11^a \in \omega(G)$, then $a = 1$.

If $31^a \in \omega(G)$, then $a = 1$ or 2 . If $a = 2$, then $s_{961} \notin \text{nse}(G)$ and so $961 \notin \omega(G)$. Hence $a = 1$.

If $2^a.3^b \in \omega(G)$, then $1 \leq a \leq 6$ and $1 \leq b \leq 3$.

If $3^a.5^b \in \omega(G)$, then $1 \leq a \leq 3$ and $1 \leq b \leq 4$.

If $2^a.3^b.5^c \in \omega(G)$, then $1 \leq a \leq 4$, $1 \leq b \leq 3$ and $1 \leq c \leq 4$.

To remove the prime 11, the fact that the prime 31 which belong to $\pi(G)$ will be proved.

Suppose that $31 \notin \pi(G)$.

- If $3, 5, 11 \notin \pi(G)$, then G is a 2-group and so $372000 + 15500k_1 + 15624k_2 + 18600k_3 + 24800k_4 + 31000k_5 + 37200k_6 + 62000k_7 + 120000k_8 = 2^m$ where $k_i, i = 1, 2, \dots, 8$ and m are non-negative integers. Since $|\omega(G)| = 8$, the equation has no solution in $\mathbb{N}$ since the number of elements of $\text{nse}(G)$ is ten. So, the following cases are considered: $\{3\}$, $\{3, 5\}$, $\{3, 11\}$, $\{3, 5, 11\}$, $\{5, 11\}$, $\{5, 7, 11\}$ and $\{11\}$,
- Let $11 \in \pi(G)$, then as $\exp(P_{11})=11$, $|P_{11}| \mid 1 + s_{11}$ and so $|P_{11}|=11$. Since $n_{11} = s_{11}/\phi(11)=1860$, then $31 \in \pi(G)$, a contradiction.

Similarly, the set which contains the prime 11 can be excluded as the the set $\{11\}$.

- Let $5 \in \pi(G)$. The exponent of P_5 is equal to 5, 25, 125, and 625.

* If $\exp(P_5)=5$, then $|P_5| \mid 1 + s_5$ and so $|P_5| \mid 5^6$.

If $|P_5| = 5$, then $n_5 = s_5/\phi(5)=3906$ and so $31 \in \pi(G)$, a contradiction.

If $|P_5| = 5^2$, then since $31, 11 \notin \pi(G)$, $372000 + 15500k_1 + 15624k_2 + 18600k_3 + 24800k_4 + 31000k_5 + 37200k_6 + 62000k_7 + 120000k_8 = 2^m.3^n.5^2$ where $k_i, i = 1, 2, \dots, 8$ and m, n are non-negative integers, and $0 \leq \sum_{i=1}^8 k_i \leq 77$. Since $372000 \leq |G| = 2^m.3^n.5^2 \leq 37200 + 77.120000$, then if $n = 0$, $m = 12, \dots, 18$; if $n = 1$, then $m = 13, \dots, 16$; if $n = 2$, then $m = 11, \dots, 15$; if $n = 3$, then $m = 10, \dots, 13$. But m is at most seven, and so these cases can be excluded. If $|P_5| = 5^3$, then similarly if $n = 0$, then $m = 12, \dots, 16$; if $n = 1$, then $m = 10, \dots, 14$; if $n = 2$, then $m = 9, \dots, 13$; if $n = 3$, then $m = 7, \dots, 11$. Therefore only $m = 7$ is considered. In this case, $372000 + 15500k_1 + 15624k_2 + 18600k_3 + 24800k_4 + 31000k_5 + 37200k_6 + 62000k_7 + 120000k_8 = |G| = 2^7.3^3.5^3$ where $k_1, \dots, k_8$ are non-negative integers. By computer computation, the equation has no solution in $\mathbb{N}$.

Similarly, the other cases $|P_5| = 5^4, 5^5, 5^6$ can be ruled out as the above methods.

- * If $\exp(P_5) = 25$, then $|P_5| \mid 1 + s_5 + s_{25}$ and so $|P_5| \mid 125$. If $|P_5| = 25$, then by (1), $n_5 = s_{25}/\pi(25)$ if $s_{25} \in \{15500, 18600, 24800, 31000, 37200\}$ and so $31 \in \pi(G)$, a contradiction.

If $s_{25} = 120000$, then $n_5 = 6000$. By Sylow's theorem, $n_5 = 5k + 1$ for some integer k , but the equation has no solution in $\mathbb{N}$.

If $|P_5| = 125$, then similarly as the case “ $\exp(P_5)=5$ and $|P_5| = 25$ ”, we can rule out this case.

- * If $\exp(P_5) = 125$, then by Lemma 2.1, $|P_5| \mid 1 + s_5 + s_{25} + s_{125}$ and so $|P_5| \mid 5^4$.

If $|P_5| = 5^3$, then by (3.1), $s_{125}=15500, 24800, 31000, 37200, 120000$. If $s_{125}=15500, 18600, 24800, 31000, 37200$, then $n_5 = s_{125}/\phi(125)=155, 186, 248, 310, 372$ and so 31 belongs to $\pi(G)$, a contradiction. Hence $s_{125}=120000$ and $n_5=1200$. But by Sylow's theorem, $n_5 = 5k + 1$ for some integer k , it is easy to see that the equation has no solution in $\mathbb{N}$. If $|P_5| = 5^4$, then similarly as the case “ $\exp(P_5)=5$ and $|P_5| = 25$ ”, we can rule out this case.

- * If $\exp(P_5) = 625$, then by Lemma 2.1, $|P_5| \mid 1 + s_5 + s_{25} + s_{125} + s_{625}$ ($s_{625} = 15500, 31000, 120000$) and so $|P_5| = 625$. If $s_{625}=15500, 31000$, $n_5 = s_{625}/\phi(625)=31, 62$ and so 31 belongs to $\pi(G)$, a contradiction. Therefore $s_{625} = 120000$ and $t=240$. But by Sylow's theorem $t = 5k + 1$ for some integer k , so the equation has no solution in $\mathbb{N}$.

Similarly, the set which contains the prime 5 can be excluded as the the set $\{5\}$.

- Let $3 \in \pi(G)$. The exponent of P_3 equal to 3, 9, 27.

- * Let $\exp(P_3)=3$. Then by Lemma 2.1, $|P_3| \mid 1 + s_3$ and so $|P_3| \mid 9$.

If $|P_3| = 3$, then $n_3 = s_3/\phi(3)$ and so 31 belongs to $\pi(G)$, a contradiction.

If $|P_3| = 9$, then $s_3 = 3t$ for some non-negative integer t . So $s_3=15624, 18600, 37200, 120000$ and $t=5208, 6200, 12400, 40000$, respectively. If $s_3=15624, 18600, 37200$, then $31 \in \pi(G)$. So $s_3 = 120000$. In this case, since $2 \in \pi(G)$ and if $31 \notin \pi(G)$, the primes $11, 5 \notin \pi(G)$, then $\pi(G) = \{2, 3\}$ and so $372000 + 15500k_1 + 15624k_2 + 18600k_3 + 24800k_4 + 31000k_5 + 37200k_6 + 62000k_7 + 120000k_8 = 2^m \cdot 9$ where $k_i, i = 1, 2, \dots, 8$ and m are non-negative integers, and $0 \leq \sum_{i=1}^8 k_i \leq 7$. Since $372000 \leq |G| = 2^m \cdot 3^2 \leq 37200 + 7 \cdot 120000$, then $m = 16$, a contradiction since m is at most 7.

- * Let $\exp(P_3)=9$. Then by Lemma 2.1, $|P_3| \mid 1 + s_3 + s_9$ and so $|P_3| \mid 81$.

If $|P_3| = 9$, then 5 or 31 belongs to $\pi(G)$, a contradiction.

If $|P_3| = 27$, then similarly $m = 16$, a contradiction. If $|P_3| = 81$, then $m > 7$, a contradiction.

- * Let $\exp(P_3)=27$. Then by Lemma 2.1, $|P_3| \mid 1 + s_3 + s_9 + s_{27}$ and so $|P_3| = 27$. So $n_3 = s_{27}/\phi(27)$ and 31 belongs to $\pi(G)$, a contradiction.

Similarly, the set which contains the prime 3 can be excluded as the the set $\{3\}$.

Therefore, $31 \in \pi(G)$.

By Lemma 2.1, $|P_{31}| \mid 1 + s_{31}$ and so $|P_{31}| = 31$.

In the following, that the prime 11 do not belong to $\pi(G)$ are proved.

Let $11 \in \pi(G)$. If $11.31 \in \omega(G)$, then by Lemma 2.3 of [15], $s_{11.31} = 10.s_{31}.t$ for some integer t . But the equation has no solution since $s_{11.31} \in \text{nse}(G)$. Hence $11.31 \notin \omega(G)$. It follows that the Sylow 11-subgroup of G acts fixed freely on the set of elements of order 31 and $11 \mid s_{31}$, a contradiction. So $11 \notin \pi(G)$.

Therefore, $\pi(G) \subseteq \{2, 3, 5, 31\}$.

From the above arguments, $2, 31 \in \pi(G)$, so the following cases will be considered: $\{2, 31\}$, $\{2, 3, 31\}$, $\{2, 5, 31\}$ and $\{2, 3, 5, 31\}$.

Case a. $\pi(G) = \{2, 31\}$.

Since $\exp(P_{31})=31$, then by Lemma 2.1, $|P_{31}| \mid 1 + s_{31}$ and so $|P_{31}|=31$. Since $n_{31} = s_{31}/\phi(31) = 4000$, then 5 belongs to $\pi(G)$, a contradiction.

Case b. $\pi(G) = \{2, 3, 31\}$.

The proof is the same as Case a.

Case c. $\pi(G) = \{2, 5, 31\}$.

Since $\exp(P_{31})=31$, then by Lemma 2.1, $|P_{31}| \mid 1 + s_{31}$ and so $|P_{31}| = 31$.

If $2.31 \in \omega(G)$, set P and Q are Sylow 31-subgroups of G , then P and Q are conjugate in G and so $C_G(P)$ and $C_G(Q)$ are conjugate in G . Then $s_{2.31} = \phi(2.31).n_{31}.k$, where k is the number of cyclic subgroups of order 2 in $C_G(P_{31})$, and so $n_{31} \mid s_{2.31}$. So $s_{2.31} = n_{31}.t$ for some integer t . Since $n_{31} = s_{31}/\phi(31)$, $s_{2.31} = 4000t$ for some integer t and so $s_{2.31} = s_{31}$. On the other hand, $2.31 \mid 1 + s_2 + s_{31} + s_{2.31} (=240776)$, a contradiction. Thus $2.31 \notin \omega(G)$. It follows that the Sylow 2-subgroup of G acts fixed point freely on the set of elements of order 31 and so $|P_2| \mid s_{31}$. So $|P_2| \mid 2^6$. Also by (3.1), $5.31 \notin \omega(G)$ and $|P_5| \mid 5^4$.

We know that $\exp(P_5) = 5, 5^2, 5^3, 5^4$.

Let $\exp(P_5) = 5$.

- If $|P_5| = 5$, then $n_5 = s_5/\phi(5) = 3906$ and so $3 \in \pi(G)$, a contradiction.
- If $|P_5| = 5^2$, then $372000 + 15500k_1 + 15624k_2 + 18600k_3 + 24800k_4 + 31000k_5 + 37200k_6 + 62000k_7 + 120000k_8 = 2^m.5^2.31$ where $k_i, i = 1, 2, \dots, 8$ and m are non-negative integers, and $0 \leq \sum_{i=1}^8 k_i \leq 3$. Since $372000 \leq |G| = 2^m.5^2.31 \leq 37200 + 3.120000$, the equation has no solution since m is at most 6.
- If $|P_5| = 5^3$, then similarly, the equation has no solution.
- If $|P_5| = 5^4$, then similarly $m = 6, 5$. Let G be a group such that $|G| = 2^6.5^4.31$ or $|G| = 2^5.5^4.31$ and $\text{nse}(G) = \text{nse}(L_3(5))$. Then by programme of [7], there is no such group.

Let $\exp(P_5) = 25$. Then by (3.1), $s_{5^2} \in \{15500, 18600, 24800, 31000, 37200, 62000, 120000\}$

- If $|P_5| = 5^2$ and $s_{5^2} \in \{15500, 18600, 37200, 120000\}$, then $n_5 = s_{5^2}/\phi(5^2)$ and $3 \in \pi(G)$, a contradiction.

If $|P_5| = 5^2$ and $s_{5^2} \in \{24800, 31000\}$, then $n_5 = 1240, 1550$. But by Sylow's theorem, $n_5 = 5k + 1$ for some integer k , the equation has no solution in $\mathbb{N}$.

- If $|P_5| \geq 5^3$, then similarly as the proof of “ $\exp(P_5) = 5$ ”, we can rule out this case.

Let $\exp(P_5) = 5^3$. Then by (3.1), $s_{5^3} \in \{15500, 24800, 31000, 37200, 120000\}$.

- If $s_{5^3} \in \{37200, 120000\}$ and $|P_5| = 5^3$, then $n_5 = s_{5^3}/\phi(5^3)$ and $3 \in \pi(G)$, a contradiction.

If $s_{5^3} \in \{15500, 24800, 31000\}$ and $|P_5| = 5^3$, then $n_5 = 155, 248, 310$. On the other hand, by Sylow's theorem, $n_5 = 5k + 1$ for some integer k , the equation has no solution in $\mathbb{N}$.

- If $|P_5| = 5^4$, then similarly as the proof of “ $\exp(P_5) = 5$ ”, we can rule out this case.

Let $\exp(P_5) = 5^4$. Then similarly as the proof of “ $\exp(P_5) = 5$ ”, we can rule out this case.

Case d. $\pi(G) = \{2, 3, 5, 31\}$.

From the above, $|P_{31}| = 31$.

If $3.31 \in \omega(G)$, set P and Q are Sylow 31-subgroups of G , then P and Q are conjugate in G and so $C_G(P)$ and $C_G(Q)$ are also conjugate in G . Therefore we have $s_{3.31} = \phi(3.31) \cdot n_{31} \cdot k$, where k is the number of cyclic subgroups of order 3 in $C_G(P_{31})$. As $n_{31} = s_{31}/\phi(31) = 120000/30 = 40000$, $240000 \mid s_{3.31}$ and so $s_{3.31} = 240000t$ for some integer t , but the equation has no solution in $\mathbb{N}$. So $3.31 \notin \omega(G)$. It follows that the Sylow 3-subgroup P_3 of G acts fixed point freely on the set of elements of order 31 and so $|P_3| \mid s_{31}$. Therefore $|P_3| \mid 3$. Similarly we can prove that $3.5 \notin \omega(G)$ and $|P_5| \mid 5^3$.

Therefore, $|G| = 2^m \cdot 3 \cdot 5^p \cdot 31$. Since $372000 = 2^5 \cdot 3 \cdot 5^3 \cdot 31 \leq |G| = 2^m \cdot 3 \cdot 5^p \cdot 31$, then $(m, p) = (5, 3), (6, 3)$.

In the following, first show that there is no group such that $|G| = 2^6 \cdot 3 \cdot 5^3 \cdot 31$ and $\text{nse}(G) = \text{nse}(L_3(5))$, second get the desired result.

Since $|\pi(G)| \geq 3$ and $\{3.31, 5.31, 3.5\} \cap \omega(G) = \Phi$, then by Lemma 2.5 of [3], G is insoluble.

Therefore, we can suppose that G has a normal series $1 \triangleleft K \triangleleft L \triangleleft G$ such that L/K is isomorphic to a simple K_i -group with $i = 3, 4$ as 9 and 961 do not divide order of G .

If L/K is isomorphic to a K_3 -simple group, then from [6], $L/K \cong A_5, U_4(2)$. From [2], $n_5(L/K) = n_5(A_5) = 6$, and so $n_5(G) = 6t$ and $5 \nmid t$. Hence the number of elements of order 5 in G is: $s_5 = 6t \cdot 4 = 24t$ for some integer t . Since

$s_5 \in \text{nse}(G)$, then $s_5=15624$ and $t = 651$. Hence $3.7.31 \mid |K| \mid 2^3 \cdot 5^3 \cdot 31$, which is a contradiction. Similarly the other group $U_4(2)$ can be ruled out as the group “ A_5 ”.

If L/K is isomorphic to a K_4 -group, then by Lemma 2.9, $L/K \cong L_2(31)$ or $L/K \cong L_3(5)$.

Let $L/K \cong L_2(31)$.

Let $\overline{G} = G/K$ and $\overline{L} = L/K$. Then

$$L_2(31) \leq \overline{L} \cong \overline{L}C_{\overline{G}}(\overline{L})/C_{\overline{G}}(\overline{L}) \leq \overline{G}/C_{\overline{G}}(\overline{L}) = N_{\overline{G}}(\overline{L})/C_{\overline{G}}(\overline{L}) \leq \text{Aut}(\overline{L})$$

Set $M = \{xK \mid xK \in C_{\overline{G}}(\overline{L})\}$, then $G/M \cong \overline{G}/C_{\overline{G}}(\overline{L})$ and so $L_2(31) \leq G/M \leq \text{Aut}(L_2(31))$. Therefore $G/M \cong L_2(31)$, or $G/M \cong 2.L_2(31)$.

If $G/M \cong L_2(31)$, then order consideration $|M| = 2.5^2$. Then there exist a group M such that M is a Frobenius subgroup with a complement of order 2 and a Frobenius kernel of order 5^2 . So there exists an element of order 3.5^2 , which is a contradiction.

If $G/M \cong 2.L_2(31)$, then $|M| = 25$ and M is a normal subgroups generated by 5-central elements or with exponent of 5^2 . If the former, then there is an element of order 5.31 , a contradiction. If the latter, also we have a contradiction.

If $L/K \cong L_3(5)$, then similarly we have that $G/M \cong L_3(5)$ or $G/M \cong 2.L_3(5)$.

If $G/M \cong L_3(5)$, then $|M| = 2$. It follows that M is a normal subgroup generated by a 2-central element and so there exists an element of order 2.31 , a contradiction.

$G/M \cong 2.L_3(5)$, then $M = 1$. Since $\text{nse}(2.L_3(5)) \neq \text{nse}(G)$, then we rule out this case.

Hence $|G| = 2^5 \cdot 3 \cdot 5^3 \cdot 31$ and by assumption, $\text{nse}(G) = \text{nse}(L_3(5))$, so by [14], $G \cong L_3(5)$.

This completes the proof of the theorem. ■

Acknowledgments. The object is supported by the Department of Education of Sichuan Province (Grant No: 12ZB085, 12ZB291 and 13ZA0119). The author is very grateful for the helpful suggestions of the referee.

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Accepted: 01.09.2013

NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR FUNCTIONS WHOSE FIRST DERIVATIVES ABSOLUTE VALUES ARE s -CONVEX

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Abstract. In this paper, some new inequalities of the left-hand side of Hermite-Hadamard-type are obtained for functions whose first derivatives absolute values are s -convex.

Keywords: Hermite-Hadamard's inequality; s -convex functions; Hölder inequality; power mean inequality.

AMS Subject Classifications: 26D15; 26D10.

1. Introduction

If $f : I \subset R_+ \rightarrow R_+$ where $R_+ = [0, \infty)$ is said to be s -convex on I if the inequality

$$(1.1) \quad f(\alpha x + (1 - \alpha)y) \leq \alpha^s f(x) + (1 - \alpha)^s f(y)$$

holds for all $x, y \in I$ and $\alpha \in [0, 1]$. It can be easily seen that for $s = 1$, s -convexity reduces to ordinary convexity of functions defined on $[0, \infty)$.

One of the most famous inequality for the class of convex functions is so called Hermite-Hadamard inequality, which states that: Let $f : I \subset R \rightarrow R$ be a convex function on the interval I , then for any $a, b \in I$ with $a \neq b$ we have the following double inequality

$$(1.2) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)dt \leq \frac{f(a) + f(b)}{2}.$$

Since then, some refinements of the Hermite-Hadamard inequality on convex functions have been extensively investigated by a number of authors (e.g., [1], [3], [4], [7], [8], [9] and [10]).

In [6], Dragomir and Fitzpatrick established a variant of Hermite-Hadamard inequality which holds for the s -convex functions.

Theorem 1.1 *Suppose that $f : I \subset [0, \infty) \rightarrow [0, \infty)$ is an s -convex function in the second sense, where $s \in (0, 1]$ and let $a, b \in [0, \infty)$, $a < b$. If $f \in L[0, 1]$, then the following inequality holds*

$$(1.3) \quad 2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a) + f(b)}{s+1}.$$

Along this paper, we consider a real interval $I \subset R$, and we denote that I^0 is the interior of I .

In [5], Dragomir and Agarwal obtained the following Hermite-Hadamard type integral inequality.

Theorem 1.2 *Let $f : I \subset R \rightarrow R$ be differentiable mapping on I^0 , where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds*

$$(1.4) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{b-a}{8} [|f'(a)| + |f'(b)|].$$

In [2], Alomari, Darus and Kirmaci proved the following inequalities of Hermite-Hadamard type for differentiable convex mappings.

Theorem 1.3 *Let $f : I \subset [0, \infty) \rightarrow R$ be a differentiable mapping on I^0 , such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is s -convex on $[a, b]$, for some fixed $s \in (0, 1]$, then the following inequality holds:*

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{b-a}{4(s+1)(s+2)} \left[|f'(a)| + 2(s+1) \left| f'\left(\frac{a+b}{2}\right) \right| + |f'(b)| \right] \\ & \leq \frac{(2^{2-s} + 1)(b-a)}{4(s+1)(s+2)} [|f'(a)| + |f'(b)|]. \end{aligned}$$

In [12], Pearce and Pečarić proved the following theorem.

Theorem 1.4 *Let $f : I \subset R \rightarrow R$ be a differentiable mapping on I^0 , such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a, b]$, for some $q \geq 1$, then the following inequality holds:*

$$(1.5) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{b-a}{4} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{1/q}.$$

If $|f'|^q$ is concave on $[a, b]$, for some $q \geq 1$, then

$$(1.6) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{b-a}{4} \left| f'\left(\frac{a+b}{2}\right) \right|.$$

For recent results and generalizations concerning Hermite-Hadamard's inequality, see [6]-[12] and the references given therein.

In this paper, we establish some new inequalities of Hadamard's type for the class of s -convex functions in the second sense.

2. Lemmas

To prove our main results, we consider the following lemma:

Lemma 2.1 *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be differentiable mapping on I° , where $a, b \in I$ with $a < b$. If $f' \in L[a, b]$, then the following inequality holds*

$$\begin{aligned} & f(\lambda a + (1 - \lambda)b) - \frac{1}{b - a} \int_a^b f(t) dt \\ &= (b - a)(1 - \lambda)^2 \int_0^1 t f' \left(t(\lambda a + (1 - \lambda)b) + (1 - t)a \right) dt \\ &+ (b - a)\lambda^2 \int_0^1 (t - 1) f' \left(tb + (1 - t)(\lambda a + (1 - \lambda)b) \right) dt. \end{aligned}$$

for each $\lambda \in [0, 1]$.

Proof. We note that

$$\begin{aligned} I_1 &= \int_0^1 t f' \left(t(\lambda a + (1 - \lambda)b) + (1 - t)a \right) dt \\ &= \frac{1}{(b - a)(1 - \lambda)} t f \left(t(\lambda a + (1 - \lambda)b) + (1 - t)a \right) \Big|_0^1 \\ &\quad - \frac{1}{(b - a)(1 - \lambda)} \int_0^1 f \left(t(\lambda a + (1 - \lambda)b) + (1 - t)a \right) dt \\ &= \frac{1}{(b - a)(1 - \lambda)} f(\lambda a + (1 - \lambda)b) \\ &\quad - \frac{1}{(b - a)(1 - \lambda)} \int_0^1 f \left(t(\lambda a + (1 - \lambda)b) + (1 - t)a \right) dt. \end{aligned}$$

Setting $x = t(\lambda a + (1 - \lambda)b) + (1 - t)a$, and $dx = (b - a)(1 - \lambda)dt$, which gives

$$I_1 = \frac{1}{(b - a)(1 - \lambda)} f(\lambda a + (1 - \lambda)b) - \frac{1}{(b - a)^2(1 - \lambda)^2} \int_a^{\lambda a + (1 - \lambda)b} f(x) dx.$$

Similarly, we can show that

$$\begin{aligned} I_2 &= \int_0^1 (t - 1) f' \left(tb + (1 - t)(\lambda a + (1 - \lambda)b) \right) dt \\ &= \frac{1}{(b - a)\lambda} f(\lambda a + (1 - \lambda)b) - \frac{1}{(b - a)^2\lambda^2} \int_{\lambda a + (1 - \lambda)b}^b f(x) dx, \end{aligned}$$

and therefore,

$$\begin{aligned} I &= (b-a)(1-\lambda)^2 I_1 + (b-a)\lambda^2 I_2 \\ &= f(\lambda a + (1-\lambda)b) - \frac{1}{b-a} \int_a^b f(x) dx, \end{aligned}$$

which completes the proof. $\blacksquare$

Remark 1. Applying Lemma 2.1 for $\lambda = \frac{1}{2}$, we get the Lemma 2.1 in [2].

3. The new Hermite-Hadamard type inequalities

Theorem 3.1 Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is s -convex on $[a, b]$, for some fixed $s \in (0, 1]$, then the following inequality holds

$$\begin{aligned} &\left| f(\lambda a + (1-\lambda)b) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ &\leq (b-a)(1-\lambda)^2 \left(\frac{1}{(s+1)(s+2)} |f'(a)| + \frac{1}{s+2} |f'(\lambda a + (1-\lambda)b)| \right) \\ &\quad + (b-a)\lambda^2 \left(\frac{1}{(s+1)(s+2)} |f'(b)| + \frac{1}{s+2} |f'(\lambda a + (1-\lambda)b)| \right), \end{aligned}$$

for each $\lambda \in [0, 1]$.

Proof. From Lemma 2.1, we have

$$\begin{aligned} &\left| f(\lambda a + (1-\lambda)b) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ &\leq (b-a)(1-\lambda)^2 \int_0^1 t |f'(t(\lambda a + (1-\lambda)b) + (1-t)a)| dt \\ &\quad + (b-a)\lambda^2 \int_0^1 (1-t) |f'(tb + (1-t)(\lambda a + (1-\lambda)b))| dt \end{aligned}$$

Because $|f'|$ is s -convex, we have

$$\begin{aligned} &\left| f(\lambda a + (1-\lambda)b) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ &\leq (b-a)(1-\lambda)^2 \int_0^1 t \left(t^s |f'(\lambda a + (1-\lambda)b)| + (1-t)^s |f'(a)| \right) dt \\ &\quad + (b-a)\lambda^2 \int_0^1 (1-t) \left(t^s |f'(b)| + (1-t)^s |f'(\lambda a + (1-\lambda)b)| \right) dt \\ &= (b-a)(1-\lambda)^2 \left(\frac{1}{(s+1)(s+2)} |f'(a)| + \frac{1}{s+2} |f'(\lambda a + (1-\lambda)b)| \right) \\ &\quad + (b-a)\lambda^2 \left(\frac{1}{(s+1)(s+2)} |f'(b)| + \frac{1}{s+2} |f'(\lambda a + (1-\lambda)b)| \right), \end{aligned}$$

which completes the proof. $\blacksquare$

Remark 2. Applying Theorem 3.1 for $\lambda = \frac{1}{2}$, we get the result in Theorem 1.3.

Theorem 3.2 Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I^0 , such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^{p/(p-1)}$ is s -convex on $[a, b]$, for some fixed $s \in (0, 1]$, $p > 1$, then the following inequality holds

$$\begin{aligned} & \left| f(\lambda a + (1 - \lambda)b) - \frac{1}{b - a} \int_a^b f(t) dt \right| \\ & \leq (b - a) \left(\frac{1}{p + 1} \right)^{1/p} \left(\frac{1}{s + 1} \right)^{1/q} \left((1 - \lambda)^2 \left[(\lambda^s + 1) |f'(a)|^q + (1 - \lambda)^s |f'(b)|^q \right]^{1/q} \right. \\ & \quad \left. + \lambda^2 \left[\lambda^s |f'(a)|^q + ((1 - \lambda)^s + 1) |f'(b)|^q \right]^{1/q} \right), \end{aligned}$$

for each $\lambda \in [0, 1]$ and p is the conjugate of q , $q = p/(p - 1)$.

Proof. From Lemma 2.1 and using the Hölder inequality, we have

$$\begin{aligned} & \left| f(\lambda a + (1 - \lambda)b) - \frac{1}{b - a} \int_a^b f(t) dt \right| \\ & \leq (b - a) (1 - \lambda)^2 \int_0^1 t |f'(t(\lambda a + (1 - \lambda)b) + (1 - t)a)| dt \\ & \quad + (b - a) \lambda^2 \int_0^1 (1 - t) |f'(tb + (1 - t)(\lambda a + (1 - \lambda)b))| dt \\ & \leq (b - a) (1 - \lambda)^2 \left(\int_0^1 t^p dt \right)^{1/p} \left(\int_0^1 |f'(t(\lambda a + (1 - \lambda)b) + (1 - t)a)|^q dt \right)^{1/q} \\ & \quad + (b - a) \lambda^2 \left(\int_0^1 (1 - t)^p dt \right)^{1/p} \left(\int_0^1 |f'(tb + (1 - t)(\lambda a + (1 - \lambda)b))|^q dt \right)^{1/q}. \end{aligned}$$

Because $|f'|^q$ is s -convex, we have

$$\int_0^1 |f'(t(\lambda a + (1 - \lambda)b) + (1 - t)a)|^q dt \leq \frac{|f'(\lambda a + (1 - \lambda)b)|^q + |f'(a)|^q}{s + 1},$$

and

$$\int_0^1 |f'(tb + (1 - t)(\lambda a + (1 - \lambda)b))|^q dt \leq \frac{|f'(b)|^q + |f'(\lambda a + (1 - \lambda)b)|^q}{s + 1}.$$

Therefore, we have

$$\begin{aligned} & \left| f(\lambda a + (1 - \lambda)b) - \frac{1}{b - a} \int_a^b f(t) dt \right| \\ & \leq (b - a) \left(\frac{1}{p + 1} \right)^{1/p} \left(\frac{1}{s + 1} \right)^{1/q} \left((1 - \lambda)^2 \left[|f'(\lambda a + (1 - \lambda)b)|^q + |f'(a)|^q \right]^{1/q} \right. \\ & \quad \left. + \lambda^2 \left[|f'(\lambda a + (1 - \lambda)b)|^q + |f'(b)|^q \right]^{1/q} \right). \end{aligned}$$

Now, since $|f'|^q$ is s -convex on $[a, b]$, for any $\lambda \in [0, 1]$, then by (1) we have

$$(3.1) \quad |f'(\lambda a + (1 - \lambda)b)|^q \leq \lambda^s |f'(a)|^q + (1 - \lambda)^s |f'(b)|^q.$$

Combining all the above inequalities, we obtain

$$\begin{aligned} & \left| f(\lambda a + (1 - \lambda)b) - \frac{1}{b - a} \int_a^b f(t) dt \right| \\ & \leq (b - a) \left(\frac{1}{p+1} \right)^{1/p} \left(\frac{1}{s+1} \right)^{1/q} \left((1 - \lambda)^2 \left[\lambda^s |f'(a)|^q + (1 - \lambda)^s |f'(b)|^q + |f'(a)|^q \right]^{1/q} \right. \\ & \quad \left. + \lambda^2 \left[\lambda^s |f'(a)|^q + (1 - \lambda)^s |f'(b)|^q + |f'(b)|^q \right]^{1/q} \right). \end{aligned}$$

This proves the theorem. ■

Theorem 3.3 *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I^0 , such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is s -convex on $[a, b]$, for some fixed $s \in (0, 1]$, $q \geq 1$, then the following inequality holds*

$$\begin{aligned} & \left| f(\lambda a + (1 - \lambda)b) - \frac{1}{b - a} \int_a^b f(t) dt \right| \\ & \leq (b - a) \left(\frac{1}{(s + 1)(s + 2)} \right)^{1/q} \left(\frac{1}{2} \right)^{1 - \frac{1}{q}} \left((1 - \lambda)^2 \left[((s + 1)\lambda^s + 1) |f'(a)|^q \right. \right. \\ & \quad \left. \left. + (s + 1)(1 - \lambda)^s |f'(b)|^q \right]^{1/q} \right. \\ & \quad \left. + \lambda^2 \left[(s + 1)\lambda^s |f'(a)|^q + ((s + 1)(1 - \lambda)^s + 1) |f'(b)|^q \right]^{1/q} \right), \end{aligned}$$

for each $\lambda \in [0, 1]$.

Proof. From Lemma 2.1 and using the well-known power-mean inequality, we have

$$\begin{aligned} & \left| f(\lambda a + (1 - \lambda)b) - \frac{1}{b - a} \int_a^b f(t) dt \right| \\ & \leq (b - a) (1 - \lambda)^2 \int_0^1 t |f'(t(\lambda a + (1 - \lambda)b) + (1 - t)a)| dt \\ & \quad + (b - a) \lambda^2 \int_0^1 (1 - t) |f'(tb + (1 - t)(\lambda a + (1 - \lambda)b))| dt \\ & \leq (b - a) (1 - \lambda)^2 \left(\int_0^1 t dt \right)^{1 - \frac{1}{q}} \left(\int_0^1 t |f'(t(\lambda a + (1 - \lambda)b) + (1 - t)a)|^q dt \right)^{1/q} \\ & \quad + (b - a) \lambda^2 \left(\int_0^1 (1 - t) dt \right)^{1 - \frac{1}{q}} \left(\int_0^1 (1 - t) |f'(tb + (1 - t)(\lambda a + (1 - \lambda)b))|^q dt \right)^{1/q}. \end{aligned}$$

Because $|f'|^q$ is s -convex, by (1) we have

$$\begin{aligned} & \int_0^1 t \left| f' \left(t(\lambda a + (1-\lambda)b) + (1-t)a \right) \right|^q dt \\ & \leq \int_0^1 \left(t^{s+1} |f'(\lambda a + (1-\lambda)b)|^q + t(1-t)^s |f'(a)|^q \right) dt \\ & = \frac{1}{(s+1)(s+2)} |f'(a)|^q + \frac{1}{s+2} |f'(\lambda a + (1-\lambda)b)|^q, \end{aligned}$$

and

$$\begin{aligned} & \int_0^1 (1-t) \left| f' \left(tb + (1-t)(\lambda a + (1-\lambda)b) \right) \right|^q dt \\ & \leq \int_0^1 \left((1-t)t^s |f'(b)|^q + (1-t)^{s+1} |f'(\lambda a + (1-\lambda)b)|^q \right) dt \\ & = \frac{1}{(s+1)(s+2)} |f'(b)|^q + \frac{1}{s+2} |f'(\lambda a + (1-\lambda)b)|^q. \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \left| f(\lambda a + (1-\lambda)b) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq (b-a) \left(\frac{1}{(s+1)(s+2)} \right)^{1/q} \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \left((1-\lambda)^2 [(s+1) |f'(\lambda a + (1-\lambda)b)|^q + |f'(a)|^q]^{1/q} \right. \\ & \quad \left. + \lambda^2 [(s+1) |f'(\lambda a + (1-\lambda)b)|^q + |f'(b)|^q]^{1/q} \right). \end{aligned}$$

Now, since $|f'|^q$ is s -convex on $[a, b]$, for any $\lambda \in [0, 1]$, then by (1) we have

$$(3.2) \quad |f'(\lambda a + (1-\lambda)b)|^q \leq \lambda^s |f'(a)|^q + (1-\lambda)^s |f'(b)|^q.$$

Combining all the above inequalities, we have

$$\begin{aligned} & \left| f(\lambda a + (1-\lambda)b) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq (b-a) \left(\frac{1}{(s+1)(s+2)} \right)^{1/q} \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \left((1-\lambda)^2 \left[((s+1)\lambda^s + 1) |f'(a)|^q \right. \right. \\ & \quad \left. \left. + (s+1)(1-\lambda)^s |f'(b)|^q \right]^{1/q} \right. \\ & \quad \left. + \lambda^2 \left[(s+1)\lambda^s |f'(a)|^q + ((s+1)(1-\lambda)^s + 1) |f'(b)|^q \right]^{1/q} \right), \end{aligned}$$

we get the desired result. ■

Theorem 3.4 Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I^0 , such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is s -concave on $[a, b]$, for some fixed $s \in (0, 1]$, $q > 1$, then the following inequality holds

$$\begin{aligned} & \left| f(\lambda a + (1-\lambda)b) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq (b-a) \left(\frac{q-1}{2q-1} \right)^{1-\frac{1}{q}} 2^{\frac{s-1}{q}} \left((1-\lambda)^2 \left| f' \left(\frac{(1+\lambda)a + (1-\lambda)b}{2} \right) \right| + \lambda^2 \left| f' \left(\frac{\lambda a + (2-\lambda)b}{2} \right) \right| \right), \end{aligned}$$

for each $\lambda \in [0, 1]$.

Proof. From Lemma 2.1 and using the Hölder inequality, we have

$$\begin{aligned}
& \left| f(\lambda a + (1 - \lambda)b) - \frac{1}{b - a} \int_a^b f(t) dt \right| \\
& \leq (b - a)(1 - \lambda)^2 \int_0^1 t |f'(t(\lambda a + (1 - \lambda)b) + (1 - t)a)| dt \\
& \quad + (b - a)\lambda^2 \int_0^1 (1 - t) |f'(tb + (1 - t)(\lambda a + (1 - \lambda)b))| dt \\
& \leq (b - a)(1 - \lambda)^2 \left(\int_0^1 t^p dt \right)^{1/p} \left(\int_0^1 |f'(t(\lambda a + (1 - \lambda)b) + (1 - t)a)|^q dt \right)^{1/q} \\
& \quad + (b - a)\lambda^2 \left(\int_0^1 (1 - t)^p dt \right)^{1/p} \left(\int_0^1 |f'(tb + (1 - t)(\lambda a + (1 - \lambda)b))|^q dt \right)^{1/q}.
\end{aligned}$$

Because $|f'|^q$ is s -concave, by the reversed direction of (3) we have

$$\int_0^1 |f'(t(\lambda a + (1 - \lambda)b) + (1 - t)a)|^q dt \leq 2^{s-1} \left| f'\left(\frac{(1 + \lambda)a + (1 - \lambda)b}{2}\right) \right|^q,$$

and

$$\int_0^1 |f'(tb + (1 - t)(\lambda a + (1 - \lambda)b))|^q dt \leq 2^{s-1} \left| f'\left(\frac{\lambda a + (2 - \lambda)b}{2}\right) \right|^q,$$

so

$$\begin{aligned}
& \left| f(\lambda a + (1 - \lambda)b) - \frac{1}{b - a} \int_a^b f(t) dt \right| \\
& \leq (b - a) \left(\frac{1}{p+1} \right)^{1/p} 2^{\frac{s-1}{q}} \left((1 - \lambda)^2 \left| f'\left(\frac{(1 + \lambda)a + (1 - \lambda)b}{2}\right) \right| + \lambda^2 \left| f'\left(\frac{\lambda a + (2 - \lambda)b}{2}\right) \right| \right) \\
& = (b - a) \left(\frac{q-1}{2q-1} \right)^{1-\frac{1}{q}} 2^{\frac{s-1}{q}} \left((1 - \lambda)^2 \left| f'\left(\frac{(1 + \lambda)a + (1 - \lambda)b}{2}\right) \right| + \lambda^2 \left| f'\left(\frac{\lambda a + (2 - \lambda)b}{2}\right) \right| \right),
\end{aligned}$$

which yields the desired result. ■

Remark 3. Applying Theorem 3.4 for $\lambda = \frac{1}{2}$, we get

$$\begin{aligned}
& \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\
& \leq \frac{b-a}{4} \left(\frac{q-1}{2q-1} \right)^{1-\frac{1}{q}} 2^{\frac{s-1}{q}} \left(\left| f'\left(\frac{3a+b}{4}\right) \right| + \left| f'\left(\frac{a+3b}{4}\right) \right| \right) \\
& \leq \frac{b-a}{4} \left(\frac{q-1}{2q-1} \right)^{1-\frac{1}{q}} \left(\left| f'\left(\frac{3a+b}{4}\right) \right| + \left| f'\left(\frac{a+3b}{4}\right) \right| \right),
\end{aligned}$$

which is an improved result comparing with Theorem 2.5 in [2].

4. Applications to special means

Now, using the results of Section 3, we give some applications to special means of real numbers.

We shall consider the means for arbitrary real numbers α, β ($\alpha \neq \beta$).

(1) Weighted mean

$$W(\alpha, \beta) = \lambda\alpha + (1 - \lambda)\beta, \quad \alpha, \beta \in R, \quad \lambda \in [0, 1].$$

(2) Generalized log-mean:

$$L_n(\alpha, \beta) = \left[\frac{\beta^{n+1} - \alpha^{n+1}}{(n+1)(\beta - \alpha)} \right]^{\frac{1}{n}}, \quad n \in Z \setminus \{0, 1\}, \quad \alpha, \beta \in R, \quad (\alpha \neq \beta).$$

Therefore, by applying the s -convex mapping $f : [0, 1] \rightarrow [0, 1]$, $f(x) = x^s$, the following inequalities hold:

Proposition 4.1 *Let $a, b \in I^0$, $a < b$ and $0 < s < 1$. Then, we have*

$$\begin{aligned} & |W^s(a, b) - L_s^s(a, b)| \\ & \leq (b - a)(1 - \lambda)^2 \left(\frac{s}{(s+1)(s+2)} |a|^{s-1} + \frac{s}{s+2} |\lambda a + (1 - \lambda)b|^{s-1} \right) \\ & \quad + (b - a)\lambda^2 \left(\frac{s}{(s+1)(s+2)} |b|^{s-1} + \frac{s}{s+2} |\lambda a + (1 - \lambda)b|^{s-1} \right), \end{aligned}$$

for each $\lambda \in [0, 1]$.

Proposition 4.2 *Let $a, b \in I^0$, $a < b$ and $0 < s < 1$. Then, for all $q > 1$, we have*

$$\begin{aligned} & |W^s(a, b) - L_s^s(a, b)| \\ & \leq s(b - a) \left(\frac{1}{p+1} \right)^{1/p} \left(\frac{1}{s+1} \right)^{1/q} \left((1 - \lambda)^2 \left[(\lambda^s + 1) |a|^{q(s-1)} + (1 - \lambda)^s |b|^{q(s-1)} \right]^{1/q} \right. \\ & \quad \left. + \lambda^2 \left[\lambda^s |a|^{q(s-1)} + ((1 - \lambda)^s + 1) |b|^{q(s-1)} \right]^{1/q} \right), \end{aligned}$$

for each $\lambda \in [0, 1]$.

Acknowledgments. This work is supported by the Youth Project of CTGU (Grant No.13QN11) and the Scientific and Technological Research Program of CMEC (Grant Nos. KJ1401006, KJ1401019).

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Accepted: 01.09.2013

ON THOMPSON'S CONJECTURE FOR $\text{Aut}(J_2)$ AND $\text{Aut}(McL)$ **Yanheng Chen****Yuming Feng**

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Abstract. Let G be a finite group and $N(G)$ be the set of conjugacy class sizes of G . In 1980s J. G. Thompson conjectured: If G is a finite group with trivial center and S is a non-abelian finite simple group satisfying that $N(G) = N(S)$, then $G \cong S$. Here, we generalize the conjecture to the automorphism groups of J_2 and McL . As a corollary this result extends the conjecture to all almost sporadic simple groups.

Keywords: finite group, almost sporadic simple groups, conjugacy class sizes, Thompson's conjecture.

AMS Mathematics Subject Classification (2010): 20D08, 20D60.

1. Introduction

All groups considered in this paper are finite and simple groups mentioned are non-abelian.

Let G be a group. For an element $x \in G$, we denote by x^G the conjugacy class of x in G and $N(G)$ the set of conjugacy class sizes of G , that is, $N(G) = \{|x^G| \mid x \in G\}$. In 1987, J.G. Thompson posed the following conjecture (ref. to [9], Problem 12.38):

Thompson's conjecture. *Let G be a group with $Z(G) = 1$ and S is a simple group satisfying that $N(G) = N(S)$, then $G \cong S$.*

The *prime graph* of a group G is defined as follows, whose vertices are the prime divisors of $|G|$ and two distinct primes p and q are joined by an edge if and only if G contains an element of order pq . J.S. Williams in [17] and A.S. Kondratiev in [18] obtained the classification of simple groups with disconnected prime graph. Based on these results, in 1994 G.Y. Chen proved that Thompson's conjecture is true for all finite simple groups with disconnected prime graph in [1], parts of the proof were given in [2], [3], [4]. The method used by G.Y. Chen requires a group with non-connected prime graph and several authors have worked on such groups in the same way as Chen.

For the simple groups with connected prime graph, there is not any progress on Thompson's conjecture for a long time. Until 2009, A.V. Vasil'ev first dealt with the groups with connected prime graph and proved that Thompson's conjecture holds for A_{10} and $L_4(4)$ (see [5]). Jiang Qinhuai et al in [7] and Gerald Pientka in [6] also independently gave a positive answer to Thompson's conjecture for A_{10} in the year of 2011 and 2012, respectively. Note that Gerald Pientka's work does not use the classification theorem of simple groups. In 2011, N. Ahanjideh in [8] proved that Thompson's conjecture is true for some projective special linear groups $PSL_n(q)$ which have connected prime graphs. Recently, I.B. Gorshkov in [9] established the validity of Thompson's conjecture for $U_4(4)$, $U_4(5)$, $S_6(4)$, $O_8^+(4)$, and A_{16} .

It is worth to mention that some authors generalize Thompson's conjecture to some almost simple groups, that is, a group M is said to be an almost simple related to S if and only if $S \leq M \leq \text{Aut}(S)$ for some simple group S . For example, in 2002 A. Khosravi and B. Khosravi in [11] generalized Thompson's conjecture to almost sporadic simple groups except $\text{Aut}(J_2)$ and $\text{Aut}(McL)$. In 2005, S.H. Alavi and A. Daneshkhah in [16] generalized Thompson's conjecture to symmetric groups S_n , where $n = p, p+1$, and p is an odd prime number. In 2011, B. Khosravi and M. Khatami in [12], [13] established the validity of Thompson's conjecture for the general projective groups $PGL(2, q)$. It is pity that they still need to assume that the groups discussed have the non-connected prime graphs. Surely, it is an interesting topic to check Thompson's conjecture for some almost simple groups which have the connected prime graphs. In this paper, we give a positive answer to Thompson's conjecture for $\text{Aut}(J_2)$ and $\text{Aut}(McL)$, both of which have the connected prime graphs. Further more, we get that Thompson's conjecture can be generalized to all the almost sporadic simple groups.

Our main contribution is the following theorem and we will give the proofs in Sections 3 and 4.

Main Theorem. *Let G be a group with $Z(G) = 1$ and M one of $\text{Aut}(J_2)$ and $\text{Aut}(McL)$ satisfying that $N(G) = N(M)$, then $G \cong M$.*

Let M be an almost simple group related to sporadic simple group S . Then, by [14], we have that $|\text{Aut}(S) : S| \leq 2$ such that $M = S$ or $\text{Aut}(S)$. G.Y. Chen proved Thompson's conjecture for all sporadic simple groups in [2]. A. Khosravi and B. Khosravi in [11] proved Thompson's conjecture for other almost sporadic

simple groups except $\text{Aut}(J_2)$ and $\text{Aut}(McL)$. Therefore, by our main conclusion the following corollary holds.

Corollary. *Thompson's conjecture holds for all the almost sporadic simple groups.*

For a group G , we denote by $\pi(G)$ the set of prime divisors of $|G|$ and denote by $\text{Soc}(G)$ the socle of G which is a subgroup generated by all minimal normal subgroups of G . In addition, we also denote by $\pi(n)$ the set of all primes dividing n where n is a positive integer, and then n_π to denote π -part of n for $\pi \subseteq \pi(n)$. The other notation and terminologies in this paper are standard and the reader is referred to [14] and [19] if necessary.

2. Preliminaries

First, we cite here some known results which are useful in the sequel.

Lemma 2.1. ([5], Lemma 5, and [9], Lemma 1.4) *Let K be a normal subgroup of G and $\overline{G} = G/K$. Then*

- (1) *If $\overline{x}$ is the image of an element x of G in the group $\overline{G}$, then $|x^K| \mid |x^G|$ and $|\overline{x}^{\overline{G}}| \mid |x^G|$.*
- (2) *If $x \in G$ and $(|x|, |K|) = 1$, then $C_{\overline{G}}(\overline{x}) = C_G(x)K/K$.*
- (3) *If $x, y \in G$, $(|x|, |y|) = 1$, and $xy = yx$, then $C_G(xy) = C_G(x) \cap C_G(y)$.*

Lemma 2.2. ([5], Lemma 4) *Let G be a group with trivial center, $p \in \pi(G)$ and p^2 not divide n for any $n \in N(G)$. Then a Sylow p -subgroup of G is elementary abelian.*

Lemma 2.3. ([9], Lemma 1.10) *Let a Sylow p -subgroup of G be of order p , x be an element of order p , and $|x^G|$ be a number that is maximal with respect to divisibility in $N(G)$. Then $C_G(x)$ is an abelian group.*

Lemma 2.4. ([9], Lemma 1.9) *Let G be a group, and p and q be numbers in $\pi(G)$. If G satisfies the following conditions:*

- (a) *$N(G)$ contains no number divisible by p^2 or q^2 ;*
- (b) *$N(G)$ contains no number except 1 co-prime to pq ;*
- (c) *$N(G)$ contains a number h_q such that any n in $N(G)$ not divisible by q does not divide h_q and $N(G)$ contains no number divisible by h_q and n ;*
- (d) *$N(G)$ contains a number h_p such that any l in $N(G)$ not divisible by p does not divide h_p and $N(G)$ contains no number divisible by h_p and l .*

Then, the Sylow p -subgroups and q -subgroups of G are cyclic groups of prime order. In addition, G has no element of order pq .

Lemma 2.5. ([9], Lemma 1.11) *Let G be a group such that $\pi(G)$ has no number greater than 17, and there are numbers p and q in $\pi(G)$ such that:*

- (a) p and q are nonadjacent in the prime graph of G ;
- (b) Sylow p -subgroups and q -subgroups of G have prime orders;
- (c) $q - 1$ is not divisible by p , while $p - 1$ is not divisible by q ;
- (d) the centralizer of any element of order p or q is abelian;
- (e) p and q are greater than 5.

Then G is insoluble and possesses a unique non-abelian composition factor S , whose order is divisible by pq . Moreover, If K is the soluble radical of G , then $S \leq G/K \leq \text{Aut}(S)$.

Lemma 2.6. ([9], Lemma 1.12) *Let G be a group, K the soluble radical of G , and $G/K = S$ a simple non-abelian group. Suppose that there exists a prime p such that $p \in \pi(G) \setminus \pi(K)$. Assume that an element g of order p of G satisfies the following conditions:*

- (a) $|g^G| = |\bar{g}^S|$, where $\bar{g}$ is the image of an element g in the group S ;
- (b) the number $|g^G|$ is maximal with respect to divisibility in $N(G)$;
- (c) the subgroup $C_G(g)$ is abelian.

Then $K \leq Z(G)$.

By [14], we know that $|\text{Out}(J_2)| = |\text{Out}(McL)| = 2$ such that M is isomorphic to $\text{Aut}(J_2) = J_2 : 2$ or $\text{Aut}(McL) = McL : 2$. Information on the set $N(M)$ and the order of M are given in the next two lemmas is obtained via [14].

Lemma 2.7. *Let $M \cong J_2 : 2$. Then*

- (1) $|M| = 2^8 \cdot 3^3 \cdot 5^2 \cdot 7$;
- (2) $N(M) = \{n_1 = 1, n_2 = 3^2 \cdot 5 \cdot 7, n_3 = 2^3 \cdot 3^2 \cdot 5 \cdot 7, n_4 = 2^4 \cdot 5 \cdot 7, n_5 = 2^5 \cdot 3 \cdot 5^2 \cdot 7, n_6 = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7, n_7 = 2^6 \cdot 3^2 \cdot 7, n_8 = 2^7 \cdot 3^3 \cdot 7, n_9 = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7, n_{10} = 2^5 \cdot 3^2 \cdot 5^2 \cdot 7, n_{11} = 2^7 \cdot 3^3 \cdot 5^2, n_{12} = 2^4 \cdot 3^3 \cdot 5^2 \cdot 7, n_{13} = 2^6 \cdot 3^3 \cdot 5 \cdot 7, n_{14} = 2^7 \cdot 3^3 \cdot 5 \cdot 7, n_{15} = 2^8 \cdot 3^2 \cdot 5 \cdot 7, n_{16} = 2^3 \cdot 3^2 \cdot 5^2, n_{17} = 2^3 \cdot 3^2 \cdot 5^2 \cdot 7, n_{18} = 2^6 \cdot 3^2 \cdot 5^2 \cdot 7, n_{19} = 2^3 \cdot 3^3 \cdot 5^2 \cdot 7\}$.

Especially,

- (3) $N(M)$ contains no number other than n_{11} and n_{16} not divisible by 7;

- (4) $N(M)$ contains no number divided by 7^2 ;
- (5) n_4 is only one in $N(M)$ not divisible by 3.

Lemma 2.8. *Let $M \cong McL : 2$. Then*

- (1) $|M| = 2^8 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$;
- (2) $N(M) = \{n_1 = 1, n_2 = 3^4 \cdot 5^2 \cdot 11, n_3 = 2^4 \cdot 5^2 \cdot 7 \cdot 11, n_4 = 2^5 \cdot 3 \cdot 5^3 \cdot 7 \cdot 11, n_5 = 2^2 \cdot 3^5 \cdot 5^3 \cdot 7 \cdot 11, n_6 = 2^6 \cdot 3^5 \cdot 7 \cdot 11, n_7 = 2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 11, n_8 = 2^4 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11, n_9 = 2^5 \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 11, n_{10} = 2^7 \cdot 3^6 \cdot 5^3 \cdot 11, n_{11} = 2^4 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11, n_{12} = 2^8 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11, n_{13} = 2^6 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11, n_{14} = 2^7 \cdot 3^6 \cdot 5^3 \cdot 7, n_{15} = 2^5 \cdot 3^5 \cdot 5^3 \cdot 7 \cdot 11, n_{16} = 2^7 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11, n_{17} = 2^3 \cdot 3^4 \cdot 5^2 \cdot 7, n_{18} = 2^3 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11, n_{19} = 2^6 \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 11, n_{20} = 2^3 \cdot 3^5 \cdot 5^3 \cdot 7 \cdot 11, n_{21} = 2^3 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11, n_{22} = 2^7 \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11, n_{23} = 2^6 \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11\}$.

In particular,

- (3) $N(M)$ contains no number other than n_2 and n_{10} not divisible by 7;
- (4) $N(M)$ contains no number other than n_{14} and n_{17} not divisible by 11;
- (5) For any $n \in N(M)$ and $p \in \{7, 11\}$, it follows that $p^2 \nmid n$;
- (6) For any $n \in N(M)$, either 7 or 11 divides n .

Lemma 2.9. *Let M be one of $J_2 : 2$ and $McL : 2$, and G be a group with trivial center. If $N(G) = N(M)$, then $|M||G|$ and $\pi(M) = \pi(G)$.*

Proof. Since the number in $N(G)$ divides $|G|$, under the hypothesis we see that $|M||G|$ by Lemmas 2.7 and 2.8. $\pi(M) = \pi(G)$ is the result of Lemma 1.2.1 in [1] or Lemma 3 in [5].

Let S be a simple group with $\pi(S) \subseteq \{2, 3, 5, 7, 11\}$. Up to isomorphism, there are finitely many finite non-abelian simple groups S with $\pi(S) \subseteq \{2, 3, 5, 7, 11\}$. Using the classification of finite simple groups one can list them all. For convenience, we list all the cases of S as well as the orders of S , the orders of the outer automorphism of S in Table 1. Especially, we have that $\pi(\text{Out}(S)) \subseteq \{2, 3\}$.

3. Proof of Main Theorem for $J_2 : 2$

In this section, we prove the Main Theorem for $J_2 : 2$ according to the information in Lemma 2.7.

Theorem 3.1. *Let $M \cong J_2 : 2$ and G be a group with trivial center. If $N(G) = N(M)$, then $G \cong M$.*

Proof. We divide the proof of this theorem into eight steps.

Table 1. Non-abelian simple groups S with $\pi(S) \subseteq \{2, 3, 5, 7, 11\}$

S	Order of S	$ \text{Out}(S) $	S	Order of S	$ \text{Out}(S) $
A_5	$2^2 \cdot 3 \cdot 5$	2	A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	2
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2
A_6	$2^3 \cdot 3^2 \cdot 5$	2^2	$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	$ S_3 $
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3	$S_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	$ D_8 $
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2	$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	2
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	$ D_{12} $	$O_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	$ S_3 $
$U_4(2)$	$2^6 \cdot 3^4 \cdot 5$	2	$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	2^2
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	2	M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	2
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	1	A_{11}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	2
$L_2(11)$	$2^2 \cdot 3 \cdot 5 \cdot 11$	2	HS	$2^9 \cdot 3^5 \cdot 5^3 \cdot 7 \cdot 11$	2
$U_5(2)$	$2^{10} \cdot 3^5 \cdot 5 \cdot 11$	2	A_{12}	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$	2
McL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	2	$U_6(2)$	$2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	$ S_3 $

Step 1. The Sylow 7-subgroup P of G is order of 7.

Using Lemma 2.2 and (2) of Lemma 2.7, we derive that P is elementary abelian. Assume that 7^2 divides G . Since $N(G) = N(M)$, the centralizer of every element of G contains an element of order 7 by (4) of Lemma 2.7. Consider an element y of G such that $|y^G| = n_8 = 2^7 \cdot 3^3 \cdot 7$.

Suppose that 7 does not divide $|y|$. Let x be an element of order 7 in $C_G(y)$. Then by (3) of Lemma 2.1, $C_G(xy) = C_G(x) \cap C_G(y)$, and so $\text{lcm}(|x^G|, |y^G|)$ divides $|(xy)^G|$. Since P is abelian, $C_G(x)$ includes P up to conjugacy. Hence 7 does not divide $|x^G|$. It follows that $|x^G|$ is equal to $n_{11} = 2^7 \cdot 3^3 \cdot 5^2$ or $n_{16} = 2^3 \cdot 3^2 \cdot 5^2$. In both cases, $2^7 \cdot 3^3 \cdot 5^2 \cdot 7$ divides $|(xy)^G|$, which is impossible by (2) of Lemma 2.7.

Suppose that 7 divides $|y|$. Let $|y| = 7t$. Since P is elementary abelian, one has that $\gcd(7, t) = 1$. Put $u = y^7$ and $v = y^t$. Then $y = uv$, and so $C_G(uv) = C_G(u) \cap C_G(v)$ by (3) of Lemma 2.1. Therefore $|v^G|$ divides $|y^G| = n_8 = 2^7 \cdot 3^3 \cdot 7$. On the other hand, the element v is order of 7, and thus $|v^G|$ is equal to $n_{11} = 2^7 \cdot 3^3 \cdot 5^2$ or $n_{16} = 2^3 \cdot 3^2 \cdot 5^2$, a contradiction.

Hence P has order of 7.

Step 2. For an element $x \in G$ of order 7, it follows that $|x^G| = n_{11} = 2^7 \cdot 3^3 \cdot 5^2$ and $C_G(x)$ is abelian.

For any $1 \neq y \in C_G(x)$, since $7 \parallel |G|$ by Step 1, one has that $7 \nmid |y^G|$, and hence $|y^G| = n_{11}$ or n_{16} by (2) of Lemma 2.7. Assume that $|x^G| = n_{16} = 2^3 \cdot 3^2 \cdot 5^2$ and let H be a Sylow 3-subgroup of $C_G(x)$. Then H is a nontrivial group of order $|G|_3/3^2$ by Lemma 2.9. Hence there exists a 3-subgroup K of G such that H is a normal subgroup of K and $|K/H| = 3$. Then $1 \neq H \cap Z(K) \leq C_G(x)$. Take $1 \neq h \in H \cap Z(K)$, we have that $K \leq C_G(h)$, and so $|h^G|_3 \leq 3$. But $|h^G| = n_{11}$ or n_{16} , a contradiction. It follows that $|x^G| = n_{11} = 2^7 \cdot 3^3 \cdot 5^2$.

Since n_{11} is maximal with respect to divisibility in $N(G)$, Lemma 2.3 implies that the group $C_G(x)$ is abelian.

Step 3. Suppose that $q \in \{2, 3, 5\}$, Q is a Sylow q -subgroup of G , and $Z(Q)$ is its center. Let $x \in Z(Q)$, then $7 \nmid |C_G(x)|$.

Let $x \in Z(Q)$. Then q does not divide $|x^G|$, and so by Lemma 2.7 we have that $|x^G| = n_2 = 3^2 \cdot 5 \cdot 7$ while $q = 2$, $|x^G| = n_4 = 2^4 \cdot 5 \cdot 7$ while $q = 3$, and $|x^G|$ is equal to $n_7 = 2^6 \cdot 3^2 \cdot 7$ or $n_8 = 2^7 \cdot 3^3 \cdot 7$ while $q = 5$. The Step 3 follows.

Step 4. G is non-soluble and $O_2, 2'(G) = O_2(G)$.

Let $K = O_2(G)$, $\bar{G} = G/K$, and denote by $\bar{x}$ the images of an element x of G in $\bar{G}$. If the statement is false, then there exists $r \in \{3, 5, 7\}$ such that $O_r(\bar{G}) \neq 1$.

If $O_7(\bar{G}) \neq 1$, then $|O_7(\bar{G})| = 7$ by Step 1. Let y be an element of the center $Z(Q)$ of a Sylow 5-subgroup Q . Obviously, the subgroup $O_7(\bar{G})\langle\bar{y}\rangle$ is cyclic. Hence 7 divides $|C_{\bar{G}}(\bar{y})|$. Since $(5, |K|) = 1$, Lemma 2.1(2) implies that 7 divides $|C_G(y)|$, which is impossible by Step 3. Thus, $O_7(\bar{G}) = 1$.

Let $q \in \{3, 5\}$, and Q be a Sylow q -subgroup of G . If $O_q(\bar{G}) \neq 1$, then $V = Z(O_q(\bar{G}))$ is a nontrivial normal subgroup of G . If x is an element of order 7 in G , then $V = C_V(\bar{x}) \times [V, \bar{x}]$ such that $\bar{x}$ acts fixed-point freely on $[V, \bar{x}]$, and then $|[V, \bar{x}]| - 1$ is divisible by 7. Lemma 2.1 (1) implies that $|\bar{x}^G|$ is a divisor of $2^7 \cdot 3^3 \cdot 5^2$, and hence $|V : C_V(\bar{x})|$ divides 5^2 or 3^3 so that $|[V, \bar{x}]| \leq q^3$. But $n = 6$ is the least number such that 7 divides $q^n - 1$, we have that $[V, x] = 1$ and $V = C_V(\bar{x})$. On the other hand, the center $Z(Q)$ of a Sylow subgroup Q has a nontrivial intersection with V . Let $\bar{z}$ be of order q from this intersection. Since $(|K|, q) = 1$, there exists a pre-image z of $\bar{z}$ in G such that z lies in the center of a Sylow q -subgroup of G by Lemma 2.1 (2). Further, the centralizer of z also contains an element of order 7, which contradicts Step 3. Therefore, $O_q(\bar{G}) = 1$. The Step 4 follows.

Step 5. Let $K = O_2(G)$, $\bar{G} = G/K$. Then, every minimal normal subgroup of $\bar{G}$ is non-soluble. Especially, $\text{Soc}(\bar{G}) \trianglelefteq \bar{G} \lesssim \text{Aut}(\text{Soc}(\bar{G}))$.

Let N be any minimal normal subgroup of $\bar{G}$ and assume that N is solvable. Then N is an elementary abelian p -group for some $p \in \{3, 5, 7\}$, and so $N \leq O_p(\bar{G})$. It follows that $O_p(\bar{G})$ is nontrivial, contradicts Step 4. Hence a very minimal normal subgroup of $\bar{G}$ is non-soluble. Let $N_1, N_2, \dots, N_s$ be all minimal normal subgroups of $\bar{G}$, where s is a positive integer. Then $\text{Soc}(\bar{G}) = N_1 \times N_2 \times \dots \times N_s$. We assert that $C_{\bar{G}}(\text{Soc}(\bar{G})) = 1$. Otherwise, there exists a minimal normal subgroup N of $\bar{G}$ so that $N \leq C_{\bar{G}}(\text{Soc}(\bar{G})) \cap \text{Soc}(\bar{G})$. Thus N is an abelian group, a contradiction. By N/C theorem, we have $\text{Soc}(\bar{G}) \trianglelefteq \bar{G} = \bar{G}/C_{\bar{G}}(\text{Soc}(\bar{G})) \lesssim \text{Aut}(\text{Soc}(\bar{G}))$, as desired.

Step 6. Let $L = \text{Soc}(\bar{G})$. Then L is a non-abelian simple group and $7 \nmid |L|$.

By Step 5, we can have that $L = S_1 \times S_2 \times \dots \times S_k$ is a direct product of non-abelian simple groups of $S_1, S_2, \dots$, and S_k . Let g be an element of order 7 of G and suppose that $7 \notin \pi(L)$. Then $\bar{g}$ is of order 7 in $\bar{G}$ and induces a nontrivial outer automorphism of the group L . Suppose that there exists i such that $S_i^g \neq S_i$. Without loss of generality, we assume that $i = 1$. Let $H = \langle s \mid s = s_1 s_1^{\bar{g}} s_1^{\bar{g}^2} \dots s_1^{\bar{g}^6}, s_1 \in S_1 \rangle$. Then H lies in the centralizer of the

element $\bar{g}$ and is isomorphic to S_1 , but the centralizer of g is abelian by Step 2, a contradiction. Hence $\bar{g}$ induces a nontrivial outer automorphism of the group S_i such that $7 \mid |\text{Out}(S_i)|$. In view of $\pi(S_i) \subseteq \pi(G) = \{2, 3, 5, 7\}$ and by Table 1, the prime divisors of $|\text{Out}(S_i)|$ are not greater than 5, a contradiction. Therefore, $7 \mid |L|$.

If $n > 1$ and $\bar{g} \in S_j$, then $S_i \leq C_{\bar{G}}(\bar{g})$ for any $1 \leq j \leq n$, $j \neq i$. On the other hand, $C_{\bar{G}}(\bar{g})$ is abelian by Step 2, a contradiction. Therefore, $n = 1$, and so L is a non-abelian simple group and $7 \mid |L|$.

Step 7. $L \cong J_2$.

By Step 6 and Step 1, we have that L is a non-abelian simple group such that $7 \mid |M|$. Then, by Table 1, L may be isomorphic to one of the following groups:

$$L_2(7), L_2(8), U_3(3), A_7, A_8, L_3(4), U_3(5), A_9, A_{10}, U_4(3), J_2, S_6(2), O_8^+(2).$$

Hence, by Table 1 and Step 5, we see that $\pi(\text{Out}(L)) \subseteq \{2, 3\}$ and $L \trianglelefteq \bar{G} \lesssim \text{Aut}(L)$. Since $K = O_2(G)$ and $|M| \mid |G|$, we have that 5^2 divides $|L|$, which implies that L can be only isomorphic to one of the following groups:

$$U_3(5), A_{10}, J_2, O_8^+(2).$$

If $L \cong U_3(5)$ or $O_8^+(2)$, then let x be an element of order 7 in G and $\bar{x}$ be its image in $\bar{G}$, of course $\bar{x} \in L$, $|\bar{x}^{\bar{G}}|$ is a multiple of 5^3 or 3^5 , so is $|x^G|$ by (1) of Lemma 2.1. This contradicts (2) of Lemma 2.7.

Assume that $L \cong A_{10}$. Then, there exists an element w of order 5 in L such that $|w^L|$ is a multiple of 3^4 , so is $|w^{\bar{G}}|$ by (2) of Lemma 2.1, which implies that $N(G)$ has one number divided by 3^4 . This is impossible by (2) of Lemma 2.7. Hence, L must be isomorphic to J_2 .

Step 8. $G \cong M$.

Let x be an element of order 7 in G and $\bar{x}$ be its image in $\bar{G}$. It is clear that $\bar{x} \in L$. By Lemma 2.1 and [14], we have that $|\bar{x}^L| = |\bar{x}^{\bar{G}}| = |x^G| = 2^7 \cdot 3^3 \cdot 5^2$ such that $K \leq C_G(x)$. If $K \neq 1$, then x centralizes an element from the center of a Sylow 2-subgroup of G , which is impossible by Step 3. Hence $G = \bar{G}$. Recall that $L \trianglelefteq \bar{G} \lesssim \text{Aut}(L)$. Then $G \cong J_2$ or $J_2 : 2$ by Table 1. If $G \cong J_2$, then $|N(G)| = |N(J_2)|$, a contradiction by [14]. Therefore $G \cong J_2 : 2 \cong M$, as claimed.

4. Proof of Main Theorem for $McL : 2$

In this section, we prove the following theorem according to Lemma 2.8.

Theorem 4.1. *Let $M \cong McL : 2$ and G be a group with trivial center. If $N(G) = N(M)$, then $G \cong M$.*

Proof. We divide the proof of this theorem into seven steps.

Step 1. Sylow 7-subgroups and Sylow 11-subgroups of G are cyclic groups of prime order and there are no elements of order 77 in G .

In view of $N(G) = N(M)$ and Lemma 2.9, we can choose $p = 7$ and $q = 11$, and take $h_7 = n_{14}$ and $h_{11} = n_{10}$ such that G meets the hypotheses of Lemma 2.4. Hence Sylow 7-subgroups and Sylow 11-subgroups of G are cyclic groups of prime order and there are no elements of order $7 \cdot 11$ in G .

Step 2. Let $g, h \in G$ be elements of orders 7 and 11, respectively. Then $|g^G| = n_{10} = 2^7 \cdot 3^6 \cdot 5^3 \cdot 11$ and $|h^G| = n_{14} = 2^7 \cdot 3^6 \cdot 5^3 \cdot 7$, and $C_G(g)$ and $C_G(h)$ are abelian.

We only state the case of $p = 7$, and the case of $q = 11$ is similar.

Since the Sylow 7-subgroup of G is order of 7 by Step 1, one has that $7 \nmid |x^G|$ for any $1 \neq x \in C_G(g)$. Hence $|x^G| = n_2$ or n_{10} . Assume that $|g^G| = n_2 = 3^4 \cdot 5^2 \cdot 11$. Let H be a Sylow 5-subgroup of $C_G(g)$. Then H is a nontrivial group of order $|G|_5/5^2$ by Lemma 2.9. Hence there exists a 5-subgroup K of G such that H is a normal subgroup of K and $|K/H| = 5$. Further, $1 \neq H \cap Z(K) \leq C_G(x)$. Taking $1 \neq h \in H \cap Z(K)$, one has that $5^2 \nmid |h^G|_5$ and $7 \nmid |h^G|$, which implies $|h^G| = 1$ by Lemma 2.8, and so $y \in Z(G)$, a contradiction. It follows that $|g^G| = n_{10} = 2^7 \cdot 3^6 \cdot 5^3 \cdot 11$.

Since n_{10} is maximal with respect to divisibility in $N(G)$, Lemma 2.3 implies that the group $C_G(g)$ is abelian.

In the following discussion, we assume that K is the soluble radical of a group G , and $\overline{G} = G/K$.

Step 3. G is non-soluble and has a unique non-abelian composition factor S such that $7 \cdot 11 \mid |S|$ and $S \leq \overline{G} \leq \text{Aut}(S)$. Moreover, S may be isomorphic to one of the following groups:

$$M_{22}, A_{11}, HS, A_{12}, McL, U_6(2).$$

In view of Step 1 and Step 2, we can choose $p = 7$ and $q = 11$ such that G satisfies the hypotheses of Lemma 2.5. Hence G is non-soluble and has a unique non-abelian composition factor S such that $7 \cdot 11 \mid |S|$ and $S \leq \overline{G} \leq \text{Aut}(S)$. Since $\{7, 11\} \subseteq \pi(S) \subseteq \{2, 3, 5, 7, 11\}$, we can easily get that S can be isomorphic to one of the groups: $M_{22}, A_{11}, HS, A_{12}, McL, U_6(2)$ by Table 1.

Step 4. $5^3 \mid |S|$.

Since $N(G)$ contains an integer divisible by 5^3 , $|G|$ is divisible by 5^3 . Assume $|S|$ is not divisible by 5^3 . By Table 1, $|\text{Out}(S)|$ is not divisible by 5. Consequently, $|K|$ is divisible by 5. For $|K|$ is not divisible by 7, there is an element g in G such that $|g| = 7$ and the image $\bar{g}$ of g in the group $\overline{G}$ is nontrivial. By virtue of Step 2, $|g^G|_5 = |n_{10}|_5 = 5^3$, the subgroup K contains an element h of order 5 such that g does not centralize h . Let $R_1 \triangleleft R_2 \triangleleft \cdots \triangleleft R_n \triangleleft K$ be an un-refinable series of characteristic subgroups in K and l a number such that $h \in R_{l+1} \setminus R_l$. The image $\tilde{h}$ of h in the group K/R_l is nontrivial and lies in H , which is a normal abelian 5-subgroup of G/R_l . Let $\tilde{G} = G/R_l$ and $\tilde{g}$ be the image of g in the group $\tilde{G}$. Then

$H = C_H(\tilde{g}) \times [\tilde{g}, H]$ and g acts fixed-point freely on $[\tilde{g}, H]$ such that $||[\tilde{g}, H]| - 1$ is divisible by 7. Thus $||[\tilde{g}, H]| \geq 5^6$, which implies that

$$5^6 \left| \frac{|H|}{|C_H(\tilde{g})|} \right| \left| \frac{|\tilde{K}|_5}{|C_{\tilde{K}}(\tilde{g})|_5} \right|.$$

By Lemma 2.1,

$$C_{\overline{G}}(\overline{g}) = C_{\tilde{G}}(\tilde{g})\tilde{K}/\tilde{K} = C_{\tilde{G}}(\tilde{g})/(C_{\tilde{G}}(\tilde{g}) \cap \tilde{K}) = C_{\tilde{G}}(\tilde{g})/C_{\tilde{K}}(\tilde{g}),$$

and then

$$|\tilde{g}^{\tilde{G}}|_5 = |\overline{G}|_5 |\tilde{K}|_5 / (|C_{\overline{G}}(\overline{g})|_5 |C_{\tilde{K}}(\tilde{g})|_5) \geq 5^6,$$

so is $|g^G|_5$. It follows a contradiction by (2) of Lemma 2.8, which implies that $N(G)$ has no number divisible by 5^6 . Hence $5^3 ||S|$.

Step 5. $S \cong McL$.

By Step 3 and Step 4, we can get that S may be isomorphic to one of groups HS and McL by Table 1. If $M \cong HS$, then let x be an element of order 7 in G and $\bar{x}$ be its image in $\overline{G}$, of course $\bar{x} \in S$, $|\bar{x}^S|$ is a multiple of 2^9 , so are $|\bar{x}^{\overline{G}}|$ and $|x^G|$ by Lemma 2.1. This contradicts (2) of Lemma 2.8. Hence $S \cong McL$.

Step 6. $G/K \cong McL : 2$.

By virtue of $S \leq G/K \leq \text{Aut}(S)$ and [14], we have $G/K \cong McL$ or $McL : 2$ and $K \neq 1$. Assume that $G/K \cong McL$. Then, let $g \in G$, $|g| = 7$, and $\bar{g} \in G/K = \overline{G}$ be the image of the element g . In view of Lemma 2.8 (2), Step 2, and [14], the number n_{10} is maximal in $N(G)$ and the number $|g^G| = |\bar{g}^S| = n_{10}$. Thus G satisfies the hypothesis of Lemma 2.6, and so $K \leq Z(G)$, a contradiction to $Z(G) = 1$. Therefore it is impossible that $G/K \cong S \cong McL$, and so G/K must be isomorphic to $McL : 2$.

Step 7. K is a trivial group such that $G \cong M \cong McL : 2$.

Also let $g \in G$, $|g| = 7$, and $\bar{g} \in G/K = \overline{G} \cong M$ be the image of the element g . By Lemma 2.8 (2), Step 2, and [14], the number n_{10} is maximal in $N(G)$ and the number $|g^G| = |\bar{g}^{\overline{G}}| = |\bar{g}^S| = n_{10}$. Thus $K \leq C_G(g)$. If $K \neq 1$ and let $p ||K|$, then g centralizes an element from the center of a Sylow p -subgroup of G , which is impossible by Lemma 2.8. Hence K is a trivial group such that $G \cong M \cong McL : 2$.

Proof of Main Theorem. It follows directly from Theorems 3.1 and 4.1.

Acknowledgement. This work was supported by National Natural Science Foundation of China (Grants Nos. 11271301, 11001226), the Scientific and Technological Research Program of Chongqing Municipal Education Commission (Grant Nos. KJ1401006, KJ1401019), the Scientific Research Foundation of Chongqing Municipal Science and Technology Commission (Grant No. cstc2014jcyjA00009), the Fundamental Research Funds for the Central Universities, and the Youth Foundation of Chongqing Three Gorges University (Grant No. 12QN-23).

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Accepted: 13.10.2013

ON SOME GROWTH PROPERTIES OF DIFFERENTIAL POLYNOMIALS IN THE LIGHT OF RELATIVE ORDER

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Abstract. In the paper we establish some newly developed results based on the growth properties of relative order (relative lower order), relative type (relative lower type) and relative weak type of differential polynomials generated by entire and meromorphic functions.

Keywords: entire function, meromorphic function, relative order (relative lower order), relative type (relative lower type), relative weak type, polynomials.

2010 Mathematics Subject Classification: 30D35, 30D30, 30D20.

1. Introduction, definitions and notations

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be a non-constant meromorphic function defined in the open complex plane $\mathbb{C}$. Also, let $n_{0j}, n_{1j}, \dots, n_{kj}$ ($k \geq 1$) be non-negative integers such that for each j , $\sum_{i=0}^k n_{ij} \geq 1$.

We call $M_j[f] = A_j(f)^{n_{0j}}(f^{(1)})^{n_{1j}} \dots (f^{(k)})^{n_{kj}}$, where $T(r, A_j) = S(r, f)$ to be a differential monomial generated by f . The numbers $\gamma_{M_j} = \sum_{i=0}^k n_{ij}$ and $\Gamma_{M_j} = \sum_{i=0}^k (i+1)n_{ij}$ are called, respectively, the degree and weight of $M_j[f]$ ([2], [8]). The expression $P[f] = \sum_{j=1}^s M_j[f]$ is called a differential polynomial generated by f . The numbers $\gamma_P = \max_{1 \leq j \leq s} \gamma_{M_j}$ and $\Gamma_P = \max_{1 \leq j \leq s} \Gamma_{M_j}$ are called, respectively, the degree and weight of $P[f]$ ([2], [8]). Also, we call the numbers $\gamma_P = \min_{1 \leq j \leq s} \gamma_{M_j}$ and k (the order of the highest derivative of f) the lower degree and the order of $P[f]$, respectively. If $\gamma_P = \gamma_P$, $P[f]$ is called a homogeneous differential polynomial. Throughout the paper we consider only the non-constant differential polynomials and we denote by $P_0[f]$ a differential polynomial not containing f , i.e., for which $n_{0j} = 0$ for $j = 1, 2, \dots, s$. We consider only those $P[f], P_0[f]$ singularities of whose individual terms do not cancel each other.

The following definitions are well known.

Definition 1. The quantity $\Theta(a; f)$ of a meromorphic function f is defined as follows

$$\Theta(a; f) = 1 - \limsup_{r \rightarrow \infty} \frac{\bar{N}(r, a; f)}{T(r, f)}.$$

Definition 2. [7] For $a \in \mathbb{C} \cup \{\infty\}$, let $n_p(r, a; f)$ denotes the number of zeros of $f - a$ in $|z| \leq r$, where a zero of multiplicity $< p$ is counted according to its multiplicity and a zero of multiplicity $\geq p$ is counted exactly p times; and $N_p(r, a; f)$ is defined in terms of $n_p(r, a; f)$ in the usual way. We define

$$\delta_p(a; f) = 1 - \limsup_{r \rightarrow \infty} \frac{N_p(r, a; f)}{T(r, f)}.$$

Definition 3. [1] $P[f]$ is said to be admissible if

- (i) $P[f]$ is homogeneous, or
- (ii) $P[f]$ is non homogeneous and $m(r, f) = S(r, f)$.

The following definitions are also well known.

Definition 4. The order ρ_f and lower order λ_f of a meromorphic function f are defined as

$$\rho_f = \limsup_{r \rightarrow \infty} \frac{\log T_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r \rightarrow \infty} \frac{\log T_f(r)}{\log r}.$$

Definition 5. The type σ_f and lower type $\bar{\sigma}_f$ of a meromorphic function f are defined as

$$\sigma_f = \limsup_{r \rightarrow \infty} \frac{T_f(r)}{r^{\rho_f}} \text{ and } \bar{\sigma}_f = \liminf_{r \rightarrow \infty} \frac{T_f(r)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

In this connection, Datta and Jha [3] gave the definition of weak type of a meromorphic function of finite positive lower order in the following way:

Definition 6. [3] The weak type τ_f of a meromorphic function f of finite positive lower order λ_f is defined by

$$\tau_f = \liminf_{r \rightarrow \infty} \frac{T_f(r)}{r^{\lambda_f}}.$$

Similarly, one can define the growth indicator $\bar{\tau}_f$ of a meromorphic function f of finite positive lower order λ_f as

$$\bar{\tau}_f = \limsup_{r \rightarrow \infty} \frac{T_f(r)}{r^{\lambda_f}}.$$

For an entire function g , the Nevanlinna's characteristic function $T_g(r)$ is defined as

$$T_g(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |g(re^{i\theta})| d\theta,$$

where $\log^+ x = \max(0, \log x)$ for $x > 0$.

If g is non-constant then $T_g(r)$ is strictly increasing and continuous and its inverse $T_g^{-1} : (T_g(0), \infty) \rightarrow (0, \infty)$ exists and is such that $\lim_{s \rightarrow \infty} T_g^{-1}(s) = \infty$.

Lahiri and Banerjee [6] introduced the definition of relative order of a meromorphic function with respect to an entire function which is as follows:

Definition 7. [6] Let f be meromorphic and g be entire. The relative order of f with respect to g denoted by $\rho_g(f)$ is defined as

$$\rho_g(f) = \inf \left\{ \mu > 0 : T_f(r) < T_g(r^\mu) \text{ for all sufficiently large } r \right\} = \limsup_{r \rightarrow \infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.$$

The definition coincides with the classical one [6] if $g(z) = \exp z$.

Similarly, one can define the relative lower order of a meromorphic function f with respect to an entire g denoted by $\lambda_g(f)$ in the following manner:

$$\lambda_g(f) = \liminf_{r \rightarrow \infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.$$

Datta and Biswas [4] gave the definition of relative type and relative weak type of a meromorphic function with respect to an entire function g which are as follows:

Definition 8. [4] The relative type $\sigma_g(f)$ of a meromorphic function f with respect to an entire function g are defined as

$$\sigma_g(f) = \limsup_{r \rightarrow \infty} \frac{T_g^{-1} T_f(r)}{r^{\rho_g(f)}}, \quad \text{where } 0 < \rho_g(f) < \infty.$$

Similarly, one can define the lower relative type $\bar{\sigma}_g(f)$ in the following way

$$\bar{\sigma}_g(f) = \liminf_{r \rightarrow \infty} \frac{T_g^{-1}T_f(r)}{r^{\rho_g(f)}}, \quad \text{where } 0 < \rho_g(f) < \infty.$$

Definition 9. [4] The relative weak type $\tau_g(f)$ of a meromorphic function f with respect to an entire function g with finite positive relative lower order $\lambda_g(f)$ is defined by

$$\tau_g(f) = \liminf_{r \rightarrow \infty} \frac{T_g^{-1}T_f(r)}{r^{\lambda_g(f)}}.$$

Analogously, one can define the growth indicator $\bar{\tau}_g(f)$ of a meromorphic function f with respect to an entire function g with finite positive relative lower order $\lambda_g(f)$ as

$$\bar{\tau}_g(f) = \limsup_{r \rightarrow \infty} \frac{T_g^{-1}T_f(r)}{r^{\lambda_g(f)}}.$$

In this paper we establish some newly developed results based on the growth properties of relative order (relative lower order) relative type (relative lower type) and relative weak type of polynomials generated by entire and meromorphic functions. We do not explain the standard notations and definitions in the theory of entire and meromorphic functions because those are available in [5] and [9].

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. [1] Let f be either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$. Then, for homogeneous $P_0[f]$,

$$\lim_{r \rightarrow \infty} \frac{T_{P_0[f]}(r)}{T_f(r)} = \gamma_{P_0[f]}.$$

Lemma 2. [1] Let $P_0[f]$ be admissible. If f is of finite order or of non zero lower order and $\sum_{a \neq \infty} \Theta(a; f) = 2$, then

$$\lim_{r \rightarrow \infty} \frac{T_{P_0[f]}(r)}{T_f(r)} = \Gamma_{P_0[f]}.$$

Lemma 3. [3] If f be a meromorphic function of regular growth, i.e., $\rho_f = \lambda_f$ then

$$\sigma_f = \bar{\sigma}_f = \tau_f = \bar{\tau}_f.$$

3. Theorems

In this section, we present the main results of the paper. It is needless to mention that in the paper, the admissibility and homogeneity of $P_0[f]$ will be needed as per the requirements of the theorems.

Theorem 1. Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$

and g be an entire function with $0 < \tau_g \leq \bar{\tau}_g < \infty$ and $0 < \bar{\sigma}_g \leq \sigma_g < \infty$. Also let $\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$. Then

$$\begin{aligned} & \max \left\{ \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}} \right)^{\frac{1}{\lambda_g}} \cdot \left(\frac{\tau_g}{\bar{\tau}_g} \right)^{\frac{1}{\lambda_g}}, \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}} \right)^{\frac{1}{\rho_g}} \cdot \left(\frac{\bar{\sigma}_g}{\sigma_g} \right)^{\frac{1}{\rho_g}} \right\} \\ & \leq \liminf_{r \rightarrow \infty} \frac{T_{P_0[g]}^{-1} T_{P_0[f]}(r)}{T_g^{-1} T_f(r)} \leq \limsup_{r \rightarrow \infty} \frac{T_{P_0[g]}^{-1} T_{P_0[f]}(r)}{T_g^{-1} T_f(r)} \\ & \leq \min \left\{ \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}} \right)^{\frac{1}{\lambda_g}} \cdot \left(\frac{\bar{\tau}_g}{\tau_g} \right)^{\frac{1}{\lambda_g}}, \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}} \right)^{\frac{1}{\rho_g}} \cdot \left(\frac{\sigma_g}{\bar{\sigma}_g} \right)^{\frac{1}{\rho_g}} \right\}, \end{aligned}$$

where $P_0[f]$ and $P_0[g]$ are homogeneous.

Proof. For any $\varepsilon(> 0)$, we get from Lemma 1, for all sufficiently large values of r ,

$$(1) \quad T_{P_0[f]}(r) \leq \{\gamma_{P_0[f]} + \varepsilon\} T_f(r)$$

and

$$(2) \quad T_{P_0[f]}(r) \geq \{\gamma_{P_0[f]} - \varepsilon\} T_f(r).$$

Also, from Lemma 1, we get for all sufficiently large values of r that

$$\begin{aligned} & T_{P_0[g]}(r) \geq \{\gamma_{P_0[g]} - \varepsilon\} T_g(r) \\ & \text{i.e., } r \geq T_{P_0[g]}^{-1} [\{\gamma_{P_0[g]} - \varepsilon\} T_g(r)] \\ (3) \quad & \text{i.e., } T_g^{-1} \left(\frac{r}{\gamma_{P_0[g]} - \varepsilon} \right) \geq T_{P_0[g]}^{-1}(r). \end{aligned}$$

and

$$\begin{aligned} & T_{P_0[g]}(r) \leq \{\gamma_{P_0[g]} + \varepsilon\} T_g(r) \\ & \text{i.e., } r \leq T_{P_0[g]}^{-1} [\{\gamma_{P_0[g]} + \varepsilon\} T_g(r)] \\ (4) \quad & \text{i.e., } T_g^{-1} \left(\frac{r}{\gamma_{P_0[g]} + \varepsilon} \right) \leq T_{P_0[g]}^{-1}(r). \end{aligned}$$

Now, from (1) and (3), it follows for all sufficiently large values of r ,

$$\begin{aligned} & T_{P_0[g]}^{-1} T_{P_0[f]}(r) \leq T_{P_0[g]}^{-1} [\{\gamma_{P_0[f]} + \varepsilon\} T_f(r)] \\ (5) \quad & \text{i.e., } T_{P_0[g]}^{-1} T_{P_0[f]}(r) \leq T_g^{-1} \left[\left(\frac{\gamma_{P_0[f]} + \varepsilon}{\gamma_{P_0[g]} - \varepsilon} \right) T_f(r) \right]. \end{aligned}$$

Again from (2) and (4) it follows for all sufficiently large values of r ,

$$(6) \quad \begin{aligned} T_{P_0[g]}^{-1} T_{P_0[f]}(r) &\geq T_{P_0[g]}^{-1} [\{\gamma_{P_0[f]} - \varepsilon\} T_f(r)] \\ \text{i.e., } T_{P_0[g]}^{-1} T_{P_0[f]}(r) &\geq T_g^{-1} \left[\left(\frac{\gamma_{P_0[f]} - \varepsilon}{\gamma_{P_0[g]} + \varepsilon} \right) T_f(r) \right]. \end{aligned}$$

Now, for the definition of type and lower type we get for all sufficiently large values of r that

$$(7) \quad \begin{aligned} T_g \left(\left\{ \frac{T_f(r)}{(\sigma_g + \varepsilon)} \right\}^{\frac{1}{\rho_g}} \right) &\leq T_f(r) \\ \text{i.e., } T_g^{-1} T_f(r) &\geq \left\{ \frac{T_f(r)}{(\sigma_g + \varepsilon)} \right\}^{\frac{1}{\rho_g}} \end{aligned}$$

and

$$(8) \quad \begin{aligned} T_g \left(\left\{ \left(\frac{\gamma_{P_0[f]} + \varepsilon}{(\gamma_{P_0[g]} - \varepsilon)(\bar{\sigma}_g - \varepsilon)} \right) T_f(r) \right\}^{\frac{1}{\rho_g}} \right) &\geq \left[\left(\frac{\gamma_{P_0[f]} + \varepsilon}{\gamma_{P_0[g]} - \varepsilon} \right) T_f(r) \right] \\ \text{i.e., } \left[\left(\frac{\gamma_{P_0[f]} + \varepsilon}{(\gamma_{P_0[g]} - \varepsilon)(\bar{\sigma}_g - \varepsilon)} \right) T_f(r) \right]^{\frac{1}{\rho_g}} &\geq T_g^{-1} \left[\left(\frac{\gamma_{P_0[f]} + \varepsilon}{\gamma_{P_0[g]} - \varepsilon} \right) T_f(r) \right]. \end{aligned}$$

Therefore, from (5) and (8), it follows for all sufficiently large values of r that

$$(9) \quad T_{P_0[g]}^{-1} T_{P_0[f]}(r) \leq \left[\left(\frac{\gamma_{P_0[f]} + \varepsilon}{(\gamma_{P_0[g]} - \varepsilon)(\bar{\sigma}_g - \varepsilon)} \right) T_f(r) \right]^{\frac{1}{\rho_g}}.$$

Therefore, from (7) and (9), it follows for all sufficiently large values of r that

$$(10) \quad \begin{aligned} \frac{T_{P_0[g]}^{-1} T_{P_0[f]}(r)}{T_g^{-1} T_f(r)} &\leq \frac{\left[\left(\frac{\gamma_{P_0[f]} + \varepsilon}{(\gamma_{P_0[g]} - \varepsilon)(\bar{\sigma}_g - \varepsilon)} \right) T_f(r) \right]^{\frac{1}{\rho_g}}}{\left\{ \frac{T_f(r)}{(\sigma_g + \varepsilon)} \right\}^{\frac{1}{\rho_g}}} \\ \text{i.e., } \frac{T_{P_0[g]}^{-1} T_{P_0[f]}(r)}{T_g^{-1} T_f(r)} &\leq \left(\frac{(\gamma_{P_0[f]} + \varepsilon)(\sigma_g + \varepsilon)}{(\gamma_{P_0[g]} - \varepsilon)(\bar{\sigma}_g - \varepsilon)} \right)^{\frac{1}{\rho_g}} \\ \text{i.e., } \limsup_{r \rightarrow \infty} \frac{T_{P_0[g]}^{-1} T_{P_0[f]}(r)}{T_g^{-1} T_f(r)} &\leq \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}} \right)^{\frac{1}{\rho_g}} \cdot \left(\frac{\sigma_g}{\bar{\sigma}_g} \right)^{\frac{1}{\rho_g}}. \end{aligned}$$

Similarly, from (6), it can be shown for all sufficiently large values of r ,

$$(11) \quad \liminf_{r \rightarrow \infty} \frac{T_{P_0[g]}^{-1} T_{P_0[f]}(r)}{T_g^{-1} T_f(r)} \geq \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}} \right)^{\frac{1}{\rho_g}} \cdot \left(\frac{\bar{\sigma}_g}{\sigma_g} \right)^{\frac{1}{\rho_g}}.$$

Therefore, from (10) and (11), we obtain that

$$(12) \quad \begin{aligned} \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}} \right)^{\frac{1}{\rho_g}} \cdot \left(\frac{\bar{\sigma}_g}{\sigma_g} \right)^{\frac{1}{\rho_g}} &\leq \liminf_{r \rightarrow \infty} \frac{T_{P_0[g]}^{-1} T_{P_0[f]}(r)}{T_g^{-1} T_f(r)} \\ &\leq \limsup_{r \rightarrow \infty} \frac{T_{P_0[g]}^{-1} T_{P_0[f]}(r)}{T_g^{-1} T_f(r)} \\ &\leq \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}} \right)^{\frac{1}{\rho_g}} \cdot \left(\frac{\sigma_g}{\bar{\sigma}_g} \right)^{\frac{1}{\rho_g}}. \end{aligned}$$

Similarly, using the weak type, one can easily verify that

$$(13) \quad \begin{aligned} \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}} \right)^{\frac{1}{\lambda_g}} \cdot \left(\frac{\tau_g}{\bar{\tau}_g} \right)^{\frac{1}{\lambda_g}} &\leq \liminf_{r \rightarrow \infty} \frac{T_{P_0[g]}^{-1} T_{P_0[f]}(r)}{T_g^{-1} T_f(r)} \\ &\leq \limsup_{r \rightarrow \infty} \frac{T_{P_0[g]}^{-1} T_{P_0[f]}(r)}{T_g^{-1} T_f(r)} \\ &\leq \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}} \right)^{\frac{1}{\lambda_g}} \cdot \left(\frac{\bar{\tau}_g}{\tau_g} \right)^{\frac{1}{\lambda_g}}. \end{aligned}$$

Thus the theorem follows from (12) and (13). ■

Theorem 2. Let f be a meromorphic function either of finite order or of non-zero lower order such that $\sum_{a \neq \infty} \Theta(a; f) = 2$ and g be an entire function with

$0 < \tau_g \leq \bar{\tau}_g < \infty$ and $0 < \bar{\sigma}_g \leq \sigma_g < \infty$. Also let $\sum_{a \neq \infty} \Theta(a; g) = 2$. Then

$$\begin{aligned} &\max \left\{ \left(\frac{\Gamma_{P_0[f]}}{\Gamma_{P_0[g]}} \right)^{\frac{1}{\lambda_g}} \cdot \left(\frac{\tau_g}{\bar{\tau}_g} \right)^{\frac{1}{\lambda_g}}, \left(\frac{\Gamma_{P_0[f]}}{\Gamma_{P_0[g]}} \right)^{\frac{1}{\rho_g}} \cdot \left(\frac{\bar{\sigma}_g}{\sigma_g} \right)^{\frac{1}{\rho_g}} \right\} \\ &\leq \liminf_{r \rightarrow \infty} \frac{T_{P_0[g]}^{-1} T_{P_0[f]}(r)}{T_g^{-1} T_f(r)} \leq \limsup_{r \rightarrow \infty} \frac{T_{P_0[g]}^{-1} T_{P_0[f]}(r)}{T_g^{-1} T_f(r)} \\ &\leq \min \left\{ \left(\frac{\Gamma_{P_0[f]}}{\Gamma_{P_0[g]}} \right)^{\frac{1}{\lambda_g}} \cdot \left(\frac{\bar{\tau}_g}{\tau_g} \right)^{\frac{1}{\lambda_g}}, \left(\frac{\Gamma_{P_0[f]}}{\Gamma_{P_0[g]}} \right)^{\frac{1}{\rho_g}} \cdot \left(\frac{\sigma_g}{\bar{\sigma}_g} \right)^{\frac{1}{\rho_g}} \right\}, \end{aligned}$$

where $P_0[f]$ and $P_0[g]$ are admissible.

With the help of Lemma 2, Theorem 2 can be carried out in the line of Theorem 1. So, the proof is omitted.

Corollary 1. *Under the same conditions of Theorem 1, if g is of regular growth, then by Lemma 3 one can easily verify that*

$$\lim_{r \rightarrow \infty} \frac{T_{P_0[g]}^{-1} T_{P_0[f]}(r)}{T_g^{-1} T_f(r)} = \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}} \right)^{\frac{1}{\rho_g}}.$$

Corollary 2. *Under the same conditions of Theorem 2, if g is of regular growth, then by Lemma 3 one can also verify that*

$$\lim_{r \rightarrow \infty} \frac{T_{P_0[g]}^{-1} T_{P_0[f]}(r)}{T_g^{-1} T_f(r)} = \left(\frac{\Gamma_{P_0[f]}}{\Gamma_{P_0[g]}} \right)^{\frac{1}{\rho_g}}.$$

Theorem 3. *Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$ and g be an entire function with $0 < \lambda_g \leq \rho_g < \infty$. Also let $\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$. Then, for homogeneous $P_0[f]$ and $P_0[g]$,*

$$\frac{\lambda_g}{\rho_g} \leq \liminf_{r \rightarrow \infty} \frac{\log T_{P_0[g]}^{-1} T_{P_0[f]}(r)}{\log T_g^{-1} T_f(r)} \leq \limsup_{r \rightarrow \infty} \frac{\log T_{P_0[g]}^{-1} T_{P_0[f]}(r)}{\log T_g^{-1} T_f(r)} \leq \frac{\rho_g}{\lambda_g}.$$

Proof. From (5) and (6), we get for all sufficiently large values of r that

$$(14) \quad \log T_{P_0[g]}^{-1} T_{P_0[f]}(r) \leq \log T_g^{-1} \left[\left(\frac{\gamma_{P_0[f]} + \varepsilon}{\gamma_{P_0[g]} - \varepsilon} \right) T_f(r) \right]$$

and

$$(15) \quad \log T_{P_0[g]}^{-1} T_{P_0[f]}(r) \geq \log T_g^{-1} \left[\left(\frac{\gamma_{P_0[f]} - \varepsilon}{\gamma_{P_0[g]} + \varepsilon} \right) T_f(r) \right].$$

Now for the definition of order and lower order we get for all sufficiently large values of r that

$$(16) \quad \begin{aligned} T_g \left(\{T_f(r)\}^{\frac{1}{\rho_g + \varepsilon}} \right) &\leq T_f(r) \\ \text{i.e., } \log T_g^{-1} T_f(r) &\geq \frac{1}{(\rho_g + \varepsilon)} \log T_f(r). \end{aligned}$$

and

$$T_g \left(\left\{ \left(\frac{\gamma_{P_0[f]} + \varepsilon}{(\gamma_{P_0[g]} - \varepsilon)(\bar{\sigma}_g - \varepsilon)} \right) T_f(r) \right\}^{\frac{1}{\lambda_g - \varepsilon}} \right) \geq \left[\left(\frac{\gamma_{P_0[f]} + \varepsilon}{\gamma_{P_0[g]} - \varepsilon} \right) T_f(r) \right]$$

$$i.e., \left[\left(\frac{\gamma_{P_0[f]} + \varepsilon}{(\gamma_{P_0[g]} - \varepsilon)(\bar{\sigma}_g - \varepsilon)} \right) T_f(r) \right]^{\frac{1}{\lambda_g - \varepsilon}} \geq T_g^{-1} \left[\left(\frac{\gamma_{P_0[f]} + \varepsilon}{\gamma_{P_0[g]} - \varepsilon} \right) T_f(r) \right]$$

$$(17) \quad i.e., \frac{1}{(\lambda_g - \varepsilon)} \log T_f(r) + O(1) \geq \log T_g^{-1} \left[\left(\frac{\gamma_{P_0[f]} + \varepsilon}{\gamma_{P_0[g]} - \varepsilon} \right) T_f(r) \right].$$

Therefore from (14) and (17) it follows for all sufficiently large values of r that

$$(18) \quad \log T_{P_0[g]}^{-1} T_{P_0[f]}(r) \leq \frac{1}{(\lambda_g - \varepsilon)} \log T_f(r) + O(1).$$

Therefore from (16) and (18) it follows for all sufficiently large values of r that

$$(19) \quad i.e., \limsup_{r \rightarrow \infty} \frac{\log T_{P_0[g]}^{-1} T_{P_0[f]}(r)}{\log T_g^{-1} T_f(r)} \leq \frac{\rho_g}{\lambda_g}.$$

Similarly from (15) it can be shown for all sufficiently large values of r that

$$(20) \quad \liminf_{r \rightarrow \infty} \frac{\log T_{P_0[g]}^{-1} T_{P_0[f]}(r)}{\log T_g^{-1} T_f(r)} \geq \frac{\lambda_g}{\rho_g}.$$

Therefore from (19) and (20) we obtain that

$$\frac{\lambda_g}{\rho_g} \leq \liminf_{r \rightarrow \infty} \frac{\log T_{P_0[g]}^{-1} T_{P_0[f]}(r)}{\log T_g^{-1} T_f(r)} \leq \limsup_{r \rightarrow \infty} \frac{\log T_{P_0[g]}^{-1} T_{P_0[f]}(r)}{\log T_g^{-1} T_f(r)} \leq \frac{\rho_g}{\lambda_g}.$$

Thus the theorem follows from above. ■

Remark 1. The conclusion of Theorem 3 can also drawn under the hypothesis $\sum_{a \neq \infty} \Theta(a; f) = 2$ and $\sum_{a \neq \infty} \Theta(a; g) = 2$ instead of “ $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$ ” and “ $\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$ ” where $P_0[f]$ and $P_0[g]$ are admissible.

Corollary 3. Under the same conditions of Theorem 3 and Remark 1 if g is of regular growth then one may get that

$$\lim_{r \rightarrow \infty} \frac{\log T_{P_0[g]}^{-1} T_{P_0[f]}(r)}{\log T_g^{-1} T_f(r)} = 1.$$

Theorem 4. *If f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$ and g be an entire function of regular growth having non zero finite order and $\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$. Then the relative order and relative lower order of $P_0[f]$ with respect to $P_0[g]$ are same as those of f with respect to g where $P_0[f]$ and $P_0[g]$ are homogeneous.*

Proof. By Corollary 3, we obtain that

$$\begin{aligned} \rho_{P_0[g]}(P_0[f]) &= \limsup_{r \rightarrow \infty} \frac{\log T_{P_0[g]}^{-1} T_{P_0[f]}(r)}{\log r} \\ &= \limsup_{r \rightarrow \infty} \frac{\log T_g^{-1} T_f(r)}{\log r} \cdot \lim_{r \rightarrow \infty} \frac{\log T_{P_0[g]}^{-1} T_{P_0[f]}(r)}{\log T_g^{-1} T_f(r)} \\ &= \rho_g(f) \cdot 1 = \rho_g(f). \end{aligned}$$

In a similar manner,

$$\lambda_{P_0[g]}(P_0[f]) = \lambda_g(f).$$

Thus, the theorem follows. ■

Theorem 5. *If f be a meromorphic function either of finite order or of non-zero lower order such that $\sum_{a \neq \infty} \Theta(a; f) = 2$ and g be an entire function of regular growth having non zero finite order and $\sum_{a \neq \infty} \Theta(a; g) = 2$. Then the relative order and relative lower order of $P_0[f]$ with respect to $P_0[g]$ are same as those of f with respect to g where $P_0[f]$ and $P_0[g]$ are admissible.*

We omit the proof of Theorem 5 because it can be carried out in the line of Theorem 4 and with the help of Corollary 3.

Theorem 6. *If f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$ and g be an entire function of regular growth having non zero finite type and $\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$. Then the relative type and relative lower type of $P_0[f]$ with respect to $P_0[g]$ are $\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}}$ times that of f with respect to g if $\rho_g(f)$ is positive finite and $P_0[f]$ and $P_0[g]$ are homogeneous.*

Proof. From Corollary 1 and Theorem 4, we get that

$$\begin{aligned} \sigma_{P_0[g]}(P_0[f]) &= \limsup_{r \rightarrow \infty} \frac{T_{P_0[g]}^{-1} T_{P_0[f]}(r)}{r^{\rho_{P_0[g]}(P_0[f])}} \\ &= \lim_{r \rightarrow \infty} \frac{T_{P_0[g]}^{-1} T_{P_0[f]}(r)}{T_g^{-1} T_f(r)} \cdot \limsup_{r \rightarrow \infty} \frac{T_g^{-1} T_f(r)}{r^{\rho_g(f)}} = \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}} \sigma_g(f). \end{aligned}$$

Similarly,

$$\bar{\sigma}_{P_0[g]}(P_0[f]) = \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}} \right)^{\frac{1}{\rho_g}} \cdot \bar{\sigma}_g(f).$$

This proves the theorem. $\blacksquare$

Theorem 7. Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$ and g be an entire function of regular growth having non zero finite type and $\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$. Then $\tau_{P_0[g]}(P_0[f])$ and $\tau_{P_0[g]}^-(P_0[f])$ are $\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}} \right)^{\frac{1}{\rho_g}}$ times that of f with respect to g , i.e., $\tau_{P_0[g]}(P_0[f]) = \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}} \right)^{\frac{1}{\rho_g}} \cdot \tau_g(f)$ and $\tau_{P_0[g]}^-(P_0[f]) = \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}} \right)^{\frac{1}{\rho_g}} \cdot \bar{\tau}_g(f)$ when $\lambda_g(f)$ is positive finite and $P_0[f]$ and $P_0[g]$ are homogeneous.

We omit the proof of Theorem 7 because it can be carried out in the line of Theorem 6 and with the help of Theorem 5 and Corollary 2.

In a similar manner, we can state the following two theorem without proof:

Theorem 8. If f be a meromorphic function either of finite order or of non-zero lower order such that $\sum_{a \neq \infty} \Theta(a; f) = 2$ and g be an entire function of regular growth having non zero finite type and $\sum_{a \neq \infty} \Theta(a; g) = 2$, then the relative type and relative lower type of $P_0[f]$ with respect to $P_0[g]$ are $\left(\frac{\Gamma_{P_0[f]}}{\Gamma_{P_0[g]}} \right)^{\frac{1}{\rho_g}}$ times that of f with respect to g if $\rho_g(f)$ is positive finite and $P_0[f]$ and $P_0[g]$ are admissible.

Theorem 9. Let f be a meromorphic function either of finite order or of non-zero lower order such that $\sum_{a \neq \infty} \Theta(a; f) = 2$ and g be an entire function of regular growth having non zero finite type and $\sum_{a \neq \infty} \Theta(a; g) = 2$. Then $\tau_{P_0[g]}(P_0[f])$ and $\tau_{P_0[g]}^-(P_0[f])$ are $\left(\frac{\Gamma_{P_0[f]}}{\Gamma_{P_0[g]}} \right)^{\frac{1}{\rho_g}}$ times that of f with respect to g i.e., $\tau_{P_0[g]}(P_0[f]) = \left(\frac{\Gamma_{P_0[f]}}{\Gamma_{P_0[g]}} \right)^{\frac{1}{\rho_g}} \cdot \tau_g(f)$ and $\tau_{P_0[g]}^-(P_0[f]) = \left(\frac{\Gamma_{P_0[f]}}{\Gamma_{P_0[g]}} \right)^{\frac{1}{\rho_g}} \cdot \bar{\tau}_g(f)$ when $\lambda_g(f)$ is positive finite and $P_0[f]$ and $P_0[g]$ are homogeneous.

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Accepted: 18.10.2013

JENSEN TYPE WEIGHTED INEQUALITIES FOR FUNCTIONS OF SELFADJOINT AND UNITARY OPERATORS

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Abstract. On making use of the spectral representations in terms of the Riemann-Stieltjes integral for the selfadjoint and unitary operators in Hilbert spaces we establish here some weighted inequalities of Jensen's type for convex, square-convex and Arg-square-convex functions. Some applications for simple functions of operators that belong to those classes are also provided.

Keywords: Jensen's inequality, Hölder's inequality, Measurable functions, Lebesgue integral, Selfadjoint operators, Unitary operators, Spectral family.

1991 Mathematics Subject Classification: Primary 26D15, 26D20; Secondary 47A63.

1. Introduction

Let A be a selfadjoint operator on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ with the spectrum $Sp(A)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family*. Then for any continuous function $f : [m, M] \rightarrow \mathbb{R}$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral* (see, for instance, [19, p. 257]):

$$(1) \quad \langle f(A)x, y \rangle = \int_{m-0}^M f(\lambda) d(\langle E_\lambda x, y \rangle),$$

and

$$(2) \quad \|f(A)x\|^2 = \int_{m-0}^M |f(\lambda)|^2 d\|E_\lambda x\|^2,$$

for any $x, y \in H$.

The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0 \text{ while } g_{x,y}(M) = \langle x, y \rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$ for any $x \in H$.

The following result that provides an operator version for the Jensen inequality is due to Mond & Pečarić [23] (see also [18, p. 5]):

Theorem 1 (Mond-Pečarić, 1993, [23]) *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If h is a convex function on $[m, M]$, then*

$$(MP) \quad h(\langle Ax, x \rangle) \leq \langle h(A)x, x \rangle$$

for each $x \in H$ with $\|x\| = 1$.

As a special case of Theorem 1 we have the following Hölder-McCarthy inequality:

Theorem 2 (Hölder-McCarthy, 1967, [21]) *Let A be a selfadjoint positive operator on a Hilbert space H . Then, for all $x \in H$ with $\|x\| = 1$,*

- (i) $\langle A^r x, x \rangle \geq \langle Ax, x \rangle^r$ for all $r > 1$;
- (ii) $\langle A^r x, x \rangle \leq \langle Ax, x \rangle^r$ for all $0 < r < 1$;
- (iii) If A is invertible, then $\langle A^r x, x \rangle \geq \langle Ax, x \rangle^r$ for all $r < 0$.

The following reverse for the Mond-Pečarić inequality that generalizes the scalar Lah-Ribarić inequality for convex functions is well known, see for instance [18, p. 57]:

Theorem 3 *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If h is a convex function on $[m, M]$, then*

$$(LR) \quad \langle h(A)x, x \rangle \leq \frac{M - \langle Ax, x \rangle}{M - m} \cdot h(m) + \frac{\langle Ax, x \rangle - m}{M - m} \cdot h(M)$$

for each $x \in H$ with $\|x\| = 1$.

We recall that the bounded linear operator $U : H \rightarrow H$ on the Hilbert space H is *unitary* iff $U^* = U^{-1}$.

It is well known that (see for instance [19, p. 275-p. 276]), if U is a unitary operator, then there exists a family of *projections* $\{E_\lambda\}_{\lambda \in [0, 2\pi]}$, called the *spectral family* of U with the following properties

- a) $E_\lambda \leq E_\mu$ for $0 \leq \lambda \leq \mu \leq 2\pi$;
- b) $E_0 = 0$ and $E_{2\pi} = 1_H$ (the identity operator on H);
- c) $E_{\lambda+0} = E_\lambda$ for $0 \leq \lambda < 2\pi$;
- d) $U = \int_0^{2\pi} e^{i\lambda} dE_\lambda$ where the integral is of Riemann-Stieltjes type.

Moreover, if $\{F_\lambda\}_{\lambda \in [0, 2\pi]}$ is a family of projections satisfying the requirements a)-d) above for the operator U , then $F_\lambda = E_\lambda$ for all $\lambda \in [0, 2\pi]$.

Also, for every continuous complex valued function $f : \mathcal{C}(0, 1) \rightarrow \mathbb{C}$ on the complex unit circle $\mathcal{C}(0, 1)$, we have

$$(3) \quad f(U) = \int_0^{2\pi} f(e^{i\lambda}) dE_\lambda$$

where the integral is taken in the Riemann-Stieltjes sense.

In particular, we have the equalities

$$(4) \quad \langle f(U)x, y \rangle = \int_0^{2\pi} f(e^{i\lambda}) d\langle E_\lambda x, y \rangle$$

and

$$(5) \quad \|f(U)x\|^2 = \int_0^{2\pi} |f(e^{i\lambda})|^2 d\|E_\lambda x\|^2 = \int_0^{2\pi} |f(e^{i\lambda})|^2 d\langle E_\lambda x, x \rangle,$$

for any $x, y \in H$.

From the above properties it follows that the function $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is *monotonic nondecreasing* and *right continuous* on $[0, 2\pi]$ for any $x \in H$.

For $z \in \mathbb{C} \setminus \{0\}$ we call the *principal value* of $\log(z)$ the complex number

$$\text{Log}(z) := \ln|z| + i\text{Arg}(z)$$

where $0 \leq \text{Arg}(z) < 2\pi$.

We observe that for $t \in [0, 2\pi)$ we have

$$\text{Log}(e^{it}) = it.$$

If we extend this equality by continuity in the point $t = 2\pi$, then we can define the operator $\text{Log}(U) : H \rightarrow H$ as

$$(6) \quad \text{Log}(U)x = \int_0^{2\pi} \text{Log}(e^{i\lambda}) dE_\lambda x = \int_0^{2\pi} (i\lambda) dE_\lambda x, \quad x \in H.$$

Utilizing these spectral representations in terms of the Riemann-Stieltjes integral for the selfadjoint and unitary operators we establish here some weighted inequalities of Jensen's type for three classes of functions: convex, square-convex and Arg-square-convex functions. Some applications for simple functions of operators that belong to those classes are also provided.

For classical and recent results concerning inequalities for continuous functions of selfadjoint operators, see [23], [24], [25], [20], [18], [6], [9], [10], [12], [11], [16], [15], [14], [13], [7], and [8].

2. Weighted inequalities for the Riemann-Stieltjes integral

We can state the following result concerning the weighted Riemann-Stieltjes integral of monotonic nondecreasing integrators:

Theorem 4 *Let $\Phi : [\gamma, \Gamma] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a continuous convex function on the interval $[\gamma, \Gamma]$, $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function on the interval $[a, b]$ and with the property that*

$$(7) \quad \gamma \leq f(t) \leq \Gamma \text{ for any } t \in [a, b]$$

and $w : [a, b] \rightarrow [0, \infty)$ be continuous on $[a, b]$. Then, for each monotonic non-decreasing function $u : [a, b] \rightarrow \mathbb{R}$ such that $\int_a^b w(t) du(t) > 0$, we have the inequalities

$$(8) \quad \begin{aligned} & \Phi \left(\frac{\int_a^b w(t) f(t) du(t)}{\int_a^b w(t) du(t)} \right) \leq \frac{\int_a^b w(t) (\Phi \circ f)(t) du(t)}{\int_a^b w(t) du(t)} \\ & \leq \frac{\Phi(\gamma) \left(\Gamma - \frac{\int_a^b w(t) f(t) du(t)}{\int_a^b w(t) du(t)} \right) + \Phi(\Gamma) \left(\frac{\int_a^b w(t) f(t) du(t)}{\int_a^b w(t) du(t)} - \gamma \right)}{\Gamma - \gamma}. \end{aligned}$$

Proof. Utilizing the gradient inequality for the convex function Φ , namely,

$$\Phi(\varsigma) - \Phi(\tau) \geq \delta_{\Phi}(\tau)(\varsigma - \tau)$$

for any $\varsigma, \tau \in [\gamma, \Gamma]$ where $\delta_{\Phi}(\tau) \in [\Phi'_-(\tau), \Phi'_+(\tau)]$, (for $\tau = \gamma$ we take $\delta_{\Phi}(\tau) = \Phi'_+(\gamma)$ and for $\tau = \Gamma$ we take $\delta_{\Phi}(\tau) = \Phi'_-(\Gamma)$), then we get

$$(9) \quad \begin{aligned} & \Phi(\varsigma) - \Phi \left(\frac{\int_a^b w(t) f(t) du(t)}{\int_a^b w(t) du(t)} \right) \\ & \geq \delta_{\Phi} \left(\frac{\int_a^b w(t) f(t) du(t)}{\int_a^b w(t) du(t)} \right) \left(\varsigma - \frac{\int_a^b w(t) f(t) du(t)}{\int_a^b w(t) du(t)} \right) \end{aligned}$$

for any $\varsigma \in [\gamma, \Gamma]$, since obviously, by (7)

$$\frac{\int_a^b w(t) f(t) du(t)}{\int_a^b w(t) du(t)} \in [\gamma, \Gamma].$$

Since f satisfies (7), then by (9) we get

$$(10) \quad (\Phi \circ f)(s) - \Phi \left(\frac{\int_a^b w(t) f(t) du(t)}{\int_a^b w(t) du(t)} \right) \\ \geq \delta_\Phi \left(\frac{\int_a^b w(t) f(t) du(t)}{\int_a^b w(t) du(t)} \right) \left(f(s) - \frac{\int_a^b w(t) f(t) du(t)}{\int_a^b w(t) du(t)} \right)$$

for any $s \in [a, b]$.

Now, if we multiply (10) by $w(s) \geq 0$ and integrate the result over the monotonic nondecreasing integrator u on the interval $[a, b]$ we obtain the first inequality in (8).

By the convexity of Φ we also have the inequality

$$\Phi(\tau) \leq \frac{(\Gamma - \tau) \Phi(\gamma) + (\tau - \gamma) \Phi(\Gamma)}{\Gamma - \gamma}$$

for any $\tau \in [\gamma, \Gamma]$, which, by (9) implies that

$$(11) \quad (\Phi \circ f)(s) \leq \frac{(\Gamma - f(s)) \Phi(\gamma) + (f(s) - \gamma) \Phi(\Gamma)}{\Gamma - \gamma}$$

for any $s \in [a, b]$.

Now, if we multiply (11) by $w(s) \geq 0$ and integrate the result over the monotonic nondecreasing integrator u on the interval $[a, b]$ we obtain the second inequality in (11).

The proof is complete. ■

Remark 1 The above inequality provides a generalization for the unweighted case, namely $w(t) = 1$, $t \in [a, b]$, which can be stated as

$$(12) \quad \Phi \left(\frac{\int_a^b f(t) du(t)}{u(b) - u(a)} \right) \leq \frac{\int_a^b (\Phi \circ f)(t) du(t)}{u(b) - u(a)} \\ \leq \frac{\Phi(\gamma) \left(\Gamma - \frac{\int_a^b f(t) du(t)}{u(b) - u(a)} \right) + \Phi(\Gamma) \left(\frac{\int_a^b f(t) du(t)}{u(b) - u(a)} - \gamma \right)}{\Gamma - \gamma}.$$

For inequalities related to the Jensen's result, see [1], [2], [3], [17], [4], [26] and [27].

Corollary 1 Let $h : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function on the interval $[a, b]$ and with the property that

$$(13) \quad 0 \leq \gamma \leq h(t) \leq \Gamma \text{ for any } t \in [a, b]$$

and $w : [a, b] \rightarrow [0, \infty)$ be continuous on $[a, b]$. Assume also that $u : [a, b] \rightarrow \mathbb{R}$ is a monotonic nondecreasing function such that $\int_a^b w(t) du(t) > 0$.

(i) If $p \geq 1$, then

$$\begin{aligned}
 (14) \quad & \left(\int_a^b w(t) h(t) du(t) \right)^p \\
 & \leq \left[\int_a^b w(t) du(t) \right]^{p-1} \int_a^b w(t) h^p(t) du(t) \\
 & \leq \frac{1}{\Gamma - \gamma} \left[\int_a^b w(t) du(t) \right]^p \\
 & \times \left[\gamma^p \left(\Gamma - \frac{\int_a^b w(t) h(t) du(t)}{\int_a^b w(t) du(t)} \right) + \Phi^p \left(\frac{\int_a^b w(t) h(t) du(t)}{\int_a^b w(t) du(t)} - \gamma \right) \right].
 \end{aligned}$$

(ii) If $p \in (0, 1)$, then the inequalities reverse in (14).

(iii) If $p < 0$ and $\gamma > 0$, then the inequality (14) also holds.

The proof follows by Theorem 4 applied for the convex (concave) function $f(t) = t^p$, $p \in (-\infty, 0) \cup [1, \infty)$ ($p \in (0, 1)$).

The following result is the well known version of the Hölder inequality for the Riemann-Stieltjes integral with monotonic nondecreasing integrators $u : [a, b] \rightarrow \mathbb{R}$:

$$(15) \quad \int_a^b |f(t) g(t)| du(t) \leq \left[\int_a^b |f(t)|^p du(t) \right]^{1/p} \left[\int_a^b |g(t)|^q du(t) \right]^{1/q},$$

where $f, g : [a, b] \subset \mathbb{R} \rightarrow \mathbb{C}$ are continuous and $p, q > 1$ with $1/p + 1/q = 1$.

Proposition 1 Let $f, g : [a, b] \subset \mathbb{R} \rightarrow \mathbb{C} \setminus \{0\}$ be continuous on $[a, b]$ and $u : [a, b] \rightarrow \mathbb{R}$ monotonic nondecreasing on $[a, b]$. Let $p, q \in \mathbb{R} \setminus \{0\}$ with $1/p + 1/q = 1$ and assume that

$$(16) \quad 0 \leq \gamma \leq \frac{|f(t)|}{|g(t)|^{q-1}} \leq \Gamma \text{ for any } t \in [a, b].$$

(i) If $p > 1$, then

$$\begin{aligned}
 (17) \quad & \int_a^b |f(t) g(t)| du(t) \\
 & \leq \left[\int_a^b |g(t)|^q du(t) \right]^{1/q} \left[\int_a^b |f(t)|^p du(t) \right]^{1/p} \\
 & \leq \frac{1}{(\Gamma - \gamma)^{1/p}} \int_a^b |g(t)|^q du(t) \\
 & \times \left[\gamma^p \left(\Gamma - \frac{\int_a^b |f(t) g(t)| du(t)}{\int_a^b |g(t)|^q du(t)} \right) + \Phi^p \left(\frac{\int_a^b |f(t) g(t)| du(t)}{\int_a^b |g(t)|^q du(t)} - \gamma \right) \right]^{1/p}.
 \end{aligned}$$

(ii) If $p \in (0, 1)$, then the inequalities in (17) reverse.

(iii) If $p < 0$ and $\gamma > 0$ then the inequalities in (17) also reverse.

Proof. Follows by Corollary 1, by choosing

$$h = \frac{|f|}{|g|^{q-1}} \text{ and } w = |g|^q$$

and performing some simple calculation.

The details are omitted. ■

Corollary 2 *Let $h : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function on the interval $[a, b]$ and with the property that*

$$(18) \quad 0 < \gamma \leq h(t) \leq \Gamma \text{ for any } t \in [a, b]$$

and $w : [a, b] \rightarrow [0, \infty)$ be continuous on $[a, b]$. Assume also that $u : [a, b] \rightarrow \mathbb{R}$ is a monotonic nondecreasing function such that $\int_a^b w(t) du(t) > 0$. Then

$$(19) \quad \frac{\int_a^b w(t) h(t) du(t)}{\int_a^b w(t) du(t)} \geq \exp \left[\frac{\int_a^b w(t) (\ln \circ h)(t) du(t)}{\int_a^b w(t) du(t)} \right] \\ \geq \gamma^{\frac{1}{\Gamma-\gamma} \left(\Gamma - \frac{\int_a^b w(t) h(t) du(t)}{\int_a^b w(t) du(t)} \right)} \Gamma^{\frac{1}{\Gamma-\gamma} \left(\frac{\int_a^b w(t) h(t) du(t)}{\int_a^b w(t) du(t)} - \gamma \right)}.$$

The proof follows by Theorem 4 applied for the convex function $\Phi(t) = -\ln t$, $t > 0$.

3. Weighted inequalities for convex functions of selfadjoint operators

We can state the following result concerning the weighted Jensen's inequality for continuous functions of selfadjoint operators:

Theorem 5 *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If $\Phi : [\gamma, \Gamma] \subset \mathbb{R} \rightarrow \mathbb{R}$ is a continuous convex function on the interval $[\gamma, \Gamma]$, $f : [m, M] \subset \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function on the interval $[m, M]$ and with the property that*

$$(20) \quad \gamma \leq f(t) \leq \Gamma \text{ for any } t \in [m, M]$$

and $w : [m, M] \rightarrow [0, \infty)$ is continuous on $[m, M]$, then

$$(21) \quad \Phi \left(\frac{\langle w(A) f(A) x, x \rangle}{\langle w(A) x, x \rangle} \right) \leq \frac{\langle w(A) (\Phi \circ f)(A) x, x \rangle}{\langle w(A) x, x \rangle} \\ \leq \frac{\Phi(\gamma) \left(\Gamma - \frac{\langle w(A) f(A) x, x \rangle}{\langle w(A) x, x \rangle} \right) + \Phi(\Gamma) \left(\frac{\langle w(A) f(A) x, x \rangle}{\langle w(A) x, x \rangle} - \gamma \right)}{\Gamma - \gamma},$$

for any $x \in H$ with $\|x\| = 1$ and $\langle w(A) x, x \rangle \neq 0$.

Proof. Let $\{E_\lambda\}_\lambda$ be the spectral family of the operator A . Let $\varepsilon > 0$ and write the inequality (8) on the interval $[a, b] = [m - \varepsilon, M]$ and for the monotonic nondecreasing function $g(t) = \langle E_t x, x \rangle$, $x \in H$ with $\|x\| = 1$, to get

$$(22) \quad \begin{aligned} & \Phi \left(\frac{\int_{m-\varepsilon}^M w(t) f(t) d \langle E_t x, x \rangle}{\int_{m-\varepsilon}^M w(t) d \langle E_t x, x \rangle} \right) \leq \frac{\int_{m-\varepsilon}^M w(t) (\Phi \circ f)(t) d \langle E_t x, x \rangle}{\int_{m-\varepsilon}^M w(t) d \langle E_t x, x \rangle} \\ & \leq \frac{\left(\Gamma - \frac{\int_{m-\varepsilon}^M w(t) f(t) d \langle E_t x, x \rangle}{\int_{m-\varepsilon}^M w(t) d \langle E_t x, x \rangle} \right) \Phi(\gamma) + \left(\frac{\int_{m-\varepsilon}^M w(t) f(t) d \langle E_t x, x \rangle}{\int_{m-\varepsilon}^M w(t) d \langle E_t x, x \rangle} - \gamma \right) \Phi(\Gamma)}{\Gamma - \gamma}. \end{aligned}$$

Letting $\varepsilon \rightarrow 0+$ and utilizing the spectral representation (1), we deduce from (22) the desired result (21). $\blacksquare$

Remark 2 If we choose $w(t) = 1$ and $f(t) = t$ with $t \in [m, M]$ then we get from (21) the inequalities (MP) and (LR).

We have the following generalization and reverse for the Hölder-McCarthy inequality:

Corollary 3 Let A be a selfadjoint positive operator on a Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If the functions $f, w : [m, M] \rightarrow [0, \infty)$ are continuous and f satisfies the condition (20) with $\gamma \geq 0$, then for any $p \geq 1$ we have

$$(23) \quad \begin{aligned} & \langle w(A) f(A) x, x \rangle^p \\ & \leq \langle w(A) f^p(A) x, x \rangle \langle w(A) x, x \rangle^{p-1} \\ & \leq \frac{1}{\Gamma - \gamma} \langle w(A) x, x \rangle^{p-1} \\ & \quad \times [\gamma^p (\langle w(A) [\Gamma 1_H - f(A)] x, x \rangle) + \Gamma^p (\langle w(A) [f(A) - \gamma 1_H] x, x \rangle)] \end{aligned}$$

where $x \in H$ with $\|x\| = 1$.

If $p \in (0, 1)$ then the inequalities reverse in (23).

If $\gamma > 0$ and $p < 0$ the inequalities in (23) also hold.

Remark 3 If we choose $w(t) = 1$ and $f(t) = t$ with $t \in [m, M] \subset [0, \infty)$ then we get from (23)

$$(24) \quad \begin{aligned} & \langle Ax, x \rangle^p \leq \langle A^p x, x \rangle \\ & \leq \frac{1}{M - m} [m^p (\langle (M 1_H - A) x, x \rangle) + M^p (\langle (A - m 1_H) x, x \rangle)] \end{aligned}$$

for any $p \geq 1$, where $x \in H$ with $\|x\| = 1$.

If $p \in (0, 1)$, then the inequalities reverse in (24).

If $m > 0$ and $p < 0$ then the inequalities in (24) also hold.

Remark 4 If we choose $w(t) = f(t) = t$ with $t \in [m, M] \subset [0, \infty)$, then we get from (23)

$$(25) \quad \begin{aligned} \langle A^2 x, x \rangle^p &\leq \langle A^p x, x \rangle \langle Ax, x \rangle^{p-1} \\ &\leq \frac{1}{M-m} \langle Ax, x \rangle^{p-1} [m^p (\langle A(M1_H - A)x, x \rangle) + M^p (\langle A(A - m1_H)x, x \rangle)] \end{aligned}$$

for any $p \geq 1$, where $x \in H$ with $\|x\| = 1$.

If $p \in (0, 1)$, then the inequalities reverse in (25).

If $m > 0$ and $p < 0$ then the inequalities in (25) also hold.

Corollary 4 Let A be a selfadjoint positive operator on a Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If the functions $f, w : [m, M] \rightarrow [0, \infty)$ are continuous and f satisfies the condition (20) with $\gamma > 0$ then

$$(26) \quad \begin{aligned} \frac{\langle w(A)f(A)x, x \rangle}{\langle w(A)x, x \rangle} &\geq \exp \left[\frac{\langle w(A)(\ln \circ f)(A)x, x \rangle}{\langle w(A)x, x \rangle} \right] \\ &\geq \gamma^{\frac{1}{\Gamma-\gamma} \left(\Gamma - \frac{\langle w(A)f(A)x, x \rangle}{\langle w(A)x, x \rangle} \right)} \Gamma^{\frac{1}{\Gamma-\gamma} \left(\frac{\langle w(A)f(A)x, x \rangle}{\langle w(A)x, x \rangle} - \gamma \right)} \end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

Remark 5 If we choose $w(t) = 1$ and $f(t) = t$ with $t \in [m, M] \subset (0, \infty)$ then we get from (26)

$$(27) \quad \begin{aligned} \langle Ax, x \rangle &\geq \exp [\langle \ln Ax, x \rangle] \\ &\geq m^{\frac{1}{M-m} \langle (M1_H - A)x, x \rangle} M^{\frac{1}{M-m} \langle (A - 1_H m)x, x \rangle} \end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

Also, if we choose $w(t) = f(t) = t$ with $t \in [m, M] \subset (0, \infty)$ then we get from (26) that

$$(28) \quad \begin{aligned} \frac{\langle A^2 x, x \rangle}{\langle Ax, x \rangle} &\geq \exp \left[\frac{\langle A \ln Ax, x \rangle}{\langle Ax, x \rangle} \right] \\ &\geq m^{\frac{1}{M-m} \left(M - \frac{\langle A^2 x, x \rangle}{\langle Ax, x \rangle} \right)} M^{\frac{1}{M-m} \left(\frac{\langle A^2 x, x \rangle}{\langle Ax, x \rangle} - m \right)} \end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

Remark 6 If we choose $w(t) = t^r$ and $f(t) = t^q$ with $t \in [m, M] \subset (0, \infty)$ where $r, q > 0$, then we get from (21) that

$$(29) \quad \begin{aligned} \Phi \left(\frac{\langle A^{r+q} x, x \rangle}{\langle A^r x, x \rangle} \right) &\leq \frac{\langle A^r \Phi(A^q)x, x \rangle}{\langle A^r x, x \rangle} \\ &\leq \frac{\Phi(\gamma^q) \left(\Gamma^q - \frac{\langle A^{r+q} x, x \rangle}{\langle A^r x, x \rangle} \right) + \Phi(\Gamma^q) \left(\frac{\langle A^{r+q} x, x \rangle}{\langle A^r x, x \rangle} - \gamma^q \right)}{\Gamma^q - \gamma^q}, \end{aligned}$$

for a continuous convex function $\Phi : [m^q, M^q] \rightarrow \mathbb{R}$ and for any $x \in H$ with $\|x\| = 1$.

We have the following Hölder type inequality for continuous functions of selfadjoint operators:

Proposition 2 *Let A be a selfadjoint positive operator on a Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If $f, g : [a, b] \subset \mathbb{R} \rightarrow \mathbb{C} \setminus \{0\}$ are continuous on $[a, b]$ and $p, q \in \mathbb{R} \setminus \{0\}$ with $1/p + 1/q = 1$ are such that*

$$(30) \quad 0 \leq \gamma \leq \frac{|f(t)|}{|g(t)|^{q-1}} \leq \Gamma \text{ for any } t \in [a, b],$$

then we have the inequalities

$$(31) \quad \begin{aligned} & \langle |f(A)g(A)|x, x \rangle \\ & \leq [\langle |g(A)|^q x, x \rangle]^{1/q} [\langle |f(A)|^p x, x \rangle]^{1/p} \\ & \leq \frac{1}{(\Gamma - \gamma)^{1/p}} \langle |g(A)|^q x, x \rangle \\ & \quad \times \left[\gamma^p \left(\Gamma - \frac{\langle |f(A)g(A)|x, x \rangle}{\langle |g(A)|^q x, x \rangle} \right) + \Gamma^p \left(\frac{\langle |f(A)g(A)|x, x \rangle}{\langle |g(A)|^q x, x \rangle} - \gamma \right) \right]^{1/p}, \end{aligned}$$

for $p > 1$ and for any $x \in H$ with $\|x\| = 1$ and $\langle |g(A)|^q x, x \rangle \neq 0$.

If $p \in (0, 1)$, then the inequalities in (31) reverse;

If $p < 0$ and $\gamma > 0$ then the inequalities in (31) also reverse.

4. Weighted inequalities for square-convex functions

We introduce the following class of complex valued functions:

Definition 1 A function $\Phi : [\gamma, \Gamma] \subset \mathbb{R} \rightarrow \mathbb{C}$ is called square-convex on $[\gamma, \Gamma]$ if the associated function $\varphi : [\gamma, \Gamma] \rightarrow [0, \infty)$, $\varphi(t) = |\Phi(t)|^2$ is convex on $[\gamma, \Gamma]$.

A simple example of such a function is the concave power function $\Phi : [\gamma, \Gamma] \subset [0, \infty) \rightarrow [0, \infty)$, $\Phi(t) = t^r$ with $r \in [\frac{1}{2}, 1]$. Also, if $h : [\gamma, \Gamma] \rightarrow [0, \infty)$ is convex then the complex valued function $\Phi : [\gamma, \Gamma] \subset \mathbb{R} \rightarrow \mathbb{C}$ given by $\Phi(t) = h^{1/2}(t)e^{it}$ is square-convex on $[\gamma, \Gamma]$.

Consider the function $f(t) = \ln(t+1)$. We observe that it is concave and positive on $(0, \infty)$ and if we define $\varphi(t) = [\ln(t+1)]^2$, then we have that

$$\varphi''(t) = \frac{2}{(t+1)^2} [1 - \ln(t+1)], \quad t > -1,$$

showing that f is square-convex on the interval $[0, e-1]$.

Another example for trigonometric functions is for instance $f(t) = \cos t$, $t \in [\frac{\pi}{4}, \frac{\pi}{2}]$. The function $\varphi(t) = \cos^2 t$ has the second derivative $\varphi''(t) = -2 \cos(2t)$ which is positive for $t \in [\frac{\pi}{4}, \frac{\pi}{2}]$. Therefore, f is square-convex on the interval $[\frac{\pi}{4}, \frac{\pi}{2}]$.

Theorem 6 Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If $\Phi : [\gamma, \Gamma] \subset \mathbb{R} \rightarrow \mathbb{C}$ is a continuous square-convex function on the interval $[\gamma, \Gamma]$, $f : [m, M] \subset \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function on the interval $[m, M]$ and with the property that

$$(32) \quad \gamma \leq f(t) \leq \Gamma \text{ for any } t \in [m, M]$$

and $w : [m, M] \rightarrow [0, \infty)$ is continuous on $[m, M]$, then

$$(33) \quad \left| \Phi \left(\frac{\langle w(A) f(A) x, x \rangle}{\langle w(A) x, x \rangle} \right) \right| \leq \left[\frac{\langle w(A) (|\Phi|^2 \circ f)(A) x, x \rangle}{\langle w(A) x, x \rangle} \right]^{1/2} \\ \leq \left[\frac{|\Phi(\gamma)|^2 \left(\Gamma - \frac{\langle w(A) f(A) x, x \rangle}{\langle w(A) x, x \rangle} \right) + |\Phi(\Gamma)|^2 \left(\frac{\langle w(A) f(A) x, x \rangle}{\langle w(A) x, x \rangle} - \gamma \right)}{\Gamma - \gamma} \right]^{1/2}$$

for any $x \in H$ with $\|x\| = 1$ and $\langle w(A) x, x \rangle \neq 0$.

The proof follows from Theorem 4 applied for the function $\varphi : [\gamma, \Gamma] \rightarrow [0, \infty)$, $\varphi(t) = |\Phi(t)|^2$ that is continuous convex on $[\gamma, \Gamma]$. The details are omitted.

Remark 7 If $w(t) = 1$, then we get from (33) the following simpler result

$$(34) \quad |\Phi(\langle f(A) x, x \rangle)| \leq \|(\Phi \circ f)(A) x\| \\ \leq \left[\frac{|\Phi(\gamma)|^2 \langle (\Gamma 1_H - f(A)) x, x \rangle + |\Phi(\Gamma)|^2 \langle (f(A) - 1_H \gamma) x, x \rangle}{\Gamma - \gamma} \right]^{1/2},$$

for any $x \in H$ with $\|x\| = 1$.

This is true since

$$\begin{aligned} \langle (|\Phi|^2 \circ f)(A) x, x \rangle &= \int_{m-0}^M |\Phi(f(t))|^2 d\langle E_t x, x \rangle \\ &= \|\Phi(f(A)) x\|^2 \end{aligned}$$

for any $x \in H$ with $\|x\| = 1$ (for the second equality see for instance [19, p. 257]).

Corollary 5 With the assumptions of Theorem 6 for A, f, w and if $\gamma > 0$, then we have

$$(35) \quad \left(\frac{\langle w(A) f(A) x, x \rangle}{\langle w(A) x, x \rangle} \right)^q \leq \left[\frac{\langle w(A) f^{2q}(A) x, x \rangle}{\langle w(A) x, x \rangle} \right]^{\frac{1}{2}} \\ \leq \left[\frac{\gamma^{2q} \left(\Gamma - \frac{\langle w(A) f(A) x, x \rangle}{\langle w(A) x, x \rangle} \right) + \Gamma^{2q} \left(\frac{\langle w(A) f(A) x, x \rangle}{\langle w(A) x, x \rangle} - \gamma \right)}{\Gamma - \gamma} \right]^{\frac{1}{2}},$$

for any $q \in [\frac{1}{2}, 1]$ and any $x \in H$ with $\|x\| = 1$ and $\langle w(A) x, x \rangle \neq 0$.

Remark 8 If we choose $w(t) = 1$ and $f(t) = t$ with $t \in [m, M] \subset (0, \infty)$ then we get from (35)

$$(36) \quad \langle Ax, x \rangle^q \leq \|A^q x\| \\ \leq \left[\frac{m^{2q} \langle (M1_H - A)x, x \rangle + M^{2q} \langle (A - 1_H m)x, x \rangle}{M - m} \right]^{1/2},$$

for any $q \in [\frac{1}{2}, 1]$ and any $x \in H$ with $\|x\| = 1$.

Also, if we choose $w(t) = f(t) = t$ with $t \in [m, M] \subset (0, \infty)$ then we get from (35)

$$(37) \quad \left(\frac{\langle A^2 x, x \rangle}{\langle Ax, x \rangle} \right)^q \leq \left[\frac{\langle A^{2q+1} x, x \rangle}{\langle Ax, x \rangle} \right]^{1/2} \\ \leq \left[\frac{m^{2q} \left(M - \frac{\langle A^2 x, x \rangle}{\langle Ax, x \rangle} \right) + M^{2q} \left(\frac{\langle A^2 x, x \rangle}{\langle Ax, x \rangle} - m \right)}{M - m} \right]^{1/2},$$

for any $q \in [\frac{1}{2}, 1]$ and any $x \in H$ with $\|x\| = 1$.

Remark 9 If we choose $w(t) = t^r$ and $f(t) = t^s$ with $t \in [m, M] \subset (0, \infty)$ where $r, s > 0$, then we get from (35) that

$$(38) \quad \left(\frac{\langle A^{r+s} x, x \rangle}{\langle A^r x, x \rangle} \right)^q \leq \left[\frac{\langle A^{r+2qs} x, x \rangle}{\langle A^r x, x \rangle} \right]^{\frac{1}{2}} \\ \leq \left[\frac{m^{2qs} \left(M^s - \frac{\langle A^{r+s} x, x \rangle}{\langle A^r x, x \rangle} \right) + M^{2qs} \left(\frac{\langle A^{r+s} x, x \rangle}{\langle A^r x, x \rangle} - m^s \right)}{M^s - m^s} \right]^{\frac{1}{2}},$$

for any $q \in [\frac{1}{2}, 1]$ and any $x \in H$ with $\|x\| = 1$.

5. Weighted inequalities for Arg-square-convex functions

The function $\Phi : \mathcal{C}(0, 1) \rightarrow \mathbb{C}$ will be called *Arg-square-convex* if the composite function $\varphi : [0, 2\pi] \rightarrow [0, \infty)$,

$$\varphi(t) := \begin{cases} |\Phi(e^{it})|^2, & t \in [0, 2\pi) \\ \lim_{s \rightarrow 2\pi-} |\Phi(e^{is})|^2, & t = 2\pi \end{cases}$$

is continuous and convex on $[0, 2\pi]$.

To make the distinction between the value $\varphi(0) = |\Phi(e^{i0})|^2 = |\Phi(1)|^2$ and the value $\varphi(2\pi) = \lim_{s \rightarrow 2\pi-} |\Phi(e^{is})|^2$, we denote by $\Phi_c(1) := \lim_{s \rightarrow 2\pi-} \Phi(e^{is})$. With this notation, we have $\varphi(2\pi) = |\Phi_c(1)|^2$.

The function $\Phi_n : \mathcal{C}(0, 1) \rightarrow \mathbb{C}$, $\Phi_n(z) = [\text{Log}(z)]^n$, where n is a positive integer, is Arg-square-convex. We have

$$\varphi_n(t) = |\Phi_n(e^{it})|^2 = |[\text{Log}(e^{it})]^n|^2 = |it|^{2n} = t^{2n}, t \in [0, 2\pi),$$

and

$$\varphi_n(2\pi) = \lim_{s \rightarrow 2\pi-} |\Phi_n(e^{is})|^2 = |\Phi_{n,c}(1)|^2 = (2\pi)^{2n}.$$

For $q \geq \frac{1}{2}$, define the function $\Phi_q : \mathcal{C}(0, 1) \rightarrow [0, \infty)$ by $\Phi_q(z) = |\text{Log}(z)|^q$. We have

$$\varphi_q(t) = |\Phi_q(e^{it})|^2 = |\text{Log}(e^{it})|^{2q} = |it|^{2q} = t^{2q}, t \in [0, 2\pi)$$

and

$$\varphi_q(2\pi) = \lim_{s \rightarrow 2\pi-} |\Phi_q(e^{is})|^2 = |\Phi_{q,c}(1)|^2 = (2\pi)^{2q}.$$

The function Φ_q for $q \geq \frac{1}{2}$ is an Arg-square-convex function.

If $g : [0, 2\pi] \rightarrow [0, \infty)$ is continuous and convex on $[0, 2\pi]$, then the composite function $\Phi : \mathcal{C}(0, 1) \rightarrow [0, \infty)$ defined by

$$\Phi(z) := [g(|\text{Log}(z)|)]^{1/2}$$

is an Arg-square-convex function on $\mathcal{C}(0, 1)$.

Theorem 7 Let $U \in B(H)$ be a unitary operator on the Hilbert space H and $\Phi : \mathcal{C}(0, 1) \rightarrow \mathbb{C}$ a continuous and Arg-square-convex function on $\mathcal{C}(0, 1)$. If $w : \mathcal{C}(0, 1) \rightarrow [0, \infty)$ is a continuous function, then we have

$$\begin{aligned} & \left| \Phi \left(\exp \left[\frac{\langle w(U) \text{Log}(U)x, x \rangle}{\langle w(U)x, x \rangle} \right] \right) \right| \leq \left[\frac{\langle w(U) |\Phi(U)|^2 x, x \rangle}{\langle w(U)x, x \rangle} \right]^{1/2} \\ (39) \quad & \leq \left[\frac{\left(2\pi - \frac{\langle w(U) |\text{Log}(U)| x, x \rangle}{\langle w(U)x, x \rangle} \right) |\Phi(1)|^2 + \frac{\langle w(U) |\text{Log}(U)| x, x \rangle}{\langle w(U)x, x \rangle} |\Phi_c(1)|^2}{2\pi} \right]^{1/2} \end{aligned}$$

for any $x \in H$, $\|x\| = 1$, where $\Phi_c(1) := \lim_{s \rightarrow 2\pi-} \Phi(e^{is})$.

Proof. We apply Theorem 4 to the function $\varphi : [0, 2\pi] \rightarrow [0, \infty)$,

$$\varphi(t) = \begin{cases} |\Phi(e^{it})|^2, & t \in [0, 2\pi) \\ \lim_{s \rightarrow 2\pi-} |\Phi(e^{is})|^2, & t = 2\pi \end{cases}$$

that is continuous and convex on $[0, 2\pi]$.

If $\{E_\lambda\}_{\lambda \in [0, 2\pi]}$ is the spectral family of the operator U , then we can write the inequality (8) on the interval $[a, b] = [0, 2\pi]$ for the monotonic nondecreasing integrator $u(t) = \langle E_t x, x \rangle$ and for the identity function $f(t) = t, t \in [0, 2\pi]$ to get

$$(40) \quad \left| \Phi \left(\exp \left[\frac{i \int_0^{2\pi} w(e^{it}) t d \langle E_t x, x \rangle}{\int_0^{2\pi} w(e^{it}) d \langle E_t x, x \rangle} \right] \right) \right|^2 \leq \frac{\int_0^{2\pi} w(e^{it}) |\Phi(e^{it})|^2 d \langle E_t x, x \rangle}{\int_0^{2\pi} w(e^{it}) d \langle E_t x, x \rangle}$$

$$\leq \frac{\left(2\pi - \frac{\int_0^{2\pi} w(e^{it}) t d \langle E_t x, x \rangle}{\int_0^{2\pi} w(e^{it}) d \langle E_t x, x \rangle} \right) |\Phi(1)|^2 + \left(\frac{\int_0^{2\pi} w(e^{it}) \Phi(t) d \langle E_t x, x \rangle}{\int_0^{2\pi} w(t) d \langle E_t x, x \rangle} \right) |\Phi_c(1)|^2}{2\pi}$$

for any $x \in H, \|x\| = 1$.

Since, by the spectral representation of functions of unitary operators (3), we have

$$i \int_0^{2\pi} w(e^{it}) t d \langle E_t x, x \rangle = \int_0^{2\pi} w(e^{it}) \operatorname{Log}(e^{it}) d \langle E_t x, x \rangle$$

$$= \langle w(U) \operatorname{Log}(U) x, x \rangle$$

$$\int_0^{2\pi} w(e^{it}) d \langle E_t x, x \rangle = \langle w(U) x, x \rangle,$$

$$\int_0^{2\pi} w(e^{it}) |\Phi(e^{it})|^2 d \langle E_t x, x \rangle = \langle w(U) |\Phi(U)|^2 x, x \rangle \text{ and}$$

$$\int_0^{2\pi} w(e^{it}) t d \langle E_t x, x \rangle = \langle w(U) |\operatorname{Log}(U)| x, x \rangle$$

for any $x \in H, \|x\| = 1$, then inequality (40) produces the desired result (39). ■

Remark 10 If $w(t) = 1$, then we get from (39) the following simpler result

$$(41) \quad |\Phi(\exp[\langle \operatorname{Log}(U) x, x \rangle])| \leq \|\Phi(U) x\|$$

$$\leq \left[\frac{\langle (2\pi 1_H - |\operatorname{Log}(U)|) x, x \rangle |\Phi(1)|^2 + \langle |\operatorname{Log}(U)| x, x \rangle |\Phi_c(1)|^2}{2\pi} \right]^{1/2}$$

for any $x \in H$ with $\|x\| = 1$.

This is true since

$$\langle |\Phi(U)|^2 x, x \rangle = \int_0^{2\pi} |\Phi(e^{it})|^2 d \langle E_t x, x \rangle = \|\Phi(U) x\|^2$$

for any $x \in H$ with $\|x\| = 1$ (for the second equality see (5)).

The interested reader may apply the inequality (39) for different examples of Arg-square-convex functions. We give here only one example, for instance if we choose the function $\Phi_q(z) = |\operatorname{Log}(z)|^q, q \geq 1/2$ as introduced above, then we get from (39)

$$\begin{aligned}
 (42) \quad \left| \operatorname{Log} \left(\exp \left[\frac{\langle w(U) \operatorname{Log}(U)x, x \rangle}{\langle w(U)x, x \rangle} \right] \right) \right|^q &\leq \left[\frac{\langle w(U) |\operatorname{Log}(U)|^{2q} x, x \rangle}{\langle w(U)x, x \rangle} \right]^{1/2} \\
 &\leq \frac{\langle w(U) |\operatorname{Log}(U)| x, x \rangle^{1/2}}{\langle w(U)x, x \rangle^{1/2}} (2\pi)^{q-1/2}
 \end{aligned}$$

for any $x \in H$ with $\|x\| = 1$ and $w : \mathcal{C}(0, 1) \rightarrow [0, \infty)$ a continuous function.

In particular, we have

$$\begin{aligned}
 (43) \quad |\operatorname{Log}(\exp[\langle \operatorname{Log}(U)x, x \rangle])|^q &\leq \| |\operatorname{Log}(U)|^q x \|^2 \\
 &\leq (2\pi)^{q-1/2} \langle |\operatorname{Log}(U)| x, x \rangle^{1/2}
 \end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

Finally, we notice that the following result providing Hölder's type inequalities for continuous functions of unitary operators can be stated:

Proposition 3 *Let $U \in B(H)$ be a unitary operator on the Hilbert space H and. If $f, g : \mathcal{C}(0, 1) \rightarrow \mathbb{C} \setminus \{0\}$ are continuous on $\mathcal{C}(0, 1)$ and $p, q \in \mathbb{R} \setminus \{0\}$ with $1/p + 1/q = 1$ are such that*

$$(44) \quad 0 \leq \gamma \leq \frac{|f(e^{it})|}{|g(e^{it})|^{q-1}} \leq \Gamma \text{ for any } t \in [0, 2\pi]$$

then we have the inequalities

$$\begin{aligned}
 (45) \quad &\langle |f(U)g(U)| x, x \rangle \\
 &\leq [\langle |g(U)|^q x, x \rangle]^{1/q} [\langle |f(U)|^p x, x \rangle]^{1/p} \\
 &\leq \frac{1}{(\Gamma - \gamma)^{1/p}} \langle |g(U)|^q x, x \rangle \\
 &\times \left[\gamma^p \left(\Gamma - \frac{\langle |f(U)g(U)| x, x \rangle}{\langle |g(U)|^q x, x \rangle} \right) + \Gamma^p \left(\frac{\langle |f(U)g(U)| x, x \rangle}{\langle |g(U)|^q x, x \rangle} - \gamma \right) \right]^{1/p},
 \end{aligned}$$

for $p > 1$ and for any $x \in H$ with $\|x\| = 1$ and $\langle |g(U)|^q x, x \rangle \neq 0$.

If $p \in (0, 1)$, then the inequalities in (45) reverse;

If $p < 0$ and $\gamma > 0$ then the inequalities in (45) also reverse.

The proof follows by Proposition 1 and the spectral representation for continuous functions of unitary operators.

If $g : [0, 2\pi] \rightarrow [0, \infty)$ is continuous and convex on $[0, 2\pi]$, then the composite function $f : \mathcal{C}(0, 1) \rightarrow [0, \infty)$ defined by

$$f(z) := [g(|\operatorname{Log}(z)|)]^{1/2}$$

is an Arg-square-convex function on $\mathcal{C}(0, 1)$.

As examples of such functions we have

$$f_{\alpha}(z) := \exp(\alpha | \operatorname{Log}(z)|)$$

which are Arg-square-convex functions on $\mathcal{C}(0, 1)$ for any real number $\alpha \neq 0$.

We also notice that the family of functions

$$f_{m,n} : \mathcal{C}(0, 1) \rightarrow \mathbb{C}, f_{m,n}(z) = z^m [\operatorname{Log}(z)]^n,$$

where $m \neq 0$ is an integer and n is a positive integer, are Arg-square-convex functions.

The reader may apply the above inequalities for these functions as well. However, the details are omitted.

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Accepted: 25.10.2013

φ J-MULTIPLIERS AND φ J-MULTIPLIERS QUADRATIC ON JORDAN BANACH ALGEBRAS

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Abstract. In this work, we define the notion of φ J-multipliers on Jordan-Banach algebras without order and investigate some of their properties. We show that a φ J-multiplier satisfies the condition $T[U_x(y)] = U_{\varphi(x)}[T(y)]$. This suggests that we define a new concept which is the φ J-multiplier quadratic. We show several algebraic or topological properties for both concepts. In particular, we extend some known results for φ -multipliers to φ J-multipliers and φ J-multipliers quadratic.

Keywords: Multiplier, J-Multiplier, J-multiplier quadratic, φ -Multiplier, φ J-Multiplier, φ J-multiplier quadratic, Idempotent homomorphism, Quadratic operator, Spectrum of an element, Product of Jordan, Jordan Banach Algebras, Algebra without order, Special Jordan algebra, Full algebra. Strong topology.

2010 Mathematics Subject Classification: 47B48.

1. Introduction

The aim of this work is to study the φ J-Multipliers and φ J-Multipliers Quadratic in the case of Jordan-Banach algebras.

In the first part, we are going to resume the following definition of a φ -Multiplier T which is due to M. Adib and A. Riazi [1]:

$$T(x \cdot y) = \varphi(x) \cdot Ty = T(x) \cdot \varphi(y).$$

We will show that some results that are true in the case of Banach algebras remain true in the case of Jordan-Banach. We will demonstrate that every φ J-multiplier of Jordan-Banach algebras verifies the relation:

$$T[U_x(y)] = U_{\varphi(x)}[T(y)], \quad \forall x, y \in A,$$

and that this condition is not sufficient for a linear operator to be a φ J-multiplier. This suggests to us to define a new notion which is that of φ J-multiplier quadratic.

In the second part, we will define φ J-multipliers quadratic of Jordan-Banach algebras. We will show that this generalized definition is the one given in the case of Banach. We will compare the two notions just defined and show some theorems of the type Wang. More particularly, we will establish some algebraic and topological relations between all φ J-multipliers quadratic $M_{\varphi JQ}(A)$ of Jordan-Banach and the algebra of continuous linear operators of A , noted $\mathcal{L}(A)$. We will also show some algebraic properties of $M_{\varphi JQ}(A)$.

2. Preliminaries

Let $\mathbb{K}$ be a commutative field of characteristic zero.

We call $\mathbb{K}$ -algebra every $\mathbb{K}$ -vector space A provided with a bilinear product $(x, y) \rightarrow x.y$ of $A \times A$ in A . If the product is associative (resp. commutative), we say that the algebra is associative (resp. commutative).

If A is a non-commutative algebra, we define the following operators: $R_x(y) = xy$ and $L_x(y) = yx$. In this case, we note A^+ the algebra of the same vector space structure as A , provided with product $\circ$ Defined by

$$x \circ y = \frac{1}{2}(xy + yx), \forall x, y \in A$$

$\circ$ is called Jordan product.

If A is an associative algebra not commutative, the subalgebras of A^+ are called special Jordan algebras.

Among the remarkable identities of these algebras, we note the two following identities:

$$x \circ y = y \circ x \quad (C)$$

$$(x^2 \circ y) \circ x = x^2 \circ (y \circ x) \quad (J)$$

More generally, an algebra $(A, \cdot)$ that verifies the two previous identities is called Jordan algebra.

In a Jordan algebra A , we define the following applications:

$$U_x(y) = 2x(xy) - x^2y$$

$$U_{x,y} = \frac{1}{2}(U_{x+y} - U_x - U_y)$$

The application $x \rightarrow U_x$ is quadratic, thus we have: $U_{ax} = a^2U_x$ for every $a \in \mathbb{K}$ and the application $(x, y) \rightarrow U_{x,y}$ is bilinear.

If A has a unity, then $R_x = U_{x,e} = U_{e,x}$.

In the case of special Jordan algebra, we have

$$U_x(y) = xyx, (\forall x, y \in A).$$

$a \in A$ is invertible if and only if U_a is invertible in $\mathcal{L}(A)$ and we have

$$U_{a^{-1}} = (U_a)^{-1}.$$

A normed Jordan algebra A is a Jordan algebra provided with a vector space norm $\|\cdot\|$ verifying

$$\|xy\| \leq \|x\|\|y\|, \quad (\forall x, y \in A).$$

A Jordan Banach algebra $(A, \|\cdot\|)$ is a normed Jordan algebra and complete for its norm.

The spectrum of an element x of Jordan algebra A is a set of scalars λ verifying that $x - \lambda e$ is not invertible in A .

The spectrum of an element of Jordan-Banach algebra is a nonempty compact.

3. The φ J-multipliers of Jordan Banach Algebra

In [2], Helgason has defined a multiplier T of Banach algebra by posing for x et y elements of A :

$$(Tx) \cdot y = x \cdot (Ty)$$

Then, he has showed that, if A is without order, then this definition is equivalent to

$$(3.1) \quad T(x \cdot y) = x \cdot Ty$$

This remark has allowed Wang to establish a number of results on multipliers [9].

As the relation (3.1) arises out quite simply of the associativity of A , we have had the idea of taking it as a definition of a J-multiplier of Jordan Banach algebra [8], in order to find in the case Jordan some results of the type Wang.

In [1], the authors defined the concept of φ -multipliers and proved a some useful results.

We propose in this section to extend the concept of φ -multiplier on Jordan Banach algebras by asking the Definition 3.2.

Definition 3.1 Let A be a Jordan-Banach algebra. A is without order if for all x element of A , $xA = 0$ implies $x = 0$, or, for all x element of A , $Ax = 0$ implies $x = 0$.

Obviously, if A has a unit or if A is semi-simple, then it is without order.

Definition 3.2 Let A be a Jordan-Banach algebra and φ an idempotent homomorphism A . A left (resp. right) φ J-multiplier on A is a bounded linear mapping T of A in A such that $T(x \cdot y) = T(x)\varphi(y)$ (resp. $T(x \cdot y) = \varphi(x)T(y)$) for all $x, y \in A$. We say T is a φ J-multiplier on A if it is both a left φ J-multiplier and right φ J-multiplier. We denote $M_{\varphi J}(A)$ the collection of all φ J-multipliers of A .

Remark 3.1 It is clear that the condition "idempotent" is not necessary in the definition of a φ J-multiplier. But it plays an important role in the following theorems.

Remark 3.2 It is obvious that φ is a φ J-multiplier of A .

To illustrate this theory, we give two important examples of idempotent homomorphisms algebra. It is therefore an example of φ -multiplier and another φ J-multiplier:

Example 3.1 Let $\omega = (\omega_n)_{n \geq 0}$ be a sequence of real numbers such that $\omega_n \geq 1$. Consider the space $c_0(\omega)$ of all sequences $(x_n)_{n \geq 0}$ of complex number for which $\lim_{n \rightarrow \infty} x_n \omega_n = 0$ (see [6]). We equip this space with the weighted supremum norm $\|\cdot\|_w$ given by

$$\|x\|_w := \sup_{n \geq 0} |x_n \omega_n|, \text{ for all } x = (x_n)_{n \geq 0}$$

Moreover, with respect to coordinatewise operations $c_0(\omega)$ is a commutative Banach algebra, consider the standard basis $(e_n)_{n \geq 0}$ where $e_n = (\delta_{nj})_{j \geq 0}$, then e_n is idempotent for all $n \geq 0$.

Now, let the multiplication operator by the element e_k

$$\begin{aligned} L_{e_k} : c_0(\omega) &\longrightarrow c_0(\omega) \\ x &\longrightarrow e_k x \end{aligned}$$

Since e_k is idempotent then L_{e_k} is idempotent homomorphism of algebra $c_0(\omega)$, indeed:

$$(L_{e_k}(xy) = e_k xy = e_k^2 xy = e_k x e_k y = L_{e_k}(x) L_{e_k}(y) \text{ because } c_0(\omega) \text{ is commutative})$$

Therefore, L_{e_k} is a φ -multiplier of $\mathbf{A}$.

Example 3.2 Let A be a non-commutative Banach algebra, $E = \{e \in \mathbf{A} : e^2 = e\}$ denote the set of idempotents in $\mathbf{A}$ and $C(\mathbf{A}) = \{x \in \mathbf{A} : ax = xa, \text{ for all } a \text{ in } \mathbf{A}\}$ the centre of $\mathbf{A}$.

The characterization of Banach algebras idempotents in $C(\mathbf{A})$ has been obtained by Zemanek ([10], Theorem 5.2), he showed that an idempotent $e \in E$ belongs to the centre $C(\mathbf{A})$ if and only if e is an isolated point in the set E . In other words, the centre of a Banach algebra meets the set of idempotents just in its isolated points.

Let $e \in E$ belongs in $C(\mathbf{A})$, and consider the multiplication operator by e . Then L_e is an idempotent homomorphism algebra for $\mathbf{A}^+$ (special Jordan algebra). Therefore, L_e is a φ J-multiplier of $\mathbf{A}^+$.

Proposition 3.1 Let A be a non-commutative Banach algebra. If T is a φ -multiplier of A , then T is a φ J-multiplier in special Jordan algebra A^+ .

Proof. Let A be a non-commutative Banach algebra and T a φ -multiplier of A . Then we have:

$$\begin{aligned} T(x \circ y) &= T\left[\frac{1}{2}(x \cdot y + y \cdot x)\right] \\ &= \frac{1}{2}[T(x \cdot y) + T(y \cdot x)] \\ &= \frac{1}{2}[\varphi(x)T(y) + T(y)\varphi(x)] \\ &= \varphi(x) \circ T(y). \end{aligned}$$

$$\begin{aligned} T(x \circ y) &= T\left[\frac{1}{2}(x \cdot y + y \cdot x)\right] \\ &= \frac{1}{2}[T(x \cdot y) + T(y \cdot x)] \\ &= \frac{1}{2}[T(x)\varphi(y) + \varphi(y)T(x)] \\ &= T(x) \circ \varphi(y). \end{aligned}$$

$$T(x \circ y) = \varphi(x) \circ T(y) = T(x) \circ \varphi(y).$$

Therefore, T is a φ J-multiplier in special Jordan algebra A^+ . ■

Remark 3.3 The definition of a φ J-multiplier of Jordan Banach algebra generalizes well that given in the case of Banach algebra.

For the adopted definition of a φ J-multiplier of Jordan Banach algebra, we have the following theorems the demonstration of which is exactly the same in the case of Banach algebras.

Theorem 3.1 *Let A be a Jordan-Banach algebra without order and φ be an idempotent homomorphism. Then the set $M_{\varphi J}(A)$ is a closed and commutative subalgebra of Banach algebra of continuous linear operators $\mathcal{L}(A)$, for the topology of simple convergence. Moreover if $A^2 = A$ and φ commutes with any φ J-multiplier of A , then $M_{\varphi J}(A)$ is commutative and without order.*

Theorem 3.2 *Let A be a Jordan-Banach algebra without order. Then $M_{\varphi J}(A)$ is complete in the strong operator topology.*

Proof. Suppose (T_λ) is a Cauchy net in the strong operator. Then, for each x in A , $(T_\lambda(x))$ is a Cauchy net in A and hence there exists $T(x)$ in A such that $\lim_{\lambda \rightarrow +\infty} \|T_\lambda(z) - T(z)\| = 0$.

If x, y in A , then:

$$\begin{aligned} \|T(xy) - \varphi(x)T(y)\| &\leq \|T(xy) - T_\lambda(xy)\| + \|T_\lambda(xy) - \varphi(x)T(y)\| \\ \|T(xy) - \varphi(x)T(y)\| &\leq \|T(xy) - T_\lambda(xy)\| + \|\varphi(x)T_\lambda(y) - \varphi(x)T(y)\| \\ \|T(xy) - \varphi(x)T(y)\| &\leq \|T - T_\lambda\| \|xy\| + \|\varphi(x)\| \|T_\lambda - T\| \|y\| \end{aligned}$$

and so $T(xy) = \varphi(x)T(y)$.

The same way, we show that $T(xy) = T(x)\varphi(y)$, for all x, y in A . Therefore, T in $M_{\varphi J}(A)$ and $M_{\varphi J}(A)$ is complete in the strong operator topology. ■

Theorem 3.3 *Let A be a Jordan-Banach algebra without order and φ be an isomorphism from A into A . If T is a φJ -multiplier of A , then the following statements are equivalent:*

- i) T is bijective.
- ii) T^{-1} exists and $T^{-1} \in M_{\varphi^{-1}J}(A)$.

Corollary 3.1 *Let A be a Jordan-Banach unitary algebra. Then, for every multiplier T , we have:*

$$Sp_{M_{\varphi J}(A)}(A) = Sp_{\mathcal{L}(A)}(T).$$

$M_{\varphi J}(A)$ is a full subalgebra de $\mathcal{L}(A)$.

Remark 3.4 Like all Jordan Banach algebra are commutative, we have the following theorem whose proof is exactly the same as in the case of Banach algebras.

Theorem 3.4 *Let A be a Jordan Banach algebra without order and φ be a homomorphism from A to A with dense range. Then T is a φJ -multiplier if, and only if, $Tx^2 = \varphi(x)T(x)$ for all x in A .*

Corollary 3.2 *Let A be a Jordan Banach Algebra without order and T a linear map from A to A . Then T is a φJ -multiplier of A if, and only if, $Tx^2 = xT(x)$ for all x in A .*

Proof. Just take $\varphi(x) = x$. ■

Proposition 3.2 *Let A be a Banach algebra and T a φ -multiplier of A . Then:*

- i) *We have*

$$(3.2) \quad T \circ U_x \circ T = U_{\varphi(x)} \circ T^2, \quad \forall x \in A.$$

- ii) *(3.2) remains true in special algebra A^+ .*

Proof. Let A be a Banach algebra and $T \in M_{\varphi J}(A)$.

- i) We have :

$$U_x(T(y)) = x^2 \cdot T(y), \quad \forall x, y \in A$$

Therefore:

$$\begin{aligned} T[U_x(T(y))] &= T[x^2 \cdot T(y)] \\ &= \varphi(x^2) \cdot T[T(y)] \\ &= [\varphi(x)]^2 \cdot T[T(y)] \\ &= U_{\varphi(x)}[T^2(y)] \end{aligned}$$

Therefore we have :

$$T[U_x(T(y))] = U_{\varphi(x)}[T^2(y)], \quad \forall x, y \in A$$

ii) In every special Jordan algebra, we have:

$$U_x(y) = x \cdot y \cdot x, \quad \forall x, y \in A$$

Therefore, we have:

$$U_x(T(y)) = x \cdot T(y) \cdot x.$$

Thus,

$$\begin{aligned} T[U_x(T(y))] &= T[x \cdot T(y) \cdot x] \\ &= \varphi(x)T[T(y) \cdot x] \\ &= \varphi(x) \cdot T[T(y)] \cdot \varphi(x) \\ &= U_{\varphi(x)}[T^2(y)] \end{aligned}$$

whence

$$U_{\varphi(x)}[T^2(y)] = T \circ U_x \circ T(y), \quad \forall x, y \in A.$$

Taking this into account, it is normal to ask ourselves the following questions: Given a Jordan-Banach algebra A and a φ J-multiplier T , is the (3.2) relation verified? This is the purpose of the following proposition:

Proposition 3.3 *Let A be a Jordan-Banach algebra. If T is φ J-multiplier of A , we have:*

$$T \circ U_x \circ T = U_{\varphi(x)} \circ T^2, \quad \forall x \in A$$

Proof. Let $T \in M_{\varphi J}(A)$ and $x, y \in A$. Then we have:

$$\begin{aligned} TU_xT(y) &= T[U_x(T(y))] \\ &= T[2x \cdot (x \cdot T(y)) - x^2 \cdot T(y)] \\ &= 2\varphi(x) \cdot T[x \cdot T(y)] - \varphi(x^2) \cdot T[T(y)] \\ &= 2\varphi(x) \cdot [\varphi(x) \cdot T[T(y)] - [\varphi(x)]^2 \cdot T^2(y)] \\ &= 2\varphi(x) \cdot [\varphi(x) \cdot T^2(y) - [\varphi(x)]^2 \cdot T^2(y)] \\ &= U_{\varphi(x)} \circ T^2(y), \quad \forall x, y \in A \end{aligned}$$

Consequently, $U_{\varphi(x)} \circ T^2 = T \circ U_x \circ T$, for all element x in A . ■

Proposition 3.4 *Let A be a Jordan-Banach algebra. If T is a φ J-multiplier of A , then for x and y elements of A , we have:*

$$(3.3) \quad T[U_x(y)] = U_{\varphi(x)}[T(y)]$$

Proof. Let T be a φ J-multiplier of A , x and y elements of A . Then we have:

$$\begin{aligned} T[U_x(y)] &= T[2x \cdot (x \cdot y) - x^2 \cdot y] \\ &= \varphi(2x) \cdot T(x \cdot y) - \varphi(x^2) \cdot T(y) \\ &= 2\varphi(x) \cdot [\varphi(x) \cdot T(y)] - \varphi(x)^2 \cdot T(y) \\ &= U_{\varphi(x)}[T(y)]. \end{aligned}$$
■

Remark 3.5 Let A be a Jordan-Banach algebra. If T is a φ J-multiplier of A , then for x and y elements of A , we have:

$$(3.4) \quad T[U_{\varphi(x)}(y)] = U_{\varphi(x)}[T(y)]$$

Proof. Let T be a φ J-multiplier of A , x and y elements of A . Then we have:

$$\begin{aligned} T[U_{\varphi(x)}(y)] &= T[2\varphi(x) \cdot (\varphi(x) \cdot y) - (\varphi(x))^2 \cdot y] \\ &= 2\varphi(\varphi(x))T(\varphi(x) \cdot y) - \varphi((\varphi(x))^2)T(y) \\ &= 2\varphi(x)[\varphi(x)T(y)] - [\varphi(x)]^2T(y) \\ &= U_{\varphi(x)}[T(y)] \quad \blacksquare \end{aligned}$$

Remark 3.6 The previous Proposition 3.3 and Remark 3.4. suggest to define a new notion which is that of φ J-multiplier quadratic; this is what we are doing in the next section.

4. φ J-Multipliers quadratic of Jordan-Banach algebra

Definition 4.3 Let A be Jordan-Banach algebra, φ an idempotent homomorphism A and $T : A \longrightarrow A$ a continuous linear application. We say that T is a φ J-multiplier quadratic of A if, we have:

$$T[U_x(y)] = U_{\varphi(x)}[T(y)], \forall x, y \in A$$

The set of a φ J-multiplier quadratic of A is noted $M_{\varphi JQ}(A)$.

Remark 4.7 φ is an element of $M_{\varphi JQ}(A)$.

Proposition 4.5 Let A be Banach algebra. If T is a φ -multiplier of A , then we have:

$$T[U_x(y)] = U_{\varphi(x)}[T(y)], \forall x, y \in A$$

Proof. T is a φ -multiplier of A , then we have:

$$T(x.y) = T(x).\varphi(y) = \varphi(x).T(y), \forall x, y \in A$$

Therefore,

$$\begin{aligned} T[U_x(y)] &= T(x^2 \cdot y) = T[x \cdot (x \cdot y)] = \varphi(x) \cdot T(x \cdot y) \\ &= \varphi(x) \cdot [\varphi(x) \cdot T(y)] = \varphi(x)^2 \cdot T(y) = U_{\varphi(x)}[T(y)]. \quad \blacksquare \end{aligned}$$

Proposition 4.6 Let A be Banach algebra. If T is a φ -multiplier of A , then T is a φ J-multiplier quadratic in the special Jordan algebra A^+ .

Proof. As in A^+ , $U_x(y) = x.y.x, \forall x, y \in A$, we have:

$$T[U_x(y)] = T(x \cdot y \cdot x) = \varphi(x) \cdot T(y \cdot x) = \varphi(x) \cdot T(y) \cdot \varphi(x) = U_{\varphi(x)}[T(y)]. \quad \blacksquare$$

Remark 4.8 The definition of φ J-multiplier quadratic well generalizes that of φ -multiplier given in the case of Banach.

Remark 4.9 Let A be Jordan-Banach algebra. Then we have

$$M_{\varphi J}(A) \subset M_{\varphi JQ}(A).$$

In what follows, $\mathcal{L}(A)$ will denote the Banach algebra of all continuous linear operators from A to A .

Theorem 4.5 *Let A be Jordan-Banach algebra. Then, $M_{\varphi JQ}(A)$ is a closed subalgebra of $\mathcal{L}(A)$, for the topology of simple convergence, which contains φ .*

Proof. Let T and S be two elements of $M_{\varphi JQ}(A)$ and $\alpha \in \mathbb{C}$. For any two elements x and y of A , we have:

$$U_{\varphi(x)}[(\alpha T)(y)] = U_{\varphi(x)}[\alpha(T(y))] = \alpha U_{\varphi(x)}(T(y)) = \alpha[T(U_x(y))].$$

Therefore, $\alpha T \in M_{\varphi JQ}(A)$.

$$(TS)[U_x(y)] = T[S(U_x(y))] = T[U_{\varphi(x)}(S(y))] = U_{\varphi(\varphi(x))}[T(S(y))] = U_{\varphi(x)}[(TS)(y)]$$

Therefore, $TS \in M_{\varphi JQ}(A)$.

$$\begin{aligned} U_{\varphi(x)}[(T + S)(y)] &= U_{\varphi(x)}[T(y) + S(y)] = U_{\varphi(x)}[T(y)] + U_{\varphi(x)}[S(y)] \\ &= T[U_x(y)] + S[U_x(y)] \\ &= (T + S)[U_x(y)] \end{aligned}$$

Therefore, $(T + S) \in M_{\varphi JQ}(A)$. Therefore, $M_{\varphi JQ}(A)$ is a subalgebra of $\mathcal{L}(A)$ which contains the φ . Let's show that $M(A)$ is closed in $\mathcal{L}(A)$ for the topology of simple convergence. $(T_\lambda)_{\lambda \in \Lambda}$ is a suite of elements of $M_{\varphi JQ}(A)$ and T an element de $\mathcal{L}(A)$ such that, for every element z de A , we have:

$$\lim_{\lambda \rightarrow +\infty} \|T_\lambda(z) - T(z)\| = 0$$

As with any $\lambda \in \Lambda$ and $T_\lambda \in M_{\varphi JQ}(A)$, we have:

$$U_{\varphi(x)}[T_\lambda(y)] = T_\lambda[U_x(y)], \quad \forall x, y \in A$$

Therefore:

$$\|T[U_x(y)] - U_{\varphi(x)}[T(y)]\| \leq \|T[U_x(y)] - T_\lambda[U_x(y)]\| + \|U_{\varphi(x)}[T_\lambda(y)] - U_{\varphi(x)}[T(y)]\|$$

Since the application $z \rightarrow U_x(z)$ is continuous on A and $(T_\lambda(z))_\lambda$ converges to $T(z)$ for every z in A then: $(\forall x, y \in A), (\forall \epsilon > 0), (\exists \lambda \in \Lambda)$ such that

$$\begin{aligned} \|T[U_x(y)] - T_\lambda[U_x(y)]\| &\leq \epsilon/2 \\ \|U_{\varphi(x)}[T_\lambda(y)] - U_{\varphi(x)}[T(y)]\| &\leq \epsilon/2. \end{aligned}$$

Therefore,

$$\|T[U_x(y)] - U_{\varphi(x)}[T(y)]\| \leq \epsilon.$$

Then, we have:

$$T[U_x(y)] = U_{\varphi(x)}[T(y)], \quad \forall x, y \in A,$$

i.e., $T \in M_{\varphi JQ}(A)$. ■

Remark 4.10 For φJ multipliers, we have shown that if $A^2 = A$, then $M_{\varphi J}$ is without order. We will show that in the quadratic case, we only assume that A is unitary. Hence we have the following proposition.

Proposition 4.7 *Let A be Jordan-Banach algebra with identity e . Then, $M_{\varphi JQ}(A)$ is without order.*

Proof. Let T a element of $M_{\varphi JQ}(A)$ such that $ST = 0$ for each $T \in M_{\varphi JQ}(A)$. We have, for all x, y in A

$$\begin{aligned} T[U_x(y)] &= U_{\varphi(x)}[T(y)] \\ &= U_{\varphi(\varphi(x))}T(y) \\ &= \varphi[U_{\varphi(x)}(T(y))] \\ &= U_{\varphi(x)}[\varphi(T(y))] \end{aligned}$$

As φ is a φJ -multiplier quadratic and the application $z \rightarrow U_x(z)$ is linear, then $\varphi(T(y)) = 0$. Therefore, $U_{\varphi(x)}[\varphi(T(y))] = 0$, where $T[U_x(y)] = 0$, $\forall x, y \in A$.

In particular, if we take $x = e$, we obtain $T(y) = 0$, for all y in A . Therefore, $T = 0$. ■

Theorem 4.6 *Let A be Jordan-Banach algebra. Then, $M_{\varphi JQ}(A)$ is complete in the strong operator topology.*

Proof. Suppose $(T_\lambda)_{\lambda \in \Lambda}$ is a Cauchy net in the strong operator topology. Then for each element z in A , $(T_\lambda(z))_{\lambda \in \Lambda}$ is a cauchy net in A that is complete and hence there exists $T(z)$ in A such that:

$$\lim_{\lambda \rightarrow +\infty} \|T_\lambda(z) - T(z)\| = 0.$$

It is conventional to show that T is linear continuous and that

$$\lim_{\lambda \rightarrow +\infty} \|T_\lambda - T\| = 0.$$

For any two elements x and y of A , we have:

$$\begin{aligned} \|T[U_x(y)] - U_{\varphi(x)}[T(y)]\| &\leq \|T[U_x(y)] - T_\lambda[U_x(y)]\| + \|U_{\varphi(x)}[T_\lambda(y) - U_{\varphi(x)}[T(y)]]\| \\ \|T[U_x(y)] - U_{\varphi(x)}[T(y)]\| &\leq \|(T - T_\lambda)(U_x(y))\| + \|(U_{\varphi(x)})(T_\lambda(y) - T(y))\| \end{aligned}$$

Since T_λ , T and $z \rightarrow U_x(z)$ are linear and continuous applications on A , then:

$$\|(T - T_\lambda)(U_x(y))\| \leq \|T - T_\lambda\| \|U_x(y)\|$$

and

$$\|(U_{\varphi(x)})(T_\lambda(y) - T(y))\| \leq \|U_{\varphi(x)}\| \|T - T_\lambda\| \|y\|.$$

We have

$$\lim_{\lambda \rightarrow +\infty} \|T_\lambda - T\| = 0.$$

Then

$$\|T[U_x(y)] - U_{\varphi(x)}[T(y)]\| = 0,$$

and so

$$T[U_x(y)] = U_{\varphi(x)}[T(y)], \forall x, y \in A.$$

Therefore, $T \in M_{\varphi JQ}(A)$ and $M_{\varphi JQ}(A)$ is complete in the strong operator topology. ■

Theorem 4.7 *Let A be a Jordan-Banach algebra without order and φ be an isomorphism from A into A . If T is a φ J-multiplier quadratic of A , then the following statements are equivalent:*

- i) T is bijective.
- ii) T^{-1} exists and $T^{-1} \in M_{\varphi^{-1}JQ}(A)$.

Proof.

ii) $\Rightarrow$ i) Obvious

i) $\Rightarrow$ ii) T^{-1} is linear. It is continuous according to the theorem of open application. On the other hand, we have by hypothesis:

$$T[U_x(y)] = U_{\varphi(x)}[T(y)], \forall x, y \in A$$

Let x and z are two elements of A . We pose $y = T^{-1}(z)$.

Then, we have

$$T[U_{\varphi^{-1}(x)}(T^{-1}(z))] = U_x[T(T^{-1}(z))] = U_x(z).$$

Thus, we obtain

$$U_{\varphi^{-1}(x)}(T^{-1}(z)) = T^{-1}[U_x(z)], \forall x, z \in A.$$

Hence $T^{-1} \in M_{\varphi^{-1}JQ}(A)$. ■

Corollary 4.3 *Let A be Jordan-Banach algebra without order and $T \in M_{\varphi JQ}(A)$. Then, we have*

$$Sp_{M_{\varphi JQ}(A)}(T) = Sp_{\mathcal{L}(A)}(T)$$

i.e., $M_{\varphi JQ}(A)$ is a full subalgebra of $\mathcal{L}(A)$.

Acknowledgement. The author would like to thank the referee for helpful suggestions and remarks.

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Accepted: 2.11.2013

A NOTE ON BOOLEAN SUBSETS OF ORTHOMODULAR POSETS¹

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Abstract. Modelling quantum systems by orthomodular posets $\mathcal{P} = (P, \leq, ', 0, 1)$ gives rise to the question, when a finite subset A of P lies within a Boolean subalgebra of $\mathcal{P}$, in which case A is called Boolean. Boolean subsets A specify the physical subsystem represented by A to be classical. We give a characterization of a subset of P to be Boolean by only taking into account terms of elements of this subset and in such a way that an inductive algorithm can be derived.

Keywords: orthomodular poset, Boolean algebra.

AMS Subject Classification: 06C15, 06E99, 03G12, 81P10

1. Introduction

Orthomodular lattices, and more general, orthomodular posets $\mathcal{P} = (P, \leq, ', 0, 1)$ have been intensively studied as models for quantum logics (cf., e.g., [2], [5], [6], [7] and [8]). It is well known that these algebraic structures correspond to classical mechanical systems if and only if they are Boolean algebras. However, if one deals with only a finite subset A of P the question arises whether the physical subsystem represented by A is classical, which then means that A lies within a Boolean subalgebra of $\mathcal{P}$. If this is the case, then A will be called Boolean. For this definition, cf. the pioneering paper [1], in which the question of Boolean subsets was settled for the special case of so-called algebras of numerical events. For the general case of arbitrary orthomodular posets, Boolean subsets have been

¹Support of the research of the second author by ÖAD, Cooperation between Austria and Czech Republic in Science and Technology, grant No. CZ 03/2013, is gratefully acknowledged.

identified by means of a generalization of the compatibility relation of two elements (cf. [3], [4] and [8]).

In this paper, we will stick to the classical commutativity relation of only two elements and will only take into account terms of elements of the given finite set of elements of P to answer the question, whether this set is Boolean. The characterization we derive will be inductive and hence will give rise to a step by step procedure.

We begin by recollecting the definition of an orthomodular poset and some of its properties.

An orthomodular poset is a ordered quintuple $\mathcal{P} = (P, \leq, ', 0, 1)$ such that $(P, \leq, 0, 1)$ is a bounded poset and $'$ is a unary operation on P satisfying the following conditions for all $x, y \in P$:

- (i) $x \leq y$ implies $x' \geq y'$.
- (ii) $(x')' = x$
- (iii) $x \vee x' = 1$
- (iv) If $x \perp y$, i. e. if $x \leq y'$, in which case x and y are said to be orthogonal (to each other), then $x \vee y$, the supremum of x and y , exists.
- (v) If $x \leq y$ then the orthomodular law $y = x \vee (y \wedge x')$ holds.

$y \wedge x'$ denotes the infimum of y and x' . It is easy to see that within an orthomodular poset the existence of $x \vee y$ implies the existence of $x' \wedge y'$ and that the de Morgan laws $(x \vee y)' = x' \wedge y'$ and $(x \wedge y)' = x' \vee y'$ hold, in the sense that if one side is defined then so is the other and they are equal. Moreover, if $x \leq y$ then $x \perp y'$ and hence $x \vee y'$ exists which in turn shows the existence of $x' \wedge y = (x \vee y')'$. The existence of the term on the right-hand side of the orthomodular law is secured by the fact that $x \perp (y \wedge x')$.

Two elements a and b of an orthomodular poset $\mathcal{P} = (P, \leq, ', 0, 1)$ are said to commute (with each other) – in short, $a \text{ C } b$ – if there exist three mutually orthogonal elements c, d, e of P such that $a = c \vee d$ and $b = d \vee e$. It is well known (cf. e. g. [8]) that the elements c, d, e are unique if they exist, namely $c = a \wedge b'$, $d = a \wedge b$ and $e = a' \wedge b$.

There are numerous characterizations and implications concerning the property $a \text{ C } b$ (cf. e. g. [2] and [8]). We list the very conditions we need for our further considerations, and when this seems essential, we will refer to them.

Properties of C

For an orthomodular poset $\mathcal{P} = (P, \leq, ', 0, 1)$ the following conditions for $a, b \in P$ are equivalent:

- (i) $a \text{ C } b$
- (ii) The subset $\{a, b\}$ of P is Boolean, i. e. there exists a Boolean subalgebra of $\mathcal{P}$ containing a and b .

- (iii) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ for three arbitrary but fixed elements of the four elements a, a', b, b'
- (iv) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$ for three arbitrary but fixed elements of the four elements a, a', b, b'

Moreover, the following holds:

- (v) If $a \mathsf{C} b$ then $x \mathsf{C} y$ for all $x, y \in \{0, a, a', b, b', 1\}$.
- (vi) If $a \leq b$ then $a \mathsf{C} b$.
- (vii) If $a \perp b$ then $a \mathsf{C} b$.
- (viii) If $a \mathsf{C} b$ then $a \vee b$ and $a \wedge b$ exist.

2. Characterizing Boolean subsets

As usual, we will write $\bigvee_{i \in I} s_i$ and $\bigwedge_{i \in I} s_i$ for the supremum and infimum, respectively, of the elements $s_i, i \in I$, and we will use the notation $\bigvee S$ and $\bigwedge S$ if $S = \{s_i \mid i \in I\}$. Assume that S is a finite subset of pairwise orthogonal elements of an orthomodular poset $\mathcal{P} = (P, \leq, ', 0, 1)$. Then it can immediately be seen that $\bigvee S$ exists.

The following result is well-known (cf. [3], [4] and [8]):

Lemma 2.1. *Let A be a finite subset of mutually orthogonal elements of an orthomodular poset $\mathcal{P} = (P, \leq, ', 0, 1)$. Then the subset $\{\bigvee D \mid D \subseteq A\}$ of P is Boolean.*

Now, we can prove our main theorem:

Theorem 2.2. *Let $\mathcal{P} = (P, \leq, ', 0, 1)$ be an orthomodular poset, $1 \leq k < n$ and A an n -element subset of P . If any k -element subset of A is Boolean and $(\bigwedge B) \mathsf{C} (\bigwedge D)$ for all k -element subsets B and D of A then also any $(k + 1)$ -element subset of A is Boolean.*

Proof. With regard to the properties of C the case $k = 1$ is obvious. Now assume $k > 1$ and let A be the set $\{a_1, \dots, a_n\}$.

First, we show by induction on s that for all $s = k, k - 1, \dots, 1$ we have

$$a_1 \wedge \dots \wedge a_s \wedge a'_{s+1} \wedge \dots \wedge a'_k \mathsf{C} a_{k+1}.$$

Since $(a_1 \wedge \dots \wedge a_k) \mathsf{C} (a_2 \wedge \dots \wedge a_{k+1})$ the infimum

$$(a_1 \wedge \dots \wedge a_k) \wedge (a_2 \wedge \dots \wedge a_{k+1}) = a_1 \wedge \dots \wedge a_{k+1}$$

exists, and due to property (iii) of C we obtain

$$\begin{aligned}
a_1 \wedge \dots \wedge a_k &= (a_1 \wedge \dots \wedge a_k) \wedge ((a_2 \wedge \dots \wedge a_{k+1}) \vee (a_2 \wedge \dots \wedge a_{k+1})') \\
&= ((a_1 \wedge \dots \wedge a_k) \wedge (a_2 \wedge \dots \wedge a_{k+1})) \vee ((a_1 \wedge \dots \wedge a_k) \wedge (a_2' \vee \dots \vee a_{k+1}')) \\
&= (a_1 \wedge \dots \wedge a_{k+1}) \vee (a_1 \wedge \dots \wedge a_k \wedge (a_2' \vee \dots \vee a_{k+1}')) \\
&= (a_1 \wedge \dots \wedge a_{k+1}) \vee (a_1 \wedge ((a_2 \wedge \dots \wedge a_k) \wedge ((a_2 \wedge \dots \wedge a_k)' \vee a_{k+1}')))) \\
&= (a_1 \wedge \dots \wedge a_{k+1}) \vee (a_1 \wedge ((a_2 \wedge \dots \wedge a_k) \wedge a_{k+1}')) \\
&= (a_1 \wedge \dots \wedge a_{k+1}) \vee (a_1 \wedge \dots \wedge a_k \wedge a_{k+1}') \\
&= ((a_1 \wedge \dots \wedge a_k) \wedge a_{k+1}) \vee ((a_1 \wedge \dots \wedge a_k) \wedge a_{k+1}').
\end{aligned}$$

Hence, again by property (iii) of C, $a_1 \wedge \dots \wedge a_k \text{ C } a_{k+1}$, which proves our assertion for $s = k$.

Now, assume $1 < s \leq k$ and

$$a_1 \wedge \dots \wedge a_i \wedge a_{i+1}' \wedge \dots \wedge a_k' \text{ C } a_{k+1}$$

for all $i = s, \dots, k$. Our goal is to show that

$$a_1 \wedge \dots \wedge a_{s-1} \wedge a_s' \wedge \dots \wedge a_k' \text{ C } a_{k+1}.$$

According to the properties of C,

$$a_1 \wedge \dots \wedge a_s \wedge a_{s+1}' \wedge \dots \wedge a_k' \text{ C } a_{k+1}'$$

and hence

$$a_1 \wedge \dots \wedge a_s \wedge a_{s+1}' \wedge \dots \wedge a_k' \wedge a_{k+1}' = (a_1 \wedge \dots \wedge a_s \wedge a_{s+1}' \wedge \dots \wedge a_k') \wedge a_{k+1}'$$

exist. In the following, let us denote a by a^1 and a' by a^{-1} . For reasons of symmetry also $a_1^{i_1} \wedge \dots \wedge a_{k+1}^{i_{k+1}}$ exists whenever

$$|\{j \in \{1, \dots, k+1\} \mid i_j = 1\}| \geq s.$$

Now, we compute by relying on the properties of C

$$\begin{aligned}
&(a_1 \wedge \dots \wedge a_{s-1} \wedge a_s' \wedge \dots \wedge a_k') \wedge (a_{k+1}' \vee (a_1 \wedge \dots \wedge a_{s-1} \wedge a_s' \wedge \dots \wedge a_k')) \\
&= (a_1 \wedge \dots \wedge a_{s-1} \wedge a_s' \wedge \dots \wedge a_k') \wedge (a_{k+1}' \vee a_1' \vee \dots \vee a_{s-1}' \vee a_s \vee \dots \vee a_k) \\
&= (a_1 \wedge \dots \wedge a_{s-1} \wedge a_s' \wedge \dots \wedge a_{k-1}') \wedge \\
&\quad \wedge (a_k' \wedge (a_k \vee (a_1' \vee \dots \vee a_{s-1}' \vee a_s \vee \dots \vee a_{k-1} \vee a_{k+1}')))) \\
&= (a_1 \wedge \dots \wedge a_{s-1} \wedge a_s' \wedge \dots \wedge a_{k-1}') \wedge \\
&\quad \wedge (a_k' \wedge (a_1' \vee \dots \vee a_{s-1}' \vee a_s \vee \dots \vee a_{k-1} \vee a_{k+1}')) \\
&= a_k' \wedge ((a_1 \wedge \dots \wedge a_{s-1} \wedge a_s' \wedge \dots \wedge a_{k-1}') \wedge \\
&\quad \wedge ((a_1 \wedge \dots \wedge a_{s-1} \wedge a_s' \wedge \dots \wedge a_{k-1}')' \vee a_{k+1}')) \\
&= a_k' \wedge ((a_1 \wedge \dots \wedge a_{s-1} \wedge a_s' \wedge \dots \wedge a_{k-1}') \wedge a_{k+1}') \\
&= a_1 \wedge \dots \wedge a_{s-1} \wedge a_s' \wedge \dots \wedge a_{k-1}' \wedge a_k' \wedge a_{k+1}' \\
&= (a_1 \wedge \dots \wedge a_{s-1} \wedge a_s' \wedge \dots \wedge a_{k-1}' \wedge a_k') \wedge a_{k+1}'
\end{aligned}$$

showing

$$a_1 \wedge \dots \wedge a_{s-1} \wedge a'_s \wedge \dots \wedge a'_k \leq a_{k+1}$$

which completes the proof by induction.

We have shown that

$$a_1 \wedge a_2^{i_2} \wedge \dots \wedge a_k^{i_k} \leq a_{k+1}.$$

for all $i_2, \dots, i_k \in \{-1, 1\}$ and we point out that the elements

$$a_1^{i_1} \wedge \dots \wedge a_{k+1}^{i_{k+1}} = (a_1^{i_1} \wedge \dots \wedge a_k^{i_k}) \wedge (a_2^{i_2} \wedge \dots \wedge a_{k+1}^{i_{k+1}})$$

$(i_1, \dots, i_{k+1} \in \{-1, 1\})$ exist and are mutually orthogonal. Further, we obtain

$$\begin{aligned} a_1 &= \bigvee_{i_2, \dots, i_k \in \{-1, 1\}} (a_1 \wedge a_2^{i_2} \wedge \dots \wedge a_k^{i_k}) \\ &= \bigvee_{i_2, \dots, i_k \in \{-1, 1\}} (((a_1 \wedge a_2^{i_2} \wedge \dots \wedge a_k^{i_k}) \wedge a_{k+1}) \vee ((a_1 \wedge a_2^{i_2} \wedge \dots \wedge a_k^{i_k}) \wedge a'_{k+1})) \\ &= \bigvee_{i_2, \dots, i_k \in \{-1, 1\}} ((a_1 \wedge a_2^{i_2} \wedge \dots \wedge a_k^{i_k} \wedge a_{k+1}) \vee (a_1 \wedge a_2^{i_2} \wedge \dots \wedge a_k^{i_k} \wedge a'_{k+1})) \\ &= \bigvee_{i_2, \dots, i_{k+1} \in \{-1, 1\}} (a_1 \wedge a_2^{i_2} \wedge \dots \wedge a_{k+1}^{i_{k+1}}). \end{aligned}$$

For reasons of symmetry, we also have

$$a_j = \bigvee_{i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_{k+1} \in \{-1, 1\}} (a_1^{i_1} \wedge \dots \wedge a_{j-1}^{i_{j-1}} \wedge a_j \wedge a_{j+1}^{i_{j+1}} \wedge \dots \wedge a_{k+1}^{i_{k+1}})$$

for all $j = 1, \dots, k+1$ which, according to Lemma 2.1, shows that $\{a_1, \dots, a_{k+1}\}$ is Boolean. By symmetry, it follows that every $(k+1)$ -element subset of A is Boolean. ■

Corollary 2.3. *Let $\mathcal{P} = (P, \leq, ', 0, 1)$ be an orthomodular poset, $n > 1$ and A an n -element subset of P . Then A is Boolean if and only if $(\bigwedge B) \leq (\bigwedge D)$ for every $k \in \{1, \dots, n-1\}$ and every k -element subsets B and D of A .*

Remark 2.4. According to Theorem 2.2, $\bigwedge B$ exists for all k -element subsets of A , $k \in \{1, 2, \dots, n-1\}$.

3. Remarks about algorithmic aspects

If one wants to check whether a subset $A = \{a_1, \dots, a_n\}$ of an orthomodular poset $\mathcal{P}$ is Boolean, one can proceed as follows:

- (1) Check whether $a_i \leq a_j$ for $i, j \in \{1, 2, \dots, n\}, i \neq j$, e. g. by looking for the existence of $a_i \wedge a_j$, $a_i \wedge a'_j$ and whether $(a_i \wedge a_j) \vee (a_i \wedge a'_j) = a_i$. If these elements exist and the equation holds (which is equivalent to $a_i \leq a_j$) and if $\mathcal{P}$ will be a lattice (as in the case of a Hilbert space logic) then one is done; A is Boolean. If these elements exist and the equation holds and $\mathcal{P}$ is not a lattice then

- (2) for $r \in \{1, \dots, \binom{n}{2}\}$, denote the elements $a_i \wedge a_j, i, j \in \{1, \dots, n\}, i \neq j$, by b_r and check, whether $b_r \leq b_s$ for $r, s \in \{1, \dots, \binom{n}{2}\}, r \neq s$. Assuming, this is the case then
- (3) replenish the elements b_r to triples $a_i \wedge a_j \wedge a_k, i, j, k \in \{1, \dots, \binom{n}{3}\}, i \neq j \neq k \neq i$, and so on,

until one comes across a pair $(\bigwedge B, \bigwedge D)$ of infima of subsets B and D of A with $|B| = |D| \leq n - 1$ which do not commute, in which case A is not Boolean; otherwise, the subset A is Boolean.

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Accepted: 23.11.2013

SOME DERIVATIONS ON THE BOUNDS FOR THE ZEROS OF ENTIRE FUNCTIONS BASED ON SLOWLY CHANGING FUNCTIONS

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Abstract. A single valued function of one complex variable which is analytic in the finite complex plane is called an entire function. The purpose of this paper is to establish the bounds for the moduli of zeros of entire functions in the light of slowly changing functions.

Keywords: zeros of entire functions, proper ring shaped region.

2000 Mathematics Subject Classification: Primary 30C15, 30C10, Secondary 26C10.

1. Introduction, definitions and notations

Let

$$P(z) = a_0 + a_1z + a_2z^2 + a_3z^3 + \dots + a_{n-1}z^{n-1} + a_nz^n; |a_n| \neq 0$$

be a polynomial of degree n . Datt and Govil [2], Govil and Rahaman [4], Marden [8], Mohammad [9], Chattopadhyay, Das, Jain and Konwer [1], Joyal, Labelle and Rahaman [5], Jain [6],[7], Sun and Hsieh [12], Zilovic, Roytman, Combettes and Swamy [14], Das and Datta [3] etc. worked in the theory of the distribution of the zeros of polynomials and obtained some newly developed results.

In this paper we intend to establish some of sharper results concerning the theory of distribution of zeros of entire functions in the light of slowly changing functions.

The following definitions are well known :

Definition 1. The order ρ and lower order λ of an entire function f are defined as

$$\rho = \limsup_{r \rightarrow \infty} \frac{\log^{[2]} M(r, f)}{\log r} \quad \text{and} \quad \lambda = \liminf_{r \rightarrow \infty} \frac{\log^{[2]} M(r, f)}{\log r},$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $k = 1, 2, 3, \dots$ and $\log^{[0]} x = x$.

Let $L \equiv L(r)$ be a positive continuous function increasing slowly, i.e., $L(ar) \sim L(r)$ as $r \rightarrow \infty$ for every positive constant a . Singh and Barker[10] defined it in the following way:

Definition 2. [10] A positive continuous function $L(r)$ is called a *slowly changing function* if, for $\varepsilon (> 0)$,

$$\frac{1}{k^\varepsilon} \leq \frac{L(kr)}{L(r)} \leq k^\varepsilon \quad \text{for } r > r(\varepsilon)$$

and uniformly for $k(\geq 1)$.

If, further, $L(r)$ is differentiable, the above condition is equivalent to

$$\lim_{r \rightarrow \infty} \frac{rL'(r)}{L(r)} = 0.$$

Somasundaram and Thamizharasi [11] introduced the notions of L -order and L -lower order for entire functions defined in the open complex plane $\mathbb{C}$ as follows:

Definition 3. [11] The L -order ρ^L and the L -lower order λ^L of an entire function f are defined as

$$\rho^L = \limsup_{r \rightarrow \infty} \frac{\log^{[2]} M(r, f)}{\log[rL(r)]} \quad \text{and} \quad \lambda^L = \liminf_{r \rightarrow \infty} \frac{\log^{[2]} M(r, f)}{\log[rL(r)]}.$$

The more generalized concept for L -order and L -lower order are L^* -order and L^* -lower order respectively. Their definitions are as follows:

Definition 4. The L^* -order ρ^{L^*} and the L^* -lower order λ^{L^*} of an entire function f are defined as

$$\rho^{L^*} = \limsup_{r \rightarrow \infty} \frac{\log^{[2]} M(r, f)}{\log[re^{L(r)}]} \quad \text{and} \quad \lambda^{L^*} = \liminf_{r \rightarrow \infty} \frac{\log^{[2]} M(r, f)}{\log[re^{L(r)}]}.$$

2. Lemmas

In this section, we present some lemmas which will be needed in the sequel.

Lemma 1. *If $f(z)$ is an entire function of L -order ρ^L , then for every $\varepsilon > 0$ the inequality*

$$N(r) \leq [rL(r)]^{\rho^L + \varepsilon}$$

holds for all sufficiently large r where $N(r)$ is the number of zeros of $f(z)$ in $|z| \leq [rL(r)]$.

Proof. Let us suppose that $f(0) = 1$. This supposition can be made without loss of generality because if $f(z)$ has a zero of order ' m ' at the origin then we may consider $g(z) = c \cdot \frac{f(z)}{z^m}$ where c is so chosen that $g(0) = 1$. Since the function $g(z)$ and $f(z)$ have the same order therefore it will be unimportant for our investigations that the number of zeros of $g(z)$ and $f(z)$ differ by m .

We further assume that $f(z)$ has no zeros on $|z| = 2[rL(r)]$ and the zeros z_i 's of $f(z)$ in $|z| < [rL(r)]$ are in non decreasing order of their moduli so that $|z_i| \leq |z_{i+1}|$. Also let ρ^L suppose to be finite.

Now, we shall make use of Jenson's formula as state below

$$(1) \quad \log |f(0)| = - \sum_{i=1}^n \log \frac{R}{|z_i|} + \frac{1}{2\pi} \int_0^{2\pi} \log |f(R e^{i\phi})| d\phi.$$

Let us replace R by $2r$ and n by $N(2r)$ in (1)

$$\log |f(0)| = - \sum_{i=1}^{N(2r)} \log \frac{2r}{|z_i|} + \frac{1}{2\pi} \int_0^{2\pi} \log |f(2r e^{i\phi})| d\phi.$$

Since $f(0) = 1$, $\log |f(0)| = \log 1 = 0$.

$$(2) \quad \therefore \sum_{i=1}^{N(2r)} \log \frac{2r}{|z_i|} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(2r e^{i\phi})| d\phi.$$

$$(3) \quad \text{L.H.S.} = \sum_{i=1}^{N(2r)} \log \frac{2r}{|z_i|} \geq \sum_{i=1}^{N(r)} \log \frac{2r}{|z_i|} \geq N(r) \log 2$$

because for large values of r ,

$$\log \frac{2r}{|z_i|} \geq \log 2.$$

$$(4) \quad \begin{aligned} \text{R.H.S} &= \frac{1}{2\pi} \int_0^{2\pi} \log |f(2r e^{i\phi})| d\phi \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \log M(2r) d\phi = \log M(2r). \end{aligned}$$

Again, by definition of order ρ^L of $f(z)$, we have for every $\varepsilon > 0$, and as $L(2r) \sim L(r)$,

$$(5) \quad \begin{aligned} \log M(2r) &\leq [2rL(2r)]^{\rho^L + \varepsilon/2} \\ \log M(2r) &\leq [2rL(r)]^{\rho^L + \varepsilon/2}. \end{aligned}$$

Hence, from (2) by the help of (3), (4) and (5), we have

$$\begin{aligned} N(r) \log 2 &\leq [2rL(r)]^{\rho^L + \varepsilon/2} \\ \text{i.e., } N(r) &\leq \frac{2^{\rho^L + \varepsilon/2}}{\log 2} \cdot \frac{(rL(r))^{\rho^L + \varepsilon}}{(rL(r))^{\varepsilon/2}} \leq [rL(r)]^{\rho^L + \varepsilon}. \end{aligned}$$

This proves the lemma. ■

In the line of Lemma 1, we may state the following lemma:

Lemma 2. *If $f(z)$ is an entire function of L^* -order ρ^{L^*} , then for every $\varepsilon > 0$ the inequality*

$$N(r) \leq [re^{L(r)}]^{\rho^{L^*} + \varepsilon}$$

holds for all sufficiently large r where $N(r)$ is the number of zeros of $f(z)$ in $|z| \leq [re^{L(r)}]$.

Proof. With the initial assumptions as laid down in Lemma 1, let us suppose that $f(z)$ has no zeros on $|z| = 2[re^{L(r)}]$ and the zeros z_i 's of $f(z)$ in $|z| < [re^{L(r)}]$ are in non decreasing order of their moduli so that $|z_i| \leq |z_{i+1}|$. Also, let ρ^{L^*} supposed to be finite.

In view of (1), (2), (3) and (4), by definition of ρ^{L^*} and as $L(2r) \sim L(r)$, we get for every $\varepsilon > 0$ that

$$\begin{aligned} (6) \quad \log M(2r) &\leq [2re^{L(2r)}]^{\rho^{L^*} + \varepsilon/2} \\ \text{i.e., } \log M(2r) &\leq [2re^{L(r)}]^{\rho^{L^*} + \varepsilon/2}. \end{aligned}$$

Hence, by the help of (3), (4) and (6), we obtain from (2) that

$$\begin{aligned} N(r) \log 2 &\leq [2re^{L(r)}]^{\rho^{L^*} + \varepsilon/2} \\ N(r) &\leq \frac{2^{\rho^{L^*} + \varepsilon/2}}{\log 2} \cdot \frac{[re^{L(r)}]^{\rho^{L^*} + \varepsilon}}{[rL(r)]^{\varepsilon/2}} \leq [re^{L(r)}]^{\rho^{L^*} + \varepsilon}. \end{aligned}$$

Thus, the lemma is established. ■

3. Theorems

In this section, we present the main results of the paper.

Theorem 1. *Let $P(z)$ be an entire function having L -order ρ^L in the disc $|z| \leq [rL(r)]$ for sufficiently large r . Also, let the Taylor's series expansion of $P(z)$ be given by*

$$P(z) = a_0 + a_{p_1} z^{p_1} + \cdots + a_{p_m} z^{p_m} + a_{N(r)} z^{N(r)}, \quad a_0 \neq 0, \quad a_{N(r)} \neq 0$$

with $1 \leq p_1 < p_2 < \cdots < p_m \leq N(r) - 1$, p_i 's are integers such that for $\rho^L > 0$,

$$|a_0| (\rho^L)^{N(r)} \geq |a_{p_1}| (\rho^L)^{N(r)-p_1} \geq \cdots \geq |a_{p_m}| (\rho^L)^{N(r)-p_m} \geq |a_{N(r)}|.$$

Then, all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{\rho^L \left(1 + \frac{|a_{p_1}|}{|a_0|(\rho^L)^{p_1}}\right)} < |z| < \frac{1}{\rho^L} \left(1 + \frac{|a_0|}{|a_{N(r)}|}(\rho^L)^{N(r)}\right).$$

Proof. Given that

$$P(z) = a_0 + a_{p_1} z^{p_1} + \cdots + a_{p_m} z^{p_m} + a_{N(r)} z^{N(r)}$$

where p_i 's are integers and $1 \leq p_1 < p_2 < \cdots < p_m \leq N(r) - 1$. Then for $\rho^L > 0$,

$$|a_0|(\rho^L)^{N(r)} \geq |a_{p_1}|(\rho^L)^{N(r)-p_1} \geq \cdots \geq |a_{p_m}|(\rho^L)^{N(r)-p_m} \geq |a_{N(r)}|.$$

Let us consider

$$\begin{aligned} Q(z) &= (\rho^L)^{N(r)} P\left(\frac{z}{\rho^L}\right) \\ &= (\rho^L)^{N(r)} \left\{ a_0 + a_{p_1} \frac{z^{p_1}}{(\rho^L)^{p_1}} + \cdots + a_{p_m} \frac{z^{p_m}}{(\rho^L)^{p_m}} + a_{N(r)} \frac{z^{N(r)}}{(\rho^L)^{N(r)}} \right\} \\ &= a_0 (\rho^L)^{N(r)} + a_{p_1} (\rho^L)^{N(r)-p_1} z^{p_1} + \cdots + a_{p_m} (\rho^L)^{N(r)-p_m} z^{p_m} + a_{N(r)} z^{N(r)}. \end{aligned}$$

Therefore,

$$(7) \quad |Q(z)| \geq |a_{N(r)} z^{N(r)}| - |a_0 (\rho^L)^{N(r)} + a_{p_1} (\rho^L)^{N(r)-p_1} z^{p_1} + \cdots + a_{p_m} (\rho^L)^{N(r)-p_m} z^{p_m}|.$$

Now, using the given condition of Theorem 1 we obtain that

$$\begin{aligned} &|a_0 (\rho^L)^{N(r)} + a_{p_1} (\rho^L)^{N(r)-p_1} z^{p_1} + \cdots + a_{p_m} (\rho^L)^{N(r)-p_m} z^{p_m}| \\ &\leq |a_0| (\rho^L)^{N(r)} + |a_{p_1}| (\rho^L)^{N(r)-p_1} |z|^{p_1} + \cdots + |a_{p_m}| (\rho^L)^{N(r)-p_m} |z|^{p_m} \\ &\leq |a_0| (\rho^L)^{N(r)} |z|^{N(r)} \left(\frac{1}{|z|^{N(r)-p_m}} + \cdots + \frac{1}{|z|^{N(r)}} \right) \text{ for } |z| \neq 0. \end{aligned}$$

Using (7), we get for $|z| \neq 0$ that

$$\begin{aligned} (8) \quad |Q(z)| &\geq |a_{N(r)}| |z|^{N(r)} - |a_0| (\rho^L)^{N(r)} |z|^{N(r)} \left(\frac{1}{|z|^{N(r)-p_m}} + \cdots + \frac{1}{|z|^{N(r)}} \right) \\ &> |a_{N(r)}| |z|^{N(r)} - |a_0| (\rho^L)^{N(r)} |z|^{N(r)} \left(\frac{1}{|z|^{N(r)-p_m}} + \cdots + \frac{1}{|z|^{N(r)}} + \cdots \right) \\ &= |a_{N(r)}| |z|^{N(r)} - |a_0| (\rho^L)^{N(r)} |z|^{N(r)} \left(\sum_{k=1}^{\infty} \frac{1}{|z|^k} \right). \end{aligned}$$

The geometric series $\sum_{k=1}^{\infty} \frac{1}{|z|^k}$ is convergent for

$$\frac{1}{|z|} < 1$$

i.e., for $|z| > 1$

and converges to

$$\frac{1}{|z|} \frac{1}{1 - \frac{1}{|z|}} = \frac{1}{|z| - 1}.$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{1}{|z|^k} = \frac{1}{|z| - 1} \quad \text{for } |z| > 1.$$

Using (8), we get from above that for $|z| > 1$

$$\begin{aligned} |Q(z)| &> |a_{N(r)}| |z|^{N(r)} - |a_0| (\rho^L)^{N(r)} |z|^{N(r)} \left(\frac{1}{|z| - 1} \right) \\ &= |z|^{N(r)} \left(|a_{N(r)}| - \frac{|a_0| (\rho^L)^{N(r)}}{|z| - 1} \right). \end{aligned}$$

Now, for $|z| > 1$,

$$\begin{aligned} |Q(z)| > 0 &\text{ if } |a_{N(r)}| - \frac{|a_0| (\rho^L)^{N(r)}}{|z| - 1} \geq 0 \\ \text{i.e., if } |a_{N(r)}| &\geq \frac{|a_0| (\rho^L)^{N(r)}}{|z| - 1} \\ \text{i.e., if } |z| - 1 &\geq \frac{|a_0| (\rho^L)^{N(r)}}{|a_{N(r)}|} \\ \text{i.e., if } |z| &\geq 1 + \frac{|a_0| (\rho^L)^{N(r)}}{|a_{N(r)}|} > 1. \end{aligned}$$

Therefore, $|Q(z)| > 0$ if

$$|z| \geq 1 + \frac{|a_0| (\rho^L)^{N(r)}}{|a_{N(r)}|}.$$

Therefore, $Q(z)$ does not vanish for

$$|z| \geq 1 + \frac{|a_0| (\rho^L)^{N(r)}}{|a_{N(r)}|}.$$

So, all the zeros of $Q(z)$ lie in

$$|z| < 1 + \frac{|a_0| (\rho^L)^{N(r)}}{|a_{N(r)}|}.$$

Let $z = z_0$ be any zero of $P(z)$. Therefore, $P(z_0) = 0$. Clearly, $z_0 \neq 0$ as $a_0 \neq 0$. Putting $z = \rho^L z_0$ in $Q(z)$, we get that

$$Q(\rho^L z_0) = (\rho^L)^{N(r)} P(z_0) = (\rho^L)^{N(r)} \cdot 0 = 0.$$

So $z = \rho^L z_0$ is a zero of $Q(z)$. Hence,

$$\begin{aligned} |\rho^L z_0| &< 1 + \frac{|a_0| (\rho^L)^{N(r)}}{|a_{N(r)}|} \\ \text{i.e., } |z_0| &< \frac{1}{\rho^L} \left(1 + \frac{|a_0| (\rho^L)^{N(r)}}{|a_{N(r)}|} \right). \end{aligned}$$

Since z_0 is an arbitrary zero of $P(z)$, therefore, all the zeros of $Q(z)$ lie in

$$(9) \quad |z| < \frac{1}{\rho^L} \left(1 + \frac{|a_0| (\rho^L)^{N(r)}}{|a_{N(r)}|} \right).$$

Again, let us consider

$$R(z) = (\rho^L)^{N(r)} z^{N(r)} P\left(\frac{1}{\rho^L z}\right).$$

Therefore,

$$\begin{aligned} R(z) &= (\rho^L)^{N(r)} z^{N(r)} \left\{ a_0 + a_{p_1} \frac{1}{(\rho^L)^{p_1} z^{p_1}} + \dots + a_{p_m} \frac{1}{(\rho^L)^{p_m} z^{p_m}} + a_{N(r)} \frac{1}{(\rho^L)^{N(r)} z^{N(r)}} \right\} \\ &= a_0 (\rho^L)^{N(r)} z^{N(r)} + a_{p_1} (\rho^L)^{N(r)-p_1} z^{N(r)-p_1} + \dots + a_{p_m} (\rho^L)^{N(r)-p_m} z^{N(r)-p_m} + a_{N(r)}. \end{aligned}$$

Now,

$$(10) \quad \begin{aligned} |R(z)| &\geq |a_0 (\rho^L)^{N(r)} z^{N(r)}| \\ &\quad - |a_{p_1} (\rho^L)^{N(r)-p_1} z^{N(r)-p_1} + \dots + a_{p_m} (\rho^L)^{N(r)-p_m} z^{N(r)-p_m} + a_{N(r)}|. \end{aligned}$$

Also

$$\begin{aligned} &|a_{p_1} (\rho^L)^{N(r)-p_1} z^{N(r)-p_1} + \dots + a_{p_m} (\rho^L)^{N(r)-p_m} z^{N(r)-p_m} + a_{N(r)}| \\ &\leq |a_{p_1} (\rho^L)^{N(r)-p_1} z^{N(r)-p_1}| + \dots + |a_{p_m} (\rho^L)^{N(r)-p_m} z^{N(r)-p_m}| + |a_{N(r)}| \\ (11) \quad &= |a_{p_1}| (\rho^L)^{N(r)-p_1} |z|^{N(r)-p_1} + \dots + |a_{p_m}| (\rho^L)^{N(r)-p_m} |z|^{N(r)-p_m} + |a_{N(r)}| \\ &\leq |a_{p_1}| (\rho^L)^{N(r)-p_1} \left(|z|^{N(r)-p_1} + \dots + |z|^{N(r)-p_m} + 1 \right). \end{aligned}$$

Using (11), we get from (10) that for $|z| \neq 0$

$$\begin{aligned} |R(z)| &\geq |a_0| (\rho^L)^{N(r)} |z|^{N(r)} - |a_{p_1}| (\rho^L)^{N(r)-p_1} \left(|z|^{N(r)-p_1} + \dots + |z|^{N(r)-p_m} + 1 \right) \\ &= |a_0| (\rho^L)^{N(r)} |z|^{N(r)} - |a_{p_1}| (\rho^L)^{N(r)-p_1} |z|^{N(r)} \left(\frac{1}{|z|^{p_1}} + \dots + \frac{1}{|z|^{p_m}} + \frac{1}{|z|^{N(r)}} \right) \\ &> |a_0| (\rho^L)^{N(r)} |z|^{N(r)} \\ &\quad - |a_{p_1}| (\rho^L)^{N(r)-p_1} |z|^{N(r)} \left(\frac{1}{|z|^{p_1}} + \dots + \frac{1}{|z|^{p_m}} + \frac{1}{|z|^{N(r)}} + \dots \right). \end{aligned}$$

Therefore, for $|z| \neq 0$,

$$(12) \quad |R(z)| > |a_0| (\rho^L)^{N(r)} |z|^{N(r)} - |a_{p_1}| (\rho^L)^{N(r)-p_1} |z|^{N(r)} \left(\sum_{k=1}^{\infty} \frac{1}{|z|^k} \right).$$

Now, the geometric series $\sum_{k=1}^{\infty} \frac{1}{|z|^k}$ is convergent for

$$\frac{1}{|z|} < 1$$

i.e., for $|z| > 1$

and converges to

$$\frac{1}{|z|} \frac{1}{1 - \frac{1}{|z|}} = \frac{1}{|z| - 1}.$$

So

$$\sum_{k=1}^{\infty} \frac{1}{|z|^k} = \frac{1}{|z| - 1} \text{ for } |z| > 1.$$

Therefore, for $|z| > 1$,

$$\begin{aligned} |R(z)| &> |a_0| (\rho^L)^{N(r)} |z|^{N(r)} - |a_{p_1}| (\rho^L)^{N(r)-p_1} |z|^{N(r)} \left(\frac{1}{|z|-1} \right) \\ &= |z|^{N(r)} (\rho^L)^{N(r)-p_1} \left(|a_0| (\rho^L)^{p_1} - \frac{|a_{p_1}|}{|z|-1} \right). \end{aligned}$$

i.e., for $|z| > 1$

$$|R(z)| > |z|^{N(r)} (\rho^L)^{N(r)-p_1} \left(|a_0| (\rho^L)^{p_1} - \frac{|a_{p_1}|}{|z|-1} \right).$$

Now,

$$\begin{aligned} |R(z)| > 0 &\text{ if } |a_0| (\rho^L)^{p_1} - \frac{|a_{p_1}|}{|z|-1} \geq 0 \\ \text{i.e., if } |a_0| (\rho^L)^{p_1} &\geq \frac{|a_{p_1}|}{|z|-1} \\ \text{i.e., if } |z|-1 &\geq \frac{|a_{p_1}|}{|a_0| (\rho^L)^{p_1}} \\ \text{i.e., if } |z| &\geq 1 + \frac{|a_{p_1}|}{|a_0| (\rho^L)^{p_1}} > 1. \end{aligned}$$

Therefore,

$$|R(z)| > 0 \text{ if } |z| \geq 1 + \frac{|a_{p_1}|}{|a_0| (\rho^L)^{p_1}}.$$

Since $R(z)$ does not vanish in

$$|z| \geq 1 + \frac{|a_{p_1}|}{|a_0| (\rho^L)^{p_1}},$$

all the zeros of $R(z)$ lie in

$$|z| < 1 + \frac{|a_{p_1}|}{|a_0|(\rho^L)^{p_1}}.$$

Let $z = z_0$ be any zero of $P(z)$. Therefore, $P(z_0) = 0$. Clearly, $z_0 \neq 0$ as $a_0 \neq 0$. Putting $z = 1/\rho^L z_0$ in $R(z)$, we obtain that

$$\begin{aligned} R\left(\frac{1}{\rho^L z_0}\right) &= (\rho^L)^{N(r)} \left(\frac{1}{\rho^L z_0}\right)^{N(r)} \cdot P(z_0) \\ &= \left(\frac{1}{z_0}\right)^{N(r)} \cdot 0 = 0. \end{aligned}$$

So

$$\begin{aligned} \left|\frac{1}{\rho^L z_0}\right| &< 1 + \frac{|a_{p_1}|}{|a_0|(\rho^L)^{p_1}} \\ \text{i.e., } \left|\frac{1}{z_0}\right| &< \rho^L \left(1 + \frac{|a_{p_1}|}{|a_0|(\rho^L)^{p_1}}\right) \\ \text{i.e., } |z_0| &> \frac{1}{\rho^L \left(1 + \frac{|a_{p_1}|}{|a_0|(\rho^L)^{p_1}}\right)}. \end{aligned}$$

As z_0 is an arbitrary zero of $P(z)$, all the zeros of $P(z)$ lie in

$$(13) \quad |z| > \frac{1}{\rho^L \left(1 + \frac{|a_{p_1}|}{|a_0|(\rho^L)^{p_1}}\right)}.$$

So, from (9) and (13), we may conclude that all the zeros of $P(z)$ lie in the proper ring shaped region

$$\frac{1}{\rho^L \left(1 + \frac{|a_{p_1}|}{|a_0|(\rho^L)^{p_1}}\right)} < |z| < \frac{1}{\rho^L} \left(1 + \frac{|a_0|}{|a_{N(r)}|}(\rho^L)^{N(r)}\right).$$

This proves the theorem. ■

In the line of Theorem 1, we may state the following theorem in view of Lemma 2:

Theorem 2. *Let $P(z)$ be an entire function having L^* -order ρ^{L^*} in the disc $|z| \leq [re^{L(r)}]$ for sufficiently large r . Also, let the Taylor's series expansion of $P(z)$ be given by*

$$P(z) = a_0 + a_{p_1} z^{p_1} + \cdots + a_{p_m} z^{p_m} + a_{N(r)} z^{N(r)}, \quad a_0 \neq 0, \quad a_{N(r)} \neq 0$$

with $1 \leq p_1 < p_2 < \cdots < p_m \leq N(r) - 1$, p_i 's are integers such that for $\rho^{L^*} > 0$,

$$|a_0| (\rho^{L^*})^{N(r)} \geq |a_{p_1}| (\rho^{L^*})^{N(r)-p_1} \geq \cdots \geq |a_{p_m}| (\rho^{L^*})^{N(r)-p_m} \geq |a_{N(r)}|.$$

Then, all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{\rho^{L^*} \left(1 + \frac{|a_{p_1}|}{|a_0|(\rho^{L^*})^{p_1}} \right)} < |z| < \frac{1}{\rho^{L^*}} \left(1 + \frac{|a_0|}{|a_{N(r)}|} (\rho^{L^*})^{N(r)} \right).$$

The proof is omitted.

Corollary 1. In view of Theorem 1, we may conclude that all the zeros of

$$P(z) = a_0 + a_{p_1} z^{p_1} + \cdots + a_{p_m} z^{p_m} + a_n z^n$$

of degree n with $1 \leq p_1 < p_2 < \cdots < p_m \leq n - 1$, p_i 's are integers such that for $\rho^L > 0$,

$$|a_0| \geq |a_{p_1}| \geq \cdots \geq |a_n|$$

lie in the ring shaped region

$$\frac{1}{\left(1 + \frac{|a_{p_1}|}{|a_0|} \right)} < |z| < \left(1 + \frac{|a_0|}{|a_n|} \right)$$

on putting $\rho^L = 1$ in Theorem 1.

Corollary 2. In view of Theorem 2, we may conclude that all the zeros of

$$P(z) = a_0 + a_{p_1} z^{p_1} + \cdots + a_{p_m} z^{p_m} + a_n z^n$$

of degree n with $1 \leq p_1 < p_2 < \cdots < p_m \leq n - 1$, p_i 's are integers such that for $\rho^{L^*} > 0$,

$$|a_0| \geq |a_{p_1}| \geq \cdots \geq |a_n|$$

lie in the ring shaped region

$$\frac{1}{\left(1 + \frac{|a_{p_1}|}{|a_0|} \right)} < |z| < \left(1 + \frac{|a_0|}{|a_n|} \right)$$

on putting $\rho^{L^*} = 1$ in Theorem 2.

Theorem 3. Let $P(z)$ be an entire function having L -order ρ^L . For sufficiently large r in the disc $|z| \leq [rL(r)]$, the Taylor's series expansion of $P(z)$ be given by $P(z) = a_0 + a_1 z + \cdots + a_{N(r)} z^{N(r)}$, $a_0 \neq 0$. Further, for $\rho^L > 0$,

$$|a_0| (\rho^L)^{N(r)} \geq |a_1| (\rho^L)^{N(r)-1} \geq \cdots \geq |a_{N(r)}|.$$

Then, all the zeros of $P(z)$ lie in the ring shaped region.

$$\frac{1}{\rho^L t'_0} < |z| < \frac{1}{\rho^L} t_0,$$

where t_0 and t'_0 are the greatest roots of

$$g(t) \equiv |a_{N(r)}| t^{N(r)+1} - (|a_{N(r)}| + (\rho^L)^{N(r)} |a_0|) t^{N(r)} + (\rho^L)^{N(r)} |a_0| = 0$$

and

$$f(t) \equiv |a_0| \rho^L t^{N(r)+1} - (|a_0| \rho^L + |a_1|) t^{N(r)} + |a_1| = 0.$$

Proof. Let

$$P(z) = a_0 + a_1 z + \cdots + a_{N(r)} z^{N(r)}$$

by applying Lemma 1 and in view of Taylor's series expansion of $P(z)$. Also

$$|a_0| (\rho^L)^{N(r)} \geq |a_1| (\rho^L)^{N(r)-1} \geq \cdots \geq |a_{N(r)}|.$$

Let us consider

$$\begin{aligned} Q(z) &= (\rho^L)^{N(r)} P\left(\frac{z}{\rho^L}\right) \\ &= (\rho^L)^{N(r)} \left\{ a_0 + a_1 \frac{z}{\rho^L} + a_2 \frac{z^2}{(\rho^L)^2} + \cdots + a_{N(r)} \frac{z^{N(r)}}{(\rho^L)^{N(r)}} \right\} \\ &= a_0 (\rho^L)^{N(r)} + a_1 (\rho^L)^{N(r)-1} z + \cdots + a_{N(r)} z^{N(r)}. \end{aligned}$$

Now

$$|Q(z)| \geq |a_{N(r)}| |z|^{N(r)} - |a_0 (\rho^L)^{N(r)} + a_1 (\rho^L)^{N(r)-1} z + \cdots + a_{N(r)-1} z^{N(r)-1}|.$$

Also, applying the condition $|a_0| (\rho^L)^{N(r)} \geq |a_1| (\rho^L)^{N(r)-1} \geq \cdots \geq |a_{N(r)}|$, we get from above that

$$\begin{aligned} &|a_0 (\rho^L)^{N(r)} + a_1 (\rho^L)^{N(r)-1} z + \cdots + a_{N(r)-1} z^{N(r)-1}| \\ &\leq |a_0| (\rho^L)^{N(r)} + |a_1| (\rho^L)^{N(r)-1} |z| + \cdots + |a_{N(r)-1}| |z|^{N(r)-1} \\ &\leq |a_0| (\rho^L)^{N(r)} (1 + |z| + \cdots + |z|^{N(r)-1}) \\ &= |a_0| (\rho^L)^{N(r)} \frac{|z|^{N(r)} - 1}{|z| - 1} \text{ for } |z| \neq 1. \end{aligned}$$

Therefore, it follows from above that

$$|Q(z)| \geq |a_{N(r)}| |z|^{N(r)} - |a_0| (\rho^L)^{N(r)} \cdot \frac{|z|^{N(r)} - 1}{|z| - 1}.$$

Now

$$|Q(z)| > 0 \text{ if } |a_{N(r)}| |z|^{N(r)} - |a_0| (\rho^L)^{N(r)} \cdot \frac{|z|^{N(r)} - 1}{|z| - 1} > 0$$

$$\text{i.e., if } |a_{N(r)}| |z|^{N(r)} > |a_0| (\rho^L)^{N(r)} \cdot \frac{|z|^{N(r)} - 1}{|z| - 1}$$

$$\text{i.e., if } |a_{N(r)}| |z|^{N(r)} (|z| - 1) > |a_0| (\rho^L)^{N(r)} (|z|^{N(r)} - 1)$$

$$\text{i.e., if } |a_{N(r)}| |z|^{N(r)+1} - (|a_{N(r)}| + |a_0| (\rho^L)^{N(r)}) |z|^{N(r)} + |a_0| (\rho^L)^{N(r)} > 0.$$

Let us consider

$$(14) \quad g(t) \equiv |a_{N(r)}| t^{N(r)+1} - (|a_{N(r)}| + |a_0| (\rho^L)^{N(r)}) t^{N(r)} + |a_0| (\rho^L)^{N(r)} = 0.$$

The maximum number of positive roots of (14) is two because maximum number of changes of sign in $g(t) = 0$ is two and if it is less, less by two. Clearly, $t = 1$ is a positive root of $g(t) = 0$. Therefore, $g(t) = 0$ must have exactly one positive root other than 1. Let the positive root of $g(t)$ be t_1 . Let us take $t_0 = \max \{1, t_1\}$. Clearly, for $t > t_0$, $g(t) > 0$. If not for some $t_2 > t_0$, $g(t_2) < 0$. Also $g(\infty) > 0$. Therefore $g(t) = 0$ has another positive root in (t_2, ∞) which gives a contradiction. So, for $t > t_0$, $g(t) > 0$. Also $t_0 \geq 1$. Therefore, $|Q(z)| > 0$ if $|z| > t_0$. So, $Q(z)$ does not vanish in $|z| > t_0$. Hence, all the zeros of $Q(z)$ lie in $|z| \leq t_0$.

Let $z = z_0$ be a zero of $P(z)$. So, $P(z_0) = 0$. Clearly, $z_0 \neq 0$ as $a_0 \neq 0$. Putting $z = \rho^L z_0$ in $Q(z)$, we get that

$$Q(\rho^L z_0) = (\rho^L)^{N(r)} P(z_0) = (\rho^L)^{N(r)} \cdot 0 = 0.$$

Therefore, $z = \rho^L z_0$ is a zero of $Q(z)$. So, $|\rho^L z_0| \leq t_0$ or $|z_0| \leq \frac{1}{\rho^L} t_0$. As z_0 is an arbitrary zero of $P(z)$,

$$(15) \quad \text{all the zeros of } P(z) \text{ lie in the region } |z| \leq \frac{1}{\rho^L} t_0.$$

In order to prove the lower bound of Theorem 3, let us consider

$$R(z) = (\rho^L)^{N(r)} z^{N(r)} P\left(\frac{1}{\rho^L z}\right).$$

Then

$$\begin{aligned} R(z) &= (\rho^L)^{N(r)} z^{N(r)} \left(a_0 + \frac{a_1}{\rho^L z} + \cdots + a_{N(r)} \frac{1}{(\rho^L)^{N(r)} z^{N(r)}} \right) \\ &= a_0 (\rho^L)^{N(r)} z^{N(r)} + a_1 (\rho^L)^{N(r)-1} z^{N(r)-1} + \cdots + a_{N(r)}. \end{aligned}$$

Now

$$|R(z)| \geq |a_0| (\rho^L)^{N(r)} |z|^{N(r)} - |a_1 (\rho^L)^{N(r)-1} z^{N(r)-1} + \cdots + a_{N(r)}|.$$

Also

$$\left| a_1(\rho^L)^{N(r)-1} z^{N(r)-1} + \dots + a_{N(r)} \right| \leq |a_1| (\rho^L)^{N(r)-1} |z|^{N(r)-1} + \dots + |a_{N(r)}|.$$

So, applying the condition $|a_0| (\rho^L)^{N(r)} \geq |a_1| (\rho^L)^{N(r)-1} \geq \dots \geq |a_{N(r)}|$, we get from above that

$$\begin{aligned} & - \left| a_1(\rho^L)^{N(r)-1} z^{N(r)-1} + \dots + a_{N(r)} \right| \\ & \geq -|a_1| (\rho^L)^{N(r)-1} |z|^{N(r)-1} - \dots - |a_{N(r)}| \\ & \geq -|a_1| (\rho^L)^{N(r)-1} \left(|z|^{N(r)-1} + \dots + 1 \right) \\ & = -|a_1| (\rho^L)^{N(r)-1} \frac{|z|^{N(r)-1} - 1}{|z| - 1} \text{ for } |z| \neq 1. \end{aligned}$$

Using (16), we get for $|z| \neq 1$ that

$$(16) \quad |R(z)| \geq (\rho^L)^{N(r)-1} \left(|a_0| \rho^L |z|^{N(r)} - |a_1| \frac{|z|^{N(r)-1} - 1}{|z| - 1} \right).$$

Now

$$\begin{aligned} |R(z)| > 0 & \text{ if } (\rho^L)^{N(r)-1} \left(|a_0| \rho^L |z|^{N(r)} - |a_1| \frac{|z|^{N(r)-1} - 1}{|z| - 1} \right) > 0 \\ \text{i.e., if } & |a_0| \rho^L |z|^{N(r)} - |a_1| \frac{|z|^{N(r)-1} - 1}{|z| - 1} > 0 \\ \text{i.e., if } & |a_0| \rho^L |z|^{N(r)} > |a_1| \frac{|z|^{N(r)-1} - 1}{|z| - 1} \\ \text{i.e., if } & |a_0| \rho^L |z|^{N(r)} (|z| - 1) > |a_1| (|z|^{N(r)-1} - 1) \\ \text{i.e., if } & |a_0| \rho^L |z|^{N(r)+1} - (|a_0| \rho^L + |a_1|) |z|^{N(r)} + |a_1| > 0. \end{aligned}$$

Let us consider

$$f(t) \equiv |a_0| \rho^L t^{N(r)+1} - (|a_0| \rho^L + |a_1|) t^{N(r)} + |a_1| = 0.$$

Clearly, $f(t) = 0$ has two positive roots, because the number of changes of sign of $f(t)$ is two. If it is less, less by two. Also, $t = 1$ is the one of the positive roots of $f(t) = 0$. Let us suppose that $t = t_2$ be the other positive root. Also, let $t'_0 = \max \{1, t_2\}$ and so $t'_0 \geq 1$. Now $t > t'_0$ implies $f(t) > 0$. If not, then there exists some $t_3 > t'_0$ such that $f(t_3) < 0$. Also, $f(\infty) > 0$. Therefore, there exists another positive root in (t_3, ∞) which is a contradiction. So, $|R(z)| > 0$ if $|z| > t'_0$. Thus $R(z)$ does not vanish in $|z| > t'_0$. In other words, all the zeros of $R(z)$ lie in $|z| \leq t'_0$.

Let $z = z_0$ be any zero of $P(z)$. So, $P(z_0) = 0$. Clearly, $z_0 \neq 0$ as $a_0 \neq 0$. Putting $z = \frac{1}{\rho^L z_0}$ in $R(z)$, we get that

$$R\left(\frac{1}{\rho^L z_0}\right) = (\rho^L)^{N(r)} \left(\frac{1}{\rho^L z_0}\right)^{N(r)} P(z_0) = \left(\frac{1}{z_0}\right)^{N(r)} \cdot 0 = 0$$

Therefore, $\frac{1}{\rho^L z_0}$ is a root of $R(z)$. So, $\left|\frac{1}{\rho^L z_0}\right| \leq t'_0$ implies $|z_0| \geq \frac{1}{\rho^L t'_0}$. As z_0 is an arbitrary zero of $P(z) = 0$,

$$(17) \quad \text{all the zeros of } P(z) \text{ lie in } |z| \geq \frac{1}{\rho^L t'_0}.$$

From (15) and (18) we have all the zeros of $P(z)$ lie in the ring shaped region given by

$$\frac{1}{\rho^L t'_0} \leq |z| \leq \frac{1}{\rho^L} t_0$$

where t_0 and t'_0 are the greatest positive roots of $g(t) = 0$ and $f(t) = 0$ respectively.

This proves the theorem. $\blacksquare$

In the line of Theorem 3, we may state the following theorem in view of Lemma 2:

Theorem 4. *Let $P(z)$ be an entire function having L^* -order ρ^{L^*} . For sufficiently large r in the disc $|z| \leq [re^{L(r)}]$, the Taylor's series expansion of $P(z)$ be given by*

$$P(z) = a_0 + a_1 z + \cdots + a_{N(r)} z^{N(r)}, \quad a_0 \neq 0.$$

Further, for $\rho^{L^*} > 0$,

$$|a_0| (\rho^{L^*})^{N(r)} \geq |a_1| (\rho^{L^*})^{N(r)-1} \geq \cdots \geq |a_{N(r)}|.$$

Then all the zeros of $P(z)$ lie in the ring shaped region.

$$\frac{1}{\rho^{L^*} t'_0} < |z| < \frac{1}{\rho^{L^*}} t_0$$

where t_0 and t'_0 are the greatest roots of

$$g(t) \equiv |a_{N(r)}| t^{N(r)+1} - (|a_{N(r)}| + (\rho^{L^*})^{N(r)} |a_0|) t^{N(r)} + (\rho^{L^*})^{N(r)} |a_0| = 0$$

and

$$f(t) \equiv |a_0| \rho^{L^*} t^{N(r)+1} - (|a_0| \rho^{L^*} + |a_1|) t^{N(r)} + |a_1| = 0.$$

The proof is omitted.

Corollary 3. *From Theorem 3, we can easily conclude that all the zeros of*

$$P(z) = a_0 + a_1z + \cdots + a_nz^n$$

of degree n with property $|a_0| \geq |a_1| \geq \cdots \geq |a_n|$ lie in the ring shaped region

$$\frac{1}{t'_0} \leq |z| \leq t_0$$

where t_0 and t'_0 are the greatest positive roots of

$$g(t) \equiv |a_n|t^{n+1} - (|a_n| + |a_0|)t^n + |a_0| = 0$$

and

$$f(t) \equiv |a_0|t^{n+1} - (|a_0| + |a_1|)t^n + |a_1| = 0$$

respectively by putting $\rho^L = 1$.

Corollary 4. *From Theorem 4, we can easily conclude that all the zeros of*

$$P(z) = a_0 + a_1z + \cdots + a_nz^n$$

of degree n with property $|a_0| \geq |a_1| \geq \cdots \geq |a_n|$ lie in the ring shaped region

$$\frac{1}{t'_0} \leq |z| \leq t_0$$

where t_0 and t'_0 are the greatest positive roots of

$$g(t) \equiv |a_n|t^{n+1} - (|a_n| + |a_0|)t^n + |a_0| = 0$$

and

$$f(t) \equiv |a_0|t^{n+1} - (|a_0| + |a_1|)t^n + |a_1| = 0$$

respectively by putting $\rho^{L^*} = 1$.

Corollary 5. *Under the conditions of Theorem 3 and*

$$P(z) = a_0 + a_{p_1}z^{p_1} + \cdots + a_{p_m}z^{p_m} + a_{N(r)}z^{N(r)}$$

with

$$1 \leq p_1 \leq p_2 \leq \cdots \leq p_m \leq N(r) - 1,$$

where p_i 's are integers and $a_0, a_{p_1}, \dots, a_{N(r)}$ are non vanishing coefficients with

$$|a_0|(\rho^L)^{N(r)} \geq |a_{p_1}|(\rho^L)^{N(r)-p_1} \geq \cdots \geq |a_{p_m}|(\rho^L)^{N(r)-p_m} \geq |a_{N(r)}|$$

then we can show that all the zeros of $P(z)$ lie in

$$\frac{1}{\rho^L t'_0} \leq |z| \leq \frac{1}{\rho^L} t_0$$

where t_0 and t'_0 are the greatest positive roots of

$$g(t) \equiv |a_{N(r)}|t^{N(r)+1} - (|a_{N(r)}| + |a_0|(\rho^L)^{N(r)})t^{N(r)} + |a_0|(\rho^L)^{N(r)} = 0$$

and

$$f(t) \equiv |a_0|(\rho^L)^{p_1}t^{N(r)+1} - (|a_0|(\rho^L)^{p_1} + |a_{p_1}|)t^{N(r)} - |a_{p_1}| = 0 \text{ respectively.}$$

Corollary 6. *Under the conditions of Theorem 4 and*

$$P(z) = a_0 + a_{p_1} z^{p_1} + \dots + a_{p_m} z^{p_m} + a_{N(r)} z^{N(r)}$$

with

$$1 \leq p_1 \leq p_2 \leq \dots \leq p_m \leq N(r) - 1,$$

where p_i 's are integers and $a_0, a_{p_1}, \dots, a_{N(r)}$ are non vanishing coefficients with

$$|a_0| (\rho^{L^*})^{N(r)} \geq |a_{p_1}| (\rho^{L^*})^{N(r)-p_1} \geq \dots \geq |a_{p_m}| (\rho^{L^*})^{N(r)-p_m} \geq |a_{N(r)}|$$

then we can show that all the zeros of $P(z)$ lie in

$$\frac{1}{\rho^{L^*} t'_0} \leq |z| \leq \frac{1}{\rho^{L^*}} t_0$$

where t_0 and t'_0 are the greatest positive roots of

$$g(t) \equiv |a_{N(r)}| t^{N(r)+1} - (|a_{N(r)}| + |a_0| (\rho^{L^*})^{N(r)}) t^{N(r)} + |a_0| (\rho^{L^*})^{N(r)} = 0$$

and

$$f(t) \equiv |a_0| (\rho^{L^*})^{p_1} t^{N(r)+1} - (|a_0| (\rho^{L^*})^{p_1} + |a_{p_1}|) t^{N(r)} - |a_{p_1}| = 0 \text{ respectively.}$$

Corollary 7. *If we put $\rho^L = 1$ in Corollary 5, then all the zeros of*

$$P(z) = a_0 + a_{p_1} z^{p_1} + \dots + a_{p_m} z^{p_m} + a_n z^n$$

lie in the ring shaped region

$$\frac{1}{t'_0} \leq |z| \leq t_0$$

where t_0 and t'_0 are the greatest positive roots of

$$g(t) \equiv |a_n| t^{n+1} - (|a_n| + |a_0|) t^n + |a_0| = 0$$

and

$$f(t) \equiv |a_0| t^{n+1} - (|a_0| + |a_{p_1}|) t^n - |a_{p_1}| = 0 \text{ respectively}$$

provided

$$|a_0| \geq |a_{p_1}| \geq \dots \geq |a_{p_m}| \geq |a_n|.$$

Corollary 8. *If we put $\rho^{L^*} = 1$ in Corollary 6, then all the zeros of*

$$P(z) = a_0 + a_{p_1} z^{p_1} + \dots + a_{p_m} z^{p_m} + a_n z^n$$

lie in the ring shaped region

$$\frac{1}{t'_0} \leq |z| \leq t_0$$

where t_0 and t'_0 are the greatest positive roots of

$$g(t) \equiv |a_n| t^{n+1} - (|a_n| + |a_0|) t^n + |a_0| = 0$$

and

$$f(t) \equiv |a_0| t^{n+1} - (|a_0| + |a_{p_1}|) t^n - |a_{p_1}| = 0 \quad \text{respectively}$$

provided

$$|a_0| \geq |a_{p_1}| \geq \dots \geq |a_{p_m}| \geq |a_n|.$$

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Accepted: 9.12.2013

SOFT INTERSECTION h -IDEALS OF HEMIRINGS AND ITS APPLICATIONS

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Abstract. In this paper, we introduce a new kind of soft hemirings called soft intersection hemirings and obtain some related properties. Some basic operations are also investigated. Finally, we describe some characterizations of h -hemiregular hemirings by means of SI - h -ideals.

Keywords and phrases: Soft set; soft intersection hemiring; soft intersection h -ideal; soft h -sum; soft h -product; h -hemiregular hemiring.

2000 Mathematics Subject Classification: 16Y60; 13E05; 16Y99.

1. Introduction

The complexities of modeling uncertain data in economics, engineering, environmental science, sociology and many other fields can not be successfully dealt with by classical methods. In order to overcome these difficulties, Molodtsov [17] introduced the concept of soft sets as a new mathematical tool for dealing with uncertainties. Maji [15] discussed further soft set theory. Ali et al. [3] proposed some new operations on soft sets. In the same time, this theory has proven useful in many different fields such as decision making [5], [6], [7], [9], [16], data analysis, forecasting and so on. Recently, the algebraic structures of soft sets have been studied increasingly, such as soft rings [1], soft-int groups [4], soft semirings [8], soft BCK/BCI-algebras [11], soft intersection near-rings [18] and so on.

On the other hand, semirings have been found useful for dealing with problems in different areas of applied mathematics and information sciences, as the semiring structure provides an algebraic framework for modeling and investigating the key factors in these problems. We know that ideals in the semiring S do not in general coincide with the usual ring ideals if S is a ring, and so many results in ring theory have no analogues in semirings using only ideals. Consequently, some

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more restricted concepts of ideals such as k -ideals and h -ideals [2], [19], [10], [15], [16], [14], [20], [21] have been investigated. Nowadays, many researchers discussed this theory including their application. In application, hemirings are useful in automata and formal languages.

In this paper, we introduce a new kind of soft hemirings called soft intersection hemirings and obtain some related properties. Some basic operations are also investigated. Finally, we describe some characterizations of h -hemiregular hemirings by means of SI - h -ideals.

2. Preliminaries

A semiring $(S, +, \cdot)$ with zero is called a hemiring if $(S, +)$ is commutative. A subhemiring of a hemiring S is a subset A of S closed under addition and multiplication. A left (resp., right) ideal of a hemiring S is a subset A of S closed under addition such that $SA \subseteq A$ (resp., $AS \subseteq A$). A subset A is called an ideal if it is both a left ideal and a right ideal. A subhemiring (left ideal, right ideal, ideal) A of S is called an h -subhemiring (left h -ideal, right h -ideal, h -ideal) of S , respectively, if for any $x, z \in S$, and $a, b \in A$, $x + a + z = b + z$ implies $x \in A$. The h -closure $\overline{A}$ of a subset A of S is defined as $\overline{A} = \{x \in S \mid x + a + z = b + z \text{ for some } a, b \in A, z \in S\}$.

Throughout this section, S is a hemiring, U is an initial universe, E is a set of parameters, $P(U)$ is the power set of U and $A, B, C \subseteq E$.

Definition 2.1 [17] A soft set f_A of U is a set defined by $f_A : E \rightarrow P(U)$ such that $f_A(x) = \emptyset$ if $x \notin A$. Here f_A is also called an approximate function. A soft set over U can be represented by the set of ordered pairs $f_A = \{(x, f_A(x)) \mid x \in E, f_A(x) \in P(U)\}$. It is clear to see that a soft set is a parameterized family of subsets of the set U . Note that the set of all soft sets over U will be denoted by $S(U)$.

Definition 2.2 [6] Let $f_A, f_B \in S(U)$, then

- (i) The intersection of f_A and f_B , denoted by $f_A \widetilde{\cap} f_B$, is defined as $f_A \widetilde{\cap} f_B = f_{A \widetilde{\cap} B}$, where $f_{A \widetilde{\cap} B}(x) = f_A(x) \cap f_B(x)$, for all $x \in E$;
- (ii) The union of f_A and f_B , denoted by $f_A \widetilde{\cup} f_B$, is defined as $f_A \widetilde{\cup} f_B = f_{A \widetilde{\cup} B}$, where $f_{A \widetilde{\cup} B}(x) = f_A(x) \cup f_B(x)$, for all $x \in E$.

Definition 2.3 [6] Let $f_A, f_B \in S(U)$. Then $\wedge$ -product and $\vee$ -product of f_A and f_B , denoted by $f_A \wedge f_B$ and $f_A \vee f_B$, are defined by $f_{A \wedge B}(x, y) = f_A(x) \cap f_B(y)$, $f_{A \vee B}(x, y) = f_A(x) \cup f_B(y)$ for all $x, y \in E$, respectively.

Definition 2.4 [4] Let f_A be a soft set over U and $\alpha \subseteq U$. Then, upper α -inclusion of f_A , denoted by $U(f_A; \alpha)$, is defined as $U(f_A; \alpha) = \{x \in A \mid f_A(x) \supseteq \alpha\}$.

3. SI -hemirings (SI - h -ideals)

In this paper, we introduce the concepts of soft intersection hemirings (soft intersection h -ideals) and obtain some related properties.

Definition 3.1 A soft set f_S over U is called a soft intersection hemiring (briefly, SI -hemiring) of S over U if it satisfies:

$$(SI_1) \quad f_S(x + y) \supseteq f_S(x) \cap f_S(y),$$

$$(SI_2) \quad f_S(xy) \supseteq f_S(x) \cap f_S(y),$$

$$(SI_3) \quad f_S(x) \supseteq f_S(a) \cap f_S(b) \text{ with } x + a + z = b + z \text{ for all } x, a, b, z \in S.$$

Example 3.2 Let $U = S = Z_6 = \{0, 1, 2, 3, 4, 5\}$ be the hemiring of non-negative integers module 6. Define a soft set f_S over U by $f_S(0) = f_S(2) = f_S(4) = \{0, 1, 2, 3, 4, 5\}$ and $f_S(1) = f_S(3) = f_S(5) = \{0, 2, 4\}$. Then one can easily check that f_S is an SI -hemiring of S over U .

From the above definition, we can obtain the following:

Proposition 3.3 If f_S is an SI -hemiring of S over U , then $f_S(0) \supseteq f_S(x)$ for all $x \in S$.

Definition 3.4 A soft set f_S over U is called a soft intersection left (right) h -ideal (briefly, SI -left(right) h -ideal) of S over U if it satisfies (SI_1) , (SI_3) and:

$$(SI_4) \quad f_S(xy) \supseteq f_S(y) \text{ } (f_S(xy) \supseteq f_S(x)), \text{ for all } x, y \in S.$$

A soft set over U is called a soft intersection h -ideal (briefly, SI - h -ideal) of S if it is both an SI -left h -ideal and an SI -right h -ideal of S over U .

Example 3.5 Assume that $U = Z^+$ is the universal set and $S = Z_6$ is the set of parameters. Define a soft set f_S as $f_S(0) = \{n | n \in Z^+\}$, $f_S(1) = f_S(5) = \{6n | n \in Z^+\}$, $f_S(2) = f_S(4) = \{2n | n \in Z^+\}$ and $f_S(3) = \{3n | n \in Z^+\}$. Then, one can easily check that f_S is an SI - h -ideal of S over U .

Proposition 3.6 Let f_{S_1} and f_{S_2} be two SI -hemirings of S_1 and S_2 over U , respectively. Then $f_{S_1} \wedge f_{S_2}$ is an SI -hemiring of $S_1 \times S_2$ over U .

Proof. Let f_{S_1} and f_{S_2} be two SI -hemirings of S_1 and S_2 over U , respectively. Then, for all $(x_1, y_1), (x_2, y_2) \in S_1 \times S_2$, we have

$$\begin{aligned} (i) \quad f_{S_1 \wedge S_2}((x_1, y_1) + (x_2, y_2)) &= f_{S_1 \wedge S_2}(x_1 + x_2, y_1 + y_2) \\ &= f_{S_1}(x_1 + x_2) \cap f_{S_2}(y_1 + y_2) \\ &\supseteq (f_{S_1}(x_1) \cap f_{S_1}(x_2)) \cap (f_{S_2}(y_1) \cap f_{S_2}(y_2)) \\ &= (f_{S_1}(x_1) \cap f_{S_2}(y_1)) \cap (f_{S_1}(x_2) \cap f_{S_2}(y_2)) \\ &= f_{S_1 \wedge S_2}(x_1, y_1) \cap f_{S_1 \wedge S_2}(x_2, y_2). \end{aligned}$$

(ii) is similar to (i).

(iii) Let $(a_1, a_2), (b_1, b_2), (x_1, x_2), (z_1, z_2) \in S_1 \times S_2$ be such that $(x_1, x_2) + (a_1, a_2) + (z_1, z_2) = (b_1, b_2) + (z_1, z_2)$, and so $x_1 + a_1 + z_1 = b_1 + z_1$ and $x_2 + a_2 + z_2 = b_2 + z_2$. Then

$$\begin{aligned} f_{S_1 \wedge S_2}(x_1, x_2) &= f_{S_1}(x_1) \cap f_{S_2}(x_2) \\ &\supseteq (f_{S_1}(a_1) \cap f_{S_1}(b_1)) \cap (f_{S_2}(a_2) \cap f_{S_2}(b_2)) \\ &= (f_{S_1}(a_1) \cap f_{S_2}(a_2)) \cap (f_{S_1}(b_1) \cap f_{S_2}(b_2)) \\ &= f_{S_1 \wedge S_2}(a_1, a_2) \cap f_{S_1 \wedge S_2}(b_1, b_2). \end{aligned}$$

Hence, $f_{S_1 \wedge S_2}$ is an SI -hemiring of $S_1 \times S_2$ over U . ■

Similarly, we can obtain the following result:

Proposition 3.7 *Let f_{S_1} and f_{S_2} be two SI -h-ideals of S_1 and S_2 over U , respectively. Then $f_{S_1} \wedge f_{S_2}$ is an SI -h-ideal of $S_1 \times S_2$ over U .*

Remark 3.8 Note that $f_{S_1} \vee f_{S_2}$ is not an SI -hemiring or SI -h-ideal over U .

Example 3.9 Let $U = S_3$, symmetric group, be the universal set, $S_1 = Z_5 = \{0, 1, 2, 3, 4\}$ and $S_2 = Z_2 = \{0, 1\}$ be two hemirings of non-negative integers module 5 and module 2, respectively. Define two soft sets f_{S_1} and f_{S_2} over U by $f_{S_1}(0) = S_3$, $f_{S_1}(1) = f_{S_1}(4) = \{(1), (12), (132)\}$, $f_{S_1}(2) = f_{S_1}(3) = \{(12), (123), (132)\}$, $f_{S_2}(0) = S_3$, $f_{S_2}(1) = \{(1), (12), (132)\}$. It is clear that f_{S_1} and f_{S_2} are two SI -hemirings over U . However, we have

$$\begin{aligned} f_{S_1 \vee S_2}((3, 1) + (1, 0)) &= f_{S_1 \vee S_2}(4, 1) \\ &= f_{S_1}(4) \cup f_{S_2}(1) \\ &= \{(1), (12), (132)\}, \end{aligned}$$

but

$$\begin{aligned} f_{S_1 \vee S_2}(3, 1) \cap f_{S_1 \vee S_2}(1, 0) &= (f_{S_1}(3) \cup f_{S_2}(1)) \cap (f_{S_1}(1) \cup f_{S_2}(0)) \\ &= \{(1), (12), (123), (132)\} \cap S_3 \\ &= \{(1), (12), (123), (132)\}. \end{aligned}$$

This implies that $f_{S_1 \vee S_2}((3, 1) + (1, 0)) \not\supseteq f_{S_1 \vee S_2}(3, 1) \cap f_{S_1 \vee S_2}(1, 0)$. Hence, $f_{S_1 \vee S_2}$ is not an SI -hemiring over U .

Theorem 3.10 *Let f_S and g_S be two SI -hemirings of S over U . Then $f_S \widetilde{\cap} g_S$ is also an SI -hemiring of S over U .*

Proof. Let f_S and g_S be two SI -hemirings of S over U . Then for all $x, y \in S$, we have

$$\begin{aligned} (i) \quad (f_S \widetilde{\cap} g_S)(x + y) &= f_S(x + y) \cap g_S(x + y) \\ &\supseteq (f_S(x) \cap f_S(y)) \cap (g_S(x) \cap g_S(y)) \\ &= (f_S(x) \cap g_S(x)) \cap (f_S(y) \cap g_S(y)) \\ &= (f_S \widetilde{\cap} g_S)(x) \cap (f_S \widetilde{\cap} g_S)(y). \end{aligned}$$

(ii) is similar to (i).

(iii) Let $a, b, x, z \in S$ be such that $x + a + z = b + z$. Then

$$\begin{aligned} (f_S \widetilde{\cap} g_S)(x) &= f_S(x) \cap g_S(x) \\ &\supseteq (f_S(a) \cap f_S(b)) \cap (g_S(a) \cap g_S(b)) \\ &= (f_S(a) \cap g_S(a)) \cap (f_S(b) \cap g_S(b)) \\ &= (f_S \widetilde{\cap} g_S)(a) \cap (f_S \widetilde{\cap} g_S)(b). \end{aligned}$$

Hence $f_S \widetilde{\cap} g_S$ is an SI -hemiring of S over U . ■

Similarly, we can obtain the following theorem:

Theorem 3.11 *Let f_S and g_S be two SI - h -ideals of S over U , then $f_S \widetilde{\cap} g_S$ is also an SI - h -ideal of S over U .*

4. h -hemiregular hemirings

In this section, we describe the characterizations of h -hemiregular hemirings by means of SI - h -ideals.

Definition 4.1 [22] A hemiring S is called h -hemiregular if for each $x \in S$, these exist $a_1, a_2, z \in S$ such that $x + xa_1x + z = xa_2x + z$.

Lemma 4.2 [22] If A and B , are, respectively, a right and a left h -ideal of S , then $\overline{AB} \subseteq A \cap B$.

Lemma 4.3 [22] A hemiring S is h -hemiregular if and only if for any right h -ideal A and any left h -ideal B , we have $\overline{AB} = A \cap B$.

Definition 4.4 Let $f_S, g_S \in S(U)$. Define soft h -sum and soft h -product of f_S and g_S as follows:

$$\begin{aligned} (1) \quad (f_S +_h g_S)(x) &= \bigcup_{x+a_1+b_1+z=a_2+b_2+z} (f_S(a_1) \cap f_S(a_2) \cap g_S(b_1) \cap g_S(b_2)) \text{ and} \\ &\quad (f_S +_h g_S)(x) = \emptyset \text{ if } x \text{ cannot be expressed as } x + a_1 + b_1 + z = a_2 + b_2 + z. \\ (2) \quad (f_S \circ_h g_S)(x) &= \bigcup_{x+a_1b_1+z=a_2b_2+z} (f_S(a_1) \cap f_S(a_2) \cap g_S(b_1) \cap g_S(b_2)) \text{ and } (f_S \circ_h \\ &\quad g_S)(x) = \emptyset \text{ if } x \text{ cannot be expressed as } x + a_1b_1 + z = a_2b_2 + z. \end{aligned}$$

Lemma 4.5 *Let f_S and g_S be an SI -right h -ideal and an SI -left h -ideal of S over U , respectively, then $f_S \circ_h g_S \subseteq f_S \widetilde{\cap} g_S$.*

Proof. If $(f_S \circ_h g_S)(x) = \emptyset$, then it is clear that $f_S \circ_h g_S \subseteq f_S \widetilde{\cap} g_S$. Otherwise, we have

$$\begin{aligned}
 (f_S \circ_h g_S)(x) &= \bigcup_{x+a_1b_1+z=a_2b_2+z} (f_S(a_1) \cap f_S(a_2) \cap g_S(b_1) \cap g_S(b_2)) \\
 &\subseteq \bigcup_{x+a_1b_1+z=a_2b_2+z} (f_S(a_1b_1) \cap f_S(a_2b_2) \cap g_S(a_1b_1) \cap g_S(a_2b_2)) \\
 &\subseteq \bigcup_{x+a_1b_1+z=a_2b_2+z} (f_S(x) \cap g_S(x)) \\
 &= f_S(x) \cap g_S(x) \\
 &= (f_S \cap g_S)(x),
 \end{aligned}$$

which implies, $f_S \circ_h g_S \subseteq f_S \widetilde{\cap} g_S$. ■

Definition 4.6 Let $A \subseteq S$. We denote $\mathcal{S}_A$ the soft characteristic function of A and define as

$$\mathcal{S}_A(x) = \begin{cases} U & \text{if } x \in A, \\ \emptyset & \text{if } x \notin A. \end{cases}$$

The following proposition is obvious and we omit the details.

Proposition 4.7 Let $A, B \subseteq S$. Then the following hold:

- (1) $A \subseteq B \Rightarrow \mathcal{S}_A \subseteq \mathcal{S}_B$.
- (2) $\mathcal{S}_A \widetilde{\cap} \mathcal{S}_B = \mathcal{S}_{A \cap B}$.
- (3) $\mathcal{S}_A \circ_h \mathcal{S}_B = \mathcal{S}_{\overline{AB}}$.

Theorem 4.8 For any hemiring S , then the following are equivalent:

- (1) S is h -hemiregular;
- (2) $f_S \circ_h g_S = f_S \widetilde{\cap} g_S$ for any SI -right h -ideal f_S and SI -left h -ideal g_S of S over U .

Proof. (1) $\Rightarrow$ (2): Let S be an h -hemiregular hemiring, f_S and g_S an SI -right h -ideal and an SI -left h -ideal of S over U , respectively. By Lemma 4.5, we have $f_S \circ_h g_S \subseteq f_S \widetilde{\cap} g_S$. Let $x \in S$, then there exist $a, a', z \in S$ such that $x + xax + z = xa'x + z$ since S is h -hemiregular. Thus, we have

$$\begin{aligned}
 (f_S \circ_h g_S)(x) &= \bigcup_{x+a_1b_1+z=a_2b_2+z} (f_S(a_1) \cap f_S(a_2) \cap g_S(b_1) \cap g_S(b_2)) \\
 &\supseteq f_S(xa) \cap f_S(xa') \cap g_S(x) \\
 &\supseteq f_S(x) \cap g_S(x) \\
 &= (f_S \cap g_S)(x),
 \end{aligned}$$

which implies $f_S \circ_h g_S \supseteq f_S \widetilde{\cap} g_S$. Thus, $f_S \circ_h g_S = f_S \widetilde{\cap} g_S$.

(2) $\implies$ (1): Let R and L be any right h -ideal and left h -ideal of S , respectively. Then, by Lemma 4.2, we have $\overline{RL} \subseteq R \cap L$. Moreover, it is easy to check that $\mathcal{S}_R$ and $\mathcal{S}_L$ are an SI -right h -ideal and an SI -left h -ideal of S over U , respectively. Let $x \in R \cap L$, then, by Proposition 4.8, we have

$$\mathcal{S}_{\overline{RL}}(x) = (\mathcal{S}_R \circ_h \mathcal{S}_L)(x) = (\mathcal{S}_R \widetilde{\cap} \mathcal{S}_L)(x) = \mathcal{S}_{R \cap L}(x) = U,$$

and so $x \in \overline{RL}$. Then, $R \cap L \subseteq \overline{RL}$. Thus, $R \cap L = \overline{RL}$. It follows from Lemma 4.3 that S is h -hemiregular. $\blacksquare$

Acknowledgements. This research is partially supported by a grant of National Natural Science Foundation of China (61175055), Natural Science Foundation of Hubei Province (2012FFB01101) and Natural Science Foundation of Education Committee of Hubei Province (D20131903).

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Accepted: 10.12.2013

CHARACTERIZATIONS OF REGULAR ABEL-GRASSMANN'S GROUPOIDS BY THE PROPERTIES OF THEIR $(\in, \in \vee q_k)$ -FUZZY IDEALS

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Abstract. In this paper, we characterize some properties of regular Abel-Grassmann's groupoid in terms of its $(\in, \in \vee q_k)$ -fuzzy ideals.

Keywords and phrases: AG-groupoid, left invertive law, medial law, paramedial law and $(\in, \in \vee q_k)$ -fuzzy ideals.

1. Introduction

Fuzzy set theory and its applications are growing day by day in various branches of Science like mathematics, computer science, engineering, physics, management sciences, medical science, operational research, artificial intelligence, robotics, expert system and various other fields of sciences. Fuzzy mappings are used in fuzzy image processing, fuzzy decision making, fuzzy linear programming and fuzzy data bases. It is used in mechanical engineering, industrial engineering, computer engineering and civil engineering. Also the uses of fuzzification can be found in fuzzy systems, genetic algorithms mechanics and economics.

Mordeson et al. [20] has discovered the grand exploration of fuzzy semigroups, where theory of fuzzy semigroups is explored along with the applications of fuzzy semigroups in fuzzy coding, fuzzy finite state mechanics and fuzzy languages and the use of fuzzification in automata and formal language has widely been explored.

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Moreover the complete l-semigroups have wide range of applications in the theories of automata, formal languages and programming. It is worth mentioning that some recent investigations of l-semigroups are closely connected with algebraic logic and non-classical logic. Fuzzy sets are also closely related to other soft computing models such as rough sets [23], random sets [7] and soft sets [5], [6], [19]. Zadeh further discussed the relationships between fuzzy set theory and probability theory [31].

An AG-groupoid is a mid structure between a groupoid and a commutative semigroup. If an AG-groupoid contains left identity then this left identity is unique. However an AG-groupoid with right identity becomes a commutative semigroup with identity. Moreover every commutative AG-groupoid becomes a commutative semigroup. Mostly an AG-groupoid works like a commutative semigroup. For instance $a^2b^2 = b^2a^2$, for all a, b holds in a commutative semigroup, while this equation also holds for an AG-groupoid with left identity e , moreover $ab = (ba)e$ for any subset $\{a, b\}$ of an AG-groupoid. Now our aim is to discover some logical investigations for regular AG-groupoids using the new generalized concept of fuzzy sets. It is therefore concluded that this research work will give a new direction for applications of fuzzy set theory particularly in algebraic logic, non-classical logics, fuzzy coding, fuzzy finite state mechanics and fuzzy languages.

In [21], Murali gave the idea of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on a fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set is defined in [27]. Bhakat and Das [1], [2], gave the idea of (α, β) -fuzzy subgroups by using the “belongs to” relation $\in$ and “quasi-coincident with” relation q between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in \vee q)$ -fuzzy subgroups, where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$ and $\alpha \neq \in \wedge q$. Davvaz defined $(\in, \in \vee q)$ -fuzzy subnearrings and ideals of a near ring in [4]. Jun and Song initiated the study of (α, β) -fuzzy interior ideals of a semigroup in [10]. In [29], regular semigroups are characterized by the properties of their $(\in, \in \vee q)$ -fuzzy ideals. In [28], semigroups are characterized by the properties of their $(\in, \in \vee q_k)$ -fuzzy ideals.

In this paper we have introduced $(\in, \in \vee q_k)$ -fuzzy ideals in a new non-associative algebraic structure, that is, in an AG-groupoid and developed some new results. We have defined a regular AG-groupoid and characterized it by the properties of its $(\in, \in \vee q_k)$ -fuzzy ideals.

2. AG-groupoids

A groupoid $(S, \cdot)$ is called AG-groupoid, if its elements satisfy left invertive law: $(ab)c = (cb)a$. In an AG-groupoid medial law [12], $(ab)(cd) = (ac)(bd)$, holds for all $a, b, c, d \in S$. It is also known that in an AG-groupoid with left identity, the paramedial law: $(ab)(cd) = (db)(ca)$, holds for all $a, b, c, d \in S$. If an AG-groupoid contains left identity, the following law holds,

$$(1) \quad a(bc) = b(ac), \text{ for all } a, b, c \in S.$$

Let S be an AG-groupoid. By AG-subgroupoid of S we mean a non-empty subset A of S such that $A^2 \subseteq A$, by a left (right) ideal of S we mean a non-empty subset L of S such that $SL \subseteq L$ ($RS \subseteq R$) and by a quasi-ideal of S we mean a non-empty subset Q of S such that $QS \cap SQ \subseteq Q$. By two-sided ideal or simply ideal, we mean a non-empty subset of S which is both a left and a right ideal of S . An AG-subgroupoid B of S is called bi-ideal of S if $(BS)B \subseteq B$. A subset B of S is called generalized bi-ideal of S if $(BS)B \subseteq B$.

A fuzzy subset f of a given set S is described as an arbitrary function $f : S \rightarrow [0, 1]$, where $[0, 1]$ is the usual closed interval of real numbers. For any two fuzzy subsets f and g of S , $f \leq g$ means that, $f(x) \leq g(x)$ for all x in S . The symbols $f \cap g$ and $f \cup g$ will mean that the following fuzzy subsets of S

$$\begin{aligned}(f \cap g)(x) &= \min\{f(x), g(x)\} = f(x) \wedge g(x) \\ (f \cup g)(x) &= \max\{f(x), g(x)\} = f(x) \vee g(x)\end{aligned}$$

for all x in S .

Let f and g be any fuzzy subsets of an AG-groupoid S , then the product $f \circ g$ is defined by

$$(f \circ g)(a) = \begin{cases} \bigvee_{a=bc} \{f(b) \wedge g(c)\}, & \text{if there exist } b, c \in S, \text{ such that } a = bc \\ 0, & \text{otherwise.} \end{cases}$$

A fuzzy subset f of an AG-groupoid S is called a fuzzy AG-subgroupoid of S if $f(xy) \geq f(x) \wedge f(y)$ for all $x, y \in S$.

A fuzzy subset f of an AG-groupoid S is called fuzzy left (right) ideal of S if $f(xy) \geq f(y)$ ($f(xy) \geq f(x)$) for all $x, y \in S$.

A fuzzy subset f of an AG-groupoid S is called fuzzy two-sided ideal of S if it is both fuzzy left and fuzzy right ideal of S .

A fuzzy AG-subgroupoid f of an AG-groupoid S is called fuzzy bi-ideal of S if $f((xy)z) \geq f(x) \wedge f(z)$, for all x, y and $z \in S$.

A fuzzy subset f of an AG-groupoid S is called fuzzy generalized bi-ideal of S if $f((xy)z) \geq f(x) \wedge f(z)$, for all x, y and $z \in S$.

A fuzzy subset f of an AG-groupoid S is called fuzzy quasi-ideal of S if $(f \circ S)(x) \wedge (S \circ f)(x) \leq f(x)$, for all $x \in S$.

Let $F(S)$ denote the collection of all fuzzy subsets of an AG-groupoid S with left identity, then $(F(S), \circ)$ becomes an AG-groupoid with left identity S , that is $(F(S), \circ)$ satisfies left invertive law, medial law, paramedial law and property (1). Note that S can be considered as a fuzzy subset of S itself and we write $S = C_S$, that is, $S(x) = 1$ for all $x \in S$. Moreover $S \circ S = S$.

The characteristic function C_A for a subset A of an AG-groupoid S is defined by

$$C_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

The proof of the following three lemmas are the same as in [20].

Definition 1 A fuzzy subset f of an AG-groupoid S is called an $(\in, \in \vee q)$ -fuzzy AG-subgroupoid of S if

$$x_t \in f, y_r \in f \Rightarrow (xy)_{t \wedge r} \in \vee q f$$

for all $x, y \in S$ and $t, r \in (0, 1]$.

Theorem 1 Let f be a fuzzy subset of S . Then f is an $(\in, \in \vee q)$ -fuzzy AG-subgroupoid of S if and only if $f(xy) \geq \min\{f(x), f(y), \frac{1-k}{2}\}$ for all $x, y \in S$.

Proof. It is similar to the proof of Theorem 12 in [28]. ■

Definition 2 A fuzzy subset f of an AG-groupoid S is called an $(\in, \in \vee q)$ -fuzzy left (resp. right) ideal of S if it satisfies the following condition:

$$y_t \in f \Rightarrow (xy)_t \in \vee q f \quad (\text{resp. } y_t \in f \Rightarrow (yx)_t \in \vee q f)$$

for all $x, y \in S$ and $t \in (0, 1]$.

A fuzzy subset f of S is called an $(\in, \in \vee q)$ -fuzzy ideal of S if it is both an $(\in, \in \vee q)$ -fuzzy left ideal and an $(\in, \in \vee q)$ -fuzzy right ideal of S .

Theorem 2 A fuzzy subset f of S is an $(\in, \in \vee q)$ -fuzzy left (resp. right) ideal of S , if and only if

$$f(xy) \geq \min\left\{f(y), \frac{1-k}{2}\right\} \quad \left(\text{resp. } f(xy) \geq \min\left\{f(x), \frac{1-k}{2}\right\}\right).$$

Proof. It is similar to the proof of Lemma 5 in [28]. ■

Definition 3 Let S be an AG-groupoid, and f be a fuzzy subset of S . Then f is an $(\in, \in \vee q)$ -fuzzy generalized bi-ideal of S , if for all $x, y, z \in S$ and $t, r \in (0, 1]$, we have

$$x_t \in f, z_r \in f \Rightarrow ((xy)z)_{t \wedge r} \in \vee q f.$$

An $(\in, \in \vee q)$ -fuzzy generalized bi-ideal of S is called an $(\in, \in \vee q)$ -fuzzy bi-ideal of S if it is also an $(\in, \in \vee q)$ -fuzzy AG-subgroupoid of S .

Definition 4 A fuzzy subset f of an AG-groupoid S is called an $(\in, \in \vee q)$ -fuzzy quasi-ideal of S if it satisfies $f(x) \geq \min(f \circ C_S(x), C_S \circ f(x), \frac{1-k}{2})$, where C_S is the fuzzy subset of S mapping every element of S on 1.

Theorem 3 Let f be a fuzzy subset of an AG-groupoid S . Then f is an $(\in, \in \vee q)$ -fuzzy bi-ideal of S if and only if it satisfies:

- (i) $f(xy) \geq \min\{f(x), f(y), \frac{1-k}{2}\}$, for all $x, y \in S$.
- (ii) $f((xy)z) \geq \min\{f(x), f(z), \frac{1-k}{2}\}$, for all $x, y, z \in S$.

Definition 5 An element a of an AG-groupoid S is called *regular* if there exist x in S such that $a = (ax)a$ and S is called *regular*, if every element of S is regular.

Definition 6 An element a of an AG-groupoid S is called *intra-regular* if there exist $x, y \in S$ such that $a = (xa^2)y$ and S is called *intra-regular*, if every element of S is intra-regular.

Lemma 1 For an AG-groupoid S , the following holds.

- (i) A non empty subset I of AG-groupoid S is an ideal if and only if $(C_I)_k$ is $(\in, \in \vee q_k)$ -fuzzy ideal.
- (ii) A non empty subset L of AG-groupoid S is left ideal if and only if $(C_L)_k$ is $(\in, \in \vee q_k)$ -fuzzy left ideal.
- (iii) A non empty subset R of AG-groupoid S is right ideal if and only if $(C_R)_k$ is $(\in, \in \vee q_k)$ -fuzzy right ideal.
- (iv) A non empty subset B of AG-groupoid S is bi-ideal if and only if $(C_B)_k$ is $(\in, \in \vee q_k)$ -fuzzy bi-ideal.
- (v) A non empty subset Q of AG-groupoid S is quasi-ideal if and only if $(C_Q)_k$ is $(\in, \in \vee q_k)$ -fuzzy quasi-ideal.

Lemma 2 Let A, B be non empty subsets of an AG-groupoid S . Then the following holds.

- (i) $(C_{A \cap B})_k = (C_A \wedge_k C_B)$.
- (ii) $(C_{A \cup B})_k = (C_A \vee_k C_B)$.
- (iii) $(C_{AB})_k = (C_A \circ_k C_B)$.

Lemma 3 Let S be an AG-groupoid. If $a = a(ax)$, for some x in S . Then $a = a^2y$, for some y in S .

Proof. Using the medial law, we get

$$a = a(ax) = [a(ax)](ax) = (aa)((ax)x) = a^2y, \text{ where } y = (ax)x. \quad \blacksquare$$

Lemma 4 Let S be an AG-groupoid with left identity. If $a = a^2x$, for some x in S . Then $a = (ay)a$, for some y in S .

Proof. Using the medial law, the left invertive law, (1), the paramedial law and the medial law, we get

$$\begin{aligned} a &= a^2x = (aa)x = ((a^2x)(a^2x))x = ((a^2a^2)(xx))x = (xx^2)(a^2a^2) \\ &= a^2((xx^2)a^2) = ((xx^2)a^2)a = ((aa^2)(xx^2))a = ((x^2x)(a^2a))a \\ &= [a^2\{(x^2x)a\}]a = [\{a(x^2x)\}(aa)]a = [a(\{a(x^2x)\}a)]a \\ &= (ay)a, \text{ where } y = \{a(x^2x)\}a. \end{aligned}$$

■

Lemma 5 *In AG-groupoid S , with left identity, the following holds.*

- (i) $(aS)a^2 = (aS)a$.
- (ii) $(aS)((aS)a) = (aS)a$.
- (iii) $S((aS)a) = (aS)a$.
- (iv) $(Sa)(aS) = a(aS)$.
- (v) $(aS)(Sa) = (aS)a$.
- (vi) $[a(aS)]S = (aS)a$.
- (vii) $[(Sa)S](Sa) = (aS)(Sa)$.
- (viii) $(Sa)S = (aS)$.
- (ix) $S(Sa) = Sa$.
- (x) $Sa^2 = a^2S$.

Proof. It is easy. ■

Lemma 6 *Every intra-regular AG-groupoid with left identity is regular but the converse is not true.*

Proof. It is easy. ■

For the converse of Lemma 6, see the following example.

Example 1 Let us consider an AG-groupoid $S = \{1, 2, 3\}$ in the following multiplication table.

$\circ$	1	2	3
1	1	1	1
2	1	1	3
3	1	2	1

It is easy to check that $\{1, 2\}$ is the quasi-ideal of S . Clearly S is regular because $1 = 1 \circ 1$, $2 = (2 \circ 3) \circ 2$ and $3 = (3 \circ 2) \circ 3$. But is not intra-regular AG-groupoid.

Let us define a fuzzy subset f on S as follows:

$$f(x) = \begin{cases} 0.9 & \text{for } x = 1 \\ 0.8 & \text{for } x = 2 \\ 0.6 & \text{for } x = 3 \end{cases}$$

Then, clearly, f is an $(\in, \in \vee q_k)$ -fuzzy ideal of S .

Theorem 4 *For an AG-groupoid S , with left identity, the following are equivalent.*

- (i) S is regular.
- (ii) For bi-ideal B , ideal I and left ideal L of S , $B \cap I \cap L \subseteq (BI)L$.
- (iii) $B[a] \cap I[a] \cap L[a] \subseteq (B[a]I[a])L[a]$, for some a in S .

Proof. (i) $\Rightarrow$ (ii) Assume that B , I and L are bi-ideal, ideal and left ideal of a regular AG-groupoid S respectively. Let $a \in B \cap I \cap L$. This implies that $a \in I$, $a \in B$ and $a \in L$. Since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law and (1), we have, $a = (ax)a = (((ax)a)x)a = ((xa)(ax))a = (a((xa)x))a = (B((SI)S))L = (BI)L$. Thus, $B \cap I \cap L \subseteq (BI)L$.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i) $B[a] = a \cup a^2 \cup (aS)a$, $I[a] = a \cup Sa \cup aS$ and $L[a] = a \cup Sa$ are principle bi-ideal, principle ideal and principle left ideal of S generated by a respectively. Thus, by (iii), Lemma 5, (1), left invertive law and paramedial law we have,

$$\begin{aligned} & (a \cup a^2 \cup (aS)a) \cap (a \cup Sa \cup aS) \cap (a \cup Sa) \\ & \subseteq ((a \cup a^2 \cup (aS)a)(a \cup Sa \cup aS))(a \cup Sa) \\ & \subseteq \{S(a \cup Sa \cup aS)\}(a \cup Sa) \\ & \subseteq \{Sa \cup S(Sa) \cup S(aS)\}(a \cup Sa) \\ & = (Sa \cup aS)(a \cup Sa) \\ & = (Sa)a \cup (Sa)(Sa) \cup (aS)a \cup (aS)(Sa) \\ & = a^2S \cup a^2S \cup (aS)a \cup (aS)a \\ & = a^2S \cup (aS)a. \end{aligned}$$

Hence S is regular. ■

Theorem 5 For an AG-groupoid S , with left identity, the following are equivalent.

- (i) S is regular.
- (ii) For $(\in, \in \vee q_k)$ -fuzzy bi-ideal f , $(\in, \in \vee q_k)$ -fuzzy ideal g , and $(\in, \in \vee q_k)$ -fuzzy left ideal h of S , $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.
- (iii) For $(\in, \in \vee q_k)$ -fuzzy generalized bi-ideal f , $(\in, \in \vee q_k)$ -fuzzy ideal g , and $(\in, \in \vee q_k)$ -fuzzy left ideal h of S , $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.

Proof. (i) $\Rightarrow$ (iii) Assume that f , g and h are $(\in, \in \vee q_k)$ -fuzzy generalized bi-ideal, $(\in, \in \vee q_k)$ -fuzzy ideal and $(\in, \in \vee q_k)$ -fuzzy left ideal of a regular AG-groupoid S , respectively. Now since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law and (1), we have, $a = (ax)a = (((ax)a)x)a = ((xa)(ax))a = (a((xa)x))a$. Thus,

$$\begin{aligned} ((f \circ_k g) \circ_k h)(a) &= \bigvee_{a=pq} (f \circ_k g)(p) \wedge h(q) \wedge \frac{1-k}{2} \\ &= \bigvee_{a=pq} \left(\left\{ \bigvee_{p=uv} f(u) \wedge g(v) \wedge \frac{1-k}{2} \right\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\ &= \bigvee_{a=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge h(q) \wedge \frac{1-k}{2} \right) \end{aligned}$$

$$\begin{aligned}
&= \bigvee_{a=(a((xa)x))a=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
&\geq \{f(a) \wedge g((xa)x)\} \wedge h(a) \wedge \frac{1-k}{2} \\
&\geq \left\{ f(a) \wedge \left(g(a) \wedge \frac{1-k}{2} \right) \right\} \wedge h(a) \wedge \frac{1-k}{2} \\
&= \{f(a) \wedge g(a) \wedge \frac{1-k}{2}\} \wedge h(a) \wedge \frac{1-k}{2} \\
&= ((f \wedge_k g) \wedge_k h)(a).
\end{aligned}$$

Therefore $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.

(iii) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i) Assume that B , I and L are bi-ideal, ideal and left ideal of S respectively. Then, by Lemma 1, $(C_B)_k$, $(C_I)_k$ and $(C_L)_k$ are $(\in, \in \vee q_k)$ -fuzzy bi-ideal, $(\in, \in \vee q_k)$ -fuzzy ideal and $(\in, \in \vee q_k)$ -fuzzy left ideal of S respectively. Therefore, by Lemma 2, we have, $(C_{B \cap I \cup L})_k = (C_B \wedge_k C_I) \wedge_k C_L \leq (C_B \circ_k C_I) \circ_k C_L = (C_{(BI)L})_k = (C_{(BI)L})_k$. Therefore $B \cap I \cap L \subseteq (BI)L$. Hence, by Theorem 4, S is regular. ■

Theorem 6 For an AG-groupoid S , with left identity, the following are equivalent.

- (i) S is regular.
- (ii) For left ideal L , ideal I and quasi-ideal Q of S , $L \cap I \cap Q \subseteq (LI)Q$.
- (iii) $L[a] \cap I[a] \cap Q[a] \subseteq (L[a]I[a])Q[a]$, for some a in S .

Proof. (i) $\Rightarrow$ (ii) Assume that L , I and Q are left ideal, ideal and quasi-ideal of regular AG-groupoid S . Let $a \in L \cap I \cap Q$. This implies that $a \in L$, $a \in I$ and $a \in Q$. Now since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law and (1), we have, $a = (ax)a = (((ax)a)x)a = ((xa)(ax))a = (a((xa)x))a \in (L((SI)S))Q \subseteq (LI)Q$. Thus $L \cap I \cap Q \subseteq (LI)Q$.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i) $L[a] = a \cup Sa$, $I[a] = a \cup Sa \cup aS$ and $Q[a] = a \cup (Sa \cap aS)$ are left ideal, ideal and quasi-ideal of S generated a respectively. Thus, by (iii), Lemma 5 and medial law we have,

$$\begin{aligned}
(a \cup Sa) \cap (a \cup Sa \cup aS) \cap (a \cup (Sa \cap aS)) &\subseteq ((a \cup Sa)(a \cup Sa \cup aS)) \\
&\quad (a \cup (Sa \cap aS)) \\
&\subseteq \{(a \cup Sa)S\}(a \cup aS) \\
&= \{aS \cup (Sa)S\}(a \cup aS) \\
&= (aS)(a \cup aS) \\
&= (aS)a \cup (aS)(aS) \\
&= (aS)a \cup a^2S.
\end{aligned}$$

Hence S is regular. ■

Theorem 7 For an AG-groupoid S , with left identity, the following are equivalent.

- (i) S is regular.
- (ii) For $(\in, \in \vee q_k)$ -fuzzy left ideal f , $(\in, \in \vee q_k)$ -fuzzy ideal g , and $(\in, \in \vee q_k)$ -fuzzy quasi-ideal h of S , $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.

Proof. (i) $\Rightarrow$ (ii) Assume that f , g and h are $(\in, \in \vee q_k)$ -fuzzy left ideal, $(\in, \in \vee q_k)$ -fuzzy ideal and $(\in, \in \vee q_k)$ -fuzzy quasi-ideal of a regular AG-groupoid S , respectively. Now, since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law and (1), we have, $a = (ax) a = (((ax) a) x) a = ((xa) (ax)) a = (a ((xa) x)) a$. Thus,

$$\begin{aligned}
 ((f \circ_k g) \circ_k h)(a) &= \bigvee_{a=pq} (f \circ_k g)(p) \wedge h(q) \wedge \frac{1-k}{2} \\
 &= \bigvee_{a=pq} \left(\left\{ \bigvee_{p=uv} f(u) \wedge g(v) \wedge \frac{1-k}{2} \right\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
 &= \bigvee_{a=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
 &= \bigvee_{a=(a((xa)x))a=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
 &\geq \{f(a) \wedge g((xa) x)\} \wedge h(a) \wedge \frac{1-k}{2} \\
 &\geq \left\{ f(a) \wedge \left(g(a) \wedge \frac{1-k}{2} \right) \right\} \wedge h(a) \wedge \frac{1-k}{2} \\
 &= \{f(a) \wedge g(a) \wedge \frac{1-k}{2}\} \wedge h(a) \wedge \frac{1-k}{2} \\
 &= ((f \wedge_k g) \wedge_k h)(a).
 \end{aligned}$$

Therefore, $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.

(ii) $\Rightarrow$ (i) Assume that L , I and Q are left ideal, ideal and quasi-ideal of S respectively. Thus, by Lemma 1, $(C_L)_k$, $(C_I)_k$ and $(C_Q)_k$ are $(\in, \in \vee q_k)$ -fuzzy left ideal, $(\in, \in \vee q_k)$ -fuzzy ideal and $(\in, \in \vee q_k)$ -fuzzy quasi-ideal of S respectively. Therefore, by Lemma 2, we have, $(C_{L \cap I \cap Q})_k = (C_L \wedge_k C_I) \wedge_k C_Q \leq (C_L \circ_k C_I) \circ_k C_Q = (C_{(LI)Q})_k = (C_{(LI)Q})_k$. Therefore $L \cap I \cap Q \subseteq (LI)Q$. Hence by Theorem 6, S is regular. ■

Theorem 8 For an AG-groupoid S , with left identity, the following are equivalent.

- (i) S is regular.
- (ii) For bi-ideal B , ideal I and quasi-ideal Q of S , $B \cap I \cap Q \subseteq (BI)Q$.
- (iii) $B[a] \cap I[a] \cap Q[a] \subseteq (B[a]I[a])Q[a]$, for some a in S .

Proof. (i) $\Rightarrow$ (ii) Assume that B , I and Q are bi-ideal, ideal and quasi-ideal of regular AG-groupoid S . Let $a \in B \cap I \cap Q$. This implies that $a \in B$, $a \in I$ and $a \in Q$. Now, since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law and (1), we have, $a = (ax)a = (((ax)a)x)a = ((xa)(ax))a = (a((xa)x))a \in (B((SI)S))Q \subseteq (BI)Q$. Thus, $B \cap I \cap Q \subseteq (BI)Q$.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i) Since $B[a] = a \cup a^2 \cup (aS)a$, $I[a] = a \cup Sa \cup aS$ and $Q[a] = a \cup (Sa \cap aS)$ are principle bi-ideal, principle ideal and principle quasi-ideal of S generated by a respectively. Thus, by (ii) and Lemma 5, (1), medial law and left invertive law, we have,

$$\begin{aligned}
 & (a \cup a^2 \cup (aS)a) \cap (a \cup Sa \cup aS) \cap (a \cup (Sa \cap aS)) \\
 & \subseteq ((a \cup a^2 \cup (aS)a)(a \cup Sa \cup aS))(a \cup (Sa \cap aS)) \\
 & \subseteq (S(a \cup Sa \cup aS))(a \cup aS) \\
 & = (Sa \cup S(Sa) \cup S(aS))(a \cup aS) \\
 & = (Sa \cup S(Sa) \cup S(aS))(a \cup aS) \\
 & = (aS \cup Sa)(a \cup aS) \\
 & = (aS)a \cup (aS)(aS) \cup (Sa)a \cup (Sa)(aS) \\
 & = (aS)a \cup a^2S \cup a(aS).
 \end{aligned}$$

Hence S is regular. ■

Theorem 9 For an AG-groupoid S , with left identity, the following are equivalent.

- (i) S is regular.
- (ii) For $(\in, \in \vee q_k)$ -fuzzy bi-ideal f , $(\in, \in \vee q_k)$ -fuzzy ideal g , and $(\in, \in \vee q_k)$ -fuzzy quasi ideal h of S , $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.
- (iii) For $(\in, \in \vee q_k)$ -fuzzy generalized bi-ideal f , $(\in, \in \vee q_k)$ -fuzzy ideal g , and $(\in, \in \vee q_k)$ -fuzzy quasi ideal h of S , $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.

Proof. (i) $\Rightarrow$ (iii) Assume that f , g and h are $(\in, \in \vee q_k)$ -fuzzy generalized bi-ideal, $(\in, \in \vee q_k)$ -fuzzy ideal and $(\in, \in \vee q_k)$ -fuzzy quasi ideal of a regular AG-groupoid S , respectively. Now, since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law and (1), we have, $a = (ax)a = (((ax)a)x)a = ((xa)(ax))a = (a((xa)x))a$. Thus,

$$\begin{aligned}
 ((f \circ_k g) \circ_k h)(a) &= \bigvee_{a=pq} (f \circ_k g)(p) \wedge h(q) \wedge \frac{1-k}{2} \\
 &= \bigvee_{a=pq} \left(\left\{ \bigvee_{p=uv} f(u) \wedge g(v) \wedge \frac{1-k}{2} \right\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
 &= \bigvee_{a=(uv)q} (\{f(u) \wedge g(v)\} \wedge h(q) \wedge \frac{1-k}{2}) \\
 &= \bigvee_{a=(a((xa)x))a=(uv)q} (\{f(u) \wedge g(v)\} \wedge h(q) \wedge \frac{1-k}{2})
 \end{aligned}$$

$$\begin{aligned}
 &\geq \{f(a) \wedge g((xa)x)\} \wedge h(a) \wedge \frac{1-k}{2} \\
 &\geq \{f(a) \wedge (g(a) \wedge \frac{1-k}{2})\} \wedge h(a) \wedge \frac{1-k}{2} \\
 &= \{f(a) \wedge g(a) \wedge \frac{1-k}{2}\} \wedge h(a) \wedge \frac{1-k}{2} \\
 &= ((f \wedge_k g) \wedge_k h)(a).
 \end{aligned}$$

Therefore, $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.

(iii) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i) Assume that B , I and Q be bi-ideal, ideal and quasi-ideal of S respectively. Then, by Lemma 1, $(C_B)_k$, $(C_I)_k$ and $(C_Q)_k$ are $(\in, \in \vee q_k)$ -fuzzy bi-ideal, $(\in, \in \vee q_k)$ -fuzzy ideal and $(\in, \in \vee q_k)$ -fuzzy quasi-ideal of S respectively. Therefore, by Lemma 2, we have, $(C_{B \cap I \cup Q})_k = (C_B \wedge_k C_I) \wedge_k C_Q \leq (C_B \circ_k C_I) \circ_k C_Q = (C_{(BI)Q})_k = (C_{(BI)Q})_k$. Therefore $B \cap I \cap Q \subseteq (BI)Q$. Hence, by Theorem 8, S is regular. ■

Theorem 10 For an AG-groupoid S , with left identity, the following are equivalent.

- (i) S is regular.
- (ii) For an ideals I_1 , I_2 and I_3 of S , $I_1 \cap I_2 \cap I_3 \subseteq (I_1 I_2) I_3$.
- (iii) $I[a] \cap I[a] \cap I[a] \subseteq (I[a] I[a]) I[a]$, for some a in S .

Proof. (i) $\Rightarrow$ (ii) Assume that I_1 , I_2 , and I_3 are ideals of a regular AG-groupoid S . Let $a \in I_1 \cap I_2 \cap I_3$. This implies that $a \in I_1$, $a \in I_2$ and $a \in I_3$. Now, since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law and (1), we have, $a = (ax)a = ((ax)a)x = ((xa)(ax))a = (a((xa)x))a \in (I_1((SI_2)S))I_3 \subseteq (I_1 I_2) I_3$. Thus, $I_1 \cap I_2 \cap I_3 \subseteq (I_1 I_2) I_3$.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i)

Since $I[a] = a \cup Sa \cup aS$ is a principle ideal of S generated by a . Thus, by (iii), Lemma 5, left invertive law, medial law and paramedial law we have,

$$\begin{aligned}
 &(a \cup Sa \cup aS) \cap (a \cup Sa \cup aS) \cap (a \cup Sa \cup aS) \\
 &\subseteq ((a \cup Sa \cup aS)(a \cup Sa \cup aS)) \\
 &\quad (a \cup Sa \cup aS) \\
 &\subseteq \{(a \cup Sa \cup aS)S\}(a \cup Sa \cup aS) \\
 &= \{aS \cup (Sa)S \cup (aS)S\}(a \cup Sa \cup aS) \\
 &= \{aS \cup Sa\}(a \cup Sa \cup aS) \\
 &= (aS)a \cup (aS)(Sa) \cup (aS)(aS) \cup (Sa)a \\
 &\quad \cup (Sa)(Sa) \cup (Sa)(aS) \\
 &= (aS)a \cup a^2 S.
 \end{aligned}$$

Hence S is regular. ■

Theorem 11 *For an AG-groupoid S , with left identity, the following are equivalent.*

- (i) S is regular.
- (ii) For quasi-ideals Q_1, Q_2 and ideal I of S , $Q_1 \cap I \cap Q_2 \subseteq (Q_1 I) Q_2$.
- (iii) $Q[a] \cap I[a] \cap Q[a] \subseteq (Q[a] I[a]) Q[a]$, for some a in S .

Proof. (i) $\Rightarrow$ (ii) Assume that Q_1 and Q are quasi-ideal and I is an ideal of a regular AG-groupoid S . Let $a \in Q_1 \cap I \cap Q_2$. This implies that $a \in Q_1$, $a \in I$ and $a \in Q_2$. Now, since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law and (1), we have, $a = (ax)a = (((ax)a)x)a = ((xa)(ax))a = (a((xa)x))a \in (Q_1((SI)S))Q_2 \subseteq (Q_1 I)Q_2$. Thus, $Q_1 \cap I \cap Q_2 \subseteq (Q_1 I)Q_2$.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i) $Q[a] = a \cup (Sa \cap aS)$ and $I[a] = a \cup Sa \cup aS$ are principle quasi-ideal and principle ideal of S generated by a respectively. Thus by (iii), left invertive law, medial law and Lemma 5, we have,

$$\begin{aligned}
 & (a \cup (Sa \cap aS)) \cap (a \cup Sa \cup aS) \cap (a \cup (Sa \cap aS)) \\
 & \subseteq ((a \cup (Sa \cap aS)) (a \cup Sa \cup aS)) \\
 & (a \cup (Sa \cap aS)) \\
 & \subseteq \{(a \cup aS) S\} (a \cap aS) \\
 & = \{aS \cup (aS) S\} (a \cap aS) \\
 & = (aS \cup Sa) (a \cap aS) \\
 & = \{(aS) a \cup (aS) (aS) \cup (Sa) a \cup (Sa) aSa \\
 & = (aS) a \cup a^2 S \cup a (aS) \}.
 \end{aligned}$$

Hence, S is regular. ■

Theorem 12 *For an AG-groupoid S , with left identity, the following are equivalent.*

- (i) S is regular.
- (ii) For $(\in, \in \vee q_k)$ -fuzzy quasi-ideals f, h , and $(\in, \in \vee q_k)$ -fuzzy ideal g , of S , $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.

Proof. (i) $\Rightarrow$ (ii) Assume that f, h are $(\in, \in \vee q_k)$ -fuzzy quasi-ideal and g is $(\in, \in \vee q_k)$ -fuzzy ideal of a regular AG-groupoid S , respectively. Now since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law and (1), we have, $a = (ax)a = (((ax)a)x)a = ((xa)(ax))a = (a((xa)x))a$. Thus,

$$\begin{aligned}
 ((f \circ_k g) \circ_k h)(a) &= \bigvee_{a=pq} (f \circ_k g)(p) \wedge h(q) \wedge \frac{1-k}{2} \\
 &= \bigvee_{a=pq} \left(\left\{ \bigvee_{p=uv} f(u) \wedge g(v) \wedge \frac{1-k}{2} \right\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
 &= \bigvee_{a=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
 &= \bigvee_{a=(a((xa)x))a=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
 &\geq \{f(a) \wedge g((xa)x)\} \wedge h(a) \wedge \frac{1-k}{2} \\
 &\geq \left\{ f(a) \wedge \left(g(a) \wedge \frac{1-k}{2} \right) \right\} \wedge h(a) \wedge \frac{1-k}{2} \\
 &= \{f(a) \wedge g(a) \wedge \frac{1-k}{2}\} \wedge h(a) \wedge \frac{1-k}{2} \\
 &= ((f \wedge_k g) \wedge_k h)(a).
 \end{aligned}$$

Therefore, $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.

(ii) $\implies$ (i) Assume that Q_1 and Q_2 are quasi-ideals and I is an ideal of S respectively. Thus, by Lemma 1, $(C_{Q_1})_k$, $(C_I)_k$ and $(C_{Q_2})_k$ are $(\in, \in \vee q_k)$ -fuzzy quasi-ideal, $(\in, \in \vee q_k)$ -fuzzy ideal and $(\in, \in \vee q_k)$ -fuzzy quasi-ideal of S respectively. Therefore, by Lemma 2, we have,

$$(C_{Q_1 \cap I \cup Q_2})_k = (C_{Q_1} \wedge_k C_I) \wedge_k C_{Q_2} \leq (C_{Q_1} \circ_k C_L) \circ_k C_{Q_2} = (C_{(Q_1 I) Q_2})_k = (C_{(Q_1 I) Q_2})_k.$$

Therefore $Q_1 \cap I \cup Q_2 \subseteq (Q_1 I) Q_2$. Hence, by Theorem 11, S is regular. ■

Theorem 13 For an AG-groupoid S with left identity, the following are equivalent.

- (i) S is regular.
- (ii) For bi-ideal B , $B = (BS) B$.
- (iii) For generalized bi-ideal B , $B = (BS) B$.

Proof. The proof is straightforward. ■

Theorem 14 For an AG-groupoid S , with left identity, the following are equivalent.

- (i) S is regular.
- (ii) For $(\in, \in \vee q_k)$ -fuzzy bi-ideal f , of S , $f_k = (f \circ_k S) \circ_k f$.
- (iii) For $(\in, \in \vee q_k)$ -fuzzy generalized bi-ideal f , of S , $f_k = (f \circ_k S) \circ_k f$.

Proof. (i) $\Rightarrow$ (iii) Assume that f is $(\in, \in \vee q_k)$ -fuzzy generalized bi-ideal of a regular AG-groupoid S . Since S is regular so for $b \in S$ there exist $x \in S$ such that $b = (bx)b$. Therefore, we have,

$$\begin{aligned}
 ((f \circ_k S) \circ_k f)(b) &= \bigvee_{b=pq} (f \circ_k S)(p) \wedge f(q) \wedge \frac{1-k}{2} \\
 &= \bigvee_{b=pq} \left(\left\{ \bigvee_{p=uv} f(u) \wedge S(v) \wedge \frac{1-k}{2} \right\} \wedge f(q) \wedge \frac{1-k}{2} \right) \\
 &= \bigvee_{b=(uv)q} \left(\{f(u) \wedge S(v)\} \wedge f(q) \wedge \frac{1-k}{2} \right) \\
 &= \bigvee_{b=(bx)b=(uv)q} \left(\{f(u) \wedge S(v)\} \wedge f(q) \wedge \frac{1-k}{2} \right) \\
 &\geq \{f(b) \wedge S(x)\} \wedge f(b) \wedge \frac{1-k}{2} \\
 &\geq f(b) \wedge 1 \wedge f(b) \wedge \frac{1-k}{2} \\
 &= f(b) \wedge \frac{1-k}{2} = f_k(b).
 \end{aligned}$$

Thus $(f \circ_k S) \circ_k f \geq f_k$. Since f is $(\in, \in \vee q_k)$ -fuzzy generalized bi-ideal of a regular AG-groupoid S . So we have,

$$\begin{aligned}
 ((f \circ_k S) \circ_k f)(b) &= \bigvee_{b=pq} (f \circ_k S)(p) \wedge f(q) \wedge \frac{1-k}{2} \\
 &= \bigvee_{b=pq} \left(\left\{ \bigvee_{p=uv} f(u) \wedge S(v) \wedge \frac{1-k}{2} \right\} \wedge f(q) \wedge \frac{1-k}{2} \right) \\
 &= \bigvee_{b=pq} \left(\left\{ \bigvee_{p=uv} f(u) \wedge 1 \right\} \wedge f(q) \wedge \frac{1-k}{2} \right) \\
 &= \bigvee_{b=pq} \left(\bigvee_{p=uv} f(u) \wedge f(q) \wedge \frac{1-k}{2} \right) \\
 &= \bigvee_{b=pq} \left\{ \bigvee_{p=uv} \left(f(u) \wedge f(q) \wedge \frac{1-k}{2} \right) \right\} \\
 &\leq \bigvee_{b=(uv)q} \left(f((uv)q) \wedge \frac{1-k}{2} \right) \\
 &= f(b) \wedge \frac{1-k}{2} = f_k(b).
 \end{aligned}$$

This implies that $(f \circ_k S) \circ_k f \leq f_k$. Thus $(f \circ_k S) \circ_k f = f_k$.

(iii) $\Rightarrow$ (ii) is obvious.

(ii) $\implies$ (i) Assume that B is a bi-ideal of S . Then, by Lemma 1, $(C_B)_k$, is an $(\in, \in \vee q_k)$ -fuzzy bi-ideal of S . Therefore, by (ii) and Lemma 2, we have, $(C_B)_k = (C_B \circ_k C_S) \circ_k C_B = (C_{(BS)B})_k$. Therefore $B = (BS)B$. Hence, by Theorem 13, S is regular. ■

Acknowledgements. This research is partially supported by a grant of National Natural Science Foundation of China (61175055), Natural Science Foundation of Hubei Province (2012FFB01101) and Natural Science Foundation of Education Committee of Hubei Province (D20131903).

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SECRET KEY DISTRIBUTION TECHNIQUE USING THEORY OF NUMBERS

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Abstract. A group key distribution protocol can enable members of a group to share a secret group key and use it for secret communications. In this article we review the number of share holders require to reconstruct the secret grouped among them. The confidentiality of our proposed protocol is unconditionally secure.

Key words: Key distribution, Secret Sharing.

AMS Classification: 94A60, 94A62.

1. Introduction

Secret sharing is a technique for protecting sensitive data, such as cryptographic keys. It is used to distribute a secret value to a number of parts or shares that have to be combined together to access the original value. These shares can then be given to separate parties that protect them using standard means, e.g., memorize, store in a computer or in a safe. Secret sharing is used in modern cryptography to lower the risks associated with compromised data.

A threshold scheme enables a secret to be shared among a group of ℓ members providing each member with a share. The scheme has a threshold $t + 1$ if any

subset with cardinality $t + 1$ out of the ℓ shares enables the secret to be recovered. We will use the notation $(t + 1, \ell)$ to refer to such a scheme. Ideally, in a $(t + 1)$ threshold scheme, t shares should not give any information on the secret. We will discuss later how to express this information. Sharing a secret spreads the risk of compromising the value across several parties. Standard security assumptions of secret sharing schemes state that if an adversary gains access to any number of shares lower than some defined threshold, it gains no information of the secret value. The first secret sharing schemes were proposed by Shamir [1] and Blakley [5].

Cryptography is the collection of methods and approaches for concealing information in communications from the access by uninvited or unauthorized parties. A logical art for dealing with this problem is known from early antiquity and it developed along the centuries, mostly in the frame in which two parties, say nobleman and general, or concealed lovers, communicated in written by sending each other messages which could only be understood when knowing some additional data, secret keys and the details for the procedure of encrypting and decrypting the messages algorithm. Algorithms were often assembled from a collection of useful basic ideas, known by tradition.

The group communication has been developed extensively in many applications currently. Ensuring the security of a group communication has become one of the most important issues in the development. Generally speaking, the security properties of a group communication include two basic aspects, that is, (i) the messages transmitted within the group can only be shared by authorized group members, but not by any unauthorized users and (ii) the transmitted messages must be able to be authenticated by members. The security properties mentioned previously imply that a group session key must be used by authorized group members to encrypt and authenticate the messages.

Suppose we have a large number of people, processes, or systems that want to communicate with one another in a secure fashion. Let's further add this group of people or processes or systems is not static, meaning that the individual entities may join or leave the group at any time. In such case, if the owner of the secret can fix the number of share holders, then it will be useful to distribute the shares.

This paper is organized as follows. In Section 2, we provide a technique to fix the number of share holders to reconstruct the distributed secret among the share holders and Conclusion is given in Section 3.

2. Main result

In this section, we consider a secret S and attain a protocol to identify the number of share holders, those who are having the distributed shares among them.

Theorem 2.1. *For any positive integer $k \geq 2$, there exists $k + \frac{k(k-1)}{2}$ share holders, sharing the common secret S .*

Proof. Select the secret positive integers α, β, α_i and β_i with $1 \leq i \leq k$. Define

$$M = \beta \left(\sum_{j=1}^k \beta_j^2 + \sum_{i < j} \beta_i \beta_j \right) - 1,$$

where $i, j \in \{1, 2, \dots, k\}$, $a = \alpha M + \beta$, $a_j = \alpha_j M + \beta_j$ for $j \in \{1, 2, \dots, k\}$ and

$$N = \frac{a \left(\sum_{j=1}^k a_j^2 + \sum_{i < j} a_i a_j \right) - 1}{M}.$$

Select a secret S such that $(S, N) = 1$.

Define $Y_i \equiv S a a_i^2 \pmod{N}$ for $i \in \{1, 2, \dots, k\}$.

$$X_{i,j} \equiv S a a_i a_j \pmod{N} \text{ for } i < j \text{ and } i, j \in \{1, 2, \dots, k\}.$$

Now, it is easy to observe

$$\sum_{i=1}^k Y_i + \sum_{i < j} X_{i,j} \equiv S \pmod{N}.$$

This completes the proof. ■

3. Conclusion

Cryptography was born in early ages as a skill of mental combinations put at the service of privacy and military protection. It developed along time into a highly mathematized discipline, which unites the science of concealing with the analysis of attacks into one single unit, cryptology. There have been numerous interesting attempts to use the large list of NP complete problems. This paper hardly deals the expected number of share holder in order to share a secret with positive integer $k \geq 2$.

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Accepted: 19.12.2013

SOME CHARACTERIZATIONS OF INTRA-REGULAR ABEL GRASSMANN'S GROUPOIDS

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Abstract. In this paper, we give some characterizations of a new non-associative structure, namely intra-regular AG-groupoids by the properties of its $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy subset, $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy left (right) ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy bi-ideals.

Keywords and phrases: AG-groupoid, left invertive law, medial law, paramedial law and $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy ideal.

1. Introduction

The fuzzy set theory was developed by Zadeh in 1965 [15]. This theory plays an important role in the real life problems involving uncertainties. Fuzzy set theory can be applied to several basic notations of algebras. Rosenfeld in 1971, introduced the concept of fuzzy set theory in groups [13]. Mordeson et, al. [7] have discussed the applications of fuzzy set theory in fuzzy coding, fuzzy automata and finite state machines. In today's world, many theories have been developed to deal with such uncertainties like fuzzy set theory, theory of vague sets, theory of soft ideals, theory of intuitionistic fuzzy sets and theory of rough sets. The theory of soft sets (see [4, 5]) has many applications in different fields such as the smoothness of functions, game theory, operations research, Riemann integration etc.

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Fuzzy set theory on semigroups has already been developed. In [8] Murali defined the concept of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on a fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set was defined in [10]. Bhakat and Das [1], [2] gave the concept of (α, β) -fuzzy subgroups by using the “belongs to” relation $\in$ and “quasi-coincident with” relation q between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in \vee q)$ -fuzzy subgroups, where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$ and $\alpha \neq \in \wedge q$. In [12] regular semigroups were characterized by the properties of their $(\in, \in \vee q)$ -fuzzy ideals. In [11] semigroups were characterized by the properties of their $(\in, \in \vee q)$ -fuzzy ideals.

An AG-groupoid is a mid structure between a groupoid and a commutative semigroup. Mostly it works like a commutative semigroup. For instance, $a^2b^2 = b^2a^2$, for all a, b holds in a commutative semigroup, while this equation also holds for an AG-groupoid with left identity e , moreover $ab = (ba)e$ for any subset $\{a, b\}$ of an AG-groupoid. Now our aim is to discover some logical investigations for regular and intra-regular AG-groupoids using the new generalized concept of fuzzy sets.

In this paper, we discuss the $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy bi-ideals in a new non-associative algebraic structure, that is, in AG-groupoids and develop some new results. We characterize intra-regular AG-groupoids by the properties of their $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy ideals.

2. Preliminaries

A groupoid $(S, .)$ is called an AG-groupoid (LA-semigroup in some articles [9]), if its elements satisfy left invertive law: $(ab)c = (cb)a$. In an AG-groupoid medial law [3], $(ab)(cd) = (ac)(bd)$, holds for all $a, b, c, d \in S$. It is also known that in an AG-groupoid with left identity, the paramedial law: $(ab)(cd) = (db)(ca)$, holds for all $a, b, c, d \in S$. If an AG-groupoid contains left identity, the following law holds,

$$(1) \quad a(bc) = b(ac), \text{ for all } a, b, c \in S.$$

Let S be an AG-groupoid. By AG-subgroupoid of S we mean a non-empty subset A of S such that $A^2 \subseteq A$, and by a left (right) ideal of S we mean a non-empty subset L of S such that $SL \subseteq L$ ($LS \subseteq L$). By two-sided ideals or simply ideal, we mean a non-empty subset of S which is both a left and a right ideal of S . An AG-subgroupoid B of S is called a bi-ideal of S if $(BS)B \subseteq B$. A non-empty subset B of S is called a generalized bi-ideal of S if $(BS)B \subseteq B$.

A fuzzy subset f of a given set S is described as an arbitrary function $f : S \rightarrow [0, 1]$, where $[0, 1]$ is the usual closed interval of real numbers. For any two fuzzy subsets f and g of S , $f \leq g$ means that, $f(x) \leq g(x)$ for all x in S . Let f and g be any fuzzy subsets of an AG-groupoid S , then the product $f \circ g$ is defined by

$$(f \circ g)(a) = \begin{cases} \bigvee_{a=bc} \{f(b) \wedge g(c)\} & \text{if there exist } b, c \in S, \text{ such that } a = bc, \\ 0 & \text{otherwise.} \end{cases}$$

The following definitions are available in [7].

A fuzzy subset f of an AG-groupoid S is called a fuzzy AG-subgroupoid of S if $f(xy) \geq f(x) \wedge f(y)$ for all $x, y \in S$.

A fuzzy subset f of an AG-groupoid S is called a fuzzy left (right) ideal of S if $f(xy) \geq f(y)$ ($f(xy) \geq f(x)$) for all $x, y \in S$.

A fuzzy subset f of an AG-groupoid S is called a fuzzy ideal of S if it is both fuzzy left and fuzzy right ideal of S .

A fuzzy subset f of an AG-groupoid S is called a fuzzy generalized bi-ideal of S if $f((xy)z) \geq f(x) \wedge f(z)$, for all x, y and $z \in S$.

A fuzzy generalized bi-ideal f of S is called a fuzzy bi-ideal of S if it is fuzzy AG-subgroupoid of S . Let $F(S)$ denote the collection of all fuzzy subsets of an AG-groupoid S with left identity, then $(F(S), \circ)$ becomes an AG-groupoid with left identity S , that is $(F(S), \circ)$ satisfies left invertive law, medial law, paramedial law and property (1). Note that S can be considered as a fuzzy subset of S itself and we write $S = C_S$, that is, $S(x) = 1$ for all $x \in S$. Moreover, $S \circ S = S$.

Definition 1 [15] Let X be a non-empty set. A fuzzy subset f of X is define as a mapping from X into $[0, 1]$, where $[0, 1]$ is the usual interval of real numbers. We denote by $\mathcal{F}(X)$ the set of all fuzzy subsets of X .

A fuzzy subset f of X of the form

$$f(y) = \begin{cases} r(\neq 0) & \text{if } y = x, \\ 0 & \text{otherwise,} \end{cases}$$

is said to be a fuzzy point with support x and value r and is denoted by x_r , where $r \in (0, 1]$.

3. $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy ideals in AG-groupoids

The following definitions are available in [14].

In what follows, let $\gamma, \delta \in [0, 1]$ be such that $\gamma < \delta$. For any $B \subseteq A$, we define $X_{\gamma B}^\delta$ be the fuzzy subset of X by $X_{\gamma B}^\delta(x) \geq \delta$ for all $x \in B$ and $X_{\gamma B}^\delta(x) \leq \gamma$ otherwise. Clearly, $X_{\gamma B}^\delta$ is the characteristic function of B if $\gamma = 0$ and $\delta = 1$.

For a fuzzy point x_r and a fuzzy subset f of X , we say that

- (1) $x_r \in_\gamma f$ if $f(x) \geq r > \gamma$.
- (2) $x_r q_\delta f$ if $f(x) + r > 2\delta$.
- (3) $x_r \in_\gamma \vee q_\delta f$ if $x_r \in_\gamma f$ or $x_r q_\delta f$.

Now, we introduce a new relation on $\mathcal{F}(X)$, denoted as " $\subseteq \vee q_{(\gamma, \delta)}$ ", as follows.

For any $f, g \in \mathcal{F}(X)$, by $f \subseteq \vee q_{(\gamma, \delta)} g$ we mean that $x_r \in_\gamma f$ implies $x_r \in_\gamma \vee q_\delta g$ for all $x \in X$ and $r \in (\gamma, 1]$. Moreover, f and g are said to be (γ, δ) -equal, denoted by $f =_{(\gamma, \delta)} g$, if $f \subseteq \vee q_{(\gamma, \delta)} g$ and $g \subseteq \vee q_{(\gamma, \delta)} f$.

The proofs of the following lemmas are the same as in [14].

Lemma 1 *Let f and g be fuzzy subsets of $\mathcal{F}(X)$. Then $f \subseteq \vee q_{(\gamma, \delta)} g$ if and only if $\max\{f(x), \gamma\} \geq \min\{g(x), \delta\}$ for all $x \in X$.*

Lemma 2 *Let $f, g, h \in \mathcal{F}(X)$. If $f \subseteq \vee q_{(\gamma, \delta)} g$ and $g \subseteq \vee q_{(\gamma, \delta)} h$, then $f \subseteq \vee q_{(\gamma, \delta)} h$.*

It is shown in [14] that “ $=_{(\gamma, \delta)}$ ” is an equivalence relation on $\mathcal{F}(X)$. It is also notified that $f =_{(\gamma, \delta)} g$ if and only if $\max\{\min\{f(x), \delta\}, \gamma\} = \max\{\min\{g(x), \delta\}, \gamma\}$ for all $x \in X$.

Lemma 3 *Let A, B be any non empty subsets of an AG-groupoid S with left identity. Then we have*

- (1) $A \subseteq B$ if and only if $X_{\gamma A}^\delta \subseteq \vee q_{(\gamma, \delta)} X_{\gamma B}^\delta$, where $r \in (\gamma, 1]$ and $\gamma, \delta \in [0, 1]$.
- (2) $X_{\gamma A}^\delta \cap X_{\gamma B}^\delta =_{(\gamma, \delta)} X_{\gamma(A \cap B)}^\delta$.
- (3) $X_{\gamma A}^\delta \circ X_{\gamma B}^\delta =_{(\gamma, \delta)} X_{\gamma(AB)}^\delta$.

Proof. It is the same in [14]. ■

Definition 2 A fuzzy subset f of an AG-groupoid S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy AG-subgroupoid of S if for all $x, y \in S$ and $t, s \in (\gamma, 1]$, it satisfies $x_t \in_\gamma f$, $y_s \in_\gamma f$ implies that $(xy)_{\min\{t, s\}} \in_\gamma \vee q_\delta f$.

Theorem 1 *Let f be a fuzzy subset of an AG groupoid S . Then f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy AG subgroupoid of S if and only if $\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\}$, where $\gamma, \delta \in [0, 1]$.*

Proof. Let f be a fuzzy subset of an AG-groupoid S which is $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy subgroupoid of S . Assume that there exist $x, y \in S$ and $t \in (\gamma, 1]$, such that

$$\max\{f(xy), \gamma\} < t \leq \min\{f(x), f(y), \delta\}.$$

Then $\max\{f(xy), \gamma\} < t$. This implies that $f(xy) < t$, which further implies that $(xy)_{\min t} \in_\gamma \vee q_\delta f$ and $\min\{f(x), f(y), \delta\} \geq t$. Therefore $\min\{f(x), f(y)\} \geq t \Rightarrow f(x) \geq t > \gamma, f(y) \geq t > \gamma$, implies that $x_t \in_\gamma f, y_s \in_\gamma f$. But $(xy)_{\min\{t, s\}} \in_\gamma \vee q_\delta f$ a contradiction to the definition. Hence

$$\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\} \text{ for all } x, y \in S.$$

Conversely, assume that there exist $x, y \in S$ and $t, s \in (\gamma, 1]$ such that $x_t \in_\gamma f, y_s \in_\gamma f$ but $(xy)_{\min\{t, s\}} \in_\gamma \vee q_\delta f$, then $f(x) \geq t > \gamma, f(y) \geq s > \gamma, f(xy) < \min\{f(x), f(y), \delta\}$ and $f(xy) + \min\{t, s\} \leq 2\delta$. It follows that $f(xy) < \delta$ and so $\max\{f(xy), \gamma\} < \min\{f(x), f(y), \delta\}$, this is a contradiction. Hence $x_t \in_\gamma f, y_s \in_\gamma f$ implies that $(xy)_{\min\{t, s\}} \in_\gamma \vee q_\delta f$ for all x, y in S . ■

Definition 3 A fuzzy subset f of an AG-groupoid S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy left (respt-right) ideal of S if for all $x, y \in S$ and $t, s \in (\gamma, 1]$ it satisfies $y_t \in_\gamma f$ implies that $(xy)_t \in_\gamma \vee q_\delta f$ (respt $x_t \in_\gamma f$ implies $(xy)_t \in_\gamma \vee q_\delta f$).

Example 1 Consider an AG-groupoid $S = \{1, 2, 3\}$ in the following multiplication table.

$\circ$	1	2	3
1	1	1	1
2	1	1	3
3	1	2	1

Define a fuzzy subset f on S as follows:

$$f(x) = \begin{cases} 0.41 & \text{if } x = 1, \\ 0.44 & \text{if } x = 2, \\ 0.42 & \text{if } x = 3. \end{cases}$$

Then, we have

- f is an $(\in_{0.1}, \in_{0.1} \vee q_{0.11})$ -fuzzy right ideal,
- f is not an $(\in, \in \vee q_{0.11})$ -fuzzy right ideal.

Example 2 Let $S = \{1, 2, 3\}$ and the binary operation $\circ$ be defined on S as follows:

$\circ$	1	2	3
1	2	2	2
2	2	2	2
3	1	2	2

Then, clearly, $(S, \circ)$ is an AG-groupoid. Define a fuzzy subset f on S as follows:

$$f(x) = \begin{cases} 0.3 & \text{if } x = 1, \\ 0.7 & \text{if } x = 2, \\ 0.8 & \text{if } x = 3. \end{cases}$$

Then, we have

- f is an $(\in_{0.4}, \in_{0.4} \vee q_{0.45})$ -fuzzy AG-subgroupoid of S ,
- f is not an $(\in_{0.4}, \in_{0.4} \vee q_{0.45})$ -fuzzy right ideal of S .

Example 3 Let $S = \{1, 2, 3\}$ and the binary operation " $\cdot$ " be defined on S as follows:

$\cdot$	1	2	3
1	1	1	1
2	1	1	1
3	1	2	1

Then clearly $(S, \cdot)$ is an AG-groupoid. Define a fuzzy subset f on S as follows:

$$f(x) = \begin{cases} 0.8 & \text{if } x = 1, \\ 0.6 & \text{if } x = 2, \\ 0.5 & \text{if } x = 3. \end{cases}$$

Then it is easy to see that f is an $(\in_{0.3}, \in_{0.3} \vee q_{0.4})$ -fuzzy ideal of S .

Theorem 2 A fuzzy subset f of an AG-groupoid S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy left (respt right) ideal of S if and only if

$$\max\{f(xy), \gamma\} \geq \min\{f(y), \delta\} \text{ (respt } \max\{f(xy), \gamma\} \geq \min\{f(x), \delta\}).$$

Proof. Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy left ideal of S . If exist $x, y \in S$ and $t \in (\gamma, 1]$ such that

$$\max\{f(xy), \gamma\} < t \leq \min\{f(y), \delta\}.$$

Then $\max\{f(xy), \gamma\} < t \leq \gamma$ this implies that $(xy)_t \bar{\in}_\gamma f$ which further implies that $(xy)_t \bar{\in}_{\gamma \vee q_\delta} f$. As $\min\{f(y), \delta\} \geq t > \gamma$ which implies that $f(y) \geq t > \gamma$, this implies that $y_t \in_\gamma f$. But $(xy)_t \bar{\in}_{\gamma \vee q_\delta} f$ a contradiction to the definition. Thus

$$\max\{f(xy), \gamma\} \geq \min\{f(y), \delta\}.$$

Conversely, assume that there exist $x, y \in S$ and $t, s \in (\gamma, 1]$ such that $y_s \in_\gamma f$ but $(xy)_t \bar{\in}_{\gamma \vee q_\delta} f$, then $f(y) \geq t > \gamma$, $f(xy) < \min\{f(y), \delta\}$ and $f(xy) + t \leq 2\delta$. It follows that $f(xy) < \delta$ and so $\max\{f(xy), \gamma\} < \min\{f(y), \delta\}$ which is a contradiction. Hence $y_t \in_\gamma f$ this implies that $(xy)_{\min\{t, s\}} \in_\gamma \vee q_\delta f$ (respt $x_t \in_\gamma f$ implies $(xy)_{\min\{t, s\}} \in_\gamma \vee q_\delta f$) for all x, y in S . ■

Definition 4 A fuzzy subset f of an AG-groupoid S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy bi-ideal of S if for all x, y and $z \in S$ and $t, s \in (\gamma, 1]$, the following conditions hold.

- (1) if $x_t \in_\gamma f$ and $y_s \in_\gamma f$ implies that $(xy)_{\min\{t, s\}} \in_\gamma \vee q_\delta f$.
- (2) if $x_t \in_\gamma f$ and $z_s \in_\gamma f$ implies that $((xy)z)_{\min\{t, s\}} \in_\gamma \vee q_\delta f$.

Example 4 Define a fuzzy subset f on S in example 2 as follows:

$$f(x) = \begin{cases} 0.44 & \text{if } x = 1, \\ 0.6 & \text{if } x = 2, \\ 0.7 & \text{if } x = 3. \end{cases}$$

Then, we have

- f is an $(\in_{0.4}, \in_{0.4} \vee q_{0.45})$ -fuzzy bi-ideal of S ,
- f is not an $(\in_{0.4}, \in_{0.4} \vee q_{0.45})$ -fuzzy right ideal of S .

Theorem 3 A fuzzy subset f of an AG-groupoid S is $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy bi-ideal of S if and only if

$$(I) \max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\}.$$

$$(II) \max\{f((xy)z), \gamma\} \geq \min\{f(x), f(z), \delta\}.$$

Proof. (1) $\Leftrightarrow$ (I) is the same as Theorem 1.

(2) $\Rightarrow$ (II) Assume that $x, y \in S$ and $t, s \in (\gamma, 1]$ such that

$$\max\{f((xy)z), \gamma\} < t \leq \min\{f(x), f(z), \delta\}.$$

Then $\max\{f((xy)z), \gamma\} < t$ which implies that $f((xy)z) < t$ this implies that $((xy)z)_t \in_\gamma f$ which further implies that $((xy)z)_t \in_\gamma \bigvee q_\delta f$. Also $\min\{f(x), f(z), \delta\} \geq t > \gamma$, this implies that $f(x) \geq t > \gamma$, $f(z) \geq t > \gamma$ implies that $x_t \in_\gamma f$, $z_t \in_\gamma f$. But $((xy)z)_t \in_\gamma \bigvee q_\delta f$, a contradiction. Hence

$$\max\{f((xy)z), \gamma\} \geq \min\{f(x), f(z), \delta\}.$$

(II) $\Rightarrow$ (2) Assume that x, y in S and $t, s \in (\gamma, 1]$, such that $x_t \in_\gamma f$, $z_s \in_\gamma f$ but $((xy)z)_{\min\{t,s\}} \in_\gamma \bigvee q_\delta f$, then $f(x) \geq t > \gamma$, $f(z) \geq s > \gamma$, $f((xy)z) < \min\{f(x), f(y), \delta\}$ and $f((xy)z) + \min\{t, s\} \leq 2\delta$. It follows that $f((xy)z) < \delta$ and so $\max\{f((xy)z), \gamma\} < \min\{f(x), f(y), \delta\}$, a contradiction. Hence $x_t \in_\gamma f$, $z_s \in_\gamma f$ implies that $((xy)z)_{\min\{t,s\}} \in_\gamma \bigvee q_\delta f$ for all x, y in S . ■

Lemma 4 Let f be any $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy AG-subgroupoid and g be $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy left ideal of an AG-groupoid S . Then $(f \circ g)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy left ideal of S .

Proof. Let f and g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy AG-subgroupoid an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy left ideal of an AG-groupoid S with left identity, respectively. So for any y in S , there exist a and b in S such that $y = ab$. Therefore, $xy = x(ab) = a(xb)$.

Then

$$\begin{aligned} \min\{f \circ g(y), \delta\} &= \min\left\{\bigvee_{y=ab} \{f(a) \wedge g(b)\}, \delta\right\} \\ &= \bigvee_{y=ab} \{\min\{\min\{f(a), \delta\}, \min\{g(b), \delta\}\}\} \\ &\leq \bigvee_{xy=a(xb)} \{\min\{\max\{f(a), \gamma\}, \max\{g(xb), \gamma\}\}\} \\ &= \bigvee_{xy=a(xb)} \{\max\{\min\{f(a), g(xb)\}, \gamma\}\} \\ &\leq \bigvee_{xy=ac} \{\max\{\min\{f(a), g(c)\}, \gamma\}\} \\ &= \max\{(f \circ g)(xy), \gamma\}. \end{aligned}$$

■

Corollary 1 Let f and g be $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy left ideals of an AG-groupoid S with left identity. Then $(f \circ g)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy left ideal of S .

Theorem 4 Let f and g be $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy right ideals of an AG-groupoid S with left identity. Then $(f \circ g)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy right ideal of S .

Proof. Let f and g be $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy right ideals of an AG-groupoid S . Since S is an AG-groupoid, there exist x, y in S such that $x = ab$ and $y = y_1y_2$, for some a, b, y_1, y_2 in S . Then by medial law we get

$$xy = (ab)y = (ab)(y_1y_2) = (ay_1)(by_2).$$

Then we have

$$\begin{aligned} \min\{f \circ g(x), \delta\} &= \min \left\{ \bigvee_{x=ab} \{f(a) \wedge g(b)\}, \delta \right\} \\ &= \bigvee_{x=ab} \{\min\{\min\{f(a), \delta\}, \min\{g(b), \delta\}\}\} \\ &\leq \bigvee_{xy=(ay_1)(by_2)} \{\min\{\max\{f(ay_1), \gamma\}, \max\{g(by_2), \gamma\}\}\} \\ &= \bigvee_{xy=(ay_1)(by_2)} \{\max\{\min\{f(ay_1), g(by_2)\}, \gamma\}\} \\ &\leq \bigvee_{xy=cd} \{\max\{\min\{f(c), g(d)\}, \gamma\}\} \\ &= \max\{(f \circ g)(xy), \gamma\}. \end{aligned}$$

■

Theorem 5 Let f and g be $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy bi-ideals of an AG-groupoid S . Then $(f \circ g)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy bi-ideal of S .

Proof. Assume that f and g are $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy bi-ideals of an AG-groupoid S . Consider on the contrary that there exist $a, b \in S$ and $t, s \in (\gamma, 1]$ such that

$$\max\{(f \circ g)(ab), \gamma\} < t \leq \min\{(f \circ g)(a), (f \circ g)(b), \delta\}.$$

Then $\min\{(f \circ g)(a), (f \circ g)(b), \delta\} \geq t$ implies that $(f \circ g)(a) \geq t > \gamma$, $(f \circ g)(b) \geq t > \gamma$, this implies that $a_t \in_\gamma (f \circ g)$, $b_t \in_\gamma (f \circ g)$ and $\max\{(f \circ g)(ab), \gamma\} < t$ implies that $(f \circ g)(ab) < t$ implies that $(f \circ g)(ab) + s < 2\delta$ further implies that $(ab)_t \in_{\overline{\in_\gamma \vee q_\delta}} (f \circ g)$. But $(ab)_t \notin_{\overline{\in_\gamma \vee q_\delta}} (f \circ g)$, a contradiction. Hence

$$\max\{(f \circ g)(ab), \gamma\} \geq \min\{(f \circ g)(a), (f \circ g)(b), \delta\}.$$

Assume that $\max\{(f \circ g)((ax)b), \gamma\} < t \leq \min\{(f \circ g)(a), (f \circ g)(b), \delta\}$, then $\min\{(f \circ g)(a), (f \circ g)(b), \delta\} \geq t$ implies that $(f \circ g)(a) \geq t$, $(f \circ g)(b) \geq t$, this implies that $a_t \in_\gamma (f \circ g)$, $b_t \in_\gamma (f \circ g)$ and $\max\{(f \circ g)((ax)b), \gamma\} < t$

implies that $(f \circ g)((ax)b) < t$ further implies that $((ax)b)_t \bar{\in}_\gamma (f \circ g)$ implies that $((ax)b)_t \bar{\in}_\gamma \vee q_\delta (f \circ g)$. But $((ax)b)_t \bar{\in}_\gamma \vee q_\delta (f \circ g)$, a contradiction. Hence

$$\max\{(f \circ g)((ax)b), \gamma\} \geq \min\{(f \circ g)(a), (f \circ g)(b), \delta\}.$$

Hence $(f \circ g)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy bi-ideal of S . ■

Lemma 5 *A subset B of an AG-groupoid S is a bi-ideal if and only if $X_{\gamma B}^\delta$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy bi-ideal of S .*

Proof. (i) Let B be a bi-ideal and assume that $x, y \in B$ then for any a in S we have $(xa)y \in B$, thus $X_{\gamma B}^\delta((xa)y) \geq \delta$. Now since $x, y \in B$ so $X_{\gamma B}^\delta(x) \geq \delta$, $X_{\gamma B}^\delta(y) \geq \delta$ which clearly implies that $\min\{X_{\gamma B}^\delta(x), X_{\gamma B}^\delta(y)\} \geq \delta$. Thus

$$\begin{aligned} \max\{X_{\gamma B}^\delta((xa)y), \gamma\} &= X_{\gamma B}^\delta((xa)y). \\ \min\{X_{\gamma B}^\delta(x), X_{\gamma B}^\delta(y), \delta\} &= \delta. \end{aligned}$$

Hence $\max\{X_{\gamma B}^\delta((xa)y), \gamma\} \geq \min\{X_{\gamma B}^\delta(x), X_{\gamma B}^\delta(y), \delta\}$.

(ii) Let $x \in B, y \notin B$, then $(xa)y \notin B$, for all a in S . This implies that $X_{\gamma B}^\delta((xa)y) \leq \gamma$, $X_{\gamma B}^\delta(x) \geq \delta$ and $X_{\gamma B}^\delta(y) < \gamma$. Therefore

$$\max\{X_{\gamma B}^\delta((xa)y), \gamma\} = \gamma \text{ and } \min\{X_{\gamma B}^\delta(x), X_{\gamma B}^\delta(y), \delta\} = X_{\gamma B}^\delta(y).$$

Hence $\max\{X_{\gamma B}^\delta((xa)y), \gamma\} \geq \min\{X_{\gamma B}^\delta(x), X_{\gamma B}^\delta(y), \delta\}$.

(iii) Let $x \notin B, y \in B$ implies that $(xa)y \notin B$, for all a in S . This implies that $\max\{X_{\gamma B}^\delta((xa)y), \gamma\} \leq \gamma$, $X_{\gamma B}^\delta(x) < \gamma$, $X_{\gamma B}^\delta(y) \geq \delta$ and $\min\{X_{\gamma B}^\delta(x), X_{\gamma B}^\delta(y), \delta\} = X_{\gamma B}^\delta(x)$. Therefore

$$\max\{X_{\gamma B}^\delta((xa)y), \gamma\} \leq \gamma \text{ and } \min\{X_{\gamma B}^\delta(x), X_{\gamma B}^\delta(y), \delta\} = X_{\gamma B}^\delta(x).$$

(iv) Let $x, y \notin B$ which implies that $(xa)y \notin B$, for all a in S . This implies that $\min\{X_{\gamma B}^\delta(x), X_{\gamma B}^\delta(y)\} \leq \gamma$, $X_{\gamma B}^\delta((xa)y) \leq \gamma$. Thus

$$\begin{aligned} \max\{X_{\gamma B}^\delta((xa)y), \gamma\} &= \gamma \text{ and} \\ \min\{X_{\gamma B}^\delta(x), X_{\gamma B}^\delta(y), \delta\} &\leq \min\{X_{\gamma B}^\delta(x), X_{\gamma B}^\delta(y)\} \leq \gamma. \end{aligned}$$

Hence $\max\{X_{\gamma B}^\delta((xa)y), \gamma\} \geq \min\{X_{\gamma B}^\delta(x), X_{\gamma B}^\delta(y), \delta\}$.

If $(xa)y \in B$, then $\min\{X_{\gamma B}^\delta(x), X_{\gamma B}^\delta(y)\} \leq \gamma$, $X_{\gamma B}^\delta((xa)y) \geq \delta$. Thus

$$\begin{aligned} \max\{X_{\gamma B}^\delta((xa)y), \gamma\} &= X_{\gamma B}^\delta((xa)y) \text{ and} \\ \min\{X_{\gamma B}^\delta(x), X_{\gamma B}^\delta(y), \delta\} &\leq \min\{X_{\gamma B}^\delta(x), X_{\gamma B}^\delta(y)\} \leq \gamma. \end{aligned}$$

Hence $\max\{X_{\gamma B}^\delta((xa)y), \gamma\} \geq \min\{X_{\gamma B}^\delta(x), X_{\gamma B}^\delta(y), \delta\}$.

Conversely, let $(b_1 s)b_2 \in (BS)B$, where $b_1, b_2 \in B$ and $s \in S$.

Now, by hypothesis $\max\{X_{\gamma B}^\delta((b_1 s)b_2), \gamma\} \geq \min\{X_{\gamma B}^\delta(b_1), X_{\gamma B}^\delta(b_2), \delta\}$.

Since $b_1, b_2 \in B$, therefore $X_{\gamma B}^\delta(b_1) \geq \delta$ and $X_{\gamma B}^\delta(b_2) \geq \delta$ which implies that $\min\{X_{\gamma B}^\delta(b_1), X_{\gamma B}^\delta(b_2), \delta\} = \delta$. Thus

$$\max\{X_{\gamma B}^\delta((b_1s)b_2), \gamma\} \geq \delta.$$

This clearly implies that $\max X_{\gamma B}^\delta((b_1s)b_2) \geq \delta$. Therefore $(b_1s)b_2 \in B$. Hence B is a bi-ideal of S . ■

Similarly, we can prove the following lemmas.

Lemma 6 *A subset A of an AG-groupoid S is closed if and only if $X_{\gamma A}^\delta$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy AG-subgroupoid of S .*

Lemma 7 *For any fuzzy subset f of an AG-groupoid S with left identity, $f \subseteq \vee q_{(\gamma, \delta)} S \circ f$.*

Proof. Let f be a fuzzy subset of an AG-groupoid S with left identity. Therefore, for any a in S there exist p and q in S such that $a = pq$. Thus

$$\begin{aligned} \max\{(S \circ f)(a), \gamma\} &= \max\left\{\bigvee_{a=pq} \{\{S(p) \wedge f(q)\}, \gamma\}\right\} \\ &\geq \max\{\min\{S(e) \wedge f(a)\}, \gamma\} \\ &= \max\{\min\{1 \wedge f(a)\}, \gamma\} \\ &= \max\{f(a), \gamma\} \\ &\geq \min\{f(a), \delta\}. \end{aligned}$$

Hence by Lemma 1, $f \subseteq \vee q_{(\gamma, \delta)} S \circ f$. ■

4. Intra-regular AG-groupoids

An element a of an AG-groupoid S is called **intra-regular** if there exist $x, y \in S$ such that $a = (xa^2)y$ and S is called **intra-regular**, if every element of S is intra-regular. In this section, we discuss the characterizations of intra-regular AG-groupoids.

Example 5 *Let $S = \{1, 2, 3, 4, 5, 6\}$, the following table shows that S is an intra-regular AG-groupoid.*

$\circ$	1	2	3	4	5	6
1	2	1	1	1	1	1
2	1	2	2	2	2	2
3	1	2	6	3	4	5
4	1	2	5	6	3	4
5	1	2	4	5	6	3
6	1	2	3	4	5	6

It is easy to see that $(S, \circ)$ is an AG-groupoid and is non-commutative and non-associative structure because $(3 \circ 4) \neq (4 \circ 3)$ and $(3 \circ 6) \circ 4 \neq 3 \circ (6 \circ 4)$. Also $1 = (3 \circ 1^2) \circ 1$, $2 = (2 \circ 2^2) \circ 2$, $3 = (4 \circ 3^2) \circ 5$, $4 = (4 \circ 4^2) \circ 6$, $5 = (6 \circ 5^2) \circ 5$, $6 = (6 \circ 6^2) \circ 6$. Therefore, $(S, \circ)$ is an intra-regular AG-groupoid.

Theorem 1 *Let S be an AG-groupoid with left identity, then the following conditions are equivalent.*

- (i) S is intra-regular.
- (ii) For every subset L and bi-ideal B of S , $L \cap B \subseteq LB$.
- (iii) For every $(\in_\gamma, \in_\gamma \vee q_\delta)$ fuzzy subset f and $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy bi-ideal g , $f \cap g \subseteq \vee q_{(\gamma, \delta)} f \circ g$.

Proof. (i) $\Rightarrow$ (iii) Let f and g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy subset and an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy bi-ideal of an intra-regular AG-groupoid S with left identity, respectively. Now since S is intra-regular, therefore, for any a in S there exist x, y in S such that $a = (xa^2)y$. Since $S = S^2$, so for any y there exists y_1, y_2 such that $y = y_1y_2$. Now, using (1) and the left invertive law, the medial law and the paramedial law, we get

$$\begin{aligned}
 a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a \\
 &= (y(xa))((xa^2)y) = (y(xa^2))((xa)y) = ((y_1y_2)(xa^2))((xa)(y_1y_2)) \\
 &= ((a^2x)(y_2y_1))((y_2y_1)(ax)) = ((a^2x)(y_2y_1))(a((y_2y_1)x)) \\
 &= a[\{(a^2x)(y_2y_1)\}\{(y_2y_1)x\}] = a[\{((y_2y_1)x)a^2\}((y_2y_1)x)] \\
 &= a[\{a^2(x(y_2y_1))\}\{(y_2y_1)x\}] = a[\{\{(y_2y_1)x\}(x(y_2y_1))\}a^2] \\
 &= a[ua^2], \text{ where } u = \{(y_2y_1)x\}(x(y_2y_1)).
 \end{aligned}$$

Now

$$\begin{aligned}
 ua^2 &= u[(xa^2)y]^2 = u[(xa^2)^2y^2] = (xa^2)^2[uy^2] = [x^2(a^2)^2][uy^2] \\
 &= [(a^2)^2x^2][uy^2] = [(uy^2)x^2](a^2a^2) = (a^2a^2)[x^2(uy^2)] \\
 &= ([x^2(uy^2)]a^2)a^2 = (a^2[x^2(uy^2)])a^2 \\
 &= (a^2q)a^2, \text{ where } q = x^2(uy^2).
 \end{aligned}$$

Thus $a = a((a^2q)a^2)$, where $q = x^2(uy^2)$ and $u = \{(y_2y_1)x\}(x(y_2y_1))$.

For any a in S , there exist p and q in S such that $a = pq$, then

$$\begin{aligned}
 \max\{(f \circ g)(a), \gamma\} &= \max\left\{\bigvee_{a=pq} \{\{f(p) \wedge g(q)\}, \gamma\}\right\} \\
 &\geq \max\{\min\{f(a), g((a^2q)a^2)\}, \gamma\} \\
 &= \min\{\max\{f(a), g((a^2q)a^2)\}, \gamma\} \\
 &= \min\{\max\{f(a), \gamma\}, \max\{g((a^2q)a^2), \gamma\}\} \\
 &\geq \min\{\min\{f(a), \delta\}, \min\{g(a), \delta\}\} \\
 &= \min\{(f \cap g)(a), \delta\}.
 \end{aligned}$$

Thus, by Lemma 1, $f \cap g \subseteq \vee q_{(\gamma, \delta)} f \circ g$.

(iii) $\Rightarrow$ (ii) Let L and B be any subset and bi-ideal of S , respectively. Then, by Lemma 3 and (iii), we get

$$X_{\gamma(L \cap B)}^\delta =_{(\gamma, \delta)} X_{\gamma L}^\delta \cap X_{\gamma B}^\delta \subseteq \vee q_{(\gamma, \delta)} X_{\gamma L}^\delta \circ X_{\gamma B}^\delta =_{(\gamma, \delta)} X_{\gamma LB}^\delta.$$

Hence, by Lemma 3, we get $L \cap B \subseteq LB$.

(ii) $\Rightarrow$ (i) Since Sa is an bi-ideal and any subset of S containing a . Therefore, by (ii) medial law, paramedial law and (1), we get

$$\begin{aligned} a \in Sa \cap Sa &\subseteq (Sa)(Sa) = (SS)(aa) = (a^2S)S \\ &= ((aa)(SS))S = ((SS)(aa))S = (Sa^2)S. \end{aligned}$$

Hence S is intra-regular. ■

Remark 1 If S is an intra-regular AG-groupoid with left identity, then for every $(\in_\gamma, \in_\gamma \vee q_\delta)$ fuzzy subset f of S , $f \subseteq \vee q_{(\gamma, \delta)} f \circ S$.

Theorem 2 Let S be an AG-groupoid with left identity, then the following conditions are equivalent.

- (i) S is intra-regular.
- (ii) For every bi-ideal B and any subset A of S , $B \cap A \subseteq BA$.
- (iii) For every $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy bi-ideal f and $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy subset g of S , $f \cap g \subseteq \vee q_{(\gamma, \delta)} f \circ g$.
- (iv) For every $(\in_\gamma, \in_\gamma \vee q_\delta)$ generalized fuzzy bi-ideal f and $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy subset g of S , $f \cap g \subseteq \vee q_{(\gamma, \delta)} f \circ g$.

Proof. (i) $\Rightarrow$ (iv) Let f and g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy generalized bi-ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy subset of an intra-regular AG-groupoid S with left identity, respectively. Now since S is intra-regular, therefore for any a in S , there exist x, y in S such that $a = (xa^2)y$. Now using (1), the left invertive law, the medial law and the paramedial law, we get

$$\begin{aligned} (xa^2)y &= (a(xa))y = (y(xa))a = [(y_1y_2)(xa)]a = [(ax)(y_2y_1)]a \\ &= [((y_2y_1)x)a]a = [((y_2y_1)x)((xa^2)y)]a \\ &= [(tx)((xa^2)y)]a, \text{ where } t = y_2y_1 \\ (tx)((xa^2)y) &= (xa^2)((tx)y) = (a(xa))((tx)y) \\ &= [((tx)y)(xa)]a = [(u(xa))a], \text{ where } u = (tx)y \\ u(xa) &= u[x\{(xa^2)y\}] = u[(xa^2)(xy)] = (xa^2)(u(xy)) \\ &= ((xy)u)(a^2x) = a^2[\{(xy)u\}x] = [x\{(xy)u\}]a^2 = a[\{x((xy)u)\}a] \\ &= a(pa), \text{ where } p = (x((xy)u))a. \end{aligned}$$

Thus $a = [(a(pa))a]a = [(aq)a]a$, where $q = pa$.

For any a in S , there exist p and q in S such that $a = pq$. Then

$$\begin{aligned}
 \max\{f \circ g(a), \gamma\} &= \max \left\{ \bigvee_{a=pq} \{\{f(p) \wedge g(q)\}, \gamma\} \right\} \\
 &= \max \left\{ \bigvee_{a=pq} \{\min\{f(p), g(q)\}, \gamma\} \right\} \\
 &\geq \min \{ \max\{f((aq)a), g(a), \gamma\} \\
 &= \min \{ \max\{f((aq)a), \gamma\}, \max\{g(a), \gamma\} \} \\
 &\geq \min \{ \min\{f(a), \delta\}, \min\{g(a), \delta\} \} \\
 &= \min \{ (f \cap g)(a), \delta \}.
 \end{aligned}$$

Thus, by Lemma 1, $f \cap g \subseteq \vee q_{(\gamma, \delta)} f \circ g$.

(iv) $\Rightarrow$ (iii) It is obvious.

(iii) $\Rightarrow$ (ii) Let A be any subset and B be a bi-ideal of S . Then, by Lemma 3 and (iii), we get

$$X_{\gamma(B \cap A)}^\delta =_{(\gamma, \delta)} X_{\gamma B}^\delta \cap X_{\gamma A}^\delta \subseteq \vee q_{(\gamma, \delta)} X_{\gamma B}^\delta \circ X_{\gamma A}^\delta =_{(\gamma, \delta)} X_{\gamma BA}^\delta$$

Hence, by Lemma 3, we get $B \cap A \subseteq BA$.

(ii) $\Rightarrow$ (i) Since Sa is a bi-ideal containing a , so using (ii), we get

$$a \in Sa \cap Sa = Sa^2 = (Sa^2)S.$$

Hence S is intra-regular. ■

Corollary 2 For an AG-groupoid S with left identity, the following conditions are equivalent.

- (i) S is intra-regular.
- (ii) For all bi-ideal B_1 and B_2 , $B_1 \cap B_2 \subseteq B_1 B_2$.
- (iii) $f \cap g \subseteq \vee q_{(\gamma, \delta)} f \circ g$, for all $(\in_\gamma, \in_\gamma \vee q_\delta)$ fuzzy bi-ideals f and g .
- (iv) $f \cap g \subseteq \vee q_{(\gamma, \delta)} f \circ g$, for all $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy generalized bi-ideals f and g .

Definition 5 A fuzzy subset f of an AG-groupoid S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy semiprime ideal if $x_t^2 \in_\gamma f$ implies that $x_t \in_\gamma \vee q_\delta f$ for all $x \in S$ and $t \in (\gamma, 1]$.

Example 6 Define a fuzzy subset f on S in Example 2 as given:

$$f(x) = \begin{cases} 0.3 & \text{if } x = 1, \\ 0.2 & \text{if } x = 2, \\ 0.4 & \text{if } x = 3. \end{cases}$$

Then, it is easy to see that f is an $(\in_{0.2}, \in_{0.2} \vee q_{0.3})$ -fuzzy semiprime ideal of S .

Example 7 Define a fuzzy subset f on S in Example 5 as given:

$$f(x) = \begin{cases} 0.7 & \text{if } x = 1, \\ 0.6 & \text{if } x = 2, \\ 0.68 & \text{if } x = 3, \\ 0.63 & \text{if } x = 4, \\ 0.52 & \text{if } x = 5, \\ 0.5 & \text{if } x = 6. \end{cases}$$

Then, it is easy to see that f is an $(\in_{0.4}, \in_{0.4} \vee q_{0.5})$ -fuzzy semiprime ideal of S .

Theorem 3 A fuzzy subset f of an AG-groupoid S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy semiprime ideal if and only if $\max\{f(x), \gamma\} \geq \min\{f(x^2), \delta\}$.

Proof. Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy semiprime ideal of S . If exist $x, y \in S$ and $t \in (\gamma, 1]$ such that $\max\{f(x), \gamma\} < t \leq \min\{f(x^2), \delta\}$. Then $\max\{f(x), \gamma\} < t$ implies that $x_t \bar{\in}_\gamma f$ implies that $x_t \bar{\in}_{\gamma \vee q_\delta} f$. As $\min\{f(x^2), \delta\} \geq t > \gamma$ this implies that $f(x^2) \geq t > \gamma$ implies that $x_t^2 \in_\gamma f$. But $x_t \bar{\in}_{\gamma \vee q_\delta} f$, a contradiction to the definition of semiprime ideals. Thus, we have $\max\{f(x), \gamma\} \geq \min\{f(x^2), \delta\}$.

Conversely, assume that there exist x, y in S and $t \in (\gamma, 1]$ such that $x_t^2 \in_\gamma f$ but $x_t \bar{\in}_{\gamma \vee q_\delta} f$, then $f(x^2) \geq t > \gamma$, $f(x) < \min\{f(x^2), \delta\}$ and $f(x) + t \leq 2\delta$. It follows that $f(x) < \delta$ and so $\max\{f(x), \gamma\} < \min\{f(x^2), \delta\}$ which is a contradiction to the definition of semiprime ideals. Hence, $x_t^2 \in_\gamma f$ implies that $(x^2)_t \in_\gamma \vee q_\delta f$, for all x, y in S . ■

Theorem 4 For a non empty subset I of an AG-groupoid S with left identity, the following conditions are equivalent.

- (i) I is semiprime.
- (ii) $X_{\gamma I}^\delta$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy semiprime.

Proof. (i) $\Rightarrow$ (ii) Let I be a semiprime ideal of an AG-groupoid S . Let a be any element of S such that $a \in I$, then I is an ideal, so $a^2 \in I$. Hence $X_{\gamma I}^\delta(a), X_{\gamma I}^\delta(a^2) \geq \delta$ which implies that $\max\{X_{\gamma I}^\delta(a), \gamma\} \geq \min\{X_{\gamma I}^\delta(a^2), \delta\}$.

Now, let $a \notin I$, since I is semiprime, thus $a^2 \notin I$. This implies that $X_{\gamma I}^\delta(a) \leq \gamma$ and $X_{\gamma I}^\delta(a^2) \leq \gamma$. Therefore, $\max\{X_{\gamma I}^\delta(a), \gamma\} \geq \min\{X_{\gamma I}^\delta(a^2), \delta\}$. Hence, we have $\max\{X_{\gamma I}^\delta(a), \gamma\} \geq \min\{X_{\gamma I}^\delta(a^2), \delta\}$ for all $a \in S$.

(ii) $\Rightarrow$ (i) Let $X_{\gamma I}^\delta$ is fuzzy semiprime. If $a^2 \in I$, for some a in S , this implies that $X_{\gamma I}^\delta(a^2) \geq \delta$. Now, since $X_{\gamma I}^\delta$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy semiprime. Thus, $\max\{X_{\gamma I}^\delta(a), \gamma\} \geq \min\{X_{\gamma I}^\delta(a^2), \delta\}$. Therefore $\max\{X_{\gamma I}^\delta(a), \gamma\} \geq \delta$. But $\delta > \gamma$, so $X_{\gamma I}^\delta(a) \geq \delta$. Thus, $a \in I$. Hence, I is semiprime. ■

Lemma 8 Let f be a fuzzy subset of an AG-groupoid S . Then f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy bi-ideal of S if and only if $\max\{f(a), \gamma\} \geq \min\{((f \circ S) \circ f)(a), \delta\}$.

Proof. Assume that f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy bi-ideal of an AG-groupoid S . If $a \in S$, then there exist c, d, p and q in S such that $a = pq$ and $p = cd$. Since f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy bi-ideal of S , we have $\max\{f((cd)q), \gamma\} \geq \min\{f(c), f(q), \delta\}$. Therefore,

$$\begin{aligned}
 \min\{((f \circ S) \circ f)(a), \delta\} &= \min\left\{\bigvee_{a=pq} \{(f \circ S)(p) \wedge f(q)\}, \delta\right\} \\
 &= \min\left\{\bigvee_{a=pq} \left\{\bigvee_{p=cd} \{f(c) \wedge S(d)\} \wedge f(q)\right\}, \delta\right\} \\
 &= \min\left\{\bigvee_{a=(cd)q} \{\{f(c) \wedge 1 \wedge f(q)\}, \delta\}\right\} \\
 &= \min\left\{\bigvee_{a=(cd)q} \{f(c) \wedge f(q)\}, \delta\right\} \\
 &= \bigvee_{a=(cd)q} \{\min\{f(p), f(q), \delta\}\} \\
 &\leq \bigvee_{a=(cd)q} \{\max\{f((cd)q), \gamma\}\} \\
 &= \max\{f(a), \gamma\}.
 \end{aligned}$$

Hence, $\max\{f(a), \gamma\} \geq \min\{((f \circ S) \circ f)(a), \delta\}$.

Conversely, assume that $\max\{f(a), \gamma\} \geq \min\{((f \circ S) \circ f)(a), \delta\}$. Let a in S , there exist c, d and q in S such that $a = (cd)q$. Then we have

$$\begin{aligned}
 \max\{f((cd)q), \gamma\} &= \max\{f(a), \gamma\} \\
 &\geq \min\{((f \circ S) \circ f)(a), \delta\} \\
 &= \min\left\{\bigvee_{a=bc} \{(f \circ S)(b) \wedge f(c), \delta\}\right\} \\
 &\geq \min\{\{(f \circ S)(cd) \wedge f(q)\}, \delta\} \\
 &= \min\left\{\bigvee_{cd=st} \{f(s) \wedge S(t)\} \wedge f(q), \delta\right\} \\
 &\geq \min\{\min\{f(c), f(q), \delta\}\} \\
 &= \min\{f(c), f(q), \delta\}.
 \end{aligned}$$

Hence $\max\{f((cd)q), \gamma\} \geq \min\{f(c), f(q), \delta\}$. ■

Lemma 9 Every $(\in_\gamma \in_\gamma \vee q_\delta)$ -fuzzy right ideal of an AG-groupoid S is an $(\in_\gamma \in_\gamma \vee q_\delta)$ -fuzzy bi-ideal of S .

Proof. Let f be an $(\in_\gamma \in_\gamma \vee q_\delta)$ -fuzzy right ideal of an AG-groupoid S . For any a in S , there exist p, q, s and t in S , such that

$$\begin{aligned}
 \min\{((f \circ S) \circ f)(a), \delta\} &= \min\left\{\bigvee_{a=pq} \{(f \circ S)(p) \wedge f(q)\}, \delta\right\} \\
 &= \min\left\{\bigvee_{a=pq} \left\{\bigvee_{p=st} \{f(s) \wedge S(t)\} \wedge f(q)\right\}, \delta\right\} \\
 &= \min\left\{\bigvee_{a=(st)q} \{f(s) \wedge f(q)\}, \delta\right\} \\
 &= \min\left\{\bigvee_{a=(st)q} [\min\{f(s), f(q)\}], \delta\right\} \\
 &= \bigvee_{a=(st)q} \min[\min\{f(s), \delta\}, \min\{f(q), \delta\}] \\
 &\leq \bigvee_{a=(st)q=(qt)s} \min\{\max\{f(st)q, \gamma\}, \max\{f(qt)s, \gamma\}\} \\
 &= \min\{\max\{f(a), \gamma\}, \max\{f(a), \gamma\}\} \\
 &= \max\{f(a), \gamma\}.
 \end{aligned}$$

Hence $\max\{f(a), \gamma\} \geq \min\{((f \circ S) \circ f)(a), \delta\}$. Hence, by Lemma 8, f is an $(\in_\gamma \in_\gamma \vee q_\delta)$ -fuzzy bi-ideal of S . $\blacksquare$

Theorem 5 *Let S be an intra-regular AG-groupoid with left identity, then the following conditions are equivalent.*

- (i) S is intra-regular.
- (ii) For every ideal of S is semiprime.
- (iii) For every $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy ideal of S is fuzzy semiprime.
- (iv) For every $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy right ideal of S is fuzzy semiprime.
- (v) For every $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy bi-ideal of S is fuzzy semiprime.

Proof. (i) $\Rightarrow$ (v) Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy right ideal of an intra-regular AG-groupoid S with left identity. Now, since S is intra-regular so for each $a \in S$ there exist x, y in S such that $a = (xa^2)y$. Now, using (1) and the left invertive law, we get

$$\begin{aligned}
 a &= (xa^2)y = [x(aa)]y = [a(xa)]y = [y(xa)]a \\
 &= [y\{x((xa^2)y)\}]a = [y\{(xa^2)(xy)\}]a \\
 &= [(xa^2)\{y(xy)\}]a = [\{(xy)y\}(a^2x)]a = [a^2((y^2x)x)]a \\
 &= [a((y^2x)x)]a^2 = [\{(xa^2)y\}((y^2x)x)]a^2 \\
 &= [\{((y^2x)x)y\}(xa^2)]a^2 = [(a^2x)\{y((y^2x)x)\}]a^2 \\
 &= [\{\{y((y^2x)x)\}x\}a^2]a^2 = [a^2\{x\{y((y^2x)x)\}\}]a^2 \\
 &= (a^2q)a^2, \text{ where } q = \{((y^2x)x)\}x\}y\}.
 \end{aligned}$$

Thus

$$\begin{aligned}\max\{f(a), \gamma\} &= \max\{f((a^2q)a^2), \gamma\} \\ &\geq \min\{f(a^2), f(a^2), \delta\} \\ &= \min\{f(a^2), \delta\}.\end{aligned}$$

(v) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (iii) are obvious.

(iii) $\Rightarrow$ (ii) It follows from Lemma 3.

(ii) $\Rightarrow$ (i) Assume that every ideal is semiprime and since Sa^2 is an ideal containing a^2 . Thus, we have

$$a \in Sa^2 = (SS)a^2 = (a^2S)S = (Sa^2)S.$$

Hence S is an intra-regular AG-groupoid. ■

5. Conclusions

In this paper, we characterize intra-regular AG-groupoids with left identity using the properties of their $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy bi-deals. In our future research work we will focus on considering other types of $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy ideals intra-regular AG-groupoids. We remark that this research work will give a new direction for applications of fuzzy set theory particularly in algebraic logic, non-classical logic, fuzzy coding, fuzzy finite state mechanics and fuzzy languages.

Acknowledgements. This research is partially supported by a grant of Natural Science Foundation of Education Committee of Hubei Province(D20131903).

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Accepted: 22.12.2013

SOME DOUBLE LACUNARY SEQUENCE SPACES

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Abstract. The purpose of this paper is to introduce some double lacunary sequence spaces defined by a sequence of Orlicz functions. We also make an effort to study some topological properties and inclusion relations between these sequence spaces.

Keywords: paranorm space, Orlicz function, solid, analytic sequences, double sequences.

2000 Mathematics Subject Classification: 40A05, 40C05, 40D05.

1. Introduction and preliminaries

The initial work on double sequences is found in Bromwich [4]. Later on, it was studied by Hardy [7], Moricz [16], Moricz and Rhoades [17], Tripathy ([30], [31]), Başarir and Sonalcan [2] and many others. Hardy [7] introduced the notion of regular convergence for double sequences. Quite recently, Zeltser [33] in her Ph.D thesis has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [21] have recently introduced the statistical convergence and Cauchy convergence for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [18] and Mursaleen and Edely [22] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M -core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{k,l})$ into one whose core is a subset of the M -core of x . More recently, Altay and Başar [1] have defined the spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_p$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$,

$\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_u$, respectively and also examined some properties of these sequence spaces and determined the α -duals of the spaces $\mathcal{BS}$, $\mathcal{BV}$, $\mathcal{CS}_{bp}$ and the $\beta(v)$ -duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Recently, Başar and Sever [3] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. By the convergence of a double sequence we mean the convergence in the Pringsheim sense, i.e., a double sequence $x = (x_{k,l})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > n$ see [25]. We shall write more briefly as P -convergent. The double sequence $x = (x_{k,l})$ is bounded if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l .

An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$.

Lindenstrauss and Tzafriri [10] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

It is shown in [10] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p ($p \geq 1$). The Δ_2 -condition is equivalent to $M(Lx) \leq kLM(x)$ for all values of $x \geq 0$, and for $L > 1$. The notion of difference sequence spaces was introduced by Kizmaz [8], who studied the difference sequence spaces $l_{\infty}(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Colak [5] by introducing the spaces $l_{\infty}(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$.

Let m, n be non-negative integers, then for $Z = c, c_0$ and l_{∞} , we have sequence spaces

$$Z(\Delta_n^m) = \{x = (x_k) \in w : (\Delta_n^m x_k) \in Z\},$$

where $\Delta_n^m x = (\Delta_n^m x_k) = (\Delta_n^{m-1} x_k - \Delta_n^{m-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_n^m x_k = \sum_{v=0}^m (-1)^v \binom{m}{v} x_{k+nv}.$$

Taking $m = n = 1$, we get the spaces $l_{\infty}(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ studied by Kizmaz [8]. Taking $n = 1$, we get the spaces $l_{\infty}(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$ studied by Et and Colak [5]. Similarly, we can define difference operators on double sequence spaces as:

$$\begin{aligned} \Delta x_{k,l} &= (x_{k,l} - x_{k,l+1}) - (x_{k+1,l} - x_{k+1,l+1}) \\ &= x_{k,l} - x_{k,l+1} - x_{k+1,l} + x_{k+1,l+1}, \end{aligned}$$

$$\Delta^m x_{k,l} = \Delta^{m-1} x_{k,l} - \Delta^{m-1} x_{k,l+1} - \Delta^{m-1} x_{k+1,l} + \Delta^{m-1} x_{k+1,l+1}$$

and

$$\Delta_n^m x_{k,l} = \Delta_n^{m-1} x_{k,l} - \Delta_n^{m-1} x_{k,l+1} - \Delta_n^{m-1} x_{k+1,l} + \Delta_n^{m-1} x_{k+1,l+1}.$$

A double sequence $x = (x_{k,l})$ of real numbers is called almost P -convergent to a limit L if

$$P - \lim_{p,q \rightarrow \infty} \sup_{m,n \geq 0} \frac{1}{pq} \sum_{k=m}^{m+p-1} \sum_{l=n}^{n+q-1} |x_{k,l} - L| = 0$$

i.e., the average value of $(x_{k,l})$ taken over any rectangle

$$\{(k, l) : m \leq k \leq m + p - 1, n \leq l \leq n + q - 1\}$$

tends to L as both p and q tends to ∞ , and this P -convergence is uniform in m and n .

By a lacunary sequence $\theta = (i_r)$, $r = 0, 1, 2, \dots$, where $i_0 = 0$, we shall mean an increasing sequence of non-negative integers $h_r = (i_r - i_{r-1}) \rightarrow \infty$ ($r \rightarrow \infty$). The intervals determined by θ are denoted by $I_r = (i_{r-1}, i_r]$ and the ratio i_r/i_{r-1} will be denoted by q_r . The space of lacunary strongly convergent sequences N_θ was defined by Freedman et al. [6] as follows:

$$N_\theta = \left\{ x = (x_k) : \lim_{r \rightarrow \infty} \frac{1}{g_r} \sum_{k \in I_r} |x_k - L| = 0 \text{ for some } L \right\}.$$

The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary if there exist two increasing sequences of integers such that

$$k_0 = 0, h_r = k_r - k_{r-1} \rightarrow \infty \text{ as } r \rightarrow \infty$$

and

$$l_0 = 0, \bar{h}_s = l_s - l_{s-1} \rightarrow \infty \text{ as } s \rightarrow \infty.$$

Let $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \bar{h}_s$ and $\theta_{r,s}$ is determined by $I_{r,s} = \{(k, l) : k_{r-1} < k \leq k_r \text{ and } l_{s-1} < l \leq l_s\}$, $q_r = \frac{k_r}{k_{r-1}}$, $\bar{q}_s = \frac{l_s}{l_{s-1}}$ and $q_{r,s} = q_r \bar{q}_s$.

A sequence space E is said to be solid if $(\alpha_{k,l} x_{k,l}) \in E$, whenever $(x_{k,l}) \in E$ and for all sequence $(\alpha_{k,l})$ of scalars with $|\alpha_{k,l}| \leq 1$, for all $k, l \in \mathbb{N}$.

A sequence space E is said to be symmetric if $(x_{k,l}) \in E$ implies $(x_{\pi(k,l)}) \in E$, where π is a permutation of $\mathbb{N}$.

A sequence space E is said to be convergence free if $(y_{k,l}) \in E$ whenever $(x_{k,l}) \in E$ and $x_{k,l} = 0$ implies $y_{k,l} = 0$.

Let X be a linear metric space. A function $p : X \rightarrow \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$, for all $x \in X$,
2. $p(-x) = p(x)$, for all $x \in X$,

3. $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$,
4. if (λ_n) is a sequence of scalars with $\lambda_n \rightarrow \lambda$ as $n \rightarrow \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \rightarrow 0$ as $n \rightarrow \infty$, then $p(\lambda_n x_n - \lambda x) \rightarrow 0$ as $n \rightarrow \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [32], Theorem 10.4.2, P-183). For more details about sequence spaces see ([9], [11], [12], [13], [14], [19], [23], [24], [26], [27], [28], [29]).

Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be any sequence of positive real numbers. Let X be a seminormed space over the field of complex numbers with the seminorm q and $w(X)$ denotes the space of all sequences $x = (x_{k,l})$, where $x_{k,l} \in X$. Now, we define the following sequence spaces in this paper:

$$[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_1 = \left\{ x = (x_{k,l}) \in w(X) : \lim_{r,s \rightarrow \infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^m x_{k,l} - L)}{\rho} \right) \right]^{p_{k,l}} = 0, \right. \\ \left. \text{for some } L \text{ and } \rho > 0 \right\},$$

$$[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0 = \left\{ x = (x_{k,l}) \in w(X) : \lim_{r,s \rightarrow \infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^m x_{k,l})}{\rho} \right) \right]^{p_{k,l}} = 0, \right. \\ \left. \text{for some } \rho > 0 \right\}$$

$$[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_\infty = \left\{ x = (x_{k,l}) \in w(X) : \sup_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^m x_{k,l})}{\rho} \right) \right]^{p_{k,l}} < \infty, \right. \\ \left. \text{for some } \rho > 0 \right\}.$$

If we take $\mathcal{M}(x) = x$, we get the following spaces:

$$[N_\theta, \Delta_n^m, p, q, u]_1 = \left\{ x = (x_{k,l}) \in w(X) : \lim_{r,s \rightarrow \infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left(\frac{q(u_{k,l} \Delta_n^m x_{k,l} - L)}{\rho} \right)^{p_{k,l}} = 0, \right. \\ \left. \text{for some } L \text{ and } \rho > 0 \right\},$$

$$[N_\theta, \Delta_n^m, p, q, u]_0 = \left\{ x = (x_{k,l}) \in w(X) : \lim_{r,s \rightarrow \infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left(\frac{q(u_{k,l} \Delta_n^m x_{k,l})}{\rho} \right)^{p_{k,l}} = 0, \right. \\ \left. \text{for some } \rho > 0 \right\}$$

$$\begin{aligned}
& [N_\theta, \Delta_n^m, p, q, u]_\infty \\
&= \left\{ x = (x_{k,l}) \in w(X) : \sup_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left(\frac{q(u_{k,l} \Delta_n^m x_{k,l})}{\rho} \right)^{p_{k,l}} < \infty, \right. \\
&\quad \left. \text{for some } \rho > 0 \right\}.
\end{aligned}$$

If we take $p = (p_{k,l}) = 1$, we get the spaces like:

$$\begin{aligned}
& [N_\theta, \mathcal{M}, \Delta_n^m, q, u]_1 \\
&= \left\{ x = (x_{k,l}) \in w(X) : \lim_{r,s \rightarrow \infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^m x_{k,l} - L)}{\rho} \right) \right] = 0, \right. \\
&\quad \left. \text{for some } L \text{ and } \rho > 0 \right\},
\end{aligned}$$

$$\begin{aligned}
& [N_\theta, \mathcal{M}, \Delta_n^m, q, u]_0 \\
&= \left\{ x = (x_{k,l}) \in w(X) : \lim_{r,s \rightarrow \infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^m x_{k,l})}{\rho} \right) \right] = 0, \right. \\
&\quad \left. \text{for some } \rho > 0 \right\}
\end{aligned}$$

$$\begin{aligned}
& [N_\theta, \mathcal{M}, \Delta_n^m, q, u]_\infty \\
&= \left\{ x = (x_{k,l}) \in w(X) : \sup_{r,s \rightarrow \infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^m x_{k,l})}{\rho} \right) \right] < \infty, \right. \\
&\quad \left. \text{for some } \rho > 0 \right\}.
\end{aligned}$$

The following inequality will be used throughout the paper.

If $0 \leq p_{k,l} \leq \sup p_{k,l} = H$, $K = \max(1, 2^{H-1})$, then

$$(1.1) \quad |a_{k,l} + b_{k,l}|^{p_{k,l}} \leq K \{ |a_{k,l}|^{p_{k,l}} + |b_{k,l}|^{p_{k,l}} \}$$

for all k, l and $a_{k,l}, b_{k,l} \in \mathbb{C}$. Also $|a|^{p_{k,l}} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

The main aim of the present paper is to study some topological properties and inclusion relations between the above defined sequence spaces.

2. Main results

Theorem 2.1. *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be a sequence of strictly positive real numbers. Then, $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_1$, $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0$ and $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_\infty$ are linear spaces over the set of complex numbers $\mathbb{C}$.*

Proof. Suppose $x = (x_{k,l})$ and $y = (y_{k,l}) \in [N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0$. Then, there exist positive numbers ρ_1, ρ_2 such that

$$\lim_{r,s \rightarrow \infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^m (x_{k,l}))}{\rho_1} \right) \right]^{p_{k,l}} = 0$$

and

$$\lim_{r,s \rightarrow \infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^m(y_{k,l}))}{\rho_2} \right) \right]^{p_{k,l}} = 0.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M} = (M_{k,l})$ is a non-decreasing and convex so by using inequality (1.1), we have

$$\begin{aligned} & \lim_{r,s \rightarrow \infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^m(\alpha x_{k,l} + \beta y_{k,l}))}{\rho_3} \right) \right]^{p_{k,l}} \\ & \leq K \lim_{r,s \rightarrow \infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \frac{1}{2^{p_{k,l}}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^m(x_{k,l}))}{\rho_1} \right) \right]^{p_{k,l}} \\ & + K \lim_{r,s \rightarrow \infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \frac{1}{2^{p_{k,l}}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^m(y_{k,l}))}{\rho_2} \right) \right]^{p_{k,l}} \\ & \leq K \lim_{r,s \rightarrow \infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^m(x_{k,l}))}{\rho_1} \right) \right]^{p_{k,l}} \\ & + K \lim_{r,s \rightarrow \infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^m(y_{k,l}))}{\rho_2} \right) \right]^{p_{k,l}} \\ & = 0. \end{aligned}$$

Thus, $\alpha x + \beta y \in [N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0$. This prove that $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0$ is a linear space. Similarly, we can prove other cases.

Theorem 2.2. Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be a sequence of strictly positive real numbers and $\theta = (k_{r,s})$ be a lacunary sequence. Then $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0$ is a paranormed spaces with the paranorm

$$g(x) = \sum_{i,j=1}^{m,n} |x_{i,j}| + \inf \left\{ \rho^{p_{k,l}/H} : \sup_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^m x_{k,l})}{\rho} \right) \right] \leq 1, \right. \\ \left. \text{for some } \rho > 0 \text{ and } r = 1, 2, 3, \dots \right\},$$

where $H = \max(1, \sup_{k,l} p_{k,l})$.

Proof. Clearly $g(x) = g(-x)$. Since $M_{k,l}(0) = 0$, for all $k, l \in \mathbb{N}$, we get $g(0) = 0$. Let $x = (x_{k,l})$ and $y = (y_{k,l}) \in [N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0$ and let us choose $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\sup_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^m x_{k,l})}{\rho_1} \right) \right] \leq 1, \quad r = 1, 2, 3, \dots,$$

and

$$\sup_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^m y_{k,l})}{\rho_2} \right) \right] \leq 1, \quad r = 1, 2, 3, \dots$$

Let $\rho = \rho_1 + \rho_2$. We have

$$\begin{aligned} & \sup_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^m(x_{k,l} + y_{k,l}))}{\rho} \right) \right] \\ & \leq \left(\frac{\rho_1}{\rho_1 + \rho_2} \right) \sup_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^m x_{k,l})}{\rho_1} \right) \right] \\ & \quad + \left(\frac{\rho_2}{\rho_1 + \rho_2} \right) \sup_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^m y_{k,l})}{\rho_2} \right) \right] \\ & \leq 1. \end{aligned}$$

Since $\rho > 1$, we have

$$\begin{aligned} & g(x + y) \\ & = \sum_{i,j=1}^{m,n} |x_{i,j} + y_{i,j}| + \inf \left\{ \rho^{\frac{p_{k,l}}{H}} : \sup_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^m(x_{k,l} + y_{k,l}))}{\rho} \right) \right] \right\} \\ & \leq \sum_{i,j=1}^{m,n} |x_{i,j}| + \inf \left\{ \rho^{\frac{p_{k,l}}{H}} : \sup_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^m(x_{k,l}))}{\rho} \right) \right] \leq 1, \quad r=1, 2, 3, \dots \right\} \\ & \quad + \sum_{i,j=1}^{m,n} |y_{i,j}| + \inf \left\{ \rho^{\frac{p_{k,l}}{H}} : \sup_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^m(y_{k,l}))}{\rho} \right) \right] \leq 1, \quad r=1, 2, 3, \dots \right\} \\ & = g(x) + g(y). \end{aligned}$$

Finally, let λ be a given non-zero scalar in $\mathbb{C}$. Then the continuity of the product follows from the following expression

$$\begin{aligned} g(\lambda x) & = \sum_{i,j=1}^{m,n} |\lambda x_{i,j}| + \inf \left\{ \rho^{\frac{p_{k,l}}{H}} : \sup_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^m(\lambda x_{k,l}))}{\rho} \right) \right] \leq 1, \right. \\ & \quad \left. \text{for some } \rho > 0 \text{ and } r = 1, 2, 3, \dots \right\} \\ & = \lambda \sum_{i,j=1}^{m,n} |x_{i,j}| + \inf \left\{ (|\lambda| \eta)^{\frac{p_{k,l}}{H}} : \sup_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^m(x_{k,l}))}{\eta} \right) \right] \leq 1, \right. \\ & \quad \left. \text{for some } \rho > 0 \text{ and } r = 1, 2, 3, \dots \right\}, \end{aligned}$$

where $\eta = \frac{\rho}{|\lambda|} > 0$. This completes the proof of the theorem. $\blacksquare$

Theorem 2.3. Let $\mathcal{M} = (M_{k,l})$ and $\mathcal{M}' = (M'_{k,l})$ be two sequences of Orlicz functions and $p = (p_{k,l})$ be a bounded sequence of positive real numbers. Then

- (i) $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_Z \subseteq [N_\theta, \mathcal{M} \circ \mathcal{M}', \Delta_n^m, p, q, u]_Z$
- (ii) $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_Z \cap [N_\theta, \mathcal{M}', \Delta_n^m, p, q, u]_Z \subseteq [N_\theta, \mathcal{M} + \mathcal{M}', \Delta_n^m, p, q, u]_Z$,
where $Z = 0, 1, \infty$.

Proof. The proof of the theorem is easy, so we omit it. $\blacksquare$

Theorem 2.4. *The inclusion $[N_\theta, \mathcal{M}, \Delta_n^{m-1}, p, q, u]_Z \subseteq [N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_Z$ hold, for $m \geq 1$. In general, $[N_\theta, \mathcal{M}, \Delta_n^i, p, q, u]_Z \subseteq [N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_Z$, for $i = 1, 2, 3, \dots, m-1$ and the inclusions are strict, where $Z = 0, 1, \infty$.*

Proof. Let $x = (x_{k,l}) \in [N_\theta, \mathcal{M}, \Delta_n^{m-1}, p, q, u]_0$. Then, there exists $\rho > 0$ such that

$$\lim_{r,s \rightarrow \infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^{m-1}(x_{k,l}))}{\rho} \right) \right]^{p_{k,l}} = 0.$$

Since $\mathcal{M}$ is non-decreasing and convex, we have

$$\begin{aligned} & \lim_{r,s \rightarrow \infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^m(x_{k,l}))}{2\rho} \right) \right]^{p_{k,l}} \\ &= \lim_{r,s \rightarrow \infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^{m-1}(x_{k,l}) - u_{k,l} \Delta_n^{m-1}(x_{k+1,l+1}))}{2\rho} \right) \right]^{p_{k,l}} \\ &\leq \lim_{r,s \rightarrow \infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^{m-1}(x_{k,l}))}{2\rho} \right) \right]^{p_{k,l}} \\ &+ \lim_{r,s \rightarrow \infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^{m-1}(x_{k+1,l+1}))}{2\rho} \right) \right]^{p_{k,l}} \\ &\leq \lim_{r,s \rightarrow \infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^{m-1}(x_{k,l}))}{\rho} \right) \right]^{p_{k,l}} \\ &+ \lim_{r,s \rightarrow \infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^{m-1}(x_{k+1,l+1}))}{\rho} \right) \right]^{p_{k,l}} \rightarrow 0 \text{ as } r, s \rightarrow \infty, \end{aligned}$$

i.e., $x = (x_{k,l}) \in [N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0$. The other cases can be proved in the similar way. $\blacksquare$

Theorem 2.5. *Let $\theta = (k_r, l_s)$ be a lacunary sequence and let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions. Then*

- (i) $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0 \subseteq [N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_1 \subseteq [N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_\infty$,
and the inclusion is strict.
- (ii) If $|u_{k,l}| \leq 1$, then $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_Z \subseteq [N_\theta, \mathcal{M}, \Delta_n^m, p, q,]_Z$
for $Z = 0, 1, \infty$.

Proof. (i) The inclusion $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0 \subseteq [N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_1$ is obvious. Let $(x_{k,l})$ be an element of $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_1$. Then, there exists $\rho > 0$ such that

$$\lim_{r,s \rightarrow \infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^m(x_{k,l} - L))}{\rho} \right) \right]^{p_{k,l}} = 0.$$

Since $(M_{k,l})$ is non-decreasing and convex, we have

$$\begin{aligned} & \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^m(x_{k,l}))}{\rho} \right) \right]^{p_{k,l}} \\ & \leq K \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} \Delta_n^m(x_{k,l} - L))}{\rho} \right) \right]^{p_{k,l}} \\ & \quad + K \max \left[1, M_{k,l} \left(q \left(\frac{L}{\rho} \right) \right) \right]^H, \end{aligned}$$

where $H = \sup_{k,l} (p_{k,l})$, $K = \max(1, 2^{H-1})$. This completes the proof of the (i).

(ii) The proof of the (ii) is easy, so we omit it. $\blacksquare$

Example. Let

$$(p_{k,l}) = \begin{cases} 4, & \text{if } k, l \text{ are even;} \\ 5, & \text{otherwise.} \end{cases}$$

Let $m, n \geq 0$ be given. Let $u_{k,l} = (kl)$, $M_{k,l}(x) = x^2$, for all $k, l \in \mathbb{N}$ and $q(x) = |x|$. Let $\theta = (2^{rs})$ be a lacunary sequence. Consider a sequence $(x_{k,l})$ defined by

$$(x_{k,l}) = \{(kl)^{(mn)}, (kl)^{(mn)}, (kl)^{(mn)}, \dots\}.$$

Thus, the sequence $(x_{k,l})$ belongs to $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_1$, but $(x_{k,l})$ does not belong to $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0$.

Theorem 2.6. Let $\mathcal{M} = (M_{k,l})$ and $\mathcal{M}' = (M'_{k,l})$ be two sequences of Orlicz functions. If $\mathcal{M}$ and $\mathcal{M}'$ are equivalent for each $k, l \in \mathbb{N}$ and $\theta = (k_r, l_s)$ be a lacunary sequence. Then

$$[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_Z = [N_\theta, \mathcal{M}', \Delta_n^m, p, q, u]_Z,$$

where $Z = 0, 1, \infty$.

Proof. The proof of the theorem is easy, so omitted. $\blacksquare$

Theorem 2.7. Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions and let q_1 and q_2 be two seminorms. Then

- (i) $[N_\theta, \mathcal{M}, \Delta_n^m, p, q_1, u]_Z \cap [N_\theta, \mathcal{M}, \Delta_n^m, p, q_2, u]_Z \subseteq [N_\theta, \mathcal{M}, \Delta_n^m, p, q_1 + q_2, u]_Z$;
- (ii) $[N_\theta, \mathcal{M}, \Delta_n^m, p, q_1, u]_Z \subseteq [N_\theta, \mathcal{M}, \Delta_n^m, p, q_2, u]_Z$, where $Z = 0, 1, \infty$.

Proof. The proof is easy, so omitted.

Proposition 2.8. The spaces $[N_\theta, \mathcal{M}, p, q, u]_0$ and $[N_\theta, \mathcal{M}, p, q, u]_\infty$ are solid as well as monotone. The spaces $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_Z$ are not solid in general, for $Z = 0, 1, \infty$.

Proof. Let $(x_{k,l}) \in [N_\theta, \mathcal{M}, p, q, u]_0$. Then, there exists $\rho > 0$ such that

$$\lim_{r,s \rightarrow \infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l} x_{k,l})}{\rho} \right) \right]^{p_{k,l}} = 0.$$

Let $(\alpha_{k,l})$ be a sequence of scalars such that $|\alpha_{k,l}| \leq 1$, for all $k, l \in \mathbb{N}$. Since $|\alpha_{k,l}| \leq \max(1, |\alpha_{k,l}|^H) \leq 1$, for all $k, l \in \mathbb{N}$, where $H = \sup_{k,l} p_{k,l} < \infty$, then, for each r, s , we have

$$\frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(\alpha_{k,l}(u_{k,l}x_{k,l}))}{\rho} \right) \right]^{p_{k,l}} \leq \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \left[M_{k,l} \left(\frac{q(u_{k,l}x_{k,l})}{\rho} \right) \right]^{p_{k,l}}.$$

Therefore, $(\alpha_{k,l}x_{k,l}) \in [N_\theta, \mathcal{M}, p, q, u]_0$. Hence $[N_\theta, \mathcal{M}, p, q, u]_0$ is solid. Therefore, the space $[N_\theta, \mathcal{M}, p, q, u]_0$ is monotone.

Hence the space $[N_\theta, \mathcal{M}, p, q, u]_\infty$ is solid as well as monotone.

In order to prove that the spaces $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_1$ and $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_\infty$ are not solid in general, we consider the following example.

Example. Let $M_{k,l}(x) = x^{st}$, for all $k, l \in \mathbb{N}$ and $s, t \geq 1$. Let $(p_{k,l}) = (\frac{1}{kl})$, $(u_{k,l}) = (kl)$, for all $k, l \in \mathbb{N}$ and $q(x) = |x|$. Let $\theta = (2^{rs})$ be a lacunary sequence, for all $k, l \in \mathbb{N}$. Consider a sequence $(x_{k,l})$ defined by

$$(x_{k,l}) = (kl)^2, \text{ for all } k, l \in \mathbb{N}.$$

Then, $(x_{k,l})$ belongs to $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_1$ and $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_\infty$ for $m, n = 1$.

Let $(\alpha_{k,l}) = (-1)^{kl}$, for all $k, l \in \mathbb{N}$. Then, $(\alpha_{k,l}x_{k,l})$ does not belong to the spaces $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_1$ and $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_\infty$.

Hence the spaces $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_1$ and $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_\infty$ are not solid. Therefore, the spaces $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_1$ and $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_\infty$ are not monotone.

Proposition 2.9. *The spaces $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_Z$ are not symmetric in general, for $Z = 0, 1, \infty$.*

Proof. The proof of the result follows from the following example. ■

Example. Let $M_{k,l}(x) = x^2$, $(p_{k,l}) = (kl)$ and $(u_{k,l}) = (kl)^2$, for all $k, l \in \mathbb{N}$ and $q(x) = |x|$. Let $\theta = (2^{rs})$ be a lacunary sequence for all $k, l \in \mathbb{N}$. Consider a sequence $(x_{k,l})$ defined by $(x_{k,l}) = (kl)^3$, for all $k, l \in \mathbb{N}$. Then $(x_{k,l})$ belong to $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0$, for $m, n = 1$. Consider the sequence $(y_{k,l})$ which is the rearrangement of the sequence $(x_{k,l})$ defined by

$$(y_{k,l}) = (x_{1,1}, x_{2,2}, x_{4,4}, x_{3,3}, x_{9,9}, x_{5,5}, x_{16,16}, x_{6,6}, x_{25,25}, x_{7,7}, \dots).$$

Then, $(y_{k,l})$ does not belong to $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_Z$.

Hence the spaces $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_Z$ are not symmetric in general.

Proposition 2.10. *The space $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0$ is not convergence free.*

Proof. The proof of the result follows from the following example.

Example. Let $M_{k,l}(x) = x$, $(p_{k,l}) = (kl)$, $(u_{k,l}) = (kl)$, for all $k, l \in \mathbb{N}$ and $q(x) = |x|$. Let $\theta = (2^{rs})$ be a lacunary sequence for all $k, l \in \mathbb{N}$. Consider a sequence $(x_{k,l})$ defined by

$$(x_{k,l}) = \begin{cases} 2, & \text{if } k, l \text{ are even;} \\ 0, & \text{otherwise.} \end{cases}$$

Then $(x_{k,l})$ belongs to $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0$, for $m, n = 2$.

Consider the sequence $(y_{k,l})$ defined by

$$(y_{k,l}) = \begin{cases} (kl)^2, & \text{if } k, l \text{ are even;} \\ 0, & \text{otherwise.} \end{cases}$$

Then, $(y_{k,l})$ does not belong to $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0$.

Hence the space $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0$ is not convergence free.

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A NEW APPROACH TO A CERTAIN GENERALIZED INTEGRAL TRANSFORM FOR CERTAIN SPACE OF BOEHMIANS

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Abstract. In this article, we introduce a generalization of Fourier and Hartely transforms. The transform we have obtained has been investigated on certain space of distributions. Two spaces of Boehmians are also established. The extended transform is then obtained and is well-defined, linear, one-to-one and onto mapping. More properties are also illustrated.

Keywords: $\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}$ Transform; Fourier Transform Pair; Distribution Space; Generalized Function; Boehmian; Hartley Transform Pair.

1. Introduction

Let $\mathbb{R}$ be the set of real numbers and g be an integrable function defined on $\mathbb{R}$. The transform we consider in this article is given by the integral equation

$$(1) \quad \iota_{\alpha, \mathbb{R}}^{\beta, \gamma}(g)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(\tau) (\alpha \cos(\gamma \xi \tau) + \beta \sin(\gamma \xi \tau)) d\tau.$$

The inversion formula is recovered from our transform (1) as

$$(2) \quad g(\tau) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \iota_{\alpha, \mathbb{R}}^{\beta, \gamma}(g)(\xi) \left(\frac{\gamma}{\alpha} \cos(\gamma \xi \tau) + \frac{\gamma}{\beta} \sin(\gamma \xi \tau) \right) d\xi.$$

Let $\gamma = \alpha = \beta = 1$, then the shortness of (1) and (2) reduces to the Hartley transform pair [2], [8], [12]. On the other hand, a substitution of $\alpha = 1$, $\beta = i$, $\gamma = 1$, in (1) and (2), describes a Fourier transform pair [9].

Denote by $E'(\mathbb{R})$ the space of distributions of compact supports. Then, the extention $\widetilde{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}$ of $\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}$ to a distribution $g \in E'(\mathbb{R})$ can be given as

$$(3) \quad \widetilde{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}(g)(\xi) = \frac{1}{\sqrt{2\pi}} \langle g(\tau), \alpha \cos(\gamma\xi\tau) + \beta \sin(\gamma\xi\tau) \rangle.$$

This definition is indeed well-defined by the smoothness of $\alpha \cos(\gamma\xi\tau) + \beta \sin(\gamma\xi\tau)$. Therefore, $\widetilde{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}$ justifies its following properties:

- (i) $\widetilde{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}$ is linear;
- (ii) $\widetilde{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}$ is continuous;
- (iii) $\widetilde{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}$ is analytic.

The justification of those properties follows in fact from basic properties of distributions.

In what follows, we spread the discussion to further space of Boehmians over two sections.

In Section 2, we construct the image and preimage spaces of Boehmians. In Section 3, we discuss the transform in the context of Boehmian spaces and obtain some properties.

2. The constructed spaces of Boehmian

We assume the reader is acquainted with the concept of Boehmian spaces. For further constructions, reader can check the citations [1], [3], [4], [6], [7], [10], [11], [13] of this article.

Let us first define some auxiliary mappings that are useful to our next investigation of Boehmian spaces.

Denote by D the Schwartz space of test functions of bounded support. Then, for every $g \in E'$ and $v \in D$ we introduce the operation $\bullet$ defined by

$$(4) \quad (g \bullet v) \ell(\varsigma) = g(v * \ell)(\varsigma),$$

where $\ell \in E$, and $*$ is the usual convolution product of two functions of first kind [9], [5].

To establish the first space of Boehmians we are requested to establish the following theorems.

Theorem 1. *Let $g \in E'$ and $v \in D$; then we have $g \bullet v \in E'$.*

Proof. Let $\ell \in E$ and $\mathbb{K}$ be a compact subset of $\mathbb{R}$; then, by (4), we have

$$(5) \quad (g \bullet v) \ell(\varsigma) = g(v * \ell)(\varsigma).$$

So, to prove the theorem, it is sufficient to show that $v * \ell \in E$. By the property [9],

$$D^k f * g = D^k f * g$$

we derive that

$$\frac{d^k}{d\varsigma^k} (v * \ell) (\varsigma) = v * \frac{d^k}{d\varsigma^k} \ell (\varsigma).$$

Hence, allowing $\mathbb{K}$ traverse the compact subsets of $\mathbb{R}$ implies

$$\begin{aligned} \sup_{\varsigma \in \mathbb{K}} \left| \frac{d^k}{d\varsigma^k} (v * \ell) (\varsigma) \right| &= \sup_{\varsigma \in \mathbb{K}} \left| v * \frac{d^k}{d\varsigma^k} \ell (\varsigma) \right| \\ &\leq \eta^* \int_{\mathbb{K}} \left| \frac{d^k}{d\varsigma^k} \ell (\varsigma - x) \right| dx, \end{aligned}$$

η^* is certain positive constant.

This gives

$$\sup_{\varsigma \in \mathbb{K}} \left| \frac{d^k}{d\varsigma^k} (v * \ell) (\varsigma) \right| < \infty.$$

Hence, $v * \ell \in E$. Thus, $g \bullet \ell \in E'$.

This completes the proof of the theorem. ■

Theorem 2. Let $g_1, g_2 \in E'$ and $\ell \in D, \eta \in \mathbb{C}$; then we have

$$\eta (g_1 + g_2) \bullet \ell = (\eta g_1 + \eta g_2) \bullet \ell.$$

The proof of this theorem is straightforward. Hence, we prefer to omit the details. ■

Theorem 3. Let $g_n \rightarrow g$ in E' and $v \in D$; then $g_n \bullet v \rightarrow g \bullet v$ as $n \rightarrow \infty$.

Proof. For $\ell \in E$, we can write

$$(6) \quad (g_n \bullet v - g \bullet v) \ell = ((g_n - g) \bullet v) \ell = (g_n - g) (v * \ell).$$

Right hand side of equation (6) is well defined by Theorem 1.

Considering the limit as $n \rightarrow \infty$, the right hand side of (6) tends to 0 as $n \rightarrow \infty$.

Therefore,

$$(g_n \bullet v \rightarrow g \bullet v) \ell \rightarrow 0$$

as $n \rightarrow \infty$.

Hence $g_n \bullet v \rightarrow g \bullet v$ as $n \rightarrow \infty$.

The proof of the theorem is completed. ■

As final in this construction, we merely need to establish the following theorem.

Theorem 4. Let $g \in E'$ and $(\delta_n) \in \Delta$; then we have $g \bullet \delta_n \rightarrow g$ as $n \rightarrow \infty$.

Proof. Let $\ell \in E$ and $(\delta_n) \in \Delta$; then, since (δ_n) is delta sequence, we have $\delta_n * \ell$ as $n \rightarrow \infty$. Hence,

$$(7) \quad (g \bullet \delta_n) \ell = g (\delta_n * \ell) \rightarrow g \text{ as } n \rightarrow \infty.$$

Thus our theorem is completely proved. $\blacksquare$

The space $\mathbf{B}(E', D, \Delta, \bullet)$ is therefore considered as a Boehmian space.

The sum and multiplication by a scalar of two Boehmians can be defined in a natural way

$$\left[\frac{f_n}{\epsilon_n} \right] + \left[\frac{g_n}{\tau_n} \right] = \left[\frac{f_n \bullet \tau_n + g_n \bullet \epsilon_n}{\epsilon_n \bullet \tau_n} \right]$$

and

$$\eta \left[\frac{f_n}{\epsilon_n} \right] = \left[\frac{\eta f_n}{\epsilon_n} \right], \eta \in \mathbb{C}.$$

The operation $\bullet$ and the differentiation are defined by

$$\left[\frac{f_n}{\epsilon_n} \right] \bullet \left[\frac{g_n}{\tau_n} \right] = \left[\frac{f_n \bullet g_n}{\epsilon_n \bullet \tau_n} \right] \text{ and } D^\alpha \left[\frac{f_n}{\epsilon_n} \right] = \left[\frac{D^\alpha f_n}{\epsilon_n} \right].$$

If $\left[\frac{f_n}{\epsilon_n} \right] \in \mathbf{B}(E', D, \Delta, \bullet)$ and $\phi \in D$, then we have

$$\left[\frac{f_n}{\epsilon_n} \right] \bullet \phi = \left[\frac{f_n \bullet \phi}{\epsilon_n} \right].$$

A sequence of Boehmians (β_n) in $\mathbf{B}(E', D, \Delta, \bullet)$ is said to be δ -convergent to a Boehmian β in $\mathbf{B}(E', D, \Delta, \bullet)$, denoted by $\beta_n \xrightarrow{\delta} \beta$, if there exists a delta sequence (ϵ_n) such that $(\beta_n \bullet \epsilon_n), (\beta \bullet \epsilon_n) \in E', \forall k, n = 1, 2, 3, \dots$, and

$$(\beta_n \bullet \epsilon_k) \rightarrow (\beta \bullet \epsilon_k) \text{ as } n \rightarrow \infty, \text{ in } E', k = 1, 2, 3, \dots$$

The following lemma is equivalent for the statement of δ -convergence:

$\beta_n \xrightarrow{\delta} \beta$ (as $n \rightarrow \infty$) in $\mathbf{B}(E', D, \Delta, \bullet)$ if and only if there is $f_{n,k}, f_k \in E'$ and $(\epsilon_k) \in \Delta$ such that $\beta_n = \left[\frac{f_{n,k}}{\epsilon_k} \right], \beta = \left[\frac{f_k}{\epsilon_k} \right]$ and for each $k = 1, 2, 3, \dots$,

$$f_{n,k} \rightarrow f_k \text{ as } n \rightarrow \infty \text{ in } E'.$$

A sequence (β_n) of Boehmians in $\mathbf{B}(E', D, \Delta, \bullet)$ is said to be a Δ -convergent to a Boehmian β in $\mathbf{B}(E', D, \Delta, \bullet)$, denoted by $\beta_n \xrightarrow{\Delta} \beta$, if there exists a $(\epsilon_n) \in \Delta$ such that

$$(\beta_n - \beta) \bullet \epsilon_n \in E',$$

$$\forall n = 1, 2, 3, \dots \text{ and } (\beta_n - \beta) \bullet \epsilon_n \rightarrow 0 \text{ as } n \rightarrow \infty \text{ in } E'.$$

Let us now consider another space of Boehmians.

Let W be the space of $\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}$ transforms of distributions in E' ; then we define a product $\circledast$ as

$$(8) \quad (w \circledast v)(\xi) = \int_{\mathbb{R}} w(\xi + x) v(x) dx,$$

$\xi \in \mathbb{R}$. Then, (8) can be simply written as

$$(w \circledast v)(\xi) = v * \tilde{w}(\xi),$$

where $\tilde{w}(\xi) = w(-\xi)$.

Theorem 5. *Let $g \in E', v \in D$; then*

$$\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}(g \bullet v) = \iota_{\alpha, \mathbb{R}}^{\beta, \gamma}(g) \circledast v.$$

Proof. Over compact subsets $\mathbb{K}$ of $\mathbb{R}$ we by (3) have that

$$\begin{aligned} \iota_{\alpha, \mathbb{R}}^{\beta, \gamma}(g \bullet v)(\xi) &= \langle (g \bullet v)(\tau), \alpha \cos \gamma \xi \tau + \beta \sin \gamma \xi \tau \rangle \\ \text{i.e.} \quad &= \langle g(\tau), \langle v(x), \alpha \cos \gamma \xi (\tau + x) + \beta \sin \gamma \xi (\tau + x) \rangle \rangle \\ \text{i.e.} \quad &= \int_{\mathbb{R}} \langle g(\tau), \alpha \cos \gamma \xi (\tau + x) + \beta \sin \gamma \xi (\tau + x) \rangle v(x) dx. \end{aligned}$$

Therefore, we have obtained

$$\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}(g \bullet v)(\xi) = \left(\iota_{\alpha, \mathbb{R}}^{\beta, \gamma} g \circledast v \right)(\xi).$$

This completes the proof of the theorem. ■

Theorem 6. *Let $w \in W, v \in D$; then $w \circledast v \in W$.*

Proof. The assumption that $w \in W$ implies $w = \iota_{\alpha, \mathbb{R}}^{\beta, \gamma} g$ for some $g \in E'$.

Therefore, we get

$$(9) \quad w \circledast v = \iota_{\alpha, \mathbb{R}}^{\beta, \gamma} g \circledast v.$$

By Theorem 5, equation (9) gives

$$(10) \quad w \circledast v = \iota_{\alpha, \mathbb{R}}^{\beta, \gamma}(g \bullet v).$$

Theorem 1, therefore, implies $w \circledast v \in E'$.

This completes the proof of the theorem. ■

The proofs of the following two theorems have similar techniques.

Theorem 7. *$w_n, w \in W, (\delta_n) \in \Delta$ and $v \in D$. Then we get*

- (1) Let $w_n \rightarrow w$ in W and $v \in D$; then $w_n \otimes v \rightarrow w \otimes v$.
 (2) Let $w \in W$ and $(\delta_n) \in \Delta$; then $w \otimes \delta_n \rightarrow w$ as $n \rightarrow \infty$.

Theorem 8. Let $w_1, w_2 \in W, v \in D$; then for $\eta_1, \eta_2 \in \mathbb{C}$, we get that

$$(\eta_1^* w_1 + \eta_2^* w_2) \otimes v = \eta_1^* (w_1 \otimes v) \times \eta_2^* (w_2 \otimes v)$$

where $v \in D$.

The Boehmian space $\mathbf{B}(W, D, \Delta, \otimes)$ is therefore constructed.

Addition, multiplication by scalars and convergence on $\mathbf{B}(W, D, \Delta, \otimes)$ are similar to that of $\mathbf{B}(E', D, \Delta, \bullet)$. Hence details are avoided.

3. The generalized $\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}$ transform of Boehmians

Let $\left[\frac{g_n}{\delta_n} \right] \in \mathbf{B}(E', D, \Delta, \bullet)$; then, we define the generalized transform $\widehat{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}}$ of $\left[\frac{g_n}{\delta_n} \right]$ as

$$(11) \quad \widehat{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}} \left[\frac{g_n}{\delta_n} \right] = \left[\frac{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}(g_n)}{\delta_n} \right]$$

in the space $\mathbf{B}(W, D, \Delta, \otimes)$.

Theorem 9. The mapping $\widehat{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}}$ is well-defined.

Proof. Let $\left[\frac{f_n}{\epsilon_n} \right] = \left[\frac{g_n}{\tau_n} \right] \in \mathbf{B}(E', D, \Delta, \bullet)$; then we have

$$\frac{f_n}{\epsilon_n} \text{ is equivalent to } \frac{g_n}{\tau_n} \text{ in } \mathbf{B}(E', D, \Delta, \bullet).$$

Therefore, $f_n \bullet \tau_m = g_m \bullet \epsilon_n, \forall m, n \in N$. The action of $\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}$ jointly with Theorem 5 imply

$$\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}(f_n) \otimes \tau_m = \iota_{\alpha, \mathbb{R}}^{\beta, \gamma}(g_m) \otimes \epsilon_n,$$

$\forall m, n \in N$. Hence

$$\frac{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}(f_n)}{\epsilon_n} \text{ is equivalent to } \frac{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}(g_n)}{\tau_n} \text{ in } \mathbf{B}(W, D, \Delta, \otimes).$$

Hence, we have obtained

$$\widehat{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}} \left[\frac{f_n}{\epsilon_n} \right] = \widehat{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}} \left[\frac{g_n}{\tau_n} \right].$$

This completes the proof of the theorem. ■

Theorem 10. *The mapping $\widehat{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}}$ is linear.*

Proof. Let $\left[\frac{f_n}{\epsilon_n} \right], \left[\frac{g_n}{\tau_n} \right] \in \mathbf{B}(E', D, \Delta, \bullet), \kappa, \eta \in \mathbb{R}$; then

$$\begin{aligned} \widehat{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}} \left(\kappa \left[\frac{f_n}{\epsilon_n} \right] + \eta \left[\frac{g_n}{\tau_n} \right] \right) &= \widehat{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}} \left(\left[\frac{\kappa f_n}{\epsilon_n} \right] + \left[\frac{\eta g_n}{\tau_n} \right] \right) \\ \text{i.e.} \quad &= \widehat{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}} \left(\left[\frac{(\kappa f_n) \bullet \tau_n + (\eta g_n) \bullet \epsilon_n}{\epsilon_n \bullet \tau_n} \right] \right) \\ \text{i.e.} \quad &= \left[\frac{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}((\kappa f_n) \bullet \tau_n + (\eta g_n) \bullet \epsilon_n)}{\epsilon_n \bullet \tau_n} \right]. \end{aligned}$$

Linearity of $\widehat{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}}$ and (11) imply

$$\widehat{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}} \left(\kappa \left[\frac{f_n}{\epsilon_n} \right] + \eta \left[\frac{g_n}{\tau_n} \right] \right) = \left[\frac{\kappa \iota_{\alpha, \mathbb{R}}^{\beta, \gamma}(f_n \bullet \tau_n) + \eta \iota_{\alpha, \mathbb{R}}^{\beta, \gamma}(g_n \bullet \epsilon_n)}{\epsilon_n \circledast \tau_n} \right].$$

Theorem 5 gives

$$\widehat{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}} \left(\kappa \left[\frac{f_n}{\epsilon_n} \right] + \eta \left[\frac{g_n}{\tau_n} \right] \right) = \left[\frac{\kappa \iota_{\alpha, \mathbb{R}}^{\beta, \gamma}(f_n) \circledast \tau_n + \eta \iota_{\alpha, \mathbb{R}}^{\beta, \gamma}(g_n) \circledast \epsilon_n}{\epsilon_n \circledast \tau_n} \right].$$

Thus, we have

$$\widehat{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}} \left(\kappa \left[\frac{f_n}{\epsilon_n} \right] + \eta \left[\frac{g_n}{\tau_n} \right] \right) = \left[\frac{\kappa \iota_{\alpha, \mathbb{R}}^{\beta, \gamma}(f_n)}{\epsilon_n} \right] + \left[\frac{\eta \iota_{\alpha, \mathbb{R}}^{\beta, \gamma}(g_n)}{\tau_n} \right]$$

Hence, we got that

$$\widehat{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}} \left(\kappa \left[\frac{f_n}{\epsilon_n} \right] + \eta \left[\frac{g_n}{\tau_n} \right] \right) = \kappa \left[\frac{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}(f_n)}{\epsilon_n} \right] + \eta \left[\frac{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}(g_n)}{\tau_n} \right]$$

The theorem is completely proved. ■

Theorem 11. *The mapping $\widehat{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}}$ is an isomorphism from $\mathbf{B}(E', D, \Delta, \bullet)$ into $\mathbf{B}(W, D, \Delta, \circledast)$.*

Proof. We first prove that $\widehat{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}}$ is injective mapping from $\mathbf{B}(E', D, \Delta, \bullet)$ into $\mathbf{B}(W, D, \Delta, \circledast)$.

Assume that

$$\widehat{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}} \left[\frac{f_n}{\epsilon_n} \right] = \widehat{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}} \left[\frac{g_n}{\tau_n} \right]$$

in $\mathbf{B}(W, D, \Delta, \otimes)$.

Then, by (11), we have

$$\left[\frac{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}(f_n)}{\epsilon_n} \right] = \left[\frac{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}(g_n)}{\tau_n} \right].$$

Hence, we derive that

$$\frac{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}(f_n)}{\epsilon_n} \text{ and } \frac{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}(g_n)}{\tau_n}$$

are equivalent quotients in $\mathbf{B}(W, D, \Delta, \otimes)$.

Thus, the concept of equivalent classes of $\mathbf{B}(W, D, \Delta, \otimes)$ implies

$$\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}(f_n) \otimes \tau_m = \iota_{\alpha, \mathbb{R}}^{\beta, \gamma}(g_m) \otimes \epsilon_n.$$

Theorem 5 then gives

$$\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}(f_n \otimes \tau_n) = \iota_{\alpha, \mathbb{R}}^{\beta, \gamma}(g_n \otimes \epsilon_n).$$

Injectivity of $\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}$ and the concept of equivalent classes of $\mathbf{B}(E', D, \Delta, \bullet)$ implies

$$\frac{f_n}{\epsilon_n} \text{ are equivalent } \frac{g_n}{\tau_n} \text{ in } \mathbf{B}(E', D, \Delta, \bullet).$$

Therefore $\left[\frac{f_n}{\epsilon_n} \right]$ and $\left[\frac{g_n}{\tau_n} \right]$ are equivalent.

This proves the first part.

Surjectivity of $\widehat{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}}$ is obvious.

The theorem is completely proved. ■

The proofs of the following theorems are obvious.

Theorem 12. If $\widehat{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}} \left[\frac{g_n}{\delta_n} \right] = 0$, then

$$\left[\frac{g_n}{\delta_n} \right] = 0$$

in the sense of $\mathbf{B}(E', D, \Delta, \bullet)$.

Theorem 13. If (β_n) is sequence in $\mathbf{B}(E', D, \Delta, \bullet)$ such that $\beta_n \xrightarrow{\Delta} \beta$ as $n \rightarrow \infty$, then

$$\widehat{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}}(\beta_n) \xrightarrow{\Delta} \widehat{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}}(\beta)$$

as $n \rightarrow \infty$ in $\mathbf{B}(W, D, \Delta, \otimes)$ on compact subsets.

Theorem 14. Let $\left[\frac{g_n}{\delta_n}\right] \in \mathbf{B}(E', D, \Delta, \bullet)$; then

$$\widehat{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}}\left(\left[\frac{g_n}{\delta_n}\right] \otimes \epsilon_n\right) = \widehat{\iota_{\alpha, \mathbb{R}}^{\beta, \gamma}}\left(\epsilon_n \otimes \left[\frac{g_n}{\delta_n}\right]\right),$$

$(\epsilon_n) \in \Delta$.

Readers can check the proofs from the citations given by the same author.

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Accepted: 02.01.2014

ON SOLITARY WAVE SOLUTIONS OF NONLINEAR TIME-FRACTIONAL FORNBERG-WHITHAM EQUATION

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Abstract. This paper witnesses the combination of an efficient transformation and Exp-function method to construct generalized solitary wave solutions of the nonlinear time-fractional Fornberg-Whitham equation. Computational work and subsequent numerical results re-confirm the efficiency of proposed algorithm. It is observed that suggested scheme is highly reliable and may be extended to other nonlinear differential equations of fractional order.

Keywords: Fornberg-Whitham equation, fractional calculus, exp-function method, modified Riemann-Liouville derivative.

1. Introduction

The subject of fractional calculus [1], [2] is a rapidly growing field of research, at the interface between chaos, probability, differential equations, and mathematical physics. In recent years, nonlinear fractional differential equations (NFDEs) have gained much interest due to exact description of nonlinear phenomena of many real-time problems. The fractional calculus is also considered as a novel topic [3], [4]; has gained considerable popularity and importance during the recent past. It has been the subject of specialized conferences, workshops and treatises or so, mainly due to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. Some of the areas of present-day applications of fractional models [5]–[8] include fluid flow, solute transport or dynamical processes in self-similar and porous structures, diffusive transport akin to diffusion, material viscoelastic theory, electromagnetic theory, dynamics of earthquakes, control theory of dynamical systems, optics and signal processing, bio-sciences, economics, geology, astrophysics, probability and statistics, chemical physics, and so on. As a consequence, there has been an intensive development of the theory of fractional differential equations, see [1–8] and the references therein.

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Recently, He and Wu [9] developed a very efficient technique which is called exp-function method for solving various nonlinear physical problems. The through study of literature reveals that Exp-function method has been applied on a wide range of differential equations and is highly reliable. The exp-function method has been extremely useful for diversified nonlinear problems of physical nature and has the potential to cope with the versatility of the complex nonlinearities of the problems. The subsequent works have shown the complete reliability and efficiency of this algorithm. He et.al. [10]–[11] used this scheme to find periodic solutions of evolution equations; Mohyud-Din [12–15] extended the same for nonlinear physical problems including higher-order BVPs; Oziz [16] tried this novel approach for Fisher's equation; Wu et.al. [17], [18] for the extension of solitary, periodic and compacton-like solutions; Yusufoglu [19] for MBBN equations, Zhang [20] for high-dimensional nonlinear evolutions; Zhu [21], [22] for the Hybrid-Lattice system and discrete m KdV lattice; Kudryashov [23] for exact soliton solutions of the generalized evolution equation of wave dynamics; Momani [24] for an explicit and numerical solutions of the fractional KdV equation; Ebaid [25] for the improvement on the Exp-function method when balancing the highest order linear and nonlinear terms. The basic motivation of this paper is the development of an efficient combination comprising an efficient transformation, exp-function method using Jumarie's derivative approach [27]–[30] and its subsequent application to construct generalized solitary wave solutions of the nonlinear Fornberg-Whitham Equation [26] of fractional-order. It is to be highlighted that Ebaid [25] proved that $c = d$ and $p = q$ are the only relations that can be obtained by applying exp-function method to any nonlinear ordinary differential equation.

Theorem 1 [25] *Suppose that $u^{(r)}$ and $(u^{(\gamma)})^\lambda$ are, respectively, the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r and γ are both positive integers. Then the balancing procedure using the Exp-function ansatz; $U(\eta) = \frac{\sum_{n=-c}^d a_n \exp(n\eta)}{\sum_{m=-p}^q b_m \exp(m\eta)}$, leads to $c = d$ and $p = q$, $\forall r, s, \lambda \geq 1$.*

Theorem 2 [25] *Suppose that $u^{(r)}$ and $u^{(s)}u^k$ are, respectively, the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r, s and Ω are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c = d$ and $p = q$, $\forall r, s, k \geq 1$.*

Theorem 3 [25] *Suppose that $u^{(r)}$ and $(u^{(s)})^\Omega$ are, respectively, the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r, s and Ω are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c = d$ and $p = q$, $\forall r, s \geq 1, \forall \Omega \geq 2$.*

Theorem 4 [25] *Suppose that $u^{(r)}$ and $(u^{(s)})^\Omega u^\lambda$ are, respectively, the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r, s, Ω and λ are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c = d$ and $p = q$, $\forall r, s, \Omega, \lambda \geq 1$.*

2. Jumarie's fractional derivative

Jumarie's fractional derivative is a modified Riemann-Liouville derivative defined as [27]–[30]

$$(1) \quad D_t^\alpha f(x) = \left\{ \begin{array}{l} \frac{1}{\Gamma(-\alpha)} \int_0^x (x-t)^{-\alpha-1} (f(t) - f(0)) dt, \alpha \leq 0, \\ \frac{1}{\Gamma(-\alpha)} \frac{d}{dx} \int_0^x (x-t)^{-\alpha} (f(t) - f(0)) dt, 0 \leq \alpha \leq 1 \\ [f^{\alpha-n}(x)^n]^n, n \leq \alpha \leq n+1, n \geq 1 \end{array} \right\}$$

where $f: \mathbb{R} \rightarrow \mathbb{R}$, $x \rightarrow f(x)$ denotes a continuous (but not necessarily differentiable) function. Some useful formulas and results of Jumarie's modified Riemann-Liouville derivative were summarized in [27]–[30].

$$(2) \quad D_x^\alpha c = 0, \quad \alpha \geq 0, \quad c = \text{constant}$$

$$(3) \quad D_x^\alpha [cf(x)] = cD_x^\alpha f(x), \quad \alpha \geq 0, \quad c = \text{constant}$$

$$(4) \quad D_x^\alpha x^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}, \quad \beta \geq \alpha \geq 0.$$

$$(5) \quad D_x^\alpha [f(x)g(x)] = [D_x^\alpha f(x)g(x) + f(x)[D_x^\alpha g(x)]].$$

$$(6) \quad D_x^\alpha f(x(t)) = f'_x(x) \cdot x^\alpha(t).$$

3. Exp-function method [31]–[34]

We consider the general nonlinear FPDE of the type

$$(7) \quad P(u, u_t, u_x, u_{xx}, u_{xxx}, \dots, D_t^\alpha u, D_x^\alpha u, D_{xx}^\alpha u, \dots) = 0, \quad 0 < \alpha \leq 1,$$

where $D_t^\alpha u, D_x^\alpha u, D_{xx}^\alpha u$ are the modified Riemann-Liouville derivative of u with respect to t, x, xx , respectively.

Using a transformation [35]

$$(8) \quad \eta = kx + \frac{\omega t^\alpha}{\Gamma(1+\alpha)} + \eta_0, \quad k, \omega, \eta_0 \text{ are all constants with } k, \omega \neq 0$$

we can rewrite equation (7) in the following nonlinear ODE;

$$(9) \quad Q(u, u', u'', u''', u^{iv}) = 0,$$

where the prime denotes derivative with respect to η .

According to the Exp-function method, we assume that the wave solution can be expressed in the following form

$$(10) \quad u(\eta) = \frac{\sum_{n=-c}^d a_n \exp[n\eta]}{\sum_{m=-p}^q b_m \exp[m\eta]}$$

where p, q, c and d are positive integers which are known to be further determined, a_n and b_m are unknown constants. We can rewrite equation (4) in the following equivalent form

$$(11) \quad u(\eta) = \frac{a_c \exp(c\eta) + \cdots + a_{-d} \exp(-d\eta)}{b_p \exp(p\eta) + \cdots + b_{-q} \exp(-q\eta)}.$$

This equivalent formulation plays an important and fundamental part for finding the analytic solution of problems. To determine the value of c and p by using [25],

$$(12) \quad p = c, \quad q = d$$

4. Solution procedure

In this section, we apply the exp-function method for nonlinear time-fractional Fornberg-Whitham equation.

Example 4.1 Consider the nonlinear time-fractional Fornberg-Whitham equation.

$$(13) \quad \begin{aligned} D_t^\alpha u + u_x - u_{xxt} - uu_{xxx} + uu_x - 3u_x u_{xx} &= 0, \quad t > 0, \quad 0 < \alpha < 1 \\ \text{subject to initial condition } u(x, t) &= e^{\frac{x}{2}} \end{aligned}$$

Using (8), equation (13) can be converted to an ordinary differential equation

$$(14) \quad \omega u' + k u'^2 \omega u''' + k u u'^3 u'' = 0,$$

where the prime denotes the derivative with respect to η . The solution of the equation (13) can be expressed in the form, equation (11). To determine the value of c and p , by using [25],

$$(15) \quad p = c, \quad q = d$$

Case 4.1.1. we can freely choose the values of c and d , but we will illustrate that the final solution does not strongly depend upon the choice of values of c and d . For simplicity, we set $p = c = 1$ and $q = d = 1$ equation (11) reduces to

$$(16) \quad u(\eta) = \frac{a_1 \exp[\eta] + a_0 + a_{-1} \exp[-\eta]}{b_1 \exp[\eta] + a_0 + b_{-1} \exp[-\eta]}.$$

Substituting equation (16) into equation (14), we have

$$(17) \quad \frac{1}{A} \left[\begin{aligned} &c_5 \exp(5\eta) + c_4 \exp(4\eta) + c_3 \exp(3\eta) + c_2 \exp(2\eta) + \\ &c_1 \exp(\eta) + c_0 + c_{-1} \exp(-\eta) + c_{-2} \exp(-2\eta) + \\ &c_{-3} \exp(-3\eta) + c_{-4} \exp(-4\eta) + c_{-5} \exp(-5\eta) \end{aligned} \right] = 0$$

$A = 5(b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta))^6$ where c_i ($i = -5, -4, \dots, 4, 5$)

are constants obtained by Maple 16.

Equating the coefficients of $\exp(n\eta)$ to be zero, we obtain

$$(18) \quad \begin{pmatrix} c_{-5} = 0, c_{-4} = 0, c_{-3} = 0, c_{-2} = 0, c_{-1} = 0, c_0 = 0, \\ c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0, c_5 = 0 \end{pmatrix}$$

By solving (18), we get different solutions of (13).

1st Solution set:

$$(19) \quad \begin{cases} \omega = \omega, a_{-1} = \frac{a_1 b_{-1}}{b_1}, \\ a_1 = a_1, b_{-1} = b_{-1}, b_0 = 0, b_1 = b_1, a_0 = 0 \end{cases}$$

Therefore, we obtained the following generalized solitary solution $u(x, t)$ of equation (13)

$$(20) \quad u(x, t) = \frac{\frac{a_1 b_{-1}}{b_1} e^{-\frac{xk + \omega t^\alpha}{\Gamma(1+\alpha)}} + a_1 e^{\frac{xk + \omega t^\alpha}{\Gamma(1+\alpha)}}}{b_{-1} e^{-\frac{xk + \omega t^\alpha}{\Gamma(1+\alpha)}} + b_1 e^{\frac{xk + \omega t^\alpha}{\Gamma(1+\alpha)}}}$$

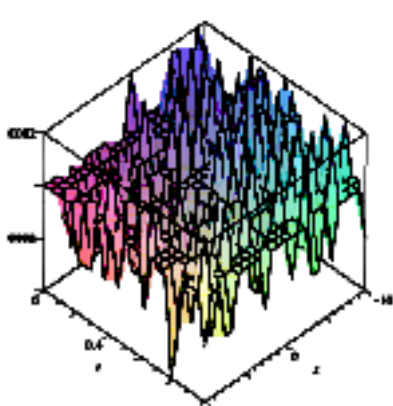


Figure 4.1(a) $\alpha = 0.25$

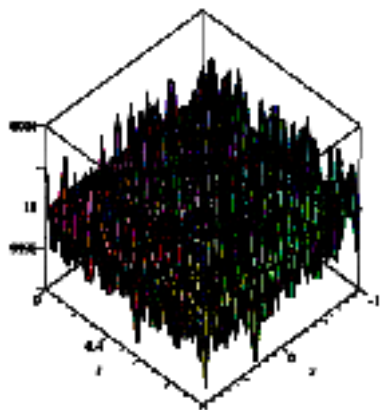


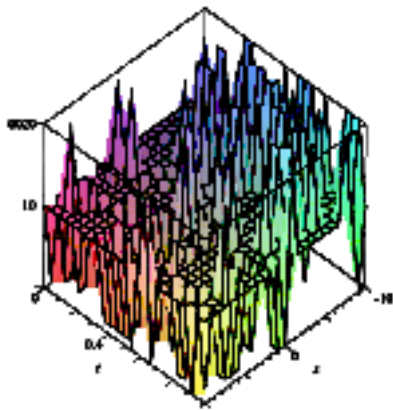
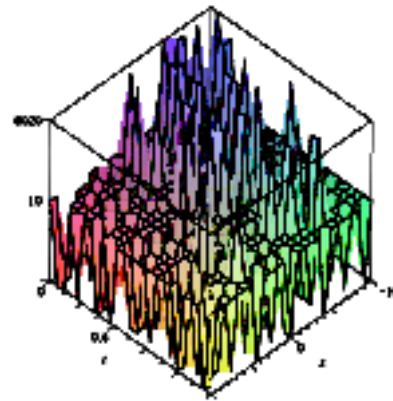
Figure 4.1(b) $\alpha = 1$

2nd Solution set:

$$(21) \quad \begin{cases} \omega = \omega, a_{-1} = \frac{a_1 b_{-1}}{b_1}, \\ a_1 = 0, b_{-1} = b_{-1}, b_0 = 0, b_1 = 0, a_0 = a_0 \end{cases}$$

We, therefore, obtained the following generalized solitary solution $u(x, t)$ of equation (13)

$$(22) \quad u(x, t) = \frac{\frac{a_1 b_{-1}}{b_1} e^{-\frac{xk + \omega t^\alpha}{\Gamma(1+\alpha)}} + a_0}{b_{-1} e^{-\frac{xk + \omega t^\alpha}{\Gamma(1+\alpha)}} + b_0}$$

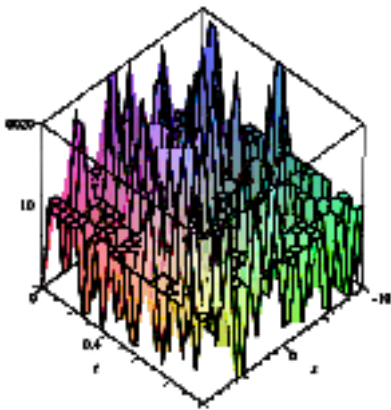
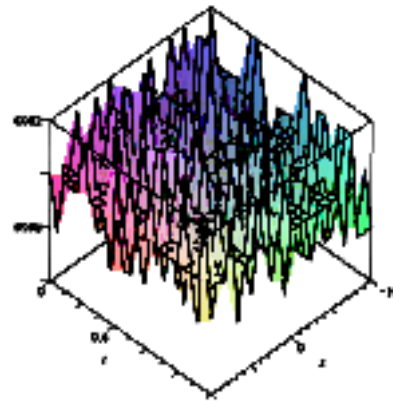
Figure 4.1(c) $\alpha = 0.25$ Figure 4.1(d) $\alpha = 1$

3rd Solution set:

$$(23) \quad \left\{ \begin{array}{l} \omega = \omega, \quad a_{-1} = a_{-1}, \quad a_1 = \frac{a_{-1}b_1}{b_{-1}}, \quad b_{-1} = b_{-1}, \\ b_0 = b_0, \quad b_1 = b_1, \quad a_0 = \frac{a_{-1}b_0}{b_{-1}} \end{array} \right\}$$

Therefore, we obtained the following generalized solitary solution $u(x, t)$ of equation (13)

$$(24) \quad u(x, t) = \frac{a_{-1}e^{-\frac{xk+\omega t^\alpha}{\Gamma(1+\alpha)}} + \frac{a_{-1}b_0}{b_{-1}} + \frac{a_{-1}b_1}{b_{-1}}e^{\frac{xk+\omega t^\alpha}{\Gamma(1+\alpha)}}}{b_{-1}e^{-\frac{xk+\omega t^\alpha}{\Gamma(1+\alpha)}} + b_0 + b_1e^{\frac{xk+\omega t^\alpha}{\Gamma(1+\alpha)}}}$$

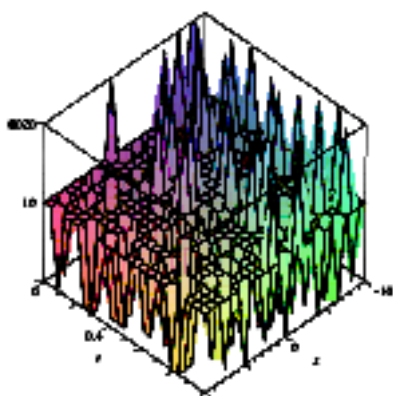
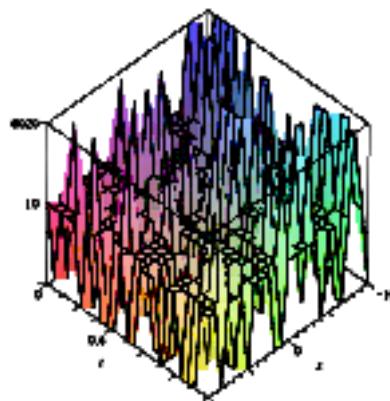
Figure 4.1(e) $\alpha = 0.25$ Figure 4.1(f) $\alpha = 1$

4th Solution set:

$$(25) \quad \left\{ \begin{array}{l} \omega = \omega, \quad a_{-1} = 0, \quad a_0 = \frac{a_1b_0}{b_1}, \\ a_1 = a_1, \quad b_{-1} = 0, \quad b_0 = b_0, \quad b_1 = b_1 \end{array} \right\},$$

Therefore, we obtained the following generalized solitary solution $u(x, t)$ of equation (13)

$$(26) \quad u(x, t) = \frac{a_1e^{\frac{xk+\omega t^\alpha}{\Gamma(1+\alpha)}} + \frac{a_1b_0}{b_1}}{b_1e^{\frac{xk+\omega t^\alpha}{\Gamma(1+\alpha)}} + b_0}$$

Figure 4.1(g) $\alpha = 0.25$ Figure 4.1(h) $\alpha = 1$

Case 4.1.2 If $p = c = 2$, and $q = d = 1$ then equation (11) reduces to (27)

$$(27) \quad u(\eta) = \frac{a_2 \exp[2\eta] + a_1 \exp[\eta] + a_0 + a_{-1} \exp[-\eta]}{b_2 \exp[2\eta] + b_1 \exp[\eta] + a_0 + b_{-1} \exp[-\eta]}.$$

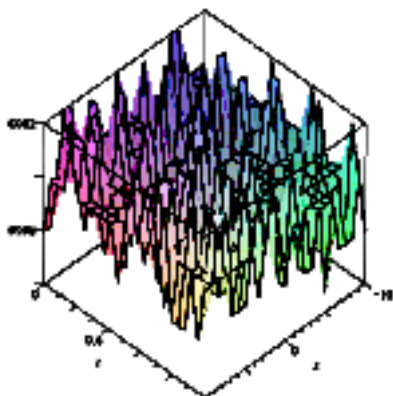
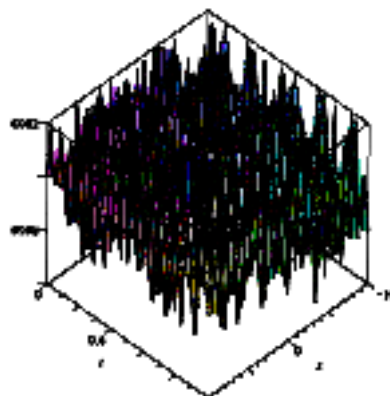
Proceeding as before, we obtain

1st Solution set:

$$(28) \quad \left\{ \begin{array}{l} b_{-1} = 0, \quad a_2 = \frac{a_0 b_2}{b_0}, \quad b_2 = b_2, \quad b_0 = b_0, \quad b_1 = b_1, \\ a_1 = \frac{a_0 b_1}{b_0}, \quad a_{-1} = 0, \quad \omega = \omega, \quad a_0 = a_0 \end{array} \right\},$$

Hence we get the generalized solitary wave solution of equation (13) as follows

$$(29) \quad u(x, t) = \frac{\frac{a_0 b_2}{b_0} e^{2\left(\frac{xk + \omega t^\alpha}{\Gamma(1+\alpha)}\right)} + \frac{a_0 b_1}{b_0} e^{\frac{xk + \omega t^\alpha}{\Gamma(1+\alpha)}} + a_0}{b_2 e^{2\left(\frac{xk + \omega t^\alpha}{\Gamma(1+\alpha)}\right)} + b_1 e^{\frac{xk + \omega t^\alpha}{\Gamma(1+\alpha)}} + b_0}$$

Figure 4.1(i) $\alpha = 0.25$ Figure 4.1(j) $\alpha = 1$

2nd Solution set:

Therefore, we obtained the Proceeding as before, we obtain

$$(30) \quad \left\{ \begin{array}{l} b_{-1} = b_{-1}, \quad a_2 = \frac{a_{-1}b_2}{b_{-1}}, \quad b_2 = b_2, \quad b_0 = b_0, \quad b_1 = b_1, \\ a_1 = \frac{a_{-1}b_1}{b_{-1}}, \quad a_{-1} = a_{-1}, \quad \omega = \omega, \quad a_0 = \frac{a_{-1}b_0}{b_{-1}} \end{array} \right\},$$

Hence, we get the generalized solitary wave solution of equation (13) as follows

$$(31) \quad u(x, t) = \frac{\frac{a_{-1}b_2}{b_{-1}}e^{2\left(\frac{xk+\omega t^\alpha}{\Gamma(1+\alpha)}\right)} + \frac{a_{-1}b_1}{b_{-1}}e^{\frac{xk+\omega t^\alpha}{\Gamma(1+\alpha)}} + \frac{a_{-1}b_0}{b_{-1}} + a_{-1}e^{-\frac{xk+\omega t^\alpha}{\Gamma(1+\alpha)}}}{b_2e^{2\left(\frac{xk+\omega t^\alpha}{\Gamma(1+\alpha)}\right)} + b_1e^{\frac{xk+\omega t^\alpha}{\Gamma(1+\alpha)}} + b_0 + b_{-1}e^{-\frac{xk+\omega t^\alpha}{\Gamma(1+\alpha)}}}$$

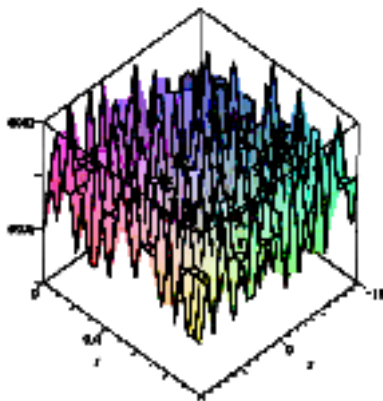


Figure 4.1(k) $\alpha = 0.25$

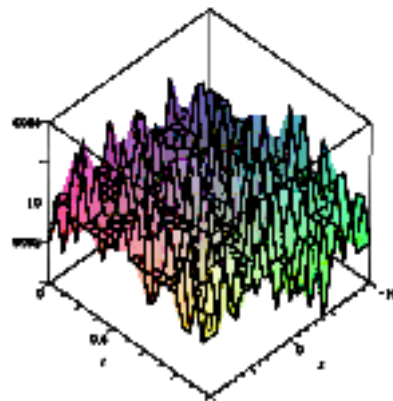


Figure 4.1(l) $\alpha = 1$

We get the same soliton solutions which clearly illustrate that final solution does not strongly depends upon these parameters.

5. Conclusion

In this paper, we applied exp-function method to construct generalized solitary solutions of the nonlinear fractional order Fornberg-Whitham equation. It is observed that the Exp-function method is very convenient to apply and is very useful for finding solutions of a wide class of nonlinear problems.

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GLOBAL EXPONENTIAL STABILITY OF IMPULSIVE HYBRID DYNAMICAL SYSTEMS WITH ANY TIME DELAY¹

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Abstract. This present paper addresses global exponential stability for a class of more general linear impulsive hybrid dynamical systems with any time delay. Combined Lyapunov function methods with the Razumikhin technique, several criteria on global exponential stability are derived, which are substantially extension and generalization of the corresponding results in recent literature. Subsequently, two application examples and its numerical simulations demonstrate that the obtained stability criteria are practical and effective.

Keywords: Impulsive dynamics, Any time delay, Hybrid systems, Global exponential stability, Razumikhin technique.

1. Introduction and notation

In past few decades, the stability problems of dynamical systems with impulses and delays have attracted a great deal of attention from a variety of applications in science and engineering, see [1]-[7] and references therein. In particular, special attention has been focused on the exponential stability of such dynamical systems because it has played an important role in practical applications of dynamical systems [8]-[10].

¹This research is supported by the Natural Science Program of Anhui Higher Education Institution under Grant, (KJ2013B216, KJ2013B217); The 2014 scientific Research Project of Hefei Normal University, (“136 Talents Project”).

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At the same time, the impulsive hybrid dynamical system is regarded as a special but important of impulsive differential systems with variable structure, and it has recently emerged as a new challenging area of research due to its potential applications in various fields of engineering areas, including control technology, industrial robotics, and communication engineering, etc [11]-[17]. There are several research works appeared in the literature on exponential stability of impulsive hybrid dynamical systems with time delay [18]-[21]. For example, Liu and Shen [19] presented some criteria on uniform stability and uniform asymptotic stability of invariant sets for impulsive hybrid dynamical systems with time delay. In [20], Li, Ma, and Feng proved some general criteria on global exponential stability for nonlinear impulsive and switching delayed dynamical systems. However, there is a restrictive condition that the time delay is less than the length of all the impulsive intervals, so they are generally inapplicable in some practical applications. In addition, Zhang and Sun [21] only given some results related to local uniformly stability for linear impulsive hybrid dynamical systems with time delay.

The main objective of this paper is further to investigate global exponential stability for a class of more general linear impulsive hybrid dynamical systems with any time delays. Based on the Lyapunov function methods combined with the Razumikhin techniques, several criteria on global exponential stability are derived analytically, which are nature extension and generalization of the corresponding results existing in the literature. Compared with some existing works, a distinctive feature of this work is to address global exponential stability for linear or nonlinear impulsive hybrid dynamical systems with any time delays. Furthermore, our results show that impulses do contribute to global exponential stability of dynamical systems with any time delays even if it may be chaotic or unstable itself, which can be usually used as an effective control strategy to stabilize the underlying delayed dynamical systems in some practical applications.

Let $R = (-\infty, +\infty)$ be the set of real numbers, $R^+ = [0, +\infty)$ be the set of nonnegative real numbers, and $N = \{0, 1, 2, \dots\}$ be the set of nonnegative integer numbers. For the vector $u \in R^n$, $u^\top$ denotes its transpose. The norm of the vector u is defined as $\|u\| = \sqrt{u^\top u}$. $R^{n \times n}$ stands for $n \times n$ the set of real matrices.

Consider the following impulsive delayed linear dynamical hybrid systems:

$$(1.1) \quad \begin{cases} \dot{x}(t) = Ax(t) + Bx(t - \tau(t)) + M_k x_k, & t \in [t_k, t_{k+1}), \\ x(t) = C_k x(t^-), & t = t_k, k \in N, \end{cases}$$

where $t \geq t_0$, $\varphi \in PC([- \tau, 0], R^n)$, $x(t) \in R^n$, $A, B, C_k, M_k \in R^{n \times n}$, $C_0 = I$, $M_k x_k$ is hybrid term.

The time sequence $\{t_k\}_{k=1}^{+\infty}$ satisfy $0 = t_0 < t_1 < t_2 < \dots < t_k < \dots$, $\lim_{k \rightarrow \infty} t_k = +\infty$, the time delay $0 \leq \tau(t) \leq \tau < +\infty$, $x(t^+) = \lim_{s \rightarrow 0^+} x(t+s)$ and $x(t^-) = \lim_{s \rightarrow 0^-} x(t+s)$.

Let $PC([- \tau, 0], R^n) = \{\phi : [- \tau, 0] \rightarrow R^n, \phi(t) \text{ is continuous everywhere except a finite number of points } \hat{t} \text{ at which } \phi(\hat{t}^+) \text{ and } \phi(\hat{t}^-) \text{ exist and } \phi(\hat{t}^+) = \phi(\hat{t})\}$.

For $\psi \in PC([- \tau, 0], R^n)$, the norm of ψ is defined by

$$\|\psi\|_\tau = \sup_{-\tau \leq s \leq 0} \|\psi(s)\|.$$

$x_t, x_{t-} \in PC([- \tau, 0], R^n)$ are defined by $x_t(s) = x(t + s)$ and $x_{t-}(s) = x(t^- + s)$ for $s \in [- \tau, 0]$, respectively.

We always assume that system (1.1) has a unique solution with respect to initial conditions. Denote by $x(t) = x(t, t_0, \varphi)$ the solution of system (1.1) such that $x_{t_0} = \varphi$. We further assume that all the solutions $x(t)$ of system (1.1) are continuous except at $t_k, k \in N$, at which $x(t)$ is right continuous. Let $x_k = x(t_k)$, $\Delta t_k = t_{k+1} - t_k, k \in N$. Obviously, $x(t) = 0$ is a solution of system (1.1), which we call the zero solution.

Definition 1.1. The zero solution of system (1.1) is said to be globally exponentially stable if there exist two constants $\lambda > 0$ and $K \geq 1$ such that for any initial value $x_{t_0} = \varphi$,

$$\|x(t, t_0, \varphi)\| \leq K \|\varphi\|_\tau e^{-\lambda(t-t_0)}, \quad t \geq t_0,$$

where $(t_0, \varphi) \in R^+ \times PC([- \tau, 0], R^n)$.

Definition 1.2. Function $V : R^+ \times R^n \rightarrow R^+$ is said to belong to the class ν_0 if

- (a) V is continuous in each of the sets $[t_k, t_{k+1}) \times R^n$, and for each $x \in R^n$, $t \in [t_k, t_{k+1}), k \in N$, $\lim_{(t,y) \rightarrow (t_k^-, x)} V(t, y) = V(t_k^-, x)$ exists, and
- (b) $V(t, x)$ is locally Lipschitzian in all $x \in R^n$, and for all $t \geq t_0, V(t, 0) \equiv 0$.

Definition 1.3. Given a function $V : R^+ \times R^n \rightarrow R^+$, the upper right-hand derivative of V with respect to system (1.1) is defined by

$$D^+V(t, x(t)) = \lim_{\delta \rightarrow 0^+} \sup \frac{1}{\delta} (V(t + \delta, x(t + \delta)) - V(t, x(t))).$$

2. Main results

In this section, by combining the Lyapunov function and Razumikhin technique, we shall present several criteria on global exponential stability for the delayed linear hybrid dynamical system (1.1).

Theorem 2.1. Let the $n \times n$ matrix P be symmetric and positive definite, λ_3 be the largest eigenvalue of $P^{-1}(A^T P + PA + 2PP)$, λ_4 be the largest eigenvalue of $P^{-1}B^T B$, λ_{M_k} be the largest eigenvalue of $P^{-1}C_k^T M_k^T M_k C_k$, λ_{C_k} be the largest eigenvalue of $P^{-1}C_k^T P C_k$, $\lambda_5 = \sup_{k \in N} \lambda_{M_k}$, $\lambda_6 = \sup_{k \in N} \lambda_{C_k}$ and $0 < \lambda_6 < 1$. Assume that there exist two constants $\lambda > 0$ and $\sigma > 0$, such that for all $k \in N$, the following conditions are satisfied:

$$(i) \quad F(\lambda) = \sigma - \lambda - \left(\lambda_3 + \frac{\lambda_4}{\lambda_6} e^{\lambda\tau} + \frac{\lambda_5}{\lambda_6} \right) \geq 0;$$

$$(ii) \quad \ln \lambda_6 < -(\sigma + \lambda)(t_{k+1} - t_k).$$

Then, for any delay $0 \leq \tau(t) \leq \tau < +\infty$, the zero solution of the linear impulsive delayed hybrid dynamical system (1.1) is globally exponentially stable with convergence rate $\frac{\lambda}{2}$.

Proof. Let $x(t) = x(t, t_0, \varphi)$ be any solution of the linear impulsive delayed hybrid dynamical system (1.1) with $x_{t_0} = \varphi$.

We construct a Lyapunov function as follow:

$$(2.1) \quad V(t, x(t)) = x^T(t)Px(t).$$

Let $\lambda_1 > 0$ and $\lambda_2 > 0$ are the smallest and the largest eigenvalues of P respectively, so we have

$$(2.2) \quad \lambda_1 \|x(t)\|^2 \leq V(t, x(t)) \leq \lambda_2 \|x(t)\|^2.$$

We shall prove that

$$(2.3) \quad V(t, x(t)) \leq \lambda_2 K \|\varphi\|_\tau^2 e^{-\lambda(t-t_0)}, \quad t \in [t_k, t_{k+1}), \quad k \in N.$$

For $t \in [t_k, t_{k+1})$, $k \in N$, we calculate the upper right derivative of $V(t, x(t))$ along the solution of system (1.1) and have

$$\begin{aligned} D^+V(t, x(t)) &= x^T(t)(A^TP + PA)x(t) \\ &\quad + 2x^T(t - \tau(t))B^TPx(t) + 2x_k^T M_k^T Px(t) \\ &\leq x^T(t)(A^TP + PA + 2PP)x(t) \\ &\quad + x^T(t - \tau(t))B^TBx(t - \tau(t)) + x_k^T M_k^T M_k x_k \\ &\leq \lambda_3 V(t, x(t)) + \lambda_4 V(t - \tau(t), x(t - \tau(t))) \\ &\quad + x_k^T M_k^T M_k x_k. \end{aligned} \quad (2.4)$$

From condition (ii), we can get

$$(2.5) \quad -\ln \lambda_6 + \lambda\tau - (\sigma + \lambda)(t_{k+1} - t_k) > 0.$$

From (2.5), we can choose $K \geq 1$, such that

$$(2.6) \quad 1 < e^{(\sigma+\lambda)(t_1-t_0)} \leq K \leq -\ln \lambda_6 e^{\lambda\tau - (\sigma+\lambda)(t_1-t_0)} e^{(\sigma+\lambda)(t_1-t_0)}.$$

Then

$$(2.7) \quad \|\varphi\|_\tau^2 < \|\varphi\|_\tau^2 e^{\sigma(t_1-t_0)} \leq K \|\varphi\|_\tau^2 e^{-\lambda(t_1-t_0)}.$$

We first prove

$$(2.8) \quad V(t, x(t)) \leq \lambda_2 K \|\varphi\|_\tau^2 e^{-\lambda(t-t_0)}, \quad t \in [t_0, t_1),$$

To do this, we only need to prove that

$$(2.9) \quad V(t, x(t)) \leq \lambda_2 K \|\varphi\|_\tau^2 e^{-\lambda(t_1-t_0)}, \quad t \in [t_0, t_1].$$

If (2.9) is not true, from (2.2) and (2.8), then there exists some $\bar{t} \in (t_0, t_1)$ such that

$$\begin{aligned} V(\bar{t}, x(\bar{t})) &> \lambda_2 K \|\varphi\|_\tau^2 e^{-\lambda(t_1-t_0)} \geq \lambda_2 \|\varphi\|_\tau^2 e^{\sigma(t_1-t_0)} \\ &> \lambda_2 \|\varphi\|_\tau^2 \geq V(t_0 + s, x(t_0 + s)), \quad s \in [-\tau, 0], \end{aligned}$$

which implies that there exists some $t^* \in (t_0, \bar{t})$ such that

$$(2.10) \quad V(t^*, x(t^*)) = \lambda_2 K \|\varphi\|_\tau^2 e^{-\lambda(t_1-t_0)},$$

and

$$(2.11) \quad V(t, x(t)) \leq V(t^*, x(t^*)), \quad t \in [t_0 - \tau, t^*],$$

then there exists $t^{**} \in [t_0, t^*)$ such that

$$(2.12) \quad V(t^{**}, x(t^{**})) = \lambda_2 \|\varphi\|_\tau^2,$$

and

$$(2.13) \quad V(t^{**}, x(t^{**})) \leq V(t, x(t)), \quad t \in [t^{**}, t^*].$$

Hence, for any $s \in [-\tau, 0]$, we have

$$\begin{aligned} V(t + s, x(t + s)) &\leq \lambda_2 K \|\varphi\|_\tau^2 e^{-\lambda(t_1-t_0)} \\ &\leq \frac{\lambda_2}{\lambda_6} e^{\lambda\tau - (\sigma + \lambda)(t_1-t_0)} e^{(\sigma + \lambda)(t_1-t_0)} \|\varphi\|_\tau^2 \\ (2.14) \quad &\leq \frac{e^{\lambda\tau}}{\lambda_6} V(t^{**}, x(t^{**})) \\ &\leq \frac{e^{\lambda\tau}}{\lambda_6} V(t, x(t)), \quad t \in [t^{**}, t^*], \end{aligned}$$

From (2.3), (2.12) and (2.13), we can get

$$\begin{aligned} x_0^T M_0^T M_0 x_0 &= x_0^T M_0^T M_0 P^{-1} P x_0 \leq \lambda_5 V(t_0, x(t_0)) \\ (2.15) \quad &\leq \lambda_5 \lambda_2 \|\varphi\|_\tau^2 \leq \lambda_5 V(t, x(t)), \quad t \in [t^{**}, t^*], \end{aligned}$$

Thus, from condition (i), (2.4), (2.14) and (2.15), we get

$$\begin{aligned} D^+ V(t, x(t)) &\leq \left(\lambda_3 + \frac{\lambda_4}{\lambda_6} e^{\lambda\tau} + \frac{\lambda_5}{\lambda_6} \right) V(t, x(t)) \\ (2.16) \quad &\leq (\sigma - \lambda) V(t, x(t)), \quad t \in [t^{**}, t^*]. \end{aligned}$$

From (2.7), (2.10), (2.11), (2.12) and (2.13) that

$$\begin{aligned} V(t^*, x(t^*)) &\leq V(t^{**}, x(t^{**}))e^{(\sigma-\lambda)(t^*-t^{**})} \\ (2.17) \quad &< \lambda_2 \|\varphi\|_\tau^2 e^{\sigma(t_1-t_0)} \leq V(t^*, x(t^*)), \end{aligned}$$

which is a contradiction. Hence (2.8) holds and then (2.3) is true for $k = 0$.

Suppose (2.3) holds for $k = 0, 1, 2, \dots, m$ ($m \in N, m \geq 0$), i.e.,

$$(2.18) \quad V(t, x(t)) \leq \lambda_2 K \|\varphi\|_\tau^2 e^{-\lambda(t-t_0)}, \quad t \in [t_k, t_{k+1}), \quad k = 0, \dots, m.$$

Next, we shall prove that (2.3) holds for $k = m + 1$, i.e.,

$$(2.19) \quad V(t, x(t)) \leq \lambda_2 K \|\varphi\|_\tau^2 e^{-\lambda(t-t_0)}, \quad t \in [t_{m+1}, t_{m+2}).$$

If (2.19) is not true, suppose

$$\bar{t} = \inf\{t \in [t_{m+1}, t_{m+2}) | V(t, x(t)) > \lambda_2 K \|\varphi\|_\tau^2 e^{-\lambda(t-t_0)}\}.$$

From condition (ii) and (2.18), we have

$$\begin{aligned} V(t_{m+1}, x(t_{m+1})) &= x^T(t_{m+1}^-) C_{m+1}^T P C_{m+1} x(t_{m+1}^-) \\ (2.20) \quad &\leq \lambda_6 V(t_{m+1}^-, x(t_{m+1}^-)) \\ &\leq \lambda_6 \lambda_2 K \|\varphi\|_\tau^2 e^{-\lambda(t_{m+1}-t_0)} \\ &< \lambda_2 K \|\varphi\|_\tau^2 e^{-\lambda(\bar{t}-t_0)}, \end{aligned}$$

and so $\bar{t} \neq t_{m+1}$. From the continuity of $V(t, x(t))$ in $[t_{m+1}, t_{m+2})$, we have

$$(2.21) \quad V(\bar{t}, x(\bar{t})) = \lambda_2 K \|\varphi\|_\tau^2 e^{-\lambda(\bar{t}-t_0)}, \quad t \in [t_{m+1}, \bar{t}]$$

By (2.20), we know that there exists some $t^* \in (t_{m+1}, \bar{t})$ such that

$$(2.22) \quad V(t^*, x(t^*)) = \lambda_6 \lambda_2 e^{\lambda(t_{m+2}-t_{m+1})} K \|\varphi\|_\tau^2 e^{-\lambda(\bar{t}-t_0)},$$

and

$$(2.23) \quad V(t^*, x(t^*)) \leq V(t, x(t)) \leq V(\bar{t}, x(\bar{t})), \quad t \in [t^*, \bar{t}].$$

From (2.19), (2.20), (2.22) and (2.23), we can obtain

$$\begin{aligned} x_{m+1}^T M_{m+1}^T M_{m+1} x_{m+1} &= x_{m+1}^{-T} C_{m+1}^T M_{m+1}^T M_{m+1} C_{m+1} x_{m+1}^- \\ (2.24) \quad &\leq \lambda_5 V(t_{m+1}^-, x(t_{m+1}^-)) \\ &= \lambda_2 e^{\lambda(t_{m+2}-t_{m+1})} K \|\varphi\|_\tau^2 e^{-\lambda(\bar{t}-t_0)} \\ &= \frac{\lambda_5}{\lambda_6} V(t^*, x(t^*)) \\ &\leq \frac{\lambda_5}{\lambda_6} V(t, x(t)), \quad t \in [t^*, \bar{t}] \end{aligned}$$

On the other hand, for any $t \in [t^*, \bar{t}]$, $s \in [-\tau, 0]$, then either $t + s \in [t_0 - \tau, t_{m+1})$ or $t + s \in [t_{m+1}, \bar{t}]$. Two cases will be discussed as follows:

If $t + s \in [t_0 - \tau, t_{m+1})$, from (2.18), we have

$$\begin{aligned} V(t + s, x(t + s)) &\leq \lambda_2 K \|\varphi\|_\tau^2 e^{-\lambda(t-t_0)} e^{-\lambda s} \\ (2.25) \qquad \qquad \qquad &\leq \lambda_2 e^{\lambda\tau} e^{\lambda(t_{m+2}-t_{m+1})} K \|\varphi\|_\tau^2 e^{-\lambda(\bar{t}-t_0)}. \end{aligned}$$

While, if $t + s \in [t_{m+1}, \bar{t}]$, from (2.21), then

$$\begin{aligned} V(t + s, x(t + s)) &\leq \lambda_2 K \|\varphi\|_\tau^2 e^{-\lambda(\bar{t}-t_0)} \\ (2.26) \qquad \qquad \qquad &\leq \lambda_2 e^{\lambda\tau} e^{\lambda(t_{m+2}-t_{m+1})} K \|\varphi\|_\tau^2 e^{-\lambda(\bar{t}-t_0)}. \end{aligned}$$

Form (2.25) and (2.26), in any case, we have for any $s \in [-\tau, 0]$,

$$\begin{aligned} V(t + s, x(t + s)) &\leq \frac{e^{\lambda\tau}}{\lambda_6} V(t^*, x(t^*)) \\ (2.27) \qquad \qquad \qquad &\leq \frac{e^{\lambda\tau}}{\lambda_6} V(t, x(t)), \quad t \in [t^*, \bar{t}]. \end{aligned}$$

Finally, from (i), (2.4), (2.24) and (2.27), we have

$$D^+V(t, x(t)) \leq \left(\lambda_3 + \frac{\lambda_4}{\lambda_6} e^{\lambda\tau} + \frac{\lambda_5}{\lambda_6} \right) V(t, x(t)) \leq (\sigma - \lambda) V(t, x(t)).$$

Thus, in view of condition (ii), we have

$$\begin{aligned} V(\bar{t}, x(\bar{t})) &\leq V(t^*, x(t^*)) e^{(\sigma-\lambda)(\bar{t}-t^*)} \\ (2.28) \qquad &< \lambda_2 e^{-(\sigma+\lambda)(t_{m+2}-t_{m+1})} e^{\lambda(t_{m+2}-t_{m+1})} K \|\varphi\|_\tau^2 e^{-\lambda(\bar{t}-t_0)} e^{(\sigma-\lambda)(\bar{t}-t^*)} \\ &< V(\bar{t}, x(\bar{t})), \end{aligned}$$

which is a contradiction. This implies the assumption is not true, and hence (2.3) holds for $k = m + 1$. Therefore, we can obtain that (2.3) holds for any $k \in N$.

By (2.2) and (2.3), we have

$$\|x(t)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}} K \|\varphi\|_\tau e^{-\frac{\lambda}{2}(t-t_0)}, \quad t \geq t_0,$$

which implies that the zero solution of the linear impulsive delayed hybrid dynamical system (1.1) is globally exponentially stable with convergence rate $\frac{\lambda}{2}$ for any time delay $0 \leq \tau(t) \leq \tau < +\infty$.

Then, we complete the proof of Theorem 2.1. ■

Corollary 2.1. *Let $\lambda_3, \lambda_4, \lambda_5, \lambda_6$ and condition (iii) be is precisely the same as that of Theorem 2.1, and assume that for all $k \in N$,*

$$(iii) \left(\lambda_3 + \frac{\lambda_4 + \lambda_5}{\lambda_6} \right) (t_{k+1} - t_k) < -\ln \lambda_6.$$

Then, for any delay $0 \leq \tau(t) \leq \tau < +\infty$, the zero solution of the linear impulsive delayed hybrid dynamical system (1.1) is globally exponentially stable.

Consider the following impulsive delayed nonlinear dynamical hybrid systems:

$$(2.29) \quad \begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t), x(t - \tau(t))) + M_k x_k, & t \in [t_k, t_{k+1}), \\ x(t) = C_k x(t^-), & t = t_k, k \in N, \\ x_{t_0} = \varphi, \end{cases}$$

where $f : R^+ \times R^n \times R^n \rightarrow R^n$ is a continuously vector-valued function, $f(t, 0, 0) = 0$, which satisfies the following condition

(H): $\|f(t, x(t), x(t - \tau(t)))\|^2 \leq l_1 \|x(t)\|^2 + l_2 \|x(t - \tau(t))\|^2$, where l_1 and l_2 are positive numbers.

In general, the condition (H) is very mild, since some general dissipative dynamical systems, such as neural networks, coupled chaos oscillators, and networks of multi-agent, etc., can be included in condition (H).

Theorem 2.2. Let the $n \times n$ matrix P be symmetric and positive definite, λ_3 be the largest eigenvalue of $P^{-1}(A^T P + PA + 2PP + l_1)$, λ_4 be the largest eigenvalue of $l_2 P^{-1}$, λ_{M_k} be the largest eigenvalue of $P^{-1} C_k^T M_k^T M_k C_k$, λ_{C_k} be the largest eigenvalue of $P^{-1} C_k^T P C_k$, $\lambda_5 = \sup_{k \in N} \lambda_{M_k}$, $\lambda_6 = \sup_{k \in N} \lambda_{C_k}$ and $0 < \lambda_6 < 1$. Assume both conditions (i) and (ii) are satisfied in Theorem 2.1.

Then, for any time delay $0 \leq \tau(t) \leq \tau < +\infty$, the zero solution of the nonlinear impulsive delayed dynamical hybrid system (2.29) is globally exponentially stable with convergence rate $\frac{\lambda}{2}$.

Proof. Similarly to the proof of Theorem 2.1, by the Lyapunov function (2.1), for $t \in [t_k, t_{k+1})$, we calculate the upper right derivative of $V(t, x(t))$ along the solution of system (2.29) and have

$$\begin{aligned} D^+ V(t, x(t)) &= x^T(t) (A^T P + PA) x(t) \\ &\quad + 2x^T(t) P f(t, x(t), x(t - \tau(t))) + 2M_k x_k x(t) \\ (2.30) \quad &\leq x^T(t) (A^T P + PA + PP + l_1 P^{-1} P) x(t) \\ &\quad + l_2 x^T x(t - \tau(t)) x(t - \tau(t)) + x_k^T M_k^T M_k x_k \\ &\leq \lambda_3 V(t, x(t)) + \lambda_4 V(t - \tau(t), x(t - \tau(t))) + x_k^T M_k^T M_k x_k. \end{aligned}$$

The rest of the proof is precisely the same as Theorem 2.1, so we omit it. Theorem 2.2 is proven. $\blacksquare$

Corollary 2.2. *Let $\lambda_3, \lambda_4, \lambda_5$ and λ_6 be is precisely the same as that of Theorem 2.2, and assume that for all $k \in N$,*

$$(iv) \left(\lambda_3 + \frac{\lambda_4 + \lambda_5}{\lambda_6} \right) (t_{k+1} - t_k) < -\ln \lambda_6.$$

Then, for any delay $0 \leq \tau(t) \leq \tau < +\infty$, the zero solution of the nonlinear impulsive delayed hybrid dynamical system (2.29) is globally exponentially stable.

Remark 2.1. Obviously, Corollary 2.1 has extended the Theorem 1 of [21] concerning the local uniform stability to the global exponential stability for the linear impulsive delayed hybrid dynamical system (1.1) under the same conditions. Therefore, Theorem 2.1 is an important improvement and generalization of the main results in [21]. In addition, it is interested that if the linear impulsive delayed hybrid dynamical systems are local uniform stable, then it must be globally exponentially stable.

Remark 2.2. Theorem 2.1, 2.2 and Corollary 2.1, 2.2 are valid for any time delay $0 \leq \tau(t) \leq \tau < +\infty$. Therefore, our results are more practically applicable than those in the literature since the restrictive condition that the time delays are less than the length of all the impulsive intervals is actually removed here (see for example: Theorem 3.1 of [20]). Furthermore, our results show that impulse and hybrid term do contribute to the global exponential stability of linear or nonlinear impulsive delayed hybrid dynamical systems even if the corresponding systems without impulses are chaotic or unstable.

Remark 2.3. Let $h(\lambda) = (-\ln \lambda_6) / \left(2\lambda + \lambda_3 + \frac{\lambda_4}{\lambda_6} e^{\lambda\tau} + \frac{\lambda_5}{\lambda_6} \right)$, in Theorem 2.1, the satisfaction of both (i) and (ii) is equivalent to $\Delta t_k < h(\lambda)$, if we require that the exponential convergence rate of system (1.1) or (2.29) is larger than or equal to any given $\frac{\lambda^*}{2} > 0$, we can choose any suitable $\Delta t_k < \Delta t_k^*$, such that system (1.1) or (2.29) is globally exponentially stable with exponential convergence rate $\frac{\lambda}{2} \geq \frac{\lambda^*}{2} > 0$, where $\Delta t_k^* = h(\lambda^*)$.

3. Application examples

In this section, we shall discuss the applications of the above theoretic criteria. Two examples and their simulations are given to show that our main results are practical.

Example 3.1. Consider the following linear impulsive delayed hybrid dynamical systems:

$$(3.1) \quad \begin{cases} \dot{x}(t) = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{4} & 5 \end{pmatrix} x(t) + \begin{pmatrix} 2 & 2 \\ \frac{1}{3} & 6 \end{pmatrix} x(t - \tau(t)) + \begin{pmatrix} \frac{1}{k+1} & 0 \\ 0 & \frac{1}{k+1} \end{pmatrix} x_k, \\ t \in [t_k, t_{k+1}), \\ x(t) = \begin{pmatrix} \frac{3k+1}{k^2+1} & 0 \\ 0 & \frac{4k+1}{k^2+1} \end{pmatrix} x(t^-), \\ t = t_k, k \in N, \end{cases}$$

where $x(t) = (x_1(t), x_2(t))^T \in R^2$.

Let $P = \begin{pmatrix} 9 & 0 \\ 0 & 12 \end{pmatrix}$, then $\lambda_3 = 34.0371$, $\lambda_4 = 3.4449$, $\lambda_5 = 0.1302$,

$\lambda_6 = 0.5208$, which implies Theorem 2.1 hold. By condition (iii) of Corollary 2.1, if $\Delta t_k < 0.0159$ the zero solution of system (3.1) is globally exponentially stable for any time delay. The numerical simulation with $\Delta t_k = 0.013$, $\tau = 0.3$ and initial functions $(\varphi_1(t), \varphi_2(t))$, where

$$(3.2) \quad \begin{cases} \varphi_1(t) = \begin{cases} 0, & t \in [-0.3, 0), \\ -1, & t = 0, \end{cases} \\ \varphi_2(t) = \begin{cases} 0, & t \in [-0.3, 0), \\ 1, & t = 0, \end{cases} \end{cases}$$

is given in Fig. 1, the global exponential convergence rate is 0.9861. The system (3.1) without impulses and hybrid term is unstable with the initial functions (3.2) (see Fig. 2).

It should be noted that the global exponential stability cannot be derived by applying existing criterion in [21], where only local uniform stability is proved under the same conditions. Therefore, Theorem 2.1 or Corollary 2.1 substantially extends and improves the main results of [21].

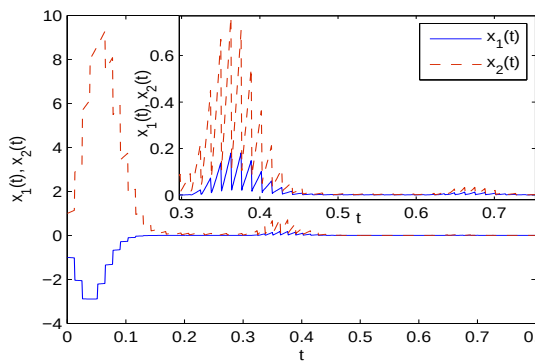


Fig. 1. Global exponential stability of system (3.1) with $\Delta t_k = 0.013$.

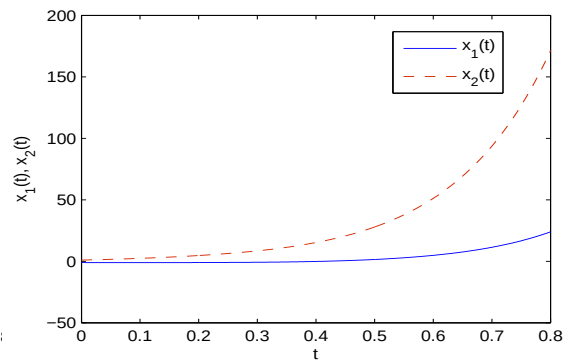


Fig. 2. Instability of system (3.1) without impulses and hybrid term.

Example 3.2. Consider the following Machey-Glass [22] nonlinear delayed impulsive hybrid model:

$$(3.3) \begin{cases} \dot{x}(t) = -0.68x(t) + \frac{1.7x(t-\tau(t))}{1+x^8(t-\tau(t))} + \frac{3}{k+2}x_k, & t \in [t_k, t_{k+1}), \\ x(t) = \frac{k+1}{2k^2+1}x(t^-), & t = t_k, k \in N, \end{cases}$$

where $0 \leq \tau(t) \leq \tau < +\infty$. The corresponding system (3.3) without impulse and hybrid term is a chaotic system. Fig. 3 shows its simulation results for $\tau = 4$ with the initial function:

$$(3.4) \quad \varphi(t) = \begin{cases} 0, & t \in [-4, 0), \\ 1.2, & t = 0, \end{cases}$$

It is easy to see that condition **(H)** holds for

$$f(t, x(t), x(t-\tau(t))) = \frac{1.7x(t-\tau(t))}{1+x^8(t-\tau(t))}$$

with $l_1 = 0$ and $l_2 = 1.7$. By taking $P = 13$, $\lambda_3 = 24.64$, $\lambda_4 = 0.0452$, $\lambda_5 = 0.1731$, $\lambda_6 = 0.0769$, then all the conditions of Theorem 2.2 are satisfied. By the condition (iv) of Corollary 2.2, if $\Delta t_k < 0.0933$ the zero solution of system (3.3) is globally exponentially stable for any time delay. The numerical simulation with $\Delta t_k = 0.092$ and the initial function (3.4) is given in Fig. 4, the global exponential convergence rate is 0.0422.

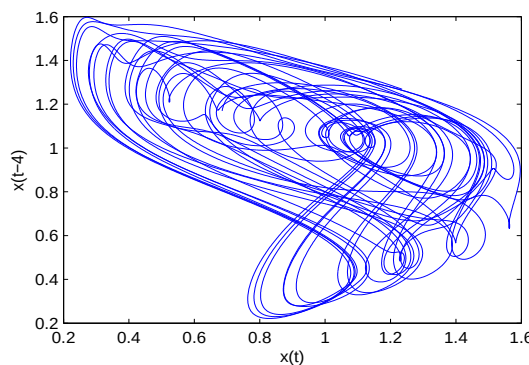


Fig. 3. Chaotic behaviors of system (3.3) without impulse and hybrid term.

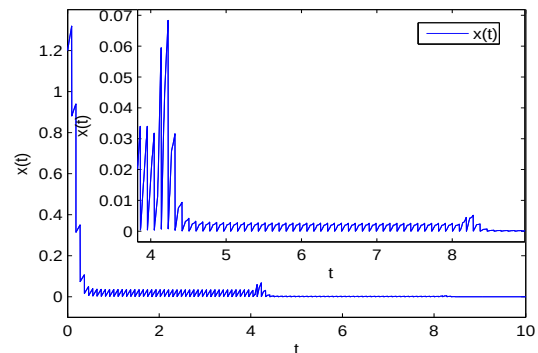


Fig. 4. Global exponential stability of system (3.3) with $\Delta t_k = 0.092$.

By Remark 2.3, let $P = 13, \tau = 4$, Fig. 5 shows the relationship between λ and Δt_k in system (3.3). By Remark 2.3, in Theorem 2.2, the satisfaction of both (i) and (ii) is equivalent to $\Delta t_k < h(\lambda)$. Hence, in Fig. 5, **D** can denote the globally exponentially stable region.

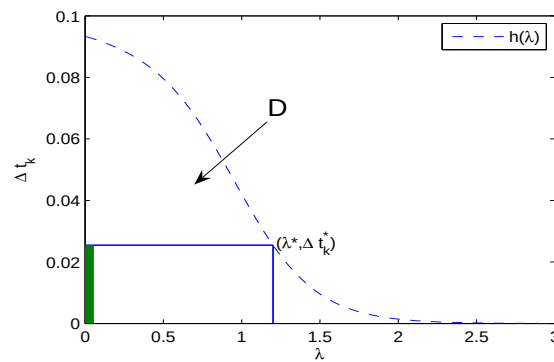


Fig. 5. Globally exponentially region of system (3.3) for $(\lambda, \Delta t_k)$.

It can be seen from Example 3.2 that the impulses hybrid terms can contribute to global exponential stability for nonlinear delayed dynamical systems even if the corresponding systems without impulses and hybrid terms may be unstable or chaotic itself, which can be usually used as an effective control strategy to stabilize the underlying delayed dynamical systems in some practical applications.

4. Conclusions

In this paper, the global exponential stability criteria of linear and nonlinear impulsive hybrid dynamical systems for any time delay are investigated. By employing the methods of Razumikhin technique and Lyapunov function, some global exponential stability criteria have been established. Our results have improved and generalized some of the known results existing in the literature. Two examples are given to illustrate the theoretical results. Furthermore, it is shown that the impulses and hybrid term can stabilize an unstable or even chaotic dynamical system, which is particularly meaningful for control and design of hybrid dynamical systems in practice.

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Accepted: 23.01.2014

FROM NEWTON TO KEPLER.

One simple derivation of Kepler's laws from Newton's ones.

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Abstract. There is plenty of ways how to deduce Kepler's laws of planetary motion from Newton's ones (law of universal gravitation and law of motion). We offer one of them which is very simple and direct. It uses only mathematical tools and is suitable for teaching purposes.

1. Kepler's and Newton's laws

Johannes Kepler (1571–1630) deduced his laws of planetary motion due to the exact measurement of planets performed by astronomer Tycho Brahe (1546–1601). The first and the second law were derived in Prague and published in work *Astronomia Nova* in 1609. The third one was formulated in Linz and published *Harmoniae Mundi* in 1619.

Sixty five years later (in 1684), Isaac Newton (1643–1727) deduced his law of universal gravitation from Kepler's laws.

Plenty of literature deals with the relation between Kepler's and Newton's laws. The most of them use physical concepts, e.g. [1] and [2]. The geometrical way of derivation was discussed in Richard Feynmann lecture presented on 13th March 1964 at Caltech (its manuscript had been lost and it was found again in 1992 at the office of author's colleague Leighton) and it is very interesting [3].

In this text, we would like to present one way how to derive Kepler's laws from the Newton's law of universal gravitation and motion. The aim and advantage of our deduction of especially the first one is its simplicity, directness, straightness and using only mathematical tools without introducing other physical variables. It can be presented maybe even in secondary schools.

2. Formulation of the laws

Let us formulate these laws at first.

- Kepler's laws of planetary motion:

(K1) The orbit of every planet is an ellipse with the Sun at one of the two foci.

(K2) A line joining a planet and the Sun sweeps out equal areas during equal intervals of time, i.e. area velocity is constant.

(K3) Quotient $\frac{T^3}{a^3}$ is constant, where T is orbital period of any planet and a is the major semi-axis of its orbit.

- Newton's laws of universal gravitation and the second law of motion:

(NG) Any point mass M attracts every single other point mass m situated in the distance r from M by a force

$$F = \kappa \frac{mM}{r^2}, \quad \text{where } \kappa = 6.672 \cdot 10^{-11} \text{ Nm}^2/\text{kg}^2$$

(acting in direction of radius-vector of both points).

(N2) The acceleration a of a body is parallel and directly proportional to the net force F acting on the body,

$$F = am, \quad \text{where } m \text{ is mass of a body.}$$

3. Derivation of Kepler's laws

At first, we will describe the situation in Figure 1. In the beginning, the point mass m (planet) in the distance r_0 from the point mass M (the Sun) has the velocity v_0 in the direction perpendicular to the radius-vector of both points (M and m).

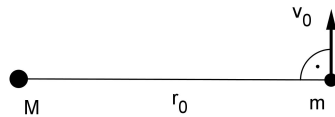


Figure 1: Situation and start condition

Let the point mass M be situated to origin of axes and let point mass m be described by the position vector $\vec{r}(t) = (x(t), y(t), z(t))$. According to (NG),

the force by which the point mass m is attracted to the point mass M can be expressed by the formula

$$\vec{F} = -\kappa \cdot \frac{mM}{|\vec{r}|^3} \cdot \vec{r} = -k \cdot \frac{m\vec{r}}{|\vec{r}|^3}, \quad \text{where } k = \kappa M. \quad (1)$$

By (N2) it holds

$$\vec{F} = m \cdot \ddot{\vec{r}}. \quad (2)$$

From these two relations (1) and (2) we obtain equation

$$\ddot{\vec{r}} + k \cdot \frac{\vec{r}}{|\vec{r}|^3} = 0. \quad (3)$$

3.1. Trajectory is a plane curve

At first, we derive that the trajectory of point mass m is situated in some plane containing the point mass M .

This fact will be proved if we show that the velocity $\dot{\vec{r}}$ of the point m is situated in the same plane as radius-vector $\vec{r}$ of that point, i.e., when $\vec{r} \times \dot{\vec{r}}$ is a constant vector. Let us calculate the derivation of this cross product using relation (3)

$$\frac{d}{dt}(\dot{\vec{r}} \times \vec{r}) = \ddot{\vec{r}} \times \vec{r} + \underbrace{\dot{\vec{r}} \times \dot{\vec{r}}}_{=0} = -\frac{k}{|\vec{r}|^3}(\vec{r} \times \vec{r}) = 0.$$

That is why the cross product $\vec{r} \times \dot{\vec{r}}$ is really constant. So, it is proved that the motion of a point in so called central-force field is a plane curve.

3.2. Trajectory is an ellipse (1. Kepler law)

We can assume for further calculation that the motion is situated in the plane xy and we shall describe this motion by position vector $\vec{r}(t) = (x(t), y(t))$ and $\vec{r}(0) = (r_0, 0)$ and $\dot{\vec{r}}(0) = (0, v_0)$ in consistency with initial conditions.

We transform the position vector to the polar coordinates r and φ

$$\vec{r} = (x, y) = (r \cos \varphi, r \sin \varphi).$$

We can differentiate twice

$$\dot{\vec{r}} = (\dot{x}, \dot{y}) = \dot{r}(\cos \varphi, \sin \varphi) + r\dot{\varphi}(-\sin \varphi, \cos \varphi),$$

$$\ddot{\vec{r}} = (\ddot{x}, \ddot{y}) = (\ddot{r} - r\dot{\varphi}^2)(\cos \varphi, \sin \varphi) + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})(-\sin \varphi, \cos \varphi)$$

and rewrite the initial conditions $\dot{r}(0) = 0$ and $\dot{\varphi}(0) = \frac{v_0}{r_0}$.

This derivatives can be put into (3) and we obtain

$$(\ddot{r} - r\dot{\varphi}^2 + \frac{k}{r^2})(\cos \varphi, r \sin \varphi) + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})(-\sin \varphi, \cos \varphi) = 0.$$

From here, we have the equations

$$\ddot{r} - r\dot{\varphi}^2 + \frac{k}{r^2} = 0 \quad \text{and} \quad 2\dot{r}\dot{\varphi} + r\ddot{\varphi} = 0. \quad (4)$$

We multiply both sides of the second part of (4) by r and get

$$0 = 2r\dot{r}\dot{\varphi} + r^2\ddot{\varphi} = \frac{d}{dt}(r^2\dot{\varphi}),$$

and so, $r^2\dot{\varphi}$ is a constant. The initial conditions give us

$$\dot{\varphi} = \frac{r_0 v_0}{r^2}. \quad (5)$$

We put this result into the first part of (4)

$$\ddot{r} - \frac{r_0^2 v_0^2}{r^3} + \frac{k}{r^2} = 0 \quad (6)$$

and we deal similarly with this equation (6). We multiply both sides of it by $\dot{r}$ and we obtain

$$0 = \dot{r}\ddot{r} - r_0^2 v_0^2 \cdot \frac{\dot{r}}{r^3} + k \cdot \frac{\dot{r}}{r^2} = \frac{d}{dt} \left(\frac{1}{2} \dot{r}^2 + \frac{1}{2} r_0^2 v_0^2 \cdot \frac{1}{r^2} - k \cdot \frac{1}{r} \right).$$

This derivative is equal to zero and it again proves that

$$\dot{r}^2 + r_0^2 v_0^2 \cdot \frac{1}{r^2} - 2k \cdot \frac{1}{r}$$

is a constant. With respect to the initial conditions we get

$$\dot{r}^2 + r_0^2 v_0^2 \cdot \frac{1}{r^2} - 2k \cdot \frac{1}{r} = v_0^2 - 2k \cdot \frac{1}{r_0}. \quad (7)$$

We can express the derivative $\dot{r}$ from (7)

$$\dot{r} = \sqrt{v_0^2 - 2k \cdot \frac{1}{r_0} - \frac{r_0^2 v_0^2}{r^2} + 2k \cdot \frac{1}{r}}. \quad (8)$$

Now, we would like to express φ as a function of r , so we are looking for the function g such that $\varphi = g(r)$ (see Figure 2).

The chain rule for derivatives gives us

$$\frac{d\varphi}{dt}(t) = \frac{dg}{dr}(r(t)) \cdot \frac{dr}{dt}(t) \quad \text{or} \quad \dot{\varphi}(t) = \frac{d\varphi}{dr}(r(t)) \cdot \dot{r}(t).$$

We put (5) and (8) into this relation and we obtain

$$\frac{r_0 v_0}{r^2} = \frac{d\varphi}{dr} \cdot \sqrt{v_0^2 - 2k \cdot \frac{1}{r_0} - \frac{r_0^2 v_0^2}{r^2} + 2k \cdot \frac{1}{r}}. \quad (9)$$

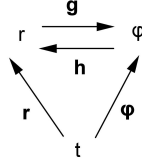


Figure 2: Functions

We can rewrite the equation (9) into form

$$\frac{d\varphi}{dr} = \frac{r_0 v_0}{r \sqrt{Ar^2 + Br + C}}, \quad (10)$$

where

$$A = v_0^2 - 2k \cdot \frac{1}{r_0}, \quad B = 2k, \quad C = -r_0^2 v_0^2 < 0$$

and the discriminant

$$D = 4(k - r_0 v_0^2)^2 > 0.$$

We shall solve (10) by integration

$$\varphi = \frac{r_0 v_0}{\sqrt{-C}} \arcsin \frac{Br + 2C}{r\sqrt{D}} + c_1 = \arcsin \frac{2kr - 2r_0^2 v_0^2}{2r(k - r_0 v_0^2)} + c_1.$$

The constant $c_1 = -\frac{\pi}{2}$ can be got from initial condition, hence

$$\varphi = \arcsin \frac{r - \frac{1}{k} r_0^2 v_0^2}{r(1 - \frac{1}{k} r_0 v_0^2)} - \frac{\pi}{2}.$$

We continue the calculation

$$\cos \varphi = \sin(\varphi + \frac{\pi}{2}) = \frac{r - \frac{1}{k} r_0^2 v_0^2}{r(1 - \frac{1}{k} r_0 v_0^2)}$$

and

$$r = \frac{\frac{1}{k} r_0^2 v_0^2}{1 - (1 - \frac{1}{k} r_0 v_0^2) \cos \varphi}. \quad (11)$$

This is an equation of conic section

$$r = \frac{p}{1 + \epsilon \cos \varphi},$$

where

$$p = \frac{1}{k} r_0^2 v_0^2 \geq 0 \quad \text{and} \quad \epsilon = \frac{1}{k} r_0 v_0^2 - 1 \geq -1.$$

Parameter $\epsilon = 0$ corresponds to the circle (and to the formula about the first cosmic speed $v_1 = \sqrt{\frac{\kappa M}{r_0}}$), parameter $\epsilon = 1$ corresponds to the parabola (and to the relations concerning the second cosmic speed $v_2 = \sqrt{\frac{2\kappa M}{r_0}}$). The elliptic trajectory remains for $|\epsilon| < 1$, i.e., $v_0 < v_2$. So we got the 1. Kepler law from here.

3.3. Area velocity is constant (2. Kepler's law)

In order to derive the 2. Kepler law, we must express the area, which is swept out by radius-vector of point mass m between time t_1 and t_2 in polar coordinates, $\Delta t = t_2 - t_1$. We use again the expression of r by φ , i.e., $r = h(\varphi)$, function h is inverse to the function g (see Figure 2).

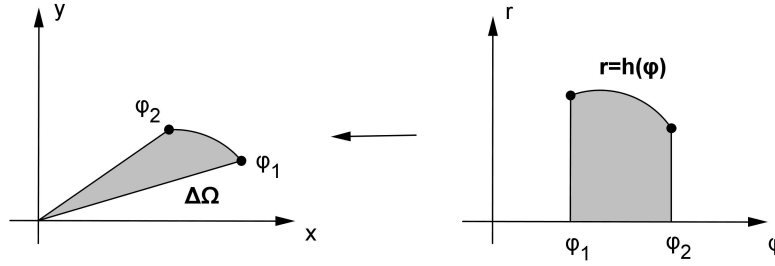


Figure 3: Area of radius-vector

Let us calculate the area of $\Delta\Omega$. We use step by step the translation to the polar coordinates (see Figure 3), the Fubini theorem and the substitution $\varphi = \varphi(t)$

$$\begin{aligned}
 |\Delta\Omega| &= \iint_{\Delta\Omega} 1 \, dx dy = \iint_D r \, dr d\varphi \\
 &= \int_{\varphi_1}^{\varphi_2} \left(\int_0^{h(\varphi)} r \, dr \right) d\varphi = \int_{\varphi_1}^{\varphi_2} \frac{1}{2} h^2(\varphi) d\varphi \\
 &= \int_{t_1}^{t_2} \frac{1}{2} h^2(\varphi(t)) \cdot \dot{\varphi}(t) dt = \int_{t_1}^{t_2} \frac{1}{2} r^2(t) \cdot \dot{\varphi}(t) dt.
 \end{aligned}$$

Now, we use relation (5) and get

$$|\Delta\Omega| = \int_{t_1}^{t_2} \frac{1}{2} v_0 r_0 dt = \frac{1}{2} v_0 r_0 (t_2 - t_1) = \frac{1}{2} v_0 r_0 \Delta t, \quad (12)$$

which gives us the 2. Kepler law about the constant area speed of radius-vector of motion.

3.4. Relation between period and semi-major axes (3. Kepler's law)

The 3. Kepler's law is a simple consequence of obtained results. We shall use (12) for $t_1 = 0$ and $t_2 = T$, where T is a period of motion of point mass m , and we can compare it with formula of ellipse area with semi-axes a and b

$$\pi ab = \frac{1}{2} r_0 v_0 T. \quad (13)$$

The relations

$$a = \frac{p}{1 - \epsilon^2} \quad \text{and} \quad b = a\sqrt{1 - \epsilon^2} \quad (14)$$

hold for parameters p, ϵ used for expression of ellipse in polar coordinates and semi-axes a, b used in cartesian coordinates. From (14) we have $b = \sqrt{ap}$ and (13) gives us $\pi a\sqrt{ap} = \frac{1}{2} r_0 v_0 T$. Because $p = \frac{1}{k} v_0^2 r_0^2$, we have

$$\pi a \sqrt{\frac{a}{k}} r_0 v_0 = \frac{1}{2} r_0 v_0 T$$

and from here we obtain

$$\frac{T^2}{a^3} = \frac{4\pi^2}{k},$$

which gives the 3. Kepler's law.

3.5. Mathematical apparatus

Finally, we summarize the mathematical apparatus used in our deductions:

1. properties of vectors, the dot (scalar, inner) product and cross (vector) product;
2. equation of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and formula of ellipse area $P = ab\pi$, where a and b are semi-axes;
3. polar coordinates $x = r \cos \varphi$, $y = r \sin \varphi$;
4. equation of conic section in polar coordinates $r(1 + \epsilon \cos \varphi) = p$, where p is parameter and ϵ is angular eccentricity, conic section is a hyperbola for $|\epsilon| > 1$, parabola for $|\epsilon| = 1$, ellipse for $|\epsilon| < 1$, (circle for $|\epsilon| = 0$); relations $b^2 = a^2(1 - \epsilon^2)$, $p = a(1 - \epsilon^2)$, where a, b are semi-axes of ellipse;
5. chain rule for derivative, derivative of cross product;
6. integral

$$\int \frac{dx}{x\sqrt{Ax^2 + Bx + C}} = \frac{1}{\sqrt{-C}} \arcsin \frac{Bx + 2C}{x\sqrt{D}} + c_1,$$

where $C < 0$, $D = B^2 - 4AC > 0$;

7. double integral, its property, Fubini theorem and substitution theorem;
8. formula of ellipse area $P = ab\pi$.

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Accepted: 30.01.2014

HOPF MODULES IN THE WEAK YETTER-DRINFELD CATEGORIES

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Abstract. Suppose that L is a weak Hopf algebra over the field k with a bijective antipode and H is a weak Hopf algebra in the weak Yetter-Drinfeld category ${}^L\mathcal{YD}$. We prove that the fundamental theorem for right H -Hopf modules in ${}^L\mathcal{YD}$.

Keywords: Weak Hopf algebra, Yetter-Drinfeld module, Fundamental theorem.

AMS 2010 Subject Classification: 16W30, 053.3.

1. Introduction

Weak Hopf algebras have been proposed by G. Böhm, F. Nill and K. Szlachanyi as a generalization of ordinary Hopf algebras in the following sense: the defining axioms are the same, but the multiplicativity of the counit and the comultiplicativity of the unit are replaced by weaker axioms. The initial motivation to study weak Hopf algebras is their connection with the theory of algebra extension [1], and another important application of weak Hopf algebras is that they provide a natural framework for the study of dynamical twists in Hopf algebras [2].

Just like finite-dimensional Hopf algebras, finite-dimensional weak Hopf algebra also obey the mathematical beauty of giving rise to a self-dual notion: the dual of it can be canonically endowed with a weak Hopf algebra structure. The notion of a weak Yetter-Drinfeld category ${}^L\mathcal{YD}$ over a weak Hopf algebra L has been introduced by Böhm in [3], and further studied by Caenepeel in [4]. The paper [5] proves that if H is a finite-dimensional weak Hopf algebra in the category ${}^L\mathcal{YD}$ over a weak Hopf algebra L , then its linear dual H^* is also a weak Hopf algebra in ${}^L\mathcal{YD}$.

In this paper, we prove the fundamental theorem for right H -Hopf modules in ${}^L\mathcal{YD}$. We also show that if H is dimensional, its dual H^* has a right H -Hopf module structure which is not analogous to usual one.

2. Preliminaries

A weak Hopf algebra is a vector space L with the structure of an associative unital algebra (L, m, μ) with multiplication $m : L \otimes L \longrightarrow L$ and unit $1 \in L$ and a coassociative coalgebra (L, Δ, ε) with comultiplication $\Delta : L \longrightarrow L \otimes L$ and counit $\varepsilon : L \longrightarrow k$ such that

(i) The comultiplication Δ is a (not necessarily unit-preserving) homomorphism of algebras such that

$$(2.1) \quad (\Delta \otimes id)\Delta(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1).$$

(ii) The counit satisfies the following identity

$$(2.2) \quad \varepsilon(kgl) = \varepsilon(kg_1)\varepsilon(g_2l) = \varepsilon(kg_2)\varepsilon(g_1l), \quad \forall k, g, l \in L.$$

(iii) There is a linear map $S_L : L \longrightarrow L$ called an antipode, such that, for all $l \in L$,

$$(2.3) \quad m(id \otimes S_L)\Delta(l) = (\varepsilon \otimes id)(\Delta(1)(l \otimes 1)),$$

$$(2.4) \quad m(S_L \otimes id)\Delta(l) = (id \otimes \varepsilon)((1 \otimes l)\Delta(1)),$$

$$(2.5) \quad S_L(l) = S_L(l_1)l_2S_L(l_3).$$

The linear map defined in (2.3) and (2.4) are called target and source counital maps and denoted by ε_t and ε_s respectively:

$$(2.6) \quad \begin{aligned} \varepsilon_t(l) &= \varepsilon(1_{(1)}l)1_{(2)} = \varepsilon(S_L(l)1_{(1)})1_{(2)}, \\ \varepsilon_s(l) &= 1_{(1)}\varepsilon(l1_{(2)}) = 1_{(1)}\varepsilon(1_{(2)}S_L(l)). \end{aligned}$$

For all $l \in L$, we have

$$(2.7) \quad \begin{aligned} l_1 \otimes \varepsilon_t(l_2) &= 1_{(1)}l \otimes 1_{(2)}, \\ \varepsilon_s(l_1) \otimes l_2 &= 1_{(1)} \otimes l1_{(2)}, \end{aligned}$$

$$(2.8) \quad \begin{aligned} l_1 \otimes \varepsilon_s(l_2) &= l1_{(1)} \otimes S_L(1_{(2)}), \\ \varepsilon_t(l_1) \otimes l_2 &= S_L(1_{(1)}) \otimes 1_{(2)}l. \end{aligned}$$

We will briefly recall the necessary definitions and notions on the weak Hopf algebras.

Definition 2.1. An algebra H is a left L -comodule algebra if H is a left L -comodule via $x \longmapsto \sigma_H(x) = x^{-1} \otimes x^0$ such that

- (1) $\sigma_H(xy) = \sigma_H(x)\sigma_H(y) = x^{-1}y^{-1} \otimes x^0y^0$,
- (2) $1^{-1} \otimes x1^0 = \varepsilon_s(x^{-1}) \otimes x^0 \quad \forall x \in H$.

Applying (2), we have the following equality

$$\begin{aligned} 1^{-1} \otimes 1^0 &= \varepsilon_s(1^{-1}) \otimes 1^0 = S_L(1_{(2)}) \otimes 1_{(1)} \longrightarrow 1, \\ &= S_L(1_{(2)}) \otimes \varepsilon_t(1_{(1)}) \longrightarrow 1, \\ &= 1_{(1)} \otimes 1_{(2)} \longrightarrow 1 \end{aligned}$$

Definition 2.2. An algebra H is a left L -module algebra if H is a left L -module via $l \otimes x \mapsto l \longrightarrow x$ such that

- (1) $l \longrightarrow xy = (l_1 \longrightarrow x)(l_2 \longrightarrow y)$,
- (2) $l \longrightarrow 1 = \varepsilon_t(l) \longrightarrow 1, \quad \forall x, y \in H, l \in L$,

the second equality is equivalent to $\varepsilon_t(l) \longrightarrow x = (l \longrightarrow 1)x$.

Definition 2.3. An algebra H is a left L -module coalgebra if H is a left L -module via $l \otimes x \mapsto l \longrightarrow x$ such that

- (1) $\Delta(l \longrightarrow x) = (l \longrightarrow x)_1 \otimes (l \longrightarrow x)_2 = (l_1 \longrightarrow x_1) \otimes (l_2 \longrightarrow x_2)$,
- (2) $\varepsilon_s(l) \longrightarrow x = x_1 \varepsilon(l \longrightarrow x_2), \quad \forall l \in L, x \in H$,

the second equation is equivalent to

$$\varepsilon(lk \longrightarrow h) = \varepsilon(lk_2) \varepsilon(k_1 \longrightarrow h), \quad \varepsilon(\varepsilon_s(l) \longrightarrow h) = \varepsilon(l \longrightarrow h), \quad l, k \in L, h \in H.$$

Definition 2.4. An algebra H is a left L -comodule coalgebra if H is a left L -comodule via $x \mapsto \sigma_H(x) = x^{-1} \otimes x^0$ such that

- (1) $x^{-1} \otimes (x^0)_1 \otimes (x^0)_2 = x_1^{-1} x_2^{-1} \otimes x_1^0 \otimes x_2^0$,
- (2) $\varepsilon(x^0) x^{-1} = \varepsilon(x^0) \varepsilon_t(x^{-1}) \quad \forall x \in H$.

3. Weak Hopf algebras in weak Yetter-Drinfeld category

Let L be a weak Hopf algebra with a bijective antipode S_L . We recall that the weak Yetter-Drinfeld category ${}^L_L\mathcal{YD}$ is the braided monoidal categories whose objects V are both left L -modules and satisfy the following conditions:

- (1) $\sigma_V(v) = v^{-1} \otimes v^0 \in L \otimes_t V = \{1_{(1)}l \otimes 1_{(2)} \longrightarrow v | \forall l \in L, v \in V\}$,
- (2) $l_1 v^{-1} \otimes l_2 \longrightarrow v^0 = (l_1 \longrightarrow v)^{-1} l_2 \otimes (l_1 \longrightarrow v)^0$ i.e.,
- (3) $\sigma_V(l \longrightarrow v) = (l \longrightarrow v)^{-1} \otimes (l \longrightarrow v)^0 = l_1 v^{-1} S_L(l_3) \otimes l_2 \longrightarrow v^0$,

where the L -module action is denoted by $l \longrightarrow v$ for $l \in L, v \in V$ and the L -comodule structure map is denoted by $\sigma_V : V \longrightarrow L \otimes V$. We use the following notation:

$$\sigma_V(v) = v^{-1} \otimes v^0, \quad (\Delta \otimes id)\sigma_V(v) = (id \otimes \sigma_V)\sigma_V(v) = v^{-2} \otimes v^{-1} \otimes v^0.$$

The braiding $\tau = \tau_{V, W} : V \otimes_t W \longrightarrow W \otimes_t V$ in this category is given by

$$\begin{aligned}\tau(1_{(1)} \longrightarrow v \otimes 1_{(2)} \longrightarrow w) &= v^{-1} \longrightarrow w \otimes v^0, \\ \tau^{-1}(1_{(1)} \longrightarrow w \otimes 1_{(2)} \longrightarrow v) &= v^0 \otimes S_L^{-1}(v^{-1}) \longrightarrow w.\end{aligned}$$

Let $V \in {}^L_L\mathcal{YD}$. Then, for all $v \in V$, we have

$$\begin{aligned}\varepsilon_s(v^{-1}) \otimes v^0 &= S_L(1_{(2)}) \otimes 1_{(1)} \longrightarrow v, \\ \varepsilon_t(v^{-1}) \otimes v^0 &= S_L(1_{(1)}) \otimes 1_{(2)} \longrightarrow v, \forall v \in V.\end{aligned}$$

In [5], Shen Bing-liang introduces the definition of weak Hopf algebra in the weak Yetter-Drinfeld category ${}^L_L\mathcal{YD}$. Moreover, they have showed that if H is a finite-dimensional weak Hopf algebra in ${}^L_L\mathcal{YD}$, then its dual H^* is a weak Hopf algebra in ${}^L_L\mathcal{YD}$.

Definition 3.1. Let L be a weak Hopf algebra with a bijective antipode S_L . An object $H \in {}^L_L\mathcal{YD}$ is called a weak bialgebra in this category if it is both an algebra and a coalgebra satisfying the following conditions:

$$\begin{aligned}(1) \quad \Delta(xy) &= x_1(x_2^{-1} \longrightarrow y_1) \otimes x_2^0 y_2, \\ \varepsilon(xyz) &= \varepsilon(xy_1)\varepsilon(y_2z) = \varepsilon(x(y_1^{-1} \longrightarrow y_2))\varepsilon(y_1^0 z), \\ \Delta^2(1) &= 1_1 \otimes 1_2 1'_1 \otimes 1'_2 = 1_1 \otimes (1_2^{-1} \longrightarrow 1'_1) 1_2^0 \otimes 1'_2.\end{aligned}$$

(2) H is both a left L -module algebra, L -comodule algebra, L -module coalgebra and L -comodule coalgebra.

(3) there exist an antipode $S : H \longrightarrow H$ (here S is both left L -linear and L -colinear i.e., S is a morphism in the category of ${}^L_L\mathcal{YD}$) satisfying

$$\begin{aligned}x_1 S(x_2) &= \varepsilon((x^{-1} \longrightarrow 1_1)x^0)1_2, \\ S(x_1)x_2 &= 1_1 \varepsilon((1_2^{-1} \longrightarrow x)1_2^0), \\ S(x_1)x_2 S(x_3) &= S(x), \quad \forall x \in H.\end{aligned}$$

Similar to the notation of weak Hopf algebra, we denote

$$\varepsilon_t(x) = \varepsilon((x^{-1} \longrightarrow 1_1)x^0)1_2, \quad \varepsilon_s(x) = 1_1 \varepsilon((1_2^{-1} \longrightarrow x)1_2^0).$$

According to the definitions of ε_t , ε_s we can obtain explicit expressions for these coproducts

$$\Delta(\varepsilon_t(x)) = \varepsilon_t(x)1_1 \otimes 1_2, \quad \Delta(\varepsilon_s(x)) = 1_1 \otimes 1_2 \varepsilon_s(x).$$

Furthermore, for $x \in H$,

$$\begin{aligned}\varepsilon(\varepsilon_t(x)) &= \varepsilon((x^{-1} \longrightarrow 1)x^0), \\ &= \varepsilon((\varepsilon_t(x^{-1}) \longrightarrow 1)x^0), \\ &= \varepsilon((S_L(1_{(1)}) \longrightarrow 1)(1_{(2)} \longrightarrow x)), \\ &= \varepsilon((\varepsilon_t(1_{(1)}) \longrightarrow 1)(1_{(2)} \longrightarrow x)), \\ &= \varepsilon((1_{(1)} \longrightarrow 1)(1_{(2)} \longrightarrow x)), \\ &= \varepsilon(x),\end{aligned}$$

in a similar way we can compute $\varepsilon(\varepsilon_s(x)) = \varepsilon(x)$. Applying Definition 3.1, one obtains immediately the following identities

$$\begin{aligned}\varepsilon(x\varepsilon_t(y)) &= \varepsilon(xy_1S(y_2)) = \varepsilon(xy_1)\varepsilon(y_2S(y_3)) = \varepsilon(xy), \\ \varepsilon(\varepsilon_s(x)y) &= \varepsilon(S(x_1)x_2y) = \varepsilon(S(x_1)x_2)\varepsilon(x_3y) = \varepsilon(xy).\end{aligned}$$

As S is both left L -linear and L -colinear, we can easily check that ε_t and ε_s are also both left L -linear and L -colinear. Moreover it is both an anti-algebra map and an anti-coalgebra map, that is

$$\begin{aligned}Sm &= m\tau_{H,H}(S \otimes S), \text{ i.e., } S(xy) = (x^{-1} \longrightarrow S(y))S(x^0), \quad x, y \in H, \\ \Delta S &= (S \otimes S)\tau_{H,H}\Delta, \text{ i.e., } \Delta(S(x)) = (x_1^{-1} \longrightarrow S(x_2)) \otimes S(x_1^0).\end{aligned}$$

In this paper, we will always assume that the antipode S is bijective. The composite-inverse S^{-1} satisfies

$$\begin{aligned}S^{-1}m &= m(S^{-1} \otimes S^{-1})\tau^{-1}, \text{ i.e.,} \\ S^{-1}(xy) &= S^{-1}(y^0)(S_L^{-1}(y^{-1}) \longrightarrow S^{-1}(x)), \\ \Delta S^{-1} &= (S^{-1} \otimes S^{-1})\tau^{-1}\Delta, \text{ i.e.,} \\ \Delta(S^{-1}(x)) &= S^{-1}(x_2^0) \otimes S_L^{-1}(x_2^{-1}) \longrightarrow S^{-1}(x_1).\end{aligned}$$

Proposition 3.2. *Suppose H is a weak Hopf algebra in ${}^L_L\mathcal{YD}$, the following identities hold*

$$\varepsilon_t \circ S = S \circ \varepsilon_s, \quad \varepsilon_s \circ S = S \circ \varepsilon_t.$$

Proof. For $x \in H$ we have

$$\begin{aligned}\varepsilon_t \circ S(x) &= [S(x)]_1 S([S(x)]_2) = (x_1^{-1} \longrightarrow S(x_2))S(S(x_1^0)), \\ &= S(S(x_1)x_2) = S \circ \varepsilon_s(x).\end{aligned}$$

In a similar way, one can verify $\varepsilon_s \circ S = S \circ \varepsilon_t$. ■

As a preparation for the theorem below, we notice that Proposition 3.2 has counterparts involving the antipode,

$$\varepsilon_t(x) = \varepsilon(S(x)1_1)1_2, \quad \varepsilon_s(x) = 1_1\varepsilon(1_2S(x)).$$

As a matter of fact,

$$\varepsilon_t(x) = \varepsilon(\varepsilon_t(x)1_1)1_2 = \varepsilon(x_1S(x_2)1_1)1_2 = \varepsilon(\varepsilon_s(x_1)S(x_2)1_1)1_2 = \varepsilon(S(x)1_1)1_2.$$

The second equality can be proven analogously. Applying Proposition 3.2, one can obtain

$$\begin{aligned}\varepsilon_s(x) &= (S \circ \varepsilon_t \circ S^{-1})(x) = S(1_{(2)}\varepsilon(S(S^{-1}(x))1_{(1)})) = S(1_{(2)})\varepsilon(x1_{(1)}), \\ \varepsilon_t(x) &= (S \circ \varepsilon_s \circ S^{-1})(x) = S(1_{(1)}\varepsilon(1_{(2)}S(S^{-1}(x)))) = S(1_{(1)})\varepsilon(1_{(2)}x).\end{aligned}$$

In a similar way, we can verify the first equation.

Proposition 3.3. Suppose H is a weak Hopf algebra in ${}^L\mathcal{YD}$. For all $x \in H$ we have the identities

$$x_1 \otimes \varepsilon_s(x_2) = x_1 \otimes S(1_2), \quad \varepsilon_t(x_1) \otimes x_2 = S(1_1) \otimes 1_2 x.$$

Proof. Using the above definitions, one obtains

$$\begin{aligned} x_1 \otimes \varepsilon_s(x_2) &= x_1 \otimes S(1_2) \varepsilon(x_2 1_1), \\ &= x_1 (x_2^{-1} \longrightarrow 1_{1'}) \varepsilon(x_2^0 1_2' 1_1) S(1_2), \\ &= x_1 (x_2^{-1} \longrightarrow 1_1) \varepsilon(x_2^0 1_2) S(1_3), \\ &= (h 1_1)_1 \varepsilon((h 1_1)_2) \otimes S(1_2), \\ &= h 1_1 \otimes S(1_2), \\ \varepsilon_t(x_1) \otimes x_2 &= S(1_1) \varepsilon(1_2 x_1) \otimes x_2, \\ &= S(1_1) \varepsilon(1_2 1_{1'} (1_2'^{-1} \longrightarrow x_1)) \otimes 1_2^0 x_2, \\ &= S(1_1) \varepsilon(1_2 (1_3^{-1} \longrightarrow x_1)) \otimes 1_3^0 x_2, \\ &= S(1_1) \otimes \varepsilon((1_2 x)_1) (1_2 x)_2, \\ &= S(1_1) \otimes 1_2 x. \blacksquare \end{aligned}$$

4. Hopf modules in the Yetter-Drinfeld categories

Since a weak Hopf algebra H in the weak Yetter-Drinfeld categories ${}^L\mathcal{YD}$ is both algebra and coalgebra, one can consider modules and comodules over H . As in the theory of Hopf algebras, an H -Hopf module is an H -module which is also an H -comodule such that these two structures are compatible (the action "commutes" with coaction):

Definition 4.1. Let H be a weak Hopf algebra in ${}^L\mathcal{YD}$. A right H -Hopf module M in ${}^L\mathcal{YD}$ such that it is both a right H -module and a right H -comodule via $\rho_M : M \longrightarrow M \otimes H$, $\rho_M(m) = m_0 \otimes m_1$ and the following equations hold:

- (1) $\rho_M(mh) = m_0(m_1^{-1} \longrightarrow h_1) \otimes m_1^0 h_2, m \in M, h \in H,$
- (2) $\sigma_M(mh) = m^{-1} h^{-1} \otimes m^0 h^0, m \in M, h \in H,$
- (3) $m^{-1} \otimes (m^0)_0 \otimes (m^0)_1 = m_0^{-1} m_1^{-1} \otimes m_0^0 \otimes m_1^0 m \in M,$
- (4) $l \longrightarrow (mh) = (l_1 \longrightarrow m)(l_2 \longrightarrow h), l \in L, m \in M, h \in H,$
- (5) $\rho_M(l \longrightarrow m) = (l_1 \longrightarrow m_0)(l_2 \longrightarrow m_1), l \in L, m \in M.$

We remark that $M \otimes_t H$ is a right H -module by $(m \otimes h)x = m(h^{-1} \longrightarrow x_1) \otimes h^0 x_2$ and a right H -comodule $\rho_{M \otimes H}(m \otimes h) = m_0 \otimes m_1^{-1} \longrightarrow h_1 \otimes m_1^0 h_2$. The condition (1) means that the H -comodule structure $\rho_M : M \longrightarrow M \otimes H$ is H -linear, or equivalently the H -module structure map $\varphi_M : M \otimes H \longrightarrow M$ is H -colinear.

Also, (2) (resp. (4)) $\iff \varphi_M$ is L -colinear (resp. L -linear); (3) (resp. (5)) $\iff \rho_M$ is L -colinear (resp. L -linear).

Example 1. H itself is a right H -Hopf module (in ${}^L_L\mathcal{YD}$) in the natural way. If V is an object in ${}^L_L\mathcal{YD}$, then so is $V \otimes_t H$ by $l \longrightarrow (v \otimes h) = (l_1 \longrightarrow v) \otimes (l_2 \longrightarrow h)$ and $\sigma_{V \otimes H}(v \otimes h) = v^{-1}h^{-1} \otimes v^0h^0$. It is also both a right H -module and a right H -comodule by $(v \otimes h)x = v \otimes hx$ and $\rho_{V \otimes H}(v \otimes h) = v \otimes h_1 \otimes h_2$. One easily checks that $V \otimes_t H$ is an H -Hopf module.

Lemma 4.2. *If H is a weak Hopf algebra in ${}^L_L\mathcal{YD}$ and M a right H -Hopf module in ${}^L_L\mathcal{YD}$, we define $M^{coH} = \{m \in M \mid \rho_M(m) = m1_1 \otimes 1_2\}$ is a L -submodule, then*

- (1) M^{coH} is a L -submodule.
- (2) M^{coH} is a L -subcomodule of M , so $M^{coH} \in {}^L_L\mathcal{YD}$.

Proof. (1) Let $n \in M^{coH}$, then

$$\begin{aligned}
 \rho_M(l \longrightarrow n) &= (l_1 \longrightarrow n1_1) \otimes (l_2 \longrightarrow 1_2), \\
 &= (l_1 \longrightarrow n)(l_2 \longrightarrow 1_1) \otimes (l_3 \longrightarrow 1_2), \\
 &= [(l_1 \longrightarrow n) \otimes 1] \Delta(\varepsilon_t(l_2) \longrightarrow 1), \\
 &= ((1_{(1)}l \longrightarrow n) \otimes 1) \Delta(1_{(2)} \longrightarrow 1), \\
 &= [1_{(1)} \longrightarrow (l \longrightarrow n)][1_{(2)} \longrightarrow (1_{(1')} \longrightarrow 1_1)] \otimes 1_{(2')} \longrightarrow 1_2, \\
 &= (l \longrightarrow n)1_1 \otimes 1_2.
 \end{aligned}$$

Hence $l \longrightarrow n \in M^{coH}$.

(2) Applying $1^{-1} \otimes 1^0 = 1_{(1)} \otimes (1_{(2)} \longrightarrow 1)$ and $\varepsilon_t(l) \longrightarrow x = (l \longrightarrow 1_H)x$, we obtain

$$\begin{aligned}
 1^{-1} \otimes (1^0)_1 \otimes (1^0)_2 &= 1_{(1)} \otimes (1_{(2)} \longrightarrow 1_1) \otimes 1_2, \\
 &= 1_{(1)} \otimes (\varepsilon_t(1_{(2)}) \longrightarrow 1_1) \otimes 1_2, \\
 &= 1_{(1)} \otimes (1_{(2)} \longrightarrow 1)1_1 \otimes 1_2, \\
 &= 1^{-1} \otimes 1^0 1_1 \otimes 1_2.
 \end{aligned}$$

For $n \in M^{coH}$ we do a calculation:

$$\begin{aligned}
 n^{-1} \otimes (n^0)_0 \otimes (n^0)_1 &= n_0^{-1} n_1^{-1} \otimes n_0^0 \otimes n_1^0, \\
 &= n^{-1} 1_1^{-1} 1_2^{-1} \otimes n^0 1_1^0 \otimes 1_2^0, \\
 &= n^{-1} 1^{-1} \otimes n^0 (1^0)_1 \otimes (1^0)_2, \\
 &= n^{-1} 1^{-1} \otimes n^0 1^0 1_1 \otimes 1_2, \\
 &= n^{-1} \otimes n^0 1_1 \otimes 1_2.
 \end{aligned}$$

This implies that $n^{-1} \otimes n^0 \in L \otimes M^{coH}$, so $M^{coH} \in {}^L_L\mathcal{YD}$. ■

Theorem 4.3. *If H is a weak Hopf algebra in ${}^L_L\mathcal{YD}$ and M a right H -Hopf module in ${}^L_L\mathcal{YD}$, M^{coH} is defined as above. Then*

- (1) Let $P(m) = m_0 S(m_1)$, $m \in M$, then $P(m) \in M^{coH}$. If $n \in M^{coH}$ and $h \in H$, then $\rho_M(nh) = nh_1 \otimes h_2$ and $P(nh) = n\varepsilon_t(h)$.
- (2) The map $F : M^{coH} \otimes_t H \longrightarrow M$, $F(n \otimes h) = nh$ is an isomorphism of Hopf modules, the inverse map is given by $G(m) = P(m_0)m_1$.

Proof. (1) Since ε_t is a left L -comodule map we have $\sigma(h_1 S(h_2)) = h^{-1} \otimes \varepsilon_t(h^0)$. Applying $x_1 \otimes \varepsilon_s(x_2) = x1_1 \otimes S(1_2)$, we obtain

$$\begin{aligned}
 \rho_M(P(m)) &= m_0(m_1^{-1}m_2^{-1} \longrightarrow S(m_3)) \otimes m_1^0 S(m_2^0), \\
 &= m_0(m_1^{-1} \longrightarrow S(m_2)) \otimes \varepsilon_t(m_1^0), \\
 &= m_0[S(m_1)]_1 \otimes S^{-1} \circ \varepsilon_s([S(m_1)]_2), \\
 &= m_0 S(m_1)1_1 \otimes S^{-1}(S(1_2)), \\
 &= m_0 S(m_1)1_1 \otimes 1_2.
 \end{aligned}$$

If $n \in M^{coH}$ and $h \in H$, then

$$\begin{aligned}
 \rho(nh) &= n1_1(1_2^{-1} \longrightarrow h_1) \otimes 1_2^0 h_2 = nh_1 \otimes h_2, \\
 P(nh) &= nh_1 S(h_2) = n\varepsilon_t(h).
 \end{aligned}$$

(2) Since

$$\begin{aligned}
 F(l \longrightarrow (n \otimes h)) &= F((l_1 \longrightarrow n) \otimes (l_2 \longrightarrow h)), \\
 &= (l_1 \longrightarrow n)(l_2 \longrightarrow h), \\
 &= l \longrightarrow nh, \\
 &= l \longrightarrow F(n \otimes h),
 \end{aligned}$$

then F is a left H -linear map. F is also left L -colinear by the following equality

$$\sigma(F(n \otimes h)) = \sigma(nh) = n1_{(1)}h^{-1} \otimes 1_{(2)}h^0 = (id \otimes F)\sigma(n \otimes h),$$

clearly F is right H -linear. It is also right H -colinear by (1). Now we have

$$\begin{aligned}
 GF(n \otimes h) &= P(nh_1) \otimes h_2 = n\varepsilon_t(h_1) \otimes h_2, \\
 &= n \otimes \varepsilon_t(h_1)h_2 = n \otimes S(1_1)1_2h, \\
 &= n \otimes h, \\
 FG(m) &= m_0 S(m_1)m_2 = m_0 \varepsilon_s(m_1), \\
 &= [m_0 \varepsilon_s(m_1)]_0 \varepsilon([m_0 \varepsilon_s(m_1)]_1), \\
 &= m_0(m_1^{-1} \longrightarrow 1_1) \varepsilon(m_1^0 1_2 \varepsilon_s(m_2)), \\
 &= m_0 \varepsilon(m_1 \varepsilon_s(m_2)), \\
 &= m_0 \varepsilon(m_1 1_1 S(1_2)), \\
 &= m_0 \varepsilon(m_1), \\
 &= m.
 \end{aligned}$$

■

Example. Let H be a weak Hopf algebra in ${}^L_L\mathcal{YD}$. $M = H$ is defined as a right H -Hopf module by Δ . Then $M^{CoH} = \{\varepsilon_t(H) | h \in H\}$.

5. Application

By [5] we make H^* into a weak Hopf algebra in ${}^L_L\mathcal{YD}$. H^* has the contragredient left L -module structure, ie.,

$$(l \longrightarrow f)(h) = f(S_L(l) \longrightarrow h), \quad l \in L, \quad f \in H^*, \quad h \in H.$$

Also, since H is a finite-dimensional left L -comodule, H^* has the transposed right L -comodule structure and so it becomes a left L -comodule via $\sigma_{H^*} : H^* \longrightarrow L \otimes H^*$, $\sigma_{H^*}(f) = f^{-1} \otimes f^0$, where

$$f^0(h)f^{-1} = f(h^0)S_L^{-1}(h^{-1}), \quad h \in H.$$

Now assume that H is finite-dimensional, we will show that H^* becomes a right H -Hopf module in ${}^L_L\mathcal{YD}$. First, H^* is a right H -module by

$$(fh)(x) = f(hx), \quad f \in H^*, \quad h \in H.$$

Second, H^* is a right H -comodule using the identification $\theta_H : H^* \otimes H \cong \text{Hom}(H, H)$, $\theta_H(f \otimes h)(l) = f(h^{-1} \longrightarrow l)h^0$ as follows:

$$\rho_{H^*} : H^* \longrightarrow H^* \otimes H \cong \text{Hom}(H, H), \quad \rho_{H^*}(f)(x) = f(x_1)S(x_2).$$

That is, $\rho_{H^*}(f) = f_0 \otimes f_1$ means

$$(5.1) \quad f(x_1)S(x_2) = \rho_{H^*}(f)(x) = \theta_H(f_0 \otimes f_1)(x) = f_0(f_1^{-1} \longrightarrow x)f_1^0, \quad x \in H.$$

Proposition 5.1. H^* is a right H -comodule by $\theta_{H \otimes H}$.

Proof. First, we check that H -comodule via above using $\theta_{H \otimes H}$. Applying σ_H to (5.1) we obtain

$$f_1^{-1} \otimes f_0(f_1^{-2} \longrightarrow x)f_1^0 = x_2^{-1} \otimes f(x_1)S(x_2^0). \quad (5.2)$$

Now for $f \in H^*$, $x \in H$, we have

$$\begin{aligned} \theta_{H \otimes H}((f_0)_0 \otimes (f_0)_1 \otimes f_1)(x) &= (f_0)_0((f_0)_1^{-1}f_1^{-1} \longrightarrow x)(f_0)_1^0 \otimes f_1^0, \\ &= f_0((f_1^{-1} \longrightarrow x)_1)S((f_1^{-1} \longrightarrow x)_2) \otimes f_1^0, \\ &= f_0(f_1^{-2} \longrightarrow x_1)(f_1^{-1} \longrightarrow S(x_2)) \otimes f_1^0, \\ &= (f_1^{-1} \longrightarrow S(x_2)) \otimes f_0(f_1^{-2} \longrightarrow x_1)f_1^0 \\ &= (x_2^{-1} \longrightarrow S(x_3)) \otimes f(x_1)S(x_2^0), \\ &= \Delta_H(f(x_1)S(x_2)), \\ &= \Delta_H(f_0(f_1^{-1} \longrightarrow x)f_1^0), \\ &= f_0(f_1^{-1} \longrightarrow x)(f_1^0)_1 \otimes (f_1^0)_2, \\ &= f_0((f_1)_1^{-1}(f_1)_2^{-1} \longrightarrow x)(f_1)_1^0 \otimes (f_1)_2^0, \\ &= \theta_{H \otimes H}(f_0 \otimes (f_1)_1 \otimes (f_1)_2)(x). \end{aligned}$$

It implies that $(f_0)_0 \otimes (f_0)_1 \otimes f_1 = f_0 \otimes (f_1)_1 \otimes (f_1)_2$.

According to $\sigma_V(l \rightarrow v) = l_1 v^{-1} S_L(l_3) \otimes l_2 \rightarrow v^0$, we calculate the following equality

$$\begin{aligned}
 & (1_{(1)} \rightarrow f_0)((1_{(2)} \rightarrow f_1)^{-1} \rightarrow x)(1_{(2)} \rightarrow f_1)^0 \\
 &= f_0((S_L(1_{(1)})1_{(2)}f_1^{-1}S_L(1_{(4)})) \rightarrow x)(1_{(3)} \rightarrow f_1^0), \\
 &= f_0((\varepsilon_s(1_{(1)})f_1^{-1}S_L(1_{(3)})) \rightarrow x)(1_{(2)} \rightarrow f_1^0), \\
 &= f_0((1_{(1')}f_1^{-1}S_L(1_{(2)})) \rightarrow x)(1_{(1)}1_{(2')} \rightarrow f_1^0), \\
 &= f_0((1_{(1)}f_1^{-1}S_L(1_{(2)})) \rightarrow x)(1_{(3)} \rightarrow f_1^0), \\
 &= f_0((1 \rightarrow f_1)^{-1} \rightarrow x)(1 \rightarrow f_1)^0, \\
 &= f_0(f_1^{-1} \rightarrow x)f_1^0,
 \end{aligned}$$

so we obtain $(1_{(1)} \rightarrow f_0) \otimes (1_{(2)} \rightarrow f_1) = f_0 \otimes f_1$.

Applying ε to (5.1), we obtain

$$\begin{aligned}
 f(x) &= f_0(f_1^{-1} \rightarrow x)\varepsilon(f_1^0), \\
 &= f_0(\varepsilon_t(f_1^{-1})\varepsilon(f_1^0) \rightarrow x), \\
 &= f_0(S_L(1_{(1)} \rightarrow x))\varepsilon(1_{(2)} \rightarrow f_1), \\
 &= (1_{(1)} \rightarrow f_0)(x)\varepsilon(1_{(2)} \rightarrow f_1), \\
 &= f_0(x)\varepsilon(f_1), \\
 &= (f_0\varepsilon(f_1))(x).
 \end{aligned}$$

Hence $(id \otimes \varepsilon)\rho_{H^*}(f) = f$, thus H^* becomes a right H -comodule. $\blacksquare$

Theorem 5.2. *With the notation as above, then H^* is a right H -Hopf module in ${}^L\mathcal{YD}$. Moreover, $(H^*)^{coH} = \{f \in H^* | f(x_1)x_2 = f(1_1x)1_2, \ x \in H\}$.*

Proof. Now, we prove that H^* is a right H -Hopf module. First, we will show that $(fh)_0 \otimes (fh)_1 = f_0(f_1^{-1} \rightarrow h_1) \otimes f_1^0 h_2$, since for $x \in H$,

$$\begin{aligned}
 & \theta_H (f_0(f_1^{-1} \rightarrow h_1) \otimes f_1^0 h_2)(x) \\
 &= (f_0(f_1^{-1} \rightarrow h_1))((f_1^0 h_2)^{-1} \rightarrow x)((f_1^0 h_2)^0), \\
 &= (f_0(f_1^{-2} \rightarrow h_1))(f_1^{-1} h_2^{-1} \rightarrow x)f_1^0 h_2^0, \\
 &= f_0((f_1^{-2} \rightarrow h_1)(f_1^{-1} h_2^{-1} \rightarrow x))f_1^0 h_2^0, \\
 &= f_0(f_1^{-1} \rightarrow (h_1(h_2^{-1} \rightarrow x)))f_1^0 h_2^0, \\
 &= f((h_1(h_2^{-1} \rightarrow x))_1)S((h_1(h_2^{-1} \rightarrow x))_2)h_2^0, \\
 &= f(h_1(h_2^{-2} h_3^{-2} \rightarrow x_1))S(h_2^{-1} h_3^{-1} \rightarrow x_2)S(h_2^0 h_3^0), \\
 &= f(h_1(h_2^{-2} \rightarrow x_1))S(h_2^{-1} \rightarrow x_2)\varepsilon_s(h_2^0), \\
 &= f(h_1((\varepsilon_s(h_2))^{-2} \rightarrow x_1))S((\varepsilon_s(h_2))^{-1} \rightarrow x_2)(\varepsilon_s(h_2))^0, \\
 &= f(h_1(1_2^{-2} \rightarrow x_1))(1_2^{-1} \rightarrow S(x_2))S(1_2^0), \\
 &= f(h_1(1_2^{-1} \rightarrow x_1))S(1_2^0 x_2), \\
 &= (f \cdot h)(1_1(1_2^{-1} \rightarrow x_1))S(1_2^0 x_2), \\
 &= (f \cdot h)(x_1)S(x_2), \\
 &= [(f \cdot h)_0 \otimes ((f \cdot h)_1)](x).
 \end{aligned}$$

To verify that $\sigma_{H^*}(fh) = f^{-1}h^{-1} \otimes f^0h^0$ for $f \in H^*$, $h \in H$, we compute for $x \in H$

$$\begin{aligned}
f^{-1}h^{-1}f^0h^0(x) &= f^{-1}h^{-1}f^0(h^0x), \\
&= f(h^0x^0)S_L^{-1}(h^{-1}x^{-1})h^{-2}, \\
&= f((1_{(1)} \longrightarrow h)x^0)S_L^{-1}(x^{-1})1_{(2)} \\
&= f((1_{(1')}1_{(1)} \longrightarrow h)(1_{(2')} \longrightarrow x^0))S_L^{-1}(x^{-1})1_{(2)}, \\
&= f((1_{(1)} \longrightarrow h)(\varepsilon_t(1_{(2)}) \longrightarrow x^0))S_L^{-1}(x^{-1})1_{(3)}, \\
&= f((1_{(1)} \longrightarrow h)(1_{(2)} \longrightarrow 1)x^0)S_L^{-1}(x^{-1})1_{(3)}, \\
&= f((1_{(1')} \longrightarrow h)(1_{(2')} \longrightarrow (1_{(1)} \longrightarrow 1))x^0)S_L^{-1}(x^{-1})1_{(2)}, \\
&= f(h(1_{(1)} \longrightarrow 1)x^0)S_L^{-1}(x^{-1})1_{(2)}, \\
&= f(h(\varepsilon_t(1_{(1)}) \longrightarrow x^0))S_L^{-1}(x^{-1})1_{(2)}, \\
&= f(h(S(1_{(1)}) \longrightarrow x^0))S_L^{-1}(x^{-1})1_{(2)}, \\
&= f(h(S(S^{-1}(1_{(2)})) \longrightarrow x^0)S_L^{-1}(x^{-1})S_L^{-1}(1_{(1)})), \\
&= f(h(1_{(2)} \longrightarrow x^0))S_L^{-1}(1_{(1)}x^{-1}), \\
&= f(hx^0)S_L^{-1}(x^{-1}), \\
&= (fh)(x^0)S_L^{-1}(x^{-1}), \\
&= (fh)^{-1}(fh)^0(x).
\end{aligned}$$

Next, we want to check $(l \longrightarrow fh) = (l_1 \longrightarrow f)(l_2 \longrightarrow h)$ for $l \in L$, $h \in H$, $f \in H^*$, since for $x \in H$

$$\begin{aligned}
((l_1 \longrightarrow f)(l_2 \longrightarrow h))(x) &= (l_1 \longrightarrow f)((l_2 \longrightarrow h)x), \\
&= f(S_L(l_1) \longrightarrow ((l_2 \longrightarrow h)x)), \\
&= f((S_L(l_2)l_3 \longrightarrow h)(S_L(l_1) \longrightarrow x)), \\
&= f((\varepsilon_s(l_2) \longrightarrow h)(S_L(l_1) \longrightarrow x)), \\
&= f((S_L(1_{(2)}) \longrightarrow h)(S_L(l1_{(1)}) \longrightarrow x)), \\
&= f((1_{(1)} \longrightarrow h)(1_{(2)} \longrightarrow (S_L(l) \longrightarrow x))), \\
&= f(h(S_L(l) \longrightarrow x)), \\
&= (fh)(S_L(l) \longrightarrow x), \\
&= (l \longrightarrow fh)(x).
\end{aligned}$$

We have $f^{-1} \otimes (f^0)_0 \otimes (f^0)_1 = f_0^{-1}f_1^{-1} \otimes f_0^0 \otimes f_1^0$ for $f \in H^*$, since for $x \in H$

$$\begin{aligned}
&f_0^{-1} f_1^{-1} \otimes \theta_H(f_0^0 \otimes f_1^0)(x) \\
&= f_0^{-1}f_1^{-2} \otimes f_0^0(f_1^{-1} \longrightarrow x)f_1^0, \\
&= f_0((f_1^{-1} \longrightarrow x)^0)S_L^{-1}((f_1^{-1} \longrightarrow x)^{-1})f_1^{-2} \otimes f_1^0, \\
&= f_0(f_1^{-2} \longrightarrow x^0)S_L^{-1}(f_1^{-3}x^{-1}S(f_1^{-1}))f_1^{-4} \otimes f_1^0, \\
&= f_0(f_1^{-2} \longrightarrow x^0)f_1^{-1}S_L^{-1}(x^{-1})\varepsilon_t(f_1^{-3}) \otimes f_1^0, \\
&= f_0((f_1^{-2}1_{(2)} \longrightarrow x^0)f_1^{-1}S_L^{-1}(1_{(1)}x^{-1}) \otimes f_1^0, \\
&= f_0(f_1^{-2} \longrightarrow x^0)f_1^{-1}S_L^{-1}(x^{-1}) \otimes f_1^0,
\end{aligned}$$

$$\begin{aligned}
&= f_1^{-1} S_L^{-1}(x^{-1}) \otimes f_1^0 f_0(f_1^{-2} \longrightarrow x^0), \\
&= (x^0)_2^{-1} S_L^{-1}(x^{-1}) \otimes f((x^0)_1) S(((x^0)_2)^0), \\
&= x_2^{-1} S_L^{-1}(x_1^{-1} x_2^{-2}) \otimes f(x_1^0) S(x_2^0), \\
&= S_L^{-1}(x_1^{-1} \varepsilon_t(x_2^{-1})) \otimes f(x_1^0) S(x_2^0), \\
&= S_L^{-1}(x_1^{-1} S_L(1_{(1)})) \otimes f(x_1^0) S(1_{(2)} \longrightarrow x_2), \\
&= S_L^{-1}((1_{(1)} \longrightarrow x_1)^{-1}) \otimes f((1_{(1)} \longrightarrow x_1)^0) S(1_{(2)} \longrightarrow x_2), \\
&= S_L^{-1}(x_1^{-1}) f(x_1^0) \otimes S(x_2), \\
&= f^0(x_1) f^{-1} \otimes S(x_2), \\
&= f^{-1} \otimes f^0(x_1) S(x_2), \\
&= f^{-1} \otimes (f^0)_0((f^0)_1^{-1} \longrightarrow x) (f_1^0)_1^0, \\
&= f^{-1} \otimes \theta_H((f^0)_0 \otimes (f^0)_1)(x).
\end{aligned}$$

Finally, we show that $\rho_{H^*}(l \longrightarrow f) = l_1 \longrightarrow f_0 \otimes l_2 \longrightarrow f_1$. Since for $l \in L$, $f \in H^*$, $x \in H$

$$\begin{aligned}
&\theta_H((l_1 \longrightarrow f_0) \otimes (l_2 \longrightarrow f_1))(x) \\
&= (l_1 \longrightarrow f_0)((l_2 \longrightarrow f_1)^{-1} \longrightarrow x)(l_2 \longrightarrow f_1)^0, \\
&= (l_1 \longrightarrow f_0)(l_2 f_1^{-1} S_L(l_4) \longrightarrow x)(l_3 \longrightarrow f_1^0), \\
&= f_0(S_L(l_1) l_2 f_1^{-1} S_L(l_4) \longrightarrow x)(l_3 \longrightarrow f_1^0), \\
&= f_0(\varepsilon_s(l_1) f_1^{-1} S_L(l_3) \longrightarrow x)(l_2 \longrightarrow f_1^0), \\
&= f_0(1_{(1)} f_1^{-1} S_L(l_2) \longrightarrow x)(l_1 1_{(2)} \longrightarrow f_1^0), \\
&= f_0(f_1^{-1} \longrightarrow (S_L(l_2) \longrightarrow x))(l_1 \longrightarrow f_1^0), \\
&= f((S_L(l_2) \longrightarrow x)_1)(l_1 \longrightarrow S((S_L(l_2) \longrightarrow x)_2)), \\
&= f(S_L(l_3) \longrightarrow x_1)(l_1 \longrightarrow S(S_L(l_2) \longrightarrow x_2)), \\
&= f(S_L(l_3) \longrightarrow x_1)(l_1 S_L(l_2) \longrightarrow S(x_2)), \\
&= f(S_L(l_2) \longrightarrow x_1)(\varepsilon_t(l_1) \longrightarrow S(x_2)), \\
&= F(S_L(1_{(2)} l) \longrightarrow x_1)(S_L(1_{(1)}) \longrightarrow S(x_2)), \\
&= f(S_L(l) \longrightarrow (1_{(1)} \longrightarrow x_1)) S(1_{(2)} \longrightarrow x_2), \\
&= f(S_L(l) \longrightarrow x_1) S(x_2), \\
&= (l \longrightarrow f)(x_1) S(x_2), \\
&= \theta_H((l \longrightarrow f)_0 \otimes (l \longrightarrow f)_1)(x).
\end{aligned}$$

From all above, H^* is a right H -Hopf module in ${}^L_L\mathcal{YD}$. ■

Applying Theorem 4.3, we can obtain the following result.

Corollary 5.3. *H^* is defined a right H -Hopf module in ${}^L_L\mathcal{YD}$ as above, then $H^{*CoH} \otimes_t H \cong H^*$.*

Corollary 5.4. *Let L be a finite-dimensional Hopf algebra. Assume that H is a finite-dimensional Hopf algebra in the Yetter-Drinfeld category in ${}^L_L\mathcal{YD}$. H^* is defined a right H -Hopf module in ${}^L_L\mathcal{YD}$ as above, then $H^{*CoH} = \text{Hom}(H, k) = \{f \in H^* | f(x_1)x_2 = f(x)1_H, \ x \in H\}$.*

Example. Recall that a groupoid G is a category in which every morphism is an isomorphism. In this section, we consider finite groupoids, i.e., groupoids which have a finite number of objects. The set of objects of G will be denoted by G_0 , and the set of morphisms by G_1 . The identity morphism on $x \in G_0$ will also be denoted by x . For $\sigma : x \rightarrow y$ in G_1 , we write $s(\sigma) = x$ and $t(\sigma) = y$, respectively for the source and the target of σ . For every $x \in G$, $G_x = \{\sigma \in G_1 | s(\sigma) = t(\sigma) = x\}$ is a group.

Let G be a groupoid, and k a commutative ring. The groupoid algebra is the direct product $kG = \bigoplus_{\sigma \in G_1} ku_\sigma$, with multiplication defined by the formula $u_\sigma u_\tau = u_{\sigma\tau}$, if $t(\tau) = s(\sigma)$ (0 if $t(\tau) \neq s(\sigma)$). The unit element is $1 = \sum_{x \in G_0} u_x$. kG is a weak Hopf algebra, with comultiplication, counit and antipode given by the formulas

$$\Delta(u_\sigma) = u_\sigma \otimes u_\sigma, \quad \varepsilon(u_\sigma) = 1, \quad S(u_\sigma) = u_{\sigma^{-1}}$$

Using the formula

$$\Delta(1) = \sum_{x \in G_0} u_x \otimes u_x,$$

suppose kG is free of finite rank as a k -module, hence $(kG)^*$ is also a weak Hopf algebra. As a k -module, $(kG)^* = \bigoplus_{\sigma \in G_1} kv_\sigma$ with $\langle v_\sigma, u_\tau \rangle = \delta_{\sigma, \tau}$. Suppose $(kG)^*$

is defined a right H -Hopf module in ${}^L_L\mathcal{YD}$ as above. Then

$$\begin{aligned} ((kG)^*)^{CoH} &= \{f \in (kG)^* | f(u_\sigma)u_\sigma = \sum_{x \in G_0} f(u_x u_\sigma)u_x\}, \\ &= \{f \in (kG)^* | f(u_\sigma)u_\sigma = \sum_{x \in G_0} f(u_\sigma)u_{t(\sigma)}\}. \end{aligned}$$

Acknowledgements. Research supported by Project of Shandong Province Higher Educational Science and Technology Program (J12LI07) and the doctoral foundation of Shandong Jianzhu University (424111).

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Accepted: 08.02.2014

RIGHT ALTERNATIVE RINGS WITH $x(yz)-y(xz)$ IN THE CENTER**K. Madhusudhan Reddy**

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Abstract. In [1] it was proved that if R is a prime right alternative ring of char. $\neq 2, 3$ with $(R, R, U) \subseteq U$ or $S(x^2, x, y) = 0$, then either $U = C$ or R is strongly $(-1, 1)$. In this paper first we prove that if R is a prime right alternative ring with $x(yz) - y(xz) \in U$, then $(R, R, U) \subseteq U$. Using this we prove that either $U = C$ or R is strongly $(-1, 1)$.

Keywords and phrases: Right alternative ring, Char. $\neq n$, Strongly $(-1, 1)$ Ring, Center, Commutative Center.

AMS Subject Classification: 17D15.

1. Introduction

Throughout this paper R represents a right alternative ring of char. $\neq 2$. We denote $S(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y)$. A right alternative ring satisfying the identity $[[R, R], R] = 0$ is called a strongly $(-1, 1)$ ring. The set C defined by $C = \{c \in N(R) / [c, R] = 0\}$ is called the center of R and $N_a(R) = \{v \in R / (x, x, v) = 0, \forall x \in R\}$ is called the alternative nucleus of R . A ring R is said to be commutative center if $U = \{u \in U / [u, R] = 0\}$.

In any right alternative ring we have the following identities:

$$\begin{aligned}
 (1.1) \quad & (y, x, z) + (y, z, x) = 0, \\
 (1.2) \quad & [xy, z] = x[y, z] + [x, z]y + 2(x, y, z) + (z, x, y), \\
 (1.3) \quad & 2S(x, y, z) = [[x, y], z] + [[y, z], x] + [[z, x], y], \\
 (1.4) \quad & (z, x^2, y) = (z, x, xy + yx), \\
 (1.5) \quad & (z, x, xy) = (z, x, y)x, \\
 (1.6) \quad & (wx, y, z) + (w, x, [y, z]) = w(x, y, z) + (w, y, z)x, \\
 (1.7) \quad & ([w, x], y, z) - [w, (x, y, z)] + [x, (w, y, z)] = (x, w, [y, z]) - (w, x, [y, z]), \\
 (1.8) \quad & \left. \begin{aligned} ((a, y, z), b, c) &= ((a, b, c), y, z) - (a, b, (c, y, z)) - (a, (b, y, z), c) \\ &+ (a, b, c)[y, z] - (a, b, c[y, z]) + (a, b, [y, z])c. \end{aligned} \right\}
 \end{aligned}$$

For $u \in U$ the following identities and relations hold in $R.[1]$

$$\begin{aligned}
 (1.9) \quad & 2[x, (z, y, u)] = [x, (u, y, z)] = (x, [y, z], u), \\
 (1.10) \quad & ([x, y], [z, w], u) = 0, \\
 (1.11) \quad & (R, [R, R], U) \subseteq U, \\
 (1.12) \quad & (a, a, (x, y, u)) = 0, \\
 (1.13) \quad & ((x, y, u), z, w) = 2(w, z, (x, y, u)), \\
 (1.14) \quad & 3(c, a, (a, y, u)) = ((a, a, c), y, u), \\
 (1.15) \quad & 3[x, (x, z, (a, b, u))] = [(x, x, z), (a, b, u)].
 \end{aligned}$$

In a right alternative ring of char. $\neq 2$, we have $(R, R, U) \subseteq U$ if and only if

$$(1.16) \quad (R, [R, R], U) = 0,$$

by (1.9). Also in any right alternative ring of char. $\neq 2$

$$(1.17) \quad (N(R) \cap U) = C.$$

First, we prove the following lemmas.

Lemma 1. *If R is a right alternative ring with $x(yz) - y(xz) \in U$, then $(R, R, U) \subseteq U$.*

Proof. By hypothesis, $x(yz) - y(xz) \in U$.

$$\begin{aligned}
 (1.18) \quad & (xy)z - (x, y, z) + (y, x, z) - (yx)z \in U. \\
 & - (x, y, z) + (y, x, z) + [x, y]z \in U.
 \end{aligned}$$

By using semi-jacobian identity, we get

$$(1.19) \quad -[xz, y] + x[z, y] + (x, z, y) \in U.$$

Now, by taking $y = u \in U$ and using the definition of U , we get $(x, z, u) \in U$. Therefore, $(R, R, U) \subseteq U$. ■

Lemma 2. *If R is a right alternative ring with $x(yz) - y(xz) \in U$, then the ideal generated by $(R, R, U) \in R$ is $\langle (R, R, U) \rangle = (R, R, U) + R(R, R, U)$.*

Proof. Since $x(yz) - y(xz) \in U$, from Lemma 1, we have $(R, R, U) \subseteq U$.

Now, we obviously have $(R, R, U)R = R(R, R, U)$ and

$$R(R(R, R, U)) \subseteq (R, R, (R, R, U)) + R^2(R, R, U) \subseteq (R, R, U) + R(R, R, U).$$

This and (1.1) gives

$$\begin{aligned} (R(R, R, U))R &\subseteq (R, (R, R, U), R) + R((R, R, U)R) \\ &\subseteq (R, R, (R, R, U)) + R(R(R, R, U)) \\ &\subseteq (R, R, U) + R(R, R, U). \end{aligned}$$

This proves Lemma 2. ■

Lemma 3. *If R is a right alternative ring of char. $\neq 2$ with $x(yz) - y(xz) \in U$, then $K = x \in R/(x, R, U) = x(R, R, U) = 0$ is an ideal of R such that*

$$K\langle (R, R, U) \rangle = 0 \text{ and } [[R, R], R] \subseteq K.$$

Proof. We let $x \in K$. Using $U \subseteq N_a(R)$ and (1.1), we first note that

$$0 = (x, R, U) = (R, x, U) = (x, U, R).$$

Since $(R, R, U) \subseteq U$, we have

$$\begin{aligned} (xR)(R, R, U) &= x(R(R, R, U)) = x((R, R, U)R) \\ &= (x(R, R, U))R = 0 \end{aligned}$$

and

$$(Rx)(R, R, U) = R(x(R, R, U)) = 0.$$

Now, $(R, R, U) \subseteq U$ means $0 = (R, [x, R], U) = ([x, R], R, U)$ by using (1.16) and $U \subseteq N_a(R)$. Then this and (1.6) shows

$$(xR, R, U) = (Rx, R, U) \subseteq R(x, R, U) + (R, R, U)x + (R, x, [R, U]) = x(R, R, U) = 0.$$

Thus it follows that K is an ideal of R . Using this, Lemma 2, and $(R, R, U) \subseteq U$, we then see

$$K\langle (R, R, U) \rangle = K(R, R, U) + R(R, R, U) = (KR)(R, R, U) \subseteq K(R, R, U) = 0.$$

That is

$$K\langle(R, R, U)\rangle = 0.$$

Finally, we show that $[[R, R], R] \subseteq K$.

First, by $U \subseteq N_a(R)$ and (1.16), we have

$$([R, R], R], R, U) = -(R, [[R, R], R], U) = 0.$$

Then, by using (1.2), $(R, R, U) \subseteq U$, (1.16) and (1.6), we see

$$\begin{aligned} [[R, R], R](R, R, U) &\subseteq [[R, R](R, R, U), R] + [R, R]([R, R, U), R] \\ &\quad + 2([R, R], (R, R, U), R) + (R, [R, R], (R, R, U)) \\ &= [[R, R](R, R, U), R] \\ &\subseteq [([R, R]R, R, U), R] + [([R, R], R, U)R, R] \\ &\quad + [[R, R], R, [R, U]], R] \\ &= [([R, R], R, U)R, R] = 0, \end{aligned}$$

by $U \subseteq N_a(R)$ and (1.16).

This completes the proof of the lemma. ■

2. Main result

Theorem 1 *If R is a prime right alternative ring of char. $\neq 2$ with $x(yz) - y(xz) \in U$, then either $U = C$ or R is strongly $(-1, 1)$.*

Proof. From Lemma 1 we have $(R, R, U) \subseteq U$.

Since $(R, R, U) \subseteq U$, from Lemma 3, we have

$$K\langle(R, R, U)\rangle = 0 \text{ and } [[R, R], R] \subseteq K.$$

Since R is prime, either $K = 0$ or $\langle(R, R, U)\rangle = 0$.

If $K = 0$, then $[[R, R], R] = 0$.

If $\langle(R, R, U)\rangle = 0$, then $U \subseteq U \cap N(R) = C$ by (1.17).

Thus either $U = C$ or R is strongly $(-1, 1)$.

This proves the theorem. ■

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Accepted: 13.02.2014

HYERS-ULAM STABILITY OF LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER WITH CONSTANT COEFFICIENT

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Abstract. Y. Li and Y. Shen [12] have proved the Hyers-Ulam stability of differential equation $y''(x) + \alpha y'(x) + \beta y(x) = 0$, in the condition that its characteristic equation has two different positive roots. In this paper, we prove that the differential equation $y''(x) + \alpha y'(x) + \beta y(x) = 0$ has the Hyers-Ulam stability, no matter whether its characteristic roots are real or complex. Therefore the results obtained in this paper improve and extend the ones of [12].

Keywords: Hyers–Ulam stability; differential equations; characteristic equation.

2010 Mathematics Subject Classification: 34K20, 39A10.

1. Introduction

In 1940, Ulam gave a wide ranging talk before the Mathematics Club of the University of Wisconsin, in which he discussed a number of important unsolved problems (see [1]). Among those was the question concerning the stability of homomorphisms: Let G_1 be a group and G_2 be a metric group with a metric $d(\cdot, \cdot)$. Given any $\delta > 0$, does there exist an $\varepsilon > 0$ such that if a function $h : G_1 \rightarrow G_2$ satisfies the inequality

$$d(h(xy), h(x)h(y)) < \varepsilon \text{ for all } x, y \in G_1,$$

then there exists a homomorphism $H : G_1 \rightarrow G_2$ with $d(h(x), H(x)) < \delta$ for all $x \in G_1$?

The problem for the case of approximately additive mappings was solved by Hyers [2] when G_1 and G_2 are Banach spaces and the result of Hyers was generalized by Rassias (see [3]). Since then, the stability problems of functional equations have been extensively investigated by several mathematicians (see [3]–[5]).

C. Alsina and R. Ger were the first authors who investigated the Hyers-Ulam stability of differential equations. In 1998, they proved in [6] the following: Assume that a differential equation $f : I \rightarrow \mathbb{R}$ is a solution of the differential inequality

$$|y'(t) - y(t)| \leq \varepsilon,$$

where I is an open subinterval of $\mathbb{R}$. Then there exists a solution $f_0 : I \rightarrow \mathbb{R}$ of the differential equation $y'(t) = y(t)$ such that $|f(t) - f_0(t)| \leq \varepsilon$, for all $t \in I$.

The result of Hyers-Ulam stability for first-order linear differential equations has been generalized by many researchers (see [7]-[11]). Jung [11] proved the generalized Hyers-Ulam stability of differential equations of the form

$$ty'(t) + \alpha y(t) + \beta t^r x_0 = 0$$

and also applied this result to the investigation of the Hyers-Ulam stability of the differential equation

$$t^2 y''(t) + \alpha t y'(t) + \beta y(t) = 0.$$

Recently, Y. Li and Y. Shen [12] discussed the Hyers-Ulam stability of linear differential equations of second order $y''(x) + \alpha y'(x) + \beta y(x) = 0$, in the condition that the characteristic equation $\lambda^2 + \alpha\lambda + \beta = 0$ has two positive roots.

The aim of this paper is to investigate the Hyers-Ulam stability of the following linear differential equations of second order.

$$(1.1) \quad y''(x) + \alpha y'(x) + \beta y(x) = 0$$

and

$$(1.2) \quad y''(x) + \alpha y'(x) + \beta y(x) = f(x),$$

where $y : [a, b] \rightarrow \mathbb{C}$ is a twice continuously differentiable function, $f : [a, b] \rightarrow \mathbb{C}$ is continuous function and $\alpha, \beta \in \mathbb{R}$.

First of all, we give the definition of the Hyers-Ulam stability.

Definition 1.1. We say that equation (1.1) has the Hyers-Ulam stability if there exists a constant $K \geq 0$ with the following property, for every $\varepsilon > 0$, if

$$||y''(x) + \alpha y'(x) + \beta y(x)|| \leq \varepsilon,$$

then there exists some $u : [a, b] \rightarrow \mathbb{C}$ which is a twice continuously differentiable function satisfying $u''(x) + \alpha u'(x) + \beta u(x) = 0$ such that

$$||y(x) - u(x)|| \leq K\varepsilon.$$

We call such K a Hyers-Ulam stability constant for equation (1.1).

2. Main results

In the following theorem, we prove the Hyers-Ulam stability of the differential equation (1.1).

Theorem 2.1. *Equation (1.1) has the Hyers-Ulam stability, where $y : [a, b] \rightarrow \mathbb{C}$ is a twice continuously differentiable function.*

Proof. Let $\varepsilon > 0$ and $y : [a, b] \rightarrow \mathbb{C}$ be a twice continuously differentiable function such that $\|y'' + \alpha y' + \beta y\| \leq \varepsilon$.

We will show that there exists a constant K independent of ε and y such that $\|y - u\| \leq K\varepsilon$ for some twice continuously differentiable function $u : [a, b] \rightarrow \mathbb{C}$ satisfying

$$u'' + \alpha u' + \beta u = 0.$$

Let λ_1 and λ_2 be the roots of the characteristic equation

$$\lambda^2 + \alpha\lambda + \beta = 0.$$

Define $g(x) = y'(x) - \lambda_1 y(x)$, then

$$\begin{aligned} \|g'(x) - \lambda_2 g(x)\| &= \|y''(x) - \lambda_1 y'(x) - \lambda_2 y'(x) + \lambda_1 \lambda_2 y(x)\| \\ &= \|y''(x) + \alpha y'(x) + \beta y(x)\| \leq \varepsilon. \end{aligned}$$

Let $Z(x) = e^{-\lambda_2(x-a)}g(x)$, for each $x \in [a, b]$, then

$$\begin{aligned} \|Z(x) - Z(t)\| &= \|e^{-\lambda_2(x-a)}g(x) - e^{-\lambda_2(t-a)}g(t)\| \\ &= \left\| \int_t^x \frac{d}{dv} (e^{-\lambda_2(v-a)}g(v)) dv \right\| \\ (2.1) \quad &= \left\| \int_t^x e^{-\lambda_2(v-a)}[g'(v) - \lambda_2 g(v)] dv \right\| \\ &\leq \varepsilon \left| \int_t^x e^{\|\lambda_2\|(v-a)} dv \right| \leq e^{\|\lambda_2\|(b-a)}(b-a)\varepsilon, \end{aligned}$$

for any $x, t \in [a, b]$.

For any $x \in [a, b]$, it follows from (2.1) that

$$\|g(x) - e^{\lambda_2(x-b)}g(b)\| = \|e^{\lambda_2(x-a)}(Z(x) - Z(b))\| \leq e^{2\|\lambda_2\|(b-a)}(b-a)\varepsilon.$$

Let $g_1(x) = e^{\lambda_2(x-b)}g(b)$, then $g_1(x)$ satisfies

$$(2.2) \quad g_1'(x) - \lambda_2 g_1(x) = 0$$

and

$$\|g(x) - g_1(x)\| \leq e^{2\|\lambda_2\|(b-a)}(b-a)\varepsilon.$$

Since $g(x) = y'(x) - \lambda_1 y(x)$, we have that

$$\|y'(x) - \lambda_1 y(x) - g_1(x)\| \leq e^{2\|\lambda_2\|(b-a)}(b-a)\varepsilon.$$

Let $W(x) = e^{-\lambda_1(x-a)}y(x) - \int_a^x e^{-\lambda_1(v-a)}g_1(v)dv$, for each $x \in [a, b]$. By an argument similar to the above, we can show that there exists

$$u(x) = e^{\lambda_1(x-b)}y(b) - e^{\lambda_1(x-a)} \int_x^b e^{-\lambda_1(v-a)}g_1(v)dv$$

such that

$$\|y - u\| \leq e^{2(\|\lambda_2\| + \|\lambda_1\|)(b-a)}(b-a)^2\varepsilon$$

and the twice continuously differentiable function $u : [a, b] \rightarrow \mathbb{C}$ satisfying

$$(2.3) \quad u'(x) - \lambda_1 u(x) = g_1(x).$$

Finally, it follows from (2.2) and (2.3) that $u'' + \alpha u' + \beta u = 0$. Thus, the proof is completed. $\blacksquare$

Theorem 2.2. Assume that $f : [a, b] \rightarrow \mathbb{C}$ is continuous function such that $f(x)$ is integrable on $[a, c]$ for each $c \in [a, b]$. If a twice continuously differentiable function $y : [a, b] \rightarrow \mathbb{C}$ satisfies the differential inequality

$$(2.4) \quad \|y''(x) + \alpha y'(x) + \beta y(x) - f(x)\| \leq \varepsilon,$$

for all $x \in [a, b]$, then (2.4) has the Hyers-Ulam stability.

Proof. Similar to the proof of Theorem 2.1. Let λ_1 and λ_2 be the roots of characteristic equation $\lambda^2 + \alpha\lambda + \beta = 0$.

Define $g(x) = y'(x) - \lambda_1 y(x)$, we have

$$\begin{aligned} \|g'(x) - \lambda_2 g(x) - f(x)\| &= \|y''(x) - \lambda_1 y'(x) - \lambda_2 y'(x) + \lambda_1 \lambda_2 y(x) - f(x)\| \\ &= \|y''(x) + \alpha y'(x) + \beta y(x) - f(x)\| \leq \varepsilon. \end{aligned}$$

Let $g_1(x) = e^{\lambda_2(x-b)}g(b) - e^{\lambda_2(x-a)} \int_x^b e^{-\lambda_2(v-a)}f(v)dv$, then $g_1(x)$ satisfies

$$g_1'(x) - \lambda_2 g_1(x) - f(x) = 0$$

and

$$\|g(x) - g_1(x)\| \leq e^{2\|\lambda_2\|(b-a)}(b-a)\varepsilon.$$

Since $g(x) = y'(x) - \lambda_1 y(x)$, we have

$$\|y'(x) - \lambda_1 y(x) - g_1(x)\| \leq e^{2\|\lambda_2\|(b-a)}(b-a)\varepsilon.$$

By an argument similar to Theorem 2.1, we can show that there exists

$$u(x) = e^{\lambda_1(x-b)}y(b) - e^{\lambda_1(x-a)} \int_x^b e^{-\lambda_1(v-a)}g_1(v)dv$$

such that

$$\|y - u\| \leq e^{2(\|\lambda_2\| + \|\lambda_1\|)(b-a)}(b-a)^2\varepsilon$$

and the twice continuously differentiable function $u : [a, b] \rightarrow \mathbb{C}$ satisfies

$$u'' + \alpha u' + \beta u - f(x) = 0.$$

Thus, the proof is completed. ■

Remark 2.3. The roots of characteristic equation $\lambda^2 + \alpha\lambda + \beta = 0$ of equation (1.1) can be real or complex.

Acknowledgements. This research was supported by Scientific Research Fund of Yunnan Provincial Education Department (No. 2013C157).

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Accepted: 19.02.2014

A THEOREM ABOUT FINITE GROUPS WITH SPECIAL CONJUGACY CLASSES¹

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Abstract. Let G be a finite group. G has the property that for any conjugacy class length, G has exactly two conjugacy classes having such length. The paper classifies all possible structure of the finite group G under a condition that G' is nilpotent.

Keywords: finite groups, conjugacy class length, character.

2010 MR Subject Classification: 20D10, 20D60.

¹Supported by Scientific and Technological Research Program of Chongqing Municipal Education Commission (Grant Nos. KJ1401006, KJ1401019) and the Fundamental Research Funds for the Central Universities.

1. Introduction

In 1973, F. Mark ([1]) began to investigate the “ S_3 -conjecture”, where S_3 is the only finite group whose conjugacy classes all have different length. Many authors also studied this problem in special cases ([3], [4]). In 1994, J.P. Zhang ([5]) proved the conjecture is true if the group is a finite solvable group. In 2004, C.M. Boner and M.B. Ward ([2]) investigated the similar problem that if G is a finite group with exactly two conjugacy classes of the same length, and G' is nilpotent, then $G \cong Z_2, D_{10}, A_4$. This paper continues this kind of problems. We will prove that if G is a finite group with the property P^* (P^* will be defined later), and G' is nilpotent, then $G \cong Z_2, G \cong Z_2 \times S_3$ or $G \cong Z_3 \rtimes Z_4$.

For the sake of convenience, we first give some notations. Let $x \in G$, we denote by $o(x)$ the order of x , x^G denotes the conjugacy class containing x , $|x^G|$ denotes the length of the conjugacy class of x , $A \text{ char } G$ means that A is a character subgroup of G . Let $\text{Irr}(G)$ be the set of all irreducible characters of G , $\text{Irr}^*(G)$ be the set of nonlinear irreducible characters of G . We call a finite group G having property P^* if G has property that for any conjugacy class length, G has exactly two conjugacy classes having such length. We assume that all groups in this paper are finite groups. All other symbols used in this paper are standard and the readers can consult [7] for more information.

We begin our work with the following lemmas.

2. Preliminaries

Lemma 2.1. ([5]) *Suppose that G is a finite solvable group. If the conjugacy classes of G all have different length, then $G \cong S_3$.*

In the following proof, we suppose that G is a finite group with property P^* , and G' is nilpotent.

Lemma 2.2. *$Z(G) \cong Z_2$ and G/G' is an Abelian 2-group.*

Proof. Because G contains exactly two conjugacy classes of length 1, and the conjugacy classes length of any elements of $Z(G)$ is 1, we have that $Z(G) \cong Z_2$. Let $z \in Z(G)$, $z \neq 1$, $x \in G$, and $o(x)$ is odd, then the length of conjugacy classes of x and zx are equal. $o(x)$ is odd implies $C_G(x) = C_G(x^2)$, which concludes that the conjugacy length of x and x^2 are equal too. Since G has property P^* , two elements among x, x^2, zx must be conjugate. The order of x and zx are different implies that x is conjugate to x^2 . Therefore, there exists $g \in G$, such that $x^2 = x^g$.

So we have that $x = x^{-1}x^g = [x, g] \in G'$, from which we know that G/G' is an Abelian 2-group.

Lemma 2.3. *Let $P_2 \in \text{Syl}_2(G)$. If P_2 is non-abelian, then $G' = Z_2H$, $P'_2 = Z_2$, where $Z_2 = Z(G)$, H is a 2'-Hall normal subgroup of G .*

Proof. By Lemma 2.2, G/G' is an Abelian 2-group. Since G' is nilpotent, we can set $G' = L \times H$, where $L \leq P_2$. Then $P'_2 \leq L \leq P_2$. Since $L \text{ char } G' \triangleleft G$, we have that $L \triangleleft G$ and $L \cap Z(P_2) \neq \emptyset$. Also, since $H \text{ char } G' \triangleleft G$, $H \triangleleft G$ and the elements of L can commute with the elements of H . Thus $Z(G) = Z_2 = L \cap Z(P_2) \leq L$.

Now, we prove $L = Z_2$. Otherwise, since $L \triangleleft G$, L consists of conjugacy classes of G , $|L| = |Z_2| + |a^G| + \dots + |b^G|$, where $a, b \in L$. Let $|a^G|$ be the minimal length of the conjugacy classes of non-unit elements of L . Clearly, the G -conjugacy classes length of L is power of 2. If $|a^G| \geq 4$, we have a contradiction immediately since $|Z_2| = 2$. Therefore, $|a^G| = 2$. But G has another conjugacy class of length 2, let it be x^G , that is $|x^G| = 2$. If $x \in L$, then $|x^G| + |a^G| = 4$. Since G has the property P^* , the other conjugacy classes length of L is at least 4, with the same reason we have a contradiction. If $x \notin L$, it is easy to know that $|x^G| = |(zx)^G|$ and $|a^G| = |(za)^G|$, where $z \in Z(G)$. G has property P^* implies that x is conjugate to zx , and a is conjugate to za . Because $|a^G| = 2$, we can conclude that for any element g of G , $a^g = az$ or $a^g = a$. With the same reason we have that $x^g = zx$, or $x^g = x$. Furthermore, it is easy to prove for any $g \in G$, $(ax)^g = ax$, or $(ax)^g = zax$. So $|(ax)^G| = 2$ or $ax \in Z(G)$. Obviously, $ax \notin Z(G)$. We assert that ax is not conjugate to a . Otherwise, there exists $g \in G$ such that $a = (ax)^g$, then $a = (ax)^g = ax$ or $a = (ax)^g = zax$, both of them are impossible. In the same way, we can prove that ax is not conjugate to x . Therefore, $(ax)^G$ is the third conjugacy class with length 2 except a^G, x^G , it contradicts to the fact that G has property P^* .

Therefore, $|L| = 2$, and $P'_2 = L = Z_2$.

Lemma 2.4. *P_2 is an Abelian subgroup.*

Proof. Let $|P_2| = 2^n$, if P_2 is not an Abelian subgroup, by Lemma 2.3, $G' = Z_2H$, $P'_2 = Z_2$. Let N is a minimal normal subgroup of G , and $N \leq H$. Since $H \leq G'$, and G' is nilpotent, N is a p -group and $N \leq Z(H)$. Let $M = P_2N$. We consider a quotient group $\bar{M} = M/Z_2 = \bar{P}_2\bar{N} \leq \bar{G} = G/Z_2$, where $\bar{P}_2 = P_2/Z_2$, $\bar{N} = NZ_2/Z_2$. Clearly, $\bar{N}$ is a minimal normal subgroup of $\bar{M}$. If $\bar{M} = M/Z_2$ is Abelian, then for any $x \in P_2$, $y \in N$, such that $y^{-1}y^x \in Z_2 \cap N = 1$ since $N \triangleleft G$. Therefore $xy = yx$ for any $x \in P_2$, $y \in N$. Since $N \leq Z(H)$, we have $N \leq Z(G) = Z_2$, a contradiction. Therefore, $\bar{M} = M/Z_2$ is non-Abelian.

$\bar{P}_2$ is Abelian and $\bar{N}$ is a minimal normal subgroup of $\bar{M}$ imply that $\bar{M}' = \bar{N}$. Let $\chi \in \text{Irr}(\bar{M})$, $\chi(1) > 1$. Since $\bar{N}$ is an Abelian normal subgroup of $\bar{M}$, we have $\chi(1) \mid |\bar{M} : \bar{N}| = 2^{n-1}$ (Ref [6]). Also, since $\bar{P}_2$ is Abelian, for any $a \in \bar{P}_2$, $(|a^{\bar{M}}|, \chi(1)) = 1$. Therefore, $\chi(a) = 0$, or $\rho(a) = \omega I$ (Ref[6]), where ρ is an irreducible representation of $\bar{M}$ which affords χ , I is a $\chi(1) \times \chi(1)$ identity matrix.

If $\rho(a) = \omega I$, since $\bar{M}' = \bar{N}$, for any $g \in \bar{N}$, $a^{-1}a^g = h \in \bar{N}$. Therefore, $\rho(a) = \rho(a)\rho(h)$, and $\rho(h) = I$ follows. So we have that $h \in \ker \chi$.

(i) if $h \neq 1$, then $1 < \bar{N} \cap \ker \chi \triangleleft \bar{M}$. $\bar{N}$ is a minimal normal subgroup of $\bar{M}$, implies that $\bar{N} \leq \ker \chi$. It means that χ may be viewed as a irreducible character of $\bar{M}/\bar{N} \cong \bar{P}_2$. But $\bar{P}_2$ is Abelian, hence χ is one degree irreducible character, it contradicts to $\chi(1) > 1$.

(ii) if $h = 1$, then $a \in C_{\bar{G}}(\bar{N})$. Let $C_{\bar{G}}(\bar{N}) = \bar{K} \times \bar{H}$ ($\bar{K} \leq \bar{P}_2$), then $\bar{K}$ char $C_{\bar{G}}(\bar{N}) \triangleleft \bar{G}$, we have $\bar{K} \triangleleft \bar{G}$. So $KZ_2 \triangleleft G$. The elements of H and KZ_2 can commute since H is 2'-Hall normal subgroup of G . Thus any $x \in KZ_2$, $|x^G|$ is power of 2. By class equation, $2^s = |KZ_2| = 2 + 2^{n_1} + \dots + 2^{n_i}$, $n_1 \leq n_2 \leq \dots \leq n_i$. if $n_1 \geq 2$, we have a contradiction immediately. If $n_1 = 1$, then KZ_2 contains an element u with $|u^G| = 2$. So G contains another element v with $|v^G| = 2$ since G has property P^* . It is easy to see that $H \leq C_G(v)$ and $v \in KZ_2$. Hence $2^s = |KZ_2| = 2 + 2 + 2 + 2^{n_3} + \dots + 2^{n_i}$, $2 \leq n_3 \leq \dots \leq n_i$, a contradiction.

Therefore, for any $a \in \bar{P}_2$, $a \neq 1$, $\chi(a) = 0$.

Now, we restrict χ on $\bar{P}_2$, we have

$$(\chi, \chi)_{\bar{P}_2} = \frac{1}{|\bar{P}_2|} \sum_{a \in \bar{P}_2} \chi(a)\chi(a^{-1}) = \frac{\chi^2(1)}{|\bar{P}_2|}.$$

Therefore, $\chi^2(1) = |\bar{P}_2|(\chi, \chi)_{\bar{P}_2} = 2^{n-1}(\chi, \chi)_{\bar{P}_2}$. Let $\chi = n_1\theta_1 + n_2\theta_2 + \dots + n_t\theta_t$, where $\theta_i \in \text{Irr}(\bar{P}_2)$ and $n_i \geq 1$. Since $\theta_i(1) = 1$, $\chi(1) = n_1 + n_2 + \dots + n_t$, by orthogonality relation of irreducible character, $(\chi, \chi)_{\bar{P}_2} = n_1^2 + n_2^2 + \dots + n_t^2 \geq n_1 + n_2 + \dots + n_t = \chi(1)$, that is $\chi^2(1) = 2^{n-1}(\chi, \chi)_{\bar{P}_2} \geq 2^{n-1}\chi(1)$. Thus, we can conclude that $\chi(1) \geq 2^{n-1}$. Since $\chi(1)$ is power of 2, we have $(\chi, \chi)_{\bar{P}_2}$ is power of 2.

$\bar{M}' = \bar{N}$ means that the number of one degree irreducible characters of $\bar{M}'$ is $|\bar{M}/\bar{M}'| = |\bar{P}_2| = 2^{n-1}$. Let $|N| = p^m$, then

$$2^{n-1}p^m = |\bar{M}| = |\bar{M}/\bar{M}'| + \sum_{\chi \in \text{Irr}^*(\bar{M})} \chi^2(1) = 2^{n-1} + 2^{n-1} \sum_{\chi \in \text{Irr}^*(\bar{M})} (\chi, \chi)_{\bar{P}_2}.$$

Therefore, $p^m = 1 + \sum_{\chi \in \text{Irr}^*(\bar{M})} (\chi, \chi)_{\bar{P}_2}$. Since $(\chi, \chi)_{\bar{P}_2} \geq \chi(1) \geq 2^{n-1}$ and $(\chi, \chi)_{\bar{P}_2}$ is power of 2, $p^m = 1 + 2^{n-1} \sum_i 2^{n_i}$. Hence $2^{n-1} \mid p^m - 1$.

We have known before that $N \triangleleft G$, and $N \leq Z(H)$. So for any $x \in N$, $|x^G|$ is power of 2.

Clearly, $|x^G| = |(zx)^G|$, they are not conjugate because $o(x) \neq o(zx)$. Since $x \in N$ and $zx \notin N$ and G has property P^* , the distinct conjugacy classes of elements of N have distinct length.

So $p^m = |N| = 1 + |x_1^G| + |x_2^G| + \dots + |x_s^G| = 1 + 2^{m_1} + 2^{m_2} + \dots + 2^{m_s}$, $m_1 < m_2 < \dots < m_s$. Because $2^{n-1} |p^m - 1|$ and $|P_2| = |\bar{P}_2||Z_2| = 2^n$, we have $s = 1$, $m_1 = n - 1$. That is $p^m - 1 = 2^{n-1}$. But $N = 1 \cup x^G$, it implies that $\bar{P}_2$ acts fixed-point-free on N by conjugacy. By [7] 7.24, $\bar{P}_2$ is a cyclic group or generalized quaternion group. Since $\bar{P}_2$ is Abelian, $\bar{P}_2 = P_2/Z(G)$ is a cyclic group, which implies that P_2 is Abelian. It contradicts to our hypothesis of P_2 non-Abelian.

In fact, we have proven that P_2 is an Abelian subgroup.

3. Main theorem

Theorem 3.1. *Suppose that G is a finite group with property P^* , and G' is nilpotent, then $G \cong Z_2$, $G \cong Z_2 \times S_3$, or $G \cong Z_3 \rtimes Z_4$.*

Proof. Since G has property P^* , $Z(G) = Z_2 = \langle z \rangle$, $o(z) = 2$. Clearly, $G \cong Z_2$, if G is Abelian.

If G is not Abelian, let $\sum_1 = \{x^G | x \in G, zx \in x^G\}$, $\sum_2 = \{x^G | x \in G, zx \notin x^G\}$. Then $G = \sum_1 \cup \sum_2$. Clearly, $x^G \in \sum_2$ if and only if $(zx)^G \in \sum_2$. Let $\sum_2 = \sum_{21} \cup \sum_{22}$, where $\sum_{21} = \{x_1^G, x_2^G, \dots, x_l^G\}$, $\sum_{22} = \{(zx_1)^G, (zx_2)^G, \dots, (zx_l)^G\}$. Because $|x_i^G| = |(zx_i)^G|$, $i = 1, 2, \dots, l$, and G has property P^* , the length of conjugacy classes of $\sum_{21}$ are distinct.

Firstly, we assert that $\sum_1 = \emptyset$. Otherwise, let $x^G \in \sum_1$. By definition of $\sum_1$, $zx \in x^G$, so there exists $g \in G$ such that $zx = x^g$. We decompose x as production of a 2-element a and a 2'-element b , $x = ab = ba$. Let $o(a) = 2^s$, $o(b) = t$, $(2, t) = 1$. Since $(zab)^t = ((ab)^g)^t = za^t = (a^t)^g$, if let $y = a^t$, then $zy \in y^G$. Since a is a 2-element and t is odd, y is a 2-element. Let $P_2 \in \text{Syl}_2(G)$, $y \in P_2$. Since P_2 is Abelian, $|y^G|$ is a odd number. On the other hand, since $zy \in y^G$, $Z(G)$ acts on conjugacy class y^G . Obviously, the action is faithful. Hence $2 \mid |y^G|$, it is a contradiction. Thus $\sum_1 = \emptyset$.

Now, considering $\bar{G} = G/Z(G)$, we will prove that every conjugacy classes in $G/Z(G)$ has distinct length.

Let $\bar{x} \in G/Z(G)$, since $\sum_1 = \emptyset$, $zx \notin x^G$. It is easy to see that $\overline{C_G(x)} \leq C_{\bar{G}}(\bar{x})$. Let $\bar{g} \in C_{\bar{G}}(\bar{x})$, then we have $x^g = x$, or $x^g = xz$. By $\sum_1 = \emptyset$, we can conclude that

$x^g = x$, that is $g \in C_G(x)$. Therefore, $C_{\bar{G}}(\bar{x}) = \overline{C_G(x)}$. We have known that the conjugacy classes of $\sum_{21}$ has distinct length. So if let $\sum_{21}^* = \{(\bar{x})^{\bar{G}} \mid x^G \in \sum_{21}\}$, then the conjugacy classes of $\sum_{21}^*$ has distinct length too. Also since $G/Z(G) = \sum_{21}^* = \{(\bar{x})^{\bar{G}} \mid x^G \in \sum_2\}$, by Lemma 2.1, $G/Z(G) \cong S_3$. Therefore, G is a group with order 12. But all finite groups of order 12 are the following:

$$Z_{12}, Z_2 \times Z_2 \times Z_3, A_4, G \cong Z_2 \times S_3, G \cong Z_3 \rtimes Z_4.$$

It is easy to check that, $G \cong Z_2 \times S_3$ or $G \cong Z_3 \rtimes Z_4$. The proof is completed.

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Accepted: 12.03.2014

ENERGY OF AN INTUITIONISTIC FUZZY GRAPH

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Abstract. In this paper, the concept of energy of fuzzy graph is extended to the energy of an intuitionistic fuzzy graph. We have defined the adjacency matrix of an intuitionistic fuzzy graph and the energy of an intuitionistic fuzzy graph is defined in terms of its adjacency matrix. These concepts are illustrated with real time example. The lower and upper bound for the energy of an intuitionistic fuzzy graph are also derived.

Keywords: fuzzy set, intuitionistic fuzzy set, fuzzy graph, energy of fuzzy graph.

1. Introduction

Fuzzy set was introduced by Zadeh [18] whose basic component is only a membership function. The generalization of Zadeh's fuzzy set, called intuitionistic fuzzy set was introduced by Atanassov [2] which is characterized by a membership function and a non-membership function. In Zadeh's fuzzy set, the sum of membership degree and a non-membership degree is equal to one. In Atanassov intuitionistic fuzzy set, the sum of membership degree and a non-membership degree does not exceed one.

The foundation for graph theory was laid in 1735 by Leonhard Euler when he solved the "Konigsberg bridges" problem. Many real life problems can be represented by graph. In computer science, graphs are used to represent networks of communications, data organization, computational devices, the flow of computation, etc. The link structure of a website could be represented by a directed graph in which the vertices are the web pages available at the website and a directed

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edge from page A to page B exists if and only if A contains a link to B [13]. A similar approach can be taken to problems in travel, biology, computer chip design and many other fields. Hence graph theory is widely used in solving real time problems. But when the system is large and complex it is difficult to extract the exact information about the system using the classical graph theory. In such cases fuzzy graph is used to analyze the system.

The first definition of fuzzy graphs was proposed by Kafmann [10] in 1973, from the Zadeh's fuzzy relations [18] [19] [20]. But Rosenfeld [14] introduced another elaborated definition including fuzzy vertex and fuzzy edges and several fuzzy analogs of graph theoretic concepts such as paths, cycles, connectedness and etc. The first definition of intuitionistic fuzzy graphs was proposed by Atanassov [3]. I. Gutman [6] in 1978 introduced the concept of "graph energy" as the sum of the absolute values of the eigen values of the adjacency matrix of the graph. Certain bounds on energy are discussed in [4], [12] and [7]. Energy of different graphs including regular [8], non-regular [9], circulant [16] and random graphs [17] is also under study. Energy is defined for signed graphs in [5] and for weighted graphs in [15]. The energy of graph is extended to the energy of fuzzy graph in [1]. In this paper, we extend the concept of energy of fuzzy graph to the energy of an intuitionistic fuzzy graph. This paper is organized as follows. In Section 2, we give all the basic definitions related to fuzzy sets, intuitionistic fuzzy sets, energy of graph and energy of fuzzy graph. In Section 3, we define the energy of an intuitionistic fuzzy graph. The lower and upper bounds for the energy of an intuitionistic fuzzy graph are also derived. In Section 4, we illustrate these concepts by taking the website of <http://www.pantechsolutions.net/>. In Section 5, we give the conclusion.

2. Preliminaries

2.1. Energy of graph

Definition 2.1. A graph G is a pair of set (V, E) , denoted by $G = (V, E)$, where V is a set of vertices and E is a set of edges. Each edge in E is a pair of vertices in V . Each edge is associated with a set consisting of either one or two vertices called its endpoints.

Definition 2.2. A graph in which the edges are unordered vertex pair is called an undirected graph. A graph in which the edges are ordered vertex pair is called a directed graph. Hence if there is an edge from v_i to v_j in G then $(v_i, v_j) \in E$.

Definition 2.3. An edge whose endpoints are the same is called a loop. A graph without loops and parallel edges is called a simple graph. Two vertices that are connected by an edge are called adjacent. The adjacency matrix $A = [a_{ij}]$ for a graph $G = (V, E)$ is a matrix with n rows and n columns, $n = |V|$ and its entries defined by $a_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in E, \\ 0, & \text{otherwise.} \end{cases}$

Definition 2.4. The spectrum of a matrix is defined as a set of its eigenvalues. Let $G = (V, E)$ be a simple graph with n vertices and m edges. Let $A = [a_{ij}]$ be the adjacency matrix of G . As in a simple graph, there can be at most 1 edge between 2 vertices. So, the entries in A are either 0 or 1. The diagonal elements are zero since there are no loops. A is symmetric and so the spectrum of A is real. An eigenvalue of A are called eigenvalues of G [4] and the spectrum of A is called the spectrum of G . Energy of a simple graph $G = (V, E)$ with adjacency matrix A is defined as the sum of absolute values of eigenvalues of A . It is denoted by $E(G)$. That is,

$$E(G) = \sum_{i=1}^n |\lambda_i|,$$

where λ_i is an eigenvalue of A , $i = 1, 2, \dots, n$. Since A is a symmetric matrix which diagonal elements are zero,

$$\sum_{i=1}^n \lambda_i = 0.$$

By comparing the coefficients of λ^{n-2} in the characteristic polynomial

$$\prod_{i=1}^n (\lambda - \lambda_i) = |A - \lambda I|,$$

we get

$$\sum_{1 \leq i < j \leq n} \lambda_i \lambda_j = -m.$$

Theorem 2.1. Let G be a simple graph with $|V| = n$ vertices and m edges and A be the adjacency matrix of G then $\sqrt{2m + n(n-1)|A|_n^{\frac{2}{n}}} \leq E(G) \leq \sqrt{2mn}$.

2.2. Fuzzy set

Definition 2.5. Let X be a nonempty set. A fuzzy set A in X is defined as $A = \{(x, \mu_A(x)) / x \in X\}$, which is characterized by a membership function $\mu_A(x) : X \rightarrow [0, 1]$ and a fuzzy set satisfying the following condition $\mu_A(x) + \gamma_A(x) = 1$, where $\gamma_A(x) = 1 - \mu_A(x)$ is the nonmembership function (see [18]).

2.3. Intuitionistic fuzzy set

Definition 2.6. Let X be a nonempty set. An intuitionistic fuzzy set A in X is defined as $A = \{(x, \mu_A(x), \gamma_A(x)) / x \in X\}$ which is characterized by a membership function $\mu_A(x) : X \rightarrow [0, 1]$ and the nonmembership function $\gamma_A(x) : X \rightarrow [0, 1]$ and satisfying the following condition

- (i) $0 \leq \mu_A(x) + \gamma_A(x) \leq 1, \forall x \in X$,
- (ii) $0 \leq \mu_A(x), \gamma_A(x), \pi_A(x) \leq 1, \forall x \in X$,
- (iii) $\pi_A(x) = 1 - \mu_A(x) - \gamma_A(x)$,

where $\pi_A(x)$ is called the intuitionistic fuzzy index of element x in A , the value denotes a measure of nondeterminacy. Obviously, if $\pi_A(x) = 0$, then the intuitionistic fuzzy set A is just a Zadeh's fuzzy set (see [2]).

2.4. Energy of fuzzy graph [1]

Definition 2.7. Let V be a nonempty set. A fuzzy subset of V is a function $\sigma : V \rightarrow [0, 1]$. σ is called the membership function and $\sigma(v)$ is called the membership of v where $v \in V$. Let V_1 and V_2 be two nonempty sets and σ_1 and σ_2 be two fuzzy subsets of V_1 and V_2 respectively. Define a fuzzy subset μ of $V_1 \times V_2$ as $\mu(v_i, v_j) \leq \min\{\sigma_1(v_i), \sigma_2(v_j)\}$. Then, μ is called a fuzzy relation, from σ_1 to σ_2 . Suppose $\sigma_1(x) = 1, \forall x \in V_1$ and $\sigma_2(y) = 1, \forall y \in V_2$. Then, μ is called a fuzzy relation from V_1 into V_2 . $\mu(v_i, v_j)$ is interpreted as the strength of relation between v_i and v_j . Suppose $V_1 = V_2 = V$ and $\sigma_1 = \sigma_2 = \sigma$. Then, μ is called a fuzzy relation on σ . Suppose $V_1 = V_2 = V$ and $\sigma_1(x) = 1, \forall x \in V_1$, and $\sigma_2(y) = 1, \forall y \in V_2$. Then μ is called a fuzzy relation on V . From the above definitions, it follows that binary relations on crisp sets are particular cases of fuzzy relations. Let V be a nonempty set and σ be a fuzzy subset of V . Let μ be a fuzzy relation on σ . μ is said to be symmetric, if $\mu(v_i, v_j) = \mu(v_j, v_i)$ for $v_i, v_j \in V$. A fuzzy relation can also be expressed by a matrix called fuzzy relation matrix $M = [m_{ij}]$, where $m_{ij} = \mu(v_i, v_j)$.

A fuzzy graph $G = (V, \sigma, \mu)$ is a nonempty set V together with a pair of function (σ, μ) where σ is a fuzzy subset of V and μ is a fuzzy relation on σ . Let $M = [m_{ij}]$ be a fuzzy relation matrix defined on μ . This represents the strength of the relation between the vertices. Hence we have the following definitions.

Definition 2.8. [1] The adjacency matrix A of a fuzzy graph $G = (V, \sigma, \mu)$ is an $n \times n$ matrix where $n = |V|$ defined as $A = [a_{ij}]$, where $a_{ij} = \mu(v_i, v_j)$. Note that A becomes the usual adjacency matrix when all the nonzero membership values are 1, i.e., when the fuzzy graph becomes a crisp graph.

Definition 2.9. [1] Let $G = (V, \sigma, \mu)$ be a fuzzy graph and A be its adjacency matrix. The eigenvalues of A are called eigenvalues of G . The spectrum of A is called the spectrum of G . It is denoted by $\text{Spec } G$.

Definition 2.10. [1] Let $G = (V, \sigma, \mu)$ be a fuzzy graph and A be its adjacency matrix. Energy of G is defined as the sum of the absolute values of the eigenvalues of A .

Theorem 2.2. [1] Let $G = (V, \sigma, \mu)$ be a fuzzy graph with $|V| = n$ and $\mu^* = \{e_1, e_2, \dots, e_m\}$. If $m_i = \mu(e_i)$ is the strength of the relation associated with the i^{th} edge, then

$$\sqrt{2 \sum_{i=1}^m m_i^2 + n(n-1)|A|^{\frac{2}{n}}} \leq E(G) \leq \sqrt{2 \left(\sum_{i=1}^m m_i^2 \right) n}.$$

3. Energy of an intuitionistic fuzzy graph

In this section, we define the energy of an intuitionistic fuzzy directed graph without loops. The link structure of a website could be represented by a directed intuitionistic fuzzy graph. The links are considered as vertices and the path between the links are considered as edges. The weightage of the each edge are considered as the number of visitors (membership value), the number of non visitors (non - membership value) and drop off (intuitionistic fuzzy index) among that link structure. The lower and upper bounds for the energy of an intuitionistic fuzzy graph are also obtained.

Definition 3.1. An intuitionistic fuzzy graph is defined as $G = (V, E, \mu, \gamma)$ where V is the set of vertices and E is the set of edges. μ is a fuzzy membership function defined on $V \times V$ and γ is a fuzzy non - membership function defined on $V \times V$. We denote $\mu(v_i, v_j)$ by μ_{ij} and $\gamma(v_i, v_j)$ by γ_{ij} such that (i) $0 \leq \mu_{ij} + \gamma_{ij} \leq 1$ (ii) $0 \leq \mu_{ij}, \gamma_{ij}, \pi_{ij} \leq 1$, where $\pi_{ij} = 1 - \mu_{ij} - \gamma_{ij}$. Hence $(V \times V, \mu, \gamma)$ is an intuitionistic fuzzy set.

Example 3.1.

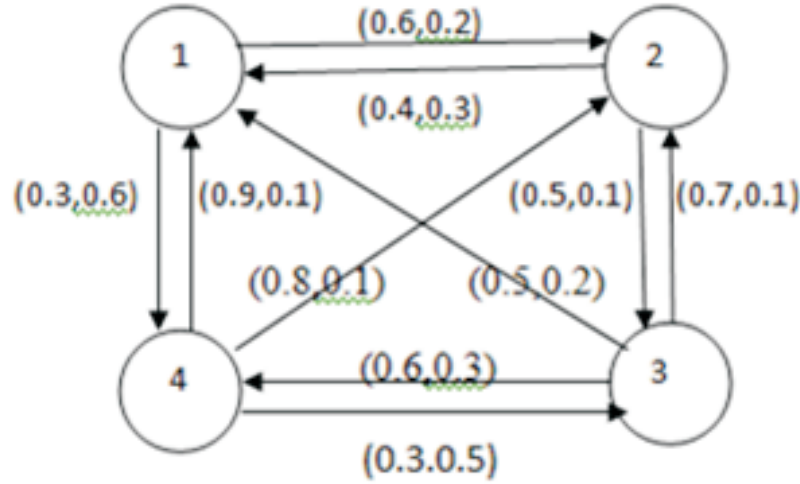


Figure 1: G1. An intuitionistic fuzzy graph

Definition 3.2. An intuitionistic fuzzy adjacency matrix of an intuitionistic fuzzy graph is defined as the adjacency matrix of the corresponding intuitionistic fuzzy graph. That is for an intuitionistic fuzzy graph $G = (V, E, \mu, \gamma)$, an intuitionistic fuzzy adjacency matrix is defined by $A(IG) = [a_{ij}]$ where $a_{ij} = (\mu_{ij}, \gamma_{ij})$. Note that μ_{ij} represents the strength of the relationship between v_i and v_j and γ_{ij} represents the strength of the non-relationship between v_i and v_j .

Example 3.2. For an intuitionistic fuzzy graph in Fig. 1, the adjacency matrix is

$$A(IG) = \begin{pmatrix} 0 & (0.6, 0.2) & 0 & (0.3, 0.6) \\ (0.4, 0.3) & 0 & (0.5, 0.1) & 0 \\ (0.5, 0.2) & (0.7, 0.1) & 0 & (0.6, 0.3) \\ (0.9, 0.1) & (0.8, 0.1) & (0.3, 0.5) & 0 \end{pmatrix}$$

Definition 3.3. The adjacency matrix of an intuitionistic fuzzy graph can be written as two matrices one containing the entries as membership values and the other containing the entries as non-membership values, i.e., $A(IG) = ((\mu_{ij}), (\gamma_{ij}))$, where

$$A(\mu_{ij}) = \begin{pmatrix} 0 & 0.6 & 0 & 0.3 \\ 0.4 & 0 & 0.5 & 0 \\ 0.5 & 0.7 & 0 & 0.6 \\ 0.9 & 0.8 & 0.3 & 0 \end{pmatrix} \text{ and } A(\gamma_{ij}) = \begin{pmatrix} 0 & 0.2 & 0 & 0.6 \\ 0.3 & 0 & 0.1 & 0 \\ 0.2 & 0.1 & 0 & 0.3 \\ 0.1 & 0.1 & 0.5 & 0 \end{pmatrix}$$

Definition 3.4. The eigenvalue of an intuitionistic fuzzy adjacency matrix $A(IG)$ is defined as (X, Y) , where X is the set eigenvalues of $A(\mu_{ij})$ and Y is the set of eigenvalues of $A(\gamma_{ij})$.

Definition 3.5. The spectrum of an intuitionistic fuzzy adjacency matrix $A(IG)$ is defined as (X, Y) , where X is the set eigenvalues of $A(\mu_{ij})$ and Y is the set of eigenvalues of $A(\gamma_{ij})$.

Definition 3.6. The energy of an intuitionistic fuzzy graph $G = (V, E, \mu, \gamma)$ is defined as

$$\left(\sum_{\lambda_i \in X} |\lambda_i|, \sum_{\delta_i \in Y} |\delta_i| \right)$$

where $\sum_{\lambda_i \in X} |\lambda_i|$ is defined as an energy of the membership matrix denoted by

$E(\mu_{ij}(G))$ and $\sum_{\delta_i \in Y} |\delta_i|$ is defined as an energy of the nonmembership matrix denoted by $E(\gamma_{ij}(G))$.

Example 3.3. For an intuitionistic fuzzy graph in Fig. 1,
 $\text{Spec}(\mu_{ij}(G_1)) = \{1.2406, -0.7153, -0.2627 + 0.2332i, -0.2627 - 0.2332i\}$
 $\text{Spec}(\gamma_{ij}(G_1)) = \{0.6441, -0.0148, -0.3146 + 0.1629i, -0.3146 - 0.1629i\}$
 $E(\mu_{ij}(G_1)) = 1.2406 + 0.7153 + 0.3513 + 0.3513 = 2.6585$
 $E(\gamma_{ij}(G_1)) = 0.6441 + 0.0148 + 0.3543 + 0.3543 = 1.3675$

Theorem 3.1. Let G be an intuitionistic fuzzy directed graph (without loops) with $|V| = n$ and $|E| = m$ and $A(IG) = ((\mu_{ij}), (\gamma_{ij}))$ be an intuitionistic fuzzy adjacency matrix of G then

$$(i) \sqrt{2 \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji} + n(n-1)|A|^{\frac{2}{n}}} \leq E(\mu_{ij}(G)) \leq \sqrt{2n \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji}},$$

where $|A|$ is the determinant of $A(\mu_{ij})$, and

$$(ii) \sqrt{2 \sum_{1 \leq i < j \leq n} \gamma_{ij} \gamma_{ji} + n(n-1)|B|^{\frac{2}{n}}} \leq E(\gamma_{ij}(G)) \leq \sqrt{2n \sum_{1 \leq i < j \leq n} \gamma_{ij} \gamma_{ji}},$$

where $|B|$ is the determinant of $A(\gamma_{ij})$.

Proof. Upper bound. Apply Cauchy's Schwartz inequality to the n numbers $1, \dots, 1$ and $|\lambda_1|, |\lambda_2|, \dots, |\lambda_n|$,

$$(1) \quad \sum_{i=1}^n |\lambda_i| \leq \sqrt{n} \sqrt{\sum_{i=1}^n |\lambda_i|^2}$$

$$(2) \quad \left(\sum_{i=1}^n \lambda_i \right)^2 = \sum_{i=1}^n |\lambda_i|^2 + 2 \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j$$

By comparing the coefficients of λ^{n-2} in the characteristic polynomial

$$\prod_{i=1}^n (\lambda - \lambda_i) = |A - \lambda I|,$$

we get

$$(3) \quad \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j = - \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji}$$

Substituting (3) in (2), we get

$$(4) \quad \sum_{i=1}^n |\lambda_i|^2 = 2 \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji}$$

Substituting (4) in (1), we get

$$\sum_{i=1}^n |\lambda_i| \leq \sqrt{n} \sqrt{2 \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji}} = \sqrt{2n \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji}}$$

$$E(\mu_{ij}(G)) \leq \sqrt{2n \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji}}$$

Lower bound

$$\begin{aligned} (E(\mu_{ij}(G)))^2 &= \left(\sum_{i=1}^n |\lambda_i| \right)^2 = \sum_{i=1}^n |\lambda_i|^2 + 2 \sum_{1 \leq i < j \leq n} |\lambda_i \lambda_j| \\ &= 2 \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji} + \frac{2n(n-1)}{2} AM \{|\lambda_i \lambda_j|\} \end{aligned}$$

$$AM \{|\lambda_i \lambda_j|\} \geq GM \{|\lambda_i \lambda_j|\}, 1 \leq i < j \leq n$$

$$\begin{aligned} E(\mu_{ij}(G)) &\geq \sqrt{2 \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji} + n(n-1) GM \{|\lambda_i \lambda_j|\}} \\ GM \{|\lambda_i \lambda_j|\} &= \left(\prod_{1 \leq i < j \leq n} |\lambda_i \lambda_j| \right)^{\frac{2}{n(n-1)}} = \left(\prod_{i=1}^n |\lambda_i|^{n-1} \right)^{\frac{2}{n(n-1)}} \\ &= \left(\prod_{i=1}^n |\lambda_i| \right)^{\frac{2}{n}} = |A|^{\frac{2}{n}} \end{aligned}$$

$$E(\mu_{ij}(G)) \geq \sqrt{2 \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji} + n(n-1) |A|^{\frac{2}{n}}}$$

Therefore,

$$\sqrt{2 \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji} + n(n-1) |A|^{\frac{2}{n}}} \leq E(\mu_{ij}(G)) \leq \sqrt{2n \sum_{1 \leq i < j \leq n} \mu_{ij} \mu_{ji}}$$

Similarly, we can prove

$$\sqrt{2 \sum_{1 \leq i < j \leq n} \gamma_{ij} \gamma_{ji} + n(n-1) |B|^{\frac{2}{n}}} \leq E(\gamma_{ij}(G)) \leq \sqrt{2n \sum_{1 \leq i < j \leq n} \gamma_{ij} \gamma_{ji}} \quad \blacksquare$$

Example 3.4. (Illustration to Theorem 3.1) For an intuitionistic fuzzy graph in Fig. 1,

$E(\mu_{ij}(G_1)) = 2.6585$ its Lower bound = 2.4599 and Upper bound = 2.8844 and $E(\gamma_{ij}(G_1)) = 1.3675$ its Lower bound = 0.9878 and Upper bound = 1.4967.

4. Numerical examples

In this section, we explain the concept for the energy of an intuitionistic fuzzy graph and lower and upper bounds for the energy of an intuitionistic fuzzy graph through a real time example.

We have taken the website <http://www.pantechsolutions.net/>. This website is modeled as an intuitionistic fuzzy graph by considering the navigation of the customer. An intuitionistic fuzzy graph of this site for four different time periods is taken for each of these periods the energy of an intuitionistic fuzzy graph and its lower and upper bounds are calculated. The energy is also represented in terms of bar diagram. We have taken the four links 1. microcontroller-boards, 2./log-in html, 3./ and 4./ project kits for our calculation.

Example 4.1. In the website <http://www.pantechsolutions.net/> we consider four links 1. microcontroller-boards, 2./log-in html, 3./ and 4./ project kits for the period July 16, 2013 to August 15, 2013.

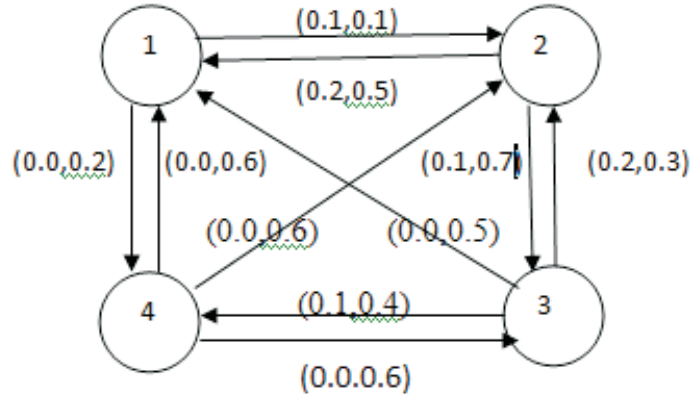


Figure 2: \$G_2\$. An intuitionistic fuzzy graph

For an intuitionistic fuzzy graph in Fig. 2,

$$\text{Spec}(\mu_{ij}(G_2)) = \{-0.2000, 0.0000, 0.2000, 0.0000\}$$

$$\text{Spec}(\gamma_{ij}(G_2)) = \{0.9927, -0.1841, -0.4043 + 0.2306i, -0.4043 - 0.2306i\}$$

$$E(\mu_{ij}(G_2)) = 0.2000 + 0.0000 + 0.2000 + 0.0000 = 0.4 \text{ its Lower bound} = 0.2828 \text{ and Upper bound} = 0.5657$$

$$E(\gamma_{ij}(G_2)) = 0.9927 + 0.1841 + 0.4655 + 0.4655 = 2.1078 \text{ its Lower bound} = 1.9047 \text{ and Upper bound} = 2.2271.$$

Example 4.2. In the same website (mentioned above in example 4.1.), we consider the same four links for the period August 16, 2013 to September 15, 2013.

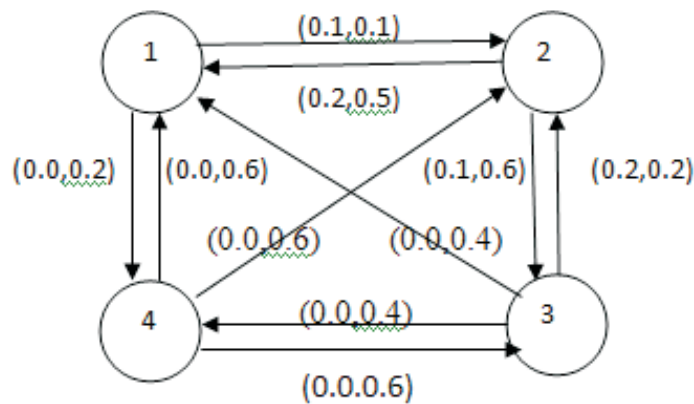


Figure 3: \$G_3\$. An intuitionistic fuzzy graph

For an intuitionistic fuzzy graph in Fig.3,

$$\text{Spec}(\mu_{ij}(G_3)) = \{-0.2000, 0.0000, 0.2000, 0.0000\}$$

$$\text{Spec}(\gamma_{ij}(G_3)) = \{0.8964, -0.1893, -0.3536 + 0.2114i, -0.3536 - 0.2114i\}$$

$$E(\mu_{ij}(G_3)) = 0.2000 + 0.0000 + 0.2000 + 0.0000 = 0.4 \text{ its Lower bound} = 0.2828 \text{ and Upper bound} = 0.5657$$

$$E(\gamma_{ij}(G_3)) = 0.8964 + 0.1893 + 0.4120 + 0.4120 = 1.9096 \text{ its Lower bound} = 2.0000 \text{ and Upper bound} = 1.7425.$$

Example 4.3. In the same website (mentioned above in example 4.1.), we consider the same four links for the period September 16, 2013 to October 15, 2013.

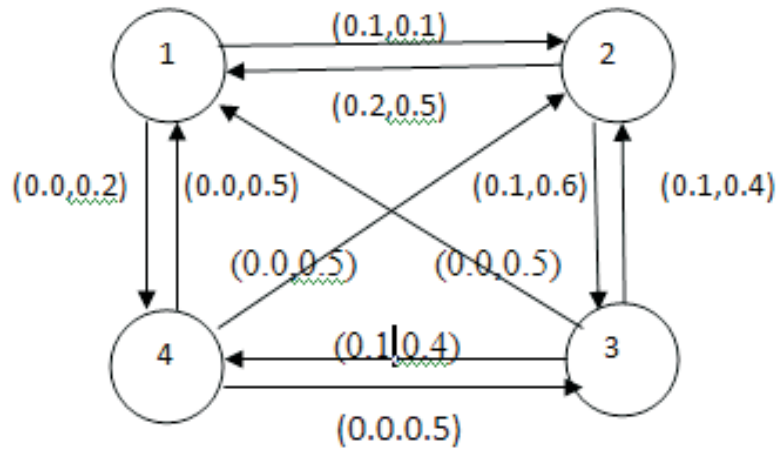


Figure 4: G4. An intuitionistic fuzzy graph

For an intuitionistic fuzzy graph in Fig.4,

$$\text{Spec}(\mu_{ij}(G_4)) = \{-0.1732, 0.1732, 0.0000, 0.0000\}$$

$$\text{Spec}(\gamma_{ij}(G_4)) = \{0.9418, -0.2000, -0.3709 + 0.1053i, -0.3709 - 0.1053i\}$$

$$E(\mu_{ij}(G_4)) = 0.1732 + 0.1732 + 0.0000 + 0.0000 = 0.3464 \text{ its Lower bound} = 0.2449 \text{ and Upper bound} = 0.4899$$

$$E(\gamma_{ij}(G_4)) = 0.9418 + 0.2000 + 0.3856 + 0.3856 = 1.913 \text{ its Lower bound} = 1.7855 \text{ and Upper bound} = 2.1726.$$

Example 4.4. In the same website (mentioned above in Example 4.1.), we consider the same four links for the period of October 16, 2013 to November 15, 2013.

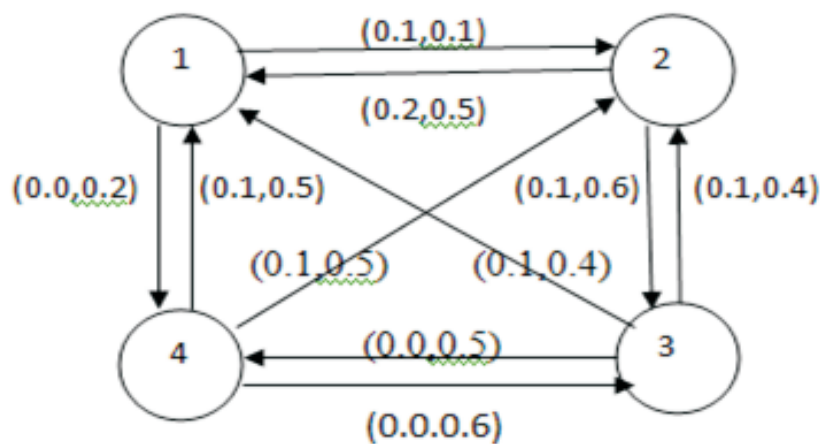


Figure 5: G5. An intuitionistic fuzzy graph

For an intuitionistic fuzzy graph in Fig. 5,

$$\text{Spec}(\mu_{ij}(G_5)) = \{0.0000, 0.1879, -0.1532, -0.0347\}$$

$$\text{Spec}(\gamma_{ij}(G_5)) = \{0.9940, -0.1302, -0.4000, -0.4638\}$$

$$E(\mu_{ij}(G_5)) = 0.0000 + 0.1879 + 0.1532 + 0.0347 = 0.3758 \text{ its Lower bound} = 0.2449 \text{ and Upper bound} = 0.4899$$

$$E(\gamma_{ij}(G_5)) = 0.9940 + 0.1302 + 0.4000 + 0.4638 = 1.988 \text{ its Lower bound} = 1.7997 \text{ and Upper bound} = 2.3495.$$

The following bar diagrams represents the energy of four links for the above four periods corresponding the membership and non-membership values respectively.

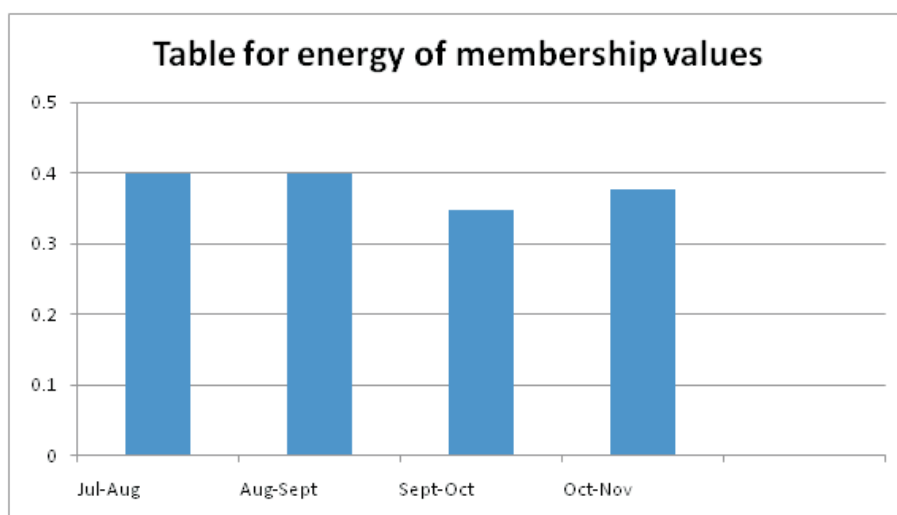


Table 1: Table for energy of membership values

From the above bar diagram the energy for the period July 16 to August 15 and August 16 to September 15 are high and same comparing the other period.

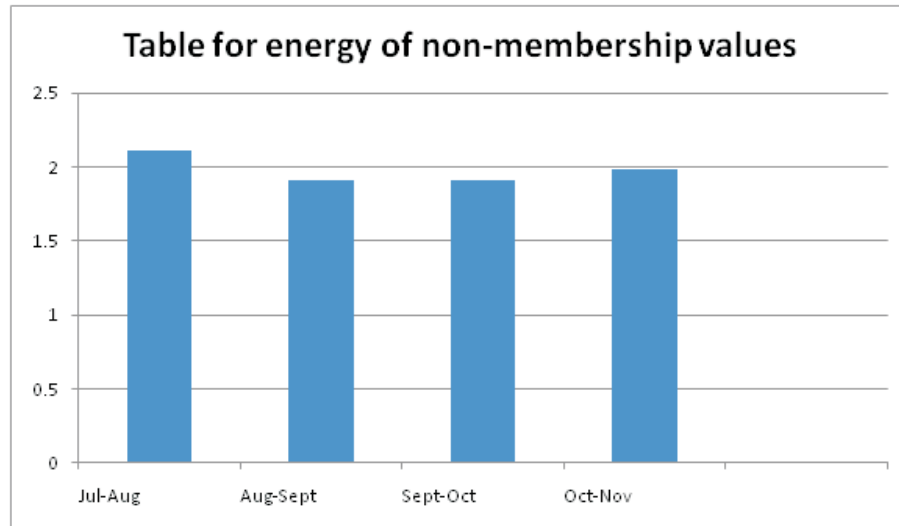


Table 2: Table for energy of non membership values

From the above bar diagram the energy for the period July 16 to August 15 is high comparing the other period.

5. Conclusion

In this paper, we defined the energy of an intuitionistic fuzzy graph in terms of its adjacency matrix. The lower and upper bounds for the energy of an intuitionistic fuzzy graph are derived. These concepts are illustrated with a real time example.

Acknowledgments. The authors thank the managements for their constant support towards the successful completion of this work.

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Accepted: 20.03.2014

ON UPPER AND LOWER ALMOST CONTRA- ω -CONTINUOUS MULTIFUNCTIONS

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Abstract. In this paper, we introduce and study the almost contra- ω -continuous multifunctions between topological spaces and obtain some characterizations and properties of such multifunctions.

Keywords: regular open set, ω -open set, almost contra- ω -continuous multifunctions.

2010 Mathematics Subject Classification: 54C60, 54C08

1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, separation axioms, etc., by utilizing generalized open sets. One of the most well known notions and also an inspiration source is the notion of ω -open [10] sets introduced by H.Z. Hdeib in 1982 and used by Al-Zoubi and Al-Nashef [1] in 2003. Various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good many of them have been extended to the setting of multifunction [2], [6],[7], [9], [17], [18]. In this paper, we introduce and study almost contra- ω -continuous multifunctions between topological spaces and obtain some characterizations of such multifunctions.

2 Preliminaries

Throughout this paper, (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces in which no separation axioms are assumed unless explicitly stated. For a subset A of (X, τ) , $\text{Cl}(A)$ and $\text{Int}(A)$ denote the closure of A with respect to τ and the interior of A with respect to τ , respectively. Recently, as generalization of closed sets, the notion of ω -closed sets were introduced and studied by Hdeib [10]. A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A is said to be ω -closed [10] if it contains all its condensation points. The complement of an ω -closed set is said to be ω -open. It is well known that a subset W of a space (X, τ) is ω -open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U \setminus W$ is countable. The family of all ω -open subsets of a topological space (X, τ) denoted by $\omega O(X, \tau)$, forms a topology on X finer than τ and the family of all ω -closed subsets of a topological space (X, τ) is denoted by $\omega C(X, \tau)$. The ω -closure and the ω -interior, that can be defined in the same way as $\text{Cl}(A)$ and $\text{Int}(A)$, respectively, will be denoted by $\omega \text{Cl}(A)$ and $\omega \text{Int}(A)$, respectively. We set $\omega O(X, x) = \{A : A \in \omega O(X) \text{ and } x \in A\}$ and $\omega C(X, x) = \{A : A \in \omega C(X) \text{ and } x \in A\}$. A subset A is said to be regular open [20] (resp. semiopen [13], preopen [14], α -open [15], semi-preopen [3]) if $A = \text{Int}(\text{Cl}(A))$ (resp. $A \subset \text{Cl}(\text{Int}(A))$, $A \subset \text{Int}(\text{Cl}(A))$, $A \subset \text{Int}(\text{Cl}(\text{Int}(A)))$, $A \subset \text{Cl}(\text{Int}(\text{Cl}(A)))$). The complement of a regular open (resp. semiopen, preopen, semi-preopen) set is called a regular closed (resp. semiclosed, preclosed, semi-preclosed). The intersection of all semiclosed (resp. preclosed, α -closed, semi-preclosed) subsets of (X, τ) containing $A \subset X$ is called the semiclosure (resp. preclosure, α -closure, semi-preclosure) of A and is denoted by $s \text{Cl}(A)$ (resp. $p \text{Cl}(A)$, $\alpha \text{Cl}(A)$, $sp \text{Cl}(A)$). The θ -semiclosure [12] of A , denoted by $s \text{Cl}_\theta(A)$, is defined to be the set of all $x \in X$ such that $A \cap \text{Cl}(U) \neq \emptyset$ for every semiopen set U containing x . A subset A is called θ -semiclosed [12] if and only if $A = s \text{Cl}_\theta(A)$. The complement of a θ -semiclosed set is called a θ -semiopen set [12]. The family of all regular open (resp. regular closed, semiopen,

semiclosed, α -open, semi-preopen, semi-preclosed) sets of (X, τ) is denoted by $RO(X)$ (resp. $RC(X)$, $SO(X)$, $SC(X)$, $\alpha O(X)$, $SPO(X)$, $SPC(X)$). By a multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$, following [4], we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X : F(x) \subset B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. In particular, $F^-(Y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$ and for each $A \subset X$, $F(A) = \cup_{x \in A} F(x)$. Then F is said to be surjection if $F(X) = Y$ and injection if $x \neq y$ implies $F(x) \cap F(y) = \emptyset$.

Definition 2.1 A multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$ is said to be:

1. upper almost ω -continuous [5] if for each point $x \in X$ and each open set V containing $F(x)$, there exists $U \in \omega O(X, x)$ such that $U \subset F^+(\text{Int}(\text{Cl}(V)))$;
2. lower almost ω -continuous [5] if for each point $x \in X$ and each open set V such that $F(x) \cap V \neq \emptyset$, there exists $U \in \omega O(X, x)$ such that $U \subset F^-(\text{Int}(\text{Cl}(V)))$.
3. upper contra- ω -continuous [6] if for each point $x \in X$ and each closed set V containing $F(x)$, there exists $U \in \omega O(X, x)$ such that $U \subset F^+(V)$;
4. lower contra- ω -continuous [6] if for each point $x \in X$ and each closed set V such that $F(x) \cap V \neq \emptyset$, there exists $U \in \omega O(X, x)$ such that $U \subset F^-(V)$;
5. upper weakly ω -continuous [7] if for each $x \in X$ and each open set V of Y such that $x \in F^+(V)$, there exists $U \in \omega O(X, x)$ such that $U \subset F^+(\text{Cl}(V))$;
6. lower weakly ω -continuous [7] if for each $x \in X$ and each open set V of Y such that $x \in F^-(V)$, there exists $U \in \omega O(X, x)$ such that $U \subset F^-(\text{Cl}(V))$.

Definition 2.2 A subset K of a space X is said to be S -closed [21] (resp. ω -compact [2]) relative to X if every cover of K by regular closed (resp. ω -open) sets of X has a finite subcover. A space X is said to be S -closed (resp. ω -compact) if X is S -closed (resp. ω -compact) relative to X .

Lemma 2.3 [1] Let A and B be subsets of a space (X, τ) .

1. If $A \in \omega O(X)$ and $B \in \tau$, then $A \cap B \in \omega O(B)$;
2. If $A \in \omega O(B)$ and $B \in \omega O(X)$, then $A \in \omega O(X)$.

Lemma 2.4 [17] For a multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$, the following holds:

1. $G_F^+(A \times B) = A \cap F^+(B)$;
2. $G_F^-(A \times B) = A \cap F^-(B)$,

for any subset A of X and B of Y , where $G_F : X \rightarrow X \times Y$ is defined as $G_F(x) = \{x\} \times F(x)$ for every $x \in X$.

Definition 2.5 [2] A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be almost contra- ω -continuous if $f^{-1}(W) \in \omega O(X)$ for every $W \in RC(Y)$.

3. On upper and lower almost contra- ω -continuous multifunctions

Definition 3.1 A multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$ is said to be:

1. upper almost contra- ω -continuous if for each point $x \in X$ and each regular closed set V with $x \in F^+(V)$, there exists $U \in \omega O(X, x)$ such that $U \subset F^+(V)$;
2. lower almost contra- ω -continuous if for each point $x \in X$ and each regular closed set V with $x \in F^-(V)$, there exists $U \in \omega O(X, x)$ such that $U \subset F^-(V)$.

Theorem 3.2 If $F : (X, \tau) \rightarrow (Y, \sigma)$ is an upper (lower) almost contra- ω -continuous multifunction, then it is upper (lower) weakly ω -continuous.

Proof. Let $x \in X$ and V be an open subset of Y with $F(x) \subset V$. This implies that $\text{Cl}(V)$ is a regular closed set with $F(x) \subset \text{Cl}(V)$. Since F is upper almost contra- ω -continuous, there exists $U \in \omega O(X, x)$ such that $U \subset F^+(\text{Cl}(V))$. Hence, F is upper weakly ω -continuous. ■

The following example shows that the converse of the above Theorem 3.2 is not true in general.

Example 3.3 Let $X = \mathfrak{R}$ with the topologies $\tau = \{\emptyset, \mathfrak{R}, \mathfrak{R} - Q\}$ and $\sigma = \{\emptyset, \mathfrak{R}, \mathfrak{R} - Q\}$. Define $F : (\mathfrak{R}, \tau) \rightarrow (\mathfrak{R}, \sigma)$ as follows: $F(x) = \{x\}$. Then F is upper weakly- ω -continuous multifunction but is not upper almost contra- ω -continuous multifunction.

Corollary 3.4 If $F : (X, \tau) \rightarrow (Y, \sigma)$ is almost contra- ω -continuous, then it is weakly ω -continuous.

Theorem 3.5 If $F : (X, \tau) \rightarrow (Y, \sigma)$ is an upper (lower) contra- ω -continuous multifunction, then it is upper (lower) almost contra ω -continuous multifunction.

Proof. The proof is obvious. ■

The following example shows that the converse of the above Theorem 3.5 is not true in general.

Example 3.6 Let $X = \mathfrak{R}$ with the topology $\tau = \{\emptyset, \mathfrak{R}, \mathfrak{R} - Q\}$. And $Y = \{a, b, c\}$ with the topology $\sigma = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Take a fixed number $e \in Q$, and define $F : (\mathfrak{R}, \tau) \rightarrow (Y, \sigma)$ as follows:

$$F(x) = \begin{cases} \{b\} & \text{if } x \in Q - \{e\} \\ \{c\} & \text{if } x \in (\mathfrak{R} - Q) \cup \{e\}. \end{cases}$$

Then F is upper almost contra- ω -continuous multifunction but is not upper contra- ω -continuous multifunction.

Corollary 3.7 [2] *If $F : (X, \tau) \rightarrow (Y, \sigma)$ is contra- ω -continuous, then F is almost contra- ω -continuous.*

Theorem 3.8 *For a multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$, the following statements are equivalent:*

1. F is upper almost contra- ω -continuous;
2. $F^+(A) \in \omega O(X)$ for every regular closed A of Y ;
3. $F^-(U) \in \omega C(X)$ for every regular open subset U of Y ;
4. $F^-(\text{Int}(\text{Cl}(A))) \in \omega C(X)$ for every open subset A of Y ;
5. $F^+(\text{Cl}(\text{Int}(A))) \in \omega O(X)$ for every closed subset A of Y ;
6. for each $x \in X$ and for each $V \in SO(Y)$ with $F(x) \subset V$, there exists $U \in \omega O(X, x)$ such that $F(U) \subset \text{Cl}(V)$;
7. $F^+(V) \subset \omega \text{Int}(F^+(\text{Cl}(V)))$ for every $V \in SO(Y)$.

Proof. (1) $\Leftrightarrow$ (2): Let $A \in RC(Y)$ and $x \in F^+(A)$. Since F is upper almost contra- ω -continuous, there exists $U \in \omega O(X, x)$ such that $U \subset F^+(A)$. Thus, $F^+(A) \in \omega O(X)$. The converse is obvious.

(2) $\Leftrightarrow$ (3) and (4) $\Leftrightarrow$ (5): It follows from the fact that $F^+(Y \setminus A) = X \setminus F^-(A)$ for every subset A of Y .

(3) $\Leftrightarrow$ (4): Let A be an open subset of Y . Since $\text{Int Cl}(A)$ is regular open, then $F^-(\text{Int}(\text{Cl}(A)))$ is ω -closed. The converse is obvious.

(5) $\Leftrightarrow$ (2): It is similar to that of (3) $\Leftrightarrow$ (4).

(6) $\Rightarrow$ (7): Let $V \in SO(Y)$ and $x \in F^+(V)$. Then $F(x) \subset V$. By (6), there exists $U \in \omega O(X, x)$ such that $F(U) \subset \text{Cl}(V)$. This implies that $x \in U \subset F^+(\text{Cl}(V))$. Hence, $x \in \omega \text{Int}(F^+(\text{Cl}(V)))$ and $F^+(V) \subset \omega \text{Int}(F^+(\text{Cl}(V)))$.

(7) $\Rightarrow$ (2): Let $A \in RC(Y)$. Since $A \in SO(Y)$, then $F^+(A) \subset \omega \text{Int}(F^+(\text{Cl}(A)))$; hence $F^+(A) \in \omega O(X)$.

(2) $\Rightarrow$ (6): Let $x \in X$ and $V \in SO(Y)$ with $F(x) \subset V$. Since $\text{Cl}(V) \in RC(Y)$, there exists $A \in \omega O(X, x)$ such that $A \subset F^+(\text{Cl}(V))$. Hence $F(A) \subset \text{Cl}(V)$. ■

Theorem 3.9 *For a multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$, the following statements are equivalent:*

1. F is lower almost contra- ω -continuous;
2. $F^-(A) \in \omega O(X)$ for every regular closed A of Y ;
3. $F^+(U) \in \omega C(X)$ for every regular open subset U of Y ;
4. $F^+(\text{Int}(\text{Cl}(A))) \in \omega C(X)$ for every open subset A of Y ;

5. $F^-(\text{Cl}(\text{Int}(A))) \in \omega O(X)$ for every closed subset A of Y ;
6. for each $x \in X$ and for each $V \in SO(Y)$ with $F(x) \cap V \neq \emptyset$, there exists $U \in \omega O(X, x)$ such that $F(u) \cap \text{Cl}(V) \neq \emptyset$ for each $u \in U$;
7. $F^-(V) \subset \omega \text{Int}(F^-(\text{Cl}(V)))$ for every $V \in SO(Y)$.

Proof. The proof is similar to that of Theorem 3.8. ■

Definition 3.10 [8] Let U be a subset of a topological space (X, τ) . The set $\cap\{V \in RO(X) : U \subset V\}$ is called the r -kernel of U and is denoted by $r\text{-Ker}(U)$.

Lemma 3.11 [8] The following properties hold for subsets U, V of a space X :

1. $x \in r\text{-Ker}(U)$ if and only if $U \cap V \neq \emptyset$ for any regular closed set V containing x .
2. $U \subset r\text{-Ker}(U)$ and $U = r\text{-Ker}(U)$ if U is regular open in X .
3. If $U \subset V$ then $r\text{-Ker}(U) \subset r\text{-Ker}(V)$.

Corollary 3.12 [2] For a function $f : (X, \tau) \rightarrow (Y, \sigma)$, the following statements are equivalent:

1. f is almost contra- ω -continuous;
2. $f^{-1}(F) \in \omega O(X)$ for every $F \in RC(Y)$;
3. for each $x \in X$ and each $F \in RC(Y, f(x))$, there exists $U \in \omega O(X, x)$ such that $f(U) \subset F$;
4. for each $x \in X$ and each $U \in RO(Y, f(x))$, there exist $V \in \omega C(X, x)$ such that $f(V) \subset U$;
5. $f^{-1}(\text{Int}(\text{Cl}(G))) \in \omega C(X)$ for every open subset G of Y ;
6. $f^{-1}(\text{Cl}(\text{Int}(F))) \in \omega O(X)$ for every closed subset F of Y ;
7. $f(\omega \text{Cl}(A)) \subset r \text{Ker}(f(A))$ for every subset A of X ;
8. $\omega \text{Cl}(f^{-1}(B)) \subset f^{-1}(r \text{Ker}(B))$ for every subset B of Y .

Theorem 3.13 For a multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$, the following statements are equivalent:

1. F is lower almost contra- ω -continuous;
2. $F^-(A) \in \omega O(X)$ for every θ -semiopen A of Y ;
3. $F^+(U) \in \omega C(X)$ for every θ -semiclosed subset U of Y ;
4. $\omega \text{Cl}(F^+(\text{Int}(\text{Cl}(B)))) \subset F^+(\omega \text{Cl}(B))$ for every subset B of Y ;

5. $\omega \text{Cl}(F^+(B)) \subset F^+(s \text{Cl}_\theta(B))$ for every subset B of Y ;

6. $F(\omega \text{Cl}(A)) \subset s \text{Cl}_\theta(F(A))$ for every subset A of X .

Proof. (1) $\Rightarrow$ (2): Let G be any θ -semiopen set of Y . There exists a family of regular closed sets $\{K_\alpha : \alpha \in \Delta\}$ such that $G = \cup\{K_\alpha : \alpha \in \Delta\}$. It follows from Theorem 3.9 (ii) that $F^-(G) = \cup\{F^-(K_\alpha) : \alpha \in \Delta\}$ is ω -open.

(2) $\Rightarrow$ (3): This is obvious.

(3) $\Rightarrow$ (4): Let B be any subset of Y . Then $\text{Int}(\text{Cl}(B))$ is regular open and it is θ -semiclosed in Y . Therefore, we have that $F^+(\text{Int}(\text{Cl}(B)))$ is ω -closed and $\omega \text{Cl}(F^+(\text{Int}(\text{Cl}(B)))) = F^+(\text{Int}(\text{Cl}(B))) \subset F^+(s \text{Cl}(B))$.

(4) $\Rightarrow$ (5): Let B be any subset of Y . For any regular open set V with $B \subset V$, we have $\omega \text{Cl}(F^+(B)) \subset \text{Cl}(F^+(V)) = \omega \text{Cl}(F^+(\text{Int}(\text{Cl}(V)))) \subset F^+(s \text{Cl}(V)) = F^+(V)$. Therefore, $\omega \text{Cl}(F^+(B)) \subset F^+(\cap\{V \in RO(Y) : B \subset V\}) = F^+(s \text{Cl}_\theta(B))$.

(5) $\Rightarrow$ (1): Let V be any semiopen set of Y . Then we have $X \setminus \omega \text{Int}(F^-(\text{Cl}(V))) = \omega \text{Cl}(F^+(Y \setminus \text{Cl}(V))) \subset F^+(s \text{Cl}_\theta(Y \setminus \text{Cl}(V))) = F^+(Y \setminus \text{Cl}(V)) = X \setminus F^-(\text{Cl}(V))$. Therefore, we obtain $F^-(V) \subset F^-(\text{Cl}(V)) \subset \omega \text{Int}(F^-(\text{Cl}(V)))$. By Theorem 3.9 (7), F is lower almost contra- ω -continuous.

(5) $\Rightarrow$ (6): Let A be a subset of X and $B = F(A)$. Then $A \subset F^+(B)$ and $\omega \text{Cl}(A) \subset \omega \text{Cl}(F^+(B)) \subset F^+(s \text{Cl}_\theta(B))$. Therefore, we have $F(\omega \text{Cl}(A)) \subset F(F^+(s \text{Cl}_\theta(B))) \subset s \text{Cl}_\theta(B) = s \text{Cl}_\theta(F(A))$.

(6) $\Rightarrow$ (5): Let B be any subset of Y . Then we have $F(\omega \text{Cl}(F^+(B))) \subset s \text{Cl}_\theta(F(F^+(B))) \subset s \text{Cl}_\theta(B)$; hence $\omega \text{Cl}(F^+(B)) \subset F^+(s \text{Cl}_\theta(B))$. ■

Corollary 3.14 For a function $f : (X, \tau) \rightarrow (Y, \sigma)$, the following properties are equivalent:

1. f is almost contra- ω -continuous;
2. $f^{-1}(V) \in \omega O(X)$ for each θ -semiopen set V of Y ;
3. $f^{-1}(F) \in \omega C(X)$ for each θ -semiclosed set F of Y .
4. for each $x \in X$ and each $U \in SO(Y, f(x))$, there exist $V \in \omega O(X, x)$ such that $f(V) \subset \text{Cl}(U)$;
5. $f^{-1}(V) \subset \omega \text{Int}(f^{-1}(\text{Cl}(V)))$ for every $V \in SO(Y)$.
6. $f(\omega \text{Cl}(A)) \subset s \text{Cl}_\theta(f(A))$ for every subset A of X ;
7. $\omega \text{Cl}(f^{-1}(B)) \subset f^{-1}(s \text{Cl}_\theta(B))$ for every subset B of Y .
8. $\omega \text{Cl}(f^{-1}(V)) \subset f^{-1}(s \text{Cl}_\theta(V))$ for every open subset V of Y .
9. $\omega \text{Cl}(f^{-1}(V)) \subset f^{-1}(s \text{Cl}(V))$ for every open subset V of Y .
10. $\omega \text{Cl}(f^{-1}(V)) \subset f^{-1}(\text{Int}(\text{Cl}(V)))$ for every open subset V of Y .

Theorem 3.15 *For a multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$, the following statements are equivalent:*

1. F is upper almost contra- ω -continuous;
2. $\omega \text{Cl}(F^-(\text{Int}(K))) \subset F^{-1}(K)$ for every semiclosed set K of Y ;
3. $\omega \text{Cl}(F^-(\text{Int}(s \text{Cl}(B)))) \subset F^-(s \text{Cl}(B))$ for every $B \subseteq Y$;
4. $F^+(s \text{Int}(B)) \subset \omega \text{Int}(F^+(\text{Cl}(s \text{Int}(B))))$ for every $B \subseteq Y$.

Proof. (1) $\Rightarrow$ (2): Let K be a semiclosed set of Y . Then $Y \setminus K$ is semiopen. By Theorem 3.8 (7), it follows that $F^+(Y \setminus K) \subset \omega \text{Int}(F^+(Y \setminus \text{Int}(K)))$. Hence $X \setminus F^-(K) \subset \omega \text{Int}(F^+(Y \setminus \text{Int}(K))) = \omega \text{Int}(X \setminus F^-(\text{Int}(K))) = X \setminus \omega \text{Cl}(F^-(\text{Int}(K)))$. Hence, $\omega \text{Cl}(F^-(\text{Int}(K))) \subset F^{-1}(K)$.

(2) $\Rightarrow$ (3): Let B be any subset of Y . Then $s \text{Cl}(B)$ is semiclosed in Y and hence $\omega \text{Cl}(F^-(\text{Int}(s \text{Cl}(B)))) \subset F^-(s \text{Cl}(B))$.

(3) $\Rightarrow$ (4): Let B be any subset of Y . Then we have

$$\begin{aligned} X \setminus F^+(s \text{Int}(B)) &= F^-(s \text{Cl}(Y \setminus B)) \supset \omega \text{Cl}(F^-(\text{Int}(s \text{Cl}(Y \setminus B)))) \\ &= \omega \text{Cl}(F^-(\text{Int}(Y \setminus s \text{Int}(B)))) = \omega \text{Cl}(F^-(Y \setminus \text{Cl}(s \text{Int}(B)))) \\ &= \omega \text{Cl}(X \setminus F^+(\text{Cl}(s \text{Int}(B)))) = X \setminus \omega \text{Int}(F^+(\text{Cl}(s \text{Int}(B)))). \end{aligned}$$

Hence, $F^+(s \text{Int}(B)) \subset \omega \text{Int}(F^+(\text{Cl}(s \text{Int}(B))))$.

(4) $\Rightarrow$ (1): Let V be any semiopen set of Y . Then $V = s \text{Int}(V)$ and hence $F^+(V) \subset \omega \text{Int}(F^+(\text{Cl}(V)))$. By Theorem 3.8 (7), F is upper almost contra- ω -continuous. ■

Theorem 3.16 *For a multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$, the following statements are equivalent:*

1. F is lower almost contra- ω -continuous;
2. $\omega \text{Cl}(F^+(\text{Int}(K))) \subset F^+(K)$ for every semiclosed set K of Y ;
3. $\omega \text{Cl}(F^+(\text{Int}(s \text{Cl}(B)))) \subset F^+(s \text{Cl}(B))$ for every $B \subseteq Y$;
4. $F^-(s \text{Int}(B)) \subset \omega \text{Int}(F^-(\text{Cl}(s \text{Int}(B))))$ for every $B \subseteq Y$.

Proof. The proof is similar to that of Theorem 3.15. ■

Recall that a topological space is said to be extremely disconnected if the closure of every open set is open in the space.

Theorem 3.17 *Let (Y, σ) be an extremely disconnected space. Then a multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$ is upper almost contra- ω -continuous if and only if it is upper almost ω -continuous.*

Proof. Let $x \in X$ and V be any regular open set of Y containing $F(x)$. Since (Y, σ) is extremely disconnected, V is regular closed and hence semiopen. By Theorem 3.8, there exists $U \in \omega O(X, x)$ such that $F(U) \subset \text{Cl}(V) = V$. Then F is upper almost ω -continuous. Conversely, let K be any regular closed subset of Y . Since (Y, σ) is extremely disconnected, K is also regular open and by Theorem 3.4 of [5], $F^+(K)$ is ω -open. By Theorem 3.8, F is upper almost contra- ω -continuous. ■

Theorem 3.18 *Let (Y, σ) be an extremely disconnected space. Then a multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$ is lower almost contra- ω -continuous if and only if F is lower almost ω -continuous.*

Proof. The proof is similar to that of Theorem 3.17. ■

Theorem 3.19 *The following statements are equivalent for a multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$:*

1. F is upper (lower) almost contra- ω -continuous;
2. $F^+(\text{Cl}(V))(F^-(\text{Cl}(V)))$ is ω -open in X for every $V \in SPO(Y)$;
3. $F^+(\text{Cl}(V))(F^-(\text{Cl}(V)))$ is ω -open in X for every $V \in SO(Y)$;
4. $F^-(\text{Int}(\text{Cl}(V)))(F^+(\text{Int}(\text{Cl}(V))))$ is ω -closed in X for every $V \in PO(Y)$.

Proof. (1) $\Rightarrow$ (2): Suppose that V is any semi-preopen set of Y . Since $\text{Cl}(V) \in RC(Y)$, by Theorem 3.8, $F^-(\text{Cl}(V))$ is ω -open in X .

(2) $\Rightarrow$ (3): This is obvious, since $SO(Y) \subset SPO(Y)$.

(3) $\Rightarrow$ (4): Let $V \in PO(Y)$. Then $Y \setminus \text{Int}(\text{Cl}(V))$ is regular closed and hence it is semiopen. Then, we have $X \setminus F^-(\text{Int}(\text{Cl}(V))) = F^+(Y \setminus \text{Int}(\text{Cl}(V))) = F^+(\text{Cl}(Y \setminus \text{Int}(\text{Cl}(V)))) \in \omega O(X)$. Hence $F^-(\text{Int}(\text{Cl}(V))) \in \omega C(X)$.

(4) $\Rightarrow$ (1): If V is any regular open set of Y . Then $V \in PO(Y)$ and hence $F^-(V) = F^-(\text{Int}(\text{Cl}(V)))$ is ω -closed in X . Therefore, F is upper almost contra- ω -continuous.

The proof of the second case is similar. ■

Lemma 3.20 [16] *For a subset V of a topological space (Y, σ) , the following properties hold:*

1. $\alpha \text{Cl}(V) = \text{Cl}(V)$ for every $V \in SPO(Y)$;
2. $p \text{Cl}(V) = \text{Cl}(V)$ for every $V \in SO(Y)$;
3. $s \text{Cl}(V) = \text{Int}(\text{Cl}(V))$ for every $V \in PO(Y)$.

Corollary 3.21 *The following statements are equivalent for a multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$:*

1. F is upper (lower) almost contra- ω -continuous;
2. $F^+(\alpha \text{Cl}(V))(F^-(\alpha \text{Cl}(V)))$ is ω -open in X for every $V \in \text{SPO}(Y)$;
3. $F^+(p \text{Cl}(V))(F^-(p \text{Cl}(V)))$ is ω -open in X for every $V \in \text{SO}(Y)$;
4. $F^-(s \text{Cl}(V))(F^+(s \text{Cl}(V)))$ is ω -closed in X for every $V \in \text{PO}(Y)$.

Proof. This is an immediate consequence of Theorem 3.19 and Lemma 3.20. ■

Theorem 3.22 *The following statements are equivalent for a multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$:*

1. F is upper almost contra- ω -continuous;
2. $\omega \text{Cl}(F^-(V)) \subset F^-(\text{Int}(\text{Cl}(V)))$ for every open subset V of Y ;
3. $\omega \text{Cl}(F^-(V)) \subset F^-(s \text{Cl}(V))$ for every open subset V of Y .

Proof. (2) $\Rightarrow$ (1): Let $V \in \text{RO}(Y)$. Then $\omega \text{Cl}(F^-(V)) \subset F^-(\text{Int}(\text{Cl}(V))) = F^-(V)$. This implies that $F^-(A)$ is ω -closed and hence F is upper almost contra- ω -continuous.

(1) $\Rightarrow$ (2): Let V be an open set. We have $\text{Int}(\text{Cl}(V)) \in \text{RO}(Y)$. By (1), $F^-(\text{Int}(\text{Cl}(V)))$ is ω -closed. Since $V \subset \text{Int}(\text{Cl}(A))$, then $F^-(A) \subset F^-(\text{Int}(\text{Cl}(A)))$. Thus, $\omega \text{Cl}(F^-(V)) \subset F^-(s \text{Cl}(V))$.

(2) $\Leftrightarrow$ (3): It follows from Lemma 3.20. ■

Theorem 3.23 *The following statements are equivalent for a multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$:*

1. F is lower almost contra- ω -continuous;
2. $\omega \text{Cl}(F^+(V)) \subset F^+(\text{Int}(\text{Cl}(V)))$ for every open subset V of Y ;
3. $\omega \text{Cl}(F^+(V)) \subset F^+(s \text{Cl}(V))$ for every open subset V of Y .

Proof. The proof is similar to that of Theorem 3.22. ■

Theorem 3.24 *Let $F : (X, \tau) \rightarrow (Y, \sigma)$ be any multifunction. If $\omega \text{Cl}(F^-(V)) \subset F^-(r\text{-Ker}(V))$ for every subset V of Y , then F is upper almost contra- ω -continuous.*

Proof. Let $V \in \text{RO}(Y)$. By Lemma 3.11, $\omega \text{Cl}(F^-(V)) \subset F^-(r\text{-Ker}(V)) = F^-(V)$. This implies that $\omega \text{Cl}(F^-(V)) = F^-(V)$; hence $F^-(V)$ is ω -closed. By Theorem 3.8, F is upper almost contra- ω -continuous. ■

Theorem 3.25 *Let $F : (X, \tau) \rightarrow (Y, \sigma)$ be any multifunction. If $F(\omega \text{Cl}(V)) \subset r\text{-Ker}(F(V))$ for every subset V of Y , then F is lower almost contra- ω -continuous.*

Proof. Let H be a regular open set of X . Then $F(\omega \text{Cl}(F^+(H))) \subset r\text{-Ker}(H)$ and $\omega \text{Cl}(F^+(H)) \subset F^+(r\text{-Ker}(H))$. By Lemma 3.11,

$$\omega \text{Cl}(F^+(H)) \subset F^+(r\text{-Ker}(H)) = F^+(H).$$

We have $\omega \text{Cl}(F^+(H)) = F^+(H)$. This implies that $F^+(H)$ is ω -closed in X . By Theorem 3.9, F is lower almost contra- ω -continuous. ■

Definition 3.26 A topological space (X, τ) is said to ω - T_2 [2], if for each pair of distinct points x and y in X , there exist disjoint ω -open sets U and V in X such that $x \in U$ and $y \in V$.

Lemma 3.27 [19] If A and B are disjoint compact subsets of an Urysohn space X , there exist open sets U and V of X such that $A \subset U$, $B \subset V$ and $\text{Cl}(U) \cap \text{Cl}(V) = \emptyset$.

Theorem 3.28 If $F : (X, \tau) \rightarrow (Y, \sigma)$ is an upper almost contra- ω -continuous injective multifunction into an Urysohn space Y and $F(x)$ is compact for each $x \in X$, then X is ω - T_2 .

Proof. For any distinct points $x_1, x_2 \in X$, we have $F(x_1) \cap F(x_2) = \emptyset$, since F is injective and $F(x_1)$ and $F(x_2)$ are disjoint compact sets, by Lemma 3.27, there exist open sets V_1 and V_2 such that $F(x_1) \subset V_1$, $F(x_2) \subset V_2$ and $\text{Cl}(V_1) \cap \text{Cl}(V_2) = \emptyset$. Since $\text{Cl}(V_1)$ and $\text{Cl}(V_2)$ are regular closed sets and F is upper almost contra- ω -continuous, there exist $U_1 \in \omega O(X, x_1)$ and $U_2 \in \omega O(X, x_2)$ such that $F(U_1) \subset \text{Cl}(V_1)$, $F(U_2) \subset \text{Cl}(V_2)$; hence $U_1 \cap U_2 = \emptyset$ and X is ω - T_2 . ■

Definition 3.29 A subset A of a topological space (X, τ) is said to be ω -dense in X if $\omega \text{Cl}(A) = X$.

Definition 3.30 A multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$ is called upper weakly continuous [19], if for each open set V containing $F(x)$ and for each $x \in X$, there exists an open set U containing x such that $F(U) \subset \text{Cl}(V)$.

Theorem 3.31 Let X be a topological space and Y an Urysohn space. If the following four conditions are satisfied:

1. $F : (X, \tau) \rightarrow (Y, \sigma)$ is an upper weakly continuous multifunction,
2. $G : X \rightarrow Y$ is an upper almost contra- ω -continuous multifunction,
3. $F(x)$ and $G(x)$ are compact sets of Y for each $x \in X$,
4. $A = \{x \in X : F(x) \cap G(x) \neq \emptyset\}$,

then A is ω -closed. Moreover if $F(x) \cap G(x) \neq \emptyset$ for each point $x \in X$ in a ω -dense set D , then $F(x) \cap G(x) \neq \emptyset$ for each $x \in X$.

Proof. Suppose that $x \in X \setminus A$. Then we have $F(x) \cap G(x) = \emptyset$. Since $F(x)$ and $G(x)$ are disjoint compact sets of an Urysohn space, by Lemma 3.27, there exist open sets V and W such that $F(x) \subset V$ and $G(x) \subset W$ and $\text{Cl}(V) \cap \text{Cl}(W) = \emptyset$. Since F is upper weakly continuous and $F(x) \subset V$, there exists an open set U_1 containing x such that $F(U_1) \subset \text{Cl}(V)$. Since $\text{Cl}(W)$ is regular closed, $G(x) \subset W$ and G is upper almost contra- ω -continuous, there exists $U_2 \in \omega O(X, x)$ such that $G(U_2) \subset \text{Cl}(W)$. Let $U = U_1 \cap U_2$, then U is ω -open and $U \cap A = \emptyset$. Therefore, $x \in X \setminus \omega \text{Cl}(A)$ and hence A is ω -closed. On the other hand, if $F(x) \cap G(x) \neq \emptyset$ on an ω -dense set D of X , then we have $X = \omega \text{Cl}(D) \subset \omega \text{Cl}(A) = A$. Therefore, we obtain $F(x) \cap G(x) \neq \emptyset$ for each $x \in X$. ■

Definition 3.32 A subset A of a space (X, τ) is said to be:

1. α -regular [11], if for each $a \in A$ and any open set U containing a , there exists an open set V of X such that $a \in V \subset \text{Cl}(V) \subset U$;
2. α -paracompact [11], if every X -open cover A has an X -open refinement which covers A and is locally finite for each point of X .

Lemma 3.33 [11] *If A is an α -regular and α -paracompact subset of a space X and U is an open neighborhood of A , then there exists an open set V of X such that $A \subset V \subset \text{Cl}(V) \subset U$.*

For a multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$, the multifunction $\text{Cl}(F) : X \rightarrow Y$ is defined by $\text{Cl}(F)(x) = \text{Cl}(F(x))$ for each point $x \in X$. Similarly, we denote $s\text{Cl}(F)$, $p\text{Cl}(F)$, $\alpha\text{Cl}(F)$, $sp\text{Cl}(F)$, $\omega\text{Cl}(F)$.

Lemma 3.34 [18] *If $F : (X, \tau) \rightarrow (Y, \sigma)$ is a multifunction such that $F(x)$ is α -paracompact α -regular for each $x \in X$, then for each open set V of Y , $G^+(V) = F^+(V)$ and for each closed set K of Y , $G^-(K) = F^-(K)$, where G denotes $\text{Cl}(F)$, $s\text{Cl}(F)$, $p\text{Cl}(F)$, $\alpha\text{Cl}(F)$, $sp\text{Cl}(F)$, $\omega\text{Cl}(F)$.*

Lemma 3.35 [18] *If $F : (X, \tau) \rightarrow (Y, \sigma)$ is a multifunction, then for each open set V of Y , $G^-(V) = F^-(V)$ and for each closed set K of Y , $G^+(K) = F^+(K)$, where G denotes $\text{Cl}(F)$, $s\text{Cl}(F)$, $p\text{Cl}(F)$, $\alpha\text{Cl}(F)$, $sp\text{Cl}(F)$, $\omega\text{Cl}(F)$.*

Theorem 3.36 *A multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$ is upper almost contra- ω -continuous if and only if G is upper almost contra- ω -continuous.*

Proof. Let K be a regular closed set of Y . By Theorem 3.8 and Lemma 3.35, $G^+(K) = F^+(K)$ is a ω -open set of X . Hence, G is upper almost contra- ω -continuous. Conversely, Let K be a regular closed set of Y . By Theorem 3.8 and Lemma 3.35, $F^+(K) = G^+(K)$ is a ω -open set of X . Hence, G is upper almost contra- ω -continuous. ■

Theorem 3.37 *Let $F : (X, \tau) \rightarrow (Y, \sigma)$ be a multifunction such that $F(x)$ is α -regular α -paracompact for each $x \in X$. Then F is lower almost contra- ω -continuous if and only if G is lower almost contra- ω -continuous.*

Proof. Let K be a regular closed set of Y . By Theorem 3.9 and Lemma 3.34, $G^-(K) = F^-(K)$ is an ω -open set of X . Hence, G is lower almost contra- ω -continuous. Conversely, Let K be a regular closed set of Y . By Theorem 3.9 and Lemma 3.34, $F^-(K) = G^-(K)$ is an ω -open set of X . Hence, G is lower almost contra- ω -continuous. ■

Theorem 3.38 *Let $F : (X, \tau) \rightarrow (Y, \sigma)$ be a multifunction and U be an open subset of X . If F is a lower (upper) almost contra- ω -continuous, then $F|_U : U \rightarrow Y$ is lower (upper) almost contra- ω -continuous.*

Proof. Let V be any regular closed set of Y . Let $x \in U$ and $x \in (F|_U)^-(V)$. Since F is a lower almost contra- ω -continuous multifunction, then there exists $G \in \omega O(X, x)$ such that $G \subset F^-(V)$. Then by Lemma 2.3, $x \in G \cap U \in \omega O(U)$ and $G \cap U \subset (F|_U)^-(V)$. This shows that $F|_U$ is a lower almost contra- ω -continuous multifunction. The proof of the second case is similar. ■

Theorem 3.39 *Let $\{U_i : i \in \Delta\}$ be an open cover of a space X . A multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$ is upper almost contra- ω -continuous if and only if the restriction $F|_{U_i} : U_i \rightarrow Y$ is upper almost contra- ω -continuous for each $i \in \Delta$.*

Proof. Suppose that F is an upper almost contra- ω -continuous multifunction. Let $i \in \Delta$, $x \in U_i$ and V be a regular closed set of Y containing $F|_{U_i}(x)$. Since F is an upper almost contra- ω -continuous multifunction and $F(x) = F|_{U_i}(x)$, there exists $G \in \omega O(X, x)$ such that $F(G) \subset V$. Set $U = G \cap U_i$, then $x \in U \in \omega O(U_i, x)$ and $F|_{U_i}(U) = F(U) \subset V$. Therefore, $F|_{U_i}$ is upper almost contra- ω -continuous. Conversely, let $x \in X$ and $V \in RC(Y)$ containing $F(x)$. There exists $i \in \Delta$ such that $x \in U_i$. Since $F|_{U_i}$ is upper almost contra- ω -continuous and $F(x) = F|_{U_i}(x)$, there exists $U \in \omega O(U_i, x)$ such that $F|_{U_i}(U) \subset V$. Then we have $U \in \omega O(X, x)$ and $F(U) \subset V$. Therefore, F is upper almost contra- ω -continuous. ■

Theorem 3.40 *Let $\{U_i : i \in \Delta\}$ be an open cover of a space X . A multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$ is lower almost contra- ω -continuous if and only if the restriction $F|_{U_i} : U_i \rightarrow Y$ is lower almost contra- ω -continuous for each $i \in \Delta$.*

Proof. The proof is similar to that of Theorem 3.39 and is thus omitted. ■

For a multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$, the graph multifunction $G_F : X \rightarrow Y$ is defined as follows: $G_F(x) = \{x\} \times F(x)$ for every $x \in X$.

Theorem 3.41 *If $G_F : X \rightarrow X \times Y$ is an upper almost contra- ω -continuous multifunction, then $F : (X, \tau) \rightarrow (Y, \sigma)$ is an upper almost contra- ω -continuous multifunction.*

Proof. Let $x \in X$ and $K \in RC(Y)$ with $F(x) \subset K$. Since $X \times K$ is regular closed in $X \times Y$ and $G_F(x) \subset X \times K$, then there exists $U \in \omega O(X, x)$ such that $G_F(U) \subset X \times K$. By Lemma 2.4, $U \subset G_F^+(X \times K) = F^+(K)$ and $F(U) \subset K$. Thus, F is an upper almost contra- ω -continuous multifunction. ■

Theorem 3.42 *If $G_F : X \rightarrow X \times Y$ is a lower almost contra- ω -continuous multifunction, then $F : (X, \tau) \rightarrow (Y, \sigma)$ is a lower almost contra- ω -continuous multifunction.*

Proof. Let $x \in X$ and $K \in RC(Y)$ with $x \in F^-(K)$. Since $X \times K$ is regular closed in $X \times Y$ and $G_F(x) \cap (X \times K) = (\{x\} \times F(x)) \cap (X \times K) = \{x\} \times (F(x) \cap K) \neq \emptyset$. Since G_F is lower almost contra- ω -continuous, then there exists $U \in \omega O(X, x)$ such that $U \subset G_F^-(X \times K)$. Then $U \subset F^-(K)$. Hence, F is a lower almost contra- ω -continuous. ■

Theorem 3.43 *Let $F : (X, \tau) \rightarrow (Y, \sigma)$ be an upper almost contra- ω -continuous surjective multifunction and $F(x)$ is a S -closed relative to Y for each $x \in X$. If A is a ω -compact relative to X , then $F(A)$ is a S -closed relative to Y .*

Proof. Let $\{V_i : i \in \Delta\}$ be any cover of $F(A)$ by regular closed sets of Y . For each $x \in A$, there exists a finite subset $\Delta(x)$ of Δ such that

$$F(x) \subset \cup \{V_i : i \in \Delta(x)\}.$$

Put $V(x) = \cup \{V_i : i \in \Delta(x)\}$. Then $F(x) \subset V(x)$ and there exists $U(x) \in \omega O(X, x)$ such that $F(U(x)) \subset V(x)$. Since $\{U(x) : x \in A\}$ is a cover of A by ω -open sets in X , there exists a finite number of points of A , say, $x_1, x_2, \dots, x_n$ such that $A \subset \cup \{U(x_i) : 1 = 1, 2, \dots, n\}$. Therefore, we obtain

$$F(A) \subset F\left(\bigcup_{i=1}^n U(x_i)\right) \bigcup_{i=1}^n F(U(x_i)) \bigcup_{i=1}^n V(x_i) \subset \bigcup_{i=1}^n \bigcup_{i \in \Delta(x_i)} V_i.$$

This shows that $F(A)$ is a S -closed relative to Y . ■

Theorem 3.44 *Let X and X_i be topological spaces for $i \in I$. If $F : X \rightarrow \prod_{i \in I} X_i$ is an upper (lower) almost contra- ω -continuous multifunction, then $P_i \circ F$ is an upper (lower) almost contra- ω -continuous multifunction for each $i \in I$, where $P_i : \prod_{i \in I} X_i \rightarrow X_i$ is the projection for each $i \in I$.*

Proof. Let H_i be a regular closed subset of X_j . We have

$$(P_j \circ F)^+(H_j) = F^+(P_j^+(H_j)) = F^+\left(H_j \times \prod_{i \neq j} X_i\right).$$

Since F is an upper almost contra- ω -continuous multifunction, $F^+\left(H_j \times \prod_{i \neq j} X_i\right)$ is ω -open in X . Hence, $P_i \circ F$ is an upper (lower) almost contra- ω -continuous. ■

Theorem 3.45 *Let X_i and Y_i be topological spaces and $F_i : X_i \rightarrow Y_i$ be a multifunction for each $i \in I$. If $F : \prod_{i \in I} X_i \rightarrow \prod_{i \in I} Y_i$, defined by $F(x_i) = \prod_{i \in I} F_i(x_i)$, is upper (lower) almost contra- ω -continuous multifunction, then F_i is upper (lower) almost contra- ω -continuous multifunction for each $i \in I$.*

Proof. Let $H_i \subset Y_i$ be a regular closed subset. Since F is upper almost contra- ω -continuous multifunction, $F^+(H_i \times \prod_{i \neq j} Y_j) = F_i^+(H_i) \times \prod_{i \neq j} X_j$ is an ω -open set. Thus, $F_i^+(H_i)$ is an ω -open set; hence F is upper almost contra- ω -continuous multifunction. ■

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Accepted: 28.03.2014

IRREDUCIBLE IDEALS IN RINGS

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Abstract. It is well known that the ideals of any ring form an algebraic lattice under the set inclusion ordering, in which the finitely generated ideals are precisely the compact elements. Strongly irreducible ideals of a ring were studied by W.J. Heinzer, L.J. Ratliff and D.E. Rush [3] and α -irreducible and α -strongly irreducible ideals of a ring were characterized by X. Lu and H. Qin [8]. In this paper, we extend these results for elements of a general algebraic lattice and obtain the results on ideals of rings and on submodules of an R -module as consequences of our general results. Also, we characterize algebraic lattices satisfying the ascending chain condition.

Keywords: α -irreducible, α -strongly irreducible, compact element, algebraic lattice, ascending chain condition, module over a ring.

2000 Mathematics Subject Classification: 06D10, 06B23.

1. Introduction and preliminaries

The concepts of irreducible elements and prime elements are crucial in the study of the structure theory of general algebraic systems, in particular, in that of distributive lattices. For example, the prime ideals play a vital role in the pioneering work of M.H. Stone [5] on the representation theory of distributive lattices. The prime ideals of a lattice are precisely the prime elements in the lattice of its ideals. In a general lattice, prime elements are also called strongly irreducible elements.

First, we recall certain elementary concepts and notations from Theory of Lattices [1]. A partially ordered set $(L, \leq)$ is called a lattice (complete lattice) if every two element subset (respectively, every arbitrary subset) of L has both infimum and supremum in L ; For any subset A of L , we write $\inf A$ or $\bigwedge A$ or $\bigwedge_{a \in A} a$ for the infimum (greatest lower bound) of A and $\sup A$ or $\bigvee A$ or $\bigvee_{a \in A} a$ for the supremum (least upper bound) of A . If A is a finite set $\{a_1, a_2, \dots, a_n\}$, then we write $\bigwedge_{i=1}^n a_i$ or $a_1 \wedge a_2 \wedge \dots \wedge a_n$ for the $\inf A$ and $\bigvee_{i=1}^n a_i$ or $a_1 \vee a_2 \vee \dots \vee a_n$ for the $\sup A$. If $(L, \leq)$ is a lattice, then $a \wedge b = \inf\{a, b\}$ and $a \vee b = \sup\{a, b\}$ give us two binary operations $\wedge$ and $\vee$ on L which are both associative, commutative and idempotent and satisfy the absorption laws $(a \wedge (a \vee b) = a = a \vee (a \wedge b))$. Conversely if $\wedge$ and $\vee$ are binary operations on a non empty set L which satisfy all the above properties and if the partial order $\leq$ on L is defined by $a = a \wedge b \Leftrightarrow a \leq b \Leftrightarrow a \vee b = b$, then $(L, \leq)$ is a lattice in which $a \wedge b$ and $a \vee b$ are respectively the infimum and supremum of any $\{a, b\}$.

A lattice $(L, \leq)$ is called bounded if it has smallest element 0 (that is, $0 \leq a$ for all $a \in L$) and the largest element 1 (that is, $a \leq 1$ for all $a \in L$). A complete lattice is necessarily bounded. Logically, the infimum and supremum of the empty set, if they exist, are the largest element 1 and smallest element 0 respectively. An element a of a complete lattice L is called compact if, for any $X \subseteq L$, $a \leq \sup X \implies a \leq \sup F$ for some finite subset F of X . A complete lattice L is called an algebraic lattice if every element of L is the supremum of a set of compact elements in L . An element m in a partially ordered set $(L, \leq)$ is called maximal if $m \leq a \in L$ implies $m = a$.

2. Algebraic Lattices with ACC

Let us recall that a partially ordered set $(P, \leq)$ is said to satisfy the ascending chain condition (ACC) if any increasing sequence in P terminates at a finite stage; that is, if $a_1 \leq a_2 \leq \dots \leq a_n \leq \dots$ is a sequence in P such that $a_n \leq a_{n+1}$ for all n , then there exists n_0 such that $a_{n_0} = a_{n_0+k}$ for all $k > 0$. It is well known that $(P, \leq)$ satisfies ACC if and only if every non empty subset of P has a maximal element. In the following, we characterize algebraic lattices satisfying the ascending chain condition.

Theorem 2.1. *Let $(L, \leq)$ be an algebraic lattice. Then $(L, \leq)$ satisfies the ACC if and only if every element of L is compact.*

Proof. Suppose that $(L, \leq)$ satisfies ACC. Let $a \in L$. We can assume that $a \neq 0$, since the smallest element 0 is always compact. Let X be a subset of L such that $a \leq \sup X$. Since $a \neq 0$, X is non empty. Consider the set

$$A = \{\sup F \mid F \text{ is a finite subset of } X\}.$$

A is a non empty subset of L . Since L satisfies ACC, A has a maximal element. Let m be a maximal element in A . Then, $m = \sup F$, for some finite subset F

of X . For any $x \in X$, we have $m = \sup F \leq \sup(F \cup \{x\}) \in A$ and, by the maximal element of m , $m = \sup(F \cup \{x\})$, so that $x \leq m$ for all $x \in X$. Therefore, $m = \sup X$ and $a \leq m = \sup F$ and F is a finite subset of X . Thus, a is compact.

Conversely, suppose that every element of L is compact. Let $a_1 \leq a_2 \leq \dots \leq a_n \leq \dots$ be an increasing sequence in L and $a = \sup_{n \in \mathbb{Z}^+} \{a_n\}$. Then a is compact and hence $a = a_{n_1} \vee a_{n_2} \vee \dots \vee a_{n_r}$ for some $n_1, n_2, \dots, n_r \in \mathbb{Z}^+$. If $n = \max\{n_1, n_2, \dots, n_r\}$, then $a = a_{n_1} \vee a_{n_2} \vee \dots \vee a_{n_r} = a_n$ and hence $a_k \leq a_n$ for all k . Therefore $a_n = a_{n+k}$ for all $k \in \mathbb{Z}^+$. Thus $(L, \leq)$ satisfies the ascending chain condition. ■

Recall that an ideal I of a commutative ring R with unity is called finitely generated if there exist $a_1, a_2, \dots, a_n \in R$ such that

$$I = \{r_1 a_1 + r_2 a_2 + \dots + r_n a_n \mid r_i \in R\}.$$

Also, R is said to be a Noetherian ring if the lattice of ideals of R satisfies ascending chain condition. It is well known that the ideals of any ring form an algebraic lattice, under the set inclusion order, in which the compact elements are precisely the compact elements. We record the following well known result as a consequence of the above theorem.

Corollary 2.2. *The following are equivalent to each other for any ring R .*

- (1) R is a Noetherian ring.
- (2) Every ideal of R is compact in the lattice of ideals of R .
- (3) Every ideal of R is finitely generated.
- (4) Any nonempty class of ideals has a maximal member.

On the same lines as discussed above, the submodule of an R -module (where R is any ring) form an algebraic lattice under the set inclusion ordering in which the finitely generated R -submodules are precisely the compact elements. An R -module is said to be Noetherian if the lattice of R -submodules satisfies the ascending chain condition.

Corollary 2.3. *The following are equivalent to each other for any module M over a ring R .*

- (1) M is a Noetherian R -module.
- (2) Every submodule of M is compact in the lattice of submodules of M .
- (3) Every submodule of M is finitely generated.
- (4) Any non empty class of submodules of M has a maximal member.

It is known that a complete lattice is algebraic if and only if it is isomorphic to the lattice of subuniverses of an (universal) algebra. In view of this, we have the following

Corollary 2.4. *The following are equivalent to each other for any universal algebra A .*

- (1) *The lattice of subuniverses of A satisfies ACC.*
- (2) *Every subuniverse of A is compact in the lattice of subuniverses of A .*
- (3) *Any subuniverse of A is finitely generated.*
- (4) *Any class of subuniverses of A has a maximal member.*

3. α -Irreducibility

In this section, we recall the notions of irreducible element and strongly irreducible element in a general complete lattice and discuss certain important properties of these in order to facilitate characterizations of these among ideals of a ring.

Definition 3.1. Let $(L, \wedge, \vee)$ be a bounded lattice and $1 \neq p \in L$.

- (1) p is said to be irreducible if, for any a and $b \in L$,

$$p = a \wedge b \implies p = a \text{ or } p = b.$$
- (2) p is said to be strongly irreducible or prime if, for any a and $b \in L$,

$$a \wedge b \leq p \implies a \leq p \text{ or } b \leq p.$$

Clearly, every prime element in any bounded lattice is irreducible and the converse is not true. However, if the lattice is distributive, then every irreducible element is prime. In the following, we write $|A|$ for the cardinality of a set A .

Definition 3.2. Let $(L, \leq)$ be a complete lattice, $1 \neq p \in L$ and α a cardinal number.

- (1) p is said to be α -irreducible if, for any $A \subseteq L$ with $|A| \leq \alpha$,

$$p = \inf A \implies p \in A.$$
- (2) p is said to be α -strongly irreducible or α -prime if, for any $A \subseteq L$ with $|A| \leq \alpha$,

$$\inf A \leq p \implies a \leq p \text{ for some } a \in A.$$

An element p is irreducible (prime) if and only if p is α -irreducible (respectively α -prime) for all positive integers α . Also, clearly every α -prime element is α -irreducible and the converse is not true; for, consider the set N of non negative integers together with the partial order defined by ' $a \leq b \iff b$ divides a '. Then $(N, \leq)$ is a complete lattice in which, for any subset A , infimum and supremum of A are precisely the LCM and GCD of A respectively. Here 2 is α -irreducible for all cardinal numbers α . But 2 is not α -prime, since the infimum of the set of all odd primes is 0 and $0 < 2$.

However, if the lattice L satisfies the infinite join distributivity (that is, $x \vee (\inf A) = \inf \{x \vee a \mid a \in A\}$ for all $x \in L$ and $A \subseteq L$), then every α -irreducible element in L is α -prime. Note that the complete lattice $(N, \leq)$ discussed above does not satisfy the infinite join distributivity. The following two theorems are proved in 2.2 and 2.7 of [6].

Theorem 3.3. *An element $p \neq 1$ in an algebraic lattice L is α -irreducible for every cardinal number α if and only if there exists a compact non zero element $c \in L$ such that p is a maximal element in the set $\{x \in L \mid c \not\leq x\}$.*

Theorem 3.4. *An element $p \neq 1$ in an algebraic lattice L is α -prime for every cardinal number α if and only if there exists a compact non zero element $c \in L$ such that p is the unique maximal element in the set $\{x \in L \mid c \not\leq x\}$.*

Also, it is proved in 2.5 of [6] that any element of an algebraic lattice L is the infimum of a set of elements in L which are α -irreducible for all cardinal numbers α .

4. Irreducible ideals of a ring

It is well known that ideals of a ring R form a complete lattice under the set inclusion order, in which the infimum of any class $\{I\}_{\alpha \in \Delta}$ of ideals of R is simply the set intersection, while the supremum is the ideal generated $\bigcup_{\alpha \in \Delta} I_\alpha$. Let us denote the set of all ideals of a ring R by $\mathcal{I}(R)$.

Theorem 4.1. *Let I be a proper ideal of a ring R . Then I is α -irreducible in the lattice $\mathcal{I}(R)$ for every cardinal number α if and only if there exists a finite non empty subset F of R such that I is a maximal member in the set of all ideals of R not containing F .*

Proof. First, observe that the lattice $\mathcal{I}(R)$ of ideals of R is an algebraic lattice in which the compact elements are precisely the finitely generated ideals of R . Also, for any ideal J of R and for any subset F of R , we have $F \subseteq J$ if and only if $\langle F \rangle \subseteq J$, where $\langle F \rangle$ is the ideal generated by F in R . Now, the theorem follows from Theorem 3.3. ■

Note that an ideal I of a ring R with unity is a proper ideal if and only if I does not contain the unity element. This, together with the above theorem, implies the following.

Corollary 4.2. *Let R be a ring with unity. Then, any maximal ideal of R is α -irreducible in $\mathcal{I}(R)$ for all cardinal numbers α .*

A proof analogous to that of Theorem 4.1 together with Theorem 3.4 yield the following.

Theorem 4.3. *A proper ideal I of a ring R is α -prime in $\mathcal{I}(R)$ for all cardinal numbers α if and only if there exists a non empty finite subset F of R such that I is the unique maximal member in the set of all ideals of R not containing F .*

Recall that a ring with unity is called a local ring if it has a unique maximal ideal. The following is an immediate consequence of the above theorem.

Corollary 4.4. *In a local ring, the maximal ideal is α -prime for all cardinal numbers α .*

Example 4.5. Consider the ring $\mathbb{Z}$ of integers. It is well known that any ideal of $\mathbb{Z}$ is of the form $n\mathbb{Z}$ for some non negative integer n . $\mathbb{Z}$ is a Noetherian ring and every ideal of $\mathbb{Z}$ is compact in the lattice $\mathcal{J}(\mathbb{Z})$ of ideals of $\mathbb{Z}$. Since

$$\{0\} = \cap \{p\mathbb{Z} \mid p \text{ is a prime number}\},$$

$\{0\}$ is not α -irreducible for any infinite cardinal α , However, $\{0\}$ is irreducible, since

$$\{0\} = n\mathbb{Z} \cap m\mathbb{Z} \implies n = 0 \text{ or } m = 0.$$

It can be proved that $\mathcal{J}(\mathbb{Z})$ is a distributive lattice and that a non zero ideal $n\mathbb{Z}$ is irreducible (and hence prime) if and only if $n = p^r$ for some prime number p and a positive integer r . Also, any ideal of $\mathbb{Z}$ is not α -prime, for an infinite cardinal α ; for, let $n\mathbb{Z}$ be a non zero proper ideal of $\mathbb{Z}$. Then $n > 1$. Consider the set

$$X = \{p\mathbb{Z} \mid p \text{ is a prime number not dividing } n\}.$$

Then $\inf X = \{0\} \subseteq n\mathbb{Z}$ and $p\mathbb{Z} \not\subseteq n\mathbb{Z}$ for all $p\mathbb{Z} \in X$. Thus $n\mathbb{Z}$ is not α -prime. On the other hand, $n\mathbb{Z}$ is α -irreducible for all cardinals α if and only if $n = p^r$ for some prime number p and a positive integer r .

Theorems 4.1 and 4.3 can be extended to submodules of an R -module M , since a submodule of M is compact in the lattice of submodules of M if and only if it is finitely generated over R .

Acknowledgement. The authors thank Professor U.M. Swamy for his help in preparing this paper.

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Accepted: 6.04.2014

INCLUSION RESULTS ON SUBCLASSES OF STARLIKE AND CONVEX FUNCTIONS ASSOCIATED WITH STRUVE FUNCTIONS

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Abstract. The present investigation our goal is to determine necessary and sufficient condition for Struve functions belonging to the classes $\mathcal{J}_\lambda^*(\alpha, \beta)$ and $\mathcal{G}_\lambda^*(\alpha, \beta)$.

Keywords: Starlike functions, Convex functions, Uniformly Starlike functions, Uniformly Convex functions, Hadamard product, Bessel function, Struve function.

2000 Mathematics Subject Classification: 30C45.

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions in the unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$$

of the form

$$(1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad z \in \mathbb{U}.$$

As usual, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are normalized by $f(0) = 0 = f'(0) - 1$ and also univalent in $\mathbb{U}$. Denote by $\mathcal{T}$ the subclass of $\mathcal{A}$ consisting of functions whose non-zero coefficients from second on, is given by

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$$(2) \quad f(z) = z - \sum_{n=2}^{\infty} a_n z^n,$$

Also, for functions $f \in \mathcal{A}$ given by (1) and $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{U}.$$

A function $f \in \mathcal{A}$ is said to be starlike of order α ($0 \leq \alpha < 1$), if and only if $\Re \left(\frac{zf'(z)}{f(z)} \right) > \alpha$ ($z \in \mathbb{U}$). This function class is denoted by $\mathcal{S}^*(\alpha)$. We also write $\mathcal{S}^*(0) \equiv \mathcal{S}^*$, where $\mathcal{S}^*$ denotes the class of functions $f \in \mathcal{A}$ that $f(\mathbb{U})$ is starlike with respect to the origin. A function $f \in \mathcal{A}$ is said to be convex of order α ($0 \leq \alpha < 1$) if and only if $\Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha$ ($z \in \mathbb{U}$). This class is denoted by $\mathcal{K}(\alpha)$. Further, $\mathcal{K}(0) = \mathcal{K}$, the well-known standard class of convex functions. It is an established fact that $f \in \mathcal{K}(\alpha) \iff zf' \in \mathcal{S}^*(\alpha)$.

It is well known that the special functions (series) play an important role in geometric function theory, especially in the solution by de Branges [6] of the famous Bieberbach conjecture. There is an extensive literature dealing with geometric properties of different types of special functions, especially for the generalized, Gaussian hypergeometric functions [5], [7], [12] and the Bessel functions [1], [2], [3], [8].

We recall here the Struve function of order p (see [10], [15]), denoted by $\mathcal{H}_p$ is given by

$$(3) \quad \mathcal{H}_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n + \frac{3}{2}) \Gamma(p + n + \frac{3}{2})} \left(\frac{z}{2} \right)^{2n+p+1}, \quad \forall z \in \mathbb{C}$$

which is the particular solution of the second order non-homogeneous differential equation

$$(4) \quad z^2 \omega''(z) + z \omega'(z) + (z^2 - p^2) \omega(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi} \Gamma(p + \frac{1}{2})}$$

where p is unrestricted real(or complex) number. The solution of the non-homogeneous differential equation

$$(5) \quad z^2 \omega''(z) + z \omega'(z) - (z^2 + p^2) \omega(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi} \Gamma(p + \frac{1}{2})}$$

is called the modified Struve function of order p and is defined by the formula

$$\mathcal{L}_p(z) = -ie^{-ip\pi/2}\mathcal{H}_p(iz) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n + \frac{3}{2}) \Gamma(p + n + \frac{3}{2})} \left(\frac{z}{2}\right)^{2n+p+1}, \forall z \in \mathbb{C}.$$

Let the second order non-homogeneous linear differential equation [15] (also see [10] and references cited therein),

$$(6) \quad z^2\omega''(z) + bz\omega'(z) + [cz^2 - p^2 + (1-b)p]\omega(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p + \frac{b}{2})}$$

where $b, p, c \in \mathbb{C}$ which is natural generalization of Struve equation. It is of interest to note that when $b = c = 1$, then we get the Struve function (3) and for $c = -1, b = 1$ the modified Struve function (5). This permit us to study Struve and modified Struve functions. Now, denote by $w_{p,b,c}(z)$ the generalized Struve function of order p given by

$$w_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n(c)^n}{\Gamma(n + \frac{3}{2}) \Gamma(p + n + \frac{b+2}{2})} \left(\frac{z}{2}\right)^{2n+p+1}, \forall z \in \mathbb{C}$$

which is the particular solution of the differential equation (6). Although the series defined above is convergent everywhere, the function $\omega_{p,b,c}$ is generally not univalent in $\mathbb{U}$. Now, consider the function $u_{p,b,c}$ defined by the transformation

$$u_{p,b,c}(z) = 2^p \sqrt{\pi} \Gamma\left(p + \frac{b+2}{2}\right) z^{-\frac{p-1}{2}} \omega_{p,b,c}(\sqrt{z}), \quad \sqrt{1} = 1.$$

By using well known Pochhammer symbol (or the shifted factorial) defined, in terms of the familiar Gamma function, by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & (n=0), \\ a(a+1)(a+2)\cdots(a+n-1) & (n \in \mathbb{N} = \{1, 2, 3, \dots\}) \end{cases}$$

we can express $u_{p,b,c}(z)$ as

$$\begin{aligned} u_{p,b,c}(z) &= \sum_{n=0}^{\infty} \frac{(-c/4)^n}{(m)_n (3/2)_n} z^n \\ &= b_0 + b_1 z + b_2 z^2 + \dots + b_n z^n + \dots, \end{aligned}$$

where $m = (p + \frac{b+2}{2}) \neq 0, -1, -2, \dots$. This function is analytic on $\mathbb{C}$ and satisfies the second-order inhomogeneous linear differential equation

$$4z^2u''(z) + 2(2p + b + 3)zu'(z) + (cz + 2p + b)u(z) = 2p + b.$$

For convenience, throughout in the sequel, we use the following notations

$$\begin{aligned} w_{p,b,c}(z) &= w_p(z), \\ u_{p,b,c}(z) &= u_p(z), \\ m &= p + \frac{b+2}{2} \end{aligned}$$

and for if $c < 0, m > 0 (m \neq 0, -1, -2, \dots)$ let,

$$zu_p(z) = z + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1} (3/2)_{n-1}} z^n = z + \sum_{n=2}^{\infty} b_{n-1} z^n$$

and

$$(7) \quad \Psi(z) = z(2 - u_p(z)) = z - \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1} (3/2)_{n-1}} z^n$$

In this paper, we introduce two new subclasses of $\mathcal{S}$ namely $\mathcal{J}_\lambda(\alpha, \beta)$ and $\mathcal{G}_\lambda(\alpha, \beta)$ to discuss some inclusion properties.

For some α ($0 \leq \alpha < 1$), λ ($0 \leq \lambda \leq 1$), $\beta > 0$ and functions of the form (1), we let $\mathcal{J}_\lambda(\alpha, \beta)$ be the subclass of $\mathcal{S}$ of satisfying the analytic criteria

$$\Re \left(\frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} - \alpha \right) > \beta \left| \frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} - 1 \right|$$

and $\mathcal{G}_\lambda(\alpha, \beta)$ the subclass of $\mathcal{S}$ of satisfying the analytic criteria

$$\Re \left(\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} - \alpha \right) > \beta \left| \frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} - 1 \right|.$$

Also denote $\mathcal{J}_\lambda^*(\alpha, \beta) = \mathcal{J}_\lambda(\alpha, \beta) \cap \mathcal{T}$ and $\mathcal{G}_\lambda^*(\alpha, \beta) = \mathcal{G}_\lambda(\alpha, \beta) \cap \mathcal{T}$, the subclasses of $\mathcal{T}$.

Example 1 [4] For some α ($0 \leq \alpha < 1$), $\beta > 0$ and choosing $\lambda = 1$ and functions of the form (2), we let $\mathcal{TS}_p(\alpha, \beta)$ be the subclass of $\mathcal{S}$ of satisfying the analytic criteria

$$\Re \left(\frac{zf'(z)}{f(z)} - \alpha \right) > \beta \left| \frac{zf'(z)}{f(z)} - 1 \right|$$

and $\mathcal{UCT}(\alpha, \beta)$ the subclass of $\mathcal{S}$ of satisfying the analytic criteria

$$\Re \left(1 + \frac{zf''(z)}{f'(z)} - \alpha \right) > \beta \left| \frac{zf''(z)}{f'(z)} \right|.$$

Note that $\mathcal{TS}_p(\alpha, 0) \equiv \mathcal{T}^*(\alpha, 0)$ and $\mathcal{UCT}(\alpha, 0) \equiv C(\alpha)$ [11], further $\mathcal{TS}_p(0, \beta) \equiv \mathcal{TS}_p(\beta)$ and $\mathcal{UCT}(0, \beta) \equiv UCT(\beta)$ [13]

Example 2 For some α ($0 \leq \alpha < 1$), $\beta > 0$ and choosing $\lambda = 0$ and functions of the form (2), we let

(i) $\mathcal{USD}(\alpha, \beta)$ the subclass of $\mathcal{S}$ of satisfying the analytic criteria

$$\Re(f'(z) - \alpha) > \beta |f'(z) - 1|$$

and

(ii) $\mathcal{UCD}(\alpha, \beta)$ the subclass of $\mathcal{S}$ of satisfying the analytic criteria

$$\Re((zf'(z))' - \alpha) > \beta |(zf'(z))' - 1|.$$

Suitably specializing the parameters we get the various subclasses studied in [9] and see the references cited therein. (also see [4], [14])

Recently, Yagmur and Orhan [15] (see [10]) have determined various sufficient conditions for the parameters p, b and c such that the functions $u_{p,b,c}(z)$ or $z \rightarrow zu_{p,b,c}(z)$ to be univalent, starlike, convex and close to convex in the open unit disk. Motivated by results on connections between various subclasses of analytic univalent functions by using hypergeometric functions (see [5], [7], [12]) and by work of Baricz [1], [2], [3]. In this paper, we obtain sufficient condition for function $h(z)$, given by

$$\begin{aligned} h_\mu(z) &= (1 - \mu)zu_p(z) + \mu zu_p'(z) \\ (8) \qquad &= z + \sum_{n=2}^{\infty} (1 + n\mu - \mu) \frac{(-c/4)^{n-1}}{(m)_{n-1} (3/2)_{n-1}} z^n. \end{aligned}$$

where $0 \leq \mu \leq 1$ in the present investigation our goal is to determine sufficient condition for function $h_\mu(z)$ belonging to the classes $\mathcal{J}_\lambda(\alpha, \beta)$ and $\mathcal{G}_\lambda(\alpha, \beta)$.

2. Main results and their consequences

We recall the following necessary and sufficient conditions for the functions $f \in \mathcal{J}_\lambda^*(\alpha, \beta)$, $f \in \mathcal{G}_\lambda^*(\alpha, \beta)$ and the subclasses stated in the Examples 1 and 2 which are relevant for our study.

Lemma 1 *A function $f(z)$ of the form (1) is in*

(i) *the class $\mathcal{J}_\lambda(\alpha, \beta)$ if*

$$(9) \qquad \sum_{n=2}^{\infty} [n(1 + \beta) - \lambda(\alpha + \beta)] |a_n| \leq 1 - \alpha.$$

(ii) *the class $\mathcal{G}_\lambda(\alpha, \beta)$ if*

$$(10) \qquad \sum_{n=2}^{\infty} n[n(1 + \beta) - \lambda(\alpha + \beta)] |a_n| \leq 1 - \alpha.$$

The above sufficient conditions are also necessary for functions f of the form (2).

Lemma 2 *A function $f(z)$ of the form (2) is in*

(i) *the class $\mathcal{TS}_p(\alpha, \beta)$ if and only if*

$$\sum_{n=2}^{\infty} [n(1 + \beta) - (\alpha + \beta)] |a_n| \leq 1 - \alpha.$$

(ii) *the class $\mathcal{UCT}(\alpha, \beta)$ if and only if*

$$\sum_{n=2}^{\infty} n[n(1 + \beta) - (\alpha + \beta)] |a_n| \leq 1 - \alpha.$$

Lemma 3 A function $f(z)$ of the form (2) is in

(i) the class $\mathcal{USD}(\alpha, \beta)$ if and only if

$$\sum_{n=2}^{\infty} n(1+\beta)|a_n| \leq 1-\alpha.$$

(ii) the class $\mathcal{UCD}(\alpha, \beta)$ if and only if

$$\sum_{n=2}^{\infty} n^2(1+\beta)|a_n| \leq 1-\alpha.$$

Theorem 1 If $c < 0, m > 0 (m \neq 0, -1, -2, \dots)$ then $h_\mu(z) \in \mathcal{J}_\lambda(\alpha, \beta)$ if

$$\begin{aligned} & \mu(1+\beta)u_p''(1) + [(2\mu+1)(1+\beta) - \mu\lambda(\alpha+\beta)]u_p'(1) \\ & + [(1+\beta) - \lambda(\alpha+\beta)]u_p(1) \\ & \leq 2 - \alpha(1+\lambda) + \beta(1-\lambda). \end{aligned} \quad (11)$$

Proof. Since $zu_p(z) = z + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1} (3/2)_{n-1}} z^n$,

$$u_p(1) - 1 = \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1} (3/2)_{n-1}}, \quad (12)$$

and differentiating $zu_p(z)$ with respect to z and taking $z = 1$ we have

$$\begin{aligned} zu_p'(z) + u_p(z) &= 1 + \sum_{n=2}^{\infty} n \frac{(-c/4)^{n-1}}{(m)_{n-1} (3/2)_{n-1}} z^{n-1} \\ u_p'(1) + u_p(1) - 1 &= \sum_{n=2}^{\infty} n \frac{(-c/4)^{n-1}}{(m)_{n-1} (3/2)_{n-1}}. \end{aligned}$$

Further, differentiating $zu_p'(z) + u_p(z)$ with respect to z and taking $z = 1$, we get

$$\begin{aligned} zu_p''(z) + 2u_p'(z) &= \sum_{n=2}^{\infty} n(n-1) \frac{(-c/4)^{n-1}}{(m)_{n-1} (3/2)_{n-1}} z^{n-2} \\ u_p''(1) + 2u_p'(1) &= \sum_{n=2}^{\infty} n(n-1) \frac{(-c/4)^{n-1}}{(m)_{n-1} (3/2)_{n-1}}. \end{aligned} \quad (13)$$

Since $h_\mu(z) \in \mathcal{J}_\lambda(\alpha, \beta)$, by virtue of Lemma 1 and (9) it suffices to show that

$$\sum_{n=2}^{\infty} (1+n\mu-\mu)[n(1+\beta) - \lambda(\alpha+\beta)] \left(\frac{(-c/4)^{n-1}}{(m)_{n-1} (3/2)_{n-1}} \right) \leq 1-\alpha. \quad (14)$$

Now, let

$$\begin{aligned}
 S(n, \lambda, \beta, \alpha) &= \sum_{n=2}^{\infty} (1 + n\mu - \mu)[n(1 + \beta) - \lambda(\alpha + \beta)] \left(\frac{(-c/4)^{n-1}}{(m)_{n-1} (3/2)_{n-1}} \right) \\
 S(n, \lambda, \beta, \alpha) &= \mu(1 + \beta) \sum_{n=2}^{\infty} n^2 \left(\frac{(-c/4)^{n-1}}{(m)_{n-1} (3/2)_{n-1}} \right) \\
 &\quad + [(1 + \beta)(1 - \mu) - \lambda\mu(\alpha + \beta)] \sum_{n=2}^{\infty} n \left(\frac{(-c/4)^{n-1}}{(m)_{n-1} (3/2)_{n-1}} \right) \\
 &\quad - \lambda(\alpha + \beta)(1 - \mu) \sum_{n=2}^{\infty} \left(\frac{(-c/4)^{n-1}}{(m)_{n-1} (3/2)_{n-1}} \right).
 \end{aligned}$$

Writing $n^2 = n(n-1) + n$, we get

$$\begin{aligned}
 S(n, \lambda, \beta, \alpha) &= \mu(1 + \beta) \sum_{n=2}^{\infty} n(n-1) \left(\frac{(-c/4)^{n-1}}{(m)_{n-1} (3/2)_{n-1}} \right) \\
 &\quad + [(1 + \beta) - \lambda\mu(\alpha + \beta)] \sum_{n=2}^{\infty} n \left(\frac{(-c/4)^{n-1}}{(m)_{n-1} (3/2)_{n-1}} \right) \\
 &\quad - \lambda(\alpha + \beta)(1 - \mu) \sum_{n=2}^{\infty} \left(\frac{(-c/4)^{n-1}}{(m)_{n-1} (3/2)_{n-1}} \right)
 \end{aligned}$$

From (12), (13), 13 and taking $z = 1$, we get

$$\begin{aligned}
 S(n, \lambda, \beta, \alpha) &\leq \mu(1 + \beta)u_p''(1) + [(1 + \beta) - \lambda\mu(\alpha + \beta)](u_p'(1) + u_p(1) - 1) \\
 &\quad - \lambda(\alpha + \beta)(1 - \mu)(u_p(1) - 1) \\
 &= \mu(1 + \beta)u_p''(1) + [(2\mu + 1)(1 + \beta) - \lambda\mu(\alpha + \beta)]u_p'(1) \\
 &\quad + [(1 + \beta) - \lambda(\alpha + \beta)](u_p(1) - 1)
 \end{aligned}$$

But this expression is bounded above by $1 - \alpha$ if (11) holds.

Thus, the proof is complete. ■

Theorem 2 If $c < 0, m > 0 (m \neq 0, -1, -2, \dots)$ then $zu_p(z) \in \mathcal{J}_\lambda(\alpha, \beta)$ if

$$(15) \quad (1 + \beta)u_p'(1) + [(1 + \beta) - \lambda(\alpha + \beta)]u_p(1) \leq 2 - \alpha(1 + \lambda) + \beta(1 - \lambda).$$

Proof. By virtue of Lemma 1 of (9), it suffices to show that

$$\sum_{n=2}^{\infty} [n(1 + \beta) - \lambda(\alpha + \beta)] \left(\frac{(-c/4)^{n-1}}{(m)_{n-1} (3/2)_{n-1}} \right) \leq 1 - \alpha.$$

Since $h_0(z) = zu_p(z)$, hence by taking $\mu = 0$ in (14) we get the above inequality. Hence by taking $\mu = 0$ in the Theorem 1, we get the desired result given in 15. ■

Theorem 3 *If $c < 0, m > 0 (m \neq 0, -1, -2, \dots)$ then $zu_p(z) \in \mathcal{G}_\lambda(\alpha, \beta)$ if*

$$(16) \quad \begin{aligned} & (1 + \beta)u_p''(1) + [3(1 + \beta) - \lambda(\alpha + \beta)]u_p'(1) \\ & + [(1 + \beta) - \lambda(\alpha + \beta)]u_p(1) \\ & \leq 2 - \beta(\lambda - 1) - \alpha(\lambda + 1). \end{aligned}$$

Proof. By virtue of Lemma 1 of (10), it suffices to show that

$$\sum_{n=2}^{\infty} n[n(1 + \beta) - \lambda(\alpha + \beta)] \left(\frac{(-c/4)^{n-1}}{(m)_{n-1} (3/2)_{n-1}} \right) \leq 1 - \alpha.$$

By definition $zu_p(z) \in \mathcal{G}_\lambda(\alpha, \beta) \Leftrightarrow zu_p'(z) \in \mathcal{J}_\lambda(\alpha, \beta)$. That is by taking $\mu = 1$ we have $h_1(z) = zu_p'(z) \in \mathcal{J}_\lambda(\alpha, \beta)$, hence by taking $\mu = 1$ in the Theorem 1, we get the desired result given in 16. ■

Remark 1 The above conditions (11) and (16) are also necessary for functions $\Psi(z)$ given by (7) and of the form

$$\begin{aligned} h_\mu^*(z) &= (1 - \mu)\Psi(z) + \mu\Psi'(z) \\ &= z - \sum_{n=2}^{\infty} (1 + n\mu - \mu) \frac{(-c/4)^{n-1}}{(m)_{n-1} (3/2)_{n-1}} z^n \end{aligned}$$

is in the classes $\mathcal{J}_\lambda^*(\alpha, \beta)$ and $\mathcal{G}_\lambda^*(\alpha, \beta)$ respectively.

Further, by taking $\lambda = 0$ (or) $\lambda = 1$ in Theorems 2 and 3, we state the following corollaries without proof.

Corollary 1 *If $c < 0, m > 0 (m \neq 0, -1, -2, \dots)$ then $z(2 - u_p(z))$,*

(i) *is in $\mathcal{TS}_p(\alpha, \beta)$ if and only if*

$$(1 + \beta)u_p'(1) + (1 - \alpha)u_p(1) \leq 2(1 - \alpha).$$

(ii) *is in $\mathcal{UCT}(\alpha, \beta)$ if and only if*

$$(1 + \beta)u_p''(1) + (3 - 2\beta - \alpha)u_p'(1) + (1 - \alpha)u_p(1) \leq 2(1 - \alpha).$$

Corollary 2 *If $c < 0, m > 0 (m \neq 0, -1, -2, \dots)$ then $z(2 - u_p(z))$,*

(i) *is in $\mathcal{USD}(\alpha, \beta)$ if and only if*

$$(1 + \beta)[u_p'(1) + u_p(1)] \leq 2 - \alpha + \beta.$$

(ii) *is in $\mathcal{UCD}(\alpha, \beta)$ if and only if*

$$(1 + \beta)[u_p''(1) + 3u_p'(1) + u_p(1)] \leq 2 - \alpha + \beta.$$

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Accepted: 6.04.2014

THE MODIFIED (w/g) -EXPANSION METHOD AND ITS APPLICATIONS FOR SOLVING THE MODIFIED GENERALIZED VAKHNENKO EQUATION

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Abstract. The modified (w/g) -expansion method for finding traveling wave solutions of nonlinear evolution equations is presented in this paper, which can be thought of as the generalization of the well-known (G'/G) -expansion method given recently by Wang et al. When the w and g are taken special choices, some familiar expansion methods can be obtained. Based on these interesting results, we further give two new forms of expansions via the modified (g'/g^2) -expansion method and the modified (g') expansion method. In order to well illustrate the effectiveness of these two modified expansion methods, they are applied to a modified generalized Vakhnenko equation.

Keywords: the modified (w/g) -expansion method; modified (g'/g^2) -expansion method; modified (g') -expansion method; nonlinear evolution equations; traveling wave solutions; a modified generalized Vakhnenko equation.

AMS Subject Classifications: 35K99, 35P05, 35P99.

1. Introduction

The investigation of the exact traveling wave solutions of nonlinear evolution equations plays an important role in the study of soliton theory. In recent years, a variety of powerful methods have been applied for constructing exact traveling and solitary wave solutions of nonlinear evolution equations, such as the tanh-function expansion method [1]–[3], Jacobi elliptic function method [4], [5], Exp-function

method [6], [7], the F-function expansion method [8], the inverse scattering method [9], Hirota bilinear transformation [10], the Backlund transform method [11], the (G'/G) -expansion method [12], [13], the sine-cosine method [6], [14], the modified simple equation method [15]–[18], the multiple exp-function method [19], [20], the transformed rational function method [21]–[23], the local fractional variation iteration method [24], the local fractional series expansion method [25], the (w/g) -expansion method [26] and so on.

The objective of this paper is to introduce the modified (w/g) -expansion method for finding exact solutions for a modified generalized Vakhnenko equation in mathematical physics. The proposed method is based on the assumption that these exact solutions can be expressed by a polynomial in $(w/g)^i$, $i = 0, 1, \dots, m$ and that w, g satisfy the following auxiliary equation

$$(1.1) \quad \left(\frac{w}{g}\right)' = a + b\left(\frac{w}{g}\right) + c\left(\frac{w}{g}\right)^2,$$

namely

$$(1.2) \quad w'g - wg' = ag^2 + bwg + cw^2,$$

where a, b, c are arbitrary constants, while $' = d/d\xi$ and $\xi = k(x - Vt)$, k and V are constants. The degree m of this polynomial can be obtained by considering the homogeneous balance between the highest-order derivatives and the nonlinear terms appearing in the given nonlinear evolution equations. The coefficients of this polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the proposed method. The rest of this paper is organized as follows. In Section 2, we describe a new modified (w/g) -expansion method. In Section 3, we apply this method to solve the modified generalized Vakhnenko equation. In Section 4, Physical explanations of some obtained solutions are obtained. In Section 5, some conclusions are given.

2. Description of a new modified (w/g) - expansion method

For a given nonlinear evolution equation

$$(2.1) \quad P(u, u_t, u_x, u_{xt}, u_{tt}, u_{xx}, \dots) = 0,$$

where P is a polynomial in the function $u(x, t)$ and its partial derivatives in which the nonlinear terms are involving. We use the wave transformation

$$(2.2) \quad u(x, t) = u(\xi), \quad \xi = k(x - Vt),$$

where k, V are constants to reduce equation (2.1) into the following ODE:

$$(2.3) \quad Q(u^{(r)}, u^{(r+1)}, \dots) = 0,$$

where $u^{(r)} = \frac{d^r}{d\xi^r}$, $r \geq 0$, and r is the least order derivatives in the given equation. Setting $u^{(r)} = v(\xi)$, where $v(\xi)$ is a new function of ξ , then equation (2.3) reduces to the following ODE:

$$(2.4) \quad F(v, v', v'', \dots) = 0,$$

where $' = d/d\xi$. We further introduce the formal solution of equation (2.4) in the following ansatz:

$$(2.5) \quad u^{(r)}(\xi) = v(\xi) = \sum_{i=0}^m \alpha_i \left(\frac{w}{g} \right)^i,$$

where w and g satisfy equation (1.2) and α_i ($i = 0, 1, \dots, m$) are constants to be determined later, while $r \geq 0$. To determine $v(\xi)$ explicitly, we take the following four steps:

Step 1. Determine the positive integer m in equation (2.5) by balancing the highest-order derivatives and the nonlinear terms in equation (2.4).

Step 2. Substitute (2.5) into (2.4) and collect all terms with the same powers of $(w/g)^i$, ($i = 0, 1, \dots, m$), together, thus the left-hand side of equation (2.4) is converted into a polynomial in $(w/g)^i$. Then set each coefficient of this polynomial to zero, to derive a set of algebraic equations for α_i, k, V .

Step 3. Solve these algebraic equations by the use of Maple or Mathematica to find the values of α_i, k, V .

Step 4. Use the results obtained in the above steps to derive a series of fundamental solutions $v(\xi)$ of equation (2.4) depending on (w/g) . Then we can obtain the exact solutions of equation (2.1) by integrating each of the obtained fundamental solutions $v(\xi)$ with respect to ξ and r times as follows:

$$(2.6) \quad u(\xi) = \int^{\xi} \int^{\xi_r} \dots \int^{\xi_2} v(\xi_1) d\xi_1 \dots d\xi_{r-1} d\xi_r + \sum_{j=1}^r d_j \xi^{r-j},$$

where d_j ($j = 1, \dots, r$) are arbitrary constants.

Remark 1. Let us now examine equation (1.2) carefully as follows:

(1) If we choose $w = g'$, $a = -\mu$, $b = -\lambda$, $c = -1$ and $r = 0$, then $u(\xi)$ can be expressed as:

$$(2.7) \quad u(\xi) = \sum_{i=0}^m \alpha_i \left(\frac{g'}{g} \right)^i,$$

where $g(\xi)$ satisfies the linear ODE:

$$(2.8) \quad g'' + \lambda g' + \mu g = 0.$$

This is just the well-known (G'/G) -expansion method proposed by Wang et al. [12].

(2) If $w = \tanh \xi$, $g = 1$, $a = 1$, $b = 0$, $c = -1$ and $r = 0$, then

$$(2.9) \quad u(\xi) = \sum_{i=0}^m \alpha_i \tanh^i(\xi),$$

which is the well-known tanh-function expansion method, see [1]–[3], [6].

(3) If $w = g'/g$, $b = 0$ and $r = 0$, then we have the new expansion

$$(2.10) \quad u(\xi) = \sum_{i=0}^m \alpha_i \left(\frac{g'}{g^2} \right)^i,$$

where $g(\xi)$ satisfies the nonlinear ODE:

$$(2.11) \quad g''g^2 - 2g(g')^2 = ag^4 + c(g')^2,$$

which is called (g'/g^2) -expansion method and proposed in [26].

(4) If $w = gg'$ and $r = 0$, then we have the new expansion

$$(2.12) \quad u(\xi) = \sum_{i=0}^m \alpha_i (g')^i,$$

where $g(\xi)$ satisfies the nonlinear ODE:

$$(2.13) \quad g'' = a + bg' + c(g')^2,$$

which is called g' -expansion method and proposed in [26]. Li et al. [26] have applied the two expansions (2.10) and (2.12) for finding exact solutions to the Vakhnenko equation

$$u_{tx} + u_x^2 + uu_{xx} + u = 0.$$

Remark 2. where $r \geq 1$, the ansatz (2.5) is new and is not reported in [26]. So, in the next section, we apply the two expansions (g'/g^2) -expansion method and the (g') -expansion method using (2.5) with $r \geq 1$ to find new exact solutions of a modified generalized Vakhnenko equation.

3. New solutions of a modified generalized Vakhnenko equation

Consider a modified generalized Vakhnenko equation (mGVE) [27]–[29]:

$$(3.1) \quad \frac{\partial}{\partial x} \left(\wp^2 u + \frac{1}{2} p u^2 + \beta u \right) + q \wp u = 0, \quad \wp = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x},$$

where p, q, β are arbitrary non-zero constants. equation (3.1) can be traced to the well-known Vakhnenko equation (VE) which was initially presented to model high-frequent wave motion in a relaxing medium [29]. Recently, equation (3.1) has been discussed using the (G'/G) -expansion method [27] and using the auxiliary equation method [28]. To calculate the exact solutions for equation (3.1), a sensible step is to transform variables. We introduce two new independent variables X and T defined by:

$$(3.2) \quad x = T + \int_{-\infty}^X U(X', T) dX' + x_0, \quad t = X,$$

where $u(x, t) = U(X, T)$ and x_0 is a constant. We introduce a new function W defined by

$$(3.3) \quad W(X, T) = \int_{-\infty}^X U(X', T) dX'.$$

Then

$$(3.4) \quad W_X(X, T) = U(X, T), \quad W_T(X, T) = \int_{-\infty}^X U_T(X', T) dX'.$$

It is easy to see that

$$(3.5) \quad \frac{\partial}{\partial T} = \frac{\partial}{\partial x} \frac{\partial x}{\partial T} + \frac{\partial}{\partial t} \frac{\partial t}{\partial T}, \quad \frac{\partial}{\partial X} = \frac{\partial}{\partial x} \frac{\partial x}{\partial X} + \frac{\partial}{\partial t} \frac{\partial t}{\partial X}.$$

From (3.2) and (3.5) we have

$$(3.6) \quad \frac{\partial}{\partial T} = (1 + W_T) \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial X} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x},$$

and hence $\wp u = U_x$, $\wp^2 u = U_{XX}$. Now, equation (3.1) reduces to

$$(3.7) \quad W_{XXXT} + pW_X W_{XT} + q(1 + W_T) W_{XX} + \beta W_{XT} = 0.$$

Assume that $W(X, T) = W(\xi)$, where ξ is given by (2.2), then equation (3.7) reduces to the equation

$$(3.8) \quad k^2 V W''' + \frac{1}{2}(p + q)kV(W')^2 + (\beta V - q)W' = 0,$$

with zero constants of integration. Setting $r = 1$ and $W' = v$, we have $W(\xi) = \int v(\xi) d\xi + d_1$, where $v(\xi)$ satisfies the following ODE:

$$(3.9) \quad k^2 V v'' + \frac{1}{2}(p + q)kV v^2 + (\beta V - q)v = 0.$$

Recently, equation (3.9) has been solved in [27] using the (G'/G) -expansion method via the expansion (2.7). Let us now solve equation (3.9) using the expansion (2.10) and (2.12).

3.1. The new modified (g'/g^2) expansion method

Balancing v'' with v^2 in equation (3.9) we get $m = 2$. Consequently, the expansion (2.10) reduces to

$$(3.10) \quad v(\xi) = \alpha_2 \left(\frac{g'}{g^2} \right)^2 + \alpha_1 \left(\frac{g'}{g^2} \right) + \alpha_0, \quad \alpha_2 \neq 0.$$

where $g(\xi)$ satisfies equation (2.11). It is easy to see that

$$(3.11) \quad v'(\xi) = 2\alpha_2 c \left(\frac{g'}{g^2} \right)^3 + \alpha_1 c \left(\frac{g'}{g^2} \right)^2 + 2\alpha_2 a \left(\frac{g'}{g^2} \right) + \alpha_1 a,$$

$$(3.12) \quad v''(\xi) = 6\alpha_2 c^2 \left(\frac{g'}{g^2} \right)^4 + 2\alpha_1 c^2 \left(\frac{g'}{g^2} \right)^3 + 8\alpha_2 ac \left(\frac{g'}{g^2} \right)^2 + 2\alpha_1 ac \left(\frac{g'}{g^2} \right) + 2\alpha_2 a^2.$$

Substituting (3.10)–(3.12) into (3.9) and collecting all terms with the same powers of $(g'/g^2)^i$, $i = 0, 1, 2, 3, 4$ together, the left hand side of equation (3.9) is converted into a polynomial in $(g'/g^2)^i$. Setting each coefficient of this polynomial to zero, we get the following algebraic equations:

$$(3.13) \quad 0 : 2\alpha_2 a^2 k^2 V + \frac{1}{2}(p+q)kV\alpha_0^2 + (V\beta - q)\alpha_0 = 0,$$

$$(3.14) \quad 1 : 2\alpha_1 ack^2 V + \alpha_0 \alpha_1 (p+q)kV + (V\beta - q)\alpha_1 = 0,$$

$$(3.15) \quad 2 : 8\alpha_2 ack^2 V + \frac{1}{2}(p+q)kV\alpha_1^2 + \alpha_0 \alpha_2 (p+q)kV + (V\beta - q)\alpha_2 = 0,$$

$$(3.16) \quad 3 : 2\alpha_1 c^2 k^2 V + \alpha_1 \alpha_2 (p+q)kV = 0,$$

$$(3.17) \quad 4 : 6\alpha_2 c^2 k^2 V + \frac{1}{2}(p+q)kV\alpha_2^2 = 0.$$

Solving the above algebraic equations, we have the following results:

Case 1.

$$(3.18) \quad \alpha_2 = -\frac{12c^2 k}{p+q}, \quad \alpha_1 = 0, \quad \alpha_0 = -\frac{12ack}{p+q}, \quad V = \frac{q}{\beta - 4ack^2}, \quad \beta \neq 4ack^2$$

Case 2.

$$(3.19) \quad \alpha_2 = -\frac{12c^2 k}{p+q}, \quad \alpha_1 = 0, \quad \alpha_0 = -\frac{4ack}{p+q}, \quad V = \frac{q}{\beta + 4ack^2}, \quad \beta \neq -4ack^2$$

where $p+q \neq 0$. It is well known [26], the solution of equation (2.11) is given, and hence g'/g^2 has the form:

(1) If $ac > 0$, then

$$(3.20) \quad \frac{g'}{g^2} = \sqrt{\frac{a}{c}} \left[\frac{c_1 \cos(\sqrt{ac}\xi) + c_2 \sin(\sqrt{ac}\xi)}{c_1 \sin(\sqrt{ac}\xi) - c_2 \cos(\sqrt{ac}\xi)} \right],$$

(2) If $ac < 0$, then

$$(3.21) \quad \frac{g'}{g^2} = \frac{1}{2c} \left[2\sqrt{|ac|} - \frac{4c_1\sqrt{|ac|} \exp(2\xi\sqrt{|ac|})}{c_1 \exp(2\xi\sqrt{|ac|}) - c_2} \right],$$

(3) If $a = 0$, $c \neq 0$, then

$$(3.22) \quad \frac{g'}{g^2} = \frac{-c_1}{c(c_1\xi + c_2)},$$

where c_1 and c_2 are arbitrary constants. Now, we have the following exact solutions of equation (3.9)

(i) If $ac > 0$, then for case1, we have

$$(3.23) \quad v(\xi) = \frac{-12kac}{p+q} \left[\frac{c_1 \cos(\sqrt{ac}\xi) + c_2 \sin(\sqrt{ac}\xi)}{c_1 \sin(\sqrt{ac}\xi) - c_2 \cos(\sqrt{ac}\xi)} \right]^2 - \frac{12kac}{p+q},$$

and consequently, we get

$$(3.24) \quad W(\xi) = \frac{-12kac}{p+q} \int \left[\frac{c_1 \cos(\sqrt{ac}\xi) + c_2 \sin(\sqrt{ac}\xi)}{c_1 \sin(\sqrt{ac}\xi) - c_2 \cos(\sqrt{ac}\xi)} \right]^2 d\xi - \frac{12kac}{p+q}\xi + d_1,$$

where d_1 is a constant of integration. Simplifying (3.24), we get two values of $W(\xi)$ as follows:

The first value is

$$(3.25) \quad W_1(\xi) = \frac{-12k\sqrt{ac}}{p+q} \tan(\xi_1 + \sqrt{ac}\xi) + d_1,$$

where $\xi_1 = \tan^{-1}\left(\frac{c_1}{c_2}\right)$, $c_1^2 + c_2^2 \neq 0$, and ξ is given by

$$(3.26) \quad \xi = k(X - VT) = k \left(t - \frac{qT}{\beta - 4ack^2} \right), \quad T = x - W_1(\xi) - x_0.$$

In this case, equation (3.1) has the general periodic solution

$$(3.27) \quad u_1(x, t) = W_{1x}(\xi) = \frac{-12k^2ac}{p+q} \sec^2(\xi_1 + \sqrt{ac}\xi).$$

The second value is

$$(3.28) \quad W_2(\xi) = \frac{-12k\sqrt{ac}}{p+q} \cot(\xi_2 - \sqrt{ac}\xi) + d_1,$$

where $\xi_2 = \cot^{-1} \left(\frac{c_1}{c_2} \right)$, $c_1^2 + c_2^2 \neq 0$, and ξ is given by (3.26), while T has the form

$$(3.29) \quad T = x - W_2(\xi) - x_0.$$

In this case, equation (3.1) has the general periodic solution

$$(3.30) \quad u_2(x, t) = W_{2x}(\xi) = \frac{-12k^2 ac}{p+q} \csc^2 (\xi_2 - \sqrt{ac} \xi),$$

(ii) If $ac < 0$, then for case 1, we have

$$(3.31) \quad v(\xi) = \frac{-12k}{p+q} \left[\sqrt{|ac|} - \frac{2c_1 \sqrt{|ac|} \exp(2\xi \sqrt{|ac|})}{c_1 \exp(2\xi \sqrt{|ac|}) - c_2} \right]^2 - \frac{12kac}{p+q},$$

and, consequently, we get

$$(3.32) \quad W(\xi) = \frac{-12k|ac|}{p+q} \int \left[\frac{c_1 \exp(\xi \sqrt{|ac|}) + c_2 \exp(-\xi \sqrt{|ac|})}{c_1 \exp(\xi \sqrt{|ac|}) - c_2 \exp(-\xi \sqrt{|ac|})} \right]^2 d\xi - \frac{12kac}{p+q} \xi + d_1.$$

Setting $c_1 = \frac{1}{2}(A+B)$, $c_2 = \frac{-1}{2}(A-B)$, $\phi = \xi \sqrt{|ac|}$, where A and B are constants, then (3.32) reduces to

$$(3.33) \quad W(\xi) = \frac{-12k|ac|}{p+q} \int \left[\frac{A \sinh \phi + B \cosh \phi}{A \cosh \phi + B \sinh \phi} \right]^2 d\xi + \frac{12k|ac|}{p+q} \xi + d_1.$$

Using the formulas (8), (10), (12) and (14) obtained in Peng [30], we have respectively the following general forms of soliton solutions:

(1) If $|A| > |B|$, then

$$(3.34) \quad W_1(\xi) = \frac{12k}{p+q} \sqrt{|ac|} \tanh[\phi + \operatorname{sgn}(AB)\psi_1] + d_1,$$

and, consequently, the solution of equation (3.1) has the form:

$$(3.35) \quad u_1(x, t) = W_{1x}(\xi) = \frac{12k^2|ac|}{p+q} \operatorname{sech}^2[\phi + \operatorname{sgn}(AB)\psi_1],$$

where

$$(3.36) \quad \xi = k(X - VT) = k \left(t - \frac{qT}{\beta - 4ack^2} \right), \quad T = x - W_1(\xi) - x_0,$$

and $\psi_1 = \tanh^{-1} \left(\frac{|B|}{|A|} \right)$, $\operatorname{sgn}(AB)$ is the sign function. Note that the solution (3.35) is in agreement with the obtained in [27].

(2) If $|B| > |A| \neq 0$, then

$$(3.37) \quad W_2(\xi) = \frac{12k\sqrt{|ac|}}{p+q} \coth[\phi + \operatorname{sgn}(AB)\psi_2] + d_1,$$

and consequently, the solution of equation (3.1) has the form

$$(3.38) \quad u_2(x, t) = W_{2x}(\xi) = \frac{-12k^2|ac|}{p+q} \csc h^2[\phi + \operatorname{sgn}(AB)\psi_2],$$

where

$$(3.39) \quad \xi = k(X - VT) = k \left(t - \frac{qT}{\beta - 4ack^2} \right), \quad T = x - W_2(\xi) - x_0,$$

and $\psi_2 = \coth^{-1} \left(\frac{|B|}{|A|} \right)$.

(3) If $|B| > |A| = 0$, then

$$(3.40) \quad W_3(\xi) = \frac{12k\sqrt{|ac|}}{p+q} \coth \phi + d_1,$$

and, consequently, the solution of equation (3.1) has the form

$$(3.41) \quad u_3(x, t) = W_{3x}(\xi) = \frac{-12k^2|ac|}{p+q} \csc h^2 \phi,$$

where

$$(3.42) \quad \xi = k(X - VT) = k \left(t - \frac{qT}{\beta - 4ack^2} \right), \quad T = x - W_3(\xi) - x_0,$$

(4) If $|A| = |B|$, then

$$(3.43) \quad W_4(\xi) = d_1,$$

and, consequently, the solution of equation (3.1) is the zero solution

$$(3.44) \quad u_4(x, t) = W_{4x}(\xi) = 0,$$

where

$$(3.45) \quad \xi = k(X - VT) = k \left(t - \frac{qT}{\beta - 4ack^2} \right), \quad T = x - W_4(\xi) - x_0.$$

(iii) If $a = 0$, $c \neq 0$, then for case 1, we have

$$(3.46) \quad v(\xi) = \frac{-12k}{p+q} \left(\frac{c_1}{c_1\xi + c_2} \right)^2,$$

and, consequently, we get

$$(3.47) \quad W_5(\xi) = \frac{12k}{p+q} \left(\frac{c_1}{c_1\xi + c_2} \right) + d_1.$$

The solution of equation (3.1) in this case has the form:

$$(3.48) \quad u_5(x, t) = W_{5x}(\xi) = \frac{-12k^2}{(p+q)} \left(\frac{c_1^2}{(c_1\xi + c_2)^2} \right),$$

where

$$(3.49) \quad \xi = k(X - VT) = k \left(t - \frac{qT}{\beta} \right), \quad T = x - W_5(\xi) - x_0,$$

Similarly, we can find the exact solutions for case 2, which are omitted here for simplicity.

3.2. The new modified (g') -expansion method

Setting $m = 2$, in (2.12) we have the formal solution of equation (3.9) in the form

$$(3.50) \quad v(\xi) = \alpha_2(g')^2 + \alpha_1(g') + \alpha_0, \quad \alpha_2 \neq 0,$$

where g satisfies equation (2.13). It is easy to see that

$$(3.51) \quad v'(\xi) = 2\alpha_2c(g')^3 + (2\alpha_2b + \alpha_1c)(g')^2 + (2\alpha_2a + \alpha_1b)(g') + \alpha_1a,$$

$$(3.52) \quad \begin{aligned} v''(\xi) &= 6\alpha_2c(g')^4 + (10\alpha_2bc + 2\alpha_1c^2)(g')^3 \\ &+ (4\alpha_2b^2 + 3\alpha_1bc + 8\alpha_2ac)(g')^2 \\ &+ (6\alpha_2ab + 2\alpha_1ac + \alpha_1b^2)(g') + (2\alpha_2a^2 + \alpha_1ab). \end{aligned}$$

Substituting (3.50)-(3.52) into (3.9) and collecting all terms with the same powers of $(g')^i$, ($i = 0, 1, 2, 3, 4$) together, the left-hand side of equation (3.9) is converted into a polynomial in $(g')^i$. Setting each coefficient of this polynomial to zero, we get the following algebraic equations:

$$(3.53) \quad 0 : k^2V(2\alpha_2a^2 + \alpha_1ab) + \frac{1}{2}(p+q)kV\alpha_0^2 + (\beta V - q)\alpha_0 = 0,$$

$$(3.54) \quad 1 : k^2V(6\alpha_2ab + 2\alpha_1ac + \alpha_1b^2) + \alpha_1\alpha_0(p+q)kV + (\beta V - q)\alpha_1 = 0,$$

$$(3.55) \quad 2 : k^2V(4\alpha_2b^2 + 3\alpha_1bc + 8\alpha_2ac) + \frac{1}{2}(p+q)kV\alpha_1^2 + \alpha_2\alpha_0(p+q)kV + (\beta V - q)\alpha_2 = 0,$$

$$(3.56) \quad 3 : k^2V(10\alpha_2bc + 2\alpha_1c^2) + \alpha_2\alpha_1(p+q)kV = 0,$$

$$(3.57) \quad 4 : 6k^2V\alpha_2c^2 + \frac{1}{2}(p+q)kV\alpha_2^2 = 0.$$

Solve these algebraic equations using the Maple or Mathematica we get the following cases:

Case 1.

$$(3.58) \quad \alpha_2 = \frac{-12kc^2}{p+q}, \quad \alpha_1 = \frac{-12kbc}{p+q}, \quad \alpha_0 = \frac{-2k}{p+q} (b^2+2ac), \quad V = \frac{q}{4ack^2+\beta-k^2b^2}.$$

Case 2.

$$(3.59) \quad \alpha_2 = \frac{-12kc^2}{p+q}, \quad \alpha_1 = \frac{-12kbc}{p+q}, \quad \alpha_0 = \frac{-3kb^2}{p+q}, \quad b = -2\sqrt{ac}, \quad V = \frac{q}{\beta}.$$

Case 3.

$$(3.60) \quad \alpha_2 = \frac{-12kc^2}{p+q}, \quad \alpha_1 = 0, \quad \alpha_0 = \frac{-12ack}{p+q}, \quad b = 0, \quad V = \frac{q}{\beta - 4ack^2}.$$

Case 4.

$$(3.61) \quad \alpha_2 = \frac{-12kc^2}{p+q}, \quad \alpha_1 = 0, \quad \alpha_0 = \frac{-4ack}{p+q}, \quad b = 0, \quad V = \frac{q}{\beta + 4ack^2}.$$

It is well-known [26], the solution of equation (2.13) is given, and hence $g'(\xi)$ has the forms:

If $\Delta = 4ac - b^2 < 0$, then

$$(3.62) \quad g'(\xi) = \frac{1}{2c} \left[\sqrt{-\Delta} \tanh \left(\frac{-\xi}{2} \sqrt{-\Delta} \right) - b \right].$$

If $\Delta = 4ac - b^2 > 0$, then

$$(3.63) \quad g'(\xi) = \frac{1}{2c} \left[\sqrt{\Delta} \tanh \left(\frac{\xi}{2} \sqrt{\Delta} \right) - b \right].$$

If $\Delta = 4ac - b^2 = 0$, then

$$(3.64) \quad g'(\xi) = \frac{-1}{c} \left(\frac{1}{\xi} + \frac{b}{2} \right).$$

Now, we deduce the following exact solutions of equation (3.9) as follows:

(i) If $\Delta < 0$, then for case 1, we have

$$(3.65) \quad v(\xi) = \frac{-3k\Delta}{p+q} \sec h^2 \left(\frac{-\xi}{2} \sqrt{-\Delta} \right) + \frac{2k\Delta}{p+q},$$

and consequently, we get

$$(3.66) \quad W_1(\xi) = \int v(\xi) d\xi + d_1,$$

which can be written as

$$(3.67) \quad W_1(\xi) = \frac{-6k\sqrt{-\Delta}}{p+q} \tanh \left(\frac{-\xi}{2} \sqrt{-\Delta} \right) + \frac{2k\Delta}{p+q} \xi + d_1.$$

Now, the solution of equation (3.1) in this case is the soliton solution:

$$(3.68) \quad u_1(x, t) = W_{1x}(\xi) = \frac{k^2 \Delta}{p+q} \left[2 - 3 \sec h^2 \left(\frac{-\xi}{2} \sqrt{-\Delta} \right) \right],$$

where

$$(3.69) \quad \xi = k(X - VT) = k \left(t - \frac{qT}{4ack^2 + \beta - k^2b^2} \right), \quad T = x - W_1(\xi) - x_0.$$

(ii) If $\Delta > 0$, then for case 1, we have

$$(3.70) \quad v(\xi) = \frac{-3k\Delta}{p+q} \sec^2 \left(\frac{\xi}{2} \sqrt{\Delta} \right) + \frac{2k\Delta}{p+q},$$

and consequently, we get

$$(3.71) \quad W_2(\xi) = \int v(\xi) d\xi + d_1,$$

which can be written as

$$(3.72) \quad W_2(\xi) = \frac{-6k\sqrt{\Delta}}{p+q} \tan \left(\frac{\xi}{2} \sqrt{\Delta} \right) + \frac{2k\Delta}{p+q} \xi + d_1.$$

Now, the solution of equation (3.1) in this case is the soliton solution:

$$(3.73) \quad u_2(x, t) = W_{2x}(\xi) = \frac{k^2 \Delta}{p+q} \left[2 - 3 \sec^2 \left(\frac{\xi}{2} \sqrt{\Delta} \right) \right],$$

where

$$(3.74) \quad \xi = k(X - VT) = k \left(t - \frac{qT}{4ack^2 + \beta - k^2b^2} \right), \quad T = x - W_2(\xi) - x_0.$$

(iii) If $\Delta = 0$, then for case 1, we have

$$(3.75) \quad v(\xi) = \left(\frac{-12k}{p+q} \right) \frac{1}{\xi^2},$$

and consequently, we get

$$(3.76) \quad W_3(\xi) = \int v(\xi) d\xi + d_1,$$

which can be written as

$$(3.77) \quad W_2(\xi) = \left(\frac{12k}{p+q} \right) \frac{1}{\xi} + d_1.$$

Now, the solution of equation (3.1) in this case is the soliton solution:

$$(3.78) \quad u_3(x, t) = W_{3x}(\xi) = \left(\frac{-12k^2}{p+q} \right) \frac{1}{\xi^2},$$

where

$$(3.79) \quad \xi = k(X - VT) = k \left(t - \frac{qT}{\beta} \right), \quad T = x - W_3(\xi) - x_0.$$

Similarly, we can find exact solutions for the other cases, which are omitted here for simplicity.

4. Physical explanations of some obtained solutions

In this section, we will present some graphs for three types of solutions of equation (3.1) namely hyperbolic, trigonometric and rational function solutions by selecting some special values of the parameters in the exact solutions using the mathematical software Maple, which can be shown below in Figures 1–6. From these explicit solutions, we see that the results (3.27), (3.30) and (3.73) are periodic solutions, the results (3.35) and (3.68) are bell-shaped soliton solutions, the results (3.38) and (3.41) are singular bell-shaped soliton solutions while the results (3.48) and (3.78) are rational function solutions.

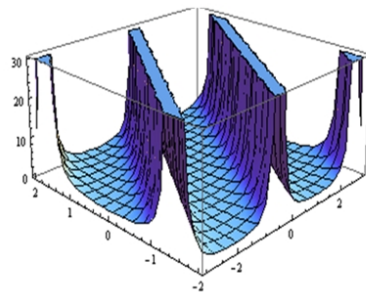


Fig1. The plot of the solution (3.27) when $k = a = c = 1, p = q = -6, \beta = -2, \xi_1 = 0$.

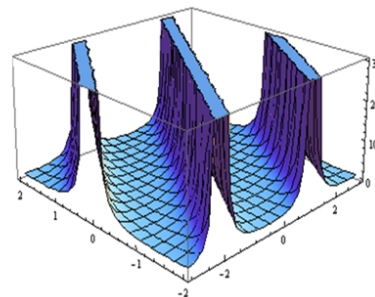


Fig2. The plot of the solution (3.30) when $k = a = c = 1, p = q = -6, \beta = -2, \xi_2 = 0$.

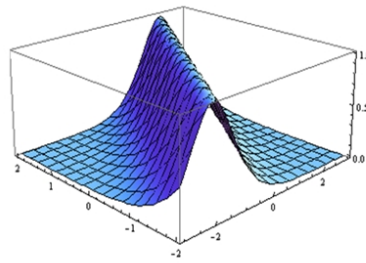


Fig3. The plot of the solution (3.35) when $k = c = 1, a = -1, p = q = 6, \beta = 2, \psi_1 = 0$.

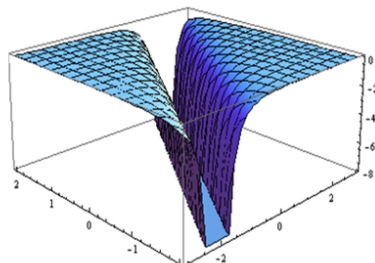


Fig4. The plot of the solution (3.38) when $k = c = 1, a = -1, p = q = 6, \beta = 2, \psi_2 = 0$.

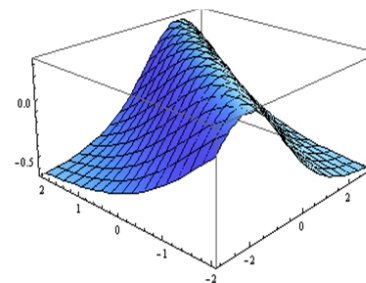


Fig5. The plot of the solution (3.68) when $k = p = 1, q = 2, \Delta = -1, \beta = 3$.

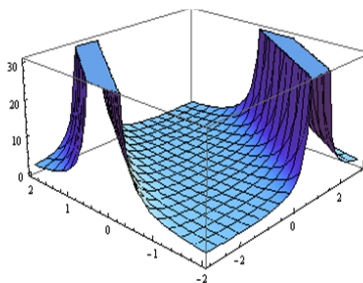


Fig6. The plot of the solution (3.73) when $k = p = 1, q = 2, \Delta = 1, \beta = 1$.

The plot of some solutions

5. Some conclusions

In this paper, we have introduced and presented the new modified (w/g) -expansion method when w and g are taken special values and then we proposed two expansion methods via the modified (g'/g^2) -expansion method and the modified (g') -expansion method. We have applied these two methods to find the exact solutions for the modified generalized Vakhnenko equation (3.1) such as the periodic, soliton and rational function solutions. Finally, with the aid of Maple or Mathematica we have assured the correctness of the obtained solutions by putting them back into the original equation (3.1).

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Accepted: 10.04.2014

ROUGH NEUTROSOPHIC SETS

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Abstract. Both neutrosophic sets theory and rough sets theory are emerging as powerful tool for managing uncertainty, indeterminate, incomplete and imprecise information. In this paper we develop an hybrid structure called *rough neutrosophic sets* and studied their properties.

Keywords: Rough set, rough neutrosophic set.

1. Introduction

In 1982, Pawlak [1] introduced the concept of rough set (RS), as a formal tool for modeling and processing incomplete information in information systems. There are two basic elements in rough set theory, crisp set and equivalence relation, which constitute the mathematical basis of RSs. The basic idea of rough set is based upon the approximation of sets by a pair of sets known as the lower approximation and the upper approximation of a set. Here, the lower and upper approximation operators are based on equivalence relation. After Pawlak, there has been many models built upon different aspect, i.e., universe, relations, object, operators by many scholars [2], [3], [4], [5], [6], [7]. Various notions that combine rough sets and fuzzy sets, vague set and intuitionistic fuzzy sets are introduced, such as rough fuzzy sets, fuzzy rough sets, generalize fuzzy rough, intuitionistic fuzzy rough sets, rough intuitionistic fuzzy sets, rough vagues sets. The theory of

rough sets is based upon the classification mechanism, from which the classification can be viewed as an equivalence relation and knowledge blocks induced by it be a partition on universe.

One of the interesting generalizations of the theory of fuzzy sets and intuitionistic fuzzy sets is the theory of neutrosophic sets introduced by F. Smarandache [8], [9]. Neutrosophic sets described by three functions: a membership function indeterminacy function and a non-membership function that are independently related. The theory of neutrosophic set have achieved great success in various areas such as medical diagnosis [10], database [11], [12], topology [13], image processing [14], [15], [16], and decision making problem [17]. While the neutrosophic set is a powerful tool to deal with indeterminate and inconsistent data, the theory of rough sets is a powerful mathematical tool to deal with incompleteness.

Neutrosophic sets and rough sets are two different topics, none conflicts the other. Recently many researchers applied the notion of neutrosophic sets to relations, group theory, ring theory, soft set theory [23], [24], [25], [26], [27], [28], [29], [30], [31], [32] and so on. The main objective of this study was to introduce a new hybrid intelligent structure called *rough neutrosophic sets*. The significance of introducing hybrid set structures is that the computational techniques based on any one of theses structures alone will not always yield the best results but a fusion of two or more of them can often give better results.

The rest of this paper is organized as follows. Some preliminary concepts required in our work are briefly recalled in Section 2. In Section 3, the concept of rough neutrosophic sets is investigated. Section 4 concludes the paper.

2. Preliminaries

In this section we present some preliminaries which will be useful to our work in the next section. For more details the reader may refer to [1], [8], [9].

Definition 2.1. [8] Let X be an universe of discourse, with a generic element in X denoted by x , the neutrosophic (NS) set is an object having the form

$$A = \{ \langle x : \mu_A(x), \nu_A(x), \omega_A(x) \rangle, x \in X \},$$

where the functions $\mu, \nu, \omega : X \rightarrow]^{-0}, 1^{+}[$ define respectively the degree of membership (or Truth), the degree of indeterminacy, and the degree of non-membership (or Falsehood) of the element $x \in X$ to the set A with the condition

$$(1) \quad ^{-}0 \leq \mu_A(x) + \nu_A(x) + \omega_A(x) \leq 3^{+}.$$

From a philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^{-0}, 1^{+}[$. So, instead of $]^{-0}, 1^{+}[$ we need to take the interval $[0, 1]$ for technical applications, because $]^{-0}, 1^{+}[$ will be difficult to apply in the real applications such as in scientific and engineering problems.

For two NS,

$$A = \{\langle x, \mu_A(x), \nu_A(x), \omega_A(x) \rangle \mid x \in X\} \text{ and} \\ B = \{\langle x, \mu_B(x), \nu_B(x), \omega_B(x) \rangle \mid x \in X\},$$

the relations are defined as follows:

- (i) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$, $\nu_A(x) \geq \nu_B(x)$, $\omega_A(x) \geq \omega_B(x)$,
- (ii) $A = B$ if and only if $\mu_A(x) = \mu_B(x)$, $\nu_A(x) = \nu_B(x)$, $\omega_A(x) = \omega_B(x)$,
- (iii) $A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x)), \max(\omega_A(x), \omega_B(x)) \rangle \mid x \in X\}$,
- (iv) $A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x)), \min(\omega_A(x), \omega_B(x)) \rangle \mid x \in X\}$,
- (v) $A^C = \{\langle x, \omega_A(x), 1 - \nu_A(x), \mu_A(x) \rangle \mid x \in X\}$
- (vi) $0_n = (0, 1, 1)$ and $1_n = (1, 0, 0)$.

As an illustration, let us consider the following example.

Example 2.2. Assume that the universe of discourse $U = \{x_1, x_2, x_3\}$, where x_1 characterizes the capability, x_2 characterizes the trustworthiness and x_3 indicates the prices of the objects. It may be further assumed that the values of x_1, x_2 and x_3 are in $[0, 1]$ and they are obtained from some questionnaires of some experts. The experts may impose their opinion in three components viz. the degree of goodness, the degree of indeterminacy and that of poorness to explain the characteristics of the objects. Suppose A is a neutrosophic set (NS) of U , such that,

$$A = \{\langle x_1, (0.3, 0.5, 0.6) \rangle, \langle x_2, (0.3, 0.2, 0.3) \rangle, \langle x_3, (0.3, 0.5, 0.6) \rangle\},$$

where the degree of goodness of capability is 0.3, degree of indeterminacy of capability is 0.5 and degree of falsity of capability is 0.6 etc.

Definition 2.3. [1] Let U be any non-empty set. Suppose R is an equivalence relation over U . For any non-null subset X of U , the sets

$$A_1(x) = \{x : [x]_R \subseteq X\} \text{ and } A_2(x) = \{x : [x]_R \cap X \neq \emptyset\}$$

are called the lower approximation and upper approximation, respectively of X , where the pair $S = (U, R)$ is called an approximation space. This equivalent relation R is called indiscernibility relation.

The pair $A(X) = (A_1(x), A_2(x))$ is called the rough set of X in S . Here $[x]_R$ denotes the equivalence class of R containing x .

Definition 2.4. [1] Let $A = (A_1, A_2)$ and $B = (B_1, B_2)$ be two rough sets in the approximation space $S = (U, R)$. Then,

$$\begin{aligned} A \cup B &= (A_1 \cup B_1, A_2 \cup B_2), \\ A \cap B &= (A_1 \cap B_1, A_2 \cap B_2), \\ A \subseteq B &\text{ if } A \cap B = A, \\ \sim A &= \{U - A_2, U - A_1\}. \end{aligned}$$

3. Rough neutrosophic sets

In this section we introduce the notion of rough neutrosophic sets by combining both rough sets and neutrosophic sets. and some operations viz. union, intersection, inclusion and equalities over them. Rough neutrosophic set are the generalization of rough fuzzy sets [2] and rough intuitionistic fuzzy sets [22].

Definition 3.1. Let U be a non-null set and R be an equivalence relation on U . Let F be neutrosophic set in U with the membership function μ_F , indeterminacy function ν_F and non-membership function ω_F . The lower and the upper approximations of F in the approximation (U, R) denoted by $\underline{N}(F)$ and $\overline{N}(F)$ are respectively defined as follows:

$$\begin{aligned} \underline{N}(F) &= \{ \langle x, \mu_{\underline{N}(F)}(x), \nu_{\underline{N}(F)}(x), \omega_{\underline{N}(F)}(x) \rangle \mid y \in [x]_R, x \in U \}, \\ \overline{N}(F) &= \{ \langle x, \mu_{\overline{N}(F)}(x), \nu_{\overline{N}(F)}(x), \omega_{\overline{N}(F)}(x) \rangle \mid y \in [x]_R, x \in U \}, \end{aligned}$$

where:

$$\begin{aligned} \mu_{\underline{N}(F)}(x) &= \bigwedge_{y \in [x]_R} \mu_F(y), \quad \nu_{\underline{N}(F)}(x) = \bigvee_{y \in [x]_R} \nu_F(y), \quad \omega_{\underline{N}(F)}(x) = \bigvee_{y \in [x]_R} \omega_F(y), \\ \mu_{\overline{N}(F)}(x) &= \bigvee_{y \in [x]_R} \mu_F(y), \quad \nu_{\overline{N}(F)}(x) = \bigwedge_{y \in [x]_R} \nu_F(y), \quad \omega_{\overline{N}(F)}(x) = \bigwedge_{y \in [x]_R} \omega_F(y). \end{aligned}$$

So

$$0 \leq \mu_{\overline{N}(F)}(x) + \nu_{\overline{N}(F)}(x) + \omega_{\overline{N}(F)}(x) \leq 3$$

and

$$\mu_{\underline{N}(F)}(x) + \nu_{\underline{N}(F)}(x) + \omega_{\underline{N}(F)}(x) \leq 3,$$

where " $\vee$ " and " $\wedge$ " mean "max" and "min" operators respectively, $\mu_F(x)$, $\nu_F(y)$ and $\omega_F(y)$ are the membership, indeterminacy and non-membership of y with respect to F . It is easy to see that $\overline{N}(F)$ and $\underline{N}(F)$ are two neutrosophic sets in U , thus the NS mappings $\overline{N}, \underline{N} : N(U) \rightarrow N(U)$ are, respectively, referred to as the upper and lower rough NS approximation operators, and the pair $(\underline{N}(F), \overline{N}(F))$ is called the rough neutrosophic set in (U, R) .

From the above definition, we can see that $\underline{N}(F)$ and $\overline{N}(F)$ have constant membership on the equivalence classes of U , if $\underline{N}(F) = \overline{N}(F)$; i.e.,

$$\mu_{\underline{N}(F)} = \mu_{\overline{N}(F)},$$

$$\nu_{\underline{N}(F)} = \nu_{\overline{N}(F)},$$

$$\omega_{\underline{N}(F)} = \omega_{\overline{N}(F)}.$$

For any $x \in U$, we call F a definable neutrosophic set in the approximation (U, R) . It is easily to be proved that Zero O_N neutrosophic set and unit neutrosophic sets 1_N are definable neutrosophic sets. Let us consider a simple example in the following.

Example 3.2. Let $U = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8\}$ be the universe of discourse. Let R be an equivalence relation its partition of U is given by

$$U/R = \{\{p_1, p_4\}, \{p_2, p_3, p_6\}, \{p_5\}, \{p_7, p_8\}\}.$$

Let

$$N(F) = \{(p_1, (0.2, 0.3, 0.4)), (p_4, (0.3, 0.5, 0.4)), (p_5, (0.4, 0.6, 0.2)), \\ (p_7, (0.1, 0.3, 0.5))\}$$

be a neutrosophic set of U . By Definition 3.1, we obtain:

$$\begin{aligned} \underline{N}(F) &= \{(p_1, (0.2, 0.5, 0.4)), (p_4, (0.2, 0.5, 0.4)), (p_5, (0.4, 0.6, 0.2))\}; \\ \overline{N}(F) &= \{(p_1, (0.2, 0.3, 0.4)), (p_4, (0.2, 0.3, 0.4)), (p_5, (0.4, 0.6, 0.2)), \\ &\quad (p_7, (0.1, 0.3, 0.5)), (p_8, (0.1, 0.3, 0.5))\}. \end{aligned}$$

For another neutrosophic sets

$$N(G) = \{(p_1, (0.2, 0.3, 0.4)), (p_4, (0.2, 0.3, 0.4)), (p_5, (0.4, 0.6, 0.2))\}.$$

The lower approximation and upper approximation of $N(G)$ are calculated as

$$\begin{aligned} \underline{N}(G) &= \{(p_1, (0.2, 0.3, 0.4)), (p_4, (0.2, 0.3, 0.4)), (p_5, (0.4, 0.6, 0.2))\}; \\ \overline{N}(G) &= \{(p_1, (0.2, 0.3, 0.4)), (p_4, (0.2, 0.3, 0.4)), (p_5, (0.4, 0.6, 0.2))\}. \end{aligned}$$

Obviously $\underline{N}(G) = \overline{N}(G)$ is a definable neutrosophic set in the approximation space (U, R) .

Definition 3.3. If $N(F) = (\underline{N}(F), \overline{N}(F))$ is a rough neutrosophic set in (U, R) , the rough complement of $N(F)$ is the rough neutrosophic set denoted $\sim N(F) = (\underline{N}(F)^c, \overline{N}(F)^c)$, where $\underline{N}(F)^c, \overline{N}(F)^c$ are the complements of neutrosophic sets $\underline{N}(F)$ and $\overline{N}(F)$, respectively,

$$\underline{N}(F)^c = \{ \langle x, \omega_{\underline{N}(F)}, 1 - \nu_{\underline{N}(F)}(x), \mu_{\underline{N}(F)}(x) \rangle \mid x \in U \},$$

and

$$\overline{N}(F)^c = \{ \langle x, \omega_{\overline{N}(F)}, 1 - \nu_{\overline{N}(F)}(x), \mu_{\overline{N}(F)}(x) \rangle \mid x \in U \}.$$

Definition 3.4. If $N(F_1)$ and $N(F_2)$ are two rough neutrosophic set of the neutrosophic sets F_1 and F_2 respectively in U , then we define the following:

- (i) $N(F_1) = N(F_2)$ iff $\underline{N}(F_1) = \underline{N}(F_2)$ and $\overline{N}(F_1) = \overline{N}(F_2)$.
- (ii) $N(F_1) \subseteq N(F_2)$ iff $\underline{N}(F_1) \subseteq \underline{N}(F_2)$ and $\overline{N}(F_1) \subseteq \overline{N}(F_2)$.
- (iii) $N(F_1) \cup N(F_2) = \langle \underline{N}(F_1) \cup \underline{N}(F_2), \overline{N}(F_1) \cup \overline{N}(F_2) \rangle$.
- (iv) $N(F_1) \cap N(F_2) = \langle \underline{N}(F_1) \cap \underline{N}(F_2), \overline{N}(F_1) \cap \overline{N}(F_2) \rangle$.
- (v) $N(F_1) + N(F_2) = \langle \underline{N}(F_1) + \underline{N}(F_2), \overline{N}(F_1) + \overline{N}(F_2) \rangle$.
- (vi) $N(F_1) \cdot N(F_2) = \langle \underline{N}(F_1) \cdot \underline{N}(F_2), \overline{N}(F_1) \cdot \overline{N}(F_2) \rangle$.

If N, M, L are rough neutrosophic set in (U, R) , then the results in the following proposition are straightforward from definitions.

Proposition 3.5.

- (i) $\sim N(\sim N) = N$
- (ii) $N \cup M = M \cup N, N \cap M = M \cap N$
- (iii) $(N \cup M) \cup L = N \cup (M \cup L)$ and $(N \cap M) \cap L = N \cap (M \cap L)$
- (iv) $(N \cup M) \cap L = (N \cap M) \cap (N \cup L)$ and $(N \cap M) \cup L = (N \cap M) \cup (N \cap L)$.

De Morgan 's Laws are satisfied for neutrosophic sets:

Proposition 3.6.

- (i) $\sim (N(F_1) \cup N(F_2)) = (\sim N(F_1)) \cap (\sim N(F_2))$
- (ii) $\sim (N(F_1) \cap N(F_2)) = (\sim N(F_1)) \cup (\sim N(F_2))$.

Proof. (i) $(N(F_1) \cup N(F_2)) = \sim (\{\underline{N}(F_1) \cup \underline{N}(F_2)\}, \{\overline{N}(F_1) \cup \overline{N}(F_2)\}) = (\sim \{\underline{N}(F_1) \cup \underline{N}(F_2)\}, \sim \{\overline{N}(F_1) \cup \overline{N}(F_2)\}) = (\{\underline{N}(F_1) \cup \underline{N}(F_2)\}^c, \{\overline{N}(F_1) \cup \overline{N}(F_2)\}^c) = (\sim \{\underline{N}(F_1) \cap \underline{N}(F_2)\}, \sim \{\overline{N}(F_1) \cap \overline{N}(F_2)\}) = (\sim N(F_1)) \cap (\sim N(F_2)).$

(ii) Similar to the proof of (i).

Proposition 3.7. *If F_1 and F_2 are two neutrosophic sets in U such that $F_1 \subseteq F_2$, then $N(F_1) \subseteq N(F_2)$*

- (i) $N(F_1 \cup F_2) \supseteq N(F_1) \cup N(F_2),$
- (ii) $N(F_1 \cap F_2) \subseteq N(F_1) \cap N(F_2).$

Proof.

$$\begin{aligned}
 \mu_{\underline{N}(F_1 \cup F_2)}(x) &= \inf\{\mu_{F_1 \cup F_2}(x) \mid x \in X_i\} \\
 &= \inf(\max\{\mu_{F_1}(x), \mu_{F_2}(x) \mid x \in X_i\}) \\
 &\geq \max\{\inf\{\mu_{F_1}(x) \mid x \in X_i\}, \inf\{\mu_{F_2}(x) \mid x \in X_i\}\} \\
 &= \max\{\mu_{\underline{N}(F_1)}(x_i), \mu_{\underline{N}(F_2)}(x_i)\} \\
 &= \mu_{\underline{N}(F_1) \cup \underline{N}(F_2)}(x_i).
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 \nu_{\underline{N}(F_1 \cup F_2)}(x_i) &\leq (\nu_{\underline{N}(F_1)} \cup \nu_{\underline{N}(F_2)})(x_i) \\
 \omega_{\underline{N}(F_1 \cup F_2)}(x_i) &\leq (\omega_{\underline{N}(F_1)} \cup \omega_{\underline{N}(F_2)})(x_i)
 \end{aligned}$$

Thus,

$$\underline{N}(F_1 \cup F_2) \supseteq \underline{N}(F_1) \cup \underline{N}(F_2).$$

We can also see that

$$\overline{N}(F_1 \cup F_2) = \overline{N}(F_1) \cup \overline{N}(F_2).$$

Hence,

$$N(F_1 \cup F_2) \supseteq N(F_1) \cup N(F_2).$$

(ii) The proof of (ii) is similar to the proof of (i).

Proposition 3.8.

- (i) $N(F) = \sim \overline{N}(\sim F)$
- (ii) $\overline{N}(F) = \sim \underline{N}(\sim F)$
- (iii) $\underline{N}(F) \supseteq \overline{N}(F)$.

Proof. According to Definition 3.1, we can obtain

$$\begin{aligned}
 \text{(i)} \quad F &= \{\langle x, \mu_F(x), \nu_F(x), \omega_F(x) \rangle \mid x \in X\} \\
 \sim F &= \{\langle x, \omega_F(x), 1 - \nu_F(x), \mu_F(x) \rangle \mid x \in X\} \\
 \overline{N}(\sim F) &= \left\{ \left\langle x, \omega_{\overline{N}(\sim F)}(x), 1 - \nu_{\overline{N}(\sim F)}(x), \mu_{\overline{N}(\sim F)}(x) \right\rangle \mid y \in [x]_R, x \in U \right\} \\
 \sim \overline{N}(\sim F) &= \left\{ \left\langle x, \mu_{\overline{N}(\sim F)}(x), 1 - (1 - \nu_{\overline{N}(\sim F)}(x)), \omega_{\overline{N}(\sim F)}(x) \right\rangle \mid y \in [x]_R, x \in U \right\} \\
 &= \left\{ \left\langle x, \mu_{\overline{N}(\sim F)}(x), \nu_{\overline{N}(\sim F)}(x), \omega_{\overline{N}(\sim F)}(x) \right\rangle \mid y \in [x]_R, x \in U \right\}
 \end{aligned}$$

where

$$\mu_{\overline{N}(\sim F)}(x) = \bigwedge_{y \in [x]_R} \mu_F(y), \quad \nu_{\overline{N}(\sim F)}(x) = \bigvee_{y \in [x]_R} \nu_F(y), \quad \omega_{\overline{N}(\sim F)}(x) = \bigvee_{y \in [x]_R} \omega_F(y).$$

Hence $\underline{N}(F) = \sim \overline{N}(\sim F)$.

(ii) The proof is similar to the proof of (i).

(iii) For any $y \in \underline{N}(F)$, we can have

$$\mu_{\underline{N}(F)}(x) = \bigwedge_{y \in [x]_R} \mu_F(y) \leq \bigvee_{y \in [x]_R} \mu_F(y), \quad \nu_{\underline{N}(F)}(x) = \bigvee_{y \in [x]_R} \nu_F(y) \geq \bigwedge_{y \in [x]_R} \nu_F(y)$$

$$\text{and } \omega_{\underline{N}(F)}(x) = \bigvee_{y \in [x]_R} \omega_F(y) \geq \bigwedge_{y \in [x]_R} \omega_F(y).$$

Hence $\underline{N}(F) \subseteq \overline{N}(F)$.

4. Conclusion

In this paper we have defined the notion of rough neutrosophic sets. We have also studied some properties on them and proved some propositions. The concept combines two different theories which are rough sets theory and neutrosophic theory. While neutrosophic set theory is mainly concerned with, indeterminate and inconsistent information, rough set theory is with incompleteness; but both the theories deal with imprecision. Consequently, by the way they are defined, it is clear that rough neutrosophic sets can be utilized for dealing with both of indeterminacy and incompleteness.

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Accepted: 22.04.2014

NEUTROSOPHIC PARAMETRIZED SOFT SET THEORY AND ITS DECISION MAKING

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Abstract. In this work, we present definition of neutrosophic parameterized (NP) soft set and its operations. Then we define NP-aggregation operator to form NP-soft decision making method which allows constructing more efficient decision processes. We also give an example which shows that they can be successfully applied to problem that contain indeterminacy.

Keywords: soft set, neutrosophic set, neutrosophic soft set, neutrosophic parameterized soft set, aggregation operator.

1. Introduction

In 1999, Smarandache firstly proposed the theory of neutrosophic set (NS) [28], which is the generalization of the classical sets, conventional fuzzy set [30] and intuitionistic fuzzy set [5]. After Smarandache, neutrosophic sets has been successfully applied to many fields such as; control theory [1], databases [2], [3], medical diagnosis problem [4], decision making problem [21], topology [22], and so on.

In 1999, a Russian researcher [27] firstly gave the soft set theory as a general mathematical tool for dealing with uncertainty and vagueness and how soft set theory is free from the parameterization inadequacy syndrome of fuzzy set theory, rough set theory, probability theory. Then, many interesting results of soft set

theory have been studied on fuzzy soft sets [8], [12], [23], on intuitionistic fuzzy soft set theory [14], [25], on possibility fuzzy soft set [7], on generalized fuzzy soft sets [26], [29], on generalized intuitionistic fuzzy soft [6], on interval-valued intuitionistic fuzzy soft sets [20], on intuitionistic neutrosophic soft set [9], on generalized neutrosophic soft set [10], on fuzzy parameterized soft set theory [17], [18], on fuzzy parameterized fuzzy soft set theory [13], on intuitionistic fuzzy parameterized soft set theory [15], on IFP-fuzzy soft set theory [16], on neutrosophic soft set [24], interval-valued neutrosophic soft set [11], [19]. In this paper, our main objective is to introduce the notion of neutrosophic parameterized soft set which is a generalization of fuzzy parameterized soft set and intuitionistic fuzzy parameterized soft set. The paper is structured as follows. In Section 2, we first recall the necessary background on neutrosophic and soft set. In Section 3, we give neutrosophic parameterized soft set theory and their respective properties. In Section 4, we present a neutrosophic parameterized aggregation operator. In Section 5, a neutrosophic parameterized decision methods is presented with example. Finally, we conclude the paper.

2. Preliminaries

Throughout this paper, let U be a universal set and E be the set of all possible parameters under consideration with respect to U , usually, parameters are attributes, characteristics, or properties of objects in U .

We now recall some basic notions of neutrosophic set and soft set. For more details, the reader could refer to [23], [27].

Definition 1. [27] Let U be a universe of discourse then the neutrosophic set A is an object having the form

$$A = \{ \langle x : \mu_{A(x)}, \nu_{A(x)}, \omega_{A(x)} \rangle, x \in U \},$$

where the functions $\mu, \nu, \omega : U \rightarrow]-0, 1^+[$ define respectively the degree of membership, the degree of indeterminacy, and the degree of non-membership of the element $x \in X$ to the set A with the condition.

$$(1) \quad -0 \leq \mu_{A(x)} + \nu_{A(x)} + \omega_{A(x)} \leq 3^+.$$

From the philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $] -0, 1^+[$. So, instead of $] -0, 1^+[$, we need to take the interval $[0, 1]$ for technical applications, because $] -0, 1^+[$ will be difficult to apply in the real applications such as in scientific and engineering problems.

For two NS,

$$A_{NS} = \{ \langle x, \mu_A(x), \nu_A(x), \omega_A(x) \rangle \mid x \in X \}$$

and

$$B_{NS} = \{ \langle x, \mu_B(x), \nu_B(x), \omega_B(x) \rangle \mid x \in X \}.$$

Then,

1. $A_{NS} \subseteq B_{NS}$ if and only if
 $\mu_A(x) \leq \mu_B(x), \nu_A(x) \geq \nu_B(x), \omega_A(x) \geq \omega_B(x)$.
2. $A_{NS} = B_{NS}$ if and only if
 $\mu_A(x) = \mu_B(x), \nu_A(x) = \nu_B(x), \omega_A(x) = \omega_B(x)$ for any $x \in X$.
3. The complement of A_{NS} is denoted by A_{NS}° and is defined by
 $A_{NS}^\circ = \{\langle x, \omega_A(x), 1 - \nu_A(x), \mu_A(x) \rangle \mid x \in X\}$.
4. $A \cap B = \{\langle x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\nu_A(x), \nu_B(x)\}, \max\{\omega_A(x), \omega_B(x)\} \rangle : x \in X\}$.
5. $A \cup B = \{\langle x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\nu_A(x), \nu_B(x)\}, \min\{\omega_A(x), \omega_B(x)\} \rangle : x \in X\}$.

As an illustration, let us consider the following example.

Example 1. Assume that the universe of discourse $U = \{x_1, x_2, x_3, x_4\}$. It may be further assumed that the values of x_1, x_2, x_3 and x_4 are in $[0, 1]$. Then, A is a neutrosophic set (NS) of U , such that,

$$A = \{\langle x_1, 0.4, 0.6, 0.5 \rangle, \langle x_2, 0.3, 0.4, 0.7 \rangle, \langle x_3, 0.4, 0.4, 0.6 \rangle, \langle x_4, 0.5, 0.4, 0.8 \rangle\}.$$

Definition 2. [23] Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denotes the power set of U . Consider a nonempty set $A, A \subset E$. A pair (K, A) is called a soft set over U , where K is a mapping given by $K : A \rightarrow P(U)$.

As an illustration, let us consider the following example.

Example 2. Suppose that U is the set of houses under consideration, say

$$U = \{h_1, h_2, h_3, h_4, h_5\}.$$

Let E be the set of some attributes of such houses, say

$$E = \{e_1, e_2, e_3, e_4, e_5\},$$

where e_1, e_2, e_3, e_4, e_5 stand for the attributes "beautiful", "costly", "in the green surroundings", "moderate" and technically, respectively. In this case, to define a soft set means to point out expensive houses, beautiful houses, and so on. For example, the soft set (K, A) that describes the "attractiveness of the houses" in the opinion of a buyer, says Thomas, and may be defined like this:

$$A = \{e_1, e_2, e_3, e_4, e_5\},$$

$$K(e_1) = \{h_2, h_3, h_5\},$$

$$K(e_2) = \{h_2, h_4\},$$

$$K(e_3) = \{h_1\},$$

$$K(e_4) = U,$$

$$K(e_5) = \{h_3, h_5\}.$$

3. Neutrosophic parameterized soft set theory

In this section, we define neutrosophic parameterized soft set and their operations.

Definition 3.1. Let U be an initial universe, $P(U)$ be the power set of U , E be a set of all parameters and K be a neutrosophic set over E . Then, a neutrosophic parameterized soft sets

$$\Psi_K = \{(\langle x, \mu_K(x), \nu_K(x), \omega_K(x), f_K(x) \rangle : x \in E)\},$$

where $\mu_K : E \rightarrow [0, 1]$, $\nu_K : E \rightarrow [0, 1]$, $\omega_K : E \rightarrow [0, 1]$ and $f_K : E \rightarrow P(U)$ such that $f_K(x) = \Phi$ if $\mu_K(x) = 0$, $\nu_K(x) = 1$ and $\omega_K(x) = 1$. Here, the functions μ_K, ν_K and ω_K are called membership function, indeterminacy function and non-membership function of neutrosophic parameterized soft set (NP-soft set), respectively.

Example 3.2. Assume that $U = \{u_1, u_2, u_3\}$ is a universal set and $E = \{x_1, x_2\}$ is a set of parameters. If $K = \{\langle x_1, 0.2, 0.3, 0.4 \rangle, \langle x_2, 0.3, 0.5, 0.4 \rangle\}$ and $f_K(x_1) = \{u_2, u_3\}$, $f_K(x_2) = U$, then, a neutrosophic parameterized soft set Ψ_K is written by

$$\Psi_K = \{(\langle x_1, 0.2, 0.3, 0.4 \rangle, \{u_2, u_3\}), (\langle x_2, 0.3, 0.5, 0.4 \rangle, U)\}.$$

Definition 3.3. Let $\Psi_K \in NP$ -soft set. If $f_K(x) = U$, $\mu_K(x) = 0$, $\nu_K(x) = 1$ and $\omega_K(x) = 1$, for all $x \in E$, then Ψ_K is called a K -empty NP-soft set, denoted by Ψ_{Φ_K} . If $K = \Phi$, then the K -empty NP-soft set is called an empty NP-soft set, denoted by Ψ_{Φ} .

Definition 3.4. Let $\Psi_K \in NP$ -soft set. If $f_K(x) = U$, $\mu_K(x) = 1$, $\nu_K(x) = 0$ and $\omega_K(x) = 0$, for all $x \in E$, then Ψ_K is called a K -universal NP-soft set, denoted by $\Psi_{\tilde{K}}$. If $K = E$, then the K -universal NP-soft set is called a universal NP-soft set, denoted by $\Psi_{\tilde{E}}$.

Definition 3.5. Ψ_K and Ω_L are two NP-soft sets. Then, Ψ_K is an NP-subset of Ω_L , denoted by $\Psi_K \sqsubseteq \Omega_L$ if and only if $\mu_K(x) \leq \mu_L(x)$, $\nu_K(x) \geq \nu_L(x)$ and $\omega_K(x) \geq \omega_L(x)$, and $f_K(x) \sqsubseteq g_L(x)$ for all $x \in E$.

Definition 3.6. Ψ_K and Ω_L are two NP-soft sets. Then, $\Psi_K = \Omega_L$, if and only if $\Psi_K \sqsubseteq \Omega_L$ and $\Omega_L \sqsubseteq \Psi_K$, for all $x \in E$.

Definition 3.7. Let $\Psi_K \in NP$ -soft set. Then, the complement of Ψ_K , denoted by Ψ_K^c , is defined by

$$\Psi_K^c = \{(\langle x, \omega_K(x), \nu_K(x), \mu_K(x) \rangle, f_{K^c}(x)) : x \in E\},$$

where $f_{K^c}(x) = U \setminus f_K(x)$.

Definition 3.8. Let Ψ_K and Ω_L be two NP-soft sets. Then, the union of Ψ_K and Ω_L , denoted by $\Psi_K \sqcup \Omega_L$, is defined by

$$\Psi_K \sqcup \Omega_L = \{(\langle x, \max\{\mu_K(x), \mu_L(x)\}, \min\{\nu_K(x), \nu_L(x)\}, \min\{\omega_K(x), \omega_L(x)\} \rangle, f_{K \cup L}(x)) : x \in E\},$$

where $f_{K \cup L}(x) = f_K(x) \cup f_L(x)$.

Definition 3.9. Let Ψ_K and Ω_L be two NP-soft sets. Then, the intersection of Ψ_K and Ω_L , denoted by $\Psi_K \sqcap \Omega_L$, is defined by

$$\Psi_K \sqcap \Omega_L = \{(\langle x, \min\{\mu_K(x), \mu_L(x)\}, \max\{\nu_K(x), \nu_L(x)\}, \max\{\omega_K(x), \omega_L(x)\} \rangle, f_{K \cap L}(x)) : x \in E\},$$

where $f_{K \cap L}(x) = f_K(x) \cap f_L(x)$.

Example 3.10. Let $U = \{u_1, u_2, u_3, u_4\}$, $E = \{x_1, x_2, x_3\}$. Then,

$$\begin{aligned} \Psi_K &= \{(\langle x_1, 0.2, 0.3, 0.4 \rangle, \{u_1, u_2\}), (\langle x_2, 0.3, 0.5, 0.4 \rangle, \{u_2, u_3\})\} \\ \Omega_L &= \{(\langle x_2, 0.1, 0.2, 0.4 \rangle, \{u_3, u_4\}), (\langle x_3, 0.5, 0.2, 0.3 \rangle, \{u_3\})\}. \end{aligned}$$

Then

$$\begin{aligned} \Psi_K \sqcup \Omega_L &= \{(\langle x_1, 0.2, 0.3, 0.4 \rangle, \{u_1, u_2\}), (\langle x_2, 0.3, 0.2, 0.4 \rangle, \{u_2, u_3, u_4\}), \\ &\quad (\langle x_3, 0.5, 0.2, 0.3 \rangle, \{u_3\})\} \\ \Psi_K \sqcap \Omega_L &= \{(\langle x_2, 0.1, 0.5, 0.4 \rangle, \{u_3, u_4\})\} \\ \Psi_K^c &= \{(\langle x_1, 0.4, 0.3, 0.2 \rangle, \{u_3, u_4\}), (\langle x_2, 0.4, 0.5, 0.3 \rangle, \{u_1, u_4\})\}. \end{aligned}$$

Remark 3.11. $\Psi_K \sqsubseteq \Omega_L$ does not imply that every element of Ψ_K is an element of Ω_L as in the definition of classical subset. For example, assume that $U = \{u_1, u_2, u_3, u_4\}$ is a universal set of objects and $E = \{x_1, x_2, x_3\}$ is a set of all parameters, if NP-soft sets Ψ_K and Ω_L are defined as

$$\begin{aligned} \Psi_K &= \{(\langle x_1, 0.2, 0.3, 0.4 \rangle, \{u_1, u_2\}), (\langle x_2, 0.3, 0.5, 0.4 \rangle, \{u_2\})\} \\ \Omega_L &= \{(\langle x_1, 0.3, 0.2, 0.4 \rangle, U), (\langle x_2, 0.5, 0.2, 0.3 \rangle, \{u_1, u_4\})\}. \end{aligned}$$

It can be seen that $\Psi_K \sqsubseteq \Omega_L$, but every element of Ψ_K is not an element of Ω_L .

Proposition 3.12. Let $\Psi_K, \Omega_L \in NP\text{-soft sets}$. Then

- (i) $\Psi_K \sqsubseteq \Psi_{\tilde{E}}$.
- (ii) $\Psi_{\Phi} \sqsubseteq \Psi_K$.
- (iii) $\Psi_K \sqsubseteq \Psi_K$.

Proof. It is clear from Definitions 3.3-3.5.

Proposition 3.13. *Let Ψ_K, Ω_L and $\gamma_M \in NP$ -soft sets. Then*

- (i) $\Psi_K = \Omega_L$ and $\Omega_L = \gamma_M \iff \Psi_K = \gamma_M$,
- (ii) $\Psi_K \sqsubseteq \Omega_L$ and $\Omega_L \sqsubseteq \Psi_K \iff \Psi_K = \Omega_L$,
- (iii) $\Psi_K \sqsubseteq \Omega_L$ and $\Omega_L \sqsubseteq \gamma_M \implies \Psi_K \sqsubseteq \gamma_M$.

Proof. It can be proved by Definitions 3.3-3.5.

Proposition 3.14. *Let $\Psi_K \in NP$ -soft set. Then*

- (i) $(\Psi_K^c)^c = \Psi_K$,
- (ii) $\Psi_\Phi^c = \Psi_{\tilde{E}}$,
- (iii) $\Psi_E^c = \Psi_{\tilde{\Phi}}$.

Proof. It is trial.

Proposition 3.15. *Let Ψ_K, Ω_L and $\gamma_M \in NP$ -soft sets. Then*

- (i) $\Psi_K \sqcup \Psi_K = \Psi_K$,
- (ii) $\Psi_K \sqcup \Psi_{\tilde{\Phi}} = \Psi_K$,
- (iii) $\Psi_K \sqcup \Psi_{\tilde{E}} = \Psi_{\tilde{E}}$,
- (iv) $\Psi_K \sqcup \Omega_L = \Omega_L \sqcup \Psi_K$,
- (v) $(\Psi_K \sqcup \Omega_L) \sqcup \gamma_M = \Psi_K \sqcup (\Omega_L \sqcup \gamma_M)$.

Proof. It is clear.

Proposition 3.16. *Let Ψ_K, Ω_L and $\gamma_M \in NP$ -soft sets. Then*

- (i) $\Psi_K \sqcap \Psi_K = \Psi_K$,
- (ii) $\Psi_K \sqcap \Psi_{\tilde{\Phi}} = \Psi_{\tilde{\Phi}}$,
- (iii) $\Psi_K \sqcap \Psi_{\tilde{E}} = \Psi_K$,
- (iv) $\Psi_K \sqcap \Omega_L = \Omega_L \sqcap \Psi_K$,
- (v) $(\Psi_K \sqcap \Omega_L) \sqcap \gamma_M = \Psi_K \sqcap (\Omega_L \sqcap \gamma_M)$.

Proof. It is clear.

Proposition 3.17. *Let Ψ_K, Ω_L and $\gamma_M \in NP$ -soft sets. Then*

- (i) $\Psi_K \sqcup (\Omega_L \sqcap \gamma_M) = (\Psi_K \sqcup \Omega_L) \sqcap (\Psi_K \sqcup \gamma_M)$,

$$(ii) \Psi_K \sqcap (\Omega_L \sqcup \gamma_M) = (\Psi_K \sqcup \Omega_L) \sqcup (\Psi_K \sqcup \gamma_M).$$

Proof. It can be proved by Definitions 3.8 and 3.9.

Proposition 3.18. Let $\Psi_K, \Omega_L \in NP$ -soft set. Then

$$(i) (\Psi_K \sqcup \Omega_L)^c = \Psi_K^c \sqcap \Omega_L^c,$$

$$(ii) (\Psi_K \sqcap \Omega_L)^c = \Psi_K^c \sqcup \Omega_L^c.$$

Proof. It is clear.

Definition 3.19. Let $\Psi_K, \Omega_L \in NP$ -soft set. Then

(i) An OR-product of Ψ_K and Ω_L denoted by $\Psi_K \underline{\vee} \Omega_L$ is defined as follows

$$\Psi_K \underline{\vee} \Omega_L = \{(\langle (x, y), (\max\{\mu_K(x), \mu_L(y)\}, \min\{\nu_K(x), \nu_L(y)\}, \min\{\omega_K(x), \omega_L(y)\}) \rangle, \Psi_K \cup \Omega_L(x, y)) : x, y \in E\},$$

$$\text{where } \Psi_K \cup \Omega_L(x, y) = \Psi_K(x) \cup \Omega_L(y).$$

(ii) An AND-product of Ψ_K and Ω_L , denoted by $\Psi_K \overline{\wedge} \Omega_L$, is defined as follows

$$\Psi_K \overline{\wedge} \Omega_L = \{(\langle (x, y), (\min\{\mu_K(x), \mu_L(y)\}, \max\{\nu_K(x), \nu_L(y)\}, \max\{\omega_K(x), \omega_L(y)\}) \rangle, \Psi_K \cap \Omega_L(x, y)) : x, y \in E\},$$

$$\text{where } \Psi_K \cap \Omega_L(x, y) = \Psi_K(x) \cap \Omega_L(y).$$

Proposition 3.20. Let Ψ_K, Ω_L and $\gamma_M \in NP$ -soft sets. Then

$$(i) \Psi_K \overline{\wedge} \Psi_\Phi = \Psi_\Phi$$

$$(ii) (\Psi_K \overline{\wedge} \Omega_L) \overline{\wedge} \gamma_M = \Psi_K \overline{\wedge} (\Omega_L \overline{\wedge} \gamma_M)$$

$$(iii) (\Psi_K \underline{\vee} \Omega_L) \underline{\vee} \gamma_M = \Psi_K \underline{\vee} (\Omega_L \underline{\vee} \gamma_M).$$

Proof. It can be proved by Definition 3.15.

4. NP-aggregation operator

In this section, we define an NP-aggregation operator of an NP-soft set to construct a decision method by which approximate functions of a soft set are combined to produce a single neutrosophic set that can be used to evaluate each alternative.

Definition 4.1. Let $\Psi_K \in NP$ -soft set. Then an NP-aggregation operator of Ψ_K , denoted by Ψ_K^{agg} , is defined by

$$\Psi_K^{agg} = \{(\langle u, \mu_K^{agg}(u), \nu_K^{agg}(u), \omega_K^{agg}(u) \rangle) : u \in U\},$$

which is a neutrosophic set over U ,

$$\begin{aligned}\mu_K^{agg} : U &\rightarrow [0, 1] & \mu_K^{agg}(u) &= \frac{1}{|U|} \sum_{\substack{x \in E \\ u \in U}} \mu_K(x) \gamma_{f_K(x)}(u), \\ \nu_K^{agg} : U &\rightarrow [0, 1] & \nu_K^{agg}(u) &= \frac{1}{|U|} \sum_{\substack{x \in E \\ u \in U}} \nu_K(x) \gamma_{f_K(x)}(u),\end{aligned}$$

and

$$\omega_K^{agg} : U \rightarrow [0, 1] \quad \omega_K^{agg}(u) = \frac{1}{|U|} \sum_{\substack{x \in E \\ u \in U}} \omega_K(x) \gamma_{f_K(x)}(u),$$

and where

$$\gamma_{f_K(x)}(u) = \begin{cases} 1, & x \in f_K(x), \\ 0, & \text{otherwise.} \end{cases}$$

$|U|$ is the cardinality of U .

Definition 4.2. Let $\Psi_K \in NP$ -soft set and Ψ_K^{agg} an aggregation neutrosophic parameterized soft set, then a reduced fuzzy set of Ψ_K^{agg} is a fuzzy set over U denoted by

$$\Psi_K^{agg} = \left\{ \frac{\mu_{\Psi_{Kf}}^{agg}(u)}{u} : u \in U \right\},$$

where $\mu_{\Psi_{Kf}}^{agg}(u) : U \rightarrow [0, 1]$ and $\mu_{\Psi_{Kf}}^{agg}(u) + \nu_K^{agg}(u) - \omega_K^{agg}(u)$.

5. NP-Decision methods

Inspired by the decision making methods regard in [13]-[15]. In this section, we also present an NP-decision method to a neutrosophic parameterized soft set. Based on Definitions 4.1 and 4.2, we construct an NP-decision making method by the following algorithm. Now, we construct an NP-soft decision making method by the following algorithm to produce a decision fuzzy set from a crisp set of the alternatives.

According to the problem, the decision maker

- (i) constructs a feasible Neutrosophic subsets K over the parameters set E ,
- (ii) constructs an NP-soft set Ψ_K over the alternatives set U ,
- (iii) computes the aggregation neutrosophic parameterized soft set Ψ_K^{agg} of Ψ_K ,
- (iv) computes the reduced fuzzy set $\mu_{\Psi_{Kf}}^{agg}(u)$ of Ψ_K^{agg} ,
- (v) chooses the element of $\mu_{\Psi_{Kf}}^{agg}(u)$ that has maximum membership degree.

Now, we can give an example for the NP-soft decision making method.

Example. Assume that a company wants to fill a position. There are four candidates who fill in a form in order to apply formally for the position. There is a decision maker (DM) that is from the department of human resources. He wants to interview the candidates, but it is very difficult to make it all of them. Therefore, by using the NP-soft decision making method, the number of candidates are reduced to a suitable one. Assume that the set of candidates $U = \{u_1, u_2, u_3, u_4\}$ which may be characterized by a set of parameters $E = \{x_1, x_2, x_3\}$. For $i = 1, 2, 3$, the parameters i stand for experience, computer knowledge and young age, respectively. Now, we can apply the method as follows:

Step i. Assume that DM constructs a feasible neutrosophic subsets K over the parameters set E as follows:

$$K = \{\langle x_1, 0.2, 0.3, 0.4 \rangle, \langle x_2, 0.3, 0.2, 0.4 \rangle, \langle x_3, 0.5, 0.2, 0.3 \rangle\}.$$

Step ii. DM constructs an NP-soft set Ψ_K over the alternatives set U as follows:

$$\Psi_K = \{(\langle x_1, 0.2, 0.3, 0.4 \rangle, \{u_1, u_2\}), (\langle x_2, 0.3, 0.2, 0.4 \rangle, \{u_2, u_3, u_4\}), \\ (\langle x_3, 0.5, 0.2, 0.3 \rangle, \{u_3\})\}.$$

Step iii. DM computes the aggregation neutrosophic parameterized soft set Ψ_K^{agg} of Ψ_K as follows:

$$\Psi_K^{agg} = \{\langle u_1, 0.05, 0.075, 0.1 \rangle, \langle u_2, 0.1, 0.125, 0.2 \rangle, \langle u_3, 0.2, 0.1, 0.175 \rangle, \\ \langle u_4, 0.125, 0.05, 0.075 \rangle\}.$$

Step iv. Computes the reduced fuzzy set $\mu_{\Psi_{Kf}^{agg}(u)}$ of Ψ_K^{agg} as follows:

$$\mu_{\Psi_{Kf}^{agg}(u_1)} = 0.025$$

$$\mu_{\Psi_{Kf}^{agg}(u_2)} = 0.025$$

$$\mu_{\Psi_{Kf}^{agg}(u_3)} = 0.125$$

$$\mu_{\Psi_{Kf}^{agg}(u_4)} = 0.1$$

Step v. Finally, DM chooses u_3 for the position from $\mu_{\Psi_{Kf}^{agg}(u)}$ since it has the maximum degree 0.125 among the others.

6. Conclusion

In this work, we have introduced the concept of neutrosophic parameterized soft set and studied some of its properties. The complement, union and intersection operations have been defined on the neutrosophic parameterized soft set. The definition of NP-aggregation operator is introduced with application of this operation in decision making problems.

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Accepted: 22.04.2014

YOUNG TYPE INEQUALITIES FOR MATRICES

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Abstract. In this note, we present a refinement of an inequality due to Hirzallah and Kittaneh [*Linear Algebra Appl.*, 308 (2000), 77-84]. Meanwhile, we also obtain an improvement of a result shown by Kittaneh and Manasrah [*Linear Multilinear Algebra*, 59 (2011), 1031-1037].

Keywords: Young type inequalities; Hilbert-Schmidt norm; positive semidefinite matrices; Kantorovich constant.

MSC (2010) Subject Classification: 15A60.

1. Introduction

Let M_n be the space of $n \times n$ complex matrices. For $A = (a_{ij}) \in M_n$, the Hilbert-Schmidt norm of A is defined by

$$\|A\|_2 = \sqrt{\sum_{i,j=1}^n |a_{ij}|^2}.$$

It is known that the Hilbert-Schmidt norm is unitarily invariant.

The classical Young inequality for scalar says that if $a, b \geq 0$ and $0 \leq v \leq 1$, then

$$a^v b^{1-v} \leq va + (1-v)b$$

with equality if and only if $a = b$. By using Young's inequality, we can obtain some results of Heinz mean. For more information on Heinz inequality for matrices the reader is referred to [1]–[3].

The Kantorovich constant is defined as

$$K(t, 2) = \frac{(t+1)^2}{4t}$$

for $t > 0$. Zuo, Shi, Fujii [4] proved that if $a, b \geq 0$ and $0 \leq v \leq 1$, then

$$(1.1) \quad K(h, 2)^r a^v b^{1-v} \leq va + (1-v)b,$$

where $h = \frac{a}{b}$, $r = \min\{v, 1-v\}$. This is a refinement of the classical Young inequality.

Let $A, X, B \in M_n$ such that A and B are positive semidefinite. Kosaki [5] and Bhatia-Parthasarathy [6] proved that if $0 \leq v \leq 1$, then

$$(1.2) \quad \|A^v X B^{1-v}\|_2^2 \leq \|vAX + (1-v)XB\|_2^2.$$

This is a matrix version of Young inequality. Hirzallah and Kittaneh [7] proved that if $0 \leq v \leq 1$, then

$$(1.3) \quad \|A^v X B^{1-v}\|_2^2 + v_0^2 \|AX - XB\|_2^2 \leq \|vAX + (1-v)XB\|_2^2,$$

where $v_0 = \min\{v, 1-v\}$. Inequality (1.3) is an improvement of inequality (1.2). Kittaneh-Manasrah [8] and He-Zou [9] proved that if $0 \leq v \leq 1$, then

$$(1.4) \quad \|vAX + (1-v)XB\|_2^2 \leq \|A^v X B^{1-v}\|_2^2 + s_0^2 \|AX - XB\|_2^2,$$

where $s_0 = \max\{v, 1-v\}$. Inequality (1.4) is a reverse inequality of (1.3).

In this note, we present refinements of inequalities (1.3) and (1.4).

2. Main results

In this section, we first give a refinement of inequality (1.3). To achieve it, we need the following lemma.

Lemma 2.1. *If $a, b \geq 0$ and $0 \leq v \leq 1$, then*

$$(2.1) \quad K(h, 2)^r (a^v b^{1-v})^2 + v_0^2 (a-b)^2 \leq (va + (1-v)b)^2,$$

where $h = \frac{a}{b}$, $v_0 = \min\{v, 1-v\}$, $r = \min\{2v_0, 1-2v_0\}$.

Proof. If $v = \frac{1}{2}$, inequality (2.1) becomes equality. If $v < \frac{1}{2}$, then by (1.1), we have

$$\begin{aligned} (va + (1-v)b)^2 - v_0^2 (a-b)^2 &= (va + (1-v)b)^2 - v^2 (a-b)^2 \\ &= 2vab + (1-2v)b^2 \\ &\geq K(h, 2)^r (a^v b^{1-v})^2. \end{aligned}$$

If $v > \frac{1}{2}$, then by (1.1), we have

$$\begin{aligned} (va + (1-v)b)^2 - v_0^2 (a-b)^2 &= (va + (1-v)b)^2 - (1-v)^2 (a-b)^2 \\ &= (2v-1)a^2 + 2(1-v)ab \\ &\geq K(h, 2)^r (a^v b^{1-v})^2. \end{aligned}$$

This completes the proof. ■

Theorem 2.2. Let $A, X, B \in M_n$ such that A and B are positive semidefinite. Suppose that the spectral decomposition of A, B are $A = U\Lambda_1U^*$, $B = V\Lambda_2V^*$ respectively, where $\Lambda_1 = \text{diag}(\lambda_1, \dots, \lambda_n)$, $\Lambda_2 = \text{diag}(\mu_1, \dots, \mu_n)$, $\lambda_i, \mu_i \geq 0$, $i = 1, \dots, n$. Let

$$K = \min \left\{ K \left(\frac{\lambda_i}{\mu_j}, 2 \right), i, j = 1, \dots, n \right\}.$$

Then

$$(2.2) \quad K^r \|A^v X B^{1-v}\|_2^2 + v_0^2 \|AX - XB\|_2^2 \leq \|vAX + (1-v)XB\|_2^2,$$

where $v_0 = \min\{v, 1-v\}$, $r = \min\{2v_0, 1-2v_0\}$.

Proof. Let $Y = U^*XV = [y_{ij}]$. We have as in [10],

$$\begin{aligned} \|vAX + (1-v)XB\|_2^2 &= \sum_{i,j=1}^n (v\lambda_i + (1-v)\mu_j)^2 |y_{ij}|^2, \\ \|A^v X B^{1-v}\|_2^2 &= \sum_{i,j=1}^n (\lambda_i^v \mu_j^{1-v})^2 |y_{ij}|^2, \\ \|AX - XB\|_2^2 &= \sum_{i,j=1}^n (\lambda_i - \mu_j)^2 |y_{ij}|^2. \end{aligned}$$

Inequality (2.2) deduces from inequality (2.1) and above equalities. This completes the proof. $\blacksquare$

Next, we show an improvement of inequality (1.4). To do this, we need the following lemma.

Lemma 2.3. If $a, b \geq 0$ and $0 \leq v \leq 1$, then

$$(2.3) \quad (va + (1-v)b)^2 \leq K(h, 2)^{-r} (a^v b^{1-v})^2 + s_0^2 (a-b)^2,$$

where $h = \frac{a}{b}$, $s_0 = \max\{v, 1-v\}$, $r = \min\{2s_0 - 1, 2 - 2s_0\}$.

Proof. If $v = \frac{1}{2}$, inequality (2.3) becomes equality. If $v < \frac{1}{2}$, then by (1.1), we have

$$\begin{aligned} s_0^2 (a-b)^2 - (va + (1-v)b)^2 &= (1-v)^2 (a-b)^2 - (va + (1-v)b)^2 \\ &= (1-2v)a^2 - 2(1-v)ab \\ &= (1-2v)a^2 + 2vab - 2ab \\ &\geq K(h, 2)^r (a^{1-v}b^v)^2 - 2ab \\ &\geq -K(h, 2)^{-r} (a^v b^{1-v})^2. \end{aligned}$$

If $v > \frac{1}{2}$, then by (1.1), we have

$$\begin{aligned} s_0^2 (a-b)^2 - (va + (1-v)b)^2 &= v^2 (a-b)^2 - (va + (1-v)b)^2 \\ &= (2v-1)b^2 - 2vab \\ &= (2v-1)b^2 + 2(1-v)ab - 2ab \\ &\geq K(h, 2)^r (a^{1-v}b^v)^2 - 2ab \\ &\geq -K(h, 2)^{-r} (a^v b^{1-v})^2. \end{aligned}$$

This completes the proof. $\blacksquare$

Theorem 2.4. Let $A, X, B \in M_n$ such that A and B are positive semidefinite. Suppose that the spectral decomposition of A, B are $A = U\Lambda_1 U^*$, $B = V\Lambda_2 V^*$ respectively, where $\Lambda_1 = \text{diag}(\lambda_1, \dots, \lambda_n)$, $\Lambda_2 = \text{diag}(\mu_1, \dots, \mu_n)$, $\lambda_i, \mu_i \geq 0$, $i = 1, \dots, n$. Let

$$K = \min \left\{ K \left(\frac{\lambda_i}{\mu_j}, 2 \right), i, j = 1, \dots, n \right\}.$$

Then

$$(2.4) \quad \|vAX + (1-v)XB\|_2^2 \leq K^{-r} \|A^v X B^{1-v}\|_2^2 + s_0^2 \|AX - XB\|_2^2,$$

where $s_0 = \max\{v, 1-v\}$, $r = \min\{2v_0, 1-2v_0\}$.

Proof. The result follows from inequality (2.3) and by using an argument similar to that used for the proof of Theorem 2.2. This completes the proof. $\blacksquare$

Since $K(t, 2) = \frac{(t+1)^2}{4t} \geq 1$ for $t > 0$, it follows that inequalities (2.2) and (2.4) are refinements of inequalities (1.3) and (1.4) respectively.

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Accepted: 06.05.2014

BOUNDS FOR THE EIGENVALUES OF MATRICES

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Abstract. In this paper, we prove that all the eigenvalues of arbitrarily complex matrix lies in a closed disk with the radius involving the sum of the squares of the absolute values of the eigenvalues. So, any known upper bound on this sum of squares yields a "new" eigenvalues inclusion region. As applications, some existing results are obtained or improved.

Keywords: eigenvalues; Frobenius norm; trace; disk.

MSC (2010) Subject Classification: 15A18; 15A60.

1. Introduction

Let M be an $n \times n$ complex matrix and λ_i ($i = 1, \dots, n$) be the eigenvalues of M . We denote by $\|M\|_F$, M^* and $\text{tr} M$ the Frobenius norm, conjugate transpose and trace of M , respectively. Let $i \in \mathbb{N}$ and $R_i(M) = \sum_{j=1, j \neq i}^n |m_{ij}|$ be the sum of the absolute values of the non-diagonal entries in the i th row of $M = [m_{ij}]_{n \times n}$. For $z = a + bi \in \mathbb{C}$, the conjugate of z is denoted by $\bar{z} = a - bi$.

Estimation of eigenvalues has been, and still is, a hot topic of matrix analysis. There are many research papers published in a variety of journals each year, and different approaches have been taken for different purposes. Here, we briefly review three types of estimation methods for eigenvalues of matrices.

The well-known upper bound for $\sum_{i=1}^n |\lambda_i|^2$ was presented by Schur in 1909 as follows:

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$$(1.1) \quad \sum_{i=1}^n |\lambda_i|^2 \leq \|M\|_F^2,$$

which is the first bound for $\sum_{i=1}^n |\lambda_i|^2$.

The famous Geršgorin disk theorem says that every eigenvalue of M lies in at least one of the Geršgorin discs centered at m_{ii} with radius $R_i(M)$; i.e.,

$$\lambda_i \in \bigcup_{i=1}^n \{z \in \mathbb{C} : |z - m_{ii}| \leq R_i(M)\}, \quad i = 1, \dots, n.$$

The Geršgorin disks are a particular class of easily computed regions in the plane that are guaranteed to include the eigenvalues of a given matrix. Many authors, perhaps attracted by the geometrical elegance of the Geršgorin theory, have generalized the ideas and methods of this theory to obtain other types of inclusion [1, Chapter 6].

In 1994, Gu [2, Theorem 1] proved that every eigenvalue of M lies in the following disk centered at $\frac{\text{tr} M}{n}$:

$$(1.2) \quad \left\{ z \in \mathbb{C} : \left| z - \frac{\text{tr} M}{n} \right| \leq \sqrt{\frac{n-1}{n} \left(\|M\|_F^2 - \frac{|\text{tr} M|^2}{n} \right)} \right\}.$$

Bhatia [3, p.24] pointed that results such as (1.2) are interesting because they give some information about the location of the eigenvalues of a matrix in terms of more easily computable function like the Frobenius norm and the trace.

In this paper, we prove that all the eigenvalues of M lies in the following disk centered at $\frac{\text{tr} M}{n}$:

$$(1.3) \quad \left\{ z \in \mathbb{C} : \left| z - \frac{\text{tr} M}{n} \right| \leq \sqrt{\frac{n-1}{n} \left(\sum_{i=1}^n |\lambda_i|^2 - \frac{|\text{tr} M|^2}{n} \right)} \right\}.$$

The radius involves $\sum_{i=1}^n |\lambda_i|^2$, thus (1.3) gives the relationship between the first and third types of estimation methods. Any known upper bound for $\sum_{i=1}^n |\lambda_i|^2$ yields a "new" estimation of eigenvalues such as (1.2). As applications, some existing estimations are directly obtained by (1.3).

2. Main result

Theorem 2.1. *Let M be an $n \times n$ complex matrix. Then all the eigenvalues of M are located in the following disk:*

$$\left\{ z \in \mathbb{C} : \left| z - \frac{\text{tr} M}{n} \right| \leq \sqrt{\frac{n-1}{n} \left(\sum_{i=1}^n |\lambda_i|^2 - \frac{|\text{tr} M|^2}{n} \right)} \right\}.$$

Proof. We show that λ_1 is located in this disk, and similarly for others. By the Cauchy–Schwarz inequality, we have

$$\left| \sum_{i=2}^n \lambda_i \right|^2 \leq \sum_{i=2}^n 1^2 \cdot \sum_{i=2}^n |\lambda_i|^2 \leq (n-1) \sum_{i=2}^n |\lambda_i|^2.$$

Then

$$\begin{aligned} (2.1) \quad (n-1) |\lambda_1|^2 + \left| \sum_{i=2}^n \lambda_i \right|^2 &\leq (n-1) |\lambda_1|^2 + (n-1) \sum_{i=2}^n |\lambda_i|^2 \\ &= (n-1) \sum_{i=1}^n |\lambda_i|^2. \end{aligned}$$

Since

$$\left| \sum_{i=2}^n \lambda_i \right|^2 = \left| \sum_{i=1}^n \lambda_i \right|^2 - \lambda_1 \sum_{i=1}^n \bar{\lambda}_i - \bar{\lambda}_1 \sum_{i=1}^n \lambda_i + |\lambda_1|^2,$$

by (2.1), we have

$$n |\lambda_1|^2 - \lambda_1 \sum_{i=1}^n \bar{\lambda}_i - \bar{\lambda}_1 \sum_{i=1}^n \lambda_i + \left| \sum_{i=1}^n \lambda_i \right|^2 \leq (n-1) \sum_{i=1}^n |\lambda_i|^2,$$

which implies

$$n |\lambda_1|^2 - \lambda_1 \sum_{i=1}^n \bar{\lambda}_i - \bar{\lambda}_1 \sum_{i=1}^n \lambda_i + \frac{1}{n} \left| \sum_{i=1}^n \lambda_i \right|^2 \leq (n-1) \sum_{i=1}^n |\lambda_i|^2 - \frac{n-1}{n} \left| \sum_{i=1}^n \lambda_i \right|^2,$$

and then

$$|\lambda_1|^2 - \frac{\lambda_1}{n} \sum_{i=1}^n \bar{\lambda}_i - \frac{\bar{\lambda}_1}{n} \sum_{i=1}^n \lambda_i + \frac{1}{n^2} \left| \sum_{i=1}^n \lambda_i \right|^2 \leq \frac{n-1}{n} \sum_{i=1}^n |\lambda_i|^2 - \frac{n-1}{n^2} \left| \sum_{i=1}^n \lambda_i \right|^2.$$

Consequently,

$$\left(\lambda_1 - \frac{1}{n} \sum_{i=1}^n \lambda_i\right) \left(\overline{\lambda_1} - \frac{1}{n} \sum_{i=1}^n \overline{\lambda_i}\right) \leq \frac{n-1}{n} \left(\sum_{i=1}^n |\lambda_i|^2 - \frac{1}{n} \left|\sum_{i=1}^n \lambda_i\right|^2\right),$$

which is equivalent to

$$\left|\lambda_1 - \frac{\operatorname{tr} M}{n}\right| \leq \sqrt{\frac{n-1}{n} \left(\sum_{i=1}^n |\lambda_i|^2 - \frac{|\operatorname{tr} M|^2}{n}\right)}.$$

The proof is completed. ■

3. Applications

In section, we give some applications of our result.

Corollary 3.1. [2, Theorem 1]. *Let M be an $n \times n$ complex matrix. Then all the eigenvalues of M are located in the following disk:*

$$\left\{z \in \mathbb{C} : \left|z - \frac{\operatorname{tr} M}{n}\right| \leq \sqrt{\frac{n-1}{n} \left(\|M\|_F^2 - \frac{|\operatorname{tr} M|^2}{n}\right)}\right\}.$$

This result is a corollary of Theorem 2.1 and the Schur's inequality (1.1).

Let M be an $n \times n$ complex matrix partitioned as

$$M = \begin{bmatrix} A_{k \times k} & B_{k \times (n-k)} \\ C_{(n-k) \times k} & D_{(n-k) \times (n-k)} \end{bmatrix}, \quad 1 \leq k \leq n-1,$$

where $A_{k \times k}$ is the principal submatrix of M . Tu [4, Theorem 1] obtained the following result:

$$(3.1) \quad \sum_{i=1}^n |\lambda_i|^2 \leq \|M\|_F^2 - \max_{1 \leq k \leq n-1} (\|B_{k \times (n-k)}\| - \|C_{(n-k) \times k}\|)^2.$$

The following result obtained by Zou and Jiang is a corollary of Theorem 2.1 and (3.1).

Corollary 3.2. [5, Theorem 2.1] *Let M be an $n \times n$ complex matrix. Then all the eigenvalues of M are located in the following disk:*

$$\left\{ z \in \mathbb{C} : \left| z - \frac{\operatorname{tr} M}{n} \right| \leq R \right\},$$

where

$$R = \sqrt{\frac{n-1}{n} \left(\|M\|_F^2 - \max_{1 \leq k \leq n-1} (\|B_{k \times (n-k)}\|_F - \|C_{(n-k) \times k}\|_F)^2 - \frac{|\operatorname{tr} M|^2}{n} \right)}.$$

Kress et al. [6, Theorem 1] obtained an upper bound for $\sum_{i=1}^n |\lambda_i|^2$ as follows:

$$(3.2) \quad \sum_{i=1}^n |\lambda_i|^2 \leq \sqrt{\|M\|_F^4 - \frac{1}{2} \|MM^* - M^*M\|_F^2}.$$

If M is non-normal, then (3.2) is obviously sharper than (1.1). The following result is a corollary of Theorem 2.1 and (3.2).

Corollary 3.3. *Let M be an $n \times n$ complex matrix. Then all the eigenvalues of M are located in the following disk:*

$$(3.3) \quad \left\{ z \in \mathbb{C} : \left| z - \frac{\operatorname{tr} M}{n} \right| \leq R \right\},$$

where

$$R = \sqrt{\frac{n-1}{n} \left(\sqrt{\|M\|_F^4 - \frac{1}{2} \|MM^* - M^*M\|_F^2} - \frac{|\operatorname{tr} M|^2}{n} \right)}.$$

Note that the inequality (3.3) is a refinement of (1.2) for non-normal matrices.

Acknowledgments. The author wishes to express his heartfelt thanks to the referee and Professor Piergiulio Corsini for their detailed and helpful suggestions for revising the manuscript. This research was supported by Scientific and Technological Research Program of Chongqing Municipal Education Commission (Grant No. KJ131122) and Scientific Research Project of Chongqing Three Gorges University (No. 12ZD-17).

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Accepted: 06.05.2014

ON THOMPSON'S CONJECTURE FOR ALTERNATING GROUP A_{26}

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Abstract. Let G be a group. Denote by $N(G)$ the set of nonidentity orders of conjugacy classes of elements in G . For groups A_{10} , A_{16} and A_{22} which are uniquely determined by $N(G)$, theses degrees are $p+3$ and $p+4$ is a prime with $p = 7, 13, 19$. If $p+4$ is composite, then whether can the groups A_{p+3} be characterized by $N(G)$. In this paper, we give an example for A_{p+3} with $p+4$ composite, namely, we proved that if G is a group with trivial center and $N(G) = N(A_{26})$, then $G \cong A_{26}$.

Keywords and phrases: element order, alternating group, thompson's conjecture, conjugacy classes, simple group.

AMS Subject Classification: 20D05, 20D06, 20D20.

1. Introduction

Let G be a finite group. Denote by $N(G)$ the set of nonidentity orders of conjugacy classes of elements in G . Related to $N(G)$, J.G. Thompson gave the following conjecture.

Thompson's Conjecture. (see [11, Question 12.38]). *If L is a finite simple non-Abelian group, G is a finite group with trivial center, and $N(G) = N(L)$, then $G \cong L$.*

Let $\pi(G)$ denote the set of all prime divisors of $|G|$. Let $GK(G)$ be a graph with vertex set $\pi(G)$ such that two primes p and q in $\pi(G)$ are joined by an edge if G has an element of order $p \cdot q$. We set $s(G)$ denote the number of connected components of the prime graph $GK(G)$. A classification of all finite simple groups with disconnected prime graph was obtained in [10], [14]. Based on these results,

Thompson's conjecture was proved valid for all finite simple groups with $s(G) \geq 2$ (see [2], [3]). So whether there is a group with connected prime graph for which Thompson's conjecture would be true? Recently, the groups A_{10} , A_{16} and A_{22} were proved valid for this conjecture (see [13], [6] and [15], respectively). As the development of this topics, we will prove that Thompson's conjecture is true for the alternating group A_{26} of degree 26.

We will introduce some notations used to the proof of the main theorem. Let A_n and S_n denote the alternating and symmetric groups of degree n , respectively. Let G be a group. Set $\text{Aut}(G)$ denotes the automorphism group of G . Let $\omega(G)$ denote the set of element order of G . The notations are standard (see [5], for instance).

2. Some lemmas

In this section we will give some preliminary results.

Lemma 2.1 [13, Lemma 1.2] and [1, Lemma 2.3] *Let $x, y \in G$, $(|x|, |y|) = 1$, and $xy = yx$. Then*

- (1) $C_G(xy) = C_G(x) \cap C_G(y)$;
- (2) $|x^G|$ divides $|(xy)^G|$;
- (3) *If $|x^G| = |(xy)^G|$, then $C_G(x) \leq C_G(y)$.*

Lemma 2.2 [13, Lemma 3] *If P and H are finite groups with trivial centers, and $N(P) = N(H)$, then $\pi(P) = \pi(H)$.*

Lemma 2.3 [13, Lemma 4] *Suppose that G is a finite group with trivial center and p is a prime from $\pi(G)$ such that p^2 does not divide $|x^G|$ for all x in G . Then a Sylow p -subgroup of G is elementary abelian.*

Lemma 2.4 [13, Lemma 5] *Let K be a normal subgroup of G , and $\overline{G} = G/K$.*

- (1) *If $\overline{x}$ is the image of an element x of G in $\overline{G}$. Then $|\overline{x}^{\overline{G}}|$ divides $|x^G|$.*
- (2) *If $(|x|, |K|) = 1$, then $C_{\overline{G}}(\overline{x}) = C_G(x)K/K$.*
- (3) *If $y \in K$, then $|y^K|$ divides $|y^G|$.*

3. Main theorem and its proof

In this section, we give the main theorem and its proof.

Theorem 3.1 *Let G be a finite group with trivial center and $N(G) = N(A_{26})$. Then G is isomorphic to A_{26} .*

Proof. We divide the proof into the following lemmas.

Lemma 3.2 *Let $L = A_{26}$. Then the following hold.*

- (1) $|L| = 2^{22} \cdot 3^{10} \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23$.
- (2) numbers $\frac{26!}{23 \cdot 3}$ and $\frac{26 \cdot 25 \cdot 24}{3}$ are maximal and minimal with respect to divisibility in the set $N(L)$, respectively.
- (3) for any $n \in N(L)$, numbers 17^2 , 19^2 and 23^2 do not divide n .
- (4) $23'$ -numbers in $N(L) \setminus \{1\}$ are

$$2^4 \cdot 5^2 \cdot 13,$$

$$2^{22} \cdot 3^9 \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19.$$
 $19'$ -numbers in $N(L) \setminus \{1\}$ are

$$2^4 \cdot 5^2 \cdot 13,$$

$$2^7 \cdot 3^2 \cdot 5^3 \cdot 11 \cdot 13 \cdot 23,$$

$$2^4 \cdot 3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13 \cdot 23,$$

$$2^2 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 23,$$

$$2^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 23,$$

$$2^5 \cdot 3 \cdot 5 \cdot 11 \cdot 13 \cdot 23,$$

$$2 \cdot 3 \cdot 5^2 \cdot 13 \cdot 23,$$

$$2^{20} \cdot 3^{10} \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 23,$$

$$2^{22} \cdot 3^8 \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 23,$$

$$2^{19} \cdot 3^9 \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23,$$

$$2^{20} \cdot 3^9 \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 23,$$

$$2^{22} \cdot 3^{10} \cdot 5^6 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 23.$$
 $17'$ -numbers in $N(L) \setminus \{1\}$ are

$$2^4 \cdot 5^2 \cdot 13,$$

$$2^7 \cdot 3^2 \cdot 5^3 \cdot 11 \cdot 13 \cdot 23,$$

$$2^4 \cdot 3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13 \cdot 23,$$

$$2^2 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 23,$$

$$2^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 23,$$

$$2^5 \cdot 3 \cdot 5 \cdot 11 \cdot 13 \cdot 23,$$

$$2 \cdot 3 \cdot 5^2 \cdot 13 \cdot 23,$$

$$2^8 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 23,$$

$$2^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 23,$$

$$2^5 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 23,$$

$$\begin{aligned}
& 2^5 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 23, \\
& 2^5 \cdot 3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 23, \\
& 2^7 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 23, \\
& 2^{22} \cdot 3^8 \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 19 \cdot 23, \\
& 2^{20} \cdot 3^9 \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 19 \cdot 23, \\
& 2^{20} \cdot 3^{10} \cdot 5^5 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 19 \cdot 23, \\
& 2^{22} \cdot 3^6 \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 19 \cdot 23, \\
& 2^{18} \cdot 3^{10} \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 19 \cdot 23, \\
& 2^{22} \cdot 3^{10} \cdot 5^6 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 19 \cdot 23, \\
& 2^{19} \cdot 3^9 \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 19 \cdot 23, \\
& 2^{21} \cdot 3^7 \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 19 \cdot 23, \\
& 2^{22} \cdot 3^9 \cdot 5^5 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 19 \cdot 23, \\
& 2^{21} \cdot 3^9 \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 19 \cdot 23, \\
& 2^{20} \cdot 3^9 \cdot 5^5 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 19 \cdot 23, \\
& 2^{17} \cdot 3^9 \cdot 5^5 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 19 \cdot 23, \\
& 2^{19} \cdot 3^8 \cdot 5^5 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 19 \cdot 23, \\
& 2^{15} \cdot 3^6 \cdot 5^5 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 19 \cdot 23.
\end{aligned}$$

(5) *the numbers*

$$\begin{aligned}
& 2^{22} \cdot 3^{10} \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 17 \cdot 19 \cdot 23, \\
& 2^{22} \cdot 3^8 \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 19 \cdot 23, \\
& 2^{22} \cdot 3^{10} \cdot 5^6 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 23, \\
& 2^{22} \cdot 3^9 \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19
\end{aligned}$$

are maximal elements of $N(L)$ by divisibility.

(6) *for any $n \in N(L) \setminus \{1\}$, n is divisible by 11 or 13.*

Proof. The information is obtained via GAP [7] or [8]. ■

Lemma 3.3 *Let G be a finite group with trivial center and $N(G) = N(L)$. Then $|L| \mid |G|$ and $\pi(G) = \pi(L) = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$.*

Proof. Since $|x^G| |C_G(x)| = |G|$, then every member form $N(G)$ divides the order of G and $|L| \mid |G|$. So by Lemmas 2.2 and 3.2, we have that $\pi(G) = \pi(L)$. ■

Lemma 3.4 *Suppose that G is a finite group with trivial center and $N(G) = N(L)$. Let $p \in \{17, 19, 23\}$. Then the Sylow p -subgroup S of G is of order p . There does not exist an element of order $17 \cdot 19$, $17 \cdot 23$ and $19 \cdot 23$.*

Proof. By Lemma 2.3, S is elementary abelian. If $|x| = p$, then we have that $|x^G|$ is a p' -number.

Suppose that $p = 23$ and $|S| \geq 23^2$. We can assume that there exists an element y of G with that $|y^G| = 2^{22} \cdot 3^{10} \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 17 \cdot 19 \cdot 23$ by Lemma 3.2.

Assume that $23 \mid |y|$. Let $|y| = 23t$. Since S is elementary abelian, the numbers 23 and t are coprime. Put $u = y^{23}$, $v = y^t$. Then $y = uv$, $C_G(uv) = C_G(u) \cap C_G(v)$ by Lemma 2.1. Therefore $|v^G| \mid |y^G| = 2^{22} \cdot 3^{10} \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 17 \cdot 19 \cdot 23$. On the other hand, the element v is of order 23, and therefore, $|v^G| = 2^4 \cdot 5^2 \cdot 13$, $2^{22} \cdot 3^9 \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19$. It follows that $13 \mid |v^G|$, a contradiction.

Assume that $23 \nmid |y|$. Let x be an element of $C_G(y)$ of order 23. Then $C_G(xy) = C_G(x) \cap C_G(y)$, and therefore $|x^G| \mid |(xy)^G|$ and $|y^G| \mid |(xy)^G|$. Since S is abelian, $S \leq C_G(x)$. It follows that $23 \nmid |x^G|$ and so, $|x^G| = 2^4 \cdot 5^2 \cdot 13$, $2^{22} \cdot 3^9 \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19$, a contradiction.

For cases “ $p = 19$ and $p = 17$ ”, we can get the same result as “ $p = 23$ ”.

There is no element of order $17 \cdot 19$, $17 \cdot 23$, or $19 \cdot 23$ in G . ■

Lemma 3.5 *Suppose that G is a finite group with trivial center and $N(G) = N(L)$. Assume that $p \in \{2, 5, 7, 11, 13\}$, P is a Sylow p -subgroup of G , and $Z(P)$ is its center. Then for every element $x \in Z(P) \setminus \{1\}$, the order of the centralizer $C_G(x)$ is coprime to 23 if $p = 2$, coprime to 23 if $p = 5$, coprime to 23 if $p = 7$, coprime to $17 \cdot 19 \cdot 23$ if $p = 11$, and coprime to $17 \cdot 19 \cdot 23$ if $p = 13$.*

Proof. By Lemma 3.4, $|G|_p = |L|_p$ for $p \in \{17, 19, 23\}$. For any $1 \neq x \in Z(P)$, we get the desired results since $N(G) = N(L)$. ■

Lemma 3.6 *Let G be a finite group and $p \in \pi(G)$. If $p^2 \nmid |G|$, then G has a normal series $1 \leq K \leq L \leq G$, such that L/K is a simple group and $p \in \pi(L/K)$.*

Proof. Since G is a finite group, G has a chief series. So let $G_0 \leq G_1 \leq G_2 \cdots \leq G_r = G$ be a chief series of G . There exists some t , such that $1 \leq t \leq r$ and $p \in \pi(G_t) \setminus \pi(G_{t-1})$. Let $L = G_t$ and $K = G_{t-1}$, then $1 \leq K \leq L \leq G$ is a normal series of G and L/K is a chief factor of G . Therefore L/K is a minimal normal subgroup of G/K . We know that the chief factors are characteristically simple. Also every characteristically simple group is a simple group or a product of isomorphic simple groups. So L/K is a simple group or a product of isomorphic simple groups. Since $p \in \pi(L/K)$ and $p^2 \nmid |L/K|$, it follows that L/K is a simple group. ■

Lemma 3.7 *There is a normal series $1 \leq K \leq L \leq G$ such that $L/K \cong A_{26}$.*

Proof. By Lemma 3.6, we have that there is a normal series $1 \leq K \leq L \leq G$ such that $M := L/K$ is isomorphic to a direct product of nonabelian simple groups $S_1, S_2, \dots, S_n$ which are listed in [16]. Since G has not a Hall $\{17, 19, 23\}$ -subgroup, then the numbers 17, 19 and 23 divide the order of exactly one of these groups. Without generality, we assume that the numbers 17, 19, 23 divide S_1 . Obviously, $S_1 \triangleleft \overline{G}$. Let $G^* = \overline{G}/S_1$ and $M^* = M/S_1$. We prove that $n = 1$.

Assume that $n \geq 2$. Then a Sylow 2-subgroup P_2 of G^* is nontrivial and its center $Z(P_2)$ has a nontrivial intersection with M^* . Let x be a nontrivial element of $S_2 \times \cdots \times S_n$ with that its image in $\overline{G}$ lies in $Z(P_2)$. Since $y \in Z(S_1)$, then there is an element of order $2 \cdot 23$. It follows that there exists an element y with $|y^G| = 2^{19} \cdot 3^{10} \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19$, which contradicts Lemma 3.2(4). So $n = 1$.

Therefore

$$L/K \leq \overline{G} \leq \text{Aut}(L/K)$$

with L/K a nonabelian simple group.

For $p \in \{17, 19, 23\}$, we have that $|G|_p = p$ by Lemma 3.4 and by Lemma 3.6, $p \in \pi(L/K)$. It follows from [16], that $L/K \cong A_n$ where $n = 23, 24, 25, 26, 27, 28$.

If $L/K \cong A_{23}$, then there exist an element $\bar{x}$ of $\overline{G}$ with that

$$\bar{x}^{\overline{G}} = 2^{18} \cdot 3^9 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19.$$

Let x be the preimage of $\bar{x}$ of $\overline{G}$ in G . Then since $|\bar{x}^{\overline{G}}|$ is maximal in $N(G)$, this forces $|x^G| = |\bar{x}^{\overline{G}}|$. It follows that $13 \mid |C_G(x)|$. Then there is an element of order $13 \cdot 23$, and by the proof of Lemma 3.4, we get a contradiction.

Similarly, we can rule out these cases “ $L/K \cong A_{24}$, and $L/K \cong A_{25}$ ”.

If $L/K \cong A_{27}$, then there exist an element $\bar{x}$ of $\overline{G}$ with that

$$\bar{x}^{\overline{G}} = 2^{20} \cdot 3^{12} \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19$$

which contradicts Lemma 3.2(4).

If $L/K \cong A_{28}$, then there exist an element $\bar{x}$ of $\overline{G}$ with that

$$\bar{x}^{\overline{G}} = 2^{22} \cdot 3^{13} \cdot 5^5 \cdot 7^5 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19$$

which contradicts Lemma 3.2(4).

Therefore, $L/K \cong A_{26}$. ■

Lemma 3.8 $G \cong A_{26}$.

Proof. By Lemma 3.7,

$$A_{26} \leq \overline{G} \leq \text{Aut}(A_{26}) \cong S_{26}.$$

If $\overline{G} \cong S_{26}$, then there exist an element $\bar{x}$ of $\overline{G}$ with that

$$\bar{x}^{\overline{G}} = 2^{19} \cdot 3^8 \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19$$

which contradicts Lemma 3.2(4).

So $\overline{G} \cong A_{26}$. Then we define the normal series $1 \leq K \leq G$ into the chief ones. We prove that $K = 1$. If $K \neq 1$, then order consideration, we have that $G/K \cong A_{26}$ and $|K| = 2$. It follows that $Z(G) = K$ and there is an element of order $2 \cdot 23$, a contradiction.

Therefore, $K = 1$ and $G \cong A_{26}$.

This completes the proof of the Lemma and also of the main theorem. ■

4. Some applications and problem

Chen et al in [4] proved that the group A_{10} can be characterized by its order and two special conjugacy classes sizes. Then obviously, we also have the following result.

Corollary 4.1 *Let G be a finite group with trivial center. Assume that $N(G) = N(A_{26})$ and $|G| = |A_{26}|$. Then $G \cong A_{26}$.*

We know that the alternating group A_n with $n = 10, 16, 22, 26$, are characterized by $N(G)$. Then, by [2], [3], we have the following.

Corollary 4.2 *Let G be a finite group with trivial center. Assume that $N(G) = N(A_n)$ with $n \leq 26$. Then $G \cong A_n$.*

Related to **Thompson's conjecture**, we give the following problem.

Problem. *Are the alternating groups A_{p+3} of degree $p+3$ where p is a prime, characterized by $N(G)$?*

Shi gave the following conjecture.

Conjecture [12] *Let G be a group and H a finite simple group. Then $G \cong H$ if and only if (a) $\omega(G) = \omega(H)$ and (b) $|G| = |H|$.*

Then, we have the following corollary.

Corollary 4.3 *Let G is a group and n an integer with $n \leq 26$. Then $G \cong A_n$ if and only if $\omega(G) = \omega(A_n)$ and $|G| = |A_n|$.*

Acknowledgments. The object was supported by the Department of Education of Sichuan Province (Grant No: 12ZB291), by the Opening Project of Sichuan Province University Key Laboratory of Bridge Non-destruction Detecting and Engineering Computing (Grant No: 2013QYJ02) and by the Scientific Research Project of Sichuan University of Science and Engineering (Grant No: 2014RC02). The authors are very grateful for the helpful suggestions of the referee especially in Lemma 10 which give the proof more simple.

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Accepted: 04.06.2014

APPLICATION OF BIPOLAR FUZZY SOFT SETS IN K -ALGEBRAS**Muhammad Akram**

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Abstract. On the basis of the concept of bipolar fuzzy soft sets, a new kind of K -algebra is introduced in this paper. The concepts of bipolar fuzzy soft K -algebras are described and some related properties are investigated. The notion of a generalized bipolar fuzzy soft K -algebra is also introduced and discussed.

Keywords: Soft K -algebras; bipolar fuzzy soft K -subalgebras; $(\in, \in \vee q)$ -bipolar fuzzy soft K -subalgebras.

2000 Mathematics Subject Classification: 20N15, 94D05.

1. Introduction

The notion of a K -algebra $(G, \cdot, \odot, e)$ was first introduced by Dar and Akram [12] in 2003 and published in 2005. A K -algebra is an algebra built on a group $(G, \cdot, e)$ by adjoining an induced binary operation $\odot$ on G which is attached to an abstract K -algebra $(G, \cdot, \odot, e)$. This system is, in general non-commutative and non-associative with a right identity e , if a group is non-commutative. For

a given group, the K -algebra is proper if group is not an elementary abelian 2-group. Thus, a K -algebra is abelian and non-abelian purely depends on the base group. In 2004, Dar and Akram further renamed a K -algebra on a group G as a $K(G)$ -algebra [13] due to its structural basis G . The $K(G)$ -algebras have been characterized by their left and right mappings in [13, 14] when group is abelian and non-abelian.

In 1994, Zhang [29] initiated the concept of bipolar fuzzy sets as a generalization of fuzzy sets [28]. Bipolar fuzzy sets are an extension of fuzzy sets [28] whose membership degree range is $[-1, 1]$. In a bipolar fuzzy set, the membership degree 0 of an element means that the element is irrelevant to the corresponding property, the membership degree $(0, 1]$ of an element indicates that the element somewhat satisfies the property, and the membership degree $[-1, 0)$ of an element indicates that the element somewhat satisfies the implicit counter-property. Although bipolar fuzzy sets and intuitionistic fuzzy sets look similar to each other, they are essentially different sets. In many domains, it is important to be able to deal with bipolar information. It is noted that positive information represents what is granted to be possible, while negative information represents what is considered to be impossible. This domain has recently motivated new research in several directions. In particular, fuzzy and possibilistic formalisms for bipolar information have been proposed [16], because when we deal with spatial information in image processing or in spatial reasoning applications, this bipolarity also occurs. For instance, when we assess the position of an object in a space, we may have positive information expressed as a set of possible places and negative information expressed as a set of impossible places.

In 1999, Molodtsov [23] initiated the novel concept of soft set theory to deal with uncertainties which can not be handled by traditional mathematical tools. Applications of soft set theory in real life problems are now catching momentum due to the general nature parametrization expressed by a soft set. Maji et al. [22] gave first practical application of soft sets in decision making problems. They also presented the definition of fuzzy soft set [21]. Following the concept of soft sets, several authors have applied this concept to algebraic structures [17-20, 30]. Al-Shehri *et al.*[3] has applied this concept to K -algebras. Akram *et al.* introduced the notions of fuzzy soft K -subalgebras in [6]. Moreover, Akram *et al.* [7] has introduced the notions of intuitionist fuzzy soft K -algebras and studied some of their properties. Bipolar fuzzy sets and soft sets are two different methods for representing uncertainty and vagueness. In this article, we apply these methods in combination to study uncertainty and vagueness in K -algebras. We first introduce the concept of bipolar fuzzy soft K -algebras and investigate some of their properties. Then we introduce the notion of a generalized bipolar fuzzy soft K -algebra and studied some of its related properties. The definitions and terminologies that we used in this paper are standard.

2. Preliminaries

In this section, we review some elementary concepts that are necessary for this paper.

Let $(G, \cdot, e)$ be a group in which each non-identity element is not of order 2. Then a K -algebra [12] is a structure $\mathcal{K} = (G, \cdot, \odot, e)$ on a group G in which induced binary operation $\odot : G \times G \rightarrow G$ is defined by $\odot(a, b) = a \odot b = a.b^{-1}$ and satisfies the following axioms:

$$(K1) \quad (a \odot b) \odot (a \odot c) = (a \odot ((e \odot c) \odot (e \odot b))) \odot a,$$

$$(K2) \quad a \odot (a \odot b) = (a \odot (e \odot b)) \odot a,$$

$$(K3) \quad (a \odot a) = e,$$

$$(K4) \quad (a \odot e) = a,$$

$$(K5) \quad (e \odot a) = a^{-1},$$

for all $a, b, c \in G$. A K -algebra $\mathcal{K}$ is called abelian if and only if $x \odot (e \odot b) = b \odot (e \odot a)$ for all $a, b \in G$. If a K -algebra $\mathcal{K}$ is abelian, then the axioms (1) and (2) can be written as:

$$(\overline{K1}) \quad (a \odot b) \odot (a \odot c) = c \odot b.$$

$$(\overline{K2}) \quad a \odot (a \odot b) = b.$$

In what follows, we denote a K -algebra by $\mathcal{K}$ unless otherwise specified.

A nonempty subset H of a K -algebra $\mathcal{K}$ is called a *subalgebra* [12] of the K -algebra $\mathcal{K}$ if $a \odot b \in H$, for all $a, b \in H$. Note that every subalgebra of a K -algebra $\mathcal{K}$ contains the identity e of the group $(G, \cdot, e)$. Naturally, the mapping $f : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ of K -algebras is called a *homomorphism* [15] if $f(a \odot b) = f(a) \odot f(b)$, for all $a, b \in \mathcal{K}_1$. We refer the reader to the book [10] for further information regarding K -algebras.

Definition 2.1. [29] Let X be a nonempty set. A *bipolar fuzzy set* B in X is an object having the form

$$B = \{(x, \mu^+(x), \mu^-(x)) \mid x \in X\}$$

where $\mu^+ : X \rightarrow [0, 1]$ and $\mu^- : X \rightarrow [-1, 0]$ are mappings.

We use the positive membership degree $\mu^+(x)$ to denote the satisfaction degree of an element x to the property corresponding to a bipolar fuzzy set B , and the negative membership degree $\mu^-(x)$ to denote the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar fuzzy set B . If $\mu^+(x) \neq 0$ and $\mu^-(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for B . If $\mu^+(x) = 0$ and $\mu^-(x) \neq 0$, it is the situation that x does not satisfy the property of B but somewhat satisfies the counter property of B . It is possible for an element x to be such that $\mu^+(x) \neq 0$ and $\mu^-(x) \neq 0$ when the membership function of the property overlaps that of its counter property over some portion of X .

For the sake of simplicity, we shall use the symbol $B = (\mu^+, \mu^-)$ for the bipolar fuzzy set $B = \{(x, \mu^+(x), \mu^-(x)) \mid x \in X\}$.

Definition 2.2. [29] For every two bipolar fuzzy sets $A = (\mu_A^+, \mu_A^-)$ and $B = (\mu_B^+, \mu_B^-)$ in X , we define

- $(A \cap B)(x) = (\min(\mu_A^+(x), \mu_B^+(x)), \max(\mu_A^-(x), \mu_B^-(x)))$,
- $(A \cup B)(x) = (\max(\mu_A^+(x), \mu_B^+(x)), \min(\mu_A^-(x), \mu_B^-(x)))$.

Definition 2.3. For $x \in X$, define $A_x = \{(y, z) \in X \times X | x = yz\}$. For two bipolar fuzzy subsets $A = (\mu_A^+, \mu_A^-)$ and $B = (\mu_B^+, \mu_B^-)$ of X , the product of two bipolar fuzzy subsets is denoted by $A \circ B$ and is defined as:

$$\mu_A^+ \circ \mu_B^+(x) = \begin{cases} \bigvee_{(s,t) \in A_x} \{\mu_A^+(s) \wedge \mu_B^+(t)\} & \text{if } A_x \neq \emptyset, \\ 0 & \text{if } A_x = \emptyset \end{cases}$$

and

$$\mu_A^- \circ \mu_B^-(x) = \begin{cases} \bigwedge_{(s,t) \in A_x} \{\mu_A^-(s) \vee \mu_B^-(t)\} & \text{if } A_x \neq \emptyset, \\ 0 & \text{if } A_x = \emptyset. \end{cases}$$

Molodtsov [23] defined the notion of soft set in the following way: Let U be an initial universe and E be a set of parameters. Let $P(U)$ denotes the power set of U and let A be non-empty subset of E .

Definition 2.4. [23] A pair (Φ, A) is called a *soft set* over U , where Φ is a mapping given by $\Phi : A \rightarrow P(U)$.

In other words, a soft set over U is a parameterized family of subsets of the universe U . For $\varepsilon \in A$, $\Phi(\varepsilon)$ may be considered as the set of ε - approximate elements of the soft set (Φ, A) . Clearly, a soft set is not just a subset of U . The concept of bipolar fuzzy soft set, which was originally proposed by [27]. Let $BF(U)$ denote the family of all bipolar fuzzy sets in U .

Definition 2.5. [27] Let U be an initial universe and $A \subseteq E$ be a set of parameters. A pair (ϕ, A) is called an *bipolar fuzzy soft set* over U , where ϕ is a mapping given by $\phi : A \rightarrow BF(U)$. A bipolar fuzzy soft set is a parameterized family of bipolar fuzzy subsets of U . For any $\varepsilon \in A$, ϕ_ε is referred to as the set of ε - approximate elements of the bipolar fuzzy soft set (ϕ, A) , which is actually a bipolar fuzzy set on U and can be

$$\phi_\varepsilon = \{(\mu_{\phi_\varepsilon}^+(x), \mu_{\phi_\varepsilon}^-(x)) \mid x \in U\}$$

where $\mu_{\phi_\varepsilon}^+(x)$ denotes the degree of x keeping the parameter ε , $\mu_{\phi_\varepsilon}^-(x)$ denotes the degree of x keeping the non-parameter ε .

Definition 2.6. [27] Let (ϕ, A) and (ψ, B) be two bipolar fuzzy soft sets over U . We say that (ϕ, A) is a *bipolar fuzzy soft subset* of (ψ, B) and write $(\phi, A) \subseteq (\psi, B)$ if

(i) $A \subseteq B$,

(ii) For any $\varepsilon \in A$, $\phi(\varepsilon) \subseteq \psi(\varepsilon)$.

(ϕ, A) and (ψ, B) are said to be bipolar fuzzy soft equal and write $(\phi, A) = (\psi, B)$ if $(\phi, A) \subseteq (\psi, B)$ and $(\psi, B) \subseteq (\phi, A)$.

Definition 2.7. [27] Let (ϕ, A) and (ψ, B) be two bipolar fuzzy soft sets over U . Then their *extended intersection* is a bipolar fuzzy soft set denoted by (φ, C) , where $C = A \cup B$ and

$$\varphi(\varepsilon) = \begin{cases} \phi(\varepsilon) & \text{if } \varepsilon \in A - B, \\ \psi(\varepsilon) & \text{if } \varepsilon \in B - A, \\ \phi(\varepsilon) \cap \psi(\varepsilon) & \text{if } \varepsilon \in A \cap B, \end{cases}$$

for all $\varepsilon \in C$. This is denoted by $(\varphi, C) = (\phi, A) \widetilde{\cap} (\psi, B)$.

Definition 2.8. [27] If (ϕ, A) and (ψ, B) are two bipolar fuzzy soft sets over the same universe U then “ $(\phi, A) \text{AND} (\psi, B)$ ” is a bipolar fuzzy soft set denoted by $(\phi, A) \wedge (\psi, B)$, and is defined by $(\phi, A) \wedge (\psi, B) = (\varphi, A \times B)$ where, $\varphi(a, b) = \varphi(a) \cap \psi(b)$ for all $(a, b) \in A \times B$. Here $\cap$ is the operation of a bipolar fuzzy intersection.

Definition 2.9. [27] Let (ϕ, A) and (ψ, B) be two bipolar fuzzy soft sets over U . Then their *extended union* denoted by (φ, C) , where $C = A \cup B$ and

$$\varphi(\varepsilon) = \begin{cases} \phi(\varepsilon) & \text{if } \varepsilon \in A - B, \\ \psi(\varepsilon) & \text{if } \varepsilon \in B - A, \\ \phi(\varepsilon) \cup \psi(\varepsilon) & \text{if } \varepsilon \in A \cap B, \end{cases}$$

for all $\varepsilon \in C$. This is denoted by $(\varphi, C) = (\phi, A) \widetilde{\cup} (\psi, B)$.

Definition 2.10. [24] Let (ϕ, A) and (ψ, B) be two fuzzy soft sets over a common universe U with $A \cap B \neq \emptyset$, then their *restricted intersection* is a bipolar fuzzy soft set $(\varphi, A \cap B)$ denoted by $(\phi, A) \cap (\psi, B) = (\varphi, A \cap B)$ where, $\varphi(\varepsilon) = \phi(\varepsilon) \cap \psi(\varepsilon)$ for all $\varepsilon \in A \cap B$.

Definition 2.11. [24] Let (ϕ, A) and (ψ, B) be two bipolar fuzzy soft sets over a common universe U with $A \cap B \neq \emptyset$, then their *restricted union* is denoted by $(\phi, A) \cup (\psi, B)$ and is defined as $(\phi, A) \cup (\psi, B) = (\varphi, C)$ where $C = A \cap B$ and for all $\varepsilon \in C$, $\varphi(\varepsilon) = \phi(\varepsilon) \cup \psi(\varepsilon)$.

Definition 2.12. [24] The *extended product* of two bipolar fuzzy soft sets (ϕ, A) and (ψ, B) over U is a fuzzy soft set, denoted by $(\phi \circ \psi, C)$, where $C = A \cup B$ and

$$(\phi \circ \psi)(\varepsilon) = \begin{cases} \phi(\varepsilon) & \text{if } \varepsilon \in A - B, \\ \psi(\varepsilon) & \text{if } \varepsilon \in B - A, \\ \phi(\varepsilon) \circ \psi(\varepsilon) & \text{if } \varepsilon \in A \cap B, \end{cases}$$

for all $\varepsilon \in C$. This is denoted by $(\phi \circ \psi, C) = (\phi, A) \widetilde{\circ} (\psi, B)$.

Definition 2.13. [24] If $A \cap B \neq \emptyset$, then the *restricted product* (φ, C) of two bipolar fuzzy soft sets (ϕ, A) and (ψ, B) over U is defined as the bipolar fuzzy soft set, $(\varphi, A \cap B)$ denoted by $(\phi, A) o_R (\psi, B)$ where $\varphi(\varepsilon) = \phi(\varepsilon) \circ \psi(\varepsilon)$, for all $\varepsilon \in A \cap B$. Here $\phi(\varepsilon) \circ \psi(\varepsilon)$ is the product of two bipolar fuzzy subsets of U .

3. Bipolar fuzzy soft K -algebras

Definition 3.1. Let (ϕ, A) be a bipolar fuzzy soft set over $\mathcal{K}$. Then (ϕ, A) is said to be a *bipolar fuzzy soft K -subalgebra* over $\mathcal{K}$ if $\phi(x)$ is a bipolar fuzzy K -subalgebra of $\mathcal{K}$ for all $x \in A$, that is, a bipolar fuzzy soft set (ϕ, A) over $\mathcal{K}$ is called a *bipolar fuzzy soft K -subalgebra* of $\mathcal{K}$ if the following conditions are satisfied:

- (1) $\mu_{\phi_\varepsilon}^+(x \odot y) \geq \min\{\mu_{\phi_\varepsilon}^+(x), \mu_{\phi_\varepsilon}^+(y)\},$
- (2) $\mu_{\phi_\varepsilon}^-(x \odot y) \leq \max\{\mu_{\phi_\varepsilon}^-(x), \mu_{\phi_\varepsilon}^-(y)\}$

for all $x, y \in G$.

Definition 3.2. Let (ϕ, A) and (ψ, B) be bipolar fuzzy soft K -subalgebras over $\mathcal{K}$. Then (ϕ, A) is a bipolar fuzzy subalgebra of (ψ, B) if (i) $A \subset B$ and (ii) $\phi(x)$ is a bipolar fuzzy subalgebra of $\psi(x)$ for all $x \in A$.

Example 3.3. Consider the K -algebra $\mathcal{K} = (G, \cdot, \odot, e)$, where $G = \{e, a, a^2, a^3\}$ is the cyclic group of order 4 and $\odot$ is given by the following Cayley's table:

$\odot$	e	a	a^2	a^3
e	e	a^3	a^2	a
a	a	e	a^3	a^2
a^2	a^2	a	e	a^3
a^3	a^3	a^2	a	e

Let $A = \{e_1, e_2, e_3\}$ and $\phi : A \rightarrow \mathcal{P}(G)$ be a set-valued function defined by

$$\begin{aligned}\phi(e_1) &= \left\{ \frac{e}{(0.7, -0.2)}, \frac{a}{(0.3, -0.4)}, \frac{a^2}{(0.6, -0.2)}, \frac{a^3}{(0.3, -0.6)} \right\}, \\ \phi(e_2) &= \left\{ \frac{e}{(0.5, -0.4)}, \frac{a}{(0.3, -0.3)}, \frac{a^2}{(0.4, -0.3)}, \frac{a^3}{(0.4, -0.5)} \right\}, \\ \phi(e_3) &= \left\{ \frac{e}{(0.5, -0.4)}, \frac{a}{(0.5, -0.2)}, \frac{a^2}{(0.5, -0.4)}, \frac{a^3}{(0.2, -0.5)} \right\}.\end{aligned}$$

Let $B = \{e_2, e_3\}$ and $\psi : B \rightarrow \mathcal{P}(G)$ be a set-valued function defined by

$$\begin{aligned}\psi(e_2) &= \left\{ \frac{e}{(0.2, -0.2)}, \frac{a}{(0.2, -0.3)}, \frac{a^2}{(0.1, -0.6)}, \frac{a^3}{(0.3, -0.0)} \right\}, \\ \psi(e_3) &= \left\{ \frac{e}{(0.2, -0.2)}, \frac{a}{(0.3, -0.3)}, \frac{a^2}{(0.3, -0.3)}, \frac{a^3}{(0.4, -0.4)} \right\}.\end{aligned}$$

Clearly, (ϕ, A) and (ψ, B) are bipolar fuzzy soft sets over $\mathcal{K}$. By routine calculations, it is easy to see that $\phi(x)$ and $\psi(x)$ are bipolar fuzzy K -subalgebras for $x \in A$ and $x \in B$, respectively. Hence (ϕ, A) and (ψ, B) are bipolar fuzzy soft K -subalgebras.

We state the following propositions without their proofs.

Proposition 3.4. *Let (ϕ, A) and (ψ, B) be bipolar fuzzy soft K -subalgebras over $\mathcal{K}$, then $(\phi, A) \widetilde{\cap} (\psi, B)$ and $(\phi, A) \wedge (\psi, B)$ are bipolar fuzzy soft K -subalgebras over $\mathcal{K}$.*

Proposition 3.5. *Let (ϕ, A) and (ψ, B) be bipolar fuzzy soft K -subalgebras over $\mathcal{K}$. If $A \cap B = \emptyset$ then $(\phi, A) \widetilde{\cup} (\psi, B)$ is a bipolar fuzzy soft K -subalgebra of $\mathcal{K}$.*

Proposition 3.6. *Let (ϕ, A) and (ψ, B) be bipolar fuzzy soft K -subalgebras over $\mathcal{K}$. If $\phi(x) \subseteq \psi(x)$ for all $x \in A$, (ϕ, A) is a bipolar fuzzy soft subalgebra of (ψ, B) .*

Theorem 3.7. *Let (ϕ, A) be a bipolar fuzzy soft K -subalgebra over $\mathcal{K}$ and let $\{(\varphi_i, B_i) \mid i \in I\}$ be a nonempty family of bipolar fuzzy soft K -subalgebras of (ϕ, A) . Then*

- (a) $\widetilde{\bigcap}_{i \in I} (\varphi_i, B_i)$ is a bipolar fuzzy soft K -subalgebra of (ϕ, A) ,
- (b) $\bigwedge_{i \in I} (\varphi_i, B_i)$ is a bipolar fuzzy soft K -subalgebra of $\bigwedge_{i \in I} (\phi, A)$,
- (c) If $B_i \cap B_j = \emptyset$ for all $i, j \in I$, then $\widetilde{\bigvee}_{i \in I} (\varphi_i, B_i)$ is a bipolar fuzzy soft K -subalgebra of $\widetilde{\bigvee}_{i \in I} (\phi, A)$.

Definition 3.8. Let (ϕ, A) be a bipolar fuzzy soft set over U . For each $s \in [0, 1]$, the set $(\phi, A)^s = (\phi^s, A)$ is called an s -level soft set of (ϕ, A) , where $\phi_\varepsilon^s = \{x \in U \mid \mu_{\phi_\varepsilon}^+(x) \geq s, \mu_{\phi_\varepsilon}^-(x) \leq -s\}$ for all $\varepsilon \in A$. Clearly, $(\phi, A)^s$ is a soft set over U .

Example 3.9. Consider the K -algebra $\mathcal{K} = (G, \cdot, \odot, e)$, where $G = \{e, a, a^2, a^3\}$ is the cyclic group of order 4 and $\odot$ is given by Cayley's table in Example 3.3. 0.2-level soft set of $(\phi, A) = \{e_1, e_2, e_3\}$.

Theorem 3.10. *Let (ϕ, A) be a bipolar fuzzy soft set over $\mathcal{K}$. (ϕ, A) is a bipolar fuzzy soft K -subalgebra if and only if $(\phi, A)^s$ is a soft K -subalgebra over $\mathcal{K}$ for each $s \in [0, 1]$.*

Proof. Suppose that (ϕ, A) is a bipolar fuzzy soft K -subalgebra. For each $s \in [0, 1]$, $\varepsilon \in A$ and $x_1, x_2 \in \phi_\varepsilon^s$, then $\mu_{\phi_\varepsilon}^+(x_1) \geq s$, $\mu_{\phi_\varepsilon}^+(x_2) \geq s$ and $\mu_{\phi_\varepsilon}^-(x_1) \leq -s$, $\mu_{\phi_\varepsilon}^-(x_2) \leq -s$. From definition 3.1, it follows that ϕ_ε^s is a bipolar fuzzy K -subalgebra over $\mathcal{K}$. Thus $\mu_{\phi_\varepsilon}^+(x_1 \odot x_2) \geq \min(\mu_{\phi_\varepsilon}^+(x_1), \mu_{\phi_\varepsilon}^+(x_2))$, $\mu_{\phi_\varepsilon}^+(x_1 \odot x_2) \geq s$, $\mu_{\phi_\varepsilon}^-(x_1 \odot x_2) \leq \max(\mu_{\phi_\varepsilon}^-(x_1), \mu_{\phi_\varepsilon}^-(x_2))$, $\mu_{\phi_\varepsilon}^-(x_1 \odot x_2) \leq -s$. This implies that

$x_1 \odot x_2 \in \phi_\varepsilon^s$, i.e., ϕ_ε^s is a K -subalgebra over $\mathcal{K}$. According to Definition 3.8, $(\phi, A)^s$ is a soft K -subalgebra over $\mathcal{K}$ for each $s \in [0, 1]$.

Conversely, assume that $(\phi, A)^s$ is a soft K -subalgebra over $\mathcal{K}$ for each $s \in [0, 1]$. For each $\varepsilon \in A$ and $x_1, x_2 \in G$, let $s = \min\{\mu_{\phi_\varepsilon}^+(x_1), \mu_{\phi_\varepsilon}^+(x_2)\}$ and let $-s = \max\{\mu_{\phi_\varepsilon}^-(x_1), \mu_{\phi_\varepsilon}^-(x_2)\}$, then $x_1, x_2 \in \phi_\varepsilon^s$. Since ϕ_ε^s is a K -subalgebra over $\mathcal{K}$, then $x_1 \odot x_2 \in \phi_\varepsilon^s$. This means that $\mu_{\phi_\varepsilon}^+(x_1 \odot x_2) \geq \min(\mu_{\phi_\varepsilon}^+(x_1), \mu_{\phi_\varepsilon}^+(x_2))$, $\mu_{\phi_\varepsilon}^-(x_1 \odot x_2) \leq \max(\mu_{\phi_\varepsilon}^-(x_1), \mu_{\phi_\varepsilon}^-(x_2))$, i.e., ϕ_ε^s is a bipolar fuzzy K -subalgebra over $\mathcal{K}$. According to Definition 3.1, (ϕ, A) is a bipolar fuzzy soft K -subalgebra over $\mathcal{K}$. This completes the proof. ■

Proposition 3.11. *Let $\mathcal{K}$ be a K -algebra built on Abelian group. Let (ϕ, A) , (ψ, B) and (φ, C) be bipolar fuzzy soft K -subalgebras over $\mathcal{K}$ where A , B and C are subsets of E . Then*

- (1) $(\phi, A) \widetilde{\circ} ((\psi, B) \widetilde{\cup} (\varphi, C)) = ((\phi, A) \widetilde{\circ} (\psi, B)) \widetilde{\cup} ((\phi, A) \widetilde{\circ} (\varphi, C)),$
- (2) $(\phi, A) \widetilde{\circ} ((\psi, B) \sqcup (\varphi, C)) = ((\phi, A) \widetilde{\circ} (\psi, B)) \sqcup ((\phi, A) \widetilde{\circ} (\varphi, C)),$
- (3) $((\phi, A) \widetilde{\cup} (\psi, B)) \widetilde{\circ} (\varphi, C) = ((\phi, A) \widetilde{\circ} (\varphi, C)) \widetilde{\cup} ((\psi, B) \widetilde{\circ} (\varphi, C)),$
- (4) $((\phi, A) \sqcup (\psi, B)) \widetilde{\circ} (\varphi, C) = ((\phi, A) \widetilde{\circ} (\varphi, C)) \sqcup ((\psi, B) \widetilde{\circ} (\varphi, C)).$

Proof. The proofs follow from the definitions of operations. ■

Proposition 3.12. *Let $\mathcal{K}$ be a K -algebra built on Abelian group. Let (ϕ, A) , (ψ, B) and (φ, C) be bipolar fuzzy soft K -subalgebras over $\mathcal{K}$ where A , B and C are subsets of E . Then*

- (5) $(\phi, A) \widetilde{\circ}_R ((\psi, B) \sqcup (\varphi, C)) \widetilde{\subset} ((\phi, A) \widetilde{\circ}_R (\psi, B)) \sqcup ((\phi, A) \widetilde{\circ}_R (\varphi, C)),$
- (6) $(\phi, A) \widetilde{\circ}_R ((\psi, B) \widetilde{\cup} (\varphi, C)) \widetilde{\subset} ((\phi, A) \widetilde{\circ}_R (\psi, B)) \widetilde{\cup} ((\phi, A) \widetilde{\circ}_R (\varphi, C)),$
- (7) $((\phi, A) \widetilde{\cup} (\psi, B)) \widetilde{\circ}_R (\varphi, C) \widetilde{\subset} ((\phi, A) \widetilde{\circ}_R (\varphi, C)) \widetilde{\cup} ((\psi, B) \widetilde{\circ}_R (\varphi, C)),$
- (8) $((\phi, A) \sqcup (\psi, B)) \widetilde{\circ}_R (\varphi, C) \widetilde{\subset} ((\phi, A) \widetilde{\circ}_R (\varphi, C)) \sqcup ((\psi, B) \widetilde{\circ}_R (\varphi, C)).$

Proof. The proofs follow from the definitions of operations. ■

Definition 3.13. Let $f : X \rightarrow Y$ and $g : A \rightarrow B$ be two functions, A and B are parametric sets from the crisp sets X and Y , respectively. Then the pair (f, g) is called a *bipolar fuzzy soft function* from X to Y .

Definition 3.14. Let (ϕ, A) and (ψ, B) be two bipolar fuzzy soft sets over G_1 and G_2 , respectively and let (f, g) be a bipolar fuzzy soft function from G_1 to G_2 .

- (1) The image of (ϕ, A) under the bipolar fuzzy soft function (f, g) , denoted by $(f, g)(\phi, A)$, is the bipolar fuzzy soft set on $\mathcal{K}_2$ defined by $(f, g)(\phi, A) = (f(\phi), g(A))$, where for all $k \in g(A)$, $y \in G_2$

$$\mu_{f(\phi)_k}^+(y) = \begin{cases} \bigvee_{f(x)=y} \bigvee_{g(a)=k} \phi_a(x) & \text{if } x \in g^{-1}(y), \\ 1 & \text{otherwise,} \end{cases}$$

$$\mu_{f(\phi)_k}^-(y) = \begin{cases} \bigwedge_{f(x)=y} \bigwedge_{g(a)=k} \phi_a(x) & \text{if } x \in g^{-1}(y), \\ -1 & \text{otherwise.} \end{cases}$$

- (2) The preimage of (ψ, B) under the bipolar fuzzy soft function (f, g) , denoted by $(f, g)^{-1}(\psi, B)$, is the bipolar fuzzy soft set over $\mathcal{K}_1$ defined by $(f, g)^{-1}(\psi, B) = (f^{-1}(\psi), g^{-1}(B))$, where for all $a \in g^{-1}(A)$, for all $x \in G_1$,

$$\mu_{f^{-1}(\psi)_a}^+(x) = \mu_{\psi_{g(a)}}^+(f(x)),$$

$$\mu_{f^{-1}(\psi)_a}^-(x) = \mu_{\psi_{g(a)}}^-(f(x)).$$

Definition 3.15. Let (f, g) be a bipolar fuzzy soft function from $\mathcal{K}_1$ to $\mathcal{K}_2$. If f is a homomorphism from $\mathcal{K}_1$ to $\mathcal{K}_2$ then (f, g) is said to be *bipolar fuzzy soft homomorphism*. If f is an isomorphism from $\mathcal{K}_1$ to $\mathcal{K}_2$ and g is one-to-one mapping from A onto B then (f, g) is said to be bipolar fuzzy soft isomorphism.

Theorem 3.16. Let (ψ, B) be a bipolar fuzzy soft K -subalgebra over $\mathcal{K}_2$ and let (f, g) be a bipolar fuzzy soft homomorphism from $\mathcal{K}_1$ to $\mathcal{K}_2$. Then $(f, g)^{-1}(\psi, B)$ is a bipolar fuzzy soft K -subalgebra over $\mathcal{K}_1$.

Proof. Let $x_1, x_2 \in G_1$, then

$$\begin{aligned} f^{-1}(\mu_{\psi_\epsilon}^+)(x_1 \odot x_2) &= \mu_{\psi_{g(\epsilon)}}^+(f(x_1 \odot x_2)) = \mu_{\psi_{g(\epsilon)}}^+(f(x_1) \odot f(x_2)) \\ &\geq \min\{\mu_{\psi_{g(\epsilon)}}^+(f(x_1)), \mu_{\psi_{g(\epsilon)}}^+(f(x_2))\} \\ &= \min\{f^{-1}(\mu_{g_\epsilon}^+)(x_1), f^{-1}(\mu_{g_\epsilon}^+)(x_2)\}, \end{aligned}$$

$$\begin{aligned} f^{-1}(\mu_{\psi_\epsilon}^-)(x_1 \odot x_2) &= \mu_{\psi_{g(\epsilon)}}^-(f(x_1 \odot x_2)) = \mu_{\psi_{g(\epsilon)}}^-(f(x_1) \odot f(x_2)) \\ &\leq \max\{\mu_{\psi_{g(\epsilon)}}^-(f(x_1)), \mu_{\psi_{g(\epsilon)}}^-(f(x_2))\} \\ &= \max\{f^{-1}(\mu_{g_\epsilon}^-)(x_1), f^{-1}(\mu_{g_\epsilon}^-)(x_2)\}. \end{aligned}$$

Hence $(f, g)^{-1}(\psi, B)$ is a bipolar fuzzy soft K -subalgebra over $\mathcal{K}_1$. ■

Remark. Let (ϕ, A) be a bipolar fuzzy soft K -subalgebra over $\mathcal{K}_1$ and let (f, g) be a bipolar fuzzy soft homomorphism from $\mathcal{K}_1$ to $\mathcal{K}_2$. Then $(f, g)(\phi, A)$ may not be a bipolar fuzzy soft K -subalgebra over $\mathcal{K}_2$.

4. Generalized bipolar fuzzy soft K -algebras

Definition 4.1. Let c be a point in a non-empty set G . If $\gamma \in (0, 1]$ and $\delta \in [-1, 0)$ are two real numbers, then the bipolar fuzzy $c(\gamma, \delta) = \langle x, c_\gamma, c_\delta \rangle$ is called a bipolar fuzzy point in G , where γ (resp, δ) is the positive degree of membership (resp, negative degree of membership) of $c(\gamma, \delta)$ and $c \in G$ is the support of $c(\gamma, \delta)$. Let $c(\gamma, \delta)$ be a bipolar fuzzy in G and let $A = \langle x, \mu_A^+, \mu_A^- \rangle$ be a bipolar fuzzy in G . Then $c(\gamma, \delta)$ is said to belong to A , written $c(\gamma, \delta) \in A$ if $\mu_A^+(c) \geq \gamma$ and $\mu_A^-(c) \leq \delta$. We say that $c(\gamma, \delta)$ is quasicoincident with A , written $c(\gamma, \delta)qA$, if $\mu_A^+(c) + \gamma > 1$ and $\mu_A^-(c) + \delta < -1$. To say that $c(\gamma, \delta) \in \vee qA$ (resp, $c(\gamma, \delta) \in \wedge qA$) means that $c(\gamma, \delta) \in A$ or $c(\gamma, \delta)qA$ (resp, $c(\gamma, \delta) \in A$ and $c(\gamma, \delta)qA$) and $c(\gamma, \delta) \overline{\in} \vee qA$ means that $c(\gamma, \delta) \in \vee qA$ does not hold.

Definition 4.2. A bipolar fuzzy set $A = (\mu_A^+, \mu_A^-)$ in $\mathcal{K}$ is called an $(\in, \in \vee q)$ -bipolar fuzzy K -algebra of $\mathcal{K}$ if it satisfies the following conditions:

- (a) $x(s, t) \in A \Rightarrow e(s, t) \in \vee qA$,
- (b) $x(s_1, t_1) \in A, y(s_2, t_2) \in A \Rightarrow (x \odot y)(\min(s_1, s_2), \max(t_1, t_2)) \in \vee qA$,

for all $x, y \in G, s, s_1, s_2 \in (0, 1], t, t_1, t_2 \in [-1, 0)$.

Example 4.3. Consider the K -algebra $\mathcal{K} = (G, \cdot, \odot, e)$, where $G = \{e, a, a^2, a^3\}$ is the cyclic group of order 4 and $\odot$ is given by Cayley's table in Example 3.3. We define a bipolar fuzzy set $A : G \rightarrow [0, 1] \times [-1, 0]$ by

$$\mu_A^+(x) := \begin{cases} 1 & \text{if } x = e, \\ 0.5 & \text{otherwise,} \end{cases}$$

$$\mu_A^-(x) := \begin{cases} 0 & \text{if } x = e, \\ -0.3 & \text{otherwise.} \end{cases}$$

Take $s = 0.4 \in (0, 1]$ and $t = -0.5 \in [-1, 0)$. By routine computations, it is easy to see that A is not an $(\in, \in \vee q)$ -bipolar fuzzy K -subalgebra of $\mathcal{K}$.

Theorem 4.4. Let A be a bipolar fuzzy set in a K -algebra $\mathcal{K}$. Then A is an $(\in, \in \vee q)$ -bipolar fuzzy K -subalgebra of $\mathcal{K}$ if and only if $\mu_A^+(x \odot y) \geq \min(\mu_A^+(x), \mu_A^+(y), 0.5)$, $\mu_A^-(x \odot y) \leq \max(\mu_A^-(x), \mu_A^-(y), -0.5)$ hold for all $x, y \in G$.

Proof. The proof is straightforward and thus omitted. ■

Theorem 4.5. Let A be a bipolar fuzzy set of K -algebra of $\mathcal{K}$. Then A is an $(\in, \in \vee q)$ -bipolar fuzzy K -subalgebra of $\mathcal{K}$ if and only if each nonempty $A_{(s,t)}$, $s \in (0.5, 1], t \in [-1, -0.5]$ is a K -subalgebra of $\mathcal{K}$.

Proof. Assume that A is an $(\in, \in \vee q)$ -bipolar fuzzy K -subalgebra of $\mathcal{K}$ and let $s \in (0.5, 1], t \in [-1, -0.5]$. If $x, y \in A_{(s,t)}$, then $\mu_A^+(x) \geq s$ and $\mu_A^+(y) \geq s$, $\mu_A^-(x) \leq t$ and $\mu_A^-(y) \leq t$. Thus,

$$\mu_A^+(x \odot y) \geq \min(\mu_A^+(x), \mu_A^+(y), 0.5) \geq \min(s, 0.5) = s,$$

$$\mu_A^-(x \odot y) \leq \max(\mu_A^-(x), \mu_A^-(y), -0.5) \leq \max(t, -0.5) = t,$$

and so $x \odot y \in A_{(s,t)}$. This shows that $A_{(s,t)}$ are K -subalgebras of $\mathcal{K}$. The proof of converse part is obvious. This ends the proof. ■

Theorem 4.6. *Let A be a bipolar fuzzy set in a K -algebra $\mathcal{K}$. Then $A_{(s,t)}$ is a K -subalgebra of $\mathcal{K}$ if and only if $\max(\mu_A^+(x \odot y), 0.5) \geq \min(\mu_A^+(x), \mu_A^+(y))$, $\min(\mu_A^-(x \odot y), -0.5) \leq \max(\mu_A^-(x), \mu_A^-(y))$ for all $x, y \in G$.*

Proof. The proof is straightforward and thus omitted. ■

Definition 4.7. Let (ϕ, A) be a bipolar fuzzy soft set over a K -algebra $\mathcal{K}$. Then (ϕ, A) is called an $(\in, \in \vee q)$ -bipolar fuzzy soft K -subalgebra if $\phi(\alpha)$ is an $(\in, \in \vee q)$ -bipolar fuzzy K -subalgebra of $\mathcal{K}$ for all $\alpha \in A$.

Theorem 4.8. *Let (ϕ, A) and (ψ, B) be two $(\in, \in \vee q)$ -bipolar fuzzy soft K -subalgebras over a K -algebra $\mathcal{K}$. Then $(\phi, A) \wedge (\psi, B)$ is an $(\in, \in \vee q)$ -bipolar fuzzy soft K -subalgebra over $\mathcal{K}$.*

Proof. Since $(\phi, A) \wedge (\psi, B) = (\varphi, C)$, where $C = A \times B$ and $\varphi(\alpha, \beta) = \phi(\alpha) \cap \psi(\beta)$ for all $(\alpha, \beta) \in C$. Now for any $(\alpha, \beta) \in C$, since (ϕ, A) and (ψ, B) are $(\in, \in \vee q)$ -bipolar fuzzy soft K -subalgebras over $\mathcal{K}$, we have both $\phi(\alpha)$ and $\psi(\beta)$ are $(\in, \in \vee q)$ -bipolar fuzzy K -subalgebras of $\mathcal{K}$. Thus $\varphi(\alpha, \beta) = \phi(\alpha) \cap \psi(\beta)$ is an $(\in, \in \vee q)$ -bipolar fuzzy K -subalgebra of $\mathcal{K}$. Therefore, $(\phi, A) \wedge (\psi, B)$ is an $(\in, \in \vee q)$ -bipolar fuzzy soft K -subalgebra over $\mathcal{K}$. ■

Theorem 4.9. *Let (ϕ, A) and (ψ, B) be two $(\in, \in \vee q)$ -bipolar fuzzy soft K -subalgebras over a K -algebra $\mathcal{K}$. If $A \cap B \neq \emptyset$, then $(\phi, A) \tilde{\cap} (\psi, B)$ is an $(\in, \in \vee q)$ -bipolar fuzzy soft K -subalgebra over $\mathcal{K}$.*

Proof. Since $(\phi, A) \tilde{\cap} (\psi, B) = (\varphi, C)$, where $C = A \cap B$ and $\varphi(\alpha) = \phi(\alpha) \cap \psi(\alpha)$ for all $\alpha \in C$. Now, for any $\alpha \in C$, since (ϕ, A) and (ψ, B) are $(\in, \in \vee q)$ -bipolar fuzzy soft K -subalgebras over $\mathcal{K}$, we have both $\phi(\alpha)$ and $\psi(\alpha)$ are $(\in, \in \vee q)$ -bipolar fuzzy K -subalgebras of $\mathcal{K}$. Thus $\varphi(\alpha) = \phi(\alpha) \cap \psi(\alpha)$ is an $(\in, \in \vee q)$ -bipolar fuzzy K -subalgebra of $\mathcal{K}$. Therefore, $(\phi, A) \tilde{\cap} (\psi, B)$ is an $(\in, \in \vee q)$ -bipolar fuzzy soft K -subalgebra over $\mathcal{K}$. ■

As a generalization of Theorem 3.7, we have the following result.

Theorem 4.10. *Let (ϕ, A) be an $(\in, \in \vee q)$ -bipolar fuzzy soft K -subalgebra over $\mathcal{K}$ and let $\{(\varphi_i, B_i) \mid i \in I\}$ be a nonempty family of $(\in, \in \vee q)$ -bipolar fuzzy soft K -subalgebras of (ϕ, A) . Then*

- (a) $\widetilde{\bigcap}_{i \in I} (\varphi_i, B_i)$ is an $(\in, \in \vee q)$ -bipolar fuzzy soft K -subalgebra of (ϕ, A) ,
- (b) $\bigwedge_{i \in I} (\varphi_i, B_i)$ is an $(\in, \in \vee q)$ -bipolar fuzzy soft K -subalgebra of $\bigwedge_{i \in I} (\phi, A)$,
- (c) If $B_i \cap B_j = \emptyset$ for all $i, j \in I$, then $\widetilde{\bigvee}_{i \in I} (\varphi_i, B_i)$ is an $(\in, \in \vee q)$ -bipolar fuzzy soft K -subalgebra of $\bigvee_{i \in I} \phi, A$.

Proof. The proof is straightforward and thus omitted. ■

Theorem 4.11. *Let (ϕ, A) and (ψ, B) be two $(\in, \in \vee q)$ -bipolar fuzzy soft K -subalgebras over a K -algebra $\mathcal{K}$. Then $(\phi, A) \tilde{\cap} (\psi, B)$ is an $(\in, \in \vee q)$ -bipolar fuzzy soft K -subalgebra over $\mathcal{K}$.*

Proof. Since $(\phi, A) \tilde{\cap} (\psi, B) = (\varphi, C)$, where $C = A \cup B$ and

$$\varphi(\alpha) = \begin{cases} \phi(\alpha) & \text{if } \alpha \in A - B, \\ \psi(\alpha) & \text{if } \alpha \in B - C, \\ \phi(\alpha) \cap \psi(\alpha) & \text{if } \alpha \in A \cap B. \end{cases}$$

for all $\alpha \in C$.

Now, for any $\alpha \in C$, we consider the following cases.

Case 1. $\alpha \in A - B$. Then $\varphi(\alpha) = \phi(\alpha)$ is an $(\in, \in \vee q)$ -bipolar fuzzy K -subalgebra of $\mathcal{K}$ since (ϕ, A) is an $(\in, \in \vee q)$ -bipolar fuzzy soft K -subalgebra over $\mathcal{K}$.

Case 2. $\alpha \in B - A$. Then $\varphi(\alpha) = \psi(\alpha)$ is an $(\in, \in \vee q)$ -bipolar fuzzy K -subalgebra of $\mathcal{K}$ since (ψ, B) is an $(\in, \in \vee q)$ -bipolar fuzzy soft K -subalgebra over $\mathcal{K}$.

Case 3. $\alpha \in A \cap B$. Then $\varphi(\alpha) = \phi(\alpha) \cap \psi(\alpha)$ is an $(\in, \in \vee q)$ -bipolar fuzzy K -subalgebra of $\mathcal{K}$ by the assumption. Thus, in any case, $\varphi(\alpha)$ is an $(\in, \in \vee q)$ -bipolar fuzzy K -subalgebra of $\mathcal{K}$. Therefore, $(\phi, A) \tilde{\cap} (\psi, B)$ is an $(\in, \in \vee q)$ -bipolar fuzzy soft K -subalgebra over $\mathcal{K}$. ■

Theorem 4.12. *Let (ϕ, A) and (ψ, B) be two $(\in, \in \vee q)$ -bipolar fuzzy soft K -subalgebras over a K -algebra $\mathcal{K}$. If A and B are disjoint, then $(\phi, A) \tilde{\cup} (\psi, B)$ is an $(\in, \in \vee q)$ -bipolar fuzzy soft K -subalgebra over $\mathcal{K}$.*

Proof. The proof is straightforward and thus omitted. ■

5. Conclusions

Molodtsov introduced the concept of soft set theory as a general mathematical tool for dealing with uncertainty and vagueness, and many researchers have created some models to solve problems in decision making and medical diagnosis. In soft computing and uncertain modeling, soft sets can be combined with other mathematical tools. Bipolar fuzzy sets and soft sets are two different mathematical tools for representing vagueness and uncertainty. In this paper we have applied these mathematical tools in combination to study vagueness and uncertainty in K -algebras. We have introduced the notion of an $(\in, \in \vee q)$ -bipolar fuzzy soft K -subalgebra. The natural extension of this research work is connected with the study of: (i) fuzzy soft rough K -algebras, and (ii) bipolar fuzzy rough K -algebras.

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Accepted: 14.06.2014

SEMIGROUP DISTANCES OF FINITE GROUPOIDS

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Abstract. The simplest cases of non-associative groupoids are presented by groupoids (so called SH-groupoids) having just one non-associative (ordered) triple of elements. In this paper, only SH-groupoids having the simplest possible non-associative triple (a, a, a) are investigated. For each positive integer n finite SH-groupoids $E_n(\cdot)$ generated by at most two elements are constructed and their semigroup distances are described. It is proved that there are finite non-associative groupoids having their semigroup distance equal just to the arbitrary given positive integer n .

Keywords: groupoid, non-associative triple, distance of groupoids, semigroup distance.

1991 Mathematics Subject Classification: 20N05.

Introduction

Groupoids having only one non-associative (ordered) triple were studied at first by G. Szász in [10] and [11], and by P. Hájek in [2] and [3]. Therefore they are called Szász-Hájek groupoids and shortly denoted as SH-groupoids. Sets of non-associative triples and semigroup distances of some finite groupoids were also investigated in [1], [4] and [5]. The structure of SH-groupoids was studied later by T. Kepka and M. Trch, and the main properties of SH-groupoids were described in [6], [7], [8] and [9] for each of possible non-associative triples. Further, it was proved in [6] that the semigroup distance of a minimal SG-groupoid having the only non-associative triple (a, a, a) is equal to the number 1 or to the number 2. In [13], a minimal SH-groupoid with the semigroup distance equal to 3 is constructed. This paper is an immediate continuation of [12], [14] and [15].

1. Preliminaries

A groupoid $G(\cdot)$ is called σ -stratified groupoid if there exists a mapping σ of the set G to the set of positive integers such that $\sigma(x \cdot y) = \sigma(x) + \sigma(y)$ for each couple $x, y \in G$. The mapping σ is called a mapping *stratifying* the underlying set G .

For each positive integer n , the corresponding subset $L_n G = \{x \in G \mid \sigma(x) = n\}$ is called n -th layer of the underlying set G . Of course, we have

$$G = L_1(G) \cup L_2(G) \cup \dots \cup L_n(G) \cup L_{n+1}(G) \dots$$

Furthermore, if $G(\cdot)$ is a σ -stratified groupoid then each subgroupoid $H(\cdot)$ of the groupoid $G(\cdot)$ is also a σ -stratified groupoid.

If $G(\cdot)$ is a σ -stratified groupoid and κ is a congruence on $G(\cdot)$ then the congruence κ is called σ -stratified congruence if $\sigma(x) = \sigma(y)$ for each $(x, y) \in \kappa$. In this case the corresponding groupoid $G/\kappa(\cdot)$ is a σ -stratified groupoid, too.

Let $G(\cdot)$ be a σ -stratified groupoid. Further, let k be an arbitrary given positive integer and, finally, let $t \in G$ be an arbitrary chosen element having $\sigma(t) \geq k$. Then there exists a groupoid $G^k(o)$ such that $x \circ y = x \cdot y$ for every $x, y \in G$ having $\sigma(x) + \sigma(y) < k$ and $x \circ y = t$ whenever $\sigma(x) + \sigma(y) \geq k$. The corresponding groupoid $G^k(o)$ is called *restriction of the k -th order* of the groupoid $G(\cdot)$. If the groupoid $G(\cdot)$ is finitely generated then the corresponding groupoid $G^k(o)$ is a finite groupoid which can be seen as a σ -stratified groupoid up to the k -layer.

If $G(\cdot)$ is a non-associative σ -stratified groupoid, then the set M containing positive integers $\sigma(x) + \sigma(y) + \sigma(z)$ for every $x, y, z \in G$ such that $x \cdot yz \neq xy \cdot z$ is non-empty. This set contains the least element m having the same property and the corresponding restriction of the $(m + 1)$ -th order is a non-associative groupoid. The corresponding groupoid will be called *fundamental restriction* of the non-associative σ -stratified groupoid and it will be denoted as $\overline{G}(\cdot)$.

The simplest type of non-associative groupoids are groupoids containing just one non-associative ordered triple of elements. A groupoid $G(\cdot)$ is called an *SH-groupoid of the type (a, a, a)* if there exists an element $a \in G$ such that the ordered triple (a, a, a) is the only non-associative triple of the groupoid $G(\cdot)$. An SH-groupoid $G(\cdot)$ of the type (a, a, a) is called *minimal SH-groupoid* if the groupoid $G(\cdot)$ is generated by the one-element set $\{a\}$.

Let $H(\cdot)$ be a subgroupoid of an SH-groupoid $G(\cdot)$ having the non-associative triple (a, a, a) . Then either $(a, a, a) \in H^3$ and $H(\cdot)$ is an SH-groupoid having the non-associative triple (a, a, a) , or $H(\cdot)$ is a semigroup in the opposite case.

Let κ be a congruence on SH-groupoid $G(\cdot)$. If (a, a, a) is the corresponding non-associative triple then either $(a \cdot aa, aa \cdot a) \in \kappa$ and then the corresponding groupoid $G(\cdot)/\kappa$ is a semigroup, or $(a \cdot aa, aa \cdot a) \notin \kappa$ and then the corresponding groupoid $G(\cdot)/\kappa$ is an SH-groupoid of the same type (a, a, a) .

The following main theorem concerning SH-groupoids was proved by G. Szász:

1.1. Theorem. *Let $G(\cdot)$ be an SH-groupoid. If (a, b, c) is the only non-associative triple of $G(\cdot)$ and let $x, y \in G$ be such that $x \cdot y \in \{a, b, c\}$. Then $x \cdot y \in \{x, y\}$.*

Let $G(\diamond)$ and $G(*)$ be a couple of groupoids having the same underlying set G . Then $\text{dist}(G(\diamond), G(*))$ denotes $\text{card}\{(x, y) \in G^2 \mid x \diamond y \neq x * y\}$.

Let $G(\cdot)$ be a groupoid. Let $\text{sdist}(G(\cdot))$ be the minimum of cardinal numbers $\text{dist}(G(\cdot), G(*))$, where $G(*)$ runs through the set of all semigroups having the underlying set G . The number $\text{sdist}(G(\cdot))$ is called *semigroup distance* of the groupoid $G(\cdot)$.

Let $G(\cdot)$ be a groupoid and let $G(\star)$ be a semigroup having the same underlying set G . If $\text{sdist}(G(\cdot)) = \text{dist}(G(\cdot), G(\star))$ then the semigroup $G(\star)$ will be further denoted as *nearest semigroup* of the groupoid $G(\cdot)$.

2. Stratified minimal SH-groupoids of the type (a, a, a)

In this part, let $F(\cdot)$ denote the absolutely free groupoid generated by a one-element set $\{a\}$. Let $\lambda(u)$ denote the length of the term u for each element $u \in F$. It is obvious that $\lambda(x \cdot y) = \lambda(x) + \lambda(y)$ for every $x, y \in F$. Thus, the groupoid $F(\cdot)$ is a λ -stratified groupoid.

If $G(\cdot)$ is an arbitrary minimal SH-groupoid of the type (a, a, a) then there exists a congruence κ on $F(\cdot)$ such that $G(\cdot)$ is an isomorphic image of the groupoid $F/\kappa(\cdot)$. Of course, the corresponding congruence κ satisfies the following two conditions: $(a \cdot aa, aa \cdot a) \notin \kappa$ and $(x \cdot yz, xy \cdot z) \in \kappa$ for every $x, y, z \in G$, $(x, y, z) \neq (a, a, a)$.

Suppose at first that there exists at least one stratified SH-groupoid $G(\cdot)$ of the type (a, a, a) . If σ is the corresponding stratifying mapping of the underlying set G then the inequality $\sigma(a \cdot a) = 2 \cdot \sigma(a) > \sigma(a)$ has to be valid.

2.1. Construction. Suppose that the elements a, b, c, d, f, g and, further, $a^4, a^5, \dots, a^n, a^{n+1}, \dots$ are pairwise different.

- (i) Put $V = \{a, b, c, d, a^4, a^5, \dots, a^k, a^{k+1}, \dots\}$. Define a mapping λ of the set V to the set of positive integers such that $\lambda(a) = 1$, $\lambda(b) = 2$, $\lambda(c) = 3 = \lambda(d)$ and let $\lambda(a^k) = k$ for each positive integer $k \geq 4$.

Define, further, a binary operation $x \cdot y$ on the set V such that $a \cdot a = b$, $a \cdot b = c$, $b \cdot a = d$ and put $x \cdot y = a^k$ whenever $\lambda(x) + \lambda(y) = k \geq 4$. Then $V(\cdot)$ becomes a groupoid and it is easy to check that $V(\cdot)$ is a λ -stratified groupoid.

- (ii) Put $W = \{a, b, c, d, f, g, a^5, a^6, \dots, a^m, a^{m+1}, \dots\}$ and define a mapping λ of the set W to the set of positive integers such that $\lambda(a) = 1$, $\lambda(b) = 2$, $\lambda(c) = 3 = \lambda(d)$, $\lambda(f) = 4 = \lambda(g)$ and let $\lambda(a^m) = m$ for each positive integer $m \geq 5$.

Define, further, a binary operation $x \cdot y$ on the set W such that $a \cdot a = b$, $a \cdot b = c$, $b \cdot a = d$, $f = a \cdot c = b \cdot b = d \cdot a$, $a \cdot d = g = c \cdot a$ and, further, put $x \cdot y = a^m$ whenever $\lambda(x) + \lambda(y) = m \geq 5$. Then $W(\cdot)$ becomes a groupoid and it is easy to check that $W(\cdot)$ is again a λ -stratified groupoid.

2.2. Lemma. *The groupoids $V(\cdot)$ and $W(\cdot)$ are the only two different λ -stratified minimal SH-groupoids of the type (a, a, a) .*

Proof. It is obvious that both groupoids $V(\cdot)$, $W(\cdot)$ are generated by the one-element set $\{a\}$ and $a \cdot aa \neq aa \cdot a$. From the definition it follows immediately that $\lambda(x \cdot y) \neq 1$. Thus, $x \cdot y \neq a$ for every $x, y \in V$. The same is also true for every $x, y \in W$.

(i) At first consider the groupoid $V(\cdot)$. It is easy to check that $x \cdot yz = xy \cdot z$ whenever $(x, y, z) \neq (a, a, a)$. From the construction it follows that $\lambda(x \cdot yz) = \lambda(xy \cdot z) \geq 4$. But each layer $L_k(V)$ contains only one element for each $k \geq 4$. Thus, $V(\cdot)$ is a λ -stratified minimal SH-groupoid of the type (a, a, a) . It is easy to see that $\text{sdist}(V(\cdot)) = 1$. In fact, if we put $a \star b = d \neq c = a \cdot b$ and $x \star y = x \cdot y$ in the remaining cases then the groupoid $V(\star)$ is a semigroup. Therefore the SH-groupoid $V(\cdot)$ of the type (a, a, a) will be further called *minimal SH-groupoid of the first kind*.

(ii) Consider, further, the groupoid $W(\cdot)$, in which $a(a \cdot aa) = aa \cdot aa = (aa \cdot a)a$, $a(aa \cdot a) = (a \cdot aa)a$ and, thus, $f = a \cdot c = b \cdot b = d \cdot a$, $a \cdot d = g = c \cdot a$. We have $\lambda(x \cdot yz) = \lambda(xy \cdot z) \geq 5$ in the remaining cases. But each layer $L_k(W)$ contains only one element for each $k \geq 5$ and, thus, $x \cdot yz = xy \cdot z$ if $(x, y, z) \neq (a, a, a)$. Therefore, $W(\cdot)$ is the other λ -stratified minimal SH-groupoid of the type (a, a, a) . Now, it is easy to check that $\text{sdist}(W(\cdot)) = 2$. In fact, if we put $c \star a = f \neq g = c \cdot a$, and $b \star a = c \neq d = b \cdot a$ and $x \star y = x \cdot y$ in the remaining cases then the groupoid $W(\star)$ is a semigroup. Therefore, the SH-groupoid $W(\cdot)$ of the type (a, a, a) will be further called *minimal SH-groupoid of the second kind*.

(iii) Finally, let κ be an arbitrary λ -stratified congruence on $F(\cdot)$. It follows from $\lambda(x \cdot y) > 1$ that $x \cdot y \neq a$ for every $x, y \in F/\kappa$. Therefore, $a \cdot f = a \cdot bb = a \cdot ac = b \cdot c = ba \cdot b = (ba \cdot a) \cdot a = f \cdot a$. Similarly, we get $f \cdot a = bb \cdot a = ac \cdot a = a \cdot ca = a \cdot g = g \cdot a = ad \cdot a = a \cdot da = a \cdot f$. Thus, the layer $L_5(F/\kappa)$ contains only one element a^5 and the rest is clear. ■

2.3. Lemma. *Let k be an arbitrary given positive integer such that $k \geq 5$. Consider the k -th restriction $V^k(\cdot)$ for the SH-groupoid $V(\cdot)$. Then the underlying set of the restricted groupoid $V^k(\cdot)$ is finite and the groupoid $V^k(\cdot)$ is an SH-groupoid of the type (a, a, a) having $\text{sdist}(V^k(\cdot)) = 1$.*

Proof. It follows from the construction that the underlying set V^k of the groupoid $V^k(\cdot)$ is finite and it contains just $k + 1$ different elements. Further, define on V^k a new binary operation $\star$ such that $b \star a = c \neq d = b \cdot a$ and $x \star y = x \cdot y$ in the remaining cases. It is easy to see that $V^k(\star)$ is a semigroup and the rest is clear. ■

2.4. Definition. Let $V(\cdot)$ be the minimal SH-groupoid of the first kind constructed in 2.1 and let $k = 4$. The corresponding 4-th restriction of the SH-groupoid $V(\cdot)$ of the type (a, a, a) will be further called *fundamental restriction of the first kind* of the minimal SH-groupoid $V(\cdot)$.

2.5. Example. The following five-element groupoid $F(\cdot)$ is just the fundamental restriction of the minimal SH-groupoid $V(\cdot)$ and $\text{sdist}(F(\cdot)) = 1$.

F	a	b	c	d	e
a	b	c	e	e	e
b	d	e	e	e	e
c	e	e	e	e	e
d	e	e	e	e	e
e	e	e	e	e	e

It is easy to check that the groupoid $F(\cdot)$ is also the fundamental restriction of the SH-groupoid $W(\cdot)$.

2.6. Lemma. Let k be an arbitrary given positive integer such that $k \geq 5$. Consider the k -th restriction $W^k(\cdot)$ for the SH-groupoid $W(\cdot)$. Then the underlying set of the restricted groupoid $W^k(\cdot)$ is finite and the groupoid $W^k(\cdot)$ is an SH-groupoid of the type (a, a, a) having $\text{sdist}(W^k(\cdot)) = 2$.

Proof. It is obvious and similar to the proof of Lemma 2.3. ■

2.7. Definition. Let $W(\cdot)$ be the minimal SH-groupoid of the second kind constructed in 2.1 and let $k = 5$. Then the corresponding 5-th restriction of the SH-groupoid $W(\cdot)$ of the type (a, a, a) will be further called *fundamental restriction of the second kind* of the minimal SH-groupoid $W(\cdot)$.

2.8. Example. The following seven-element groupoid $H(\cdot)$ is the fundamental restriction of the minimal SH-groupoid $W(\cdot)$ of the second kind and $\text{sdist}(H(\cdot)) = 2$.

G	a	b	c	d	f	g	h
a	b	c	f	g	h	h	h
b	d	f	h	g	h	h	h
c	g	h	h	h	h	h	h
d	f	h	h	h	h	h	h
f	h	h	h	h	h	h	h
g	h	h	h	h	h	h	h
h	h	h	h	h	h	h	h

3. Stratified SH-groupoids generated by two-element set

In this part, let $F(\cdot)$ denote an absolutely free groupoid generated by a two-element set $\{a, p\}$ and let $\lambda(u)$ denote the length of an arbitrary element $u \in G$.

3.1. Construction. Denote by κ the least congruence on $F(\cdot)$ such that $a \cdot aa \neq aa \cdot a$, $aa \cdot aa = a(aa \cdot a)$ and $x \cdot yz = xy \cdot z$ for every $x, y, z \in G$ whenever $(a, a, a) \neq (x, y, z)$. Consider the groupoid $F/\kappa(\cdot)$. It is obvious that $\lambda(x \cdot y) = \lambda(x) + \lambda(y) \neq 1$ and, thus, $x \cdot y \neq a$ for every $x, y \in F$.

It follows from this that $(tu \cdot v) \cdot w = tu \cdot vw = t \cdot (u \cdot vw)$ for every $t, u, v, w \in F$. Therefore, $x_1 \cdot x_2 \dots x_n = x_1 x_2 \cdot x_3 \dots x_n = \dots = x_1 x_2 \dots x_{n-1} \cdot x_n$, for each positive integer $n \geq 5$ and $x_1, x_2, \dots, x_n \in F$.

Denote, for the simplicity, the groupoid $F/\kappa(\cdot)$ as $E(\cdot)$. The groupoid $E(\cdot)$ contains a proper subgroupoid generated by the one-element set $\{a\}$. It is obvious that this is just the λ -stratified SH-groupoid $V(\cdot)$ constructed in 2.1.

Similarly, $E(\cdot)$ contains a proper subgroupoid $P(\cdot)$ generated by the one-element set $\{p\}$. It follows immediately from the construction that it is an infinite semigroup generated by the one-element set $\{p\}$. Further, it is easy to see that for each positive integer $m \geq 2$ and each m -element ordered set $(x_1, x_2, \dots, x_m) \in \{a, p\}^m$ such that $(a, a, \dots, a) \neq (x_1, x_2, \dots, x_m) \neq (p, p, \dots, p)$ the underlying set E contains just $2^m - 2$ different elements $x_1 x_2 \dots x_m$ of the length m . Therefore, for each positive integer $n \geq 4$ the corresponding layer $L_n(E)$ contains 2^n elements. Further, it is obvious that $\text{card}(L_3(E)) = 9$, $\text{card}(L_2(E)) = 4$ and $\text{card}(L_1(E)) = 2$.

3.2. Lemma. *Let $E(\cdot)$ be the groupoid constructed in 3.1 Then*

- (i) $E(\cdot)$ is the λ -stratified groupoid;
- (ii) $E(\cdot)$ is a primitive extension of the SH-groupoid $V(\cdot)$, see [14];
- (iii) $E(\cdot)$ is an SH-groupoid of the type (a, a, a) ;
- (iv) $\text{sdist}(E(\cdot)) = 1$.

Proof. It follows immediately from Construction 3.1. ■

3.3. Construction. Let $E(\cdot)$ be the λ -stratified SH-groupoid constructed in 3.1. For each positive integer $n \geq 2$ and $(a, a, \dots, a) \neq (x_1, x_2, \dots, x_n) \neq (p, p, \dots, p)$, consider the element $u = x_1 x_2 \dots x_n$ of the length $\lambda(u) = n$. Let $\alpha(u)$ denote the number of all placements of the element a in the corresponding ordered set $(x_1, x_2, \dots, x_n)$. Similarly, let $\pi(u)$ denote the number of all placements of the element p in the corresponding ordered set $(x_1, x_2, \dots, x_n)$.

For arbitrary given positive integers k, m , define a mapping σ of the set E to the set of positive integers such that $\sigma(u) = k \cdot \alpha(u) + m \cdot \pi(u)$. Especially, we have $\sigma(a) = k$, $\sigma(b) = 2k$, $\sigma(c) = 3k = \sigma(d)$ and $\sigma(a^n) = n \cdot k$ for each $n \geq 4$. Finally, we have $\sigma(p^n) = n \cdot m$ for each positive integer n . It is easy to check that the mapping σ is also a mapping stratifying the underlying set E . Of course, some of the corresponding σ -layers $L_n(E)$ could be empty subsets of the set E .

4. Reduced stratified SH-groupoids of type (a, a, a)

In this part, of the paper consider the groupoid $E(\cdot)$ constructed in 3.1. Let n be an arbitrary given positive integer. Suppose, at first, that there is at least one stratified groupoid $E_n(\cdot)$ satisfying the condition $a^2 = p^n$. If σ is a mapping stratifying the underlying set E then $2 \cdot \sigma(a) = n \cdot \sigma(p)$.

4.1. Construction. Let $n = 2k$ where k is an arbitrary positive integer and let $E(\cdot)$ be the groupoid constructed in 3.1. Put $\sigma(a) = k$, $\sigma(p) = 1$ and, further, let $\sigma(u) = k \cdot \alpha(u) + \pi(u)$ in the remaining cases. Now, consider the least congruence κ on E containing the ordered pair (b, p^{2k}) . Then $\sigma(b) = \sigma(p^{2k})$ and, thus, κ is a σ -stratified congruence on the groupoid $E(\cdot)$. Denote, for the simplicity, the corresponding groupoid $E/\kappa(\cdot)$ as $E_{2k}(\cdot)$.

4.2. Construction. Let $n = 2k + 1$ where k is an arbitrary positive integer and let $E(\cdot)$ be the groupoid constructed in 3.1. Put $\sigma(a) = 2k + 1$, $\sigma(p) = 2$ and, further, let $\sigma(u) = (2k + 1) \cdot \alpha(u) + 2 \cdot \pi(u)$ in the remaining cases. Now, consider the least congruence κ on $E(\cdot)$ containing the ordered pair (b, p^{2k+1}) . Then $\sigma(b) = \sigma(p^{2k+1})$ and, thus, κ is a σ -stratified congruence on the groupoid $E(\cdot)$. Denote, for the simplicity, the corresponding groupoid $E/\kappa(\cdot)$ as $E_{2k+1}(\cdot)$.

4.3. Lemma. *Let $n \geq 2$ and let $E_n(\cdot)$ be the groupoid constructed in 4.1, 4.2. Then*

- (i) $E_n(\cdot)$ is a groupoid generated by the two-element set $\{a, p\}$;
- (ii) $E_n(\cdot)$ is a σ -stratified SH-groupoid containing the non-associative triple (a, a, a) ;
- (iii) the equation $x \cdot y = b$ is solvable in $E_n(\cdot)$ and it has just n different solutions;
- (iv) the equation $x \cdot y = c$ is solvable in $E_n(\cdot)$ and it has just n different solutions;
- (v) the equation $x \cdot y = d$ is solvable in $E_n(\cdot)$ and it has just n different solutions.

Proof. (i) It is obvious that $c = a \cdot b = a \cdot aa \neq aa \cdot a = b \cdot a = d$. Further, it follows from the construction that $x \cdot yz = xy \cdot z$ whenever $x, y, z \in H_n$ are such that $(a, a, a) \neq (x, y, z)$.

(ii) The mapping σ is a mapping stratifying the underlying set E_n . Thus, $E_n(\cdot)$ is a σ -stratified SH-groupoid of the type (a, a, a) .

(iii) It is obvious that $b = a \cdot a = p \cdot p^{n-1} = p^2 \cdot p^{n-2} = \dots = p^{n-1} \cdot p$. Thus, the equation $x \cdot y = b$ has just only n solutions $(a, a), (p, p^{n-1}), \dots, (p^{n-1}, p)$.

(iv) Similarly, $c = a \cdot b = ap \cdot p^{n-1} = ap^2 \cdot p^{n-2} = \dots = ap^{n-1} \cdot p$. Thus, the equation $x \cdot y = c$ has just only n solutions $(a, b), (ap, p^{n-1}), \dots, (ap^{n-1}, p)$.

(v) It is obvious that $d = b \cdot a = p \cdot p^{n-1}a = p^2 \cdot p^{n-2}a = \dots = p^{n-1} \cdot pa$. The equation $x \cdot y = d$ has just only n solutions $(b, a), (p, p^{n-1}a), \dots, (p^{n-1}, pa)$. ■

4.4. Lemma. *Let $n \geq 2$ be a positive integer. Then $\text{sdist}(E_n(\cdot)) \leq n$.*

Proof. Define on the set E_n a new binary operation $\circ$ such that $b \circ a = c \neq d = b \cdot a$ and $p^k \circ p^{n-k}a = c \neq d = p^k \cdot p^{n-k}a$ for each $1 \leq k < n$ and, further, let $x \circ y = x \cdot y$ in the remaining cases. Then $a \circ (a \circ a) = (a \circ a) \circ a$. It is possible to check that $x \circ (y \circ z) = (x \circ y) \circ z$ whenever $x, y, z \in E_n$ are such that $(x, y, z) \neq (a, a, a)$. Thus, $E_n(\circ)$ is a semigroup and the rest is clear. ■

4.5. Lemma. *Let $E_n(\cdot)$ be the groupoid constructed in 4.1, 4.2. Further, let $E_n(\star)$ be an arbitrary semigroup having $\text{sdist}(E_n(\cdot)) = \text{dist}(E_n(\cdot), (E_n(\star)))$. Then*

- (i) *either $d = a \star b \neq a \cdot b = c$, or $c = b \star a \neq b \cdot a = d$;*
- (ii) *if $b \star a \neq b \cdot a$ and $x \cdot y = b$, then $c = b \star a = x \star (y \star a) \neq x \cdot (y \cdot a)$;*
- (iii) *if $a \star b \neq a \cdot b$ and $x \cdot y = b$, then $d = a \star b = (a \star x) \star y \neq (a \cdot x) \cdot y$.*

Proof. (i) The subgroupoid $V(\cdot)$ generated by the one-element set $\{a\}$ is a minimal SH-groupoid of the type (a, a, a) and the equation $x \cdot y = b$ is solvable in $V(\cdot)$ only if $(x, y) = (a, a)$. Therefore, either $d = a \star b \neq a \cdot b = c$, or $c = b \star a \neq b \cdot a = d$.

(ii) Now, if $c = b \star a$ then also $c = p^k \star (p^{n-k} \star a) \neq p^k \cdot (p^{n-k} \cdot a) = d$ for each $1 \leq k < n$. Therefore, $\text{dist}(E_n(\star), E_n(\cdot)) \geq n$ and the rest is clear.

(iii) The other case $d = a \star b$ is similar to (ii). ■

4.6. Lemma. *Let $E_n(\cdot)$ be the groupoid constructed in 4.1, 4.2. Then*

$$\text{sdist}(E_n(\cdot)) = n.$$

Proof. It follows from proof of the previous lemma. ■

5. Fundamental restrictions of reduced stratified SH-groupoids

In this part, consider the reduced stratified SH-groupoids $E_n(\cdot)$ constructed in 4.1 and 4.2. For each positive integer m there are finite groupoids $F_{(n,m)}(\cdot)$ constructed as a m -th restriction of the reduced stratified SH-groupoid $E_n(\cdot)$.

Put, further, $m = 3 \cdot \sigma(a) + 1$ and consider the fundamental restriction of the reduced σ -stratified SH-groupoid $E_n(\cdot)$.

Especially, for each even positive integer $n = 2k$ let σ be a mapping stratifying the underlying set E_{2k} such that $\sigma(a) = k$ and $\sigma(p) = 1$. Then we have $m = 3k + 1$. Denote the corresponding fundamental restriction of the SH-groupoid $E_{2k}(\cdot)$ as $\overline{E}_{2k}(\cdot)$.

Similarly, for an arbitrary given odd positive integer $n = 2k + 1$, $n \geq 3$, let σ be a mapping stratifying the underlying set E_{2k+1} such that $\sigma(a) = 2k + 1$ and $\sigma(p) = 2$. Then $m = 6k + 4$. Denote the corresponding fundamental restriction of the SH-groupoid $E_{2k+1}(\cdot)$ as $\overline{E}_{2k+1}(\cdot)$.

It is obvious that the fundamental restrictions $\overline{E}_{2k}(\cdot)$ and $\overline{E}_{2k+1}(\cdot)$ are finite non-associative SH-groupoids. In this section, it will be proved that

$$\text{sdist}(\overline{E}_{2k}(\cdot)) = 2k \text{ and } \text{sdist}(\overline{E}_{2k+1}(\cdot)) = 2k + 1.$$

5.1 Lemma. *For each positive integer $n \geq 2$ we have $\text{sdist}(\overline{E}_n(\cdot)) \leq n$.*

Proof. The proof is similar to the proof of Lemma 4.4. ■

5.2 Proposition. *For each positive integer k we have $\text{sdist}(\overline{E}_{2k+1}(\cdot)) = 2k + 1$.*

Proof. Consider the fundamental restriction $\overline{E}_{2k+1}(\cdot)$ and suppose that there exists a semigroup $\overline{E}_{2k+1}(\nabla)$ having $\text{dist}(\overline{E}_{2k+1}(\cdot), \overline{E}_{2k+1}(\nabla)) < 2k + 1$. In this case we have $a^2 = p^{2k+1}$ and the corresponding stratifying mapping σ satisfies the condition $\sigma(a) = 2k + 1$ and $\sigma(p) = 2$.

Denote by D_p the set describing "different products" and containing just only all ordered pairs (x, y) satisfying the condition $x \cdot y \neq x \star y$. An unknown semigroup $\overline{E}_{2k+1}(\nabla)$ will be investigated step by step. For each reconstructing step denote by K_p the set of so far known pairs (x, y) having $x \cdot y \neq x \star y$ and let U_p denote the set of the remaining (still unknown) ordered pairs from the set D_p .

It is obvious that at least one of the following three inequalities has to be valid: $a \nabla a \neq a \cdot a$, $a \nabla b \neq a \cdot b$, $b \nabla a \neq b \cdot a$. Thus, the set K_p contains at least one element at the starting step. Therefore, the set U_p can contain at most $2k - 1$ elements at the same time.

Finally, consider the following $2k$ -element set I_u of "further investigated" ordered pairs $(p, u), (p^2, u), \dots, (p^k, u), (u, p^k), (u, p^{k-1}), \dots, (u, p)$ for $u \in U_p$. It is obvious that there is at least one positive integer $1 \leq i \leq k$ such that $p^i \nabla u = p^i \cdot u$ or $u \nabla p^i = u \cdot p^i$.

(i) In the first part of the proof suppose that there exists an element $u \in \overline{E}_{2k+1}$ such that $u = a \nabla a \neq b = a \cdot a$. Let, for example, $u \nabla p^i = u \cdot p^i$. If $a \nabla p^i = a \cdot p^i = r_i$ and $a \nabla r_i = a \cdot r_i$ then $\sigma(u \nabla p^i) = \sigma(u) + \sigma(p^i) = 2i + \sigma(u)$. But we also have $\sigma(a \nabla a \nabla p^i) = \sigma(a) + \sigma(a \nabla p^i) = \sigma(a) + \sigma(a \cdot p^i) = 2\sigma(a) + \sigma(p^i) = 2\sigma(a) + 2i$. It follows from this that $\sigma(u) = 2\sigma(a)$. But the corresponding layer contains only one element and this is just only the element $b = a \cdot a$, a contradiction. Therefore, at least one of the following two conditions has to be valid: $a \nabla p^i \neq a \cdot p^i = r_i$ or $a \nabla r_i \neq a \cdot r_i$. Thus, after this step, the set K_p contains at least two different elements and the set U_p can contain at most $2k - 2$ elements, now.

It follows from this that $p^i \nabla u = p^i \cdot u$ or in the set I_u there has to be another positive integer j that $p^j \nabla u = p^j \cdot u$ or $u \nabla p^j = u \cdot p^j$. Suppose, for example, that $p^j \nabla u = p^j \cdot u$. Now, if $p^j \nabla a = p^j \cdot a = t_j$ and $t_j \nabla a = t_j \cdot a$ then we obtain again $\sigma(p^j \nabla u) = \sigma(u) + \sigma(p^j) = 2j + \sigma(u)$.

But we also have $\sigma(p^j \nabla a \nabla a) = \sigma(p^j \cdot a \nabla a) = \sigma(t_j \nabla a) = \sigma(t_j \cdot a) = \sigma(p^j \cdot b) = 2j + 2\sigma(a)$. Therefore, we get $\sigma(u) = \sigma(b)$. But the corresponding layer contains only one element and, thus, $u = b$, a contradiction. It follows from this again that $p^j \nabla a \neq p^j \cdot a = t_j$ or $t_j \nabla a \neq t_j \cdot a$. That means, that after this step the set K_p contains at least three different elements and the set U_p can contain at most $2k - 3$ elements, now.

It is possible to continue in this way till the set U_p is empty. But then the set I_u still contains another positive integer m such that it is valid $u \nabla p^m = u \cdot p^m$ or $p^m \nabla u = p^m \cdot u$. In this case we get $a \nabla p^m = a \cdot p^m = r_m$, $a \nabla r_m = a \cdot r_m$ and, also, $p^m \nabla a = p^m \cdot a = t_m$, $t_m \nabla a = t_m \cdot a$. It is easy to see that from this we get a contradiction. Thus, $a \nabla a = a \cdot a$ in any case.

(ii) In this part of the proof suppose, that there exists an element $v \in \overline{E}_{2k+1}$ such that $v = a \nabla b \neq c = a \cdot b$ and let $a \nabla a = a \cdot a$. Of course, the set D_p contains

at most $2k$ elements and the set U_p can contain at most $2k - 1$ elements, now. Consider the following set $I_p = \{(p, p^{2k}), (p^2, p^{2k-1}), \dots, (p^{2k}, p)\}$. It is obvious that there exists a positive integer i such that $p^i \nabla p^{2k+1-i} = p^{2k+1} = b = a \cdot a$. But then $v = a \nabla b = a \nabla p^{2k+1} = a \nabla p^i \nabla p^{2k+1-i}$.

Further, if $a \nabla p^i = a \cdot p^i = q_i$ and $q_i \nabla p^{2k+1-i} = q_i \cdot p^{2k+1-i}$, then $v = (a \nabla p^i) \nabla p^{2k+1-i} = q_i \cdot p^{2k+1-i} = (ap^i) \cdot p^{2k+1-i} = a \cdot p^{2k+1} = a \cdot b = c$. This is a contradiction. Thus, at least one of the following two inequalities has to be valid: $a \nabla p^i \neq a \cdot p^i$ or $q_i \nabla p^{2k+1-i} \neq q_i \cdot p^{2k+1-i}$. Therefore, the set K_p contains at least two different elements and the set U_p can contain at most $2k - 2$ elements, now. It follows from this that the set I_u contains another positive integer $j \neq m$ such that again $p^j \nabla p^{2k+1-j} = p^{2k+1} = b = a \cdot a$. Now, similarly as above, both equalities $a \nabla p^j = a \cdot p^j = q_j$ and $q_j \nabla p^{2k+1-j} = q_j \cdot p^{2k+1-j}$ cannot be valid at the same time. The process of the reconstruction of the expected semigroup can continue step by step in this way till the time when the set U_p is empty. But, the set I_u contains again another positive integer m such that $p^m \nabla p^{2k+1-m} = p^{2k+1} = b = a \cdot a$. It is obvious that we get immediately another contradiction. Thus, $a \nabla b = a \cdot b$.

(iii) Finally, it is clear that the case $w = b \nabla a \neq d = b \cdot a$ and $a \nabla a = a \cdot a$ is similar to (ii). Thus, there is no semigroup $\overline{E}_{2k+1}(\nabla)$ having $\text{dist}(\overline{E}_{2k+1}(\cdot), \overline{E}_{2k+1}(\nabla)) < 2k + 1$ and the proof is finished. ■

5.3 Proposition. *For each positive integer k we have $\text{sdist}(\overline{E}_{2k}(\cdot)) = 2k$.*

Proof. Consider the fundamental restriction $\overline{E}_{2k}(\cdot)$ and suppose that there exists a semigroup $\overline{E}_{2k}(\nabla)$ having $\text{dist}(\overline{E}_{2k}(\cdot), \overline{E}_{2k}(\nabla)) < 2k$. In this case we get $a^2 = p^{2k}$ and, thus, the corresponding stratifying mapping σ satisfies the condition $\sigma(a) = k$ and $\sigma(p) = 1$.

Denote by D_p the set containing just only all ordered pairs (x, y) satisfying the condition $x \cdot y \neq x \nabla y$. An unknown semigroup $\overline{E}_{2k}(\nabla)$ will be investigated step by step during the proof. For each reconstructing step denote by K_p the set of all so far known pairs (x, y) having $x \cdot y \neq x \nabla y$ and denote by U_p the set of the remaining (still unknown) ordered pairs from the set D_p .

It is obvious that at least one of the following three inequalities has to be valid: $a \nabla a \neq a \cdot a$, $a \nabla b \neq a \cdot b$, $b \nabla a \neq b \cdot a$. The set K_p contains at least one element in the beginning and, thus, the set U_p can contain at most $2k - 2$ elements at the same time.

(i) In the first part of the proof suppose that there exists an element $u \in \overline{E}_{2k}$ such that $u = a \nabla a \neq b = a \cdot a$ and $\sigma(u) \neq 2\sigma(a)$. Now consider the $2k$ -element set I_u containing the ordered pairs $(p, u), (p^2, u), \dots, (p^k, u), (u, p^k), (u, p^{k-1}), \dots, (u, p)$. It is obvious that there has to be a positive integer $1 \leq i \leq k$ such that $p^i \nabla u = p^i \cdot u$ or $u \nabla p^i = u \cdot p^i$. Let, for example, $p^i \nabla u = p^i \cdot u$. Further, suppose that $p^i \nabla a = p^i \cdot a = t_i$ and $t_i \nabla a = t_i \cdot a$. Then $\sigma(p^i \nabla u) = \sigma(p^i \cdot u) = i + \sigma(u) \neq i + 2\sigma(a)$. But we also have $\sigma(p^i \nabla u) = \sigma((p^i \nabla a) \nabla a) = \sigma((p^i \cdot a) \nabla a) = \sigma(t_i \nabla a) = \sigma(t_i \cdot a) = \sigma(p^i \cdot (aa)) = i + 2\sigma(a)$, which is a contradiction. Thus, at least one of the following two conditions $p^i \nabla a \neq p^i \cdot a = t_i$, $t_i \nabla a \neq t_i \cdot a$ holds. Now the set K_p contains at least two elements and, thus, the set U_p can contain at most $2k - 3$ elements. It is easy to see that $u \nabla p^i = u \cdot p^i$ or there is another positive integer

$j \neq i$ such that $u \nabla p^j = u \cdot p^j$ or $p^j \nabla u = p^j \cdot u$. Let, for example, $u \nabla p^j = u \cdot p^j$. Similarly as above, it is easy to check that $a \nabla p^j \neq a \cdot p^j = r_j$ or $a \nabla r_j \neq a \cdot r_j$.

Of course, it is possible to continue in this way till the time when the set U_p is empty. In this case there exists again another positive integer m such that $p^m \nabla u = p^m \cdot u$ or $u \nabla p^m = u \cdot p^m$. But now we get $p^m \nabla a \neq p^m \cdot a = t_m$, $t_m \nabla a \neq t_m \cdot a$ and $a \nabla p^j = a \cdot p^j = r_j$, $a \nabla r_j = a \cdot r_j$. It is clear that from this we get a contradiction again. Thus, we have either $\sigma(u) = 2\sigma(a)$ or $a \nabla a = a \cdot a$.

(ii) In this part of the proof suppose that there is an element $u = a \nabla a \neq a \cdot a$ having $\sigma(u) = 2\sigma(a)$. Then there exists such a positive integer m that $u = p^m a p^{k-m}$. The set D_p contains at most $2k - 1$ elements including the ordered pair (a, a) . The set K_p contains now one element and, thus, the set U_p can contain at most $2k - 2$ other elements.

The set of the ordered pairs $(p, u), (p^2, u), \dots, (p^k, u), (u, p^k), \dots, (u, p)$ contains at least one ordered pair (p^i, u) or (u, p^i) such that $p^i \nabla u = p^i \cdot u$ or $u \nabla p^i = u \cdot p^i$. Let, for example, $p^i \nabla u = p^i \cdot u$ and suppose, furthermore, that $p^i \nabla a = p^i \cdot a = t_i$ and $t_i \nabla a = t_i \cdot a$. Then we get $p^i \nabla u = p^i \nabla (a \nabla a) = (p^i \nabla a) \nabla a = p^i a \cdot a = p^i \cdot aa = p^i b = p^i \cdot p^{2k}$. But we also have $p^i \nabla u = p^i \cdot p^m a p^{k-m}$ and $p^{2k+i} \neq p^{i+m} a p^{k-m}$ in $\overline{E}_{2k}$. Thus, at least one of the following inequalities has to be valid: $p^i \nabla a \neq p^i a = t_i$ or $t_i \nabla a \neq t_i \cdot a$. Then the set K_p contains at least two different elements and, thus, the set U_p can contain at most $2k - 3$ elements.

Therefore $u \nabla p^i = u \cdot p^i$ or there has to be another positive integer $j \neq i$ such that $u \nabla p^j = u \cdot p^j$ or $p^j \nabla u = p^j \cdot u$. Let, for example, $u \nabla p^j = u p^j$. Now, suppose, furthermore, that $a \nabla p^j = a p^j = r_j$ and $a \nabla r_j = a \cdot r_j = a \cdot a p^j$. It is easy to check that, similarly as above, we get $a \nabla p^j \neq a p^j$ or $a \nabla r_j \neq a \cdot r_j$. It is possible to continue in this way till the set U_p is empty and, then, we obtain again a contradiction. It is proved now that $a \nabla a = b = a \cdot a$.

(iii) This part of the proof is similar to (ii) of the proof of the previous proposition. Let $b = a \nabla a = a \cdot a = p^{2k}$ and put $v = a \nabla b \neq a \cdot b = c$. Then $a \nabla b = a \nabla a \nabla a = b \nabla a$. The set K_p contains one element and, thus, the set U_p can contain at most $2k - 2$ elements. Now, consider $2k - 1$ of the following ordered pairs $(p, p^{2k-1}), (p^2, p^{2k-2}), \dots, (p^{2k-1}, p)$. It is obvious that there is a positive integer i such that $p^i \nabla p^{2k-i} = p^i \cdot p^{2k-i} = p^{2k} = b$. But then we get $v = a \nabla p^{2k} = a \nabla (p^i \nabla p^{2k-i})$.

Now, suppose, that $a \nabla p^i = a p^i = r_i$ and $r_i \nabla p^{2k-i} = r_i \cdot p^{2k-i} = a p^i \cdot p^{2k-i} = a \cdot p^{2k} = a \cdot b = c$. In this case we get $v = a \nabla p^{2k} = a \nabla (p^i \nabla p^{2k-i}) = (a p^i) \nabla p^{2k-i} = r_i \cdot p^{2k-i} = (a p^i) \cdot p^{2k-i} = a \cdot (p^i p^{2k-i}) = a \cdot p^{2k} = a \cdot b = c$, a contradiction. Thus $a \nabla p^i \neq a p^i$ or $r_i \nabla p^{2k-i} \neq r_i \cdot p^{2k-i}$. But then the set K_p contains at least two different elements and, thus, the set U_p can contain at most $2k - 3$ elements. Therefore, there exists another positive integer $j \neq i$ such that $p^j \nabla p^{2k-j} = p^j \cdot p^{2k-j} = p^{2k} = b$. It is obvious that this process can continue again in the same way till the the set U_p is empty. But then there has to be another positive integer m such that $p^m \nabla p^{2k-m} = p^m \cdot p^{2k-m} = p^{2k} = b$ and we again obtain a contradiction.

(iv) In the remaining part let $b = a \nabla a = a \cdot a = p^{2k}$ and put $w = b \nabla a \neq b \cdot a = d$. It is clear that this case is similar to (iii). We can again, step by step,

obtain a system of contradictions and, finally, it is clear that there is no semigroup having the corresponding property.

It follows immediately from this that $\text{sdist}(\bar{E}_{2k}(\cdot)) \geq 2k$ and the rest is clear. ■

5.4 Theorem. *For an arbitrary given positive integer n there exists a finite non-associative groupoid $G(\cdot)$ generated by the two-element set $\{a, p\}$ satisfying the condition $a \neq x \cdot y \neq p$ and having $\text{sdist}(G(\cdot)) = n$.*

Proof. It follows immediately from the construction and from 5.2 and 5.3. ■

5.5 Example. It is easy to check that the following groupoid $H(\cdot)$ is just the least minimal SH-groupoid of the type (a, a, a) and of the second kind having $\text{sdist}(H(\cdot)) = 2$, see Lemma 2.2.

H	a	b	c	d	f	g
a	b	c	f	g	f	f
b	d	f	f	f	f	f
c	g	f	f	f	f	f
d	f	f	f	f	f	f
f	f	f	f	f	f	f
g	f	f	f	f	f	f

5.6 Example. The following table is just the table of the 11-element SH-groupoid $\bar{E}_2(\cdot)$ having $\text{sdist}(\bar{E}_2(\cdot)) = 2$.

2	a	p	b	r	t	c	d	u	v	w	e
a	b	r	c	u	v	e	e	e	e	e	e
p	t	b	u	w	d	e	e	e	e	e	e
b	d	u	e	e	e	e	e	e	e	e	e
r	v	c	e	e	e	e	e	e	e	e	e
t	u	w	e	e	e	e	e	e	e	e	e
c	e	e	e	e	e	e	e	e	e	e	e
d	e	e	e	e	e	e	e	e	e	e	e
u	e	e	e	e	e	e	e	e	e	e	e
v	e	e	e	e	e	e	e	e	e	e	e
w	e	e	e	e	e	e	e	e	e	e	e
e	e	e	e	e	e	e	e	e	e	e	e

5.7 Remark. The minimal SH-groupoid $G(\cdot)$ constructed in 2.8 and having the same semigroup distance has a seven-element underlying set. It follows from 5.5 that there are finite non-associative groupoids with the semigroup distance equal to 2 and having smaller underlying set. Does there exist a finite non-associative groupoid having at most five-element underlying set and having its semigroup distance equal to 2?

5.8 Example. The following table is just the table of the 16-element SH-groupoid $\bar{E}_3(\cdot)$ having $\text{sdist}(\bar{E}_3(\cdot)) = 3$.

3	p	a	f	g	h	b	q	r	s	t	u	c	d	v	w	e
p	f	h	b	r	q	t	v	w	d	e	e	e	e	e	e	e
a	g	b	q	t	u	c	e	e	e	e	e	e	e	e	e	e
f	b	s	t	w	d	e	e	e	e	e	e	e	e	e	e	e
g	q	u	c	e	e	e	e	e	e	e	e	e	e	e	e	e
h	r	t	t	e	e	e	e	e	e	e	e	e	e	e	e	e
b	t	d	e	e	e	e	e	e	e	e	e	e	e	e	e	e
q	c	e	e	e	e	e	e	e	e	e	e	e	e	e	e	e
r	v	e	e	e	e	e	e	e	e	e	e	e	e	e	e	e
s	w	e	e	e	e	e	e	e	e	e	e	e	e	e	e	e
t	e	e	e	e	e	e	e	e	e	e	e	e	e	e	e	e
u	e	e	e	e	e	e	e	e	e	e	e	e	e	e	e	e
c	e	e	e	e	e	e	e	e	e	e	e	e	e	e	e	e
d	e	e	e	e	e	e	e	e	e	e	e	e	e	e	e	e
v	e	e	e	e	e	e	e	e	e	e	e	e	e	e	e	e
w	e	e	e	e	e	e	e	e	e	e	e	e	e	e	e	e
e	e	e	e	e	e	e	e	e	e	e	e	e	e	e	e	e

6. Comments and open problems

6.1. It was proved in 2.5, 2.6 and 2.8 that there are finite non-associative groupoids having their semigroup distance equal to 1 resp. to 2. It follows from 5.5 that the construction using a fundamental restriction of stratified SH-groupoids $E_n(\cdot)$ is not minimal with respect to the number of elements contained in the underlying set $\overline{E}_n$.

6.2. Examples 5.7 and 5.9 show that the number of elements of the underlying set $\overline{E}_n$ quickly grows up with the increasing positive integer n .

6.3. The construction of finite non-associative groupoids $\overline{E}_n(\cdot)$ is based on the use of primitive extensions of the minimal SH-groupoids $V(\cdot)$ (see 2.1) having the semigroup distance equal to 1. It is possible to use also primitive extensions of the minimal SH-groupoid $W(\cdot)$ (see 2.1) and having semigroup distance equal to 2. It seems to be clear that a lesser number of elements in the underlying set can bring a bigger semigroup distance in this case. But, in that case, the proofs should be more complicated and the result should be the same.

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ON HOW TO CONSTRUCT LEFT SEMIMODULES FROM THE RIGHT ONES

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Abstract. In the paper, various constructions of left semimodules from the right ones are investigated.

Keywords: semiring, semimodule, ideal, congruence-simple.

2000 Mathematics Subject Classification: 16Y60.

1. Introduction

(Congruence-)simple semirings are studied, e.g., in [1], [2], [4], [5], [6], [7] and some of them are constructed as endomorphism semirings of commutative semigroups. That is, these semirings are characterized via semimodules. If S is a semiring and ${}_S M$ a left semimodule then the mapping $a \mapsto \lambda_a$, $\lambda_a(x) = ax$, $a \in S$, $x \in M$, is a (semiring) homomorphism of the semiring S into the full endomorphism semiring of the additive semigroup $M(+)$. This “canonical” homomorphism is injective if and only if the semimodule ${}_S M$ is faithful. Of course, if the semiring S is simple then the homomorphism is either constant or injective. Now, in order to get various more or less regular representations of (simple) semirings via endomorphisms, we have to find “nice” left semimodules (once we compose mappings from the right to the left). Unfortunately, it may happen that left semimodules of such kind are not easily available but, contrarywise, useful right semimodules are at hand. Therefore, we have to find a passage from the right side to the left side.

1. Preliminaries

A semiring is an algebraic structure possessing two associative binary operations (most frequently denoted as addition and multiplication) where the addition is commutative and the multiplication distributes over the addition from either side. Basic specimens are endomorphism semirings of commutative semigroups and basic pieces of information on semirings are available from [3].

A subset I of a semiring S is a left (right) ideal if $SI \cup (I + I) \subseteq I$ ($IS \cup (I + I) \subseteq I$). We put $\underline{R}(S) = \{a \in S \mid Sa = \{a\}\}$. If $\underline{R}(S) \neq \emptyset$ then $\underline{R}(S)$ is additively idempotent and it is the smallest (right) ideal of S . If, moreover, the right S -semimodule $\underline{R}(S)$ is faithful then the semiring S is additively idempotent.

A semiring S is said to be (congruence-)simple if it has just two congruence relations. If S is simple and $|\underline{R}(S)| \geq 2$ then $\underline{R}(S)_S$ is faithful (see 7.1) and S is additively idempotent.

In the sequel, all semirings and all semimodules are assumed to be additively idempotent. If $M(+)$ is a semilattice then the basic order is defined on M by $x \leq y$ if and only if $x + y = y$. An element $w \in M$ is the greatest element in the ordered set $M(\leq)$ iff $x + w = w$ for every $x \in M$. That is, $w = o_M$ is the (only) additively absorbing element. The existence of such an element will be denoted by $o_M \in M$. On the other hand, $o_M \notin M$ means that M has no absorbing element. Symmetrically, w is the smallest element iff $w = 0_M$ is additively neutral. Again, $0_M \in M$ means that such an element is present and $0_M \notin M$ means that not.

2. From the right to the left (a)

Let S be a non-trivial semiring and $M (= M_S(+, \cdot))$ a non-trivial right S -semimodule. For $x, y \in M$, let $P_{x,y} = \{z \in M \mid z \leq x, z \leq y\}$. If this set is non-empty then it is a subsemilattice of $M(+)$. Anyway, we put $W_1 = \{(x, y) \mid x, y \in M, P_{x,y} \neq \emptyset\}$ and $W_2 = \{(x, y) \in W_1 \mid o_P \in P = P_{x,y}\}$. Clearly, the ordered set $M(\leq)$ is a lattice iff $W_2 = M \times M$.

Now, we define a (possibly partial) binary operation $*$ on M by $x * y = o_P$, $P = P_{x,y}$, for every pair $(x, y) \in W_2$. Some easy observations follow:

2.1 Lemma.

- (i) $(x, x) \in W_2$ and $x * x = x$ for every $x \in M$.
- (ii) $x * y = y * x$ for every $(x, y) \in W_2$.
- (iii) $(x, y) \in W_2$ and $x * y = x$ iff $x \leq y$.
- (iv) $(x, x + y) \in W_2$ and $x * (x + y) = x$ for all $x, y \in M$.
- (v) If $(x, y) \in W_2$ then $x + (x * y) = x$.
- (vi) If $(x, y) \in W_2$, $(y, z) \in W_2$ and $(x * y, z) \in W_2$ then $(x, y * z) \in W_2$ and $x * (y * z) = (x * y) * z$.
- (vii) If $o_M \in M$ then $(x, o_M) \in W_2$ and $x * o_M = x$ for every $x \in M$.
- (viii) If $0_M \in M$ then $W_1 = M \times M$, $(x, 0_M) \in W_2$ and $x * 0_M = 0_M$ for every $x \in M$.

For $a \in S$ and $x \in M$, let $Q_{x,a} = \{y \in M \mid ya \leq x\}$. If this set is non-empty then it is a subsemilattice of $M(+)$. Anyway, we put $W_3 = \{(x, a) \mid Q_{x,a} \neq \emptyset\}$ and $W_4 = \{(x, a) \in W_3 \mid o_Q \in Q = Q_{x,a}\}$.

Now, we define a (possibly partial) left S -scalar multiplication on M by $a \circ x = o_Q$, $Q = Q_{x,a}$, $(x, a) \in W_4$. The following observations are quite easy to check:

2.2 Lemma.

- (i) $(xa, a) \in W_3$ for all $a \in S$ and $x \in M$.
- (ii) If $(xa, a) \in W_4$ then $x \leq a \circ (xa)$ and $xa = (a \circ xa)a$.
- (iii) If $o_M \in M$ then $(o_M, a) \in W_4$ for every $a \in S$ and $a \circ o_M = o_M$.
- (iv) If $o_M \in M$ then $(o_M a, a) \in W_4$ for every $a \in S$ and $a \circ (o_M a) = o_M$.

2.3 Lemma. Let $a \in S$ and $x, y \in M$.

- (i) If $(x, y) \in W_2$ then $Q_{x*y,a} = Q_{x,a} \cap Q_{y,a}$.
- (ii) If $(x, y) \in W_2$ and $(x * y, a) \in W_3$ then $Q_{x,a} \cap Q_{y,a} \neq \emptyset$.
- (iii) If $(x, a), (y, a) \in W_4$ and $(a \circ x, a \circ y) \in W_1$ then $Q_{x,a} \cap Q_{y,a} \neq \emptyset$.
- (iv) If $Q_{x,a} \cap Q_{y,a} \neq \emptyset$ then $(x, y) \in W_1$ and $(x, a), (y, a) \in W_3$.
- (v) If $Q_{x,a} \cap Q_{y,a} \neq \emptyset$ and $(x, y) \in W_2$ then $(x * y, a) \in W_3$.
- (vi) If $(x, y) \in W_2$ and $(x, a), (y, a), (x * y, a) \in W_4$ then $(a \circ x, a \circ y) \in W_2$ and $a \circ (x * y) = (a \circ x) * (a \circ y)$.

2.4 Lemma. Let $a, b \in S$ and $x \in M$.

- (i) $Q_{x,a+b} = Q_{x,a} \cap Q_{x,b}$.
- (ii) If $(x, a), (x, b) \in W_4$ and $(a \circ x, b \circ x) \in W_1$ then $Q_{x,a+b} \neq \emptyset$.
- (iii) If $Q_{x,a+b} \neq \emptyset$ then $(x, a), (x, b) \in W_3$.
- (iv) If $(x, a), (x, b), (x, a+b) \in W_4$ then $(a \circ x, b \circ x) \in W_2$ and $(a+b) \circ x = (a \circ x) * (b \circ x)$.

2.5 Lemma. Let $a, b \in S$ and $x \in M$.

- (i) If $(x, b) \in W_4$ then $Q_{b \circ x, a} = Q_{x, ab}$.
- (ii) If $(x, b), (b \circ x, a) \in W_4$ then $(x, ab) \in W_4$ and $a \circ (b \circ x) = (ab) \circ x$.
- (iii) If $(x, ab) \in W_3$ then $(x, b) \in W_3$.
- (iv) If $(x, b), (x, ab) \in W_4$ then $(b \circ x, a) \in W_4$ and $a \circ (b \circ x) = (ab) \circ x$.
- (v) If $(x, b) \notin W_3$ then $(x, ab) \notin W_3$.

2.6 Remark. Assume that $W_3 = W_4$ and the (partial) operation $\circ$ can be completed to a full one (denoted again by $\circ$) in such a way that $a \circ (b \circ x) = (ab) \circ x$ for all $a, b \in S$ and $x \in M$. Furthermore, assume that for every $x \in M$ there is $a_x \in S$ with $Ma_x = \{x\}$.

- (i) We have $a_x \circ x = o_M \in M$ for every $x \in M$. Moreover, if the right semi-module M_S is faithful then $a_x \in \underline{R}(S)$.

- (ii) Assume that the semiring S is simple. If M_S is not faithful then $xa = xb$ for all $a, b \in S$ and $x \in M$, and hence $xa = xa_x = x$ and $Ma_x = M$, a contradiction. Consequently, M_S is faithful.
- (iii) Assume that $a_x \in \underline{R}(S)$ for every $x \in M$ (see (i) and (ii)). Then $b \circ o_M = b \circ (a_x \circ x) = (ba_x) \circ x = a_x \circ x = o_M$ for every $b \in S$. If $a \in \underline{R}(S)$, $x \in M$ and $y = a \circ x$ then $o_M = a_y \circ y = a_y(a \circ x) = (a_y a) \circ x = a \circ x$. Thus $\underline{R}(S) \circ M = \{o_M\}$.
- (iv) Assume, finally, that the semiring S is simple and there is a binary operation $\boxplus$ defined on M such that $M(\boxplus, \circ)$ is a left S -semimodule. Combining (i), (ii) and (iii), we see that $\underline{R}(S) \circ M = \{o_M\}$, and hence the left S -semimodule ${}_S M = M(\boxplus, \circ)$ is not faithful. Since S is simple, it follows that $a \circ x = b \circ x$ for all $a, b \in S$ and $x \in M$. In particular, $a \circ x = a_x \circ x = o_M$, $S \circ M = \{o_M\}$ and $a \circ (xa) = o_M$. The latter equality implies $xa = o_M a$. Thus $xa = o_M a = xba$ for all $a, b \in S$ and $x \in M$. Since M_S is faithful, $a = ba$ and, S being simple, $|S| = 2$.

2.7 Remark. Assume that $W_1 = W_2 \neq M \times M$, choose $\alpha \in M$ and put $x * y = \alpha$ for all $(x, y) \in (M \times M) \setminus W_1$. Now, the binary operation $*$ is defined on M and it is idempotent and commutative. If $u < \alpha$ and $(x, \alpha) \notin W_1$ then $(x * \alpha) * u = \alpha * u = u \neq \alpha = x * u = x * (\alpha * u)$. If $(x, y) \notin W_1$ then either $\alpha \not\leq x$ or $\alpha \not\leq y$. If $\alpha \not\leq x$ and $(x, \alpha) \in W_1$ then $(y * x) * x = \alpha * x \neq \alpha = y * x = y * (x * x)$. Now, it is easy to see that $M(*)$ is a semilattice if and only if α is a minimal element of the ordered set $M(\leq)$. Anyway, if α is minimal then $\alpha \not\leq v$ for at least one $v \in M$ (otherwise $\alpha = 0_M \in M$ and $W_1 = M \times M$) and $(v * \alpha) + v = \alpha + v \neq v$. It means that $M(+, *)$ is not a lattice.

2.8 Remark. If $0_M \in M$ then $W_1 = M \times M$ and $W_3 = \{(x, a) \mid 0_M a \leq x\}$.

3. From the right to the left (b)

The foregoing section is immediately continued. Here, we assume that $W_1 = W_2$ and $W_3 = W_4$. Choose and fix an element ω such that $\omega \notin S \cup M$, put $M^+ = M \cup \{\omega\}$ and extend the (partial) operations $*$ and $\circ$ defined on M in the following way: $x * u = z * \omega = \omega * z = \omega * \omega = \omega$ for all $x, y, z \in M$, $(x, y) \notin W_1$, and $a \circ u = b \circ \omega = \omega$ for all $a, b \in S$, $u \in M$, $(u, a) \notin W_3$. Furthermore, put $x + \omega = \omega + x = x$ for every $x \in M$, $\omega + \omega = \omega$ and $\omega a = \omega$ for every $a \in S$.

3.1 Proposition. *The algebraic structure $M^+(+, *)$ is a lattice.*

Proof. First, it follows from 2.1(i),(ii) that the binary structure $M^+(*)$ is both idempotent and commutative. If $\alpha, \beta, \gamma \in M^+$ and $\omega \in \{\alpha, \beta, \gamma\}$ then $(\alpha * \beta) * \gamma = \omega = \alpha * (\beta * \gamma)$ (use 2.1(vi) and the equality $W_1 = W_2$). We have proved that $M^+(*)$ is a semilattice. Using 2.1(iv),(v), we see that $M^+(+, *)$ is a lattice. ■

3.2 Theorem. *The algebraic structure ${}_S M^+ = M^+(*, \circ)$ is a left S -semimodule.*

Proof. As we already know, $M^+(*)$ is a semilattice (see 3.1). It remains to show that $a \circ (\alpha * \beta) = (a \circ \alpha) * (a \circ \beta)$, $(a + b) \circ \alpha = (a \circ \alpha) * (b \circ \alpha)$ and $a \circ (b \circ \alpha) = (ab) \circ \alpha$

for all $a, b \in S$ and $\alpha, \beta \in M^+$. The case $\alpha = \omega$ (or $\beta = \omega$) is clear and we can assume $\alpha = x, \beta = y, x, y \in M$. Now, we distinguish the following cases:

(i) Let $(x, y) \in W_1$ and $(x * y, a) \in W_3$. By 2.3(i),(ii), we have $Q_{x*y,a} = Q_{x,a} \cap Q_{y,a} \neq \emptyset$ and, by 2.3(iv) we have $(x, a), (y, a) \in W_3$. Now, $(a \circ x, a \circ y) \in W_1$ and $a \circ (x * y) = (a \circ x) * (a \circ y)$ by 2.3(vi).

(ii) Let $(x, y) \in W_1$ and $(x * y, a) \notin W_3$. By 2.3(v), $Q_{x,a} \cap Q_{y,a} = \emptyset$, and hence $(a \circ x, a \circ y) \notin W_1$ follows from 2.3(ii). Thus $a \circ (x * y) = \omega = (a \circ x) * (a \circ y)$.

(iii) Let $(x, y) \notin W_1$. Then $a \circ (x * y) = a \circ \omega = \omega$. On the other hand, $Q_{x,a} \cap Q_{y,a} = \emptyset$ by 2.3(iv), and hence, due to 2.3(iii), $(a \circ x, a \circ y) \notin W_1$. Then $(a \circ x) * (a \circ y) = \omega$.

(iv) Let $(x, a + b) \in W_3$ and $(x, a), (x, b) \in W_3$. By 2.4(iv), $(a + b) \circ x = (a \circ x) * (b \circ x)$.

(v) Let $(x, a + b) \in W_3$ and $(x, a) \notin W_3$. Then $Q_{x,a} = \emptyset$, and hence $Q_{x,a+b} = Q_{x,a} \cap Q_{x,a} = \emptyset$, a contradiction.

(vi) Let $(x, a + b) \notin W_3$. Then $(a + b) \circ x = \omega$. We have $Q_{x,a+b} = \emptyset$, and therefore $(a \circ x, b \circ x) \notin W_1$ follows from 2.4(ii). Thus $(a \circ x) * (b \circ x) = \omega$.

(vii) Let $(x, b) \in W_3$ and $(b \circ x, a) \in W_3$. Then $a \circ (b \circ x) = (ab) \circ x$ by 2.5(ii).

(viii) Let $(x, b) \in W_3$ and $(b \circ x, a) \notin W_3$. Then $a \circ (b \circ x) = \omega$, $(x, ab) \notin W_3$ by 2.5(i) and $(ab) \circ x = \omega$.

(ix) Let $(x, b) \notin W_3$. Then $a \circ (b \circ x) = \omega = (ab) \circ x$ by 2.5(v). ■

3.3 Proposition. *The algebraic structure $M_S^+ = M^+(+, \cdot)$ is a right S -semimodule.*

Proof. It is easy. ■

3.4 Remark. Clearly, $(a \circ x)b \leq a \circ (xb)$ for all $a, b \in S$ and $x \in M^+$.

3.5 Lemma.

- (i) $S \circ \omega = \{\omega\}$.
- (ii) If $o_M \in M$ then $S \circ o_M = \{o_M\}$.
- (iii) If $0_M \in M$ then $S \circ 0_M = \{0_M\}$ iff $0_M S = \{0_M\}$ and $xa \neq 0_M$ for all $a \in S$ and $x \in M \setminus \{0_M\}$.
- (iv) If $x \in M$ then $S \circ x = \{x\}$ iff $xS \leq x$ and $ya \not\leq x$ for all $a \in S, y \in M, y \not\leq x$.

3.6 Lemma. *The following conditions are equivalent for $a, b \in S$:*

- (i) $a \circ x = b \circ x$ for every $x \in M$.
- (ii) $xa = xb$ for every $x \in M$.

3.7 Corollary. *The left semimodule ${}_S M^+$ is faithful if and only if the right semimodule M_S is so.*

3.8 Lemma. *Let $a \in S$ and $w \in M$ be such that $Ma \leq w$. Then $a \circ w = o_M \in M$.*

3.9 Proposition. *Assume that for every $x \in M$ there is $a_x \in S$ with $Ma_x = \{x\}$. Then:*

- (i) $a_x \circ y = o_M \in M$ for every $y \in M^+$, $x \leq y$ (or $x * y = x$).
- (ii) $a_x \circ z = \omega$ for every $z \in M^+$, $x \not\leq z$ (or $x * z \neq x$).
- (iii) $a_x * M^+ = \{o_M, \omega\}$.
- (iv) The semimodule ${}_S M^+$ is simple.

Proof. Only (iv) needs a short proof. Let $\varrho \neq \text{id}$ be a congruence of ${}_S M^+$. We have $o_M \in M$ by 3.8 and if $(\omega, o_M) \in \varrho$ then $\varrho = M \times M$. If $(\omega, x) \in \varrho$ for some $x \in M$ then $(\omega, o_M) = (a_x \circ \omega, a_x \circ x) \in \varrho$. If $(x, y) \in \varrho$, where $x, y \in M$, $x \not\leq y$, then $(o_M, \omega) = (a_x \circ x, a_x \circ y) \in \varrho$. Thus $\varrho = M \times M$ anyway. ■

The following assertions are easy.

3.10 Proposition. Assume that $0_M \in M$ and put $L^+ = M^+ \setminus \{0_M\}$. Then $W_1 = W_2 = M \times M$ and L^+ is a subsemimodule of the left S -semimodule ${}_S M^+$ if and only if the following three conditions are satisfied:

- 1. For all $x, y \in L = M \setminus \{0_M\}$ there is at least one $z \in L$ with $z \leq x$ and $z \leq y$;
- 2. If $a \in S$ is such that $0_M a = 0_M$ then for every $x \in L$ there is at least one $y \in L$ with $ya \leq x$;
- 3. If $a \in S$ is such that $0_M a \neq 0_M$ then there is at least one $v \in L$ with $0_M a = va$.

3.11 Lemma. Assume that $0_M \in M$ and that the set $L = M \setminus \{0_M\}$ has the smallest element w . Let $a, b \in S$ be such that $Ma = \{0_M\}$ and $Mb = \{w\}$. Then:

- (i) $a \neq b$ and $a \circ x = b \circ x$ for every $x \in L^+ = L \cup \{w\}$.
- (ii) If L^+ is a subsemimodule of ${}_S M^+$ (see 3.10) then L^+ is not faithful.

3.12 Remark.

- (i) We have $P'_{x,y} = \{z \in M^+ \mid x \leq z, y \leq z\} = \{z \in M \mid x+y \leq z\}$, $P'_{x,\omega} = \{z \in M^+ \mid x \leq z, \omega \leq z\} = \{z \in M \mid x \leq z\}$ and $P'_{\omega,\omega} = \{z \in M^+ \mid \omega \leq z\} = M^+$ for all $x, y \in M$.
- (ii) $Q'_{x,a} = \{y \in M^+ \mid x \leq a \circ y\} = \{y \in M \mid xa \leq y\}$ and $Q'_{\omega,a} = \{y \in M^+ \mid ya \leq \omega\} = \{\omega\}$ for all $a \in S$ and $x \in M$.

3.13 Remark. M is a subsemimodule of ${}_S M^+$ iff the following two conditions are satisfied:

- 1. For all $x, y \in M$ there is $z \in M$ with $z \leq x$ and $z \leq y$;
- 2. For all $a \in S$ and $x \in M$ there is $y \in M$ with $ya \leq x$.

If the condition (2) is true then the set Ma is downwards cofinal in $M(\leq)$ (cf. 3.9).

4. From the right to the left (c)

The second section is continued. We will assume here that $(W_1 =) W_2 = M \times M$ (cf. 2.6 and 2.7),

4.1 Proposition. *The algebraic structure $M(+, *)$ is a lattice.*

Proof. Use 2.1. ■

Now, choose $\alpha \in M$ and put $W'_3 = \{(x, a) \in W_3 \mid x \neq \alpha\}$. Furthermore, assume that $W_3 = W_4$ and put $a \triangle x = a \circ x$ for every pair $(x, a) \in W'_3$ and $b \triangle y = \alpha$ for every pair $(b, y) \in (S \times M) \setminus W'_3$.

4.2 Lemma. *$a \triangle (\alpha * x) = (a \triangle \alpha) * (a \triangle x)$ for all $a \in S$ and $x \in M$ iff the following two conditions are satisfied:*

1. *If $a \in S$ and $x \in M$ are such that $\alpha a \leq x$, $\alpha \not\leq x$ and $(\alpha, a) \in W_3$ then $(a \circ \alpha) * (a \circ x) = \alpha$;*
2. *If $a \in S$ and $x \in M$ are such that $x \neq \alpha$, $\alpha a \not\leq x$ and $(x, a) \in W_3$ then $(\alpha, a) \in W_3$, $\alpha \not\leq x$ and $(a \circ \alpha) * (a \circ x) = \alpha * (a \circ x)$.*

Proof. (i) Let $a \in S$ and $x \in M$ be such that $\alpha a \leq x$, $\alpha \not\leq x$ and $(\alpha, a) \in W_3$. We have $\alpha \in R_{x,a}$ and we get $a \triangle x = a \circ x$. Now, $(a \triangle \alpha) * (a \triangle x) = \alpha * (a \circ x) = \alpha$, since $\alpha \leq a \circ x$. On the other hand, $(\alpha, a), (x, a) \in W_3$, $\alpha * x \neq \alpha$, and hence $(\alpha * x, a) \in W_3$ and $a \circ (\alpha * x) = (a \circ \alpha) * (a \circ x)$ by 2.3(iii),(v). Of course, $a \circ (\alpha * x) = a \triangle (\alpha * x)$.

(ii) Let $a \in S$ and $x \in M \setminus \{\alpha\}$ be such that $\alpha a \not\leq x$ and $(x, a) \in W_3$. We have $(a \triangle \alpha) * (a \triangle x) = \alpha * (a \circ x) \neq \alpha$, since $\alpha a \not\leq x$. Now, if $a \triangle (\alpha * x) = (a \triangle \alpha) * (a \triangle x)$ then $a \triangle (\alpha * x) \neq \alpha$, and hence $a \triangle (\alpha * x) = a \circ (\alpha * x)$, $(\alpha * x, a) \in W'_3$. Consequently, $(\alpha, a) \in W_3$ and $a \circ (\alpha * x) = (a \circ \alpha) * (a \circ x)$ by 2.3(vi). If $\alpha \leq x$ then $\alpha * x = \alpha$ and $a \triangle (\alpha * x) = \alpha$.

(iii) Assume that both conditions (1) and (2) are satisfied. We wish to show that $y = z$, where $y = a \triangle (\alpha * x)$ and $z = (a \triangle \alpha) * (a \triangle x) = \alpha * (a \triangle x)$.

If $x = \alpha$ then $y = \alpha = z$. If $(x, a) \notin W_3$ then $(\alpha * x, a) \notin W_3$ and $y = \alpha = z$ again. Consequently, assume that $x \neq \alpha$ and $(x, a) \in W_3$, so that $(x, a) \in W'_3$ and $a \triangle x = a \circ x$, $z = \alpha * (a \circ x)$.

Let $\alpha a \leq x$. That is, $\alpha \leq a \circ x$ and $z = \alpha$. If $\alpha \leq x$ or $(\alpha * x, a) \notin W_3$ then $y = \alpha$. If $\alpha \not\leq x$ and $(\alpha * x, a) \in W_3$ then $(\alpha, a) \in W_3$, $\alpha * x \neq \alpha$ and $y = a \circ (\alpha * x) = (a \circ \alpha) * (a \circ x) = \alpha$ by (1) and 2.3(vi).

Let $\alpha a \not\leq x$. Then $\alpha \not\leq a \circ x$ and $z = \alpha * (a \circ x) \neq \alpha$. By (2), $(\alpha, a) \in W_3$, $\alpha \not\leq x$ and $z = \alpha * (a \circ x) = (a \circ \alpha) * (a \circ x)$. On the other hand, $(\alpha * x, a) \in W_3$ by 2.3(iii),(v) and $y = a \triangle (\alpha * x) = a \circ (\alpha * x) = (a \circ \alpha) * (a \circ x) = z$ by 2.3(vi). ■

4.3 Lemma. *$a \triangle (x * y) = (a \triangle x) * (a \triangle y)$ for all $a \in S$ and $x, y \in M \setminus \{\alpha\}$ such that $x * y \in M \setminus \{\alpha\}$ iff the following condition is satisfied:*

1. *If $(z, b) \in W'_3$ is such that $(u, b) \notin W_3$ for some $u \in M \setminus \{\alpha\}$, $z * u \neq \alpha$, then $\alpha b \leq z$.*

Proof. (i) As concerns (1), we have $(z * u, b) \notin W_3$, and hence $\alpha = b \triangle (z * u)$. On the other hand, $(b \triangle z) * (b \triangle u) = (b \circ z) * \alpha$.

(ii) Assume that (1) is true and put $u = a \triangle (x * y)$ and $v = (a \triangle x) * (a \triangle y)$. If $(x, a) \notin W_3$ and $(y, a) \notin W_3$ then $(x * y, a) \notin W_3$ and we get $u = \alpha = v$. If $(x, a), (y, a) \in W_3$ then $(x * y, a) \in W_3$ (use 2.3) and $u = a \circ (x * y) = (a \circ x) *$

$(a \circ y) = v$ (use 2.3 again). On the other hand, if $(x, a) \in W_3$ and $(y, a) \notin W_3$ then $(x * y, a) \notin W_3$ and $u = \alpha$, $v = (a \circ x) * \alpha$. By (1), $\alpha a \leq x$, $\alpha \leq a \circ x$ and $v = \alpha$. ■

4.4 Lemma. $(a + b) \triangle x = (a \triangle x) * (b \triangle x)$ for all $a, b \in S$ and $x \in M$ iff the following condition is satisfied:

1. If $(x, a) \in W_3$, $x \neq \alpha$, and $(x, b) \notin W_3$ then $\alpha a \leq x$.

Proof. (I) As concerns (1), we have $(x, a + b) \notin W_3$ by 2.4(i),(iii), and hence $(a + b) \triangle x = \alpha$. On the other hand, $(a \triangle x) * (b \triangle x) = (a \circ x) * \alpha$ and if $(a \circ x) * \alpha = \alpha$ then $\alpha a \leq x$.

(ii) Assume that (1) is true and put $y = (a + b) \triangle x$ and $z = (a \triangle x) * (b \triangle x)$. If $x = \alpha$ then $y = \alpha = z$, so that we will assume that $x \neq \alpha$. If $(x, a + b) \notin W_3$ then $y = \alpha$ and, by 2.4(ii), either $(x, a) \notin W_3$ or $(x, b) \notin W_3$. Assume the former case, the latter one being symmetric. If $(x, b) \in W_3$ then $z = \alpha * (b \circ x)$ $\alpha b \leq x$ by (1), $\alpha \leq b \circ x$ and $z = \alpha$. If $(x, b) \notin W_3$ then $z = \alpha * \alpha = \alpha$. Finally, let $(x, a + b) \in W_3$, so that $y = (a + b) \circ x$. By 2.4(i),(iii), we have $(x, a), (x, b) \in W_3$, and hence $z = (a \circ x) * (a \circ y) = (a + b) \circ x = y$ by 2.4(iv). ■

4.5 Lemma. $(ab) \triangle x = a \triangle (b \triangle x)$ for all $a, b \in S$ and $x \in M$ iff the following condition is satisfied:

1. If $(\alpha, c) \in W_3$, $(x, d) \in W_3$, $x \neq \alpha$ and $d \circ x = \alpha$ then $c \circ \alpha = \alpha$.

Proof. (i) As concerns (1), we have $(x, cd) \in W_4$ and $c \circ \alpha = c \circ (d \circ x) = (cd) \circ x = (cd) \triangle x$ by 2.5(ii). On the other hand, $c \triangle (d \triangle x) = c \triangle (d \circ x) = c \triangle \alpha = \alpha$.

(ii) Assume that (1) is true. We wish to show that $y = z$, where $y = (ab) \triangle x$ and $z = a \triangle (b \triangle x)$. If $x = \alpha$ then $y = \alpha = z$, and hence we assume $x \neq \alpha$. If $(x, b) \notin W_3$ then $(x, ab) \notin W_3$ and $y = \alpha = z$ again. If $(x, b) \in W_3$ and $(b \circ x, a) \notin W_3$ then the same is true. If $(x, b) \in W_3$, $(b \circ x, a) \in W_3$ and $b \circ x \neq \alpha$ then $(x, ab) \in W_3$ and $y = (ab) \circ x = a \circ (b \circ x) = a \triangle (b \circ x) = a \triangle (b \triangle x) = z$ by 2.5(ii). Finally, if $(x, b), (b \circ x, a) \in W_3$ and $b \circ x = \alpha$ then $y = (ab) \circ x = a \circ (b \circ x) = a \circ \alpha = \alpha = a \triangle \alpha = a \triangle (b \circ x) = a \triangle (b \triangle x) = z$ by (1). ■

4.6 Lemma. Assume that 4.5(1) is true. Let $(x, a) \in W_3$, $x \neq \alpha = a \circ x$. Then:

- (i) If $b \in S$ and $y \in M$ are such that $yb \leq \alpha$ then $y \leq \alpha$ and $ya \leq x$.
- (ii) If $c \in S$ is such that $Mc \leq \alpha$ then $\alpha = o_M$ and $Ma \leq x$.

Proof. It is easy. ■

4.7 Lemma. Let $a \in S$ and $x, y \in M \setminus \{\alpha\}$ be such that $x * y = \alpha$ and $a \triangle (x * y) = (a \triangle x) * (a \triangle y)$. Then:

- (i) If $z \in M$ is such that $za \leq \alpha$ then $z \leq \alpha$ and $a \circ \alpha \leq \alpha$.
- (ii) $va \not\leq \alpha$ for at least one $v \in M$.

Proof. (i) First, $(a \triangle x) * (a \triangle y) = a \triangle (x * y) = a \triangle \alpha = \alpha$. Then $za \leq \alpha < x, y$, so that $z \leq a \circ x$ and $z \leq a \circ y$. We conclude that $z \leq (a \circ x) * (a \circ y) = (a \triangle x) * (a \triangle y) = \alpha$. If $Ma \leq \alpha$ then $M \leq \alpha$ by (i), and hence $0_M = \alpha < x, a$ contradiction. ■

In the remaining part of this section, assume that $\alpha = 0_M \in M$.

4.8 Lemma. $a \triangle (x * y) = (a \triangle x) * (a \triangle y)$ for all $a \in S$ and $x, y \in M$ iff the following condition is satisfied:

1. If $a \in S$ and $x, y, w \in M \setminus \{0_M\}$ are such that $wa \leq x$ and $wa \leq y$ then $z \leq x$ and $z \leq y$ for at least one $z \in M \setminus \{0_M\}$.

Proof. First, $0_M \leq x$ for every $x \in M$, and hence the condition 4.2(1) is satisfied. Further, if $(x, a) \in W_3$ then $0_M a \leq x$ and 4.2(2) is true as well. By 4.2, $a \triangle (x * y) = (a \triangle x) * (a \triangle y)$ whenever $0_M \in \{x, y\}$. Similarly, 4.3(1) is true and $a \triangle (x * y) = (a \triangle x) * (a \triangle y)$ whenever $x * y \neq 0_M$.

Now, let $x, y \in M \setminus \{0_M\}$ and $x * y = 0_M$. Then $a \triangle (x * y) = 0_M$ and, if $(x, a) \notin W_3$ or $(y, a) \notin W_3$ then $(a \triangle x) * (a \triangle y) = 0_M$. Assume, therefore, that $(x, a), (y, a) \in W_3$. Then $(a \triangle x) * (a \triangle y) = (a \circ x) * (a \circ y)$, and so $(a \triangle x) * (a \triangle y) = 0_M$ iff $wa \leq x$ and $wa \leq y$ implies $w = 0_M$. ■

4.9 Lemma. Assume that $wa_0 = 0_M$ for some $a_0 \in S$ and $w \in M$, $w \neq 0_M$. Then $a \triangle (x * y) = (a \triangle x) * (a \triangle y)$ for all $a \in S$ and $x, y \in M$ iff the following condition is satisfied:

1. For all $x, y \in M \setminus \{0_M\}$ there is $z \in M \setminus \{0_M\}$ with $z \leq x$ and $z \leq y$.

Proof. See 4.8 (and (4.7)). ■

4.10 Lemma. $(a + b) \triangle x = (a \triangle x) * (b \triangle x)$ for all $a, b \in S$ and $x \in M$.

Proof. Apparently, the condition 4.4(1) is true. ■

4.11 Lemma. $(ab) \triangle x = a \triangle (b \triangle x)$ for all $a, b \in S$ and $x \in M$ iff the following condition is satisfied:

1. If $xa = 0_M$ for some $a \in S$ and $x \in M \setminus \{0_M\}$ then for all $b \in S$ and $y \in M \setminus \{0_M\}$ such that $0_M b \leq y$ there is $z \in M \setminus \{0_M\}$ with $zb \leq y$.

Proof. Use 4.5, where $\alpha = 0_M$ (see also 4.6). ■

4.12 Theorem. The algebraic structure $M(*, \triangle)$ is a left S -semimodule if and only if the conditions 4.8(1) and 4.11(1) are satisfied.

Proof. See 4.8, 4.10 and 4.11. ■

4.13 Proposition. (cf. 3.10) Assume that $w_0 a_0 = 0_M$ for some $a_0 \in S$ and $w_0 \in M \setminus \{0_M\}$. Then ${}_S M = M(*, \triangle)$ is a left S -semimodule if and only if the following three conditions are satisfied:

1. For all $x, y \in M \setminus \{0_M\}$ there is $z \in M \setminus \{0_M\}$ such that $z \leq x$ and $z \leq y$;
2. If $a \in S$ is such that $0_M a = 0_M$ then, for every $x \in M \setminus \{0_M\}$, there is at least one $y \in M \setminus \{0_M\}$ such that $ya \leq x$;

3. If $a \in S$ is such that $0_M a \neq 0_M$ then there is at least one $v \in M \setminus \{0_M\}$ such that $0_M a = va$.

Proof. See 4.9, 4.10 and 4.11 (clearly, (2) and (3) are equivalent to 4.11(1) under our assumptions). ■

4.14 Proposition. Assume that $xa \neq 0_M$ for all $a \in S$ and $x \in M \setminus \{0_M\}$ (i.e., $L = M \setminus \{0_M\}$ is a subsemimodule of M_S). Then $M(*, \triangle)$ is a left S -semimodule.

Proof. See 4.8, 4.10 and 4.11. ■

4.15 Lemma.

- (i) $S \triangle 0_M = \{0_M\}$.
- (ii) If $o_M \in M$ then $S \triangle o_M = \{o_M\}$.
- (iii) If $x \in M \setminus \{0_M\}$ then $S \triangle x = \{x\}$ iff $xs \leq x$ and $yz \not\leq x$ for all $a \in S$, $y \in M$, $y \not\leq x$.

4.16 Lemma. Let $a, b \in S$ be such that $a \triangle x = b \triangle x$ for every $x \in M$. If $y \in M \setminus \{0_M\}$ is such that $ya \neq 0_M \neq yb$ then $ya = yb$.

Proof. We have $a \circ ya = a \triangle ya = b \triangle ya$ and $a \circ ya \geq y > 0_M$. Thus $b \triangle ya = b \circ ya \geq y$ and $yb \geq ya$. Symmetrically, $ya \geq yb$, so that $ya = yb$. ■

4.17 Corollary. Assume that ${}_S M$ is a semimodule and for all $a, b \in S$, $a \neq b$, there is $x \in M \setminus \{0_M\}$ such that $0_M \neq xa \neq xb \neq 0_M$. Then the left S -semimodule ${}_S M$ is faithful.

4.18 Lemma. Let $a, b \in S$ be such that $a \triangle x = b \triangle x$ for every $x \in M$. If $0_M a \neq 0_M \neq a \circ 0_M a$ and $0_M b \neq 0_M \neq b \circ 0_M b$ then $0_M a = 0_M b$.

Proof. We can proceed similarly as in the proof of 4.16. ■

4.19 Lemma. Let $a \in S$ and $x \in M \setminus \{0_M\}$ be such that $Ma \leq x$. Then:

- (i) $a \triangle x = o_M \in M$.
- (ii) If $S \triangle x = \{o_M\}$ then $o_M S \leq x$.

Proof. It is easy. ■

4.20 Corollary. Assume that the semiring S is simple, ${}_S M$ is a semimodule and there are $a, b \in S$ and $x \in M \setminus \{0_M\}$ such that $Ma \leq x$ and $o_M b \not\leq x$. Then the left S -semimodule ${}_S M$ is faithful.

4.21 Lemma. Let $x \in M \setminus \{0_M\}$ be such that $Ma_x = \{x\}$ for some $a_x \in S$. Then:

- (i) $a_x \triangle y = o_M \in M$ for every $y \in M$, $x \leq y$ (or $x * y = x$).
- (ii) $a_x \triangle z = 0_M$ for every $z \in M$, $x \not\leq z$ (or $x * z \neq x$).
- (iii) $a_x \triangle M = \{o_M, 0_M\}$.

4.22 Lemma. Let $a \in S$ be such that $Ma = \{0_M\}$. Then:

- (i) $a \triangle x = o_M \in M$ for every $x \in M \setminus \{0_M\}$.
- (ii) $a \triangle 0_M = 0_M$.

4.23 Proposition. Assume that ${}_S M = M(*, \triangle)$ is a left S -semimodule and that for every $x \in M \setminus \{0_M\}$ there is $a_x \in S$ with $Ma_x = \{x\}$. Then the left semimodule ${}_S M$ is simple.

Proof. We can proceed similarly as in the proof of 3.9(iv) (where ω is replaced by 0_M). ■

4.24 Lemma. Assume that the set $L = M \setminus \{0_M\}$ has the smallest element w . Let $a, b \in S$ be such that $Ma = \{0_M\}$ and $Mb = \{w\}$. Then:

- (i) $a \neq b$ and $a \triangle x = b \triangle x$ for every $x \in M$.
- (ii) If ${}_S M$ is a left S -semimodule then it is not faithful.

4.25 Proposition. Assume that ${}_S M$ is a faithful left S -semimodule and that for every $x \in M$ there is $a_x \in S$ with $Ma_x = \{x\}$. Then both S and M are infinite and the set $L = M \setminus \{0_M\}$ has no minimal element.

Proof. Let $w \in L$ be minimal in L . We have $wa_0 = 0 (= 0_M)$ and it follows from 4.13(1) that w is the smallest element of L . Now, ${}_S M$ is not faithful due to 4.24(ii). ■

4.26 Remark.

- (i) Assume that the conditions 4.13(1),(2),(3) are satisfied. Then ${}_S M = M(*, \triangle)$ is a left S -semimodule (see 4.13, 4.14) and, by 3.10, the set $L^+ = (M \setminus \{0_M\}) \cup \{\omega\}$ is a subsemimodule of the left S -semimodule ${}_S M^+$. Now, it is easy to see that the mapping $x \mapsto x$, $x \in L = M \setminus \{0_M\}$ and $0_M \mapsto \omega$ is an isomorphism of the semimodule ${}_S M$ onto the semimodule ${}_S L^+$.
- (ii) Assume that ${}_S M$ is a (left S -)semimodule. Then either all the three conditions 4.13(1),(2),(3) are true or $0_M \notin LS$ and $L = M \setminus \{0_M\}$ is a subsemimodule of the right semimodule M_S .

5. From the right to the left (d)

The second section is continued. Here, we assume that $0_M \in M$ (then $W_1 = M \times M$) and that $W_1 = W_2$ (then $W_2 = M \times M$ and $M(+, *)$ is a lattice – see 4.1). Furthermore, assume that $W'_3 = \{(x, a) \in W_3 \mid x \neq 0_M\} = \{(x, a) \mid x \neq 0_M, 0_M a \leq x\} \subseteq W_4$. Similarly as in the fourth section, we put $a \triangle x = a \circ x$ for $(x, a) \in W'_3$ and $a \triangle x = 0_M$ for $(x, a) \in (S \times M) \setminus W'_3$.

5.1 Remark. Let $(x, a) \in W_3 \setminus W'_3$. Then $x = 0_M$, $Q = Q_{0_M, a} = \{y \mid ya = 0_M\} \neq \emptyset$ and we get $0_M \in Q$. If $Q = \{0_M\}$ then $o_Q = 0_M$ and $(0_M, a) \in W_4$. Of course, $Q \subseteq Q_{z, a}$ for every $z \in M$.

If $o_Q \notin Q$ (i.e., $(0_M, a) \notin W_4$) then $Q \neq Q_{u, a}$ for every $u \in M \setminus \{0_M\}$ and there is $v_u \in Q_{u, a}$ such that $0_M \neq v_u a \leq u$, $v_u \neq 0_M$. We have shown that for every $u \in L = M \setminus \{0_M\}$ there is $v \in L$ such that $va \leq u$, $va \in L$.

5.2 Lemma. $a \triangle (x * y) = (a \triangle x) * (a \triangle y)$ for all $a \in S$ and $x, y \in M$ iff the condition 4.8(1) is satisfied.

Proof. If $0_M \in \{x, y\}$ then $a \triangle (x * y) = a \triangle 0_M = 0_M = (a \triangle x) * (a \triangle y)$. Assume, therefore, that $x \neq 0_M \neq y$.

If $wa \leq x$ and $wa \leq y$ for some $w \in M \setminus \{0_M\}$ then $wa \leq x * y$, $w \leq a \triangle x$, $w \leq a \triangle y$ and $w \leq (a \triangle x) * (a \triangle y)$. Moreover, $(a \triangle x)a \leq x$, $(a \triangle y)a \leq y$, and hence $((a \triangle x) * (a \triangle y))a \leq x * y$. Then $(a \triangle x) * (a \triangle y) = a \triangle (x * y)$, provided that $x * y \neq 0_M$. On the other hand, if $x * y = 0_M$ then $a \triangle (x * y) = 0_M \neq (a \triangle x) * (a \triangle y)$.

If $wa \not\leq x * y$ for every $w \in M \setminus \{0_M\}$ then $a \triangle (x * y) = 0_M = (a \triangle x) * (a \triangle y)$. ■

5.3 Lemma. $(a + b) \triangle x = (a \triangle x) * (b \triangle x)$ for all $a, b \in S$ and $x \in M$.

Proof. Easy to check. ■

5.4 Lemma. $(ab) \triangle x = a \triangle (b \triangle x)$ for all $a, b \in S$ and $x \in M$ iff the condition 4.11(1) is satisfied.

Proof. Easy to check. ■

5.5 Theorem. The algebraic structure ${}_S M = M(*, \triangle)$ is a left S -semimodule if and only if the conditions 4.8(1) and 4.11(1) are satisfied.

Proof. Combine 5.2, 5.3 and 5.4. ■

5.6 Proposition. Assume that $w_0 a_0 = 0_M$ for some $a_0 \in S$ and $w_0 \in L = M \setminus \{0_M\}$ (equivalently, L is not a subsemimodule of M_S). Then ${}_S M = M(*, \triangle)$ is a left S -semimodule if and only if the conditions 4.13(1),(2),(3) are satisfied.

Proof. See the proof of 4.13. ■

5.7 Lemma.

- (i) $S \triangle 0_M = \{0_M\}$.
- (ii) If $o_M \in M$ then $S \triangle o_M = \{o_M\}$.

5.8 Proposition. Assume that the semiring S is simple, ${}_S M$ is a semimodule and that there are $a, b \in S$ and $x \in M \setminus \{0_M\}$ such that $Ma \leq x$ and $o_M b \not\leq x$. Then the left S -semimodule ${}_S M$ is faithful.

Proof. See 4.19 and 4.20. ■

5.9 Lemma. Let $x \in M \setminus \{0_M\}$ be such that $Ma_x = \{x\}$ for some $a_x \in S$. Then:

- (i) $a_x \triangle y = o_M \in M$ for every $y \in M$, $x \leq y$ (or $x * y = x$).
- (ii) $a_x \triangle z = 0_M$ for every $z \in M$, $x \not\leq z$ (or $x * z \neq x$).

5.10 Lemma. Let $a \in S$ be such that $Ma = \{0_M\}$. Then:

- (i) $a \triangle x = o_M \in M$ for every $x \in M \setminus \{0_M\}$.
- (ii) $a \triangle 0_M = 0_M$.

5.11 Proposition. Assume that ${}_S M = M(*, \triangle)$ is a left S -semimodule and that for every $x \in M \setminus \{0_M\}$ there is $a_x \in S$ with $Ma_x = \{x\}$. Then the left S -semimodule ${}_S M$ is simple.

Proof. See the proof of 4.23. ■

5.12 Proposition. *Assume that ${}_S M$ is a faithful left S -semimodule and that for every $x \in M$ there is $a_x \in S$ with $Ma_x = \{x\}$. Then both S and M are infinite and the set $M \setminus \{0_M\}$ has no minimal element.*

Proof. See the proof of 4.25. ■

6. A few conditions

Let $M (= M(+))$ be a non-trivial semilattice and let $x \leq y$ iff $x + y = y$. Let $N = M \setminus \{o_M\}$ ($N = M$ iff $o_M \notin M$) and $N' = N \setminus \{o_N\}$ ($N' = N$ iff $o_N \notin N$). Consider the following conditions:

- (C1) If $x_1 < x_2 < x_3 < \dots$ is an infinite strictly increasing sequence of elements from M then for every $x \in N'$ there is $i \geq 1$ with $x \leq x_i$;
- (C2) If $x_1 < x_2 < x_3 < \dots$ is an infinite strictly increasing sequence of elements from M then for every $x \in N$ there is $i \geq 1$ with $x \leq x_i$;
- (C3) If $x_1 < x_2 < x_3 < \dots$ is an infinite strictly increasing sequence of elements from M then for every $x \in M$ there is $i \geq 1$ with $x \leq x_i$;
- (C4) There is no infinite strictly increasing sequence of elements from M .

Clearly, (C4) implies (C3), (C3) implies (C2) and (C2) implies (C1). If M is finite then (C4) is true. If $o_M \notin M$ then (C1), (C2) and (C3) are equivalent. If $o_N \notin N$ then (C1) and (C2) are equivalent. If $o_M \in M$ then (C3) and (C4) are equivalent. Finally, if $o_N \in N$ then (C2), (C3) and (C4) are equivalent.

6.1 Lemma. *Let M_S be a right S -semimodule satisfying (C1). Then $W_1 = W_2$.*

Proof. Let $x, y \in M$ and $P = P_{x,y}$. If $x \leq y$ or $y \leq x$ then $P \neq \emptyset$ and $o_P \in P$ trivially. Assume, therefore, that $x \not\leq y$ and $y \not\leq x$. Then $x, y \in N'$. Now, if $z_1 < z_2 < z_3 < \dots$ is an infinite strictly increasing sequence of elements from P then $x \leq z_i$ for some $i \geq 1$, so that $x = z_i$ and $z_i = z_{i+1}$, a contradiction. Consequently, P satisfies (C4) and for every $z \in P$ there is $v \in P$ such that $z \leq v$ and v is maximal in P . But $v + P \subseteq P$, and so $v = o_P \in P$. ■

6.2 Lemma. *Let M_S be a right S -semimodule satisfying (C1) and let $a \in S$ and $x \in M$ be such that $Q = Q_{x,a} \neq \emptyset$ and $o_Q \in Q$. Then:*

- (i) $Q = N'$ (i.e., $N'a \leq x$).
- (ii) *The set N' has no maximal element.*
- (iii) $N' + N' = N'$ and $N + N = N$.
- (iv) *If $o_N \in N$ then M does not satisfy (C2).*
- (v) *If $o_M \in M$ then M does not satisfy (C3).*
- (vi) *M does not satisfy (C4).*

Proof. If $v \in Q$ is maximal in Q then $v + Q \subseteq Q$ implies $v = o_Q \in Q$, a contradiction. Therefore, the set Q has no maximal element and (C1) implies $N' \subseteq Q$. Of course, $o_M \notin Q$ (otherwise $o_M = o_Q$), and hence $o_N \notin Q$ either. Thus $Q = N'$ and the rest is clear. ■

6.3 Lemma. *Let M_S be a right S -semimodule satisfying (C1) and $0_M \in M$. Let $a \in S$ be such that $0_M a = 0_M$ and $o_Q \notin Q = Q_{0_M, a}$. Then:*

- (i) $N'a = \{0_M\}$.
- (ii) If $o_M \in M$ and $o_N \in N$ then $Ma = \{0_M, o_N a, o_M a\}$.
- (iii) If $o_M \in M$ and $o_N \notin N$ then $Na = \{0_M\}$ and $Ma = \{0_M, o_M a\}$.
- (iv) If $o_M \notin M$ then $Ma = \{0_M\}$.

Proof. Use 6.2. ■

6.4 Proposition. Let M_S be a right S -semimodule satisfying (C1) and $o_M \notin M$. Then:

- (i) M satisfies (C3).
- (ii) M does not satisfy (C4).
- (iii) $W_1 = W_2$.
- (iv) $W_3 = W_4$ if and only if for all $a \in S$ and $x \in M$ there is at least one $y \in M$ with $ya \not\leq x$.
- (v) If $0_M \in M$ and $W'_3 \subseteq W_4$ then $W_3 = W_4$.

Proof. Combine 6.1 and 6.2. ■

6.5 Proposition. Let M_S be a right S -semimodule satisfying (C1) and such that $o_M \in M$ and $o_N \notin N$. Then:

- (i) M satisfies (C2).
- (ii) M satisfies (C3) if and only if M satisfies (C4).
- (iii) $W_1 = W_2$.
- (iv) $W_3 = W_4$, provided that either $x + y = o_M$ for some $x, y \in N$ or the set N has at least one maximal element.
- (v) If $0_M \in M$ and $W'_3 \subseteq W_4$ then $W_3 = W_4$.

Proof. Combine 6.1 and 6.2. ■

6.6 Proposition. Let M_S be a right S -semimodule satisfying (C1) and such that $o_M \in M$ and $o_N \in N$. Then:

- (i) M satisfies (C2) iff M satisfies (C3) iff M satisfies (C4).
- (ii) $W_1 = W_2$.
- (iii) $W_3 = W_4$, provided that either $x + y = o_N$ for some $x, y \in N$ or the set N' has at least one maximal element.
- (iv) If $0_M \in M$ and $W'_3 \subseteq W_4$ then $W_3 = W_4$.

Proof. Combine 6.1 and 6.2. ■

7. One example

In this section, let S be a semiring such that $|\underline{R}(S)| \geq 2$. Then $\underline{R}(S)_S$ is a non-trivial right S -semimodule and the semiring S is non-trivial. In fact, $\underline{R}(S)$ is the smallest (right) ideal of S .

7.1 Proposition. *If the semiring S is simple then the right semimodule $\underline{R}(S)_S$ is faithful.*

Proof. If $\underline{R}(S)_S$ is not faithful then, using the fact that S is simple, we see that $ab = ac$ for all $a \in \underline{R}(S)$ and $b, c \in S$. In particular, if $b, c \in \underline{R}(S)$ then $b = bb = bc = c$, a contradiction. ■

If $a, b \in \underline{R}(S)$ then $P_{a,b} = \{c \in \underline{R}(S) \mid a+c=a, b+c=b\}$. If $a \in S$ and $b \in \underline{R}(S)$ then $Q_{b,a} = \{c \in \underline{R}(S) \mid ca+b=b\}$. As we have already defined in the second part of this note, we put $W_1 = \{(a, b) \mid P_{a,b} \neq \emptyset\}$, $W_2 = \{(a, b) \in W_1 \mid o_P \in P = P_{a,b}\}$, $W_3 = \{(b, a) \mid Q_{b,a} \neq \emptyset\}$ and $W_4 = \{(ba) \in W_3 \mid o_Q \in Q = Q_{ba}\}$.

7.2 Assume that $W_1 = W_2$ and $W_3 = W_4$.

7.2.1 Theorem. *The algebraic structure ${}^1_S \underline{R}(S) = \underline{R}(S)^+(*, \circ)$ is a simple left S -semimodule.*

Proof. See 3.2 and 3.9(iv). ■

7.2.2 Proposition.

(i) *The left semimodule ${}^1_S \underline{R}(S)$ is faithful if and only if the right semimodule $\underline{R}(S)_S$ is so.*

(ii) *If the semiring S is simple then the left semimodule ${}^1_S \underline{R}(S)$ is faithful.*

Proof. See 3.7 and 7.1. ■

7.2.3 Proposition. $o = o_{\underline{R}(S)} \in \underline{R}(S)$.

Proof. $a \circ a = o_{\underline{R}(S)}$ for every $a \in \underline{R}(S)$. ■

7.2.4 Proposition.

(i) $S \circ \omega = \{\omega\}$.

(ii) $S \circ o = \{o\}$.

(ii) *If $a \in \underline{R}(S) \setminus \{o\}$ then $S \circ a \neq \{a\}$.*

Proof. See 3.5. ■

7.2.5 Proposition. *Let $a \in \underline{R}(S)$. Then:*

(i) $a \circ b = o$ for every $b \in \underline{R}(S)$ such that $a + b = b$.

(ii) $a \circ a = o$.

(iii) $a \circ c = \omega$ for every $c \in \underline{R}(S)$ such that $a + c \neq c$.

(iv) $a \circ \omega = \omega$.

(v) $a \circ \underline{R}(S)^+ = \{o, \omega\}$.

Proof. See 3.9. ■

7.2.6 Proposition. *Assume that $0 = 0_{\underline{R}(S)} \in \underline{R}(S)$ and put $L = \underline{R}(S) \setminus \{0\}$. Then:*

(i) $W_1 = W_2 = \underline{R}(S) \times \underline{R}(S)$.

(ii) $W_3 = W_4 = \{(b, a) \mid a \in S, b \in \underline{R}(S), 0a + b = b\}$.

(iii) $L^+ = L \cup \{\omega\}$ is a subsemimodule of the left semimodule ${}_S\underline{R}(S)$ iff the following three conditions are satisfied:

1. For all $a, b \in L$ there is at least one $c \in L$ with $c + a = a$ and $c + b = b$;
2. If $a \in S$ is such that $0a = 0$ then for every $b \in L$ there is at least one $c \in L$ with $ca + b = B$;
3. If $a \in S$ is such that $0a \neq 0$ then there is at least one $b \in L$ with $0a = ba$.

(iv) If L^+ is a faithful semimodule then the set L has no minimal element.

Proof. See 3.10 and 3.11. ■

7.3 Assume that $0 = 0_{\underline{R}(S)} \in \underline{R}(S)$ and that $(W_1 =) W_2 = \underline{R}(S) \times \underline{R}(S)$. Furthermore, put $\alpha = 0$ and assume that $W'_3 \subseteq W_4$ (see 7.2.6 and the fifth section of this note).

7.3.1 Theorem. *The algebraic structure ${}_S^2\underline{R}(S) = \underline{R}(S)(*, \triangle)$ is a left S -semimodule if and only if the three conditions 7.2.6(iii)(1),(2),(3) are satisfied.*

Proof. See 5.6 and 7.2.6(iii). ■

In the rest of 7.3, assume that ${}_S^2\underline{R}(S)$ is a left semimodule (see 7.3.1).

7.3.2 Proposition. *The left semimodule ${}_S^2\underline{R}(S)$ is simple.*

Proof. See 4.23. ■

7.3.3 Proposition. *The semimodule ${}_S^2\underline{R}(S)$ is faithful in each of the following two cases:*

- (1) For all $a, b \in S$, $a \neq b$, there is at least one $c \in \underline{R}(S)$ with $c \neq 0 \neq ca \neq cb \neq 0$.
- (2) $|\underline{R}(S)| \geq 3$ and the semiring S is simple.

Proof. See 4.18 and 4.20. ■

7.3.4 Proposition. *If the left semimodule ${}_S^2\underline{R}(S)$ is faithful then $\underline{R}(S)$ is infinite and the set $L = \underline{R}(S) \setminus \{0\}$ has no minimal element.*

Proof. See 4.25. ■

7.3.5 Proposition. *Let $a \in \underline{R}(S)$, $a \neq 0$. Then:*

- (i) $a \triangle b = o = o_{\underline{R}(S)} \in \underline{R}(S)$ for every $b \in \underline{R}(S)$ such that $a + b = b$.
- (ii) $a \triangle a = o$.
- (iii) $a \triangle c = 0$ for every $c \in \underline{R}(S)$ such that $a + c \neq c$.
- (iv) $a \triangle \underline{R}(S) = \{0, o\}$.

Proof. See 4.21. ■

7.3.6 Proposition.

- (i) $0 \triangle a = o$ for every $a \in \underline{R}(S) \setminus \{0\}$.
- (ii) $0 \triangle 0 = 0$.

Proof. See 4.22. ■

7.3.7 Remark. By 7.2.6(iii), if the conditions (1),(2),(3) hold then the set $L^+ = L \cup \{\omega\}$, $L = \underline{R}(S) \setminus \{0\}$, is a subsemimodule of the left semimodule ${}^1_S \underline{R}(S)$. Now, the mapping $a \mapsto a$ for $a \in L$ and $0 \mapsto \omega$ is an isomorphism of ${}^2_S \underline{R}(S)$ onto ${}_S L^+$.

7.4 Proposition. Assume that $o_{\underline{R}(S)} \notin \underline{R}(S)$ and that the right semimodule $\underline{R}(S)_S$ satisfies (C1). Then:

- (i) $\underline{R}(S)$ satisfies (C2) and (C3) and does not satisfy (C4).
- (ii) $W_1 = W_2$.
- (iii) $W_3 \neq W_4$.

Proof. See 6.4 (we have $(a, b) \in W_3 \setminus W_4$ for all $a, b \in \underline{R}(S)$, $a \leq b$). ■

7.5 Proposition. Assume that $o = o_{\underline{R}(S)} \in \underline{R}(S)$, $o_T \notin T$, $T = \underline{R}(S) \setminus \{o\}$ and that $\underline{R}(S)_S$ satisfies (C1). Then:

- (i) $\underline{R}(S)$ satisfies (C2).
- (ii) $W_1 = W_2$.
- (iii) $W_3 = W_4$, provided that either $a + b = o$ for some $a, b \in T$ or the set T has at least one maximal element.

Proof. See 6.5. ■

7.6 Proposition. Assume that $o = o_{\underline{R}(S)} \in \underline{R}(S)$, $o_T \in T = \underline{R}(S) \setminus \{o_S\}$ and that $\underline{R}(S)_S$ satisfies (C1). Then:

- (i) $W_1 = W_2$.
- (ii) $W_3 = W_4$, provided that either $a + b = o_T$ for some $a, b \in T \setminus \{o\}$ or the set $T \setminus \{o_T\}$ has at least one maximal element.

Proof. See 6.6. ■

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Accepted: 01.07.2014

TOWARD A NEW ALGORITHM FOR SYSTEMS OF FRACTIONAL DIFFERENTIAL-ALGEBRAIC EQUATIONS

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Abstract. This paper is concerned with the development of an efficient algorithm for the analytic solutions of systems of fractional differential -algebraic equations (FDAE). The proposed algorithm is an elegant combination of the Laplace transform method (LTM) with the homotopy analysis method (HAM). The biggest advantage of the Laplace homotopy analysis method (LHAM) over the existing standard analytical techniques is that it overcomes the difficulty arising in calculating complicated terms. Numerical examples are examined to highlight the significant features of this method.

Keywords: analytic solution; Laplace transform; HAM; fractional differential-algebraic equations.

Introduction

The fractional calculus has a long history from 30 September 1695, when the derivative of order $\alpha = 1/2$ has been described by Leibniz [18], [21]. The theory of derivatives and integrals of non-integer order goes back to Leibniz, Liouville, Grünwald, Letnikov and Riemann. There are many interesting books about fractional calculus and fractional differential equations [18], [21], [6], [22]. Differential equations of fractional order have been found to be effective to describe some physical phenomena such as rheology, damping laws, fluid flow and so on [15],

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[11]. Recently, many important mathematical models can be expressed in terms of systems of algebraic differential equations of fractional order.

Derivatives of non-integer order can be defined in different ways, e.g. Riemann–Liouville, Grünwald–Letnikov, Caputo and Generalized Functions Approach [21]. In this paper we focus attention on Caputo's definition which turns out to be more useful in real-life applications since it can be coupled with initial conditions having a clear physical meaning.

DAEs arise in the mathematical modeling of a wide variety of problems from engineering and science such as in multibody and flexible body mechanics, electrical circuit design, optimal control, incompressible fluids, molecular dynamics, chemical kinetics (quasi steady state and partial equilibrium approximations), and chemical process control. The index concept plays an important role in the analysis of DAEs. The index is a measure of the degree of difficulty in the numerical solution. In general, the higher the index is, the more difficult it is to solve the DAE. While many different concepts exist to assign an index to a DAE such as the differentiation index [3], [5], the perturbation index [4], the tractability index [23], and the nilpotency index [4]. In the case of linear DAEs with constant coefficients, all these indices are equal. In order to transform a DAE into an alternative form easier to solve, some index reduction methods have been developed [10]. These methods introduce additional variables, which leads to a drawback that the resulting DAE is a larger system than the original one.

There are only a few techniques for the solution of fractional differential-algebraic equations, since it is relatively a new subject in mathematics. The solution of fractional differential equations is much involved. In general, there exists no method that yields exact solutions for fractional differential equations. Only approximate solutions can be derived using linearization or perturbation method. In recent years, much research has been focused on the numerical solution of systems of ordinary differential equations and algebraic differential equations. Some numerical methods have been developed, such as implicit Runge Kutta method [1], Pade approximation method [12], [7], homotopy perturbation method (HPM) [16], [20], Adomian decomposition method (ADM) [8], [9], variation iteration method (VIM) [19], [17] and homotopy analysis method HAM [25].

The ADM and VIM are limited in that the former has complicated algorithms in calculating Adomian polynomials for nonlinear problems, and the later has an inherent inaccuracy in identifying the Lagrange multiplier for fractional operators, which is necessary for constructing variational iteration formula. The HPM is indeed a special case of the HAM [13], [14], [2]. However, mostly, the results given by HPM converge to the corresponding numerical solutions in a rather small region. Although the HAM provides us with a simple way to adjust and control the convergence region of solution series by choosing a proper value for the auxiliary parameters h , we face the difficulty in calculating complicated integrals that arise when dealing with strongly nonlinear problems.

Therefore, in this work we will introduce a new alternative procedure to eliminate these disadvantages in solving FDAE. The newly developed technique by no means depends on complicated tools from any field. This can be the most

important advantage over the other methods. It is worth mentioning that the proposed algorithm is an elegant combination of Laplace transform method and the homotopy analysis method. Some FDAE are examined to illustrate the effectiveness, accuracy and convenience of this method. and in all cases, the presented technique performed excellently.

1. Preliminaries and notations

In this section, let us recall essentials of fractional calculus. The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the notions of integer-order differentiation and n -fold integration. For the purpose of this paper the Caputo's definition of fractional differentiation will be used, taking the advantage of Caputo's approach that the initial conditions for fractional differential equations with Caputo's derivatives take on the traditional form as for integer-order differential equations.

Definition 1.1 Caputo's definition of the fractional-order derivative is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau,$$

where $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, α is the order of the derivative.

For the Caputo's derivative we have:

$$\begin{aligned} D^\alpha C &= 0, \quad C \text{ is constant,} \\ D^\alpha t^\beta &= \begin{cases} 0 & \beta \leq \alpha - 1, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha} & \beta > \alpha - 1. \end{cases} \end{aligned}$$

Caputo's fractional differentiation is a linear operation and if $f(\tau)$ is continuous in $[a, t]$ and $g(\tau)$ has $n+1$ continuous derivatives in $[a, t]$, it satisfies the so-called Leibnitz rule:

$$D^\alpha(f(t)g(t)) = \sum_{k=0}^{\infty} \binom{\alpha}{k} g^{(k)}(t) D^{\alpha-k} f(t)$$

For establishing our results, we also necessarily introduce the following Riemann–Liouville fractional integral operator.

Definition 1.2 A real function $f(x)$, $x > 0$, is said to be in the space C_μ , $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty)$. Clearly $C_\mu \subset C_\beta$ if $\beta \leq \mu$.

Definition 1.3 The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau$$

We mention only some properties of the operator J^α :

For $f \in C_\mu$, $\mu, \gamma \geq -1$, $\alpha, \beta \geq 0$:

1. $J^0(t) = f(t)$,
2. $J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t) = J^\beta J^\alpha f(t)$,
3. $J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\gamma+\alpha}$, $\alpha > 0$, $\gamma > -1$, $t > 0$.

Also, we need here two of its basic properties. If $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, and $f \in C_\mu^m$, $\mu \geq -1$, then

$$D^\alpha J^\alpha f(t) = f(t), \quad J^\alpha D^\alpha f(t) = f(t) - \sum_{i=0}^{m-1} f^{(i)}(0^+) \frac{t^i}{i!}, \quad t > 0.$$

For more information on the mathematical properties of fractional derivatives and integrals, one can consult [22].

Lemma 1.4 If $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, and $f \in C_\mu^m$, $\mu \geq -1$, then the Laplace transform of the fractional derivative $D^\alpha f(t)$ is

$$\mathcal{L}(D^\alpha f(t)) = s^\alpha F(s) - \sum_{i=1}^{m-1} f^{(i)}(0^+) s^{\alpha-i-1}, \quad t \geq 0$$

Here $\mathcal{L}(f(t)) = F(s)$; for more details, see [26].

2. Laplace Homotopy analysis method

In this section, we employ the Laplace homotopy analysis method to the discussed problem. To show the basic idea, let us consider the FDAEs

$$(2.1a) \quad D^{\alpha_i} u_i(t) = f_i(t, u_1, u_2, \dots, u_n, u'_1, u'_2, \dots, u'_n)$$

$$(2.1b) \quad 0 = g(t, u_1, u_2, \dots, u_n), \quad i = 1, 2, \dots, n-1, \quad 0 < \alpha_i \leq 1,$$

subject to the initial conditions

$$u_i(0) = a_i, \quad i = 1, 2, \dots, n$$

where f_i are known analytical functions.

Applying the Laplace transform to both sides of (2.1a) and using the linearity of Laplace transforms, we get

$$\begin{aligned}\mathcal{L}[D^{\alpha_i} u_i(t)] &= \mathcal{L}[f_i(t, u_1, u_2, \dots, u_n, u'_1, u'_2, \dots, u'_n)], \quad i = 1, 2, \dots, n-1, \quad 0 < \alpha_i \leq 1, \\ 0 &= g(t, u_1, u_2, \dots, u_n)\end{aligned}$$

Using Lemma 1.4, and applying the formulas of the Laplace transform, we get

$$\begin{aligned}U_i(s) &= \frac{a_i}{s} + \frac{1}{s^{\alpha_i}} \mathcal{L}[f_i(t, u_1, u_2, \dots, u_n, u'_1, u'_2, \dots, u'_n)], \\ (2.2) \quad 0 &= g(t, u_1, u_2, \dots, u_n), \quad i = 1, 2, \dots, n-1, \quad 0 < \alpha_i \leq 1,\end{aligned}$$

where $U_i(s) = \mathcal{L}(u_i(t))$.

The so-called zeroth-order deformation equations of equations (2.2) are

$$\begin{aligned}(1-q)L_i[\Phi_i(s, q) - U_{i,0}(s)] &= qh_i[\Phi_i(s, q) - \frac{a_i}{s} \\ (2.3) \quad &- \frac{1}{s^{\alpha_i}} \mathcal{L}[f_i(t, \phi_1(t; q), \dots, \phi_n(t; q), \frac{\partial}{\partial t} \phi_1(t; q), \dots, \frac{\partial}{\partial t} \phi_n(t; q))], \\ &\quad i = 1, 2, \dots, n-1, \\ (1-q)L_n[\phi_n(t; q) - u_{n,0}(t)] &= -qh_n g(t, \phi_1(t; q), \dots, \phi_n(t; q)),\end{aligned}$$

where $q \in [0, 1]$ is an embedding parameter, when $q = 0$ and $q = 1$, we have

$$\begin{aligned}\Phi_i(s, 0) &= U_{i,0}(s), \quad \Phi_i(s, 1) = U_i(s), \quad i = 1, 2, \dots, n-1, \\ \phi_n(t; 0) &= u_{n,0}(t), \quad \phi_n(t; 1) = u_n(t).\end{aligned}$$

Expanding $\Phi_i(s, q)$, $i = 1, 2, \dots, n-1$ and $\phi_n(t; q)$ in Taylor series with respect to q , we get

$$\begin{aligned}\Phi_i(s; q) &= U_{i,0}(s) + \sum_{m=1}^{\infty} U_{i,m}(s) q^m, \quad i = 1, 2, \dots, n-1, \\ (2.4) \quad \phi_n(t; q) &= u_{n,0}(t) + \sum_{m=1}^{\infty} u_{n,m}(t) q^m,\end{aligned}$$

where

$$\begin{aligned}U_{i,m}(s) &= \frac{1}{m!} \frac{\partial^m \Phi_i(s; q)}{\partial q^m} \Big|_{q=0}, \quad i = 1, 2, \dots, n-1, \\ u_{n,m}(t) &= \frac{1}{m!} \frac{\partial^m \phi_n(t; q)}{\partial q^m} \Big|_{q=0}.\end{aligned}$$

If the initial guesses, the auxiliary linear operator L_i , $i = 1, 2, \dots, n$ and the nonzero auxiliary parameter h_i are properly chosen so that the power series (2.4) converges at $q = 1$.

Then, we have under these assumptions the solution series

$$U_i(s) = \Phi_i(s; 1) = U_{i,0}(s) + \sum_{m=1}^{\infty} U_{i,m}(s), \quad i = 1, 2, \dots, n-1,$$

$$u_n(t) = \phi_n(t; 1) = u_{n,0}(t) + \sum_{m=1}^{\infty} u_{i,m}(t)$$

For brevity, define the vector

$$\vec{U}_{i,m}(s) = \{U_{i,0}(s), U_{i,1}(s), U_{i,2}(s), \dots, U_{i,m}(s)\}, \quad i = 1, 2, \dots, n-1,$$

$$\vec{u}_{n,m}(t) = \{u_{n,0}(t), u_{n,1}(t), u_{n,2}(t), \dots, u_{n,m}(t)\},$$

Differentiating the zero-order deformation equation (2.3) m times with respect to q and then dividing by $m!$ and finally setting $q = 0$, we have the so-called high-order deformation equation

$$(2.5) \quad U_{i,m}(s) = \chi_m U_{i,m-1}(s) + h_i \Re_{i,m}(\vec{U}_{i,m-1}(s)), \quad i = 1, 2, \dots, n-1,$$

$$u_{n,m}(t) = \chi_m u_{n,m-1}(t) + h_n \Re_{n,m}(\vec{u}_{n,m-1}(t))$$

where

$$\Re_{i,m}(\vec{U}_{i,m-1}(s)) = U_{i,m-1}(s) - \frac{1}{s^{\alpha_i}} \left[\frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} (\mathcal{L}[f_i(t, \phi_1(t; q), \dots, \phi_n(t; q), \right.$$

$$\left. \frac{\partial}{\partial t} \phi_1(t; q), \dots, \frac{\partial}{\partial t} \phi_n(t; q)])|_{q=0} \right] - \frac{a_i}{s} (1 - \chi_m),$$

$$\Re_{n,m}(\vec{u}_{n,m-1}(t)) = \frac{-1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} [g(t, \phi_1(t; q), \dots, \phi_n(t; q))] |_{q=0}, \quad i = 1, 2, \dots, n-1,$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

Finally, applying the inverse Laplace transforms of (2.5), then we have a power series solution

$$(2.6) \quad u_i(t) = \sum_{m=0}^{\infty} u_{i,m}(t), \quad i = 1, 2, \dots, n.$$

Note that we have great freedom to choose the value of the auxiliary parameter h_i . Mathematically, the value of $u_i(t)$ at any finite order of approximation is dependent upon the auxiliary parameter h_i , because the zeroth and high order deformation equations contain h_i . Let R_{h_i} denote the set of all values of h_i which ensure the convergence of the LHAM series solution (2.6) of $u_i(t)$. Let h_i be the variable of the horizontal axis and the limit of the series solution (2.6) of $u_i(t)$ be the variable of vertical axis. Plot the curve $u_i(t) \sim h_i$, where $u_i(t)$ denotes the limit of the series (2.6). Because the limit of all convergent series solutions

(2.6) is the same for a given a , there exists a horizontal line segment above the region $h \in R_{h_i}$. So, by plotting the curve $u_i(t) \sim h_i$ at a high enough order approximation, one can find an approximation of the set R_{h_i} [13].

3. Applications

In this part, we introduce some applications on LHAM to solve differential-algebraic equations with fractional derivatives.

Example 1. Consider the following fractional differential-algebraic equations

$$(3.1) \quad \begin{aligned} D^\alpha u(t) &= -2tu^2(t) + u(t)v(t) - 1, \quad 0 < \alpha \leq 1, \\ v(t) &= u(t)v(t) + t^2, \quad u(0) = v(0) = 1, \end{aligned}$$

for $\alpha = 1$, the exact solution $u(t) = \frac{1}{t^2 + 1}$, $v(t) = t^2 + 1$.

To derive the solution, take the Laplace transform of both sides of (3.1), we get

$$\begin{aligned} \mathcal{L}[D^\alpha u(t)] &= -2\mathcal{L}(tu^2(t)) + \mathcal{L}(u(t)v(t)) - \frac{1}{s}, \\ v(t) &= u(t)v(t) + t^2, \end{aligned}$$

so

$$\begin{aligned} s^\alpha U(s) - s^{\alpha-1} &= -2\mathcal{L}(tu^2(t)) + \mathcal{L}(u(t)v(t)) - \frac{1}{s}, \\ v(t) &= u(t)v(t) + t^2, \end{aligned}$$

or

$$\begin{aligned} U(s) &= -\frac{2}{s^\alpha} \mathcal{L}(tu^2(t)) + \frac{1}{s^\alpha} \mathcal{L}(u(t)v(t)) + \left(\frac{1}{s} - \frac{1}{s^{\alpha+1}} \right), \\ v(t) &= u(t)v(t) + t^2. \end{aligned}$$

Hence, the m th-order deformation equations can be given by

$$(3.2) \quad U_m(s) = \chi_m U_{m-1}(s) + h_1 \mathfrak{R}_{1,m}(\vec{U}_{m-1}(s)),$$

$$(3.3) \quad v_m(t) = \chi_m v_{m-1}(t) + h_2 \mathfrak{R}_{2,m}(\vec{v}_{m-1}(t)), \quad m = 1, 2, 3, \dots$$

subject to the initial condition

$$u_m(0) = v_m(0) = 0, \quad m = 1, 2, 3, \dots$$

where

$$\begin{aligned} \mathfrak{R}_{1,m}(\vec{U}_{m-1}(s)) &= U_{m-1}(s) + \frac{2}{s^\alpha} \mathcal{L} \left(t \sum_{i=0}^{m-1} u_i(t) u_{m-i-1}(t) \right) \\ &\quad - \frac{1}{s^\alpha} \mathcal{L} \left(\sum_{i=0}^{m-1} u_i(t) v_{m-i-1}(t) \right) - \left(\frac{1}{s} - \frac{1}{s^{\alpha+1}} \right) (1 - \chi_m), \\ \mathfrak{R}_{2,m}(\vec{v}_{m-1}(t)) &= v_{m-1}(t) - \sum_{i=0}^{m-1} u_i(t) v_{m-i-1}(t) - t^2 (1 - \chi_m), \quad m = 1, 2, 3, \dots \end{aligned}$$

According to the initial condition in (3.1), we can choose the initial guess of $U(s)$ and $v(t)$ as follows:

$$U_0(s) = \frac{1}{s}, \quad v_0(t) = 1$$

Using the m th-order deformation equations (3.2), (3.3) we can find that

$$\begin{aligned} U_0(s) &= \frac{1}{s}, \\ u_0(t) &= 1 \\ U_1(s) &= \frac{2h_1}{s^{\alpha+2}}, \\ u_1(t) &= \frac{2h_1}{\Gamma(2+\alpha)} t^{\alpha+1} \\ U_2(s) &= \frac{2h_1}{s^{\alpha+2}} + \frac{2h_1^2}{s^{\alpha+2}} + \frac{8h_1^2(2+\alpha)}{s^{2\alpha+3}} + \frac{2h_1h_2}{s^{\alpha+3}} - \frac{2h_1^2}{s^{2\alpha+2}}, \\ u_2(t) &= \frac{2h_1}{\Gamma(2+\alpha)} t^{\alpha+1} + \frac{2h_1^2}{\Gamma(2+\alpha)} t^{\alpha+1} + \frac{8h_1^2(2+\alpha)}{\Gamma(3+2\alpha)} t^{2\alpha+2} + \frac{2h_1h_2}{\Gamma(3+\alpha)} t^{\alpha+2} \\ &\quad - \frac{2h_1^2}{\Gamma(2+2\alpha)} t^{2\alpha+1} \\ &\vdots \\ v_0(t) &= 1 \\ v_1(t) &= -h_2 t^2 \\ v_2(t) &= -h_2 t^2 - \frac{2h_1h_2}{\Gamma(2+\alpha)} t^{\alpha+1} \\ &\vdots \end{aligned}$$

Then, using a mathematical software the solution, we successfully obtain

$$u(t) = \sum_{m=0}^{\infty} u_m(t), \quad v(t) = \sum_{m=0}^{\infty} v_m(t).$$

The convergence of these series is strongly depends upon the values of the auxiliary parameters h_i . In order to find the range of admissible values of h_i , the h_i -curves are plotted in Fig.1 for the 8th-order of approximation. We can see that the range for values of h_1, h_2 are in the range $-1.3 \leq h_1, h_2 \leq -0.4$. Using $h_1 = h_2 = -1$, $\alpha = 1$ we get

$$\begin{aligned} u_0(t) &= 1, u_1(t) = -t^2, u_2(t) = t^4, u_3(t) = -t^6, u_4(t) = t^8, \dots \\ v_0(t) &= 1, v_1(t) = t^2, v_2(t) = 0, v_3(t) = 0, v_4(t) = 0, \dots \end{aligned}$$

Proceeding in the same manner, we get that the solution $u(t)$ and $v(t)$ is given in series form by

$$u(t) = u_0(t) + \sum_{m=1}^{\infty} u_m(t) = 1 - t^2 + t^4 - t^6 + t^8 - t^{10} + \dots$$

$$v(t) = v_0(t) + \sum_{m=1}^{\infty} v_m(t) = 1 + t^2$$

and so in closed form

$$u(t) = \frac{1}{t^2 + 1}, v(t) = t^2 + 1$$

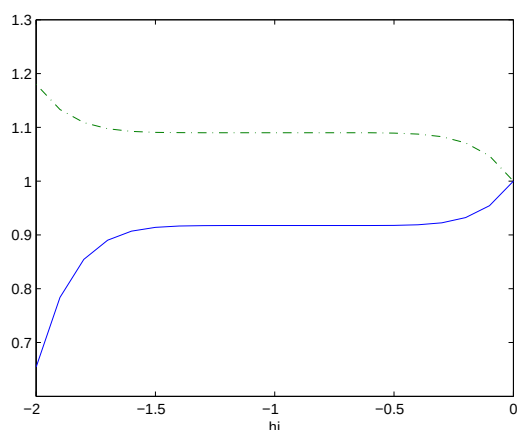


Fig.1. The h_i -curve of 8th-order approximation for $u(0.3)$ and $v(0.3)$, $\alpha = 1$ of Example 1

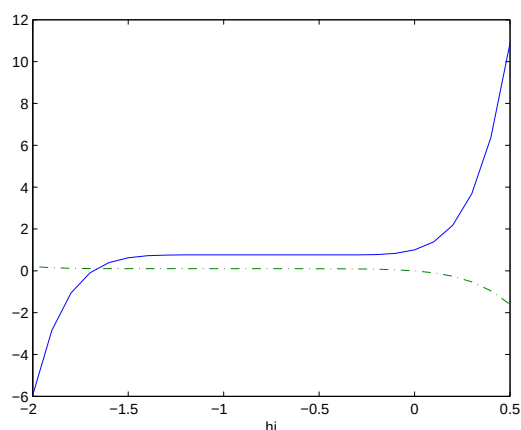


Fig.2. The h_i -curve of 8th-order approximation for $u(0.1)$ and $v(0.1)$, $\alpha = 0.5$ of Example 2

Example 2. Consider the following system of fractional differential-algebraic equations

$$(3.4) \quad \begin{aligned} D^\alpha u(t) &= tv'(t) - u(t) + (1+t)v(t), 0 < \alpha \leq 1, \\ v(t) &= \sin(t), u(0) = 1, v(0) = 0, \end{aligned}$$

for $\alpha = 1$, the exact solution $u(t) = e^{-t} + \sin(t)$, $v(t) = \sin(t)$.

Take the Laplace transform of both sides of (3.4) we get

$$\begin{aligned} U(s) &= \frac{1}{s^\alpha} \mathcal{L}(tv'(t)) - \frac{1}{s^\alpha} U(s) + \frac{1}{s} V(s) + \frac{1}{s^\alpha} \mathcal{L}(tv(t)) + \frac{1}{s}, \\ v(t) &= \sin(t), \end{aligned}$$

Hence, the m th-order deformation equations can be given by

$$(3.5) \quad U_m(s) = \chi_m U_{m-1}(s) + h_1 \mathfrak{R}_{1,m}(\vec{U}_{m-1}(s)),$$

$$(3.6) \quad v_m(t) = \chi_m v_{m-1}(t) + h_2 \mathfrak{R}_{2,m}(\vec{v}_{m-1}(t)), m = 1, 2, 3, \dots$$

subject to the initial condition

$$u_m(0) = v_m(0) = 0.$$

where

$$\begin{aligned}\Re_{1,m}(\vec{U}_{m-1}(s)) &= U_{m-1}(s) - \frac{1}{s^\alpha} \mathcal{L}(tv'_{m-1}) + \frac{1}{s^\alpha} U_{m-1}(s) - \frac{1}{s} V_{m-1}(s) \\ &\quad - \frac{1}{s^\alpha} \mathcal{L}(tv_{m-1}(t)) + \frac{1}{s} (1 - \chi_m), \\ \Re_{2,m}(\vec{v}_{m-1}(t)) &= v_{m-1}(t) - \sin(t)(1 - \chi_m), \quad m = 1, 2, 3, \dots\end{aligned}$$

According to the initial condition in (3.4), we can choose the initial guess of $U(s)$ and $v(t)$ as follows:

$$U_0(s) = \frac{1}{s}, \quad v_0(t) = 0$$

In order to find range of admissible values of h_i , the h_i -curve is plotted for 8th-order approximation when see Fig. 2. We can see that the range of values for h_i is between $-1.3 \leq h_i \leq -0.4$, using the m th-order deformation equations (3.5), (3.6) and $h_1 = h_2 = -1$, we can find that

$$\begin{aligned}(3.7) \quad u(t) &= u_0(t) + \sum_{m=1}^{\infty} u_m(t) = 1 + \frac{1}{\Gamma(1+\alpha)} t^\alpha + \frac{2}{\Gamma(2+\alpha)} t^{\alpha+1} \\ &\quad + \left[\frac{-2}{\Gamma(1+2\alpha)} + \frac{3.4^{-\alpha} \sqrt{\pi}}{\Gamma(1+\alpha) \Gamma(\frac{1}{2} + \alpha)} \right] t^{2\alpha} \\ &\quad + \frac{2}{\Gamma(3+\alpha)} t^{\alpha+2} - \frac{2}{\Gamma(2+2\alpha)} t^{2\alpha+1} - \frac{1}{\Gamma(1+3\alpha)} t^{3\alpha} + \dots \\ v(t) &= v_0(t) + \sum_{m=1}^{\infty} v_m(t) = t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{720} + \dots\end{aligned}$$

Setting $\alpha = 1$ in (3.7), we obtain the following series solution

$$\begin{aligned}u(t) &= u_0(t) + \sum_{m=1}^{\infty} u_m(t) = 1 - t + \frac{3t^2}{2} - \frac{t^3}{3!} - \frac{t^4}{8} - \frac{t^5}{5!} + \frac{7t^6}{720} - \frac{t^7}{7!} - \frac{t^8}{5760} \\ &\quad - \frac{t^9}{9!} + \frac{11t^{10}}{362880} - \frac{t^{11}}{11!} + \dots \\ v(t) &= v_0(t) + \sum_{m=1}^{\infty} v_m(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} - \frac{t^{11}}{11!} + \dots\end{aligned}$$

which is the exact solution for system (3.4), which better than solutions given by Celik, Bayram and Yeloglu [8] using Adomian decomposition method and than solutions given by Zurigat, Momani, and Alawneh [25] using HAM. However, the results given by the Adomian decomposition method converge to the corresponding numerical solutions in a rather small region. But in this method, the Laplace homotopy analysis method gives a greater region of convergence with the exact solution.

Fig. 3 shows the LHAM approximate solutions for various values of α which have the same trajectories.

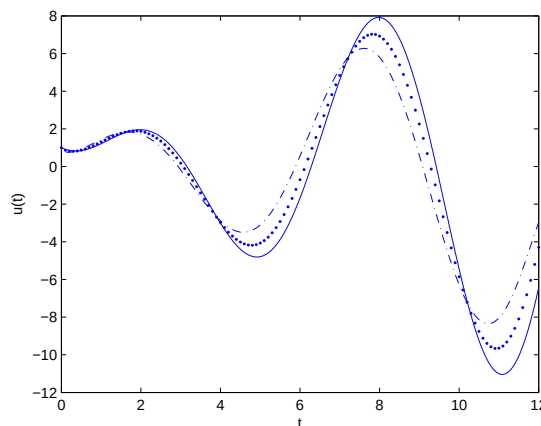


Fig.3. Plots of solution of system (10) when $h_1 = h_2 = -1$.
Solid line: $\alpha = 1$, dotted line: $\alpha = 0.8$, dash dotted line: $\alpha = 0.5$

Example 3. Consider the following system of fractional differential-algebraic equations

$$\begin{aligned}
 (3.8) \quad & D^\alpha u_1(t) = u_1(t) - u_2(t)v(t) + \sin(t) + t \cos(t), 0 < \alpha \leq 1, \\
 & D^\alpha u_2(t) = tv(t) + u_1^2(t) + \sec^2(t) - t^2(\sin^2(t) + \cos(t)) \\
 & v(t) = u_1(t) + t(\cos(t) - \sin(t)), u_1(0) = u_2(0) = v(0) = 0,
 \end{aligned}$$

for $\alpha = 1$, the exact solution $u_1(t) = t \sin(t)$, $u_2(t) = \tan(t)$, $v(t) = t \cos(t)$.

To derive the solution, we take the Laplace transform of both sides of (3.8) and we get

$$\begin{aligned}
 s^\alpha U_1(s) &= U_1(s) - \mathcal{L}(u_2(t)v(t)) + \mathcal{L}(\sin(t) + t \cos(t)), \\
 s^\alpha U_2(s) &= \mathcal{L}(tv(t)) + \mathcal{L}(u_1^2(t)) + \mathcal{L}(\sec^2(t) - t^2(\sin^2(t) + \cos(t))), \\
 v(t) &= u_1(t) + t(\cos(t) - \sin(t)),
 \end{aligned}$$

or

$$\begin{aligned}
 U_1(s) &= \frac{1}{s^\alpha} U_1(s) - \frac{1}{s^\alpha} \mathcal{L}(u_2(t)v(t)) + \frac{1}{s^\alpha} \mathcal{L}(\sin(t) + t \cos(t)), \\
 U_2(s) &= \frac{1}{s^\alpha} \mathcal{L}(tv(t)) + \frac{1}{s^\alpha} \mathcal{L}(u_1^2(t)) + \frac{1}{s^\alpha} \mathcal{L}(\sec^2(t) - t^2(\sin^2(t) + \cos(t))), \\
 v(t) &= u_1(t) + t(\cos(t) - \sin(t)).
 \end{aligned}$$

Hence, the m th-order deformation equations can be given by

$$\begin{aligned}
 (3.9) \quad & U_{i,m}(s) = \chi_m U_{i,m-1}(s) + h_i \mathfrak{R}_{i,m}(\vec{U}_{m-1}(s)), \quad i = 1, 2, \\
 & v_m(t) = \chi_m v_{m-1}(t) + h_3 \mathfrak{R}_{3,m}(\vec{v}_{m-1}(t)), \quad m = 1, 2, 3, \dots
 \end{aligned}$$

subject to the initial condition

$$U_{i,m}(0) = v_m(0) = 0, \quad i = 1, 2,$$

where

$$\begin{aligned}\Re_{1,m}(\vec{U}_{1,m-1}(s)) &= U_{1,m-1}(s) - \frac{1}{s^\alpha} U_{1,m-1}(s) \\ &\quad + \frac{1}{s^\alpha} \mathcal{L} \left(\sum_{i=0}^{m-1} u_{2,i}(t) v_{m-i-1}(t) \right) \\ &\quad - \frac{1}{s^\alpha} \mathcal{L}(\sin(t) + t \cos(t))(1 - \chi_m), \\ \Re_{2,m}(\vec{U}_{2,m-1}(s)) &= U_{2,m-1}(s) - \frac{1}{s^\alpha} \mathcal{L}(t v_{m-1}(t)) \\ &\quad - \frac{1}{s^\alpha} \mathcal{L} \left(\sum_{i=0}^{m-1} u_{1,i}(t) u_{1,m-i-1}(t) \right) \\ &\quad - \frac{1}{s^\alpha} \mathcal{L}(\sec^2(t) - t^2(\sin^2(t) + \cos(t)))(1 - \chi_m), \\ \Re_{3,m}(\vec{v}_{m-1}(t)) &= v_{m-1}(t) - u_{1,m-1}(t) - t(\cos(t) - \sin(t))(1 - \chi_m), \\ &\quad m = 1, 2, 3, \dots\end{aligned}$$

According to the initial condition in (3.8), we can choose the initial guess of $U(s)$ and $v(t)$ as follows:

$$U_{1,0}(s) = 0, U_{2,0}(s) = 0, v_0(t) = 0$$

The proper values of h_1, h_2, h_3 found from the h_i -curve shown in Fig.4, it is clear that the series of $u_i(t), v(t)$ convergent when $-1.4 \leq h_i \leq -0.7, i = 1, 2, 3$, if we set $\alpha = 1, h_1 = h_2 = h_3 = -1$ in (3.9), then we obtain the following series solution

$$\begin{aligned}u_1(t) &= u_{1,0}(t) + \sum_{m=1}^{\infty} u_{1,m}(t) = t^2 - \frac{t^4}{6} + \frac{t^6}{120} - \frac{t^8}{5040} + \frac{t^{10}}{362880} + \dots \\ u_2(t) &= u_{2,0}(t) + \sum_{m=1}^{\infty} u_m(t) = t + \frac{t^3}{3} + \frac{2t^5}{15} + \frac{17t^7}{315} + \frac{62t^9}{2835} + \dots \\ v(t) &= v_0(t) + \sum_{m=1}^{\infty} v_m(t) = t - \frac{t^3}{2} + \frac{t^5}{24} - \frac{t^7}{720} + \frac{t^9}{40320} + \dots\end{aligned}$$

which the same solutions given by F. Soltanian, S.M. Karbassi, M.M. Hosseini [24] using He's variational iteration method.

Figs. 5, 6 and 7 shows the LHAM approximate solutions for various values of α which have the same trajectories.

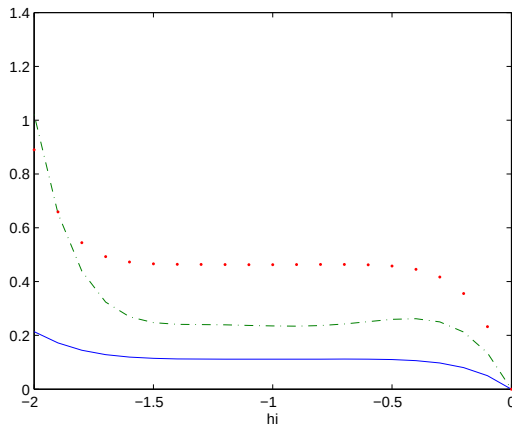


Fig.4. The h_i -curve of 8th-order approximation for $u_1(0.1)$, $u_2(0.4)$ and $v(0.3)$, $\alpha = 0.75$ of Example 3

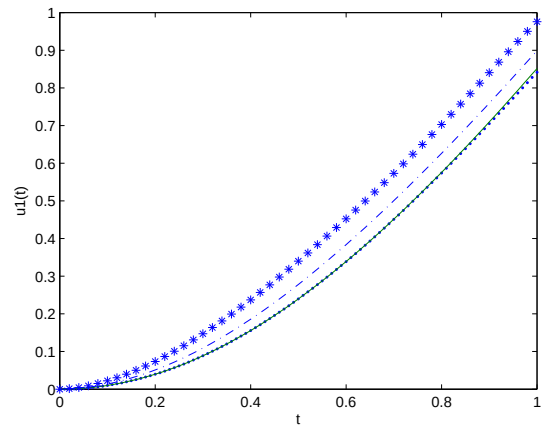


Fig.5. Plots of solution of system (14) when $h_1 = h_2 = -1$. dotted line: exact solution when $\alpha = 1$, solid line: $\alpha = 1$, dash dotted line: $\alpha = 0.9$, star line: $\alpha = 0.75$

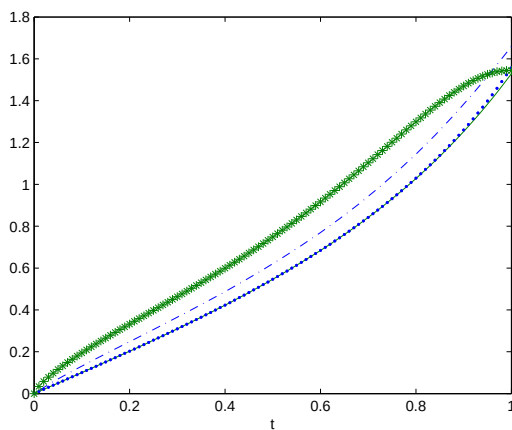


Fig.6. Plots of solution of system (14) when $h_1 = h_2 = -1$. dotted-line: exact solution when $\alpha = 1$, solid line: $\alpha = 1$, dash dotted line: $\alpha = 0.9$, star line: $\alpha = 0.75$

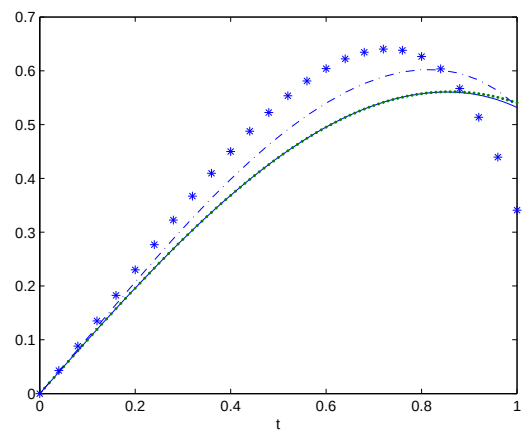


Fig.7. Plots of solution of system (14) when $h_1 = h_2 = -1$. dotted-line: exact solution when $\alpha = 1$, solid line: $\alpha = 1$, dash dotted line: $\alpha = 0.9$, star line: $\alpha = 0.75$

4. Conclusion

A combined form of the Laplace transform method with Homotopy analysis method is effectively used to handle linear and nonlinear fractional differential-algebraic equations. The main advantage of the method is its fast convergence to the solution. Moreover, it avoids the volume of calculations that required by other existing analytical methods. In practice, the utilization of the method is straightforward if some symbolic software as Matlab is used to implement the calculations. The new method leads to higher accuracy and simplicity, and in all cases the solutions obtained are easily programmable approximants to the analytic solutions of the original problems with the accuracy required. The proposed scheme can be applied for other nonlinear equations.

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Accepted: 25.01.2012

A CHARACTERIZATION OF HIGHER DERIVATIONS ON BANACH ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a Banach algebra and let every module-valued derivation from $\mathcal{A}$ to any Banach $\mathcal{A}$ -bimodule be continuous. We show that if $\{d_m\}$ is a higher derivation from $\mathcal{A}$ to a Banach algebra $\mathcal{B}$ with continuous d_0 , then there exist a continuous left $\mathcal{A}$ -module homomorphism $U : \mathfrak{B}(\mathcal{A}_1, \mathcal{B}) \rightarrow \mathcal{B}$ and a sequence $\{D_m\}$ of module-valued derivations from $\mathcal{A}$ into $\mathfrak{B}(\mathcal{A}_1, \mathcal{B})$ such that $d_m = U \circ D_m$ ($m \geq 1$), and as a consequence $\{d_m\}$ is automatically continuous. We also obtain a partial result concerning innerness of higher derivations on W^* -algebras.

Keywords: derivation, higher derivation, intertwining map, inner derivation.

2010 Mathematics Subject Classification: 46H40, 47B47.

1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be algebras. A family of linear mappings $\{d_m\}_{m=0}^k$ (k might be ∞) from $\mathcal{A}$ into $\mathcal{B}$ is called a *higher derivation of rank k* if

$$d_m(ab) = \sum_{j=0}^m d_j(a)d_{m-j}(b) \quad (a, b \in \mathcal{A}, \quad m = 0, 1, 2, \dots, k).$$

If there is no cause of ambiguity, a higher derivation will be simply denoted by $\{d_m\}$. It is obvious that for a higher derivation $\{d_m\}$, d_0 is a homomorphism and d_1 is a d_0 -derivation that is, $d_1(ab) = d_0(a)d_1(b) + d_1(a)d_0(b)$. A standard example

of a higher derivation of rank k is $\{\frac{D^m}{m!}\}_{m=0}^k$, where $D : \mathcal{A} \longrightarrow \mathcal{A}$ is a derivation. A higher derivation $\{d_m\}$ is said to be continuous if d_m is continuous for all $m \geq 0$.

Higher derivations were introduced by Hasse and Schmidt [5] and algebraists sometimes call them Hasse-Schmidt derivations. The reader may find more about the algebraic properties of higher derivations in [1], [4], [5], [15], [18], [21], [19], [6], [12]. Loy [11], Jewell [9] and Villena [20] proved the automatic continuity of higher derivations in certain cases. In [7] and [8], the authors proved some results concerning higher derivations on JB^* -algebras and Banach algebras. If $\{d_m\}$ is a higher derivation from $\mathcal{A}$ to $\mathcal{A}$ such that d_0 is the identity map on $\mathcal{A}$, then d_1 is a derivation and $\{d_m\}$ is called a *strong* higher derivation. In [10] Jun and Lee proved the Singer-Wrerner theorem for strong higher derivations. Mirzavaziri in [13] gives a characterization of a strong higher derivation defined on an algebra.

Let $\{d_m\}$ be a higher derivation from a Banach algebra $\mathcal{A}$ to a Banach algebra $\mathcal{B}$. Define

$$(1.1) \quad a.x = d_0(a)x, \quad x.a = x d_0(a) \quad (a \in \mathcal{A}, x \in \mathcal{B}).$$

Since d_0 is a homomorphism, $\mathcal{B}$ is an $\mathcal{A}$ -bimodule with respect to the mappings

$$(a, x) \rightarrow a.x, \quad (a, x) \rightarrow x.a, \quad \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{B}.$$

It is easy to see that $\mathcal{B}$ is a Banach $\mathcal{A}$ -bimodule provided that d_0 is continuous. In section 2 we give a characterization for higher derivations on certain Banach algebras. We show that if every module-valued derivation on a Banach algebra $\mathcal{A}$ is continuous, then each higher derivation $\{d_m\}$ from $\mathcal{A}$ to a Banach algebra $\mathcal{B}$ with continuous d_0 , is of the form $d_m = U \circ D_m$ ($m \geq 1$), where U is a continuous left $\mathcal{A}$ -module homomorphism and each D_m is a module-valued derivation. Therefore $\{d_m\}$ is continuous. As a consequence every higher derivation from a C^* -algebra, with continuous d_0 , is continuous. In section 3 we define an inner higher derivation. We show that if $\mathcal{A}$ is a commutative W^* -subalgebra of a W^* -algebra $\mathfrak{M}$, then each strong higher derivation from $\mathcal{A}$ to $\mathfrak{M}$ is inner.

2. Characterization

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ a Banach $\mathcal{A}$ -bimodule. A linear map $S : \mathcal{A} \longrightarrow \mathcal{X}$ is said to be left-intertwining if the map

$$b \longmapsto aS(b) - S(ab), \quad \mathcal{A} \longrightarrow \mathcal{X},$$

is continuous for each $a \in \mathcal{A}$, and right-intertwining if the map

$$a \longmapsto S(a)b - S(ab), \quad \mathcal{A} \longrightarrow \mathcal{X},$$

is continuous for all $b \in \mathcal{A}$. A linear map $S : \mathcal{A} \longrightarrow \mathcal{X}$ is intertwining if it is both left- and right-intertwining. For more about these objects see [2, Section 2.7].

Remark 2.1

- (i) Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. Suppose that $\{d_m\}$ is a higher derivation from $\mathcal{A}$ into $\mathcal{B}$ for which d_0 is continuous. Consider $\mathcal{B}$ as a Banach $\mathcal{A}$ -bimodule as in (1.1). Then it is easy to see that for every integer $m \geq 1$, $d_m : \mathcal{A} \rightarrow \mathcal{B}$ is an intertwining map whenever $d_0, \dots, d_{m-1}$ are continuous.
- (ii) Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ a Banach $\mathcal{A}$ -bimodule. Consider $\mathcal{A}_1 = \mathbb{C} \oplus \mathcal{A}$ to be the Banach algebra unitization of $\mathcal{A}$. Even if $\mathcal{A}$ is unital, $\mathcal{A}_1 \neq \mathcal{A}$ and $\mathcal{A}_1$ is a unital Banach algebra containing $\mathcal{A}$ as a closed ideal. The identity $(1, 0)$ of $\mathcal{A}_1$ will be denoted by 1. Set $\mathcal{F} = \mathfrak{B}(\mathcal{A}_1, \mathcal{X})$, the Banach space of all bounded linear operators from $\mathcal{A}_1$ to $\mathcal{X}$. For $a \in \mathcal{A}$ and $T \in \mathcal{F}$, define

$$(a.T)(b) = aT(b), \quad (T.a)(b) = T(ab) \quad (b \in \mathcal{A}_1).$$

Then $\mathcal{F}$ is an $\mathcal{A}$ -bimodule with respect to the maps

$$(a, T) \rightarrow a.T, \quad (a, T) \rightarrow T.a, \quad \mathcal{A} \times \mathcal{F} \rightarrow \mathcal{F}.$$

Now, the map

$$U : T \mapsto T(1), \quad \mathcal{F} \rightarrow \mathcal{X}$$

is a continuous linear operator and clearly

$$U(a.T) = aU(T) \quad (a \in \mathcal{A}, \quad T \in \mathcal{F}),$$

so that U is a left $\mathcal{A}$ -module homomorphism.

Dales and Villena in [3, Theorem 2.1] proved that $\mathcal{F}$ is a Banach $\mathcal{A}$ -bimodule. Also in the same theorem it has been shown that each left-intertwining map $S : \mathcal{A} \rightarrow \mathcal{X}$ is of the form $S = U \circ D$, where U is defined as above and $D : \mathcal{A} \rightarrow \mathcal{F} = \mathfrak{B}(\mathcal{A}_1, \mathcal{X})$ is a derivation defined by

$$D(a)(\beta, b) = S(\beta a + ab) - a.S(b) \quad (\beta \in \mathbb{C}, \quad a, b \in \mathcal{A}).$$

Theorem 2.2 *Let $\mathcal{A}$ be a Banach algebra for which every derivation from $\mathcal{A}$ into an arbitrary Banach $\mathcal{A}$ -bimodule is continuous. Suppose that $\{d_m\}$ is a higher derivation from $\mathcal{A}$ to a Banach algebra $\mathcal{B}$ with a continuous d_0 . Then there exists a sequence $\{D_m\}_{m \geq 1}$ of derivations from $\mathcal{A}$ to $\mathfrak{B}(\mathcal{A}_1, \mathcal{B})$ such that $d_m = U \circ D_m$ ($m \geq 1$), where $U : \mathfrak{B}(\mathcal{A}_1, \mathcal{B}) \rightarrow \mathcal{B}$ is the continuous left $\mathcal{A}$ -module homomorphism defined by $U(T) = T(1)$ for all $T \in \mathfrak{B}(\mathcal{A}_1, \mathcal{B})$. Moreover, $\{d_m\}$ is automatically continuous.*

Proof. By continuity of d_0 , $\mathcal{B}$ is a Banach $\mathcal{A}$ -bimodule with module operations defined in (1.1). Therefore d_1 is a module-valued derivation to a Banach $\mathcal{A}$ -bimodule and also an intertwining by Remark 2.1 (i). Now, by Remark 2.1 (ii) there exists a derivation $D_1 : \mathcal{A} \rightarrow \mathfrak{B}(\mathcal{A}_1, \mathcal{B})$ such that $d_1 = U \circ D_1$, where

$U : \mathfrak{B}(\mathcal{A}_1, \mathcal{B}) \rightarrow \mathcal{B}$ is defined by $U(T) = T(1)$ ($T \in \mathfrak{B}(\mathcal{A}_1, \mathcal{B})$) which is a continuous left $\mathcal{A}$ -module homomorphism. Continuity of d_1 is obvious by the assumption. Now by induction assume that for $i = 1, \dots, m-1$, $d_i = U \circ D_i$, where each D_i is a derivation from $\mathcal{A}$ to $\mathfrak{B}(\mathcal{A}_1, \mathcal{B})$ which is continuous by the hypothesis and $U : \mathfrak{B}(\mathcal{A}_1, \mathcal{B}) \rightarrow \mathcal{B}$ is as before. Now we have $d_0, d_1, \dots, d_{m-1}$ are continuous and hence d_m is an intertwining map and again by Remark 2.1 (ii) it is of the form $U \circ D_m$, where U is as before and $D_m : \mathcal{A} \rightarrow \mathfrak{B}(\mathcal{A}_1, \mathcal{B})$ is a derivation. Since by the assumption each D_m is continuous, the last assertion follows easily. ■

Corollary 2.3 *Every higher derivation $\{d_m\}$ from a C^* -algebra to a Banach algebra, with continuous d_0 , is continuous.*

Proof. Since every module-valued derivation from a C^* -algebra is continuous [14], then by Theorem 2.2 we have the result. ■

We recall that a derivation δ from a Banach algebra $\mathcal{A}$ to a Banach $\mathcal{A}$ -bimodule $\mathcal{X}$ is said to be inner if there exists $x \in \mathcal{X}$ such that $\delta(a) = ax - xa$ ($a \in \mathcal{A}$). A Banach algebra $\mathcal{A}$ for which every bounded module-valued derivation to an arbitrary Banach $\mathcal{A}$ -bimodule is inner is called *super-amenable* [16].

Corollary 2.4 *Let $\mathcal{A}$ be a super-amenable Banach algebra satisfying the hypothesis of Theorem 2.2. Then for every higher derivation $\{d_m\}$ from $\mathcal{A}$ to a Banach algebra $\mathcal{B}$, with continuous d_0 , we have $d_m = U \circ \delta_m$ ($m \geq 1$), where each δ_m is an inner derivation from $\mathcal{A}$ to $\mathfrak{B}(\mathcal{A}_1, \mathcal{B})$ and U is defined as in Remark 2.1 (ii).*

3. Inner higher derivations

We recall the definition of an inner higher derivation from [15].

Definition 3.1 Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and let $\{d_m\}$ be a higher derivation from $\mathcal{A}$ into $\mathcal{B}$. Then $\{d_m\}$ is called *inner* if for each $m \in \mathbb{N}$, there are $u_1, \dots, u_m \in \mathcal{B}$ such that

$$d_m(a) = d_0(a)u_m - \sum_{i=0}^{m-1} u_{m-i}d_i(a) \quad (a \in \mathcal{A}, m \in \mathbb{N}).$$

Note that if d_0 is continuous, then the inner higher derivation $\{d_m\}$ is also continuous.

Example 3.2 If $\{d_m\}$ is a higher derivation from a unital Banach algebra $\mathcal{A}$ to a Banach algebra $\mathcal{B}$ such that $d_0(\mathcal{A})\mathcal{B} = 0$, then $\{d_m\}$ is inner. To see this, suppose $a \in \mathcal{A}$ and let e be the identity element of $\mathcal{A}$. Then

$$d_m(a) = d_m(ea) = \sum_{i=0}^m d_i(e)d_{m-i}(a) = d_1(e)d_{m-1}(a) + \dots + d_{m-1}(e)d_1(a) + d_m(e)d_0(a).$$

For $1 \leq i \leq m$ put $u_i = -d_i(e)$, then we have

$$d_m(a) = d_0(a)u_m - u_md_0(a) - u_{m-1}d_1(a) - \dots - u_1d_{m-1}(a).$$

Therefore $\{d_m\}$ is inner.

It is well known that every derivation from a W^* -algebra $\mathfrak{M}$ to itself is inner [17, Theorem 2.5.3]. Also by [17, Corollary 2.5.4], if $\mathcal{A}$ is a C^* -subalgebra of $\mathfrak{B}(\mathcal{H})$ where $\mathfrak{B}(\mathcal{H})$ is the C^* -algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$, then every derivation $\delta : \mathcal{A} \rightarrow \mathcal{A}$ is inner when we consider it as a derivation from $\mathcal{A}$ to $\mathfrak{B}(\mathcal{H})$. More precisely, there exists $x \in \mathfrak{B}(\mathcal{H})$ such that $\delta(a) = ax - xa$ ($a \in \mathcal{A}$).

Proposition 3.3 *Let $\mathfrak{M}$ be a commutative W^* -algebra and $\{d_m\}$ a strong higher derivation from $\mathfrak{M}$ to $\mathfrak{M}$. Then each d_m ($m \geq 1$) is zero.*

Proof. By [17, Theorem 2.5.3] the result is obvious. ■

We are far from a proof of Sakai's result [17, Theorem 2.5.3] for higher derivations, but we can prove some partial results concerning higher derivations on W^* -algebras.

We recall the well known Markov-Kakutani theorem.

Theorem 3.4 *Let K be a non-empty convex compact subset of a locally convex space and let $\mathcal{S}$ be a commutative semigroup of continuous affine maps on K . Then $\mathcal{S}$ has a fixed point.*

Theorem 3.5 *Let $\mathfrak{M}$ be a W^* -algebra with identity element e and $\{d_m\}$ a strong higher derivation from $\mathfrak{M}$ to $\mathfrak{M}$. Let $\mathcal{A}$ be a commutative W^* -subalgebra of $\mathfrak{M}$ containing e . Then for each $m \in \mathbb{N}$ there are $u_0 = e, u_1, \dots, u_m$ in $\mathfrak{M}$ such that*

$$d_m(a) = au_m - \sum_{i=0}^{m-1} u_{m-i}d_i(a) \text{ for all } a \in \mathcal{A} \text{ and}$$

$$\|u_m\| \leq \|d_m\| + \|u_{m-1}\|\|d_1\| + \dots + \|u_1\|\|d_{m-1}\|.$$

Proof. By [17, Lemma 2.5.1] the result holds for $m = 1$. Now suppose that for each $j \in \{1, \dots, m-1\}$ there exist $u_0 = e, u_1, \dots, u_j$ in $\mathfrak{M}$ such that, $d_j(a) =$

$$au_j - \sum_{i=0}^j u_{j-i}d_i(a) \text{ for all } a \in \mathcal{A} \text{ and}$$

$$\|u_{m-1}\| \leq \|d_{m-1}\| + \|u_{m-2}\|\|d_1\| + \dots + \|u_1\|\|d_{m-2}\|.$$

Let $\mathcal{A}^u$ be the group of all unitary elements in $\mathcal{A}$. Since each element of $\mathcal{A}$ is a finite linear combination of elements in $\mathcal{A}^u$, so it is enough to show that the result holds for $\mathcal{A}^u$. For $a \in \mathcal{A}^u$, define $T_a(x) = [ax - d_m(a) - u_{m-1}d_1(a) - \dots - u_1d_{m-1}(a)]a^{-1}$ ($x \in \mathfrak{M}$). Then each T_a is an affine map. If $a, b \in \mathcal{A}^u$, then we have

$$\begin{aligned}
T_a T_b(x) &= T_a[(bx - d_m(b) - u_{m-1}d_1(b) - \dots - u_1d_{m-1}(b))b^{-1}] \\
&= (abxb^{-1} - ad_m(b)b^{-1} - au_{m-1}d_1(b)b^{-1} - \dots - au_1d_{m-1}(b)b^{-1} \\
&\quad - d_m(a) - u_{m-1}d_1(a) - \dots - u_1d_{m-1}(a))a^{-1} \\
&= (abx - d_m(ab) + d_m(a)b + d_{m-1}(a)d_1(b) + \dots + d_1(a)d_{m-1}(b))(ab)^{-1} \\
&\quad - au_{m-1}d_1(b)(ab)^{-1} - au_{m-2}d_2(b)(ab)^{-1} - \dots - au_1d_{m-1}(b)(ab)^{-1} \\
&\quad - d_m(a)a^{-1} - u_{m-1}d_1(a)a^{-1} - u_{m-2}d_2(a)a^{-1} - \dots - u_2d_{m-2}(a)a^{-1} - u_1d_{m-1}(a)a^{-1}.
\end{aligned}$$

Now, by the induction hypothesis, we have

$$\begin{aligned}
T_a T_b(x) &= abx(ab)^{-1} - d_m(ab)(ab)^{-1} - (au_{m-1} - d_{m-1}(a))d_1(b)(ab)^{-1} \\
&\quad - \dots - (au_1 - d_1(a))d_{m-1}(b)(ab)^{-1} - u_{m-1}d_1(a)a^{-1} - u_{m-2}d_2(a)a^{-1} \\
&\quad - \dots - u_2d_{m-2}(a)a^{-1} - u_1d_{m-1}(a)a^{-1} = abx(ab)^{-1} - d_m(ab)(ab)^{-1} \\
&\quad - (au_{m-1} - au_{m-1} + u_{m-1}a + u_{m-2}d_1(a) + \dots + u_1d_{m-2}(a))d_1(b)(ab)^{-1} \\
&\quad - \dots - (au_1 - u_1a + u_1a)d_{m-1}(b)(ab)^{-1} - u_{m-1}d_1(a)a^{-1} \\
&\quad - u_{m-2}d_2(a)a^{-1} - \dots - u_2d_{m-2}(a)a^{-1} - u_1d_{m-1}(a)a^{-1} \\
&= abx(ab)^{-1} - d_m(ab)(ab)^{-1} - u_{m-1}(ad_1(b) + d_1(a)b)(ab)^{-1} \\
&\quad - u_{m-2}(d_1(a)d_1(b) + d_2(a)b + ad_2(b))(ab)^{-1} \\
&\quad - \dots - u_1(ad_{m-1}(b) + d_1(a)d_{m-2}(b) + \dots + d_{m-2}(a)d_1(b) + d_{m-1}(a)b)(ab)^{-1} \\
&= (abx - d_m(ab) - u_{m-1}d_1(ab) - u_{m-2}d_2(ab) - \dots - u_1d_{m-1}(ab))(ab)^{-1} \\
&= T_{ab}(x) = T_{ba}(x) = T_b T_a(x).
\end{aligned}$$

Let σ denote the weak operator topology on $\mathfrak{M}$ and let K_m be the σ -closed convex hull of $\{T_a(0) : a \in \mathcal{A}^u\}$. Since each T_a is σ -continuous and $T_a T_b(x) = T_{ab}(x)$ ($a, b \in \mathcal{A}^u, x \in \mathfrak{M}$), it is easy to show that $T_a(K_m) \subseteq K_m$. On the other hand

$$\begin{aligned}
\|T_a(0)\| &= \|(-d_m(a) - u_{m-1}d_1(a) - \dots - u_1d_{m-1}(a))a^{-1}\| \\
&\leq \|d_m\| + \|u_{m-1}\|\|d_1\| + \dots + \|u_1\|\|d_{m-1}\|,
\end{aligned}$$

and it follows that

$$\sup_{x \in K_m} \|x\| \leq \|d_m\| + \|u_{m-1}\|\|d_1\| + \dots + \|u_1\|\|d_{m-1}\|.$$

Therefore, K_m is σ -compact and $\{T_a : a \in \mathcal{A}^u\}$ is a commutative semigroup of σ -continuous affine maps on K_m . Thus by Theorem 3.4 there is an element $u_m \in \mathfrak{M}$ such that

$$T_a(u_m) = u_m \quad (a \in \mathcal{A}^u).$$

Therefore

$$d_m(a) = au_m - \sum_{i=0}^{m-1} u_{m-i}d_i(a) \quad (a \in \mathcal{A}^u),$$

and clearly

$$\|u_m\| \leq \|d_m\| + \|u_{m-1}\|\|d_1\| + \dots + \|u_1\|\|d_{m-1}\|.$$

■

Corollary 3.6 *If $\mathcal{A}$ is a commutative W^* -subalgebra of a W^* -algebra $\mathfrak{M}$ containing the identity element, then each strong higher derivation from $\mathcal{A}$ to $\mathfrak{M}$ is inner.*

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Accepted: 01.07.2014

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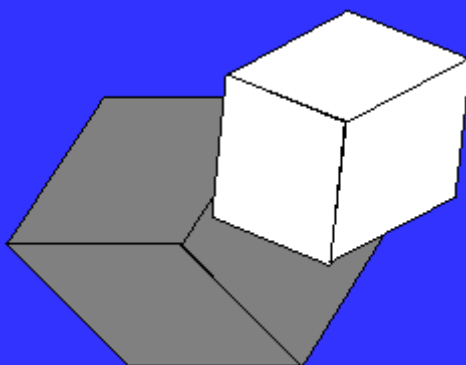
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IJPAM – Italian Journal of Pure and Applied Mathematics

Issue n° 32-2014



Publisher

Forum Editrice Universitaria Udinese Srl
Via Larga 38 - 33100 Udine
Tel: +39-0432-26001, Fax: +39-0432-296756
forum@forumeditrice.it

This journal is published with the financial support of:

- 1 - Hafez University, Zahedan, Iran*
- 2 - Authors of the papers of the single number*

Rivista semestrale: Autorizzazione Tribunale di Udine n. 8/98 del 19.3.98 - Direttore responsabile: Piergiulio Corsini

ISSN 2239-0227