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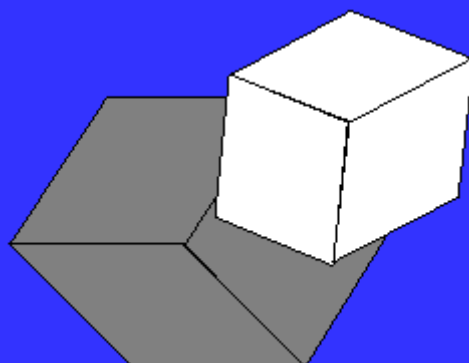
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Italian Journal of Pure and Applied Mathematics – N. 31–2013

On holomorphically decomposable Fredholm operators Abdeslam El Bakkali, Abdelaziz Tajmouati	pp.	7-14
On binomial operator representations of some polynomials K.S. Nisar, Mumtaz Ahmad Khan	pp.	15-20
On the external path length of random recursive k-ary trees Mehri Javanian	pp.	21-26
Ternary superring Yuming Feng, Baojie Zhang, Xiaoqian Song	pp.	27-34
Some fixed point theorems for multivalued weakly increasing operators Ishak Altun, Duran Turkoglu	pp.	35-42
Non-factorizable groups M.R. Darafsheh, G.R. Rezaeezadeh, M.R. Dehghan Koruki	pp.	43-48
Influence of micropolar parameters on the stability domain in a Rayleigh-Bénard convection problem - A reliable numerical study F.I. Dragomirescu, P.G. Siddheshwar, R.D. Ene	pp.	49-62
On the subgroups of torsion-free groups which are subrings in every ring A.M. Aghdam, F. Karimi, A. Najafzadeh	pp.	63-76
On hyper EQ-algebras R.A. Borzooei, B. Ganji Saffar, R. Ameri	pp.	77-96
Antipodal bipolar fuzzy graphs Muhammad Akram, Sheng-Gang Li, K.P. Shum	pp.	97-110
Compactification of a soft fuzzy product C-space T. Yogalakshmi, E. Roja, M.K. Uma	pp.	111-124
Describing Green's relations in ordered Γ-groupoids using a new concept: fuzzy subsets A. Iampan, M. Siripitukdet	pp.	124-140
Properties of bipolar fuzzy hypergraphs M. Akram, W.A. Dudek, S. Sarwar	pp.	141-160
Variation of parameters method for thin film flow of a third grade fluid down an inclined plane Z.A. Zaidi, S.U. Jan, N. Ahmed, U. Khan, S.T. Mohyud-Din	pp.	161-168
A novel approach in homotopy perturbation method for the expansion of non-linear terms Z.A. Zaidi, N. Ahmed, U. Khan, S.T. Mohyud-Din	pp.	169-176
Strongly paracompact mappings Murad Arar, Hasan Hdeib	pp.	177-186
A practical method for matrix inversion Mario Reali	pp.	187-204
Fail-stop designated recipient signature scheme based on elliptic curve discrete logarithm problem Nedal Tahat	pp.	205-218
Properties of (γ, γ')-semiopen sets C. Carpintero, N. Rajesh, E. Rosas	pp.	219-226
Operation via-regular open sets C. Carpintero, N. Rajesh, E. Rosas	pp.	227-238
Lie-admissible hyperalgebras R.M. Santilli, T. Vougiouklis	pp.	239-254
Finite p-groups determined by an inequality of the order of any two-elements generated subgroups Dapeng Yu, Heng Lv, Guiyun Chen, Jinbao Li	pp.	255-262
Finite groups with their automorphism groups having orders $4pq^2$ ($2 < p < q$) Yu Zeng, Guiyun Chen	pp.	263-276
Finite groups with $6PQ$ elements of the largest order Yong Xu, Juanjuan Gao, Hailong Hou	pp.	277-284
The black box of economic interdependence in the-process of structural change. EU and EA on the stage T.F.M. Chang, L.C. Piccinini, L. Iseppi, M.A. Lepellere	pp.	285-306
A commutative regular monoid on rough sets B. Praba, V.M. Chandrasekaran, A. Manimaran	pp.	307-318
Controllability results for second order impulsive neutral functional integrodifferential inclusions with infinite delay F. Paul Samuel, T. Gunasekar, M. Mallika Arjunan	pp.	319-332
An implicit method for solving nonconvex variational inequalities Eman Al-Shemas	pp.	333-342
Fixed point theorems on intuitionistic fuzzy quasi-metric spaces with application to the domain of words Bhavana Deshpande, Rohit Pathak	pp.	343-354

Exchanges

Up to July 2011 this journal is exchanged with the following periodicals:

1.	Acta Cybernetica - Szeged	H
2.	Acta Mathematica et Informatica Universitatis Ostraviensis	CZ
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121.	Mathematica Moravica - Cacak	SRB
122.	Mathematica Pannonica - Miskolc - Egyetemvaros	H
123.	Mathematica Scandinavica - Aarhus - Copenhagen	DK
124.	Mathematica Slovaca - Bratislava	CS
125.	Mathematicae Notae - Universidad Nacional de Rosario	AR
126.	Mathematical Chronicle - Auckland	NZ
127.	Mathematical Journal - Academy of Sciences - Uzbekistan	CSI
128.	Mathematical Journal of Okayama University - Okayama	J
129.	Mathematical Preprint - Dep. of Math., Computer Science, Physics – University of Amsterdam	NL
130.	Mathematical Reports - Kyushu University - Fukuoka	J
131.	Mathematics Applied in Science and Technology – Sangyo University, Kyoto	J
132.	Mathematics Reports Toyama University - Gofuku	J
133.	Mathematics for Applications - Institute of Mathematics of Brno University of Technology, Brno	CZ
134.	MAT - Prepublicacions - Universidad Austral	AR
135.	Mediterranean Journal of Mathematics – Università di Bari	I
136.	Memoirs of the Faculty of Science - Kochi University - Kochi	J
137.	Memorias de Mathematica da UFRJ - Instituto de Matematica - Rio de Janeiro	BR

138.	Memorie lincee - Matematica e applicazioni - Accademia Nazionale dei Lincei	I
139.	Mitteilungen der Naturforschenden Gesellschaften beider Basel	CH
140.	Monografii Matematice - Universitatea din Timișoara	RO
141.	Monthly Bulletin of the Mathematical Sciences Library – Abuja	WAN
142.	Nagoya Mathematical Journal - Nagoya University, Tokyo	J
143.	Neujahrsblatt der Naturforschenden Gesellschaft - Zürich	CH
144.	New Zealand Journal of Mathematics - University of Auckland	NZ
145.	Nieuw Archief voor Wiskunde - Stichting Mathematicae Centrum – Amsterdam	NL
146.	Nihonkai Mathematical Journal - Niigata	J
147.	Notas de Algebra y Analisis - Bahia Blanca	AR
148.	Notas de Logica Matematica - Bahia Blanca	AR
149.	Notas de Matematica Discreta - Bahia Blanca	AR
150.	Notas de Matematica - Universidad de los Andes, Merida	YV
151.	Notas de Matematicas - Murcia	E
152.	Note di Matematica - Lecce	I
153.	Novi Sad Journal of Mathematics - University of Novi Sad	SRB
154.	Obzornik za Matematiko in Fiziko - Ljubljana	SLO
155.	Octagon Mathematical Magazine - Braşov	RO
156.	Osaka Journal of Mathematics - Osaka	J
157.	Periodica Matematica Hungarica - Budapest	H
158.	Periodico di Matematiche - Roma	I
159.	Pliska - Sofia	BG
160.	Portugaliae Mathematica - Lisboa	P
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165.	Proceeding of the Indian Academy of Sciences - Bangalore	IND
166.	Proceeding of the School of Science of Tokai University - Tokai University	J
167.	Proceedings - Institut Teknologi Bandung - Bandung	RI
168.	Proceedings of the Academy of Sciences Tasked – Uzbekistan	CSI
169.	Proceedings of the Mathematical and Physical Society of Egypt – University of Cairo	ET
170.	Publicaciones del Seminario Matematico Garcia de Galdeano - Zaragoza	E
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178.	Publications Mathematiques - Besançon	F
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191.	Rendiconti lincei - Matematica e applicazioni - Accademia Nazionale dei Lincei	I
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208.	Ricerche di Matematica - Napoli	I
209.	Rivista di Matematica - Università di Parma	I
210.	Sains Malaysiana - Selangor	MAL

211.	Saitama Mathematical Journal - Saitama University	J
212.	Sankhya - Calcutta	IND
213.	Sarajevo Journal of Mathematics	BIH
214.	Sciences Bulletin, DPRK, Pyongyang	KR
215.	Scientific Review - Beograd	SRB
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221.	Serdica Mathematical Journal - Bulgarian Academy of Sciences, University of Sofia	BG
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223.	Sitzungsberichte der Mathematisch Naturwissenschaftlichen Klasse Abteilung II – Wien	A
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231.	Thai Journal of Mathematics – Chiang Mai	TH
232.	The Journal of the Academy of Mathematics Indore	IND
233.	The Journal of the Indian Academy of Mathematics - Indore	IND
234.	The Journal of the Nigerian Mathematical Society (JNMS) - Abuja	WAN
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238.	Trabalhos do Departamento de Matematica Univ. - San Paulo	BR
239.	Travaux de Mathématiques – Bruxelles	B
240.	Tsukuba Journal of Mathematics - University of Tsukuba	J
241.	UCNW Math. Preprints Prifysgol Cymru - University of Wales – Bangor	GB
242.	Ukranii Matematiskii Journal – Kiev	RU
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248.	Yugoslav Journal of Operations Research – Beograd	SRB
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ON HOLOMORPHICALLY DECOMPOSABLE FREDHOLM OPERATORS

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Abstract. Let X be a complex Banach space and $\mathcal{B}(X)$ the algebra of all bounded linear operators on X , T is said decomposably Fredholm if there exists an Fredholm operator S such that $TST = T$.

In this paper we consider the subset $\rho_{hF}(T)$ of $\mathbb{C} : \mu_0 \in \rho_{hF}(T)$ if and only if there is a neighbourhood U_0 of μ_0 and an analytic function $W : U_0 \rightarrow \mathcal{B}(X)$ such that $W(\lambda)$ is Fredholm and $(T - \lambda I)W(\lambda)(T - \lambda I) = T - \lambda I$. for all $\lambda \in U_0$, consequently we define a new spectrum $\sigma_{hF}(T) = \mathbb{C} \setminus \rho_{hF}(T)$ said holomorphically decomposable Fredholm spectrum of T . We prove that $\sigma_{hF}(T)$ is a non-empty compact subset of the classical spectrum $\sigma(T)$, and not satisfies the spectral mapping theorem.

Keywords: Fredholm operator, Kato spectrum, holomorphically Decomposable Fredholm operator.

1. Introduction and terminology

Throughout this paper, X will denote an infinite-dimensional complex Banach space. By $\mathcal{B}(X)$ we denote the Banach algebra of all bounded linear operators on X . Let $T \in \mathcal{B}(X)$. By $N(T)$, $T(X)$, $T^\infty(X)$, $\sigma(T)$, $\rho(T)$, $\rho_K(T)$, and $\sigma_K(T)$, we denote respectively the kernel, range, hyper-range, spectrum, resolvent set, Kato resolvent set, and Kato spectrum of T . Moreover, these sets are defined by:

$$\begin{aligned} N(T) &= \{u \in X / Tu = 0\} \\ T(X) &= \{Tu / u \in X\} \\ T^\infty(X) &= \bigcap_{k \geq 1} T^k(X) \\ \sigma(T) &= \{\lambda \in \mathbb{C} / T - \lambda I \text{ is not invertible on } \mathcal{B}(X)\} \\ \rho(T) &= \mathbb{C} \setminus \sigma(T) \\ \rho_K(T) &= \{\lambda \in \mathbb{C} / (T - \lambda I)(X) \text{ is closed and } N(T - \lambda I) \subset (T - \lambda I)^\infty(X)\} \\ \sigma_K(T) &= \mathbb{C} \setminus \rho_K(T) \end{aligned}$$

Recall that $T \in \mathcal{B}(X)$ is said to be relatively regular if there exists an operator $S \in \mathcal{B}(X)$ for which $TST = T$; S is called a pseudo-inverse of T , the necessary and sufficient condition for which T is relatively regular if and only if $N(T)$ and $T(X)$ are complemented in X , see [1], [2], [5] and [8].

Denote by $\alpha(T)$ the dimension of $N(T)$, and by $\beta(T)$ the codimension of the range $T(X)$. Next define the following classes of operators by:

$$\Phi(X) = \{T \in \mathcal{B}(X) : \alpha(T) \text{ and } \beta(T) \text{ are finite}\}$$

the class of all Fredholm operators, where $\text{ind}(T) := \alpha(T) - \beta(T)$ is the index of $T \in \Phi(X)$;

$$\mathcal{R}(X) = \{T \in \mathcal{B}(X) : \exists S \in \mathcal{B}(X) \text{ such that } TST = T\}$$

the class of all relatively regular operators;

$$\mathcal{J} = \{T \in \mathcal{B}(X) : T \text{ invertible}\}$$

the group of invertible operators;

$$\mathcal{JR}(X) = \{T \in \mathcal{B}(X) : \exists S \in \mathcal{J}, \text{ such that } TST = T\}$$

the class of all decomposably regular operators;

$$\Phi\mathcal{R}(X) = \{T \in \mathcal{B}(X) : TST = T \text{ for some } S \in \Phi(X)\}$$

the class of all decomposably Fredholm operators, (see Definition 2.1, below).

Note that $\mathcal{J}, \Phi(X)$ are open subsets of $\mathcal{B}(X)$ but $\mathcal{R}(X), \mathcal{JR}(X)$ and $\Phi\mathcal{R}(X)$ are neither open nor closed, see [10].

Recently, Schmoegele [10] introduced and studied the holomorphically decomposably regular resolvent set of $T \in \mathcal{B}(X)$, $\rho_{gr}(T)$, where:

$$(*) \quad \rho_{gr}(T) = \{\lambda \in \mathbb{C} : \exists U, \text{ a neighbourhood of } \lambda, \exists F : U \rightarrow \mathcal{B}(X) \text{ analytic such that } (T - \lambda I)F(\lambda)(T - \lambda I) = T - \lambda I, \text{ and } F(\lambda) \in \mathcal{J}, \forall \lambda \in U.\}$$

The holomorphically decomposably regular spectrum of T is defined by

$$\sigma_{gr}(T) = \mathbb{C} \setminus \rho_{gr}(T).$$

In [10], the author proved that $\sigma_{gr}(T)$ is a non empty compact set and does not satisfy the spectral mapping theorem, [10, Proposition 3.3 and Theorem 3.6]. We do have, however, $\sigma_{gr}(f(T)) \subseteq f(\sigma_{gr}(T))$ for $f \in \mathcal{H}(\sigma(T))$, see Example 3.4 in [10], for $f \in \mathcal{H}(\sigma(T))$, where $\mathcal{H}(\sigma(T))$ is the set of all complex-valued functions which are analytic in some neighbourhood of $\sigma(T)$, and $f(T) \in \mathcal{B}(X)$ defined by Riesz-Dunford functional calculus, see [1], [5] and [8]. Notice that $\mathcal{J} \subsetneq \Phi(X)$. Hence, it is natural to see what happened if we replace $\mathcal{J}$ by $\Phi(X)$ in (*). To this end, let

$$(*) \quad \rho_{hF}(T) = \{\lambda \in \mathbb{C} / \exists U, \text{ a neighbourhood of } \lambda, \exists F : U \rightarrow \mathcal{B}(X) \text{ analytic such that } (T - \lambda I)F(\lambda)(T - \lambda I) = T - \lambda I, \text{ and } F(\lambda) \in \Phi(X) \text{ for all } \lambda \in U.\}$$

It is clear that $\rho_{gr}(T) \subset \rho_{hF}(T)$; and setting $\sigma_{hF}(T) = \mathbb{C} \setminus \rho_{hF}(T)$, it follows that the new spectrum set satisfies $\sigma_{hF}(T) \subseteq \sigma_{gr}(T) \subseteq \sigma(T)$ (see Definition 2.2. below).

Our aim in this paper is to tackle the following problem:

Let $T \in \mathcal{B}(X)$ and consider the part $\sigma_{hF}(T)$ of the spectrum of T .

1. *Does $\sigma_{hF}(T)$ a non-empty compact subset of $\mathbb{C}$?*
2. *Does $\sigma_{hF}(T)$ satisfy the spectrum mapping theorem?*

We answer positively the first question and negatively the second.

In Section 2, we present the class of holomorphically decomposable Fredholm operators: we show that $\sigma_{hF}(T)$ is a non-empty compact subset of the spectrum, and we characterize the set $\rho_{hF}(T)$ (see Proposition 2.1).

In Section 3, we show that $\sigma_{hF}(f(T)) \subset f(\sigma_{hF}(T))$ for every $f \in \mathcal{H}(\sigma(T))$ (Theorem 3.1.) and if f is univalent then this inclusion is indeed an equality (Theorem 3.2.).

Finally, Example 1 shows that the inclusion may be proper and conclude that $\sigma_{hF}(T)$ does not satisfy the spectral mapping theorem.

2. Class of holomorphically decomposable Fredholm operators

We first begin with some definitions and examples.

Definition 2.1 Let $T \in \mathcal{B}(X)$, T is said to be decomposably Fredholm if $TST = T$ for some $S \in \Phi(X)$.

Let the set of decomposably Fredholm operators be denoted by $\Phi\mathcal{R}(X)$.

Examples. We give some classes of $\Phi\mathcal{R}(X)$.

1. If $T \in \mathcal{J}\mathcal{R}(X)$, then T is decomposably regular and there exists $S \in \mathcal{B}(X)$ invertible such that $TST = T$, clearly $S \in \Phi(X)$; in particular, $\mathcal{J}\mathcal{R}(X) \subset \Phi\mathcal{R}(X)$.
2. For $n \in \mathbb{Z}$, denote by $\Phi_n(X) = \{T \in \Phi(X) : \text{ind}(T) = \alpha(T) - \beta(T) = n\}$ and $\Phi_n\mathcal{R}(X) = \{T \in \mathcal{B}(X) : TST = T \text{ such that } S \in \Phi_n(X)\}$, then $\Phi\mathcal{R}(X) = \bigcup_{n \in \mathbb{Z}} \Phi_n\mathcal{R}(X)$ (see [10]).
3. Rakocevic in [9, Theorem 3] shows that $\Phi\mathcal{R}(X) = \mathcal{R}(X) \cap \overline{\Phi(X)}$.
4. Schmoegele in [10] proved that $\bigcirc_{\Phi_n\mathcal{R}(X)} = \Phi_{-n}(X)$ and $\bigcirc_{\Phi\mathcal{R}(X)} = \Phi(X)$.
5. If T is generalized Fredholm operator, i.e., if T is relatively regular and $(I - ST - TS) \in \Phi(X)$ for some pseudo-inverse S of T [15], we have $T \in \overline{\Phi(X)}$, and consequently $T \in \Phi\mathcal{R}(X)$.

Definition 2.2 Let $T \in \mathcal{B}(X)$, the holomorphically decomposable Fredholm resolvent set of T is defined by:

$\rho_{hF}(T) = \{\lambda \in \mathbb{C} : \exists U, \text{ a neighbourhood of } \lambda, \exists F : U \rightarrow \mathcal{B}(X) \text{ analytic such that } (T - \lambda I)F(\lambda)(T - \lambda I) = T - \lambda I \text{ and } F(\lambda) \in \Phi(X) \text{ for all } \lambda \in U\}.$

The holomorphically decomposable Fredholm spectrum of T is defined by

$$\sigma_{hF}(T) := \mathbb{C} \setminus \rho_{hF}(T).$$

An operator $T \in \mathcal{B}(X)$ for which $0 \in \rho_{hF}(T)$ is called a holomorphically decomposable Fredholm operator.

Obviously,

$$\rho(T) \subset \rho_{gr}(T) \subset \rho_{hF}(T) \quad \text{and} \quad \sigma_{hF}(T) \subset \sigma_{gr}(T) \subset \sigma(T).$$

Remarks.

1. Let $0 \in \rho_K(T)$. Then T is said Kato operator (or semi-regular/singular operator) see [1], [4], [5], [7], [8]) and, if T is relatively regular, then T is called Saphar operator. We denote the Saphar resolvent set by $\rho_{rr}(T) = \{\lambda \in \rho_K(T) / T - \lambda I \in \mathcal{R}(X)\}$ and the Saphar spectrum $\sigma_{rr}(T) := \mathbb{C} \setminus \rho_{rr}(T)$. It is well known that (see [6], [8], [12]):

$$\lambda_0 \in \rho_{rr}(T) \Leftrightarrow \exists U, \text{ a neighbourhood of } \lambda, \exists F : U \rightarrow \mathcal{B}(X) \text{ such that } F \text{ is analytic and } (T - \lambda I)F(\lambda)(T - \lambda I) = T - \lambda I, \forall \lambda \in U.$$

Consequently, if $\lambda_0 \in \rho_{hF}(T)$, then $\lambda_0 \in \rho_{rr}(T)$, we conclude that:

$$\begin{aligned} \rho(T) &\subseteq \rho_{gr}(T) \subseteq \rho_{hF}(T) \subseteq \rho_{rr}(T) \subseteq \rho_K(T) \text{ and} \\ \sigma_K(T) &\subseteq \sigma_{rr}(T) \subseteq \sigma_{hF}(T) \subseteq \sigma_{gr}(T) \subseteq \sigma(T). \end{aligned}$$

2. In [13] Schmoegeer proved that $\partial\sigma(T) \subseteq \sigma_K(T)$, see also [1], [5], [7] and [8]. Hence, it follows that $\partial\sigma(T) \subseteq \sigma_K(T) \subseteq \sigma_{rr}(T) \subseteq \sigma_{hF}(T) \subseteq \sigma_{gr}(T) \subseteq \sigma(T)$. In particular, $\sigma_{hF}(T)$ is a non empty compact subset of $\sigma(T)$.
3. If X is a Hilbert space, then T is relatively regular if and only if $T(X)$ is closed, in this special case $\rho_{rr}(T) = \rho_K(T)$ and $\sigma_K(T) = \sigma_{rr}(T) = \sigma_g(T)$ where $\sigma_g(T)$ is the generalized spectrum introduced by Mbekhta in [6], see also [1], [5], [7] and [8].

In general, $\sigma_K(T) \subsetneq \sigma_{rr}(T)$ (see [14]) and, from the strict inclusion $\mathcal{J} \subsetneq \Phi(X)$, we conclude that $\sigma_{hF}(T) \subsetneq \sigma_{gr}(T)$.

Proposition 2.1 *For an operator $T \in \mathcal{B}(X)$ the following statements are equivalent:*

- (i) $\mu_0 \in \rho_{hF}(T)$
- (ii) $\mu_0 \in \rho_K(T)$ and $T - \mu_0 I \in \Phi\mathcal{R}(X)$.

Proof. Suppose first that $\mu_0 \in \rho_{hF}(T)$, then by definition there exists a neighborhood U_0 of μ_0 and analytic function $W : U_0 \rightarrow \mathcal{B}(X)$ satisfying the equation of pseudo-inverse of $T - \lambda I$ which $W(\lambda)$ is Fredholm. For all $\lambda \in U_0$, we obtain $\mu_0 \in \rho_{rr}(T) \subset \rho_K(T)$ and since $(T - \mu_0 I)W(\mu_0)(T - \mu_0 I) = T - \mu_0 I$, we conclude that $T - \mu_0 I \in \Phi\mathcal{R}(X)$.

Conversely, assume that $\mu_0 \in \rho_K(T)$ and $T - \mu_0 I \in \Phi\mathcal{R}(X)$. By replacing T with $T - \mu_0 I$ we may assume without loss of generality $\mu_0 = 0$ and $T \in \Phi\mathcal{R}(X)$. Since $T \in \Phi\mathcal{R}(X)$ then there is $S \in \mathcal{B}(X)$ that $TST = T$ and S is Fredholm. Next, we consider the function

$$\begin{aligned} W : \mathbb{D}(0, \|S\|^{-1}) &\rightarrow \mathcal{B}(X) \\ \lambda &\mapsto W(\lambda) \end{aligned}$$

where $\mathbb{D}(0, \|S\|^{-1})$ is the open disc centred at 0 and radius $\|S\|^{-1} > 0$, defined by $W(\lambda) := (I - \lambda S)^{-1}S = S(I - \lambda S)^{-1}$ for all $\lambda \in \mathbb{D}(0, \|S\|^{-1})$, then, by Corollary 1.5 of [12], W is analytic and verifies the equation of pseudo-inverse $(T - \lambda I)W(\lambda)(T - \lambda I) = T - \lambda I$ for every $\lambda \in \mathbb{D}(0, \|S\|^{-1})$. On the other hand, since S is Fredholm and $(I - \lambda S)^{-1}$ is invertible, we have that $(I - \lambda I)^{-1}S$ is Fredholm, consequently, $0 \in \rho_{hF}(T)$; this completes the proof of the theorem. ■

Let $T \in \mathcal{B}(X)$ and $g \in \mathcal{H}(\sigma(T))$, conditions on $g(T)$ to be element of $\Phi\mathcal{R}(X)$ are given by Schmoegeer in Proposition 3.8. of [10].

Proposition 2.2 [10, Proposition 3.8] *Assume the following conditions:*

- a. $T \in \mathcal{B}(X)$ and $g \in \mathcal{H}(\sigma(T))$ has only a finite number of zeros in $\sigma(T)$
- b. $\mu_1, \dots, \mu_m$ are the zeros of g in $\sigma(T)$ with respective orders of multiplicity, $n_1, \dots, n_m$ ($\mu_i \neq \mu_j$ for $i \neq j$)
- c. $(T - \mu_j I)^{n_j} \in \Phi\mathcal{R}(X)$ for $j = 1, \dots, m$, then $g(T) \in \Phi\mathcal{R}(X)$. To be more precise, if $(T - \mu_j I)^{n_j} \in \Phi_{k_j}\mathcal{R}(X)$ and $k = \sum_{i=1}^m k_i$.

Then, $g(T) \in \Phi_k\mathcal{R}(X)$.

3. Spectral Mapping Theorem

We now discuss the spectral theorem for $\sigma_{hF}(T)$, of $T \in \mathcal{B}(X)$. Generally, the following theorem holds.

Theorem 3.1 *Let $T \in \mathcal{B}(X)$, and $f \in \mathcal{H}(\sigma(T))$. We have: $\sigma_{hF}(f(T)) \subset f(\sigma_{hF}(T))$.*

Proof. Suppose $\lambda_0 \notin f(\sigma_{hF}(T))$, we have to show $\lambda_0 \in \rho_{hF}(f(T))$. Let us consider $g(z) = f(z) - \lambda_0$, g is analytic and $g \in \mathcal{H}(\sigma(T))$, since

$$(*) \quad 0 \notin g(\sigma_{hF}(T)).$$

Since $\sigma_K(T) \subset \sigma_{hF}(T)$, it follows that $0 \notin g(\sigma_K(T))$; by the spectral mapping theorem for the Kato spectrum we have

$$(**) \quad 0 \notin g(\sigma_K(T)) = \sigma_K(g(T)).$$

Case 1. Assume that g does not vanish in $\sigma(T)$; in this case $g(T) = f(T) - \lambda_0 I$ is invertible, we then obtain $\lambda_0 \in \rho(f(T)) \subset \rho_{hF}(f(T))$.

Case 2. Assume that g has zeros in $\sigma(T)$: Since $\sigma_K(T) \subset \sigma(T)$ and (**), by Proposition 3 [13], g may have only a finite number of zeros in $\sigma(T)$, say $\mu_1, \dots, \mu_m$, where $\mu_i \neq \mu_j$ for $i \neq j$, and $n_i \geq 0$ denote the order of multiplicity of μ_i .

By (*) we have $\mu_i \notin \sigma_{hF}(T), \forall i = 1, \dots, m$, and then $\mu_i \in \rho_{hF}(T)$ for all i . Consequently, there exists $W_i(\mu_i) \in \Phi_{k_i}(X)$ such that

$$(T - \mu_i I)W_i^{n_i}(T - \mu_i I) = T - \mu_i I, \text{ or } \rho_{hF}(T) \subset \rho_{rr}(T),$$

we also have $T - \mu_i I$ Saphar having $W_i(\mu_i)$ as a pseudo-inverse, and by Lemma 5-c [11], we have

$$(T - \mu_i I)^{n_i} W_i^{n_i}(\mu_i) (T - \mu_i I)^{n_i} = (T - \mu_i I)^{n_i},$$

since $W_i^{n_i}(\mu_i) \in \Phi_{n_i k_i}(X)$, we obtain $(T - \mu_i I)^{n_i} \in \Phi_{n_i k_i} \mathcal{R}(X) \subset \Phi \mathcal{R}(X)$.

We have:

- (1) $g \in \mathcal{H}(\sigma(T))$ has only a finite number of zeros in $\sigma(T)$.
- (2) $\mu_1, \dots, \mu_m$ are the zeros of g in $\sigma(T)$ with respective orders of multiplicity, $n_1, \dots, n_m$ ($\mu_i \neq \mu_j$ for $i \neq j$)
- (3) $(T - \mu_i I)^{n_i} \in \Phi \mathcal{R}(X)$ for $i = 1, \dots, m$.

Then, by Proposition 2.2, $g(T) \in \Phi \mathcal{R}(X)$; on the other hand, by (**), $0 \notin \sigma_K(g(T))$, we have $0 \in \rho_K(g(T))$.

We conclude that $0 \in \rho_K(g(T))$ and $g(T) \in \Phi \mathcal{R}(X)$.

Consequently, by Proposition 2.1, $0 \in \rho_{hF}(g(T))$ and $\lambda_0 \in \rho_{hF}(f(T))$, hence

$$\sigma_{hF}(f(T)) \subset f(\sigma_{hF}(T)). \quad \blacksquare$$

Theorem 3.2 *Let $T \in \mathcal{B}(X)$, then $\sigma_{hF}(f(T)) = f(\sigma_{hF}(T))$, for every $f \in \mathcal{H}(\sigma(T))$, injective.*

Proof. By Theorem 3.1, we have $\sigma_{hF}(f(T)) \subset f(\sigma_{hF}(T))$, and hence it suffices to prove that $f(\sigma_{hF}(T)) \subset \sigma_{hF}(f(T))$.

To see this, let $\lambda_0 \in f(\sigma_{hF}(T))$ and suppose that $\lambda_0 \notin \sigma_{hF}(f(T))$.

Hence, $\lambda_0 \in \rho_{hF}(f(T)) \subset \rho_K(f(T))$, by spectral theorem for $\sigma_K(T)$ we have $\lambda_0 \in f(\rho_K(T))$, and there exists $\mu_0 \in \rho_K(T)$ (*) such that $f(\mu_0) = \lambda_0$.

Define $h(z) = f(z) - \lambda_0$, then $h(\lambda_0) = 0$ and $h \in \mathcal{H}(\sigma(T))$.

Consider

$$g(z) := \begin{cases} \frac{h(z)}{z - \mu_0} & \text{if } z \neq \mu_0, \\ f'(\mu_0) & \text{if } z = \mu_0. \end{cases}$$

Then $h(z) = (z - \mu_0)g(z)$ and $g \in \mathcal{H}(\sigma(T))$ such that $g(z) \neq 0$ for all $z \in \sigma(T)$ since f is injective.

We have $h(T) = (T - \mu_0 I)g(T) = g(T)(T - \mu_0 I)$ and $g(T)$ is invertible. By $\lambda_0 \in \rho_{hF}(f(T))$ we have $0 \in \rho_{hF}(h(T))$ and there exists a Fredholm operator $W(0)$ such that $h(T)W(0)h(T) = h(T)$.

Consequently, $(T - \mu_0 I)g(T)W(0)g(T)(T - \mu_0 I) = (T - \mu_0 I)g(T)$.

From invertibility of $g(T)$, we obtain that $(T - \mu_0 I)g(T)W(0)(T - \mu_0 I) = T - \mu_0 I$, and, since $(T - \mu_0 I) \in \Phi\mathcal{R}(X)$, $(g(T)W(0))$ is Fredholm, since $g(T)$ invertible and $W(0)$ Fredholm).

From $(*)$ and $(T - \mu_0 I) \in \Phi\mathcal{R}(X)$, we obtain, by Proposition 2.1, $\mu_0 \in \rho_{hF}(T)$; hence, $\mu_0 \notin \sigma_{hF}(T)$ and $\lambda_0 = f(\mu_0) \notin f(\sigma_{hF}(T))$, which contradicts the assumption $\lambda_0 \in f(\sigma_{hF}(T))$. Therefore, $f(\sigma_{hF}(T)) \subset \sigma_{hF}(f(T))$ is proved. This completes the proof of the theorem. $\blacksquare$

In the following example, we find an operator T , and $f \in \mathcal{H}(\sigma(T))$ which f is not injective, and $\sigma_{hF}(f(T)) \neq f(\sigma_{hF}(T))$.

Example 1 Denote by $\mathcal{K}(X) = \{T \in \mathcal{B}(X) : T(X) \text{ is closed and } N(T) \subset T^\infty(X)\}$, then

$$T \in \mathcal{K}(X) \Leftrightarrow 0 \in \rho_K(T).$$

On the other hand, we consider

$$\Phi\mathcal{R}(X) = \{T \in \mathcal{B}(X) : \exists S \in \mathcal{B}(X), TST = T \text{ such that } S \in \Phi(X)\}.$$

Let $T \in \mathcal{K}(X) \setminus \Phi\mathcal{R}(X)$ satisfying there exists $k_0 \geq 2$ such that $T^{k_0} \in \Phi\mathcal{R}(X)$ or, equivalently, $T^{k_0}ST^{k_0} = T^{k_0}$, $k_0 \geq 2$ for S Fredholm.

Notice that, by $T \in \mathcal{K}(X) \setminus \Phi\mathcal{R}(X)$ we have $0 \in \rho_K(T)$ and $T \notin \Phi\mathcal{R}(X)$.

This implies, by Proposition 2.1, that $0 \notin \rho_{hF}(T)$ i.e.,

$$(*) \quad 0 \in \sigma_{hF}(T).$$

Let $\mu_0 \notin \sigma(T)$, then $\mu_0 \neq 0$ (since $\sigma_{hF}(T) \subset \sigma(T)$). Let us show that

$$T^{k_0}(T - \mu_0 I) \in \Phi\mathcal{R}(X).$$

$T^{k_0} \in \Phi\mathcal{R}(X)$ implies that there exists $S \in \Phi(X)$, such that $T^{k_0}ST^{k_0} = T^{k_0}$. Since T^{k_0} and $T - \mu_0 I$ commute, we have $T^{k_0}(T - \mu_0 I)ST^{k_0} = T^{k_0}(T - \mu_0 I)$. Put $Q = S(T - \mu_0 I)^{-1}$. Then Q is Fredholm and $T^{k_0}(T - \mu_0 I)QT^{k_0}(T - \mu_0 I) = T^{k_0}(T - \mu_0 I)$. Thus,

$$T^{k_0}(T - \mu_0 I) \in \Phi\mathcal{R}(X).$$

Let $P(z) = z^{k_0}(z - \mu_0)$, then P is not injective and $P(T) = T^{k_0}(T - \mu_0 I)$, $P(T) \in \Phi\mathcal{R}(X)$. Since $0 \in \rho_K(T)$, and $P(0) = 0$, by the spectral mapping theorem

of $\sigma_K(T)$, we have $0 \in \rho_K(P(T))$. From $0 \in \rho_K(P(T))$ and since $P(T) \in \Phi\mathcal{R}(X)$. We have, by Proposition 2.1,

$$(**) \quad 0 \in \rho_{hF}(P(T)).$$

Next, if we assume that $P(\sigma_{hF}(T)) \subset \sigma_{hF}(P(T))$, by (*) we have $0 \in \sigma_{hF}(T)$, and, consequently, $0 \in \sigma_{hF}(P(T))$, which contradicts (**). Thus, $P(\sigma_{hF}(T)) \not\subset \sigma_{hF}(P(T))$. Therefore, $\sigma_{hF}(T)$ does not satisfy the spectral mapping theorem.

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ON BINOMIAL OPERATOR REPRESENTATIONS OF SOME POLYNOMIALS

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Abstract. Based on the technique used by M.A. Khan and A.K. Shukla [2] here finite series representations of binomial partial differential operators have been used to establish operator representations of various polynomials not considered in the earlier mentioned paper. The results obtained are believed to be new.

Keywords: Binomial operator, Operational representations.

AMS (MOS) Subject Classification (2000): Primary 47F05, Secondary 33C45.

1. Introduction

Recently, in 2009, M.A. Khan and A.K. Shukla [2] evolved a new technique to give operator representations of certain polynomials. They give binomial and trinomial operator representation of certain polynomials. The aim of the present paper is to strengthen the technique evolved by obtaining binomial operator representations of some more polynomials not considered in the above mentioned paper.

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2. The definitions, notations, and results used

In deriving the operational representations of various polynomials use has been made of the fact that

$$(2.1) \quad D^\mu x^\lambda = \frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda-\mu)} x^{\lambda-\mu}, \quad D \equiv \frac{d}{dx}$$

where λ and μ , $\lambda \geq \mu$ are arbitrary real numbers.

In particular, use has been made of the following results:

$$(2.2) \quad D^r e^{-x} = (-1)^r e^{-x}$$

$$(2.3) \quad D^r x^{-\alpha} = (\alpha)_r (-1)^r x^{-\alpha-r}, \quad \alpha \text{ is not an integer}$$

$$(2.4) \quad D^r x^{-\alpha-n} = (\alpha+n)_r (-1)^r x^{-\alpha-n-r}$$

$$(2.5) \quad D^{n-r} x^{\alpha-1+n} = \frac{(\alpha)_n}{(\alpha)_r} x^{\alpha-1+r}$$

$$(2.6) \quad D^{n-r} x^{-\alpha} = \frac{(\alpha)_n (-1)^n}{(1-\alpha-n)_r} x^{-\alpha-n+r}, \quad \alpha \text{ is not an integer,}$$

where n and r are denoting the positive integers and

$$(a)_n = a(a+1)\cdots(a+n-1); \quad (a)_0 = 1.$$

We also need the definitions of the following polynomials in terms of hypergeometric function and also their notations (see [1], [3], [4], [5]).

Cesaro polynomials

It is denoted by the symbol $g_n^{(s)}(x)$ and is defined as

$$(2.7) \quad g_n^{(s)}(x) = \binom{s+n}{n} {}_2F_1 \left[\begin{matrix} -n, & 1 \\ -s-n & \end{matrix} ; x \right]$$

Miexner polynomials

It is denoted by the symbol $M_n(x; \beta, c)$ and is defined as

$$(2.8) \quad M_n(x; \beta, c) = {}_2F_1 \left[\begin{matrix} -n, & -x \\ \beta & \end{matrix} ; 1-c^{-1} \right],$$

$$\beta > 0, \quad 0 < c < 1, \quad x = 0, 1, 2, \dots, N$$

Krawchouk polynomials

It is denoted by the symbol $K_n(x; P, N)$ and is defined as

$$(2.9) \quad K_n(x; P, N) = {}_2F_1 \left[\begin{matrix} -n, & -x; \\ & -N; \end{matrix} P^{-1} \right],$$

$$0 < P < 1, \quad x = 0, 1, 2, \dots, N$$

Hahn polynomials

It is denoted by the symbol $Q_n(x; \alpha, \beta, N)$ and is defined as

$$(2.10) \quad Q_n(x; \alpha, \beta, N) = {}_3F_2 \left[\begin{matrix} -n, & -x, & \alpha + \beta + n + 1; \\ & -N, & \alpha + 1; \end{matrix} 1 \right]$$

$$\alpha, \beta > -1, \quad n, x = 0, 1, 2, \dots, N$$

Sylvester polynomials

It is denoted by the symbol $\varphi_n(x)$ and is defined as

$$(2.11) \quad \varphi_n(x) = \frac{x^n}{n!} {}_2F_0 \left[\begin{matrix} -n, & x; \\ & -; \end{matrix} x^{-1} \right]$$

Gottlieb polynomials

It is denoted by the symbol $l_n(x; \lambda)$ and is defined as

$$(2.12) \quad l_n(x; \lambda) = e^{-n\lambda} {}_2F_1 \left[\begin{matrix} -n, & -x; \\ & 1; \end{matrix} 1 - e^\lambda \right]$$

Charlier polynomials

It is denoted by the symbol $C_n^a(x)$ and is defined as

$$C_n^a(x) = (-a)^n {}_2F_0 \left[\begin{matrix} -n, & -x; \\ & -; \end{matrix} \frac{1}{a} \right]$$

Mittag-Leffler polynomials

It is denoted by the symbol $g_n(z, \gamma)$ and is defined as

$$(2.14) \quad g_n(z, \gamma) = \frac{(-\gamma)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, & z; \\ & -\gamma; \end{matrix} 2 \right]$$

Shively's pseudo Laguerre polynomials

It is denoted by the symbol $R_n(a, x)$ and is defined as

$$(2.15) \quad R_n(a, x) = \frac{(a)_{2n}}{n!(a)_n} {}_1F_1 \left[\begin{matrix} -n ; \\ a+n ; \end{matrix} x \right]$$

3. Operational representations

If $D_x \equiv \frac{\partial}{\partial x}$ and $D_y \equiv \frac{\partial}{\partial y}$, M.A. Khan and A.K. Shukla [2] wrote the binomial expansion for $(D_x + D_y)^n$ as

$$(3.1) \quad (D_x + D_y)^n \equiv \sum_{r=0}^n {}^nC_r D_x^{n-r} D_y^r$$

where ${}^nC_r = \frac{n!}{r!(n-r)!}$.

By writing the finite series on the right of (3.1) M.A. Khan and A.K. Shukla [2] wrote (3.1) also as

$$(3.2) \quad (D_x + D_y)^n \equiv \sum_{r=0}^n {}^nC_r D_x^r D_y^{n-r}$$

If $F(x, y)$ is a function of x and y , they obtained the following from (3.1) and (3.2)

$$(3.3) \quad (D_x + D_y)^n F(x, y) \equiv \sum_{r=0}^n \frac{(-n)_r (-1)^r}{r!} D_x^{n-r} D_y^r F(x, y)$$

$$(3.4) \quad (D_x + D_y)^n F(x, y) \equiv \sum_{r=0}^n \frac{(-n)_r (-1)^r}{r!} D_x^r D_y^{n-r} F(x, y)$$

In particular, if $F(x, y) = f(x)g(y)$ then (3.3) and (3.4) in the form

$$(3.5) \quad (D_x + D_y)^n f(x)g(y) \equiv \sum_{r=0}^n \frac{(-n)_r (-1)^r}{r!} D_x^{n-r} f(x) D_y^r g(y)$$

$$(3.6) \quad (D_x + D_y)^n f(x)g(y) \equiv \sum_{r=0}^n \frac{(-n)_r (-1)^r}{r!} D_x^r f(x) D_y^{n-r} g(y)$$

Now, by taking special values of $f(x)$ and $g(y)$ in (3.5), we obtain the following partial differential operator representations of the polynomials given above:

$$(3.7) \quad x^{\nu+1}y(D_x + D_y)^n \{x^{\nu-1}y^{-1}\} = (-1)^n n! g_n^{(\nu)}\left(\frac{x}{y}\right)$$

$$(3.8) \quad (D_w + D_y D_z)^n \left\{ w^{\beta-1+n} y^x e^{(1-c^{-1})z} \right\} \\ = (\beta)_n w^{\beta-1} y^x e^{(1-c^{-1})z} M_n(x; \beta, c(w/y))$$

$$(3.9) \quad (D_w + D_y D_z)^n \left\{ w^{-N-1+n} y^x e^{p^{-1}z} \right\} \\ = (-N)_n w^{-N-1} y^x e^{-p^{-1}z} K_n(x; P, N), \text{ if } w/y = 1$$

$$(3.10) \quad (D_v D_w - D_y D_z)^n \left\{ v^{-N-1+n} w^{\alpha+n} y^x z^{-1-\alpha-\beta-n} \right\} \\ = (-N)_n (1+\alpha)_n v^{-N-1} w^{\alpha} y^x z^{-1-\alpha-\beta-n} Q_n\left(x; \alpha, \beta, N \frac{vw}{yz}\right)$$

$$(3.11) \quad x y e^{z/x} (1 + D_y D_z)^n \left\{ y^{-x} e^{\frac{-1}{x}z} \right\} = n! \varphi_n(y/x)$$

$$(3.12) \quad (D_w + D_y D_z)^n \left\{ w^n y^x e^{(1-e^\lambda)z} \right\} = n! y^x e^{(1-\lambda)z} l_n(x; \lambda), \text{ when } w/y = 1$$

$$(3.13) \quad (1 + D_y D_z)^n \left\{ y^x e^{\frac{1}{a}z} \right\} = \frac{e^{\frac{1}{a}z}}{(-a)^n} C_n^{(a)}(x) \text{ for } y = 1$$

$$(3.14) \quad (1 - D_y)^n \{y^{a+2n-1}\} = n! y^{a+2n-1} R_n(a, y)$$

To validate the above mentioned equations, here we illustrate some proofs:

Proof of (3.7).

$$(D_x + D_y)^n \{x^{-s-1}y^{-1}\} \\ = \sum_{r=0}^n \frac{(-n)_r (-1)^r}{r!} \frac{(1+s)_n (-1)^n}{(-s-n)_r} x^{-s-1-n+r} (1)_r (-1)^r y^{-1-r} \\ = x^{\nu+1}y(-1)^n n! g_n^{(\nu)}\left(\frac{x}{y}\right)$$

Proof of (3.8).

$$(D_w + D_y D_z)^n \left\{ w^{\beta-1+n} y^x e^{(1-c^{-1})z} \right\} \\ = \sum_{r=0}^n \frac{(-n)_r (-1)^r}{r!} D_w^{n-r} w^{\beta-1+n} D_y^r y^x D_z^r e^{(1-c^{-1})z} \\ = (\beta)_n w^{\beta-1} y^x e^{(1-c)^{-1}z} \sum_{r=0}^n \frac{(-n)_r (-x)_r (1-c^{-1})^r w^r}{r! (\beta)_r y^r} \\ = (\beta)_n w^{\beta-1} y^x e^{(1-c^{-1})z} {}_2F_1 \left[\begin{matrix} -n, & -x \\ & \beta \end{matrix}; (1-c)^{-1}(w/y) \right] \\ = (\beta)_n w^{\beta-1} y^x e^{(1-c)^{-1}z} M_n(x; \beta, c(w/y))$$

Proof of (3.13).

$$\begin{aligned} (D_w + D_y D_z)^n \{w^{-\gamma-1+n} y^{-z} e^{2z}\} &= \sum_{r=0}^n \frac{(-n)_r (-1)^r}{r!} D_w^{n-r} w^{-\gamma-1+n} D_y^r y^{-z} D_z^r e^{2z} \\ &= (-\gamma)_n w^{-\gamma-1} y^{-z} e^{2z} {}_2F_1 \left[\begin{matrix} -n, & z ; \\ & -\gamma ; \end{matrix} 2(w/y) \right] \end{aligned}$$

Proof of (3.14).

$$\begin{aligned} (1 - D_y)^n \{y^{a-1+2n}\} &= \sum_{r=0}^n \frac{(-n)_r (-1)^{n-r}}{r!} \frac{(a+n)_n}{(a+n)_r} y^{a+n-1+r} \quad \text{by (3.6)} \\ &= n! \frac{(a)_{2n}}{(a)_n} x^{a-1+n} {}_1F_1 \left[\begin{matrix} -n ; \\ a+n ; \end{matrix} y \right] \\ &= n! y^{a-1+n} R_n(a, y) \end{aligned}$$

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ON THE EXTERNAL PATH LENGTH OF RANDOM RECURSIVE k -ARY TREES

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Abstract. In this paper, we determine the expectation and variance of X_n the external path length in a random recursive k -ary tree of size n .

Key words and phrases: Recursive trees, path length.

2000 Mathematics Subject Classification: 05C05.

1. Introduction

The analysis of the length of paths in tree families has received a lot of attention, see, e.g., [1], [4], [10], [11], [13], often due to their importance in the analysis of algorithms. In [4], [11], [13] the total path length is investigated in random recursive trees. However, up to now there is no result about the external path length of random recursive k -ary trees. Here we obtain the expectation and variance of the external path length in random recursive k -ary trees.

By a *recursive tree* we mean a labeled rooted tree such that each path from the root to any node of the tree is labeled with an increasing sequence of labels.

Note that there is no restriction on the outdegrees of the nodes of a recursive tree. A recursive tree where outdegrees are restricted to k , is called a *recursive k -ary tree*.

The possible insertion positions to join a new node to a tree, are called *external nodes*. In a recursive k -ary tree, the number of nodes can be attached to a node ν of outdegree d_ν is $k - d_\nu$. Therefore the number of all external nodes

in a recursive k -ary tree of size n is,

$$\sum_{\nu=1}^n (k - d_\nu) = kn - (n - 1) = (k - 1)n + 1.$$

A *random recursive k -ary tree* of size n is constructed as follows. One starts from a root node holding the label 1; at stage i ($i = 2, 3, \dots, n$) a new node holding label i (the i th node) is attached to any previous node ν of outdegree d_ν of the already grown tree T_{i-1} of size $i - 1$ with probability $\frac{k - d_\nu}{(k - 1)(i - 1) + 1}$ (the number of remaining external nodes for the node ν , $k - d_\nu$, is divided by $(k - 1)(i - 1) + 1$, the number of all external nodes). This function implies that the higher outdegree nodes possess a lower attraction for new neighbors and no node of outdegree greater than k is introduced. These are two properties of interest in chemical applications [2], [5]. The outdegree of a node is investigated in [7] and [9].

A survey of applications and results on recursive trees is given in [12]. These trees are used, e.g., to model chain letters and pyramid schemes [6], and as a simplified growth model of the word wide web [3].

Let D_j be the depth of j th node in a random recursive k -ary tree of size n . The first cumulative random variable is the *internal path length* $I_n = \sum_{j=1}^n D_j$.

Suppose the external nodes are indexed by $1, 2, \dots, (k - 1)n + 1$, and x_j be the depth of the j th external node. The second cumulative random variable is

$X_n = \sum_{j=1}^{(k-1)n+1} x_j$. This random variable is called the *external path length*. By the proof of Theorem 1, the relation

$$(1) \quad X_n = (k - 1)I_n + nk,$$

and then $X_n = (k - 1) \sum_{j=1}^n D_j + nk$ can be deduced. The strong dependence between the random variables D_j makes it difficult to compute the exact distribution of X_n .

Throughout this paper we use the term *tree* instead of recursive k -ary tree.

2. Expectation and variance

In this section, the following results for D_n , will be used (see [8]):

$$\mathbf{E}[D_n] = \sum_{x=1}^{n-1} \frac{k}{(k - 1)x + 1},$$

and

$$\mathbf{Var}[D_n] = \sum_{x=2}^{n-1} \frac{k(k - 1)(x - 1)}{((k - 1)x + 1)^2}.$$

Theorem 1

$$\mathbf{E}[X_n] = \sum_{x=1}^n \frac{((k-1)n+1)k}{(k-1)x+1},$$

and

$$\mathbf{Var}[X_n] = \sum_{x=1}^n \frac{k(k-1)^3(n-x)(x-1)((k-1)n+1)}{((k-1)x+1)^2((k-1)(x+1)+1)}.$$

Proof. By inserting the n th node at level D_n , a tree T_n of size n is obtained from a tree T_{n-1} of size $n-1$. The n th node may replace any of the $(k-1)(n-1)+1$ external nodes of T_{n-1} with probability $1/((k-1)(n-1)+1)$. The new node gives the tree k new external nodes, but one of the external nodes of T_{n-1} is lost in the process. Therefore

$$X_n = X_{n-1} + k(D_n + 1) - D_n = X_{n-1} + (k-1)D_n + k.$$

Let $\mathcal{F}_n$ denote the sigma field generated by the tree T_n . When the shape of the tree T_{n-1} is available, the levels $x_1, \dots, x_{(k-1)(n-1)+1}$ of the external nodes are completely determined. Thus D_n may assume any of the values $x_1, \dots, x_{(k-1)(n-1)+1}$ with equal probability $1/((k-1)(n-1)+1)$. We can now formulate a conditional expectation,

$$\begin{aligned} \mathbf{E}[X_n | \mathcal{F}_{n-1}] &= \frac{1}{(k-1)(n-1)+1} \sum_{j=1}^{(k-1)(n-1)+1} (X_{n-1} + (k-1)x_j + k) \\ &= X_{n-1} + k + \frac{k-1}{(k-1)(n-1)+1} \sum_{j=1}^{(k-1)(n-1)+1} x_j. \end{aligned}$$

But the remaining sum is the external path length of T_{n-1} , i.e.,

$$(2) \quad \mathbf{E}[X_n | \mathcal{F}_{n-1}] = \frac{(k-1)n+1}{(k-1)(n-1)+1} X_{n-1} + k.$$

Taking expectations of the last relation we get the following recurrence on expected external path length

$$(3) \quad \mathbf{E}[X_n] = \frac{(k-1)n+1}{(k-1)(n-1)+1} \mathbf{E}[X_{n-1}] + k,$$

which can be easily solved under the initial condition $\mathbf{E}[X_1] = k$ to yield the first required result.

To compute the variance of X_n we formulate a recurrence for

$$Q_n := \frac{\mathbf{Var}[X_n]}{((k-1)n+1)((k-1)(n+1)+1)}$$

as follows. Let $Z_n = \frac{X_n - \mathbf{E}[X_n]}{(k-1)n+1}$. Replace X_n by $X_{n-1} + (k-1)D_n + k$ in the definition of Z_n and write

$$\begin{aligned} Z_n &= \frac{X_{n-1} + (k-1)D_n + k - \mathbf{E}[X_{n-1} + (k-1)D_n + k]}{(k-1)n+1} \\ &= \frac{(k-1)(n-1)+1}{(k-1)n+1} Z_{n-1} + \frac{k-1}{(k-1)n+1} (D_n - \mathbf{E}[D_n]). \end{aligned}$$

By squaring the latter relation and taking expectations we get

$$\begin{aligned} \mathbf{E}[Z_n^2] &= \left(\frac{(k-1)(n-1)+1}{(k-1)n+1} \right)^2 \mathbf{E}[Z_{n-1}^2] + \left(\frac{k-1}{(k-1)n+1} \right)^2 \mathbf{Var}[D_n] \\ (4) \quad &+ \frac{2(k-1)((k-1)(n-1)+1)}{((k-1)n+1)^2} \mathbf{E}[Z_{n-1}(D_n - \mathbf{E}[D_n])]. \end{aligned}$$

Since the component $\mathbf{E}[Z_{n-1}\mathbf{E}[D_n]]$ is zero, in the last term we need only to find $\mathbf{E}[Z_{n-1}D_n]$. For the required term we compute

$$\mathbf{E}[Z_{n-1}D_n] = \mathbf{E}[\mathbf{E}[Z_{n-1}D_n | \mathcal{F}_{n-1}]] = \mathbf{E}[Z_{n-1}\mathbf{E}[D_n | \mathcal{F}_{n-1}]].$$

But according to the algorithmic development,

$$\mathbf{E}[D_n | \mathcal{F}_{n-1}] = \sum_{j=1}^{(k-1)(n-1)+1} \frac{x_j}{(k-1)(n-1)+1} = \frac{X_{n-1}}{(k-1)(n-1)+1}.$$

So,

$$\mathbf{E}[Z_{n-1}D_n] = \mathbf{E}[Z_{n-1}^2].$$

Put this relation into (4) we arrive at the recurrence

$$\begin{aligned} \mathbf{E}[Z_n^2] &= \frac{((k-1)(n+1)+1)((k-1)(n-1)+1)}{((k-1)n+1)^2} \mathbf{E}[Z_{n-1}^2] \\ (5) \quad &+ \frac{(k-1)^2}{((k-1)n+1)^2} \mathbf{Var}[D_n]. \end{aligned}$$

The substitution Q_n linearizes the recurrence (5) into the simple recurrence

$$Q_n = Q_{n-1} + \frac{(k-1)^2}{((k-1)n+1)((k-1)(n+1)+1)} \mathbf{Var}[D_n].$$

By the relation for the variance of D_n , the solution to the last recurrence gives

$$Q_n = \sum_{m=3}^n \frac{(k-1)^2}{((k-1)m+1)((k-1)(m+1)+1)} \sum_{x=2}^{m-1} \frac{k(k-1)(x-1)}{((k-1)x+1)^2}.$$

Expanding $1/((k-1)m+1)((k-1)(m+1)+1)$ by partial fractions and collapsing the resulting telescopic sums, we have

$$(6) \quad Q_n = \sum_{x=2}^{n-1} \frac{k(k-1)^3(n-x)(x-1)}{((k-1)x+1)^2((k-1)(x+1)+1)((k-1)(n+1)+1)}.$$

So, by definition of Q_n , the proof is complete. $\blacksquare$

Remark. By (1) the expectation of internal path length I_n is

$$\mathbf{E}[I_n] = \frac{k}{k-1} \sum_{j=2}^{n-1} \frac{(k-1)n+1}{(k-1)j+1} - \frac{n-1}{k-1}.$$

So the average external path length is asymptotically $k-1$ times as much as the average internal path length $\mathbf{E}[I_n] \sim \frac{k}{k-1}n \ln n$.

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TERNARY SUPERRING

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Abstract. We introduce a new algebraic system, which is called ternary semisuperring. A ternary semisuperring is a nonempty set with a binary hyperoperation's addition and a ternary hyperoperation's multiplication. It is a generalization of ternary semihypererring and semisuperring. We also introduce the notions of fuzzy superideal in ternary semisuperrings and give characterizations of fuzzy superideals.

Keywords: ternary semisuperring; ternary hyperoperation; fuzzy superideal; ternary semihypererring; semisuperring.

1. Introduction

The theory of semiring was first developed by H.S. Vandiver. Semiring constitute a fairly natural generalization of ring, with board applications in the mathematical foundation of computer science. Also, semiring theory has many other applications, for example, automata theory, optimization theory, algebra of formal process, combinatorial optimization, Baysian networks and belief propagation.

The literature of ternary algebric system was introduced by D.H. Lehmer [12] in 1932. He investigated certain ternary semigroup called triplexes which turn out

to be ternary groups. The notion of ternary semigroups was known to S. Banach. He showed by an example that a ternary semigroup does not necessarily reduce to an ordinary semigroup. Dutta and Kar [8] introduced the notion of ternary semiring which is a generalization of the notion of ternary ring.

Hyperstructure theory was first introduced in 1934 by Marty at the 8th Congress of Scandinavian Mathematicians (see [13]). Later on, hyperstructures have been developed in both pure and applied sciences. A comprehensive review of the theory of hyperstructures can be found in [2], [3], [6]. Hyperstructures are generalizations of classic structures. For example, hypergroup [2] is a generalization of group, hyperlattice[11] and superlattice[10] are generalizations of lattice and so on.

The concept of fuzzy sets was first introduced by Zadeh [21] in 1965. The theory of fuzzy sets has been developed fast and has many applications in many branches of sciences. In mathematics, the study of fuzzy algebraic structures was first initiated by pioneer paper of Rosenfeld [14]. He first studied the fuzzy subgroup of a group and since then, many researchers have been engaged in extending the concepts and results of abstract algebra based on fuzzy sets.

In a ternary semihyperring [5], the addition is a hyperoperation and the multiplication is a ternary operation. Of course, it is a generalization of semiring. In this paper, we introduce a new algebraic system, which is called ternary semisuperring. A ternary semisuperring is a nonempty set with a binary hyperoperation's addition and a ternary hyperoperation's multiplication. It is a generalization of ternary semihyperring and semisuperring. We also introduce the notions of fuzzy superideal in ternary semisuperrings and give characterizations of fuzzy superideals.

2. Semiring, semihyperring and semisuperring

A *semiring* [5] is a system consisting of a nonempty set S together with two binary operations on S called *addition* and *multiplication* (denoted in the usual manner) such that

- (1) $(S, +)$ is a (commutative) semigroup;
- (2) $(S, *)$ is a semigroup;
- (3) $a * (b + c) = a * b + a * c$ and $(a + b) * c = a * c + b * c$ for all $a, b, c \in S$.

There are many examples of semirings, for example, $(\mathbf{R}, +, \cdot)$, where $\mathbf{R}$ is the set of all real numbers, $+$ and $\cdot$ is the usual addition and multiplication, is a semiring.

A *left (right) ideal* of a semiring S is a subset I of S such that

- (1) $a + b \in I$ for all $a, b \in I$;
- (2) $r * a \in I$ ($a * r \in I$) for all $r \in S$ and $a \in I$.

An ideal of a semiring S is a subset I of S such that it is both a left and a right ideal of S .

For example, under the usual addition and multiplication, $\mathbf{R}^+$, where $\mathbf{R}^+$ is the set of all positive integers, is a left (right) ideal of $\mathbf{R}$, it is also an ideal of $\mathbf{R}$.

A *partial hypergroupoid* $(H, *)$ is a nonempty set H with a function from $H \times H$ to the set of subsets of H . I.e.,

$$\begin{aligned} * : H \times H &\longrightarrow \mathbf{P}(H) \\ (x, y) &\longrightarrow x * y. \end{aligned}$$

A *hypergroupoid* is a nonempty set H , endowed with a hyperoperation, that is a function from $H \times H$ to the set of nonempty subsets of H .

If $A, B \in \mathbf{P}(H) - \{\emptyset\}$, then we define $A * B = \cup\{a * b \mid a \in A, b \in B\}$, $x * B = \{x\} * B$ and $A * y = A * \{y\}$.

A *semihypergroup* is a hypergroupoid $(H, *)$ which satisfies $a * (b * c) = (a * b) * c$.

A *semihyperring* is algebraic system $(S, \circ, +)$, where S is a nonempty set and the followings are satisfied:

- (1) $(S, +)$ is a (commutative) semihypergroup;
- (2) $(S, \circ)$ is a semigroup;
- (3) $a \circ (b + c) = a \circ b + a \circ c$ and $(a + b) \circ c = a \circ c + b \circ c$ for all $a, b, c \in S$.

In [1], Ameri and Hedayati studied this algebraic system.

In the previous definition, if the first condition is replaced by $(R, +)$ is a (commutative) semigroup and the second condition is replaced by $(R, \circ)$ is a semihypergroup, we obtain the definition of a *dual semihyperring*.

The following is the definition of a *semisuperring* (Vougiouklis in [15] and Davvaz in [7] studied this system, and called it a semihyperring).

Definition 1 A *semisuperring* is a system consisting of a nonempty set R together with two binary hyperoperations $+$ and $*$ on R such that

- (1) $(R, +)$ is a (commutative) semihypergroup;
- (2) $(R, *)$ is a semihypergroup;
- (3) $a * (b + c) = a * b + a * c$ and $(a + b) * c = a * c + b * c$ for all $a, b, c \in R$.

3. Ternary semihyperring and ternary semisuperring

Let H be a nonempty set and $P^*(H)$ denote the set of all nonempty subsets of H . A map f from $H \times H \times H$ to $P^*(H)$ is called a *ternary hyperoperation*. The pair (H, f) is called a *ternary hypergroupoid*. If $A, B, C \in P^*(H)$, then we define

$$f(A, B, C) = \bigcup_{a \in A, b \in B, c \in C} f(a, b, c).$$

Definition 2 ([4]) The ternary hypergroupoid (H, f) is called a *ternary semihypergroup* if for any $a, b, c, d, e \in H$, we have

$$f(f(a, b, c), d, e) = f(a, f(b, c, d), e) = f(a, b, f(c, d, e)).$$

Definition 3 ([5]) A nonempty set R together with a binary hyperoperation $+$ and a ternary operation f is said to be a *ternary semihyperring* if $(R, +)$ is a (commutative) semihypergroup, (R, f) is a ternary semigroup, and for any $a, b, c, d, e \in R$, the following conditions are satisfied:

- (1) $f(a + b, c, d) = f(a, c, d) + f(b, c, d)$,
- (2) $f(a, b + c, d) = f(a, b, d) + f(a, c, d)$,
- (3) $f(a, b, c + d) = f(a, b, c) + f(a, b, d)$.

Example 1 ([5]) Let $\mathbf{Z}$ be the set of all integers. We define a binary hyperoperation and a ternary multiplication on $\mathbf{Z}$ in the following way: $x \oplus y = \{x, y\}$ and $f(x, y, z) = x \cdot y \cdot z$, for all $x, y, z \in \mathbf{Z}$. Then $(\mathbf{Z}, \oplus, f)$ is a ternary semihypergroup.

Definition 4 A nonempty set R together with a binary hyperoperation $+$ and a ternary hyperoperation f is said to be a *ternary semisuperring* if $(R, +)$ is a (commutative) semihypergroup, (R, f) is a ternary semihypergroup, and for any $a, b, c, d, e \in R$, the following conditions are satisfied:

- (1) $f(a + b, c, d) = f(a, c, d) + f(b, c, d)$,
- (2) $f(a, b + c, d) = f(a, b, d) + f(a, c, d)$,
- (3) $f(a, b, c + d) = f(a, b, c) + f(a, b, d)$.

Obviously, ternary semisuperring is a generalization of ternary semihyperring and semisuperring.

Example 2 Let $\mathbf{Z}$ be the set of all integers. We define a binary hyperoperation and a ternary hyperoperation on $\mathbf{Z}$ in the following way: $x \oplus y = \{x, y\}$ and $f(x, y, z) = \{x, y, z\}$, for any $x, y, z \in \mathbf{Z}$. Then $(\mathbf{Z}, \oplus, f)$ is a ternary semisuperring.

Example 3 Let $\mathbf{Z}$ be the set of all integers. We define a binary hyperoperation as in the previous example and a ternary hyperoperation on $\mathbf{Z}$ as $g(x, y, z) = \{n_1x + n_2y + n_3z | n_1, n_2, n_3 \in \mathbf{Z}\}$, for any $x, y, z \in \mathbf{Z}$. Then $(\mathbf{Z}, \oplus, g)$ is a ternary semisuperring.

Example 4 Let $\mathbf{I}$ be the real interval $[0, 1]$. We define $x \oplus y = \{x, y\}$ and $f(x, y, z) = \{t | \min\{x, y, z\} \leq t \leq \max\{x, y, z\}\}$, for any $x, y, z \in \mathbf{I}$. Then $(\mathbf{I}, \oplus, f)$ is a ternary semisuperring.

Example 5 Let $(R, +, *)$ be a semisuperring, we can define a ternary semisuperring $(R, +, f)$ where $f(x, y, z) = x * y * z$ for all $x, y, z \in R$.

Definition 5 Let $(R, +, *)$ be a ternary semisuperring. A is a nonempty subset of R . $(A, +, *)$ is called a ternary sub-semisuperring of R if $a + b \subseteq A$ and $f(a, b, c) \subseteq A$, for all $a, b, c \in A$.

Definition 6 $(I, +, *)$ is a ternary sub-semisuperring of a ternary semisuperring $(R, +, *)$. I is called a

- (1) *left superideal* of R if $f(a, b, i) \subseteq I$, for all $a, b \in R$ and $i \in I$,
- (2) *right superideal* of R if $f(i, a, b) \subseteq I$, for all $a, b \in R$ and $i \in I$,
- (3) *lateral superideal* of R if $f(a, i, b) \subseteq I$, for all $a, b \in R$ and $i \in I$.

If I is both a left and a right superideal of R , then I is called a *two-sided superideal* of R . I is called a *superideal* if it is a left, right and a lateral superideal of R .

4. Fuzzy ternary semisuperring

Let X be a non-empty set. A map $\mu : X \rightarrow [0, 1]$ is called a *fuzzy subset* of X . If μ is a fuzzy subset of X , then for any $t \in [0, 1]$, the set $\mu_t = \{x \in X \mid \mu(x) \geq t\}$ is called a *level subset* of μ .

For any $a, b \in [0, 1]$, we will use $a \wedge b$ to denote $\min\{a, b\}$ and use $a \vee b$ to denote $\max\{a, b\}$.

Definition 7 A fuzzy subset μ of a ternary semisuperring $(R, +, *)$ is called a fuzzy ternary semisuperring of R if for any $x, y, z \in R$ we have:

- (1) $\mu(x) \wedge \mu(y) \leq \inf_{z \in x+y} \mu(z)$;
- (2) $\mu(x) \wedge \mu(y) \wedge \mu(z) \leq \inf_{z \in f(x, y, z)} \mu(z)$.

Definition 8 A fuzzy subset μ of a ternary semisuperring $(R, +, *)$ is called a fuzzy superideal of R if for any $x, y, z, v \in R$ we have:

- (1) $\mu(x) \wedge \mu(y) \leq \inf_{z \in x+y} \mu(z)$;
- (2) $\mu(x) \vee \mu(y) \vee \mu(z) \leq \inf_{v \in f(x, y, z)} \mu(v)$.

Theorem 1 A fuzzy subset μ of a ternary semisuperring $(R, +, f)$ is a fuzzy superideal if and only if each its nonempty level subset is a superideal of R .

Proof. Let μ be a fuzzy superideal of a ternary semisuperring $(R, +, f)$. If $x, y \in \mu_t$ for some $t \in [0, 1]$, then $\mu(x) \geq t$ and $\mu(y) \geq t$. Thus

$$t \leq \mu(x) \wedge \mu(y) \leq \inf_{z \in x+y} \mu(z),$$

which implies that $\mu(z) \geq t$ for any $z \in x + y$. So $x + y \subseteq \mu_t$.

Suppose that $x, y, z \in R$ and $x \in \mu_t$. Then $\mu(x) \geq t$. And from $\mu(x) \vee \mu(y) \vee \mu(z) \leq \inf_{v \in f(x, y, z)} \mu(v)$ we know that $t \leq \mu(x) \leq \inf_{v \in f(x, y, z)} \mu(v)$, which implies that $f(x, y, z) \subseteq \mu_t$. Similarly, we can obtain that $f(y, x, z) \subseteq \mu_t$ and $f(y, z, x) \subseteq \mu_t$. Hence, μ_t is a superideal of R .

Conversely, suppose that every nonempty level subset μ_t is a superideal of R . Let $t_0 = \mu(x) \wedge \mu(y)$ for any $x, y \in R$. Then $x, y \in \mu_{t_0}$, consequently, $x + y \subseteq \mu_{t_0}$. Thus

$$\mu(x) \wedge \mu(y) = t_0 \leq \inf_{v \in x+y} \mu(v).$$

Assume that $\mu(x) = t_1$. Then $x \in \mu_{t_1}$. Since μ_{t_1} is a superideal of R , we have that $f(x, y, z) \subseteq \mu_{t_1}$. Hence, $\mu(x) = t_1 \leq \inf_{v \in f(x, y, z)} \mu(v)$. Similarly, we obtain that $\mu(y) \leq \inf_{v \in f(x, y, z)} \mu(v)$ and $\mu(z) \leq \inf_{v \in f(x, y, z)} \mu(v)$. Thus $\mu(x) \vee \mu(y) \vee \mu(z) \leq \inf_{v \in f(x, y, z)} \mu(v)$.

This completes the proof. ■

Yuan [20] introduced the concept of fuzzy subgroup with thresholds. A fuzzy subgroup with thresholds λ and μ is also called a (λ, μ) -fuzzy subgroup.

Yao continued to research (λ, μ) -fuzzy normal subgroups, (λ, μ) -fuzzy quotient subgroups and (λ, μ) -fuzzy subrings in [16, 17, 18, 19]. Feng [9] investigated (λ, μ) -fuzzy sublattices and (λ, μ) -fuzzy subhyperlattices. Davvaz studied ternary semihyperring with thresholds in [5].

In the following, we study fuzzy superideal with thresholds of a ternary semisuperring.

Definition 9 Let $\alpha, \beta \in [0, 1]$ and $\alpha < \beta$. Let μ be a fuzzy subset of a ternary semisuperring $(R, +, f)$. Then μ is called a *fuzzy superideal with thresholds* of R , if

- (1) $\mu(x) \wedge \mu(y) \wedge \beta \leq \inf_{z \in x+y} \mu(z) \vee \alpha$;
- (2) $(\mu(x) \vee \mu(y) \vee \mu(z)) \wedge \beta \leq \inf_{v \in f(x, y, z)} \mu(v) \vee \alpha$.

Theorem 2 A fuzzy subset μ of a ternary semisuperring $(R, +, f)$ is a fuzzy superideal with thresholds (α, β) of R if and only if $\mu_t (\neq \emptyset)$ is a superideal of R for all $t \in (\alpha, \beta)$.

Proof. Similar to the proof of Theorem 4.14 of [5]. ■

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SOME FIXED POINT THEOREMS FOR MULTIVALUED WEAKLY INCREASING OPERATORS

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Abstract. In this paper, some common fixed point theorems for a pair of multivalued weakly increasing operators in partially ordered metric space and in partially ordered Banach space are proved.

Keywords. Banach spaces, complete metric space, fixed point, partial order.

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1. Introduction

The study on the existence of fixed points for single valued increasing operators is bounteous and successful, the results obtained are widely used to investigate the existence of solutions to the ordinary and partial differential equations (see [7], [8] and reference therein).

It is natural to extend this study to multivalued case. Recently in [5], [8], [9], the authors have verified some fixed point theorems for multivalued increasing operators. Also in [1], the authors proved some fixed point theorems for multivalued weakly uniform increasing operators.

Motivated by [1], [4], [5], [10], [11], [12] and [13], we prove some fixed point theorems for multivalued operators satisfying some weakly increasing properties in metric space and Banach space, respectively.

2. Preliminaries

In this section some notations and preliminaries are given.

Let X be a topological space and $\leq$ be a partial order endowed on X .

In order to define the multivalued increasing and multivalued weakly increasing operators, we give the following relations between two subsets of X .

Definition 1. ([5]) Let A, B be two nonempty subsets of X , the relations between A and B are defined as follows:

- (1) If, for every $a \in A$, there exists $b \in B$ such that $a \leq b$, then $A \prec_1 B$.
- (2) If, for every $b \in B$, there exists $a \in A$ such that $a \leq b$, then $A \prec_2 B$.
- (3) If, $A \prec_1 B$ and $A \prec_2 B$, then $A \prec B$.

Remark 1. $\prec_1$ and $\prec_2$ are different relations between A and B . For example, let $X = \mathbb{R}$, $A = [\frac{1}{2}, 1]$, $B = [0, 1]$, $\leq$ be usual order on X , then $A \prec_1 B$ but $A \not\prec_2 B$; if $A = [0, 1]$, $B = [0, \frac{1}{2}]$, then $A \prec_2 B$ while $A \not\prec_1 B$.

Remark 2. $\prec_1$, $\prec_2$ and $\prec$ are reflexive and transitive, but are not antisymmetric. For instance, let $X = \mathbb{R}$, $A = [0, 3]$, $B = [0, 1] \cup [2, 3]$, $\leq$ be usual order on X , then $A \prec B$ and $B \prec A$, but $A \neq B$. Hence, they are not partial orders on 2^X .

We can show that $\prec$ is a partial order on $CC(\mathbb{R})$, which denotes the family of closed and convex subsets of $\mathbb{R}$.

Let us recall some basic definitions and facts from multivalued analysis (for details, see [3]).

Let 2^X denote the family of all nonempty subset of X . A multivalued map $T : X \rightarrow 2^X$ is called upper semi continuous (u.s.c.) on X if for each $x_0 \in X$, the set Tx_0 is a nonempty, closed subset of X , and if for each open set V of X containing Tx_0 , there exists an open neighborhood U of x_0 such that $T(U) \subseteq V$.

T is said to be completely continuous if $T(B)$ is relatively compact for every bounded subset $B \subset X$.

If the multivalued map T is completely continuous with nonempty compact values, then T is u.s.c. if and only if T has a closed graph (i.e. $u_n \rightarrow u_0, v_n \rightarrow v_0, v_n \in Tu_n$ imply $v_0 \in Tu_0$).

Definition 2. ([5]) If $\{x_n\} \subset X$ satisfies $x_1 \leq x_2 \leq \dots \leq x_n \leq \dots$ or $x_1 \geq x_2 \geq \dots \geq x_n \geq \dots$, then $\{x_n\}$ is called a monotone sequence.

Definition 3. ([5]) A multivalued operator $T : X \rightarrow 2^X \setminus \emptyset$ is called order closed if for monotone sequences $\{u_n\}, \{v_n\} \subset X, u_n \rightarrow u_0, v_n \rightarrow v_0$ and $v_n \in Tu_n$ imply $v_0 \in Tu_0$.

Definition 4. ([5]) A function $f : X \rightarrow \mathbb{R}$ is called order upper (lower) semi continuous, if, for monotone sequence $\{u_n\} \subset X$, and $u_0 \in X$

$$u_n \rightarrow u_0 \implies \limsup_{n \rightarrow \infty} f(u_n) \leq f(u_0), \quad (f(u_0) \leq \liminf_{n \rightarrow \infty} f(u_n)).$$

Remark 3. An operator with closed graph must be an order closed operator and an upper (lower) semi continuous function is an order upper (lower) semi continuous. There exist some examples in [5] showing that the converse is not true.

Let X be a real Banach space with a norm $\|\cdot\|_X$. A non-empty closed subset P of X is said to be an order cone in X if

- (i) $P + P \subseteq P$,
- (ii) $\lambda P \subseteq P$, for $\lambda \geq 0$, and
- (iii) $P \cap -P = \{\theta\}$, where θ denotes the zero element of X .

Then, the relation $x \leq y$ if and only if $y - x \in P$ defines the partial ordering in X . P is called regular, if every nondecreasing and bounded above in order sequence in X has a limit, i.e., if $\{x_n\} \subset X$ satisfies $x_1 \leq x_2 \leq \dots \leq x_n \leq \dots \leq y$ ($y \in X$), then there exists $x \in X$ such that $\|x_n - x\|_X \rightarrow 0$ as $n \rightarrow \infty$. We do not require any property of the cones in the present discussion, however, the details of order cones and their properties may be found in Guo and Lakshmikantham [6].

Now we introduce the following definition.

Definition 5. Let X be a topological space and $\leq$ be a partial order endowed on X . Two maps $S, T : X \rightarrow 2^X$ are said to be weakly increasing with respect to $\prec_1$ if for any $x \in X$ we have $Sx \prec_1 Ty$ for all $y \in Sx$ and $Tx \prec_1 Sy$ for all $y \in Tx$. Similarly two maps $S, T : X \rightarrow 2^X$ are said to be weakly increasing with respect to $\prec_2$ if for any $x \in X$ we have $Ty \prec_2 Sx$ for all $y \in Sx$ and $Sy \prec_2 Tx$ for all $y \in Tx$.

Now, we give some examples.

Example 1. Let $X = [1, \infty)$ and $\leq$ be usual order on X . Consider two mappings $S, T : X \rightarrow 2^X$ defined by $Sx = [1, x^2]$ and $Tx = [1, 2x]$ for all $x \in X$. Then the pair of mappings S and T are weakly increasing with respect to $\prec_2$ but not $\prec_1$. Indeed, since

$$Ty = [1, 2y] \prec_2 [1, x^2] = Sx \text{ for all } y \in Sx$$

and

$$Sy = [1, y^2] \prec_2 [1, 2x] = Tx \text{ for all } y \in Tx$$

so S and T are weakly increasing with respect to $\prec_2$ but $S2 = [1, 4] \not\prec_1 [1, 2] = T1$ for $1 \in S2$, so S and T are not weakly increasing with respect to $\prec_1$.

Example 2. Let $X = [0, 1]$ and $\leq$ be usual order on X . Consider two mappings $S, T : X \rightarrow 2^X$ defined by $Sx = \{0, 1\}$ and $Tx = [x, 1]$ for all $x \in X$. Then the pair of mappings S and T are weakly increasing with respect to $\prec_1$ but not $\prec_2$. Indeed, since

$$Sx = \{0, 1\} \prec_1 [y, 1] = Ty \text{ for all } y \in Sx$$

and

$$Tx = [x, 1] \prec_1 \{0, 1\} = Sy \text{ for all } y \in Tx$$

so S and T are weakly increasing with respect to $\prec_1$ but $T1 = \{1\} \not\prec_2 \{0, 1\} = S1$ for $1 \in S1$, so S and T are not weakly increasing with respect to $\prec_2$.

3. Fixed point theorems

In this section, we prove some fixed point theorems for multivalued weakly increasing operators with respect to $\prec_1$ and $\prec_2$ in partial order metric space and partial order Banach space.

We will use the following lemma in our theorems.

Lemma 1. ([2]) *Let (X, d) be a metric space, $\varphi : X \rightarrow \mathbb{R}$. Define the relation $\leq$ on X as follows:*

$$x \leq y \iff d(x, y) \leq \varphi(x) - \varphi(y).$$

Then $\leq$ is a partial order on X and X is a partial ordered metric space.

Now we give a fixed point theorem.

Theorem 1. *Let (X, d) be a complete metric space, $\varphi : X \rightarrow \mathbb{R}$ be a bounded below function, $\leq$ be a partial order induced by φ and $S, T : X \rightarrow 2^X$ be two order closed with respect to $\leq$ and weakly increasing operator with respect to $\prec_1$. Then (A1) there exists a monotone sequence $\{x_n\}_{n=1}^\infty \subset X$, $x_{2n+1} \in Sx_{2n}$, $x_{2n+2} \in Tx_{2n+1}$, $n = 0, 1, 2, \dots$, such that $\{x_n\}_{n=1}^\infty$ converges to $z \in X$, which is a common fixed point of S and T .*

(A2) if φ is order lower semi continuous on X , then $\{x_n\}_{n=1}^\infty$ satisfies $x_n \leq z$, $n = 1, 2, \dots$.

Proof. First, we prove (A1). Let $x_0 \in X$ be an arbitrary point. Since $Sx_0 \neq \emptyset$, we can choose $x_1 \in Sx_0$. Now since S and T are weakly increasing with respect to $\prec_1$, we have $x_1 \in Sx_0 \prec_1 Tx_1$. Thus there exist some $x_2 \in Tx_1$ such that $x_1 \leq x_2$. Again since S and T are weakly increasing with respect to $\prec_1$ we have $x_2 \in Tx_1 \prec_1 Sx_2$. Thus there exist some $x_3 \in Sx_2$ such that $x_2 \leq x_3$. Continuing this process, we will get an increasing sequence $\{x_n\}_{n=1}^\infty$ which satisfies

$$x_{2n+1} \in Sx_{2n}, x_{2n+2} \in Tx_{2n+1}, n = 0, 1, 2, \dots$$

By the definition $\leq$, we have

$$\varphi(x_1) \geq \varphi(x_2) \geq \dots \geq \varphi(x_n) \geq \dots$$

Since φ is bounded from below, then $\{\varphi(x_n)\}_{n=1}^\infty$ is convergent. Thus, for $\varepsilon > 0$, there exists N , if $n > m > N$ then

$$d(x_m, x_n) \leq \varphi(x_m) - \varphi(x_n) < \varepsilon$$

which indicates $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence in X . Due to completeness of X , we know there is $z \in X$, $x_{2n+1} \rightarrow z$ and $x_{2n+2} \rightarrow z$. Since S and T are order closed, $\{x_n\}_{n=1}^\infty$ monotone and $x_{2n+1} \in Sx_{2n}$, $x_{2n+2} \in Tx_{2n+1}$, we deduce that $z \in Sz$ and $z \in Tz$, i.e. z is a common fixed point of S and T .

Next, we verify (A2). If φ is order lower semi continuous on X , then, for any m , we have

$$\begin{aligned} d(x_m, z) &= \lim_{n \rightarrow \infty} d(x_m, x_n) \leq \limsup_{n \rightarrow \infty} (\varphi(x_m) - \varphi(x_n)) \\ &= \varphi(x_m) - \liminf_{n \rightarrow \infty} \varphi(x_n) \leq \varphi(x_m) - \varphi(z) \end{aligned}$$

which indicates $x_m \leq z$.

The proof of the following theorem carries over in the same manner.

Theorem 2. *Let (X, d) be a complete metric space, $\varphi : X \rightarrow \mathbb{R}$ be a bounded above function, $\leq$ be a partial order induced by φ and $S, T : X \rightarrow 2^X$ be two order closed with respect to $\leq$ and weakly increasing operator with respect to $\prec_2$. Then*

- (B1) *there exists a monotone sequence $\{x_n\}_{n=1}^\infty \subset X$, $x_{2n+1} \in Sx_{2n}, x_{2n+2} \in Tx_{2n+1}, n = 0, 1, 2, \dots$, such that $\{x_n\}_{n=1}^\infty$ converges to $z \in X$, which is a common fixed point of S and T .*
- (B2) *if φ is order upper semi continuous on X , then $\{x_n\}_{n=1}^\infty$ satisfies $z \leq x_n$, $n = 1, 2, \dots$.*

Proof. First, we prove (B1). Let $x_0 \in X$ be an arbitrary point. Since $Sx_0 \neq \emptyset$, we can choose $x_1 \in Sx_0$. Now since S and T are weakly increasing with respect to $\prec_2$, we have $Tx_1 \prec_2 Sx_0$. Thus there exist some $x_2 \in Tx_1$ such that $x_2 \leq x_1$. Again since S and T are weakly increasing with respect to $\prec_2$ we have $Sx_2 \prec_2 Tx_1$. Thus, there exist some $x_3 \in Sx_2$ such that $x_3 \leq x_2$. Continuing this process, we will get an decreasing sequence $\{x_n\}_{n=1}^\infty$, which satisfies $x_{2n+1} \in Sx_{2n}, x_{2n+2} \in Tx_{2n+1}, n = 0, 1, 2, \dots$

By the definition $\leq$, we have

$$\varphi(x_1) \leq \varphi(x_2) \leq \dots \leq \varphi(x_n) \leq \dots$$

Since φ is bounded from above, then $\{\varphi(x_n)\}_{n=1}^\infty$ is convergent. Thus, for $\varepsilon > 0$, there exists N , if $n > m > N$, then

$$d(x_m, x_n) \leq \varphi(x_n) - \varphi(x_m) < \varepsilon,$$

which indicates $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence in X . Due to completeness of X , we know there is $z \in X$, $x_{2n+1} \rightarrow z$ and $x_{2n+2} \rightarrow z$. Since S and T are order closed, $\{x_n\}_{n=1}^\infty$ monotone and $x_{2n+1} \in Sx_{2n}, x_{2n+2} \in Tx_{2n+1}$, we deduce that $z \in Sz$ and $z \in Tz$, i.e. z is a common fixed point of S and T .

Next, we verify (B2). If φ is order upper semi continuous on X , then, for any m , we have

$$d(x_m, z) = \lim_{n \rightarrow \infty} d(x_m, x_n) \leq \limsup_{n \rightarrow \infty} (\varphi(x_n) - \varphi(x_m)) = \varphi(z) - \varphi(x_m),$$

which indicates $z \leq x_m$.

Now, we give some fixed point theorem in ordered Banach space.

Theorem 3. *Let E be a Banach space, $P \subset E$ be a regular cone, $\leq$ be a partial order induced by P , $X \subset E$ be a closed subset and $S, T : X \rightarrow 2^X$ be two order closed with respect to $\leq$ and weakly increasing operator with respect to $\prec_1$. If the following condition hold:*

- (i) *there exists some $u \in X$ such that $Sx \prec_1 \{u\}$ and $Tx \prec_1 \{u\}$ for all $x \in X$.*

Then, there exists a monotone sequence $\{x_n\}_{n=1}^\infty \subset X$, $x_{2n+1} \in Sx_{2n}, x_{2n+2} \in Tx_{2n+1}, n = 0, 1, 2, \dots$, such that $\{x_n\}_{n=1}^\infty$ converges to $z \in X$, which is a common fixed point of S and T .

Proof. Let $x_0 \in X$ be an arbitrary point. Since $Sx_0 \neq \emptyset$, we can choose $x_1 \in Sx_0$. Now, since S and T are weakly increasing with respect to $\prec_1$, we have $x_1 \in Sx_0 \prec_1 Tx_1$. Thus, there exist some $x_2 \in Tx_1$ such that $x_1 \leq x_2$. Again, since S and T are weakly increasing with respect to $\prec_1$, we have $x_2 \in Tx_1 \prec_1 Sx_2$.

Thus, there exist some $x_3 \in Sx_2$ such that $x_2 \leq x_3$. Continuing this process, we will get an increasing sequence $\{x_n\}_{n=1}^\infty$, which satisfies $x_{2n+1} \in Sx_{2n}, x_{2n+2} \in Tx_{2n+1}, n = 0, 1, 2, \dots$. By (i), and the definition of $\prec_1$, we have $x_n \leq u, n = 1, 2, \dots$. Since P is regular and $\{x_n\}_{n=1}^\infty$ is nondecreasing and bounded above in order, there is $z \in X$ such that $x_n \rightarrow z$ and so $x_{2n+1} \rightarrow z$ and $x_{2n+2} \rightarrow z$. Since S and T are order closed, $\{x_n\}_{n=1}^\infty$ is monotone and $x_{2n+1} \in Sx_{2n}, x_{2n+2} \in Tx_{2n+1}$, we get $z \in Sz$ and $z \in Tz$, i.e. z is a common fixed point of S and T .

The following theorem can be similarly verified.

Theorem 4. *Let E be a Banach space, $P \subset E$ be a regular cone, $\leq$ be a partial order induced by P , $X \subset E$ be a closed subset and $S, T : X \rightarrow 2^X$ be two order closed with respect to $\leq$ and weakly increasing operator with respect to $\prec_2$. If the following condition hold:*

(ii) *there exists some $u \in X$ such that $\{u\} \prec_2 Sx$ and $\{u\} \prec_2 Tx$ for all $x \in X$.*

Then there exists a monotone sequence $\{x_n\}_{n=1}^\infty \subset X$, $x_{2n+1} \in Sx_{2n}, x_{2n+2} \in Tx_{2n+1}, n = 0, 1, 2, \dots$, such that $\{x_n\}_{n=1}^\infty$ converges to $z \in X$, which is a common fixed point of S and T .

Remark 1. By Theorems 1, 2, 3 and 4, we give some improved version of Theorems 3.2, 3.3, 4.2 and 4.3 of [5], respectively. See, also, the related theorems in [1].

Now, we introduce the following condition.

Condition I. Let E be a Banach space and let $X \subset E$ be a nonempty subset. Two maps $S, T : X \rightarrow 2^X$ are said to satisfy Condition I on X if for any countable subset A of X and for any fixed $a \in X$ the condition

$$A \subseteq \{a\} \cup S(A) \cup T(A)$$

implies $\overline{A}$ is compact; here $T(A) = \cup_{x \in A} Tx$.

Now, we give the following theorems.

Theorem 5. *Let E be a Banach space, $P \subset E$ be a cone (not necessary regular), $\leq$ be a partial order induced by P , $X \subset E$ be a closed subset and $S, T : X \rightarrow 2^X$ be two order closed with respect to $\leq$ and weakly increasing operator with respect to $\prec_1$. Further, if S and T are satisfying Condition I on X , then there exists a monotone sequence $\{x_n\}_{n=1}^\infty \subset X$, $x_{2n+1} \in Sx_{2n}, x_{2n+2} \in Tx_{2n+1}, n = 0, 1, 2, \dots$, such that $\{x_n\}_{n=1}^\infty$ converges to $z \in X$, which is a common fixed point of S and T .*

Proof. Let $x_0 \in X$ be an arbitrary point. Since $Sx_0 \neq \emptyset$, we can choose $x_1 \in Sx_0$. Now, since S and T are weakly increasing with respect to $\prec_1$, we have $x_1 \in Sx_0 \prec_1 Tx_1$. Thus, there exist some $x_2 \in Tx_1$ such that $x_1 \leq x_2$. Again, since S and T are weakly increasing with respect to $\prec_1$ we have $x_2 \in Tx_1 \prec_1 Sx_2$. Thus, there exist some $x_3 \in Sx_2$ such that $x_2 \leq x_3$. Continuing this process, we will get an increasing sequence $\{x_n\}_{n=1}^\infty$ which satisfies $x_{2n+1} \in Sx_{2n}, x_{2n+2} \in Tx_{2n+1}, n = 0, 1, 2, \dots$.

Now let $A = \{x_0, x_1, \dots\}$. Since A is countable,

$$A = \{x_0\} \cup \{x_1, x_3, \dots\} \cup \{x_2, x_4, \dots\} \subseteq \{x_0\} \cup S(A) \cup T(A),$$

and S and T are satisfying Condition I, then $\overline{A}$ is compact. Thus, $\{x_n\}_{n=1}^\infty$ has a convergent subsequence which converges to say $z \in X$. However, $\{x_n\}_{n=1}^\infty$ is increasing, so the original sequence $\{x_n\}_{n=1}^\infty$ converges to $z \in X$. Since S and T are order closed, $\{x_n\}_{n=1}^\infty$ is monotone and $x_{2n+1} \in Sx_{2n}$, $x_{2n+2} \in Tx_{2n+1}$, we get $z \in Sz$ and $z \in Tz$, i.e., z is a common fixed point of S and T .

The following theorem can be similarly verified.

Theorem 6. *Let E be a Banach space, $P \subset E$ be a cone (not necessary regular), $\leq$ be a partial order induced by P , $X \subset E$ be a closed subset and $S, T : X \rightarrow 2^X$ be two order closed with respect to $\leq$ and weakly increasing operator with respect to $\prec_2$. Further, if S and T are satisfy the Condition I on X , then there exists a monotone sequence $\{x_n\}_{n=1}^\infty \subset X$, $x_{2n+1} \in Sx_{2n}$, $x_{2n+2} \in Tx_{2n+1}$, $n = 0, 1, 2, \dots$, such that $\{x_n\}_{n=1}^\infty$ converges to $z \in X$, which is a common fixed point of S and T .*

Remark 2. By Theorems 5 and 6, we give some improved version of Theorem 3.1 of [4]. See, also, the related theorems in [13].

Now, we give some examples.

Example 3. Let $X = [0, 1]$, $d(x, y) = |x - y|$, $\varphi(x) = -x$, $x, y \in X$, then X is a complete metric space and the order $\leq$ induced by φ :

$$x \leq y \iff d(x, y) \leq \varphi(x) - \varphi(y).$$

It is easy to know that this partial order is the usual order on X . Define two mappings $S, T : X \rightarrow 2^X$ as $Sx = \{0, 1\}$ and $Tx = [\frac{x}{2}, 1]$ for all $x \in X$. It is easy to verify S and T are satisfies the following:

- (1) S and T are weakly increasing operator with respect to $\prec_1$,
- (2) S and T are order closed on X .

Hence, we know that S and T have a common fixed point on X according to Theorem 1.

Example 4. Let $E = \mathbb{R}^2$, $X = [0, 1] \times [0, 1]$, $P = \{(x, y) \in X : x \geq 0, y \geq 0\}$, then P is a regular cone. For $(x, y) \in X$, defining two multivalued operators $S, T : X \rightarrow 2^X$ as follows:

$$\begin{aligned} S(x, y) &= \{(0, 0), (1, 1)\} \text{ and} \\ T(x, y) &= \begin{cases} [0, y] \times [0, y] & \text{if } x \leq y \\ [0, x] \times [0, x] & \text{if } y < x \end{cases} \end{aligned}$$

Since S and T are weakly increasing operator with respect to $\prec_2$, order closed with respect to $\leq$ and $\{(0, 0)\} \prec_2 S(x, y)$ and $\{(0, 0)\} \prec_2 T(x, y)$ for all $(x, y) \in X$, then S and T have a common fixed point on X according to Theorem 4.

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NON-FACTORIZABLE GROUPS

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Abstract. A group G is called a factorizable group if there are proper subgroups A and B of G such that $G = AB$. If G is non-trivial and no such a factorization exists G is called a non-factorizable group. In this paper we will show that if G is a non-factorizable group with Frattini subgroup as a minimal normal subgroup of order p^n , $1 \leq n \leq 5$, where p is a prime number, then G is a quasi-simple group.

Keywords and phrases: non-factorizable group, simple group, quasi-simple group.

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1. Introduction and preliminaries

Let G be a group and A be a subgroup of G . If there is a subgroup B of G such that $G = AB$, then B is called a supplement of A in G . If A and B are non-trivial subgroups of G such that $G = AB$, then G is called a factorizable group. A non-factorizable group is a non-trivial group G such that for every subgroup A of G there is no subgroup B of G such that $G = AB$. The question of whether a group is factorizable or not is raised in [1] page 13. But finding all the factorizable groups is a difficult task, moreover there are groups which are not factorizable, for example finite cyclic groups of prime order or the Prüfer p -group $\mathbb{Z}(p^\infty)$. If $G = AB$ is a factorizable group with A and B maximal subgroup of G , then $G = AB$ is called a maximal factorization of G . If $G = AB$ is a finite factorizable group, then A and B are contained in maximal subgroups M and N of G respectively, hence $G = MN$ is a maximal factorization of G . Therefore, if a finite group does not have maximal factorization, then G is a non-factorizable

group. In [8], the maximal factorizations of all the almost finite simple groups are found from which we can find all the non-factorizable finite simple groups. If $G = AB$ is a factorizable group, then the impact of A and B on the structure of G is studied extensively by some researcher. In [5] it is proved that if A and B are abelian, then G is metabelian. A generalization of Ito's result is given in [14] in which it is proved that if G is a finite group with A and B nilpotent and $(|A|, |B|) = 1$, then G is solvable. O. Kegel in [6] generalized this result of Wielandt without the assumption $(|A|, |B|) = 1$. Chernikov in [2] without the condition of finiteness on G proved that if A and B are nilpotent subgroups of G satisfying the minimum condition, then G is a solvable group.

Recently, factorizable groups which can be factored as product of two simple groups or an alternating group and a simple group have been studied. To mention a few result, one can see [13] and [3].

In [7] non-factorizable groups have been investigated. In particular the following result has been proved:

Result 1. Let G be a non-factorizable group, then G is one of the following three types of groups:

- (i) a cyclic p -group,
- (ii) a non-factorizable simple group,
- (iii) a perfect group with $\Phi(G) \neq 1$ and $G/\Phi(G)$ a non-factorizable simple group.

In the above result $\Phi(G)$ denotes the Frattini subgroup of G which by definition is the intersection of all the maximal subgroups of G . And a perfect group is a group G with $G = G'$. Referring to Result 1, if G is a finite non-factorizable group, then $G \cong \mathbb{Z}_{p^n}$, $n \in \mathbb{N}$, or G is isomorphic to one of the simple groups listed in Table 4.1 of [7] or that a group satisfying condition (iii) of the Result 1, which are called type (iii) groups. In this paper our aim is to investigate type (iii) groups and obtain some properties of these groups.

2. Main result

In this section, our results are concerned with the type (iii) groups G as mentioned in part (iii) of Result 1. It is worth mentioning that if G is an arbitrary group and $G/\Phi(G)$ is a non-abelian simple group, then $G = G'$, i.e. G is a perfect group. Therefore to find type (iii) group G it is enough to investigate the possibility of the equality $G/\Phi(G) = S$ where S is a non-abelian non-factorizable simple group.

In [7] page 50, it is mentioned that the groups $SL_2(q)$ for $q \neq 7, 11, 19, 23, 29, 59$ or $q \equiv 1 \pmod{4}$ from a collection of type (iii) groups. But by [8] the group $PSL_2(q) = SL_2(q)/\Phi(SL_2(q))$ for all q 's are factorizable, hence $SL_2(q)$ can not a type (iii) group. In Table I, using [8] we give the list of all the non-abelian non-factorizable simple groups:

Table I: non-abelian non-factorizable simple groups

Classical group	$U_{2m+1}(q)$ except $U_3(3), U_3(5), U_3(8), U_9(2)$
Exceptional groups of Lie type	$E_6(q), E_7(q), E_8(q), F_4(q)$ (q odd), $G_2(q)$ ($q \neq 3^n$ or $q \neq 4$), ${}^3D_4(q^3)$, ${}^2E_6(q^2)$, ${}^2B_2(2^{2n+1})$, ${}^2F_4(2^{2n+1})$, ${}^2G_2(3^{2n+1})$
Sporadic groups	$M_{22}, Mc, J_1, J_3, J_4, Ly, O'N, CO_2, CO_3,$ $Fi_{23}, Fi'_{24}, HN, Th, BM, M$

Let $G = SU_{2m+1}(q)$, $(m, q) \neq (1, 2)$. Then $G = G'$, $\Phi(G) = Z(G)$ is a cyclic group of order $(2m+1, q-1)$, and moreover $PSU_{2m+1}(q) = U_{2m+1}(q) = G/Z(G)$ is an example of a type (iii) group. In this case G is called a quasi-simple group, or a covering group for the simple group $U_{2m+1}(q)$.

If G is a type (iii) group, then every proper normal subgroup N of G must be contained in every maximal subgroup of G , hence $N \leq \Phi(G)$ and $\Phi(G)$ is the unique maximal normal subgroup of G , in particular $Z(G) \leq \Phi(G)$. The case of equality, i.e. $Z(G) = \Phi(G)$, is interesting because in this case G would be the covering group of a simple group. In [7] and [9] the case that $|\Phi(G)| = p$ or p^2 are investigated and it is proved that $Z(G) = \Phi(G)$, although some part of the proof of Theorem 1 in [9] is in error.

Lemma 2.1 *Let H be the non-abelian group of order pq , where p, q are prime numbers, $p > q$, $q \mid p-1$. Then $\text{Aut}(H)$ has order $p(p-1)$ and is isomorphic to the group $\mathbb{Z}_p^* \rtimes \mathbb{Z}_p$ with the composition law*

$$(i, j)(k, l) = (ik, il + j)$$

where $i, k \in \mathbb{Z}_p^* = \mathbb{Z}_p - \{0\}$ and $j, l \in \mathbb{Z}_p$. Hence $\text{Aut}(H)$ is a metabelian group.

Proof. Any group of order pq , p and q prime, $p > q$, $q \mid p-1$, is isomorphic to the group H with the following presentation, $H = \langle x, y \mid x^p = y^q = 1, y^{-1}xy = x^\alpha \rangle$, where $\alpha \equiv 1(\text{mod } p)$ and $\alpha^q \equiv 1(\text{mod } p)$. Since a Sylow p -subgroup $\langle x \rangle$ of H is normal in H it should be invariant under any automorphism ϕ of H . Let us set $\phi(x) = x^i$, $1 \leq i < p$. To define ϕ completely we must determine $\phi(y)$. Let $\phi(y) = x^j y^k$, where $0 \leq j < p$ and $0 \leq k < p$. We extend ϕ to the whole of G by being a homomorphism, so that $\phi(y^{-1}xy) = \phi(y)^{-1}\phi(x)\phi(y)$. Therefore $x^{i\alpha} = (x^j y^k)^{-1} x^i (x^j y^k) = y^{-k} x^i y^k = x^{i\alpha^k}$. Hence $x^{i\alpha(\alpha^{k-1}-1)} = 1$ which implies $p \mid i\alpha(\alpha^{k-1}-1)$. Since p is prime to i and α , we obtain $p \mid \alpha^{k-1}-1$, hence $\alpha^{k-1} \equiv 1(\text{mod } p)$. Using the condition $\alpha^q \equiv 1(\text{mod } p)$ we obtain $q \mid k-1$, implying $y^k = y$. Finally we obtain $\phi(y) = x^j y$ where $0 \leq j < p$. With this definition the equality $\phi(y^q) = 1$ is also satisfied, hence: $\text{Aut}(H) = \{\phi_{i,j} \mid \phi_{i,j}(x) = x^i, \phi_{i,j}(y) = x^j y, 1 \leq i < p, 0 \leq j < p\}$. Now it is easy to verify that

$$\phi_{i,j}\phi_{k,l} = \phi_{ik,il+j}$$

and the isomorphism stated in the lemma holds. ■

Theorem 2.2 *Let G be a type (iii) group, p and q distinct prime numbers. If $|\Phi(G)| = p^2$ or pq , then $\Phi(G) = Z(G)$, i.e. G is a quasi-simple group.*

Proof. In any case we have $G/C_G(\Phi(G))$ isomorphic to a subgroup of $\text{Aut}(\Phi(G))$. If $\Phi(G)$ is abelian, then $\Phi(G) \trianglelefteq C_G(\Phi(G)) \trianglelefteq G$, and because $\Phi(G)$ is a maximal normal subgroup of G we obtain $C_G(\Phi(G)) = \Phi(G)$ or G .

If $C_G(\Phi(G)) = G$ then $\Phi(G) \leq Z(G)$ and then $\Phi(G) = Z(G)$, the theorem holds.

Next, we consider the case $C_G(\Phi(G)) = \Phi(G)$. Hence $G/\Phi(G)$ is a simple non-abelian non-factorizable subgroup of $\text{Aut}(\Phi(G))$. If $\Phi(G) \cong \mathbb{Z}_{p^2}$ is a cyclic group of order p^2 , then $\text{Aut}(\Phi(G))$ is abelian and does not contain non-abelian simple group. If $\Phi(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$, then $\text{Aut}(\Phi(G)) = GL_2(p)$ and it is easy to see that any simple subgroup of $GL_2(p)$ is isomorphic to a simple subgroup of $SL_2(p)$. But by [11] page 404, the group $SL_2(p)$ does not contain any simple subgroup. If $\Phi(G) \cong \mathbb{Z}_{pq}$ is a cyclic group of order pq , then $\text{Aut}(\Phi(G))$ is abelian, hence $G/C_G(\Phi(G))$ can not be isomorphic to a subgroup of $\text{Aut}(\Phi(G))$, a contradiction. Finally we consider the case that $\Phi(G)$ is a non-abelian group of order pq , where p and q are distinct primes, $p > q$. In this case $G/C_G(\Phi(G))$ is isomorphic to a subgroup of $\text{Aut}(\Phi(G))$. Let us set $C = C_G(\Phi(G))$. By Lemma 2.1, $\text{Aut}(\Phi(G))$ is a metabelian group, and using the fact that $(G/C)'$ is abelian and $G' = G$, it follows that G/C is abelian. Hence $G' \leq C$ or $G \leq C$, i.e. $G = C_G(\Phi(G))$. It follows that $\Phi(G) \leq Z(G)$ which implies $\Phi(G) = Z(G)$ and again the theorem holds. ■

Let G be a type (iii) group with $|G|$ as small as possible, i.e. if H is any type (iii) group with order less than $|G|$, then or equal to $H = G$. We know that $\Phi(G) \neq 1$ and $\Phi(G)$ is a nilpotent group. If $\Phi(G)$ contains a non-trivial subgroup N of G properly, then $\Phi(G/N) = \Phi(G)/N$ is non-trivial and $(G/N)' = G/N$, moreover

$$(G/N)/\Phi(G/N) = (G/N)/\Phi(G)/N \cong G/\Phi(G)$$

is a non-abelian non-factorizable simple group. Therefore G/N is a type (iii) group whose order is less than $|G|$. Hence we may assume that N is characteristically simple, so that it does not contain any normal subgroup of G . Thus $N = \mathbb{Z}_p \times \dots \times \mathbb{Z}_p = \mathbb{Z}_p^n$, the direct product of n copies of cyclic groups of order p . In this case $\Phi(G)$ is a minimal normal subgroup of G . Again we have $G/C_G(G)$ isomorphic to a subgroup of $\text{Aut}(\Phi(G)) \cong GL_n(p)$. Since $\Phi(G)$ is abelian we obtain $\Phi(G) \leq C_G(\Phi(G))$ hence $G = C_G(\Phi(G))$ or $\Phi(G)$. In the first case we obtain $\Phi(G) = Z(G)$, hence $\Phi(G) = Z(G)$ and G is a quasi-simple group. In the second case we obtain $G/\Phi(G) \leq GL_n(p)$. Therefore either $\Phi(G) = Z(G)$ and G is a quasi-simple group or $\Phi(G) \neq Z(G)$ and $G/\Phi(G)$ is isomorphic to a non-factorizable simple subgroups of $GL_n(p)$. In the next lemma we discuss such subgroups of $GL_n(p)$ in case $n = 3, 4$ and 5 .

Lemma 2.3 *Let p be a prime number and $n \in \mathbb{N}, n \leq 5$. Then $GL_n(p)$ does not have a non-abelian non-factorizable simple group.*

Proof. It is obvious to see that any simple subgroup of $GL_n(p)$ should be contained in $SL_n(p)$. The case $n = 1$ is easy to see, and the case $n = 2$ follows from [11] page 404. If G is a simple non-abelian subgroup of $SL_n(p)$, then G is isomorphic to a subgroup of the simple group $PSL_n(p)$, $n \geq 2$. From the other hand since we assume that G is non-factorizable, hence G is one of groups listed in *Table I*. Now using the subgroup structure of the groups $PSL_n(p)$, $n \leq 5$, given in [12], [4] and [10] no possibilities for G exists. ■

Theorem 2.4 *Let G be a type (iii) group. If $|\Phi(G)| = p^n$, $n \leq 5$, p a prime number, and $\Phi(G)$ is an elementary abelian p -group, then G is a quasi-simple group.*

Proof. Since $\Phi(G)$ is abelian we obtain $C_G(\Phi(G)) = \Phi(G)$ or G . If $C_G(\Phi(G)) = G$, then $\Phi(G) = Z(G)$. Hence G is a quasi-simple group. Next we will consider the case $C_G(\Phi(G)) = \Phi(G)$. We have $G/\Phi(G)$ isomorphic to a non-factorizable non-abelian simple subgroup of $GL_n(p)$. But by *Lemma 1.4* such subgroups don't exist in $GL_n(p)$ for $n \leq 5$, the theorem is proved. ■

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INFLUENCE OF MICROPOLAR PARAMETERS ON THE STABILITY DOMAIN IN A RAYLEIGH-BÉNARD CONVECTION PROBLEM – A RELIABLE NUMERICAL STUDY

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Abstract. The main objective of this paper is the determination in the parameter space of the neutral hypersurface which separates the domain of stability from the instability domain in a problem of stationary convection in a micropolar fluid [1]. The influence of each of the micropolar parameters on the eigenparameter represented by the Rayleigh number is investigated using a spectral-Galerkin method based on expansion functions proved to assure an exponential convergence and small computational time expenses.

Keywords and phrases: micropolar fluids, spectral methods, Rayleigh number.

Mathematics Subject Classification: 65L10, 76M22.

1. Introduction

Micropolar fluids were first presented as a particular case of microfluids in which the coupling between the spin of each particle and the macroscopic velocity field was taken into account [6]. The constitutive equations for micropolar fluids are considered in fact non-Newtonian fluid models [6] since apart from the classical velocity field, a microrotation vector and a gyration parameter are introduced in order to perform an analysis of the kinematics of microrotation. All materials,

whether natural or synthetic, posses microstructure. Micropolar fluid models are very useful in explaining the flow of colloidal suspensions, liquid crystal, fluids with additives and animal bloods. The theory of micropolar fluids takes into account the microscopic effects arising from their local structure and microrotation of the fluids elements. Extensive reviews on this theory and its application can be found in recent books of Eringen [7] and Lukaszewicz [9].

Thermal effects in micropolar fluid flow problems have been extensively investigated due also to a large number of applications in engineering problems. It is now well known ([12]-[20]) that stationary convection is the preferred mode for onset of Rayleigh-Bénard convection in Eringen's micropolar fluids. Thus, the determination of a criterion for the onset of convection in a horizontal micropolar fluid layer heated from below follows a classical problem procedure. Ramma Rao [12] studied the onset of convection for a heat conducting micropolar fluid layer between two rigid boundaries. Sastry and Rao [13] and also Qin and Kaloni [10] report on the instability of a rotating micropolar fluid. Siddheshwar and Pranesh [19] investigated the individual effects of a nonuniform temperature gradient and a magnetic field on convection in micropolar fluids.

In most of the aforementioned papers, the linear stability of the micropolar fluid layer has been investigated in the rigid boundaries case or for free boundaries case using severe truncations in the series (in the Galerkin method used). With this in mind, the purpose of the present paper is to numerically examine the influence of the micropolar parameters on the stability of a micropolar fluid layer heated from below in the case of free, isothermal, spin-vanishing boundaries. The results provide important characteristics of the stability domain for various values of the micropolar parameters and show a good agreement with existing results in the classical case.

The outline of the paper is as follows: in Section 2 the eigenvalue problem governing the linear stability of the flow is presented along with the proposed numerical method (the spectral-Galerkin method based on a certain class of polynomial approximation). Section 3 is devoted to numerical results that can be used to better understand the specific features of the stability domain at a qualitative level. The paper ends with a conclusions section summarizing the main results of the paper.

2. The mathematical model: the analytical and numerical approaches

The onset of convection in the horizontal layer of micropolar fluid heated from below is governed by the conservation of mass, balance of linear momentum, balance of moment of momentum, respectively [1]

$$\begin{aligned}
 & -\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + (1 + K)\nabla^2 \mathbf{u} + K\nabla \times \phi + R\theta \mathbf{k} = 0, \\
 (1) \quad & \mathbf{j} \left(\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi \right) = K(\nabla \times \mathbf{u} - 2\phi) - C_0 \nabla \times \nabla \times \phi + C_1 \nabla \nabla \cdot \phi, \\
 & Pr \left(\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta \right) = \nabla^2 \theta + Rw,
 \end{aligned}$$

with $\mathbf{u} = (u, v, w)$, p , θ , ϕ the perturbations of the governing fields of velocity, pressure, temperature and spin, $C_0 = \frac{\gamma}{\mu l^2}$ the couple stress parameter accounts for the spin diffusion, $C_1 = \frac{\alpha + \beta + \gamma}{\mu l^2}$, K the parameter coupling vorticity and spin effects, $Ra = R^2$ is the Rayleigh number, $\mathbf{k}$ the unit vector in the z -direction, μ , α , β, γ coefficients of viscosity. The curl operator is then applied on $(1)_2$, and the z -component is taken. In [1] it is proven that the principle of exchange of stability holds, so the marginal stability is characterized by non-oscillatory motion, i.e., the following form of solution for the perturbations can be assumed

$$(2) \quad (w, \theta, \xi) = (W, \Theta, Z)(z)e^{i(a_x x + a_y y)},$$

where $\xi = (\nabla \times \phi)_z = \frac{\partial \phi_y}{\partial x} - \frac{\partial \phi_x}{\partial y}$ and $a^2 = a_x^2 + a_y^2$. In these conditions, the neutral stability of the thermal conduction against normal mode perturbations (2) is governed by an eigenvalue problem consisting of a system of linear ordinary differential equations with constant coefficients and a set of boundary conditions taken on the two boundaries. Following [1], this problem is reduced to the eigenvalue problem

$$(3) \quad \begin{cases} (1 + K)(D^2 - a^2)^2 W + K(D^2 - a^2)Z - a^2 R \Theta = 0, \\ K[(D^2 - a^2)W + 2Z] - C_0(D^2 - a^2)Z = 0, \\ (D^2 - a^2)\Theta + RW = 0, \end{cases}$$

$$(4) \quad W = D^2 W = Z = \Theta = 0 \text{ at } z = 0, 1,$$

where $D = \frac{d}{dz}$ represents the derivative with respect to the spatial independent variable, a is the wavenumber, W, Θ, Z are the perturbations amplitudes for the vertical component of the velocity, temperature and spin, respectively.

The main goal in the numerical investigation of the two-point boundary value problem (3)–(4) is to determine the critical eigenvalues Ra_{cr} represented by the first eigenvalues and separating the domain of stability from the instability domain.

Spectral methods arise from the fundamental problem of approximation of a function on an interval and are very much successful in obtaining the numerical solution of ordinary differential equations. They can be used successfully to solve classes of equations by expanding the solution function as a finite series of very smooth basis functions. In practice, summations in the expansions of the unknown functions are truncated to some finite number of terms in the approximation, N , for which the higher order terms become essentially negligible. During the last few decades optimization became a very important objective which has expanded in many directions. For this purpose new algorithmic and theoretical techniques have been developed.

Polynomials based spectral methods are widely used compared to trigonometric functions based ones since they have the characteristic of a very rapid

convergence and also reduce the computational evaluations especially in the orthogonal polynomials case. The eigenvalues are computed here by a Galerkin type spectral method involving combinations of Chebyshev polynomials which are defined on $[-1, 1]$. The first step of the numerical investigation consisted in shifting the interval $[a, b] = [0, 1]$ into $[-1, 1]$ using the transformation $x = \frac{2z - (a + b)}{b - a}$, i.e. $x = 2z - 1$. This method would transform the problem into the following generalized algebraic eigenvalue problem

$$(5) \quad \mathbf{A}\mathbf{u}_N = R\mathbf{B}\mathbf{u}_N,$$

where $\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{1}_{bc}^+ \\ \mathbf{1}_{bc}^- \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} \mathbf{B}_1 \\ \mathbf{0}_{bc} \\ \mathbf{0}_{bc} \end{pmatrix}$ corresponding to

$$\mathbf{A}_1 = \begin{pmatrix} (1+K)(4D^2 - a^2\mathcal{I})^2 & K(4D^2 - a^2\mathcal{I}) & \mathcal{O} \\ K(4D^2 - a^2\mathcal{I}) & -C_0(4D^2 - a^2\mathcal{I}) + 2K\mathcal{I} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & 4D^2 - a^2\mathcal{I} \end{pmatrix},$$

$$\mathbf{B}_1 = \begin{pmatrix} \mathcal{O} & \mathcal{O} & a^2\mathcal{I} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} \\ -\mathcal{I} & \mathcal{O} & \mathcal{O} \end{pmatrix}$$

completed with the boundary conditions for ± 1 . The D^2 , $\mathcal{I}$, $\mathcal{O}$ represent the second order differentiation matrix, the unitary matrix and the zero entries matrix of order $N+1$ and the vector $\mathbf{u}_N$ contains the values of the approximations coefficients of W, Z, Θ with respect to the approximation functions in the discretization process, i.e., $\mathbf{u}_N = (\widehat{W}_0, \dots, \widehat{W}_N, \widehat{Z}_0, \dots, \widehat{Z}_N, \widehat{\Theta}_0, \dots, \widehat{\Theta}_N)^T$. In the following, this $5(N+1) \times 3(N+1)$ system is modified in order to have only Dirichlet type boundary conditions.

By taking into account the order of differentiation in (3) - (4) we introduce the new variable $\Psi := (D^2 - a^2)W$, a " D^2 " strategy that we successfully applied in previous papers [4], [8]. Multiplying the first equation by $(1+K)$ and the last one by $-a^2$, the two-point boundary value problem (3) - (4) can be rewritten as the second order system described by a differential operator defined on the Sobolev space $[H^2(-1, 1)]^4$, $L : \mathcal{A} \rightarrow [H^2(-1, 1)]^4$ with

$$\mathcal{A} = \{\mathbf{U} = (\Psi, W, Z, \Theta), \mathbf{U} \in [H^2(-1, 1)]^4, \mathbf{U} = \mathbf{0} \text{ at } x = \pm 1\}$$

and

$$(6) \quad L\mathbf{U} = \mathbf{0},$$

$$L = \begin{pmatrix} -(1+K)\mathcal{I} & (1+K)(4D^2 - a^2\mathcal{I}) & \mathcal{O} & \mathcal{O} \\ (1+K)(4D^2 - a^2\mathcal{I}) & \mathcal{O} & K(4D^2 - a^2\mathcal{I}) & -a^2 R\mathcal{I} \\ \mathcal{O} & K(4D^2 - a^2\mathcal{I}) & 2K\mathcal{I} - C_0(4D^2 - a^2\mathcal{I}) & \mathcal{O} \\ \mathcal{O} & -a^2 R\mathcal{I} & \mathcal{O} & -(4D^2 - a^2\mathcal{I}) \end{pmatrix}.$$

The operator L is selfadjoint. Let us consider the inner product $(L\mathbf{U}, \mathbf{V})$ obtained by multiplying the equation from (6) by $\mathbf{V} = (\Psi^*, W^*, Z^*, \Theta^*) \in \mathcal{A}$ respectively, by adding the results and then integrating the obtained sum over $(-1, 1)$. Integrating by parts in this expression and taking into account the boundary conditions, we are lead to $(L\mathbf{U}, \mathbf{V}) = (\mathbf{U}, L^*\mathbf{V})$ with L^* defined as L , i.e., $L = L^*$.

We define the functional $J : \mathcal{A} \rightarrow \mathbb{R}$, $J(\mathbf{U}) = (L\mathbf{U}, \mathbf{U})$ with

$$\begin{aligned} J(\mathbf{U}) = & -(1+K) \int_{-1}^1 \Psi^2 dx + 2K \int_{-1}^1 Z^2 dx + C_0 \int_{-1}^1 (4(DZ)^2 + a^2 Z^2) dx \\ & + \int_{-1}^1 4(D\Theta)^2 + a^2 \Theta^2 dx - 2(1+K) \int_{-1}^1 (4D\Psi DW + a^2 W\Psi) dx \\ & - 2K \int_{-1}^1 (4DZ DW + a^2 ZW) dx - 2a^2 R \int_{-1}^1 \Theta W dx. \end{aligned}$$

Then the following variational principle

$$L\mathbf{U} = 0$$

holds iff $\delta J(U) = 0$, in other words $\mathbf{U}$ makes the functional stationary, where δ defines the Lagrange variation of J .

In this case, the finite dimensional space X_N on which we choose to construct the eigensolution (Ψ, W, Z, Θ) would be defined by functions $(\phi_k)_{k=0, \overline{N}}$ satisfying Dirichlet type boundary conditions in ± 1 such that this eigensolution belongs to $\mathcal{A}$, i.e.,

$$X_N = \text{span}\{\phi_0, \phi_1, \dots, \phi_N\}.$$

An approximation $\mathbf{U}_N \in X_N \setminus \{\mathbf{0}\}$ of the solution has the form $\mathbf{U}_N = (\Psi_N W_N Z_N \Theta_N)^T$, with the N index related to the discretization process. The approximations Ψ_N , W_N , Z_N , Θ_N of the unknown functions are then defined as truncated series of trial functions

$$(7) \quad \begin{aligned} \Psi_N &= \sum_{k=0}^N \widehat{\Psi}_k \phi_k(x), & W_N &= \sum_{k=0}^N \widehat{W}_k \phi_k(x), \\ Z_N &= \sum_{k=0}^N \widehat{Z}_k \phi_k(x), & \Theta_N &= \sum_{k=0}^N \widehat{\Theta}_k \phi_k(x). \end{aligned}$$

with the unknown Fourier coefficients $\widehat{\Psi}_k, \widehat{W}_k, \widehat{Z}_k, \widehat{\Theta}_k, (k = 0, 1, 2, \dots, N)$. The spectral approximation $\mathbf{U}_N$ is obtained by imposing the vanishing of the projection of the residual on the same finite dimensional space X_N

$$(8) \quad (L\mathbf{U}_N, \mathbf{U}_N) = 0.$$

Exponential convergence can be achieved with an appropriate expansion functions basis in the discretization process. The Shen basis functions [14] defined by

$$S_{1k}(x) = T_k(x) - T_{k+2}(x), \quad k = 0, 1, \dots, N,$$

are considered to define the space X_N in our case, which means that

$$\phi_k(x) = S_{1k}(x), \quad k = \overline{0, N}.$$

Then we have

$$X_N = \text{span}\{\phi_0(x) = 2 - 2x^2, \phi_1(x) = 4x - 4x^3, \phi_2(x) = -8x^4 + 10x^2 - 2, \dots\}.$$

The confidence in this choice is based on the accuracy offered by these functions in our previous investigations [5]. These functions form an orthogonal set of polynomials on $L^2(-1, 1)$ leading to convenient sparse matrices of the algebraic system obtained by minimizing the residual.

For the spectral-Galerkin method the algebraic equation allowing the recovery of the critical value of the Rayleigh number, i.e., the secular equation, has the form

$$(9) \quad \begin{vmatrix} -(1+K)g_{ik}^{00} & (1+K)(4g_{ik}^{20} - a^2 g_{ik}^{00}) & \mathcal{O} & \mathcal{O} \\ (1+K)(4g_{ik}^{20} - a^2 g_{ik}^{00}) & \mathcal{O} & K(4g_{ik}^{20} - a^2 g_{ik}^{00}) & -a^2 R g_{ik}^{00} \\ \mathcal{O} & K(4g_{ik}^{20} - a^2 g_{ik}^{00}) & 2K g_{ik}^{00} - C_0(4g_{ik}^{20} - a^2 g_{ik}^{00}) & \mathcal{O} \\ \mathcal{O} & -a^2 R g_{ik}^{00} & \mathcal{O} & -(4g_{ik}^{20} - a^2 g_{ik}^{00}) \end{vmatrix} = 0,$$

$i, k = 0, \dots, N$, where

$$g_{ik}^{lm} = \int_{-1}^1 D^l S_{1i}(x) D^m S_{1k}(x) \omega(x) dx,$$

the weight ω defined as usual for the Chebyshev polynomials case, i.e.,

$$\omega(x) = \frac{1}{\sqrt{1-x^2}}.$$

This equation is obtained by imposing the condition that nonzero eigenvectors exists. Clearly, the matrices simplifies since, based on the orthogonality property of the Shen basis polynomials, g_{ik}^{00} defines the identity matrix.

Before discussing the results we note below the physical significance of the parameters:

Parameter	Nature of effect	Physical reason
K $0 \leq K \leq 1$	Stabilizing	Increase in K indicates the increase in the concentration of microelements. This elements consume the greater part of the energy of the system in developing the gyrational velocities of the fluid and as a result the onset of convection is delayed.
C_0 $0 \leq C_0 \leq m$ m : finite, real	Destabilizing (as C_0 increase)	Increase in C_0 decrease the couple stress of the fluid which causes a decrease in microrotation and hence makes the system more unstable.

Table 1. Why and how of the stabilizing/destabilizing effects of the micropolar parameters ([13]).

3. Numerical analysis results

There are several physical and numerical aspects of the problem we are focused on. First, we are interesting in obtaining a good approximate value of the eigenparameter represented by the Rayleigh number for all values of the physical parameters. This was made possible by solving the secular equation, which is an algebraic equation, defining the hypersurface involving Ra, a, K, C_0 . Then, the critical value of the Rayleigh number at which the instability sets in is to be obtained in each parameters space.

As used by many, severely truncated (simple polynomials based) series offer approximate values of the Rayleigh number. We employed the spectral-Galerkin method to examine the nature of the eigenparameter Ra using a family of simple polynomials (denoted by p_1) which are constructed so as to satisfy the boundary conditions of the problem on $0, 1$, without satisfying an orthogonality relation. We write

$$\begin{aligned}\Psi &= a_1 x(1-x), & W &= a_2 x(1-x), \\ Z &= a_3 x(1-x), & \Theta &= a_4 x(1-x).\end{aligned}$$

Replacing these expressions in (6) and imposing the condition that the left-hand side of the obtained equations be orthogonal to the same expansion polynomials, we get an algebraic system of equation in a_i , $i = 1, 2, 3, 4$, of which we seek a nontrivial solution. This condition leads us to the secular equation of the form $F(Ra, a, K, C_0) = 0$. Due to the severe truncation in the expansion terms used by many, the numerical results are not the best ones.

a	K	$Ra(p1)$	$Ra(SB1)$	realative error in $Ra(p1)$
2	0.00001	686.0034307	667.0170434	2.846462
2.22	0.1	708.7990715	690.0629974	2.715125
2.22	0.01	678.4219683	660.4888697	2.715125
2.22	0	675.0467348	657.5153226	2.666312
2.22	0.00001	675.0501098	657.5186120	2.666312
2.22	0.000001	675.0470726	657.5156523	2.666312
2.233	0.00001	675.0050713	657.5418665	2.655831
2.3	0.00001	675.7233956	658.5858859	2.602167
2	0.00003	686.0102898	667.0237150	2.846461
2.233	0.00003	675.0118211	657.5484428	2.655831
2.3	0.00003	675.7301527	658.5924715	2.602167
2	0.00005	686.0171505	667.0303855	2.846461
2.233	0.00005	675.0185710	657.5550171	2.655831
2.3	0.00005	675.7369098	658.5990561	2.602167
2	0.00007	686.0240097	667.0370551	2.846461
2.233	0.00007	675.0253209	657.5615924	2.655831
2.3	0.00007	675.7436674	658.6056429	2.602167
2	0.0001	686.0343006	667.0470600	2.846461
2.233	0.0001	675.0354464	657.5714558	2.655831
2.3	0.0001	675.7538029	658.6155207	2.602167

Table 2. Numerical values of the Rayleigh number for very small values of the micropolar parameter K and for $C_0 = 0$.

Table 1 documents comparison of results obtained using the Shen basis with those of the conventional polynomials-based numerical solution. The relative error shown in Table 1 clearly points out the fact that we cannot obtain the best numerical results by severely truncated representations of ordinary polynomials.

Spectral methods are well known for their exponential convergence which imply that only a small number of terms are necessary in the approximation in order to get good numerical results. In spectral methods, the main target is to minimize the residual term (8) as much as possible. In order to validate the accuracy of the method for the proposed expansion Shen basis with respect to the spectral parameter N , we considered the classical case of Rayleigh–Bénard convection (here $K = 0$, $C_0 = 0$). The linear stability of the motionless state of an infinite layer of homogeneous fluid heated from below has been studied by Chandrasekhar [2] by means of classical normal modes and found that the critical Rayleigh number for the case of stress-free boundaries is $Ra_c = 657.511$ defined for a rounded value of a , i.e., $a = 2.22$. The investigation of the classical case gives us the following results: at $N = 3$ the relative error is around $4.66E - 4$, at $N = 4$ this error is less than $6.08E - 6$. Most important, these computations require small computational time. A larger value of the spectral parameter increases the computational time without offering a major decrease of the relative error. Clearly, the error is rapidly decreasing (Fig. 1).

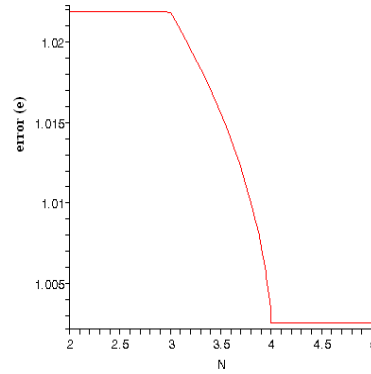


Fig. 1. The relative error (e) in the case $K = 0$, $C_0 = 0$, $a = 2.22$.

More than that, with this choice of the spectral parameter N , no convergence problems were encountered. As a result, all the numerical results presented in the paper are for $N = 4$.

Graphs depicting the stability characteristics for the case of stationary convection are presented for all the physical parameters. The sensitiveness of the critical Rayleigh number Ra_{cr} to changes in the micropolar parameters K , C_0 are presented in Figs. 2, 3, 4. The graphs in Figs. 2, 3 show a linear dependence of the Rayleigh number on K .

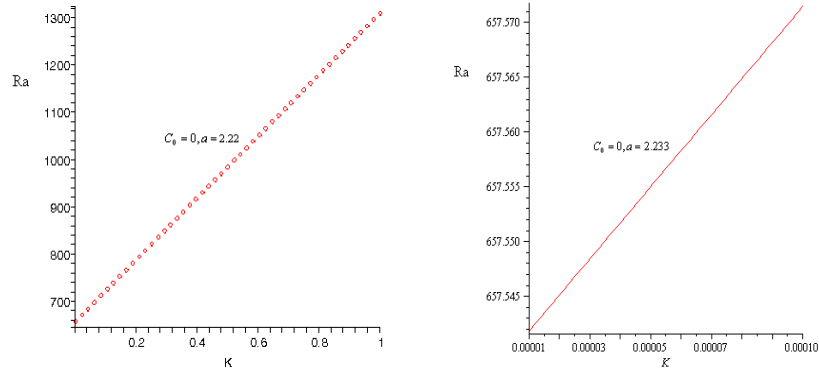


Fig. 2. Linear dependence of the critical Rayleigh number versus K for various values of C_0 and a .

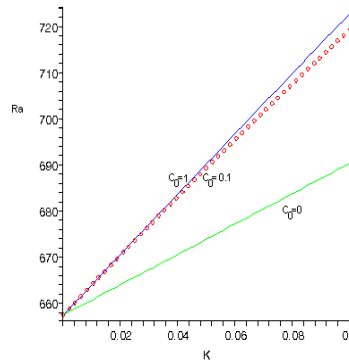


Fig. 3. The small difference in neutral curves in the (K, Ra) domain for various values of C_0 and $a = 2.233$.

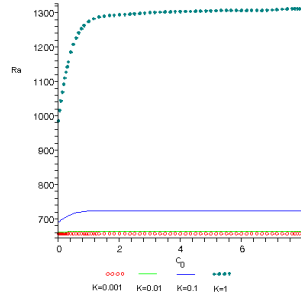


Fig. 4. Variations of the Rayleigh numbers with K in the (C, Ra) domain for a fixed $a = 2.233$.

The large influence of the coupling parameter K on Ra_c is also emphasized in Fig. 4. The neutral curves defining the dependence of the Rayleigh number on the values of the wavenumber a have a similar shape as the ones of classical Rayleigh–Bénard convection (Fig. 5). The curves are almost identical for small values of K .

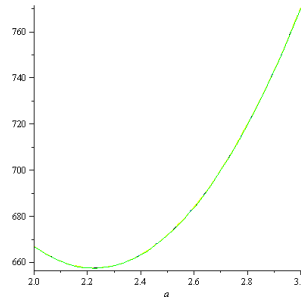


Fig. 5. Comparative neutral curves in the (a, Ra) domain for a fixed $C_0 = 0$ and $K = 0$ and $K = 0.1$.

Computations reveal that an increase in the values of C_0 , say $C_0 > 100$, do not greatly influence the values of the critical Rayleigh number. For instance, for values between $C_0 = 100$ and $C_0 = 600$ we have an increase in the value of the Rayleigh number by less than 10^{-1} . This is documented in Table 3.

C_0	$Ra(K = 0)$	$Ra(K = 1)$
100	657.5153231	1314.586918
200	657.5153236	1314.808633
300	657.5153236	1314.882604
400	657.5153231	1314.919603
500	657.5153246	1314.941806
600	657.5153231	1314.956608

Table 3. Numerical values of the critical Rayleigh number showing the small influence of the C_0 parameter, for $a = 2.22$, $K = 0$ and $K = 1$.

The critical values of the Rayleigh number for various values of the micropolar parameters are tabulated in Table 4. It can be emphasized that all values of C_0 , $K \in [0, 1]$ the critical wavenumber is $a = 2.22$.

K	C_0	Ra	a_{critic}	C_0	Ra	a_{critic}	C_0	Ra	a_{critic}
0.001	0	660.8029006	2.22	1	658.1727934	2.22	2	658.1724999	2.22
0.003	0	667.3780520	2.22	1	659.4874691	2.22	2	659.4873509	2.22
0.005	0	673.9532056	2.22	1	660.8017901	2.22	2	660.8020266	2.22
0.007	0	680.5283613	2.22	1	662.1157553	2.22	2	662.1165241	2.22
0.009	0	687.1035123	2.22	1	663.4293672	2.22	2	663.4308431	2.22
0.011	0	693.6876653	2.22	1	664.746242	2.22	2	664.7449895	2.22
0.2	0	723.2668569	2.22	1	787.2878606	2.22	2	788.1415850	2.22
0.4	0	789.0183884	2.22	1	913.7768538	2.22	2	917.0603873	2.22
0.6	0	854.7699250	2.22	1	1037.228602	2.22	2	1044.338290	2.22
0.8	0	920.5214526	2.22	1	1157.865368	2.22	2	1170.038385	2.22
1	0	986.2729905	2.22	1	1275.888240	2.22	2	1294.220609	2.22
0.001	3	658.1728242	2.22	10	658.1727934	2.22	100	658.1728386	2.22
0.003	3	659.4877356	2.22	10	659.4874691	2.22	100	659.4878656	2.22
0.005	3	660.8025294	2.22	10	660.8017901	2.22	100	660.8028903	2.22
0.007	3	662.1172045	2.22	10	662.1157553	2.22	100	662.1179090	2.22
0.009	3	663.4317616	2.22	10	663.4293672	2.22	100	663.4329258	2.22
0.011	3	664.7462007	2.22	10	664.746242	2.22	100	664.7479385	2.22
0.2	3	788.4312423	2.22	10	788.84113636	2.22	100	789.0006196	2.22
0.4	3	918.1936604	2.22	10	919.8143517	2.22	100	920.4503999	2.22
0.6	3	1046.832936	2.22	10	1050.437810	2.22	100	1051.864689	2.22
0.8	3	1174.378368	2.22	10	1180.714314	2.22	100	1183.243519	2.22
1	3	1300.858256	2.22	10	1310.646636	2.22	100	1314.586919	2.22

Table 4. Critical values of a in various parameters spaces.

For Dirichlet boundary conditions case, methods based on trigonometric Fourier series in which the unknown functions are expanded upon complete sets of functions which satisfy all boundary conditions can also be considered for numerical evaluations. The problem can be investigated in this case directly on the physical interval $[0, 1]$. Taking into account the order of differentiation of the equations and in the boundary conditions, the solutions for the unknown functions separate into completely even or odd solution functions. The lowest eigenvalue is usually obtained for the even solution, so we study the problem only for this case. The most simple solution function, $\sin(\pi z)$, satisfying the boundary conditions that can be used to represent such an approximative solution is

$$(1) \quad (\Psi, W, Z, \Theta) = (\widehat{\Psi}, \widehat{W}, \widehat{Z}, \widehat{\Theta}) \sin(\pi z),$$

where $\widehat{\Psi}, \widehat{W}, \widehat{Z}, \widehat{\Theta}$ are the unknown Fourier coefficients. The secular equation leads to:

$$Ra = R^2 = \frac{\{(\pi^2 + a^2)[C_0(1 + K)(\pi^2 + a^2) + K^2 + 2K]\}(\pi^2 + a^2)}{a(2K + C_0(\pi^2 + a^2))}$$

The above relation enables us to evaluate the value of $Ra = R^2$ at which the instability sets for various values of a and for various values of the micropolar

parameters. Numerical results for Rayleigh-Bénard convection ($K = 0$, $C_0 = 0$) cannot be obtained from the equation. Yet, numerical evaluations performed for instance for $C_0 = 0$, $K = 10^{-19}$ leads to a numerical value of $Ra_c = 657.5117337$ for $a = 2.22$ which is in good agreement with the one obtained by us, $Ra = 657.5153231$. The work on modifying the Shen-basis-reliant numerical approach is being presently pursued by the authors in a study of oscillatory convection and in a study of finite amplitude convection in different continua.

4. Conclusions

The conditions for the onset of convection in a linear stability problem of a micropolar fluid have been analyzed in this paper. Using the Boussinesq approximation, the linearized stability theory and the normal mode analysis, the eigenvalue problem for the case of two free boundaries is obtained.

We have described and applied a general spectral-Galerkin method for the numerical investigation of this eigen-boundary-value problem. The influences of all physical parameters have been systematically examined. Comparison with the corresponding results in the classical case were also performed and an excellent agreement is noted for existing results. The computations were performed with the dimensionless perturbation wavenumber ranging from 2 to 3 since the critical Rayleigh numbers are obtainable for a in this interval. The following conclusions can be drawn from the numerical results obtained by the Shen-basis-reliant approach:

- (a) The Shen-basis-reliant (SBR) approach that has exponential convergence and that which uses small computational time, compared to severely truncated Galerkin expansion methods or trigonometric Fourier series method, yields a better approximation of the eigenvalue of the problem. Table 1 lends confidence on SBR approach. Truncation beyond the 4th term in the SBR approach gives excellent results.
- (b) The relative error in the computation involving Newtonian liquids is very small for a four term spectral expansion. Our computations reveal that this is true in the micropolar fluid problem as well. The very nature of the method forces the first few terms in the SBR expansion to hold key information on the problem. This is the reason for having exponential convergence in the method.
- (c) The SBR approach very neatly captures the small changes in the eigenvalues for small variations in the parameters.
- (d) All the micropolar effects increase the stability domains of the problem.
- (e) The influence of C_0 on Ra_c only for small range of values and the linear variation of Ra_c with $K \in [0, 1]$ are also captured by SBR approach.
- (f) The micropolar effects have a marginal influence on the cell size at the onset of convection.

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ON THE SUBGROUPS OF TORSION-FREE GROUPS WHICH ARE SUBRINGS IN EVERY RING

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Abstract. In this paper we study the subgroups of a torsion-free Abelian group A which are subrings in every ring over A . In particular, we get a necessary and sufficient condition for the case of rank one and rank two torsion-free groups. Moreover, we introduce the notion of SR-group and obtain some related results.

Keywords: ring on a group, nil group, idempotent type, SR-group.

2000 Mathematics Subject Classification: 20K15.

1. Introduction

All groups considered in this paper are Abelian, with addition as the group operation. Given an Abelian group A , we call R a ring over A if the group A is isomorphic to the additive group of R . In this situation we write $R = (A, *)$, where $*$ denotes the ring multiplication. This multiplication is not assumed to be associative. Every group may be turned into a ring in a trivial way, by setting all products equal to zero; such a ring is called a zero-ring. If this is the only

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multiplication over A , then A is said to be a nil group. Stratton [9] studies the type set of a torsion-free Abelian group of rank two which supports a non-zero ring and determines all the cases for the type set of such a group. Aghdam [1] uses the cases determined by Stratton to find the rings which may be defined on rank two torsion-free groups. We use the type set and the rings over a torsion-free Abelian group A of rank two to give necessary and sufficient conditions for the subgroups of A which are subrings in every ring on A . Moreover, we give such necessary and sufficient conditions for the subgroups of completely decomposable torsion-free groups with the given typeset. Finally, we introduce the notion of an SR-group and obtain some related results.

2. Notations and preliminary results

All groups in this paper are torsion-free Abelian and in general, we will follow the notation and conventions of [8]. For a group A and $x \in A$, the p -height, height sequence and the type of x are respectively denoted by $h_p^A(x)$, $\chi_A(x)$, $t(x)$ and if $\chi = (k_1, k_2, \dots, k_n, \dots)$ and $\mu = (l_1, l_2, \dots, l_n, \dots)$ are two height sequences, then their product is defined as

$$\chi\mu = (k_1 + l_1, k_2 + l_2, \dots, k_n + l_n, \dots).$$

Moreover, we will write $\langle x \rangle_*$ for the pure subgroup of A generated by x and if $T(A) = \{t(a) \mid 0 \neq a \in A\}$, then for any $t \in T(A)$ we consider $A(t) = \{a \in A \mid t(a) \geq t\}$, which is a fully invariant pure subgroup of A .

Furthermore, following [5], suppose that $\{x, y\}$ is any independent set of a rank two torsion-free group A , then each element a of A has the unique representation $a = ux + vy$, where u, v are rational numbers. Consider

$$U_0 = \{u_0 \in \mathbb{Q} : u_0x \in A\}, \quad U = \{u \in \mathbb{Q} : ux + vy \in A \text{ for some } v \in \mathbb{Q}\}$$

$$V_0 = \{v_0 \in \mathbb{Q} : v_0y \in A\}, \quad V = \{v \in \mathbb{Q} : ux + vy \in A \text{ for some } u \in \mathbb{Q}\}.$$

Clearly, U_0 and V_0 are subgroups of U and V respectively and U, U_0, V, V_0 are called the groups of rank one *belonging* to the independent set $\{x, y\}$.

We generalize these notions for any subgroup C of A as follows:

$$U_0^C = \{u_0 \in \mathbb{Q} : u_0x \in C\}, \quad U^C = \{u \in \mathbb{Q} : ux + vy \in C \text{ for some } v \in \mathbb{Q}\}$$

$$V_0^C = \{v_0 \in \mathbb{Q} : v_0y \in C\}, \quad V^C = \{v \in \mathbb{Q} : ux + vy \in C \text{ for some } u \in \mathbb{Q}\},$$

and for any subgroups R, S of $\mathbb{Q}$ we define

$$Rx \dot{+} Sy = \{rx + sy \in A \mid r \in R, \quad s \in S\}.$$

For any subgroup C of a group A , let

$$I(C) = \langle \varphi(C) \mid \varphi \in \text{Hom}(A, E(A)) \rangle,$$

where $E(A)$ is the endomorphism group of A , i.e. $I(C)$ is the subgroup of $E(A)$ generated by all homomorphic images of C in $E(A)$. Now by an argument similar to that in Theorem 117.2 in [8], it is straightforward to see that:

Proposition 2.1 *A subgroup C of an abelian group A is a subring of every ring on A exactly if $I(C).C \leq C$.*

Finally, by Theorem 3.3 of [9], we know if A is a non-nil torsion-free group of rank two, then $T(A)$ contains a unique minimal member and at most three elements.

Using this fact we can obtain the next proposition which describes the possible rings on rank two completely decomposable non-homogeneous groups and will be needed in Theorem 5.9:

Proposition 2.2 *Let $A = A_1 \oplus A_2$ be a completely decomposable non-homogeneous group of rank two with $t(A_1) = t_1, t(A_2) = t_2$. Let x and y be non-zero elements of A_1 and A_2 respectively. If A is non-nil, then any ring on A satisfies one of the following cases:*

1. $T(A) = \{t_0, t_1, t_2\}$ with $t_0 < t_1, t_0 < t_2$.
 - (a) t_1, t_2 are incomparable and in general $x^2 = ax, y^2 = by, xy = yx = 0$ for some $a \in U_0$ and $b \in V_0$.
 - (b) If $t_1^2 = t_1, t_2^2 \neq t_2$, then $x^2 = ax, y^2 = xy = yx = 0$ for some $a \in U_0$.
 - (c) If $t_1^2 = t_1, t_2^2 = t_2$, then $x^2 = ax, y^2 = by, xy = yx = 0$ for some $a \in U_0$ and $b \in V_0$.
2. $T(A) = \{t_1, t_2\}$ with $t_1 < t_2$.
 - (d) If $t_1^2 \neq t_1, t_2^2 = t_2$, then $x^2 = ay, y^2 = by, xy = cy, yx = dy$ for some $a, b, c, d, f \in V_0$.
 - (e) If $t_1^2 \neq t_1, t_2^2 \neq t_2$, then $x^2 = ay, y^2 = 0, xy = cy, yx = dy$, for some $a, c, d \in V_0$.
 - (f) If $t_1^2 = t_1, t_2^2 = t_2$, then $x^2 = a'x + by, y^2 = cy, xy = dy, yx = fy$, for some $a' \in U_0$ and $b, c, d, f \in V_0$, in which if $b \neq 0$, then $a' \neq 0$.
 - (g) If $t_1^2 = t_1, t_2^2 \neq t_2$, then $x^2 = a'x + by, y^2 = 0, xy = dy, yx = fy$, for some $a' \in U_0$ and $a, b, c, d, f \in V_0$, in which if $b \neq 0$, then $a' \neq 0$.

Proof. 1) See [4, Proposition 2.7, Lemma 3.1 and Lemma 3.3].

2) (d) Clearly, $t(x^2) \geq t(x)^2 = t_1^2 > t_1$. Now the hypothesis that $T(A)$ contains two elements, implies that $t(x^2) = t_2$. This yields $x^2 = ay$ for some $a \in V_0$. By the same reasoning, the other parts are obtained. ■

3. Rank one and indecomposable rank two groups

In this section, we give for rank one and rank two torsion-free groups a necessary and sufficient condition for subgroups to be subrings in every ring.

Let A be a torsion-free group and C be a subgroup of A . We define the nucleus of C by:

$$N(C) = \{\alpha \in \mathbb{Q} \mid \alpha.x \in C, \forall x \in C\}.$$

Proposition 3.1 *Let A be a torsion-free group of rank one and C be a subgroup of A . Then C is a subring of every ring on A exactly if $I(C) \subseteq N(C)$.*

Proof. $\implies$) Suppose that C is a subring of every ring on A , then by Proposition 2.1 we have $I(C).C \leq C$. But C , being a group of rank 1, is isomorphic to a subgroup of $\mathbb{Q}$, which means every generator of $I(C)$ acts on C as multiplication by a rational number. Therefore $I(C).C \leq C$ implies that $r.c \in C$ for all $c \in C$ and $r \in I(C)$. Consequently, $I(C) \subseteq N(C)$.

$\impliedby$) Obvious. In fact, if $I(C) \subseteq N(C)$, then $I(C).C \subseteq N(C).C \subseteq C$. $\blacksquare$

Now, we consider rank two groups. Our arguments are based on the following cases for the typeset of a non-nil rank two group A which is a consequence of Theorem 3.3 in [9]:

- (a) $|T(A)| = 1$; in this case the type must be idempotent.
 - (b) $|T(A)| = 2$; one of the types is minimal and the other is maximal.
 - (c) $|T(A)| = 3$; one of the types is minimal and the other two types are maximal.
- In this case, at least one of the maximal types is idempotent.

Theorem 3.2 *Let A is a torsion-free group of rank two and $T(A) = \{t_0, t_1, t_2\}$ such that $t_0 < t_1$ and $t_0 < t_2$. Let $x, y \in A$, such that $t(x) = t_1$ and $t(y) = t_2$. If $t_1^2 = t_1$, $t_2^2 \neq t_2$, then $x^2 = ax$, $y^2 = xy = yx = 0$ for some rational number a .*

Proof. See [4, Lemma 3]. $\blacksquare$

Now, consider a subgroup T of U_0 defined as follows:

$$T = \{r \in \mathbb{Q} \mid x^2 = rx, y^2 = xy = yx = 0 \text{ yields a ring on } A\},$$

then we have:

Theorem 3.3 *Let A be a torsion-free group of rank two and $T(A) = \{t_0, t_1, t_2\}$ such that $t_0 < t_1, t_2$ and $t_1^2 = t_1$, $t_2^2 \neq t_2$. Let $x, y \in A$ such that $t(x) = t_1$ and $t(y) = t_2$. Then a subgroup C of A is a subring of every ring on A exactly if $TU^C U^C \subseteq U_0^C$. Moreover, if $r(C) = 1$, then C is a subring of every ring on A exactly if $C \leq V_0 y$ or $C = U_0^C(nx)$ for some integer n .*

Proof. Suppose that C is a subgroup which is a subring of any ring on A and let $c_1 = \alpha x + \beta y$, $c_2 = \gamma x + \nu y$ be two arbitrary elements of C . We know that every $\varphi \in \text{Hom}(A, \text{End}(A))$ gives rise to a ring multiplication by defining the product of $a, a' \in A$ as $a.a' = (\varphi(a))(a')$. Therefore, in view of Theorem 3.2, $(\varphi(c_1))(c_2) = c_1.c_2 = \alpha\gamma rx$ for some $r \in T$. This implies that $I(C).C = TU^C.U^C x$. Now, the first part is obtained from Proposition 2.1. But, if $r(C) = 1$, then suppose R is an arbitrary non-zero ring over A . So there exists a non-zero $a \in T$ such that $x^2 = ax$, $y^2 = xy = yx = 0$. Now, if $c = \alpha x + \beta y \in C$ be such that $\alpha \neq 0$ and $\beta \neq 0$, then $c^2 = \alpha^2 ax \neq 0$. This implies that a non-zero multiple of x is in C . Therefore, a non-zero multiple of y is in C . Hence C must be of rank two, which is a contradiction. Therefore, $\alpha = 0$ or $\beta = 0$. Consequently, $C \leq V_0 y$ or $C = U_0^C(nx)$ in which n is the smallest positive integer such that $nx \in C$. $\blacksquare$

Theorem 3.4 *Let A be a torsion-free group of rank two and $T(A) = \{t_0, t_1, t_2\}$ such that $t_0 < t_1$ and $t_0 < t_2$. Let $x, y \in A$ such that $t(x) = t_1$ and $t(y) = t_2$. If $t_1^2 = t_1$, $t_2^2 = t_2$, then $x^2 = ax$, $y^2 = by$, $xy = yx = 0$ for some rational numbers a, b which are not both zero.*

Proof. See [1, Proposition 9]. ■

In this case, consider the following subgroups of U_0 and V_0 respectively:

$$T' = \{r \in \mathbb{Q} \mid x^2 = rx, y^2 = sy, xy = yx = 0 \text{ yields a ring on } A \text{ for some } s \in \mathbb{Q}\},$$

$$S = \{s \in \mathbb{Q} \mid x^2 = rx, y^2 = sy, xy = yx = 0 \text{ yields a ring on } A \text{ for some } r \in \mathbb{Q}\},$$

then we have:

Theorem 3.5 *Let A be a torsion-free group of rank two and $T(A) = \{t_0, t_1, t_2\}$ such that $t_0 < t_1, t_2$ and $t_1^2 = t_1$, $t_2^2 = t_2$. If $x, y \in A$ such that $t(x) = t_1$, $t(y) = t_2$, then any subgroup C of A is a subring of every ring on A exactly if $T'U^CU^Cx \dot{+} SV^CV^Cy \subseteq C$. Moreover, if C is a rank one subgroup of A which is a subring of every ring on A , then $C = U_0^C(nx)$ or $C = V_0^C(my)$ for some integers m and n .*

Proof. The proof is similar to the proof of Theorem 3.3. In fact, if $c_1 = \alpha x + \beta y$ and $c_2 = ux + vy$ be two arbitrary elements of C and $\varphi \in \text{Hom}(A, \text{End}(A))$, then

$$(\varphi c_1)(c_2) = c_1.c_2 = \alpha uax + \beta vsy,$$

for some $a \in T'$ and $s \in S$. This implies $I(C).C = T'U^CU^C \dot{+} SV^CV^C$. Therefore, the first assertion follows from Proposition 2.1. Now, let C be a rank one subgroup of A which is a subring of every ring on A . Suppose that $c = \alpha x + \beta y \in C$, with $0 \neq \alpha, 0 \neq \beta$ and R be a ring on A with $x^2 = rx, y^2 = sy, xy = yx = 0$ for some $r \neq 0, s \neq 0$. Then $c^2 = \alpha^2 rx + \beta^2 sy \in C$. This yields a non-zero multiple of y and therefore a non-zero multiple of x lies in C . We conclude that C is of rank two, that is a contradiction. Hence $\alpha = 0$ or $\beta = 0$ and a similar argument to the used in Theorem 3.3 yields the result. ■

Theorem 3.6 *Let A be an indecomposable torsion-free group of rank two and $T(A) = \{t_1, t_2\}$ such that $t_1 < t_2$. If $\{x, y\}$ be an independent set such that $t(x) = t_1$, $t(y) = t_2$, then all non-trivial rings on A satisfy $x^2 = by$, $xy = yx = y^2 = 0$, for some rational number b .*

Proof. See [1, Lemma 3]. ■

In this situation, we define the following subgroup of V_0 as follows

$$W = \{r \in \mathbb{Q} \mid x^2 = ry, y^2 = xy = yx = 0 \text{ yields a ring on } A\}.$$

Theorem 3.7 *Let A be an indecomposable torsion-free group of rank two and $T(A) = \{t_1, t_2\}$ such that $t_1 < t_2$. Let $\{x, y\}$ be an independent set such that $t(x) = t_1, t(y) = t_2$. Then any subgroup C of A is a subring of every ring on A exactly if $WU^CU^C \subseteq V_0^C$. Moreover, any rank one subgroup of A which is a subring of every ring on A , is a subgroup of V_0y .*

Proof. Let C be a subgroup of A . Then, in view of Theorem 3.6, for any arbitrary $\varphi \in \text{Hom}(A, \text{End}(A))$, $c_1 = ux + vy$, $c_2 = \alpha x + \beta y$ in C we have $(\varphi c_1)(c_2) = \alpha r y$ for some $r \in W$. This means $I(A).C = WU^C U^C y$, thus the first assertion follows from Proposition 2.1. Now let C be a rank one subgroup of A which is a subring of every ring on A . Let $\alpha x + \beta y \in C$ be a non-zero element of C . We claim that $\alpha = 0$. By the way of contradiction, suppose that $\alpha \neq 0$. If R is any non-zero ring on A then there exists a non-zero $r \in W$ such that $x^2 = ry$, $xy = yx = y^2 = 0$. Hence $0 \neq \alpha^2 ry = c^2 \in C$. Therefore non-zero multiples of x and y are in C which means C is of rank two, a contradiction. ■

4. Completely decomposable groups

By [8], any element a of a torsion-free group A with maximal independent set $\{x_1, x_2, \dots\}$, has a unique representation $a = u_1 x_1 + u_2 x_2 + \dots + u_n x_n$, for some positive integer n and $u_1, \dots, u_n \in \mathbb{Q}$. Let

$$U_i = \{u_i \in \mathbb{Q} \mid \sum_{j=1}^k u_j x_j \in A, \text{ for some } u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_k \in \mathbb{Q}\} \setminus \{0\}$$

and

$$U_{0i} = \{u_i \in \mathbb{Q} \mid u_i x_i \in A\}.$$

So, if $A = \bigoplus_{i=1}^{\infty} A_i$ is a completely decomposable group and $x_i \in A_i, i = 1, 2, \dots$, then $U_i = U_{0i}$ and it is straightforward to see that $t(U_{0i}) = t(A_i)$.

Moreover, since the typeset of a completely decomposable group has at most $r(A)$ maximal element, we can reduce our consideration of such groups to three cases:

- (1) Homogeneous completely decomposable groups.
- (2) Completely decomposable groups where the types of all elements in a maximal independent set of a group are maximal and incomparable.
- (3) Completely decomposable groups where some types of elements in a maximal independent set are equal or are not maximal.

Theorem 4.1 *Let $A = \bigoplus_{i \in I} A_i$ be a homogeneous completely decomposable group. If A is non-nil, then A contains no non-trivial subgroup of rank less than $r(A)$, which is a subring of every ring on A .*

Proof. Let $t(A) = t$. If $x_i \in A_i$ and $\{x_1, x_2, \dots\}$ is a maximal independent set of A , then $t(U_i) = t = t(U_i U_j)$ for all $i, j \in \{1, 2, \dots\}$, because A is non-nil and hence $t^2 = t$. By the way of contradiction, suppose that C is a non-trivial subgroup of A with $r(C) < r(A)$ such that C is a subring of every ring on A . Let $0 \neq c = \sum_{i=1}^n \alpha_i x_i$ be an element of C . Then there exists $i \in \{1, 2, \dots, n\}$ such that $\alpha_i \neq 0$. Without loss of generality suppose that $\alpha_1 \neq 0$. Since $t(U_1^2) = t(U_1) = t(U_2) = \dots = t(U_n)$,

there exist some non-zero integer numbers $m_1, m_2, \dots, m_n, k_1, k_2, \dots, k_n$ such that: $m_1 U_1^2 = k_1 U_1, m_2 U_1^2 = k_2 U_2, \dots, m_n U_1^2 = k_n U_n$. Now we define $*_1$ as follows

$$x_i *_1 x_j = \begin{cases} m_1 x_1 & \text{if } i = j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

If $u_1 = \beta_1 x_1 + \dots + \beta_n x_n$ and $u_2 = \gamma_1 x_1 + \dots + \gamma_n x_n$ are two arbitrary elements of A , then $u_1 *_1 u_2 = m_1 \beta_1 \gamma_1 x_1$. But $m_1 \beta_1 \gamma_1 \in m_1 U_1^2 = k_1 U_1 \subseteq U_1$, hence $*_1$ is actually a ring on A . Now we have $c *_1 c = m_1 \alpha_1^2 x_1 \in C$. Similarly we may define multiplications $*_2, *_3, \dots$ on A such that for all $l = 2, 3, \dots$ we have

$$0 \neq c *_l c = m_l \alpha_l x_l \in C.$$

This implies that $r(C) = r(A)$, a contradiction. $\blacksquare$

Now, if A is a homogeneous separable group of infinite rank and $\{x_1, x_2, \dots\}$ be a maximal independent set of A , then, by Proposition 87.2 of [8], for all $i = 1, 2, \dots, \langle x_i \rangle_*$ is a direct summand of A and hence $U_{0i} = U_i$. So, by a similar proof to the previous theorem we will have:

Proposition 4.2 *If A be a non-nil homogeneous separable group (of rank $\aleph_1$), then A contains no non-trivial subgroup of rank less than $r(A)$, which is a subring of every ring on A .*

Theorem 4.3 *Let $A = \bigoplus_{i \in I} A_i$ be a completely decomposable group and $S = \{x_i \mid x_i \in A_i, i \in I\}$ a maximal independent set of A such that $t(x_i) = t_i s$ are maximal and incomparable in $T(A)$ for all $i \in I$. Then*

- (i) *Any rank one subgroup C which is a non-zero subring of every ring on A , is of the form $C = U_i^C(m x_i)$ with $t_i^2 = t_i, m \in \mathbb{Z} \setminus \{0\}$.*
- (ii) *Any subgroup C of rank k which is a subring of every ring on A , is generated by $l (\leq k)$ rational multiples of some elements in S with idempotent types and $k - l$ combinations with rational coefficients of some elements in S with non-idempotent types. Moreover, if $l \neq 0$ then C is a nonzero subring.*

Proof. (i) Let C be any rank one subgroup of A which is a subring of every ring on A and let $c = \sum_{i=1}^n \alpha_i x_i$ be a non-zero element of C . We consider two cases. First suppose that $\alpha_i \neq 0$, for some $i \in \{1, 2, \dots, n\}$ with $t^2(x_i) = t(x_i)$. For example let $\alpha_1 \neq 0$ and $t^2(x_1) = t(x_1)$. This implies $t(U_1^2) = t(U_1)$. Hence, similar to the proof of Theorem 4.1, there exists a non-zero integer m such that

$$x_i * x_j = \begin{cases} m x_1 & \text{if } i = j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

is a ring on A . Clearly, $0 \neq c * c = \alpha_1 m x_1 \in C$. Now if $\alpha_j \neq 0$ for some $j \neq 1$, then $r(C) \geq 2$, which is a contradiction. Consequently, $C = U_1^C(m x_1)$ for some $m \in \mathbb{Z} \setminus \{0\}$.

If $c = \sum_{i=1}^n \beta_i x_i$, $t^2(x_i) \neq t(x_i)$, for all $i = 1, 2, \dots, n$, then we know $t(x_i)$ is maximal and non-idempotent, hence for any ring on A we must have: $x_i x_i = x_i x_j = 0$ which yields C is zero subring.

(ii) Let C be a rank $k > 1$ subgroup of A which is a subring of every ring on A and let $\{c_1 = \alpha_{11}x_1 + \dots + \alpha_{1n}x_n, c_2 = \alpha_{21}x_1 + \dots + \alpha_{2n}x_n, \dots, c_k = \alpha_{k1}x_1 + \dots + \alpha_{kn}x_n\}$ be a maximal independent set of C . If there exist $i \in \{1, 2, \dots, k\}$ and $j \in \{1, 2, \dots, n\}$ such that $\alpha_{ij} \neq 0$ and $t^2(x_j) = t(x_j)$, then as in case (i) there exist a non-zero integer m and a ring on A with $0 \neq c_i^2 = \alpha_{ij}^2 m x_j \in C$. Let $\alpha_{ij}^2 m = \beta_j$, hence there exist $c'_2, \dots, c'_k \in C$ such that $\{\beta_j x_j, c'_2, \dots, c'_k\}$ is an independent set in C and for all $i = 1, 2, \dots, k$,

$$c'_i = \alpha'_{i1}x_1 + \dots + \alpha'_{ij-1}x_{j-1} + \alpha'_{ij+1}x_{j+1} + \dots + \alpha'_{in}x_n.$$

Repeating this procedure we get a maximal independent set in C

$$\{\beta_{j_1}x_{j_1}, \dots, \beta_{j_l}x_{j_l}, c''_1, \dots, c''_{k-l}\},$$

such that $t^2(x_{j_1}) = t(x_{j_1}), \dots, t^2(x_{j_l}) = t(x_{j_l})$ and $c''_1, \dots, c''_{k-l}$ are rational combinations of some elements in S with non-idempotent types.

In fact if C is a rank k subgroup of A which is a non-zero subring of every ring on A , then there exists an element $c = \sum_{i=1}^n \alpha_i x_i$ with $\alpha_i \neq 0$ and $t^2(x_i) = t(x_i)$. For otherwise, the maximality of types implies that $x_i x_j = x_j x_i = 0$ for all $i, j = 1, 2, \dots, n$. This implies that C is a zero subring, a contradiction.

Now, our final assertion is obvious. ■

Theorem 4.4 *Let $A = \oplus_{i \in I} A_i$ be a completely decomposable group and $S = \{x_i \mid x_i \in A, i \in I\}$ a maximal independent set of A such that some of $t(x_i)$ s are equal or are not maximal, Then any rank one subgroup C which is a non-zero subring of every ring on A , is of the form $C = U_i^C(lx_i)$ for some non-zero integer l with $t(x_i)$ idempotent.*

Proof. Let C be any rank one subgroup of A which is a subring of every ring over A and $c = \sum_{i=1}^n \alpha_i x_i \in C$. If $\alpha_i \neq 0$ for some i in which $t^2(x_i) = t(x_i)$, then by the proof of Theorem 4.1, there exists a non-zero integer m such that

$$x_r * x_s = \begin{cases} mx_i & \text{if } r = s = i, \\ 0 & \text{otherwise,} \end{cases}$$

yields a ring on A such that $c^2 = m\alpha_i^2 x_i \in C$. Moreover, by $r(C) = 1$ we obtain $\alpha_j = 0$ for all $j \neq i$. Consequently, $C = U_i^C(lx_i)$, for some non-zero integer l .

Now, suppose that C is a subring and any arbitrary element of C is of the form αc for some $\alpha \in \mathbb{Q}$ and $c = \sum_{j=1}^n \alpha_j x_j$ such that $t(x_j)$ s are not idempotent.

Suppose there exists exactly one index $i \in J$ such that $\alpha_i \neq 0$. If there is a ring on A with $x_i^2 \neq 0$ then $t(x_i) < t^2(x_i) \leq t(x_i^2)$ and so $x_i^2 = \sum_k \beta_k x_k$, for some $\beta_k \in \mathbb{Q}$ and $x_k \in S$ with $t(x_k) > t(x_i)$. Hence $c^2 = \alpha_i^2 x_i^2 = \sum_k \beta_k \alpha_i^2 x_k \notin C$ which means C is not a subring of every ring over A .

This yields that if $C = \langle \alpha_i x_i \rangle$, with $t(x_i)$ non-idempotent, then C is a subring of any ring over A exactly if C be the zero subring. Moreover, if $C = \langle \alpha x_i + \beta x_j \rangle$ and α, β are both non-zero, then for any ring on A we must have

$$c^2 = \gamma c = \gamma(\alpha x_i + \beta x_j). \quad (*)$$

But

$$c^2 = \alpha^2 x_i^2 + \alpha\beta x_i x_j + \beta\alpha x_j x_i + \beta^2 x_j^2. \quad (**)$$

Now if $c^2 \neq 0$, for a ring on A , then we must have $(*) = (**)$. Hence if $x_i^2 \neq 0$, it must be equal to a rational multiple of x_j (because $t^2(x_i) \neq t(x_i)$) which yields

$$(1) \quad t(x_i) < t(x_j).$$

By the same reason, if $x_j^2 \neq 0$, then it must be a rational multiple of x_i , which yields

$$(2) \quad t(x_j) < t(x_i).$$

Moreover, if $x_i x_j \neq 0$ or $x_j x_i \neq 0$, then similarly they must be non-zero multiples of x_i or x_j . But in this case if, for example, $x_i x_j$ be a non-zero rational multiples of x_i , then $t(x_i) = t(x_i x_j) \geq t(x_i) t(x_j)$. Now if $t(x_i) = t(x_j)$, then $t(x_i) t(x_j) > t(x_i)$; because $t(x_i)$ is not idempotent, and if $t(x_i) \neq t(x_j)$ then it is clear that $t(x_i) t(x_j) > t(x_i)$. This yields $t(x_i) = t(x_i x_j) \geq t(x_i) t(x_j) > t(x_i)$ which is a contradiction. Thus, $x_i x_j = 0$. Similarly, $x_j x_i = 0$; and so $c^2 = \alpha x_i^2 + \beta x_j^2$, which never could be of the form $(*)$.

Thus, $C = \langle \alpha x_i + \beta x_j \rangle$ with $\alpha, \beta \neq 0$ and $t(x_i), t(x_j)$ non-idempotent, is a subring of A iff $x_i^2 = x_j^2 = 0$ for every ring on A and in this case C is a zero subring. [For example if $t(x_i)$ and $t(x_j)$ are maximal and non-idempotent, then for every ring on A we have $x_i^2 = x_j^2 = 0$ and C will be a zero subring of any ring over A .]

So, by induction we could see that A doesn't contain any subgroup of the form $C = \left\langle \sum_{j=1}^n \alpha_j x_j \right\rangle$ with the $t(x_j)$ s non-idempotent and C is a non-zero subring of any ring on A . ■

5. SR-Groups

We begin with the definition:

A group A is called an SR-group if every subgroup of A is a subring in every ring over A .

The class of SR-groups is extensive since every nil group is an SR-group and a wide range of countable groups are nil; in fact it is even possible to exhibit a proper class (i.e., not a set) of nil groups. This is achieved by utilizing the following:

Proposition 5.1

- (i) *If R is any countable reduced torsion-free ring with identity which has no zero divisors, then there is a countable reduced torsion-free group A which is a nil group and $\text{End}(A) \cong R$.*
- (ii) *If R is a cotorsion-free ring (i.e. R^+ is torsion-free and reduced and contains no non-trivial pure-injective subgroup) with identity which has no zero divisors, then there exist arbitrarily large torsion-free groups A which are nil and $\text{End}(A) \cong R$.*

Proof. For part (i), we utilize Corner's famous realization theorem [[2], Theorem A] to produce a group A with $R \leq_* A \leq_* \hat{R}$, where $A = \langle R, e_r R \rangle_*$ for suitable elements $e_r = \alpha_r 1 + \beta_r r$, where α_r, β_r are algebraically independent over a certain subring of $\hat{\mathbb{Z}}$ containing $\mathbb{Z}$. To see that A is nil we compute $\text{Hom}(A, \text{End}(A))$. Since $\text{End}(A) \cong R$ and $R \leq A$, we can construe any homomorphism $\varphi \in \text{Hom}(A, \text{End}(A))$ as an endomorphism of A and hence it is equivalent to scalar multiplication on A by an element $s \in R$. Consequently $e_r s \in R$ for each $e_r \in A$ and so $(\alpha_r 1 + \beta_r r)s \in R$, giving $\beta_r r s \in R$. The algebraic independence of the β_r force $rs = 0$ for all r , and this in turn, by the assumption on R , forces $s = 0$, so that $\varphi = 0$. Since $\varphi \in \text{Hom}(A, \text{End}(A))$ was arbitrary, we conclude that $\text{Hom}(A, \text{End}(A)) = 0$ and A is nil, as required.

The proof of (ii) is similar except that one uses a more powerful “Black Box”-type realization theorem (see e.g., [3]) to obtain the group A which is now sandwiched between a large free R -module B and its natural completion $\hat{B}$. Specifically, if $\lambda = \lambda^{\aleph_0}$ is a given cardinal, we chose B to be the direct sum of λ copies of R so that $|A| \geq \lambda$. The assumption that R has no zero divisors means that B and $\hat{B}$ are both torsion-free R -modules and hence so also is A . Thus every endomorphism of A is monic. That A is nil now follows from Proposition 121.2 of [8] or alternatively the argument in part(i) may be replaced by a support argument on the “branch” elements used in the construction of A . Finally, since there exist arbitrary large cardinals λ with $\lambda^{\aleph_0} = \lambda$, we can construct the desired group A to have cardinality exceeding an arbitrarily large cardinal; clearly this results in a proper class of such groups. ■

Moreover, we have criteria to realize some groups which could not be SR-group.

Proposition 5.2 *Let M be a module over a commutative ring R . Suppose that $R^+ = A$, and $M^+ = B$. If there exist $m \in M$ and $r \in R$ such that $r^2 m \neq 0$, then $A \oplus B$ is not an SR-group.*

Proof. For $r_1, r_2 \in R$, and $m_1, m_2 \in M$ define $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1 + r_1 r_2 m)$. This multiplication induces a ring structure S on $A \oplus B$. However the subgroup $T = \{(r, 0) \mid r \in R\}$ is not a subring of S , because $(r, 0)(r, 0) = (r^2, r^2 m) \notin T$. ■

Corollary 5.3 *For every torsion-free group $B \neq 0$, the group $\mathbb{Z} \oplus B$ is not an SR-group.*

Proposition 5.4 *A direct summand of an SR-group is an SR-group.*

Proof. Let $A = B \oplus C$ is a SR-group and suppose that $S = (B, *)$ is a ring on B such that there exists a subgroup H of B which is not a subring of S . Now define $R = (A, \cdot)$ by a multiplication $(b_1, c_1) \cdot (b_2, c_2) = (b_1 * b_2, 0)$. Then $\{(b, 0) \mid b \in H\}$ is a subgroup of A which is not a subring of R . This contradiction yields the result. ■

Corollary 5.5 *If $\mathbb{Q}$ is a subgroup of an abelian group A , then A is not an SR-group.*

Proof. Follows from Proposition 5.4 and the fact that $\mathbb{Q}$ is not a SR-group. ■

Corollary 5.6 *A torsion-free SR-group is reduced.*

Proposition 5.7 *A non-nil homogeneous completely decomposable group of rank greater than one is not an SR-group.*

Proof. By Theorem 4.1, every subgroup of rank one is not a subring of every ring on this group. ■

Lemma 5.8 *Let $A = \bigoplus_{i \in I} A_i$ be a completely decomposable group with $r(A_i) = 1$. Then A is nil if and only if $t(A_i)t(A_j) \not\leq t(A_k)$ for all $i, j, k \in I$.*

Proof. See [6, Corollary 2.1.3]. ■

Now, we want to show that a non-nil completely decomposable group with rank greater than one is not SR-group.

So, let $A = \bigoplus_{i \in I} A_i$ be a completely decomposable group with $r(A_i) = 1$ and $|I| > 1$. By Lemma 5.8 we know that, if A is non-nil, then there exist $i, j, k \in I$ such that $t(A_i)t(A_j) \leq t(A_k)$ and so there exists $0 \neq \varphi \in \text{Hom}(A_i \otimes A_j, A_k)$. Therefore, we will have one of the following cases:

- (I) $i \neq j \neq k$;
- (II) $(i = j) \neq k$;
- (III) $i = j = k$;
- (IV) $i \neq j, j = k$.

Now, in each case we will find a direct summand of A which is not a SR-group and then by Proposition 5.4, A is not a SR-group.

- (I) For convenience let $i = 1, j = 2, k = 3$. Then, by Proposition 5.4, it suffices to show that $A' = A_1 \oplus A_2 \oplus A_3$ is not a SR-group. As we saw there exist $0 \neq \varphi \in \text{Hom}(A_1 \otimes A_2, A_3)$. Let $a_i \in A_i$, $i = 1, 2$, be such that $\varphi(a_1 \otimes a_2) \neq 0$. Now suppose R is a ring on A' which is defined by the multiplication $(b_1, b_2, b_3)(c_1, c_2, c_3) = (0, 0, \varphi(b_1 \otimes c_2 + c_1 \otimes b_2))$ for all $b_i, c_i \in A_i$, $i = 1, 2, 3$.

Let B be the subgroup of A' generated by $\{(a_1, 0, 0), (0, a_2, 0) \mid a_1 \in A_1, a_2 \in A_2\}$. Then $(a_1, 0, 0)(0, a_2, 0) = (0, 0, \varphi(a_1 \otimes a_2)) \notin B$. Therefore B is a subgroup of A' which is not a subring of R . So A' , and therefore A , is not an SR-group.

- (II) ($i = j$) $\neq k$; in this case we have $t^2(A_i) \leq t(A_k)$ and hence there exists $0 \neq \varphi \in \text{Hom}(A_i \otimes A_i, A_k)$. Let $a_1, a_2 \in A_i$ be such that $\varphi(a_1 \otimes a_2) \neq 0$. Let R be a ring on $A' = A_i \oplus A_k$ which is defined by $(a_i, a_k)(a'_i, a'_k) = (0, \varphi(a_i \otimes a'_i))$. Then $R \neq 0$ and $(a_i, 0)(a'_i, 0) = (0, \varphi(a_i \otimes a'_i)) \notin A_i$. Hence A_i is a subgroup of A' which is not subring of A' and this completes this part.

- (III) Let $i = j = k = 1$. Then $t(A_1)t(A_1) \leq t(A_1)$, because $t(A_i)t(A_j) \leq t(A_k)$, hence $t^2(A_1) = t(A_1)$.

If $t(A_1) = t(\mathbb{Z})$, then we have two cases:

- (a) there exists some $l \in I$ such that $l \neq 1$ and $t(A_l) = t(\mathbb{Z})$. Then $B = A_1 \oplus A_l$ is a direct summand of A and by Proposition 5.7, B is not SR-group and so A is not SR-group too.
- (b) if there exists some $l \neq 1$ such that $t(A_1) < t(A_l)$, then $t^2(A_1) = t(A_1)t(A_1) < t(A_1)t(A_l) = t(A_l)$, because $t(A_1) = (0, 0, \dots)$. So, $t^2(A_1) \leq t(A_l)$ and by (II), $B = A_1 \oplus A_l$ is not SR-group and A will not be SR-group, too.

If at least one component of $t(A_1)$ is ∞ , for example suppose the p -component of $t(A_1)$ is ∞ . In this case, by Theorem 121.1, if R be a ring on A_1 , then there exists an integer m such that $(m, p) = 1$, $R^2 = mR$ and $R \cong m\mathbb{Z}(p^{-1})$. Now consider the subgroup $C = \langle m/p \rangle$ of $(m\mathbb{Z}(p^{-1}))^+$. Since C is not a subring of $m\mathbb{Z}(p^{-1})$, the additive group of $m\mathbb{Z}(p^{-1})$ which is isomorphic to A_1 is not an SR-group and this implies that A is not an SR-group.

- (IV) Let $i \neq (j = k)$ and so $t(A_i)t(A_k) \leq t(A_k)$ which means $t(A_i) \leq t(A_k)$ and therefore if $A' = A_i \oplus A_k$, then $|T(A')| = 2$.

If $t(A_i) = t(A_k)$ then A' is homogeneous and is not an SR-group. So consider $t(A_i) < t(A_k)$, which implies $t(A_k) > t(\mathbb{Z})$.

Now if at least one of the $t(A_i)$ or $t(A_k)$ is idempotent:

- (a) $t(A_k)$ is idempotent. By $t(A_k) > t(\mathbb{Z})$, at least one of its component is ∞ and (III) yields the result.

- (b) $t(A_i)$ is idempotent;
 If $t(A_i) = t(\mathbb{Z})$, then $t^2(A_i) = t(A_i)t(A_i) \leq t(A_i)t(A_k) = t(A_k)$ and (II) yields the result.
 If $t(A_i) \neq t(\mathbb{Z})$, then $t(A_i)$ has infinity as a component and (III) yields the result.
- (c) If $t(A_i)$ and $t(A_k)$ are not idempotent then by Proposition 2.2, $B = \langle x + y \rangle; x \in A_i, y \in A_k$ is a subgroup of A' which is not a subring. This means A' , and therefore A is not an SR-group.

As a result of (I) – (IV) we are in a position to prove:

Theorem 5.9 *Let $A = \bigoplus_{i \in I} A_i$ be a completely decomposable torsion-free group with $r(A_i) = 1$. Then A is an SR-group exactly if $A \cong \mathbb{Z}$ or A is nil.*

Proof. It is clear that if $A \cong \mathbb{Z}$ or A is nil, then A is an SR-group.

Now, suppose that A is an SR-group. If A is a rank 1 group then A is nil or $A \cong \mathbb{Z}$; in fact, if A is a rank one non-nil group, and $t(A) = (k_1, k_2, \dots, k_n, \dots)$ then $k_i = \infty$ or $k_i = 0$ for all i . If there exists an i such that $k_i = \infty$, then A contains a subgroup which is isomorphic with a subgroup of $\mathbb{Q}^{(p_i)}$ which is clearly not a subring of A . Consequently, $t(A) = (0, 0, \dots, 0, 0, \dots)$ hence $A \cong \mathbb{Z}$.

Moreover, if $r(A) > 1$, then by (I) – (IV) it couldn't be a non-nil SR-group. ■

It would be interesting to know if there exists a non-nil SR-group other than the group of integers $\mathbb{Z}$.

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Abstract. In this paper, we applied the hyper structures theory to EQ -algebras and we introduced the notion of hyper EQ -algebra which is a generalization of EQ -algebra. In the following, we define the notions of good and separated hyper EQ -algebras and state and prove some properties of (good, separated) hyper EQ -algebras. Moreover, by define the concept of (pre)filter, we construct the quotient hyper EQ -algebra. Finally, we investigate the relation between hyper EQ -algebras and hyper BCK -algebras and (weak) hyper residuated lattices.

Keywords: hyper EQ -algebra, EQ -algebra, good and separated hyper EQ -algebras, (pre-)filter, hyper BCK -algebra, (weak) hyper residuated lattice.

1. Introduction

Recently, a special algebra called EQ -algebra has been introduced by Vilém Novák in [6]. These algebras are intended to become algebras of truth values for a higher-order fuzzy logic (a fuzzy type theory, FTT). An EQ -algebra has three basic binary operations (meet, multiplication and a fuzzy equality) and a top

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element. The implication is defined from the fuzzy equality " $\sim$ " by the formula $a \rightarrow b = (a \wedge b) \sim a$. Its implication and multiplication are no more closely tied by the adjunction and so, this algebra generalizes commutative residuated lattice. From the point of view of potential application, it seems very interesting that unlike (Hájek 2003), we can have non-commutativity without necessity to introduce, two kinds of implication. The concept of hyper structure (called also multialgebra) was introduced by Marty in [5], at first. That was 8th Congress of Scandinavian mathematician 1934. Till now, the hyper structures are studied from the theoretical point of view for their applications to many subject of pure and applied mathematics. Now, in this paper, we follow the study of EQ -algebras and get some results as mentioned in the abstract.

2. Preliminaries

Definition 2.1. [8] An EQ -algebra is an algebra $\varepsilon = (E, \wedge, \otimes, \sim, 1)$ of type $(2,2,2,0)$ such that, for all $x, y, z, t \in E$:

(E1) $\langle E, \wedge, 1 \rangle$ is a commutative idempotent monoid (i.e. $\wedge$ -semilattice with top element 1);

(E2) $\langle E, \otimes, 1 \rangle$ is a commutative monoid and $\otimes$ is isotone w.r.t. " $\leq$ ";

(where $x \leq y$ is defined as $x \wedge y = x$);

(E3) $x \sim x = 1$; (reflexivity axiom)

(E4) $((x \wedge y) \sim z) \otimes (t \sim x) \leq z \sim (t \wedge y)$; (substitution axiom)

(E5) $(x \sim y) \otimes (z \sim t) \leq (x \sim z) \sim (y \sim t)$; (congruence axiom)

(E6) $(x \wedge y \wedge z) \sim x \leq (x \wedge y) \sim x$; (monotonicity axiom)

(E7) $(x \wedge y) \sim x \leq (x \wedge y \wedge z) \sim (x \wedge z)$; (monotonicity axiom)

(E8) $x \otimes y \leq x \sim y$. (boundedness axiom)

Remark 2.2. In Definition 2.1, we can omit (E3). Since by (E8), we have $1 \otimes 1 \leq 1 \sim 1$. Now, since 1 is a top element of E , then $1 \sim 1 = 1$. Hence by (E7), for any $x \in E$,

$$1 = 1 \sim 1 = (1 \wedge 1) \sim 1 \leq (1 \wedge 1 \wedge x) \sim (1 \wedge x) = x \sim x$$

and so $x \sim x = 1$.

Proposition 2.3. [8] Let ε be an EQ -algebra, $x \rightarrow y := (x \wedge y) \sim x$ and $\bar{x} = x \sim 1$. Then the following properties hold, for all $x, y, z \in E$:

(i) $x \otimes y \leq x, y, \quad x \otimes y \leq x \wedge y$;

(ii) $z \otimes (x \wedge y) \leq (z \otimes x) \wedge (z \otimes y)$;

(iii) $x \sim y \leq x \rightarrow y$;

(iv) $x \rightarrow x = 1$;

(v) $(x \sim y) \otimes (y \sim z) \leq x \sim z$;

(vi) $(x \rightarrow y) \otimes (y \rightarrow z) \leq x \rightarrow z$;

(vii) $x \leq \bar{x}, \quad \bar{\bar{x}} = x$;

- (viii) $x \otimes (x \sim y) \leq \bar{y}$;
- (ix) $(z \rightarrow (x \wedge y)) \otimes (x \sim t) \leq z \rightarrow (t \wedge y)$;
- (x) $(x \rightarrow y) \otimes (y \rightarrow x) \leq x \sim y$;
- (xi) if $x \leq y \rightarrow z$, then $x \otimes y \leq \bar{z}$;
- (xii) if $x \leq y \leq z$, then $z \sim x \leq z \sim y$ and $x \sim z \leq x \sim y$.

The hyperstructure theory was introduced by Marty [5], at the 8th Congress of Scandinavian Mathematicians. In his definition, a function $\circ : A \times A \rightarrow P^*(A)$, of the set $A \times A$ into the set of all non-empty subsets of A , is called a binary hyperoperation, and the pair $(A, \circ)$ is called a hypergroupoid. If $\circ$ is associative, then A is called a semihypergroup, and it is said to be commutative if $\circ$ is commutative. Also, an element $1 \in A$ is called an identity element if $x \in 1 \circ x$, for all $x \in A$.

Note that if $A, B \subseteq H$, then

- (i) $x \circ B = \bigcup_{b \in B} (x \circ b)$, $B \circ x = \bigcup_{b \in B} (b \circ x)$,
- (ii) $A \circ B = \bigcup_{a \in A} \left(\bigcup_{b \in B} a \circ b \right)$.

3. Hyper EQ -algebra

In this section, by considering the notion of EQ -algebra, we define the notion of hyper EQ -algebra which is a generalization of EQ -algebra. Moreover, we give some examples and some important properties of this structure.

Definition 3.1. A *hyper EQ -algebra* $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a non-empty set H with a binary operations $\wedge$ and two binary hyper operations $\otimes, \sim$ and top element "1" satisfying the following conditions, for all $x, y, z, t \in H$:

- (HEQ1) $\langle H, \wedge, 1 \rangle$ is a commutative idempotent monoid with top element "1",
- (HEQ2) $\langle H, \otimes, 1 \rangle$ is a commutative semihypergroup with "1" as an identity and $\otimes$ is isotone w.r.t. $\leq$, i.e. if $x \leq y$, then $x \otimes z \ll y \otimes z$ (where $x \leq y$ if and only if $x \wedge y = x$),
- (HEQ3) $((x \wedge y) \sim z) \otimes (t \sim x) \ll z \sim (t \wedge y)$,
- (HEQ4) $(x \sim y) \otimes (z \sim t) \ll (x \sim z) \sim (y \sim t)$,
- (HEQ5) $(x \wedge y \wedge z) \sim x \ll (x \wedge y) \sim x$,
- (HEQ6) $(x \wedge y) \sim x \ll (x \wedge y \wedge z) \sim (x \wedge z)$,
- (HEQ7) $x \otimes y \ll x \sim y$,

where $A \ll B$, means that, for all $a \in A$ there exists $b \in B$ such that $a \leq b$.

Example 3.2.

- (i) In any hyper EQ -algebra $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$, if $x \sim y$ and $x \otimes y$ are singleton, for any $x, y \in H$, then $\mathcal{H}$ is an EQ -algebra.

- (ii) Let $H = \{0, 1\}$ such that, $0 < 1$. Define $\wedge, \otimes$, and $\sim$ on H as follows:

$$x \wedge y = \min\{x, y\}, \quad x \otimes y = \min\{x, y\}, \quad \begin{array}{c|cc} \sim & 0 & 1 \\ \hline 0 & \{1\} & \{0, 1\} \\ 1 & \{0, 1\} & \{1\} \end{array}$$

Then $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a hyper EQ -algebra.

- (iii) Let $H = \{0, a, 1\}$, which $0 < a < 1$, and $\wedge, \otimes$ and $\sim$ are defined on H as follows,

$$\begin{array}{c|ccc} \wedge & 0 & a & 1 \\ \hline 0 & 0 & 0 & 0 \\ a & 0 & a & a \\ 1 & 0 & a & 1 \end{array} \quad \begin{array}{c|ccc} \otimes & 0 & a & 1 \\ \hline 0 & \{0\} & \{0\} & \{0\} \\ a & \{0\} & \{0, a\} & \{0, a\} \\ 1 & \{0\} & \{0, a\} & \{1\} \end{array} \quad \begin{array}{c|ccc} \sim & 0 & a & 1 \\ \hline 0 & \{1\} & \{a\} & \{a\} \\ a & \{a\} & \{1\} & \{1, a\} \\ 1 & \{a\} & \{1, a\} & \{1\} \end{array}$$

Routine calculation shows that $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a hyper EQ -algebra.

- (iv) Let $H = \{0, a, b, 1\}$, such that $0 < a < b < 1$. Define $\wedge, \otimes$ and $\sim$ on H as follows:

$$\begin{array}{c|cccc} \otimes & 0 & a & b & 1 \\ \hline 0 & \{0\} & \{0\} & \{0\} & \{0\} \\ a & \{0\} & \{0, a\} & \{0, a\} & \{0, a\} \\ b & \{0\} & \{0, a\} & \{0, b\} & \{0, b\} \\ 1 & \{0\} & \{0, a\} & \{0, b\} & \{0, 1\} \end{array} \quad \begin{array}{c|cccc} \sim & 0 & a & b & 1 \\ \hline 0 & \{1\} & \{a, b, 1\} & \{a, 1\} & \{0, 1\} \\ a & \{a, b, 1\} & \{a, 1\} & \{a, b, 1\} & \{a, 1\} \\ b & \{a, 1\} & \{a, b, 1\} & \{b, 1\} & \{b, 1\} \\ 1 & \{0, 1\} & \{a, 1\} & \{b, 1\} & \{1\} \end{array}$$

$$x \wedge y = \min\{x, y\}$$

It is easy to see that $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a hyper EQ -algebra.

- (v) Let $H = [0, 1]$. Then we define $\wedge, \otimes, \sim$, on H as follows:

$$x \otimes y = \begin{cases} 0, & x + y \leq 1 \\ x \wedge y, & o.w. \end{cases}, \quad x \sim y = \{1, x \wedge y\}, \quad x \wedge y = \min\{x, y\}$$

Then $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a hyper EQ -algebra.

- (vi) Let $H = [0, 1]$. Define $x \wedge y = \min\{x, y\}$, $x \otimes y = xy$ and $x \sim y = [x \wedge y, 1]$, for all $x, y \in H$. Then it is not difficult to check that $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a hyper EQ -algebra.

Proposition 3.3. *Let $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ be a hyper EQ -algebra such that $x \rightarrow y = (x \wedge y) \sim x$ and $\bar{x} = x \sim 1$. Then the following conditions hold, for all $x, y, z \in H$ and $A, B, C \subseteq H$:*

- (i) $x \ll x \otimes 1, \quad A \ll A \otimes 1$;
- (ii) $1 \in x \sim x, \quad 1 \ll x \rightarrow x$ and $1 \in A \sim A$;
- (iii) $z \otimes (x \wedge y) \ll (z \otimes x) \wedge (z \otimes y)$;
- (iv) if $A \ll B$ and $B \ll C$, then $A \ll C$;

- (v) $x \sim y \ll y \sim x, A \sim B \ll B \sim A$;
- (vi) $x \sim y \ll x \rightarrow y$;
- (vii) if $x \leq y$, then $1 \in x \rightarrow y$;
- (viii) $x \ll \bar{x}$;
- (ix) if $A \ll B$, then $A \otimes C \ll B \otimes C$;
- (x) $(x \sim y) \otimes (y \sim z) \ll x \sim z$ and $(x \rightarrow y) \otimes (y \rightarrow z) \ll x \rightarrow z$;
- (xi) $(x \rightarrow y) \otimes (y \rightarrow x) \ll x \sim y$;
- (xii) $\bar{x} \otimes \bar{y} \ll x \sim y$;
- (xiii) $x \otimes (x \sim y) \ll \bar{y}$;
- (xiv) if $x \leq y$, then $\bar{x} \ll \bar{y}$, $x \sim y = y \rightarrow x$, $z \rightarrow x \ll z \rightarrow y$, $y \rightarrow z \ll x \rightarrow z$;
- (xv) if $A \ll B$, then $C \rightarrow A \ll C \rightarrow B$;
- (xvi) $y, \bar{y} \ll x \rightarrow y$;
- (xvii) $(z \rightarrow (x \wedge y)) \otimes (x \sim t) \ll z \rightarrow (t \wedge y)$;
- (xviii) if $x \ll y \rightarrow z$, then $x \otimes y \ll \bar{z}$;
- (xix) if $x \leq y \leq z$, then $z \sim x \ll z \sim y$ and $x \sim z \ll x \sim y$;
- (xx) $(x \sim y) \otimes (z \sim t) \ll (x \wedge z) \sim (y \wedge t)$;
- (xxi) $x \sim y \ll (x \wedge z) \sim (y \wedge z)$;
- (xxii) $x \sim y \ll ((x \wedge t) \sim z) \sim ((y \wedge t) \sim z)$ and $x \sim y \ll (z \sim (x \wedge t)) \sim (z \sim (y \wedge t))$;
- (xxiii) $x \rightarrow y = x \rightarrow (x \wedge y)$;
- (xxiv) $x \sim y \ll (x \sim z) \sim (y \sim z)$ and $x \sim y \ll (z \sim x) \sim (z \sim y)$;
- (xxv) $x \sim y \ll (z \rightarrow x) \sim (z \rightarrow y)$;
- (xxvi) $x \rightarrow y \ll (z \rightarrow x) \rightarrow (z \rightarrow y)$.

Proof. (i): By definition of identity, $x \in x \otimes 1$. Since $x \leq x$, then $x \ll x \otimes 1$. The second part is clear.

(ii): By (HEQ7), $1 \in 1 \otimes 1 \leq 1 \sim 1$. Since 1 is a top element, then $1 \in 1 \sim 1$. Now, by (HEQ6), $1 \sim 1 = ((1 \wedge 1) \sim 1) \ll (1 \wedge 1 \wedge x) \sim (1 \wedge x) = x \sim x$. Since 1 is a top element, then $1 \in x \sim x$. Hence $1 \ll x \sim x$. The rest is clear.

(iii): Since $x \wedge y \ll x, y$ and $\otimes$ is isotone, then $z \otimes (x \wedge y) \ll (z \otimes x)$ and $(z \otimes y)$. Hence $z \otimes (x \wedge y) \ll (z \otimes x) \wedge (z \otimes y)$.

(iv): The proof is straightforward.

(v): By (i), (ii), (iv) and (HEQ3),

$$\begin{aligned} x \sim y \ll (x \sim y) \otimes 1 &= ((x \wedge 1) \sim y) \otimes 1 \ll ((x \wedge 1) \sim y) \otimes (x \sim x) \\ &\ll y \sim (x \wedge 1) = y \sim x \end{aligned}$$

The second part is clear.

(vi): By (i), (ii), (HEQ3), (iv) and (v),

$$\begin{aligned} x \sim y \ll 1 \otimes (x \sim y) &\ll ((x \wedge x) \sim x) \otimes (x \sim y) \ll ((x \wedge x) \sim x) \otimes (y \sim x) \\ &\ll x \sim (x \wedge y) \ll (x \wedge y) \sim x = x \rightarrow y \end{aligned}$$

(vii): If $x \leq y$, then $x \rightarrow y = (x \wedge y) \sim x = x \sim x$. Hence, by (ii), $1 \in x \rightarrow y$.

(viii): By (i), (iv) and (HEQ7), $x \ll x \otimes 1 \ll x \sim 1 = \bar{x}$.

(ix): The proof is straightforward.

(x): By (HEQ3), (HEQ2), (iv) and (v),

$$\begin{aligned}(x \sim y) \otimes (y \sim z) &= (y \sim z) \otimes (x \sim y) = ((y \wedge 1) \sim z) \otimes (x \sim y) \ll (z \sim (x \wedge 1)) \\ &= (z \sim x) \ll x \sim z\end{aligned}$$

Now, by (HEQ5), (HEQ1), (iv) and (v),

$$\begin{aligned}(x \rightarrow y) \otimes (y \rightarrow z) &= ((x \wedge y) \sim x) \otimes ((y \wedge z) \sim y) \ll (x \sim (x \wedge (y \wedge z))) \\ &\ll ((x \wedge (y \wedge z)) \sim x) = (((x \wedge z) \wedge y) \sim x) \\ &\ll (x \wedge z) \sim x = x \rightarrow z.\end{aligned}$$

(xi): By (iv), (v), (x) and (HEQ3),

$$\begin{aligned}(x \rightarrow y) \otimes (y \rightarrow x) &= ((x \wedge y) \sim x) \otimes ((x \wedge y) \sim y) \\ &\ll (x \sim (x \wedge y)) \otimes ((x \wedge y) \sim y) \ll x \sim y\end{aligned}$$

(xii): By (iv), (v) and (x), $\bar{x} \otimes \bar{y} = (x \sim 1) \otimes (y \sim 1) \ll (x \sim 1) \otimes (1 \sim y) \ll x \sim y$.

(xiii): By (viii), (iv), (v), (ix) and (x),

$$x \otimes (x \sim y) \ll (x \sim 1) \otimes (x \sim y) \ll (1 \sim x) \otimes (x \sim y) \ll 1 \sim y \ll y \sim 1 = \bar{y}$$

(xiv): Let $x \leq y$. By (HEQ5), $(x \wedge y \wedge 1) \sim 1 \ll (y \wedge 1) \sim 1$. Then $x \sim 1 \ll y \sim 1$ i.e. $\bar{x} \ll \bar{y}$. Again, by (HEQ5),

$$z \rightarrow x = (z \wedge x) \sim z = (z \wedge x \wedge y) \sim z \ll (z \wedge y) \sim z = z \rightarrow y$$

The proof of other part is similar.

(xv): The proof is straightforward.

(xvi): Since $x \leq 1$, then by (xiv), $1 \rightarrow y \ll x \rightarrow y$. Hence, $\bar{y} = y \sim 1 = 1 \rightarrow y \ll x \rightarrow y$. Moreover by (viii) and (iv), $y \ll x \rightarrow y$.

(xvii): By (HEQ3), (iv) and (v),

$$\begin{aligned}(z \rightarrow (x \wedge y)) \otimes (x \sim t) &\ll (z \rightarrow (x \wedge y)) \otimes (t \sim x) \\ &= ((z \wedge (x \wedge y)) \sim z) \otimes (t \sim x) \\ &\ll z \sim (z \wedge (t \wedge y)) \ll z \rightarrow (t \wedge y)\end{aligned}$$

(xviii): Let $x \ll y \rightarrow z$. By (viii), (iv), (ix), (x) and (HEQ5),

$$\begin{aligned}x \otimes y &\ll (y \rightarrow z) \otimes \bar{y} = ((y \wedge z) \sim y) \otimes (y \sim 1) \ll (y \wedge z) \sim 1 \\ &= (y \wedge z \wedge 1) \sim 1 \ll (z \wedge 1) \sim 1 = z \sim 1 = \bar{z}\end{aligned}$$

(xix): If $x \leq y \leq z$, then by (HEQ5) and (iv),

$$z \sim x \ll (x \wedge z) \sim z = (x \wedge y \wedge z) \sim z \ll (y \wedge z) \sim z = y \sim z$$

For the other case, by (HEQ6),

$$x \sim z = (x \wedge z) \sim z \ll (x \wedge y \wedge z) \sim (z \wedge y) = x \sim y$$

(xx): By (HEQ6), $x \sim y = (x \wedge 1) \sim y \ll (x \wedge 1 \wedge z) \sim (y \wedge z) = (x \wedge z) \sim (y \wedge z)$ and in the same way $z \sim t \ll (y \wedge z) \sim (y \wedge t)$. Now, by (x), (ix) and (iv),

$$(x \sim y) \otimes (z \sim t) \ll ((x \wedge z) \sim (y \wedge z)) \otimes ((y \wedge z) \sim (y \wedge t)) \ll (x \wedge z) \sim (y \wedge t)$$

(xxi): By (i), (ii), (iv), (v) and (HEQ3),

$$x \sim y \ll y \sim x \ll 1 \otimes (y \sim x) \ll ((x \wedge z) \sim (x \wedge z)) \otimes (y \sim x) \ll (x \wedge z) \sim (y \wedge z)$$

(xxii): By (i), (ii), (iv), (xxi) and (HEQ4),

$$\begin{aligned} x \sim y &\ll (x \wedge t) \sim (y \wedge t) \ll ((x \wedge t) \sim (y \wedge t)) \otimes 1 \\ &\ll ((x \wedge t) \sim (y \wedge t)) \otimes (z \sim z) \\ &\ll ((x \wedge t) \sim z) \sim ((y \wedge t) \sim z) \end{aligned}$$

The proof of second part is similar.

(xxiii): The proof is clear.

(xxiv): By (xxii), it is enough to take $t = 1$.

(xxv): Let $t = z$. Then by (xxii), the proof is clear.

(xxvi): By (xxv), (iv), (v), (vi) and (xiv),

$$\begin{aligned} x \rightarrow y = (x \wedge y) \sim x &\ll (z \rightarrow (x \wedge y)) \sim (z \rightarrow x) \ll (z \rightarrow x) \sim (z \rightarrow (x \wedge y)) \\ &\ll (z \rightarrow x) \rightarrow (z \rightarrow (x \wedge y)) \ll (z \rightarrow x) \rightarrow (z \rightarrow y). \quad \blacksquare \end{aligned}$$

Proposition 3.4. *Let $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ be a hyper EQ-algebra with bottom element "0". If we set $\neg x = 0 \sim x$, then, for all $x, y, t \in H$;*

- (i) $1 \in \neg 0$;
- (ii) if $x \leq y$, then $\neg y \ll \neg x$;
- (iii) $x \otimes \neg x \ll \bar{0}$, $\bar{x} \otimes \bar{0} \ll \neg x$, $\neg x \otimes \bar{0} \ll \bar{x}$;
- (iv) $\neg x \otimes \neg y \ll x \rightarrow y$;
- (v) $\bar{0} \ll \neg x$;
- (vi) $x \sim y \ll \neg x \sim \neg y$;
- (vii) $(x \sim y) \otimes \neg y \ll \neg x$, $(x \rightarrow y) \otimes \neg y \ll \neg x$;
- (viii) $\neg y \ll y \rightarrow z$;
- (ix) $x \sim y \ll \neg(x \wedge t) \sim \neg(y \wedge t)$.

Proof. The proof of (i) and (ii) are clear.

(iii): Straightforward.

(iv): By Proposition 3.3 (iv), (v), (vi) and (x),

$$\neg x \otimes \neg y = (0 \sim x) \otimes (0 \sim y) \ll (x \sim 0) \otimes (0 \sim y) \ll x \sim y \ll x \rightarrow y.$$

(v): It follows from Proposition 3.3 (xix) and $0 \leq x \leq 1$.

(vi): By Proposition 3.3 (xxiv), $x \sim y \ll (0 \sim x) \sim (0 \sim y) = \neg x \sim \neg y$.

(vii): By Proposition 3.3 (iv), (v) and (x),

$$(x \sim y) \otimes \neg y = (x \sim y) \otimes (0 \sim y) \ll (x \sim y) \otimes (y \sim 0) \ll x \sim 0 \ll \neg x$$

The proof of other case is similar.

(viii): Since $0 \leq z$, then by Proposition 3.3 (xiv), $\neg y = 0 \sim y \ll y \rightarrow 0 \ll y \rightarrow z$.

(ix): By Proposition 3.3(xxii), $x \sim y \ll (0 \sim (x \wedge t)) \sim (0 \sim (y \wedge t)) = \neg(x \wedge t) \sim \neg(y \wedge t)$. ■

Theorem 3.5. *Each hyper EQ-algebra of order n , can be extend to a hyper EQ-algebra of order $n + 1$, for any $2 < n \in \mathbb{N}$.*

Proof. Let $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ be a hyper EQ-algebra of order n , e be an element such that $e \notin H$, and $\mathcal{H}_1 = H \cup \{e\}$. Then we define a binary operations $\wedge_1$ and two hyper operation $\sim_1$ and $\otimes_1$ on $\mathcal{H}_1$ by:

$$x \wedge_1 y = \begin{cases} x \wedge y & , \quad x, y \in H \\ x & , \quad x \in H, y = e \\ y & , \quad y \in H, x = e \\ e & , \quad x = e, y = e \end{cases}$$

$$x \otimes_1 y = \begin{cases} x \otimes y & , \quad x, y \in H \\ \{x, e\} & , \quad x \in H, y = e \\ \{y, e\} & , \quad y \in H, x = e \\ e & , \quad x = e, y = e \end{cases}$$

$$x \sim_1 y = x \sim y \cup \{e\}$$

We define $x \leq y$ if and only if $x \wedge_1 y = x$. It is clear that for any $x \in H$, $x \leq e$. Now, by some modification we can show that $\mathcal{H}_1 = (H_1, \wedge_1, \otimes_1, \sim_1, e)$ is a hyper EQ-algebra. ■

Example 3.6. Let $H = \{1\}$. Then $1 \wedge 1 := 1, 1 \sim 1 := \{1\}, 1 \otimes 1 := \{1\}$. Hence it is clear that $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a hyper EQ-algebra.

Corollary 3.7. *There exists at least one hyper EQ-algebra of order n , for any $n \in \mathbb{N}$.*

Proof. By Examples 3.6, 3.2(ii) and Proposition 3.5, the proof is clear. ■

Definition 3.8. Let $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ be a hyper EQ-algebra. Then H is called,

- (i) *separated* if $1 \in x \sim y$, then $x = y$, for all $x, y \in H$, (in other words $1 \in x \sim y$ if and only if $x = y$);
- (ii) *good* if $x \sim 1 = x = 1 \sim x$, for all $x \in H$.

Example 3.9. (i) Let $H = \{0, a, 1\}$, which $0 < a < 1$, and $\wedge, \otimes$ and $\sim$ are defined on H as follows,

$\wedge$	0	a	1	$\otimes$	0	a	1	$\sim$	0	a	1
0	0	0	0	0	{0}	{0}	{0}	0	{1}	{0,a}	{a}
a	0	a	a	a	{0}	{0,a}	{0,a}	a	{0,a}	{1}	{a}
1	0	a	1	1	{0}	{0,a}	{1}	1	{0,a}	{0,a}	{1}

Routine calculation shows that $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a separated hyper EQ -algebra but it is not good.

(ii) Let $H = \{0, a, 1\}$, which $0 < a < 1$, and $\wedge, \otimes$ and $\sim$ are defined on H as follows,

	$\otimes$	0	a	1		$\sim$	0	a	1
$x \wedge y = \min\{x, y\},$	0	$\{0\}$	$\{0\}$	$\{0\}$		0	$\{1\}$	$\{0, a\}$	$\{0\}$
	a	$\{0\}$	$\{0\}$	$\{0, a\}$		a	$\{0, a\}$	$\{1\}$	$\{a\}$
	1	$\{0\}$	$\{0, a\}$	$\{1\}$		1	$\{0\}$	$\{a\}$	$\{1\}$

Then $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a good and separated hyper EQ -algebra.

Proposition 3.10. *Let $\mathcal{H}$ be a good hyper EQ -algebra. Then the following conditions hold, for all $x, y \in H$,*

- (i) $x \ll (x \sim y) \sim y, x \ll (y \sim x) \sim y;$
- (ii) $x \ll (x \rightarrow y) \rightarrow y;$
- (iii) $x \otimes (x \sim y) \ll y;$
- (iv) $x \otimes (x \rightarrow y) \ll y;$
- (v) *if $x \ll y \rightarrow z$, then $x \otimes y \ll z$;*

Proof. (i): By Proposition 3.3 (i), (ii), (iv), (HEQ4) and goodness,

$$x \ll x \otimes 1 \ll (x \sim 1) \otimes (y \sim y) \ll (x \sim y) \sim (1 \sim y) = (x \sim y) \sim y.$$

The proof of second part is similar.

(ii): By (i) and Proposition 3.3(iv), (vi) and (xiv),

$$x \ll ((x \wedge y) \sim x) \sim (x \wedge y) = (x \rightarrow y) \sim (x \wedge y) \ll (x \rightarrow y) \rightarrow (x \wedge y) \ll (x \rightarrow y) \rightarrow y.$$

(iii), (iv): By Proposition 3.3 (xiii) and (iv), the proof are straightforward.

(v): Let $x \ll y \rightarrow z$, for some $x, y, z \in H$. Then by (iv), $x \otimes y \ll (y \rightarrow z) \otimes y = y \otimes (y \rightarrow z) \ll z$. Hence $x \otimes y \ll z$. ■

Proposition 3.11. *Every good hyper EQ -algebra is separated.*

Proof. Let $1 \in x \sim y$. Then by Proposition 3.3 (i), (ii), (ix) and Proposition 3.10 (iii), $x \ll x \otimes 1 \ll x \otimes (x \sim y) \ll y$. Therefore, $x \leq y$. By the similar way, $y \leq x$. Thus $x = y$. Therefore, $\mathcal{H}$ is separated. ■

4. Filters in hyper EQ -algebra

In this section we introduce the concept of prefilters and filters in hyper EQ -algebras and we give some related results.

In what follows, let $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ denote a hyper EQ -algebra, unless otherwise state.

Definition 4.1. Let F be a subset of H such that $1 \in F$. Then F is called a
 (i) *prefilter* of H , if $x \rightarrow y \subseteq F$ and $x \in F$, imply $y \in F$, for some $x, y \in H$ and $(x \otimes y) \subseteq F$, for all $x, y \in F$.
 (ii) *filter* of H , if F is a prefilter and $x \rightarrow y \subseteq F$, imply $(x \otimes z) \rightarrow (y \otimes z) \subseteq F$, for all $x, y, z \in H$.

Example 4.2. (i) Let $H = \{0, a, b, 1\}$, such that $0 < a < b < 1$. Define $\wedge, \otimes$ and $\sim$ on H as follows:

$\otimes$	0	a	b	1		$\sim$	0	a	b	1
0	{0}	{0}	{0}	{0}		0	{1}	{1}	{1}	{0,1}
a	{0}	{0,a}	{0,a}	{0,a}	,	a	{1}	{1}	{a,b,1}	{a,1}
b	{0}	{0,a}	{0,b}	{0,b}		b	{1}	{a,b,1}	{1}	{b,1}
1	{0}	{0,a}	{0,b}	{1}		1	{1}	{a,1}	{b,1}	{1}

$$x \wedge y = \min\{x, y\}$$

It is easy to see that $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a hyper EQ -algebra. It is clear that $F_1 = \{1\}$ is (pre)filter. Moreover, since $0 \rightarrow a = \{1\} \subseteq F$ but $a \notin F$, then $F_2 = \{0, b, 1\}$ is not a prefilter.

(ii) According to Example 3.9(i), $F = \{1, a\}$ is prefilter. Since $1 \rightarrow a = \{a\} \subseteq F$ and $1 \otimes a \rightarrow a \otimes a = \{0, a, 1\} \not\subseteq F$, then F is not a filter.

(iii) Let $H = \{0, a, b, 1\}$ such that $0 < a < b < 1$. Define $\wedge, \otimes$ and $\sim$ on H as follows;

$\sim$	0	a	b	1	
0	{1}	{a,b,1}	{a,1}	{1}	
a	{a,b,1}	{a,1}	{a,b,1}	{1}	
b	{a,1}	{a,b,1}	{b,1}	{1}	
1	{1}	{1}	{1}	{1}	

$x \wedge y = \min\{x, y\}$, $x \otimes y = x \wedge y$

It is easy to see that $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a hyper EQ -algebra. Let $F = \{1\}$. Since $1 \rightarrow a = a \sim 1 = \{1\} \subseteq F$ and $1 \in F$, but $a \notin F$, then F is not a (pre)filter.

Definition 4.3. A non-empty subset D of H is said to be $S_{\rightarrow}$ reflexive ($S_{\sim}$ reflexive) if $(x \rightarrow y) \cap D \neq \emptyset ((x \sim y) \cap D \neq \emptyset)$, then $x \rightarrow y \subseteq D (x \sim y \subseteq D)$, for all $x, y \in H$.

Example 4.4. According of Example 3.2(iii), $F = \{1, a\}$ is $S_{\rightarrow}$ reflexive and $S_{\sim}$ reflexive.

Remark 4.5. Let D be $S_{\sim}$ reflexive and $(x \rightarrow y) \cap D \neq \emptyset$, for $x, y \in D$. Then $((x \wedge y) \sim x) \cap D \neq \emptyset$. Since D is $S_{\sim}$ reflexive, then $((x \wedge y) \sim x) \subseteq D$ or $(x \rightarrow y) \subseteq D$. Therefore, D is $S_{\rightarrow}$ reflexive.

Lemma 4.6. Let F be $S_{\sim}$ reflexive and (pre)filter of H . Then the following conditions hold, for all $x, y, z \in H$,

- (i) if $x \in F$ and $x \leq y$, then $y \in F$, and if $A \subseteq F$ and $A \ll B$, then $B \cap F \neq \emptyset$;
- (ii) if $x \in F$ and $(x \sim y) \cap F \neq \emptyset$, then $y \in F$;
- (iii) if $x, y \in F$, then $(x \wedge y) \in F$;
- (iv) if $A \subseteq F$ and $(A \sim B) \cap F \neq \emptyset$, then $B \cap F \neq \emptyset$;
- (v) if $(x \sim y) \subseteq F$ and $(y \sim z) \subseteq F$, then $(x \sim z) \cap F \neq \emptyset$;
- (vi) if $(x \rightarrow y) \subseteq F$ and $(y \rightarrow z) \subseteq F$, then $(x \rightarrow z) \cap F \neq \emptyset$;
- (vii) if $x \ll y \rightarrow z$ and $x, y \in F$, then $z \in F$.

Proof. (i); Let $x \leq y$ and $x \in F$, for some $x, y \in H$. By Proposition 3.3 (vii), $1 \in x \rightarrow y$, then $(x \rightarrow y) \cap F \neq \emptyset$. Since F is $S_{\sim}$ reflexive, then by Remark 4.5, F is $S_{\rightarrow}$ reflexive, and so $x \rightarrow y \subseteq F$. Now since F is a (pre)filter and $x \in F$, then $y \in F$. The proof of second part is clear.

(ii): Since $(x \sim y) \cap F \neq \emptyset$, then $(x \sim y) \subseteq F$ and since $x \sim y \ll x \rightarrow y$ then by (i), $(x \rightarrow y) \cap F \neq \emptyset$. By Remark 4.5, F is $S_{\rightarrow}$ reflexive, then $x \rightarrow y \subseteq F$. Now, since F is a (pre)filter and $x \in F$, thus $y \in F$.

(iii): Since $y \ll x \rightarrow y$ and $y \in F$, then by (i), $(x \rightarrow y) \cap F \neq \emptyset$. Since F is $S_{\sim}$ reflexive, then by Remark 4.5, $x \rightarrow y \subseteq F$. By Proposition 3.3(xxiii), $x \rightarrow y = x \rightarrow (x \wedge y)$ and since $x \in F$, then $x \wedge y \in F$.

(iv): The proof is straightforward.

(v): From Proposition 3.3(xxiv), $x \sim y \ll (y \sim z) \sim (x \sim z)$. Then by (i), $((y \sim z) \sim (x \sim z)) \cap F \neq \emptyset$ and $y \sim z \subseteq F$. Hence, by (iv), $(x \sim z) \cap F \neq \emptyset$.

(vi): Let $x \rightarrow y \subseteq F$ and $(y \rightarrow z) \subseteq F$, then by Definition 4.1, $(x \rightarrow y) \otimes (y \rightarrow z) \subseteq F$. By Proposition 3.3 (x), $(x \rightarrow y) \otimes (y \rightarrow z) \ll x \rightarrow z$. Then by (i), $(x \rightarrow z) \cap F \neq \emptyset$.

(vii): Let $x \ll y \rightarrow z$. Since $x \in F$, then by (i), $y \rightarrow z \cap F \neq \emptyset$. Thus by Remark 4.5, $(y \rightarrow z) \subseteq F$ and $y \in F$. Hence $z \in F$. ■

Proposition 4.7. Let F be an $S_{\sim}$ reflexive filter of H . If $(x \sim y) \cap F \neq \emptyset$ and $(z \sim t) \cap F \neq \emptyset$, for all $x, y, z, t \in H$. Then the following conditions hold:

- (i) $((x \wedge z) \sim (y \wedge t)) \cap F \neq \emptyset$;
- (ii) $((x \sim z) \sim (y \sim t)) \cap F \neq \emptyset$;
- (iii) $((x \rightarrow z) \sim (y \rightarrow t)) \cap F \neq \emptyset$;
- (iv) $((x \otimes z) \sim (y \otimes t)) \cap F \neq \emptyset$.

Proof. (i): By Proposition 3.3 (xxi), $(x \sim y) \ll ((x \wedge z) \sim (y \wedge z))$. Since $(x \sim y) \cap F \neq \emptyset$ and F is $S_{\sim}$ reflexive, then $(x \sim y) \subseteq F$. Hence by Lemma 4.6 (i), $((x \wedge z) \sim (y \wedge z)) \cap F \neq \emptyset$. By the similar way, since $(z \sim t) \ll ((z \wedge y) \sim (t \wedge y))$, we have $((z \wedge y) \sim (t \wedge y)) \cap F \neq \emptyset$. Since F is $S_{\sim}$ reflexive then, $((x \wedge z) \sim (y \wedge z)) \subseteq F$ and $((z \wedge y) \sim (t \wedge y)) \subseteq F$. Now, by Lemma 4.6 (v), $((x \wedge z) \sim (y \wedge t)) \cap F \neq \emptyset$.

(ii), (iii): The proof are similar to the proof of (i).

(iv): Let $(x \sim y) \cap F \neq \emptyset$. Since F is a $S_{\sim}$ reflexive filter and $x \sim y \ll x \rightarrow y$, then by Remark 4.5, $(x \rightarrow y) \subseteq F$ and by Definition 4.1, $((x \otimes z) \rightarrow (y \otimes z)) \subseteq F$. By the similar way from $(z \sim t) \cap F \neq \emptyset$, we have $((z \otimes y) \rightarrow (t \otimes y)) \subseteq F$. Since F is a filter, then $((x \otimes z) \rightarrow (y \otimes z)) \otimes ((z \otimes y) \rightarrow (t \otimes y)) \subseteq F$. Hence by Proposition 3.3(x) and Lemma 4.6(i), $((x \otimes z) \sim (y \otimes t)) \cap F \neq \emptyset$. ■

Now, let F be an $S_{\sim}$ reflexive filter of H . We define, for all $x, y \in H$:

$$x \equiv_F y \quad \text{iff} \quad (x \sim y) \cap F \neq \emptyset$$

By Proposition 3.3 (ii), $1 \in x \sim x$. Since F is a filter, then $1 \in F$. Hence $(x \sim x) \cap F \neq \emptyset$, and so $x \equiv_F x$. If $x \equiv_F y$, then $(x \sim y) \cap F \neq \emptyset$. By Proposition 3.3 (v) and Lemma 4.6 (i), $(y \sim x) \cap F \neq \emptyset$, Thus $y \equiv_F x$. Now, let $x \equiv_F y$ and $y \equiv_F z$. Then $(x \sim y) \cap F \neq \emptyset$ and $(y \sim z) \cap F \neq \emptyset$. Since F is $S_{\sim}$ reflexive, then $(x \sim y) \subseteq F$ and $(y \sim z) \subseteq F$. Hence by Lemma 4.6 (v), $(x \sim z) \cap F \neq \emptyset$, and so $x \equiv_F z$. Therefore, " $\equiv_F$ " is an equivalent relation on H .

Let $x \equiv_F y$. Then $(x \sim y) \cap F \neq \emptyset$. Since F is $S_{\sim}$ reflexive, then $(x \sim y) \subseteq F$. By Proposition 3.4(vi), $x \sim y \ll \neg x \sim \neg y$. Then by Lemma 4.6 (i), $(\neg x \sim \neg y) \cap F \neq \emptyset$, and so $\neg x \equiv_F \neg y$. Also, if $x \equiv_F y$ and $z \equiv_F t$, then $(x \sim y) \cap F \neq \emptyset$ and $(z \sim t) \cap F \neq \emptyset$, and so by Proposition 4.7, $((x \wedge z) \sim (y \wedge t)) \cap F \neq \emptyset$, $((x \sim z) \sim (y \sim t)) \cap F \neq \emptyset$, $((x \rightarrow z) \sim (y \rightarrow t)) \cap F \neq \emptyset$ and $((x \otimes z) \sim (y \otimes t)) \cap F \neq \emptyset$. Hence, $x \wedge z \equiv_F y \wedge t$, $x \sim z \equiv_F y \sim t$, $x \rightarrow z \equiv_F y \rightarrow t$ and $x \otimes z \equiv_F y \otimes t$. In the other words, " $\equiv_F$ " is a congruence relation on H . Now, let $\frac{H}{\equiv_F} = \{[x] | x \in H\}$. Define operation $\bar{\wedge}$ and hyper operations $\bar{\otimes}$ and $\bar{\sim}$ on $\frac{H}{\equiv_F}$ as follows:

$$\begin{aligned} [x] \bar{\wedge} [y] &= [x \wedge y] \\ [x] \bar{\otimes} [y] &= \{[t] | t \in x \otimes y\} \\ [x] \bar{\sim} [y] &= \{[t] | t \in x \sim y\} \end{aligned}$$

According to the above mention, it is clear $\bar{\wedge}$, $\bar{\otimes}$ and $\bar{\sim}$ are well defined.

Now, we define $[x] \leq [y]$ if and only if $[x] \bar{\wedge} [y] = [x]$ if and only if $[x \wedge y] = [x]$ if and only if $x \wedge y \equiv_F x$ if and only if $(x \wedge y) \sim x \cap F \neq \emptyset$ if and only if $x) y \cap F \neq \emptyset$ and $A \ll B$ is defined as before. Then we have the following theorem.

Theorem 4.8. *Let F be an $S_{\sim}$ reflexive filter of $\mathcal{H}$. Then $\frac{\mathcal{H}}{\equiv_F} = \left(\frac{H}{\equiv_F}, \bar{\wedge}, \bar{\otimes}, \bar{\sim}, [1] \right)$ is a hyper EQ-algebra which is separated.*

Proof. (HEQ1), (HEQ2): It is easy to see that $\left(\frac{H}{\equiv_F}, \bar{\wedge} \right)$ is an idempotent commutative monoid, and $\left(\frac{H}{\equiv_F}, \bar{\otimes} \right)$ is a commutative semihypergroup with identity $[1]$ and $\bar{\otimes}$ is isotone.

(HEQ3): We show that $(([x] \bar{\wedge} [y]) \bar{\sim} [z]) \bar{\otimes} ([t] \bar{\sim} [x]) \ll ([z] \bar{\sim} ([t] \bar{\wedge} [y]))$ or, equivalently, $([x \wedge y] \bar{\sim} [z]) \bar{\otimes} ([t] \bar{\sim} [x]) \ll ([z] \bar{\sim} [t \wedge y])$. Let $[t] \in ([u] \bar{\otimes} [v]) \subseteq (([x \wedge y] \bar{\sim} [z]) \bar{\otimes} ([t] \bar{\sim} [x]))$, such that $[u] \in [x \wedge y] \bar{\sim} [z]$ and $[v] \in [t] \bar{\sim} [x]$. Then $u \in (x \wedge y) \sim z$ and $v \in (t \sim x)$. So by (HEQ3), $t \in (u \otimes v) \ll ((x \wedge y) \sim z) \otimes (t \sim x) \ll z \sim (t \wedge y)$. Thus there exists $w \in z \sim (t \wedge y)$ such that $t \in (u \otimes v) \ll w$ or $t \leq w$. Hence $1 \in t \rightarrow w$, or $1 \in (u \otimes v) \rightarrow w$, i.e. $((u \otimes v) \rightarrow w) \cap F \neq \emptyset$. Then $[u] \bar{\otimes} [v] \ll [w]$. Since $[w] \in [z \sim (t \wedge y)] = [z] \bar{\sim} ([t] \bar{\wedge} [y])$, Therefore, (HEQ3) holds. By the same way, other conditions hold.

If $[1] \in ([x] \sim [y])$, then $[1] \in \{[t] | t \in x \sim y\}$. Hence there exists $t \in x \sim y$ such that $[t] = [1]$, and so $t \sim 1 \cap F \neq \emptyset$. Then $(x \sim y) \sim 1 \cap F \neq \emptyset$. By Proposition 3.3 (ii), $((x \sim y) \sim 1) \ll (1 \sim (x \sim y))$. Then by Lemma 4.6 (i), $(1 \sim (x \sim y)) \cap F \neq \emptyset$. Since F is $S_{\sim}$ reflexive, then $(1 \sim (x \sim y)) \subseteq F$. Now, since $1 \in F$, and F is a filter, then by Lemma 4.6 (iv), $(x \sim y) \cap F \neq \emptyset$. Hence $x \equiv_F y$, i.e. $[x] = [y]$. Therefore, $\frac{\mathcal{H}}{\equiv_F}$ is separated. ■

5. Relation between hyper EQ-algebras and some other hyper structures

Definition 5.1. [4] Let H be a non-empty set and " $\circ$ " be a hyper operation on H . Then H is called a *hyper BCK-algebra* if it contains a constant " 0 " and satisfies the following axioms, for all $x, y, z \in H$:

- (HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$;
- (HK2) $(x \circ y) \circ z = (x \circ z) \circ y$;
- (HK3) $x \circ H \ll \{x\}$ (or $x \circ y \ll \{x\}$);
- (HK4) $x \ll y$ and $y \ll x$ implies $x = y$;

where, $x \ll y$ is defined by $0 \in x \circ y$, and for every $A, B \subseteq H$, $A \ll B$ means, for all $a \in A$ there exists $b \in B$ such that $a \ll b$. We say H is *bounded hyper BCK-algebra* if there exists an element $e \in H$ such that, for all $x \in H$, $x \ll e$ and such e is called the *unit* of H . If $x \wedge y = y \wedge x$, where $x \wedge y = y \circ (y \circ x)$, for all $x, y \in H$, then H is called a *commutative hyper BCK-algebra*.

Proposition 5.2. [4] Let $\mathcal{H} = (H, \circ, 0)$ be a hyper BCK-algebra. Then the following conditions hold, for all $x, y, z \in H$ and $A, B, C \subseteq H$:

- (i) $x \ll x$;
- (ii) $A \ll A$;
- (iii) $x \in x \circ 0$;
- (iv) $A \circ 0 = A$;
- (v) $y \ll z$ imply $x \circ z \ll x \circ y$;
- (vi) $0 \circ x = 0$.

Theorem 5.3. Let $\mathcal{H}$ be a separated hyper EQ-algebra such that $z \rightarrow (y \rightarrow x) = y \rightarrow (z \rightarrow x)$, for all $x, y, z \in H$. Then $\mathcal{H}$ is a hyper BCK-algebra.

Proof. By considering $x \circ y := y \rightarrow x$, $0 := 1$, $x \ll_1 y$ if and only if $1 \in x \circ y = (y \rightarrow x)$ and $A \ll_1 B$ means $B \ll A$, we have:

- (HK1): Since by Proposition 3.3(xxvi), $y \rightarrow x \ll (z \rightarrow y) \rightarrow (z \rightarrow x)$, then $(x \circ z) \circ (y \circ z) \ll_1 x \circ y$.
- (HK2): Since by assumption, $z \rightarrow (y \rightarrow x) = y \rightarrow (z \rightarrow x)$, then $(x \circ y) \circ z = (x \circ z) \circ y$.
- (HK3): Since by Proposition 3.3(xvi), $x \ll y \rightarrow x$, then $x \circ y \ll_1 \{x\}$, for all $x, y \in H$.
- (HK4): If $x \ll_1 y$ and $y \ll_1 x$, then $1 \in y \rightarrow x = ((x \wedge y) \sim y)$ and $1 \in x \rightarrow y =$

$((x \wedge y) \sim x)$. Since H is separated, then $x \wedge y = y$ and $x \wedge y = x$. Thus $x \leq y$ and $y \leq x$, i. e. $x = y$.

Therefore, $(H, \circ, 0)$ is a hyper BCK -algebra. $\blacksquare$

Example 5.4. Since in Example 3.9(ii), $\mathcal{H}$ is a separated hyper EQ -algebra such that, $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$, for all $x, y, z \in H$, then $\mathcal{H}$ is a hyper BCK -algebra.

Definition 5.5. [9] A *super lattice* is a partially ordered set $(S, \leq)$ endowed with two binary hyperoperations $\vee$ and $\wedge$ satisfying the following properties, for all $x, y, z \in S$,

- (SL1) $x \in (x \vee x) \cap (x \wedge x)$;
- (SL2) $x \vee y = y \vee x$, $x \wedge y = y \wedge x$;
- (SL3) $(x \vee y) \vee z = x \vee (y \vee z)$, $(x \wedge y) \wedge z = x \wedge (y \wedge z)$;
- (SL4) $x \in ((x \vee y) \wedge x) \cap ((x \wedge y) \vee x)$;
- (SL5) $x \leq y$ implies $y \in x \vee y$ and $x \in x \wedge y$;
- (SL6) if $x \in x \wedge y$ or $y \in x \vee y$ then $x \leq y$.

Definition 5.6. [9] By a *hyper residuated lattice* we mean a non-empty set L endowed with four binary hyperoperations $\vee, \wedge, \otimes, \rightarrow$ and two constants 0 and 1 satisfying the following conditions:

- (HRL1) $(L, \leq, \vee, \wedge, 0, 1)$ is a bounded super lattice;
- (HRL2) $(L, \otimes, 1)$ is a commutative semihypergroup with 1 as the identity;
- (HRL3) $x \otimes z \ll y$ if and only if $z \ll x)y$;

where $A \ll B$ means that $x \leq y$, for some $x \in A$ and $y \in B$.

Note. Let $\mathcal{H}$ be a hyper BCK -algebra with unite e . If $e \circ (e \circ x) = x$, for any $x \in H$, then $e \circ x$ is singleton for any $x \in H$. Since, let $u, v \in e \circ x$. Then $e \circ u \subseteq e \circ (e \circ x) = x$, and so $e \circ u = x$. Since $v \in e \circ x \subseteq e \circ (e \circ u) = u$, then $u = v$. Therefore $e \circ x$ is singleton.

Theorem 5.7. Let $\mathcal{H} = (H, \circ, 0)$ be a commutative and bounded hyper BCK -algebra with unit e and $e \circ (e \circ x) = x$, for all $x \in H$. Then $\mathcal{H}$ is a hyper residuated lattice.

Proof. Let $\mathcal{H} = (H, \circ, 0)$ be a commutative and bounded hyper BCK -algebra with unit e and $e \circ (e \circ x) = x$, for all $x \in H$. We define $x' := e \circ x$, $x \rightarrow y := y \circ x$, $1 := 0$, $0 := e$, $x \otimes y := (x \rightarrow y')'$, $x \vee y := (y \rightarrow x) \rightarrow x = (x \rightarrow y) \rightarrow y$, $x \wedge y := (x' \vee y')'$. Moreover, we define $x \leq y$ if and only if $y \in x \vee y$ and $x \ll_1 y$ if and only if $1 \in y \circ x = x \rightarrow y$. By these definitions and assumption we have $x'' = x$ and by Proposition 5.2 (v), if $x \ll_1 y$, then $y' \ll_1 x'$ and by (HK2),

$$\begin{aligned} y' \rightarrow x' &= e \circ y \rightarrow e \circ x = (y \rightarrow e) \rightarrow (x \rightarrow e) = x \rightarrow ((y \rightarrow e) \rightarrow e) \\ &= x \rightarrow y'' = x \rightarrow y. \end{aligned}$$

Now, we prove that $\mathcal{H} = (H, \wedge, \vee, \otimes, 0, 1)$ is a hyper residuated lattice.

(HRL1): We show that $(H, \leq, \vee, \wedge, 0, 1)$ is a bounded super lattice. By Proposition 5.2 (i) and (iii), $x \in 1 \rightarrow x \subseteq (x \rightarrow x) \rightarrow x = x \vee x$, then $x \leq x$.

Hence $\leq$ is reflexive. Now, let $x \leq y$ and $y \leq x$. Then by assumption $x, y \in x \vee y = (x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$. By (HK3), $(x \rightarrow y) \rightarrow y \ll y$ and $(y \rightarrow x) \rightarrow x \ll x$. Thus $x \ll y$ and $y \ll x$, and so by (HK4), $x = y$. Then $\leq$ is antisymmetric. Let $x \leq y$ and $y \leq z$. Since $y \in x \vee y = (y \rightarrow x) \rightarrow x$, then $z \in y \vee z = (y \rightarrow z) \rightarrow z \subseteq (((y \rightarrow x) \rightarrow x) \rightarrow z) \rightarrow z$. By (HK3), $(y \rightarrow x) \rightarrow x \ll x$. Thus by Proposition 5.2(v), $x \rightarrow z \ll ((y \rightarrow x) \rightarrow x) \rightarrow z$, and so $((((y \rightarrow x) \rightarrow x) \rightarrow z) \rightarrow z) \ll ((x \rightarrow z) \rightarrow z) = ((z \rightarrow x) \rightarrow x) \ll x$. Thus $1 \in x \rightarrow z$ and so $z \in (1 \rightarrow z) \subseteq ((x \rightarrow z) \rightarrow z) = x \vee z$. Hence $z \in x \vee z$, i.e. $x \leq z$. Then $\leq$ is transitive. Therefore, $(H, \leq)$ is a partially ordered set.

(SL1): According to the above mention $x \in x \vee x$. Since $x \wedge x = (x' \vee x')' = ((x' \rightarrow x') \rightarrow x')'$ and x', x'' are singleton and $1 \in x \rightarrow x = x' \rightarrow x'$, then $(1 \rightarrow x')' \subseteq ((x' \rightarrow x') \rightarrow x')'$. By Proposition 5.2(iv), $(1 \rightarrow x')' = (x')' = x$. Hence $x \in ((x' \rightarrow x') \rightarrow x')' = x \wedge x$, Therefore, $x \in (x \vee x) \cap (x \wedge x)$.

(SL2): Since H is commutative, then $x \vee y = (x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x = y \vee x$ and $x \wedge y = (x' \vee y')' = ((x' \rightarrow y') \rightarrow y')' = ((y' \rightarrow x') \rightarrow x')' = y \wedge x$.

(SL3): By (HK2) and commutativity,

$$\begin{aligned}
x \vee (y \vee z) &= (x \rightarrow ((z \rightarrow y) \rightarrow y)) \rightarrow ((z \rightarrow y) \rightarrow y) \\
&= ((z \rightarrow y) \rightarrow (x \rightarrow y)) \rightarrow ((z \rightarrow y) \rightarrow y) \\
&= (z \rightarrow y) \rightarrow (((z \rightarrow y) \rightarrow (x \rightarrow y)) \rightarrow y) \\
&= (z \rightarrow y) \rightarrow (y' \rightarrow ((z \rightarrow y) \rightarrow (x \rightarrow y)))' \\
&= y' \rightarrow ((z \rightarrow y) \rightarrow ((z \rightarrow y) \rightarrow (x \rightarrow y)))' \\
&= y' \rightarrow (((z \rightarrow y) \rightarrow (x \rightarrow y)) \rightarrow (z \rightarrow y))' \\
&= y' \rightarrow (((x \rightarrow y)' \rightarrow (z \rightarrow y))' \rightarrow (z \rightarrow y))' \\
&= y' \rightarrow (((z \rightarrow y)' \rightarrow (x \rightarrow y))' \rightarrow (x \rightarrow y))' \\
&= y' \rightarrow (((x \rightarrow y) \rightarrow (z \rightarrow y)) \rightarrow (x \rightarrow y))' \\
&= ((x \rightarrow y) \rightarrow (z \rightarrow y)) \rightarrow (y' \rightarrow (x \rightarrow y))' \\
&= ((x \rightarrow y) \rightarrow (z \rightarrow y)) \rightarrow ((x \rightarrow y) \rightarrow y) \\
&= (z \rightarrow ((x \rightarrow y) \rightarrow y)) \rightarrow (x \vee y) \\
&= (z \rightarrow (x \vee y)) \rightarrow (x \vee y) \\
&= (x \vee y) \vee z
\end{aligned}$$

By the similar way we can prove the second part.

(SL4): By (HK2) and commutativity,

$$\begin{aligned}
(x \vee y) \wedge x &= ((x \rightarrow y) \rightarrow y) \wedge x = (((x \rightarrow y) \rightarrow y)' \vee x')' \\
&= (((x \rightarrow y) \rightarrow y)' \rightarrow x') \rightarrow x' \\
&= ((x \rightarrow ((x \rightarrow y) \rightarrow y)) \rightarrow x')' = (((x \rightarrow y) \rightarrow (x \rightarrow y)) \rightarrow x')'.
\end{aligned}$$

By Proposition 5.2(ii), $1 \in (x \rightarrow y) \rightarrow (x \rightarrow y)$, and so $(1 \rightarrow x')' \subseteq (((x \rightarrow y) \rightarrow (x \rightarrow y)) \rightarrow x')'$. By Proposition 5.2(iv), $(1 \rightarrow x')' = (x')' = x$. Then $x \in (((x \rightarrow y) \rightarrow (x \rightarrow y)) \rightarrow x')'$, i.e. $x \in (x \vee y) \wedge x$. By the similar way we can prove the second part.

(SL5): Let $x \leq y$. Then $y \in x \vee y = (y \rightarrow x) \rightarrow x$. By (HK3), $(y \rightarrow x) \rightarrow x \ll x$ and so $y \ll x$. Hence $1 \in x \rightarrow y$, and so by Proposition 5.2(iii), $y \in (1 \rightarrow y) \subseteq (x \rightarrow y) \rightarrow y = x \vee y$. By the definition,

$$x \wedge y = (x' \vee y')' = ((x' \rightarrow y') \rightarrow y')' = ((y' \rightarrow x') \rightarrow x')' = ((x \rightarrow y) \rightarrow x')'$$

Since $1 \in x \rightarrow y$, we have $(1 \rightarrow x')' \subseteq ((x \rightarrow y) \rightarrow x')'$. By Proposition 5.2(iv), $(1 \rightarrow x')' = (x')' = x$. Then $x \in ((x \rightarrow y) \rightarrow x')' = x \wedge y$.

(SL6): Let $y \in x \vee y$. Then $x \leq y$. Now, let $x \in x \wedge y = (x' \vee y')'$. Since x' is singleton, then $x' \in ((x' \vee y')')' = x' \vee y' = (x' \rightarrow y') \rightarrow y'$. By (HK3), $(x' \rightarrow y') \rightarrow y' \ll y'$, then $x' \ll y'$, and so by Proposition 5.2(v), $(y')' \ll (x')'$, i.e. $y \ll x$. Thus $1 \in x \rightarrow y$. By Proposition 5.2(vi), $y \in 1 \rightarrow y \subseteq (x \rightarrow y) \rightarrow y = x \vee y$. Hence $x \leq y$. Therefore, (HRL1) is hold.

(HRL2): By definition and assumption, $x \otimes y = (x \rightarrow y')' = ((y')' \rightarrow x')' = (y \rightarrow x')' = y \otimes x$ and by (HK2),

$$\begin{aligned} x \otimes (y \otimes z) &= (x \rightarrow ((y \rightarrow z')')')' = (x \rightarrow (y \rightarrow z'))' \\ &= (y \rightarrow (x \rightarrow z'))' = (y \rightarrow (z \rightarrow x'))' \\ &= (z \rightarrow (y \rightarrow x'))' = ((y \rightarrow x')' \rightarrow z')' = (x \otimes y) \otimes z. \end{aligned}$$

By Proposition 5.2(iii) and since x' is singleton, then $x' \in 1 \rightarrow x'$. Hence $x \otimes 1 = (x \rightarrow 1')' = ((1')' \rightarrow x')' = (1 \rightarrow x')'$. Thus $(x')' \in (1 \rightarrow x')'$ or $x \in 1 \otimes x$. Therefore, $(H, \otimes, 1)$ is a commutative semihypergroup with 1 as a identity.

(HRL3): Let $x \otimes z \ll_1 y$. Then $y \ll (x \rightarrow z')'$. Hence there exists $t \in (x \rightarrow z')'$ such that $y \ll t$, and so $1 \in t \rightarrow y$. Thus $1 \in (x \rightarrow z')' \rightarrow y$. By (HK2),

$$1 \in (x \rightarrow z')' \rightarrow y = y' \rightarrow (x \rightarrow z') = y' \rightarrow (z \rightarrow x') = z \rightarrow (y' \rightarrow x') = z \rightarrow (x \rightarrow y)$$

Then there exists $u \in x \rightarrow y$ such that $1 \in z \rightarrow u$, and so $u \ll z$. Hence $z \ll_1 x \rightarrow y$. Now let $z \ll_1 x \rightarrow y$, then $x \rightarrow y \ll z$ and so, for all $t \in x \rightarrow y$, $t \ll z$. Hence

$$\begin{aligned} 1 \in z \rightarrow t &\subseteq z \rightarrow (x \rightarrow y) = x \rightarrow (z \rightarrow y) = x \rightarrow (y' \rightarrow z') \\ &= y' \rightarrow (x \rightarrow z') = (x \rightarrow z')' \rightarrow y = (x \otimes z) \rightarrow y \end{aligned}$$

Then there exists $v \in x \otimes z$ such that $1 \in v \rightarrow y$, and so $x \otimes z \ll_1 y$.

Therefore, $\mathcal{H} = (H, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ is a hyper residuated lattice. $\blacksquare$

Definition 5.8. [2] By a *weak hyper residuated lattice*, we mean a non-empty set L endowed with two binary operations $\vee, \wedge$, and two binary hyperoperations $\otimes, \rightarrow$ and two constants 0 and 1 satisfying the following conditions:

(WHRL1) $(L, \leq, \vee, \wedge, 0, 1)$ is a bounded lattice;

(WHRL2) $(L, \otimes, 1)$ is a commutative semihypergroup with 1 as the identity;

(WHRL3) $x \otimes z \ll y$ if and only if $z \ll x \rightarrow y$;

where $A \ll B$ means that $x \leq y$, for some $x \in A$ and $y \in B$. Also, $A \leq B$ means that for any $x \in A$, there exists $y \in B$ such that $x \leq y$.

Proposition 5.9. [2] Let $\mathcal{L} = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ be a weak hyper residuated lattice. Then the following conditions hold, for all $x, y, z \in L$ and $A, B, C \subseteq L$:

- (i) $x \leq y$ implies $1 \in x \rightarrow y$ and $A \ll B$ implies $1 \in A \rightarrow B$;
- (ii) $A \ll B \rightarrow C$ if and only if $A \otimes B \ll C$ if and only if $B \ll A \rightarrow C$;
- (iii) $x \otimes y \ll x, y$ and $A \otimes B \ll A, B$;
- (iv) $x \ll y \rightarrow x, A \ll B \rightarrow A$ and $1 \in x \rightarrow (y \rightarrow x)$;
- (v) $x \otimes (x \rightarrow y) \ll x, y$;
- (vi) $x \leq y$ implies $x \otimes z \ll y \otimes z, z \rightarrow x \leq z \rightarrow y, y \rightarrow z \leq x \rightarrow z$;
- (vii) $(x \rightarrow y) \otimes (y \rightarrow z) \ll x \rightarrow z$ and $y \rightarrow z \ll (x \rightarrow y) \rightarrow (x \rightarrow z)$.

Proposition 5.10. Let $\mathcal{L} = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ be a weak hyper residuated lattice such that $A \ll B$ implies $A \leq B$ and " $\leftrightarrow$ " is biresiduation operation, which is defined by

$$x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x)$$

Then the following conditions holds, for all $x, y, z \in L$;

- (i) $x \leftrightarrow y \ll (z \leftrightarrow x) \rightarrow (z \leftrightarrow y)$;
- (ii) $x \leftrightarrow y \ll (z \leftrightarrow x) \leftrightarrow (z \leftrightarrow y)$;
- (iii) $(x \leftrightarrow y) \otimes (y \leftrightarrow z) \ll x \leftrightarrow z$;
- (iv) $(x \leftrightarrow y) \otimes (z \leftrightarrow t) \ll (x \leftrightarrow z) \leftrightarrow (y \leftrightarrow t)$;
- (v) $((x \wedge y) \leftrightarrow z) \otimes (t \leftrightarrow x) \ll (z \leftrightarrow (t \wedge y))$.

Proof. At first notice that by Proposition 5.9(vi) and assumption, we have $z \otimes (x \wedge y) \ll (z \otimes x) \wedge (z \otimes y)$ and if $A \ll B$ and $B \ll C$, then $A \ll C$.

(i): By Proposition 5.9(iii) and (vii),

$$\begin{aligned} & (x \leftrightarrow y) \otimes (z \leftrightarrow x) \\ &= ((x \rightarrow y) \wedge (y \rightarrow x)) \otimes ((z \rightarrow x) \wedge (x \rightarrow z)) \\ &\ll ((x \rightarrow y) \otimes (z \rightarrow x)) \wedge ((x \rightarrow y) \otimes (x \rightarrow z)) \wedge ((y \rightarrow x) \\ &\quad \otimes (z \rightarrow x)) \wedge ((y \rightarrow x) \otimes (x \rightarrow z)) \\ &\ll ((z \rightarrow y) \wedge ((x \rightarrow y) \otimes (x \rightarrow z))) \wedge ((y \rightarrow x) \otimes (z \rightarrow x)) \wedge (y \rightarrow z) \\ &\ll (z \rightarrow y) \wedge (y \rightarrow z) = y \leftrightarrow z \end{aligned}$$

Then by (WHRL3), $(x \leftrightarrow y) \ll (z \leftrightarrow x) \rightarrow (z \leftrightarrow y)$.

(ii): By (i), $x \leftrightarrow y \ll (z \leftrightarrow x) \rightarrow (z \leftrightarrow y)$ and $y \leftrightarrow x \ll (z \leftrightarrow y) \rightarrow (z \leftrightarrow x)$. Since $x \leftrightarrow y = y \leftrightarrow x$, then $x \leftrightarrow y \ll ((z \leftrightarrow x) \rightarrow (z \leftrightarrow y)) \wedge ((z \leftrightarrow y) \rightarrow (z \leftrightarrow x)) = ((z \leftrightarrow x) \leftrightarrow (z \leftrightarrow y))$.

(iii): By Proposition 5.9 (iii) and (vii),

$$\begin{aligned} & (x \leftrightarrow y) \otimes (y \leftrightarrow z) \\ &= ((x \rightarrow y) \wedge (y \rightarrow x)) \otimes ((y \rightarrow z) \wedge (z \rightarrow y)) \\ &\ll ((x \rightarrow y) \otimes (y \rightarrow z)) \wedge ((x \rightarrow y) \otimes (z \rightarrow y)) \wedge ((y \rightarrow x) \\ &\quad \otimes (y \rightarrow z)) \wedge ((y \rightarrow x) \otimes (z \rightarrow y)) \\ &\ll (x \rightarrow z) \wedge ((x \rightarrow y) \otimes (z \rightarrow y)) \wedge ((y \rightarrow x) \otimes (y \rightarrow z)) \wedge (z \rightarrow x) \\ &\ll (x \rightarrow z) \wedge (z \rightarrow x) = (x \leftrightarrow z). \end{aligned}$$

(iv): By (iii),

$$(x \leftrightarrow y) \otimes (z \leftrightarrow t) \otimes (x \leftrightarrow z) \ll (x \leftrightarrow y) \otimes (x \leftrightarrow t) = (t \leftrightarrow x) \otimes (x \leftrightarrow y) \\ \ll t \leftrightarrow y = y \leftrightarrow t$$

Then by (WHRL3), $(x \leftrightarrow y) \otimes (z \leftrightarrow t) \ll (x \leftrightarrow z) \rightarrow (y \leftrightarrow t)$. By the same way we have, $(x \leftrightarrow y) \otimes (z \leftrightarrow t) \ll (y \leftrightarrow t) \rightarrow (x \leftrightarrow z)$. Hence $(x \leftrightarrow y) \otimes (z \leftrightarrow t) \ll (x \leftrightarrow z) \leftrightarrow (y \leftrightarrow t)$.

(v): First, we show that:

$$((x \wedge y) \rightarrow z) \otimes (t \rightarrow x) \ll (t \wedge y) \rightarrow z \quad (1)$$

$$(z \rightarrow (x \wedge y)) \otimes (x \rightarrow t) \ll z \rightarrow (t \wedge y) \quad (2)$$

(1): By Proposition 5.9(iii) and (v),

$$(t \wedge y) \otimes (t \rightarrow x) \ll (t \otimes (t \rightarrow x)) \wedge (y \otimes (t \rightarrow x)) \ll x \wedge y.$$

Then by Proposition 5.9(v) and (vi),

$$(t \wedge y) \otimes ((x \wedge y) \rightarrow z) \otimes (t \rightarrow x) \ll (x \wedge y) \otimes (x \wedge y \rightarrow z) \ll z,$$

and so, $((x \wedge y) \rightarrow z) \otimes (t \rightarrow x) \ll (t \wedge y) \rightarrow z$.

By the similar way, we can prove (2).

Now the same as (iii) and (iv), we can prove (v). ■

Theorem 5.11. *Let $\mathcal{L} = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ be a weak hyper residuated lattice such that $A \ll B$ means $A \leq B$. Then $\mathcal{L} = (L, \wedge, \otimes, \leftrightarrow, 1)$ is a separated hyper EQ-algebra.*

Proof. (HEQ1): By Definition 5.8, it is clear that $(L, \wedge, 1)$ is an idempotent commutative monoid ($x \leq y$ if and only if $x \wedge y = x$).

(HEQ2): By Definition 5.8, $(L, \otimes, 1)$ is a commutative semihypergroup with identity "1" and by Proposition 5.9(vi), $\otimes$ is isotone.

Let $x \sim y := x \leftrightarrow y$, for all $x, y \in H$. Then

(HEQ3): Since $((x \wedge y) \sim z) \otimes (t \sim x) = ((x \wedge y) \leftrightarrow z) \otimes (t \leftrightarrow x)$. Then by Proposition 5.10(v), $((x \wedge y) \leftrightarrow z) \otimes (t \leftrightarrow x) \ll (z \leftrightarrow (t \wedge y))$. Then

$$((x \wedge y) \sim z) \otimes (t \sim x) \ll (z \leftrightarrow (t \wedge y)) = (z \sim (t \sim y)).$$

(HEQ4): By Proposition 5.10(iv),

$$(x \sim y) \otimes (z \sim t) = (x \leftrightarrow y) \otimes (z \leftrightarrow t) \ll (x \leftrightarrow z) \leftrightarrow (y \leftrightarrow t) = (x \sim z) \sim (y \sim t)$$

(HEQ5): Since $x \wedge y \leq x$, then by Proposition 5.9(i), $1 \in ((x \wedge y) \rightarrow x)$. Hence by definition $\ll$, we have $(x \wedge y \wedge z) \rightarrow x \ll (x \wedge y) \rightarrow x$. By Proposition 5.9(vi), $x \rightarrow (x \wedge y \wedge z) \ll x \rightarrow (x \wedge y)$. Hence,

$$(x \wedge y \wedge z) \sim x = (x \wedge y \wedge z) \leftrightarrow x = ((x \wedge y \wedge z) \rightarrow x) \wedge (x \rightarrow (x \wedge y \wedge z)) \\ \ll ((x \wedge y) \rightarrow x) \wedge (x \rightarrow (x \wedge y)) = (x \wedge y) \leftrightarrow x \\ = (x \wedge y) \sim x$$

(HEQ6): Since $x \wedge y \wedge z \leq x \wedge z$, then by Proposition 5.9(i), $1 \in (x \wedge y \wedge z) \rightarrow x \wedge z$. Hence $((x \wedge y) \rightarrow x) \ll ((x \wedge y \wedge z) \rightarrow (x \wedge z))$. By Proposition 5.9 (iii) and (v),

$$(x \rightarrow (x \wedge y)) \otimes (x \wedge z) \ll (x \otimes (x \rightarrow (x \wedge y))) \wedge (z \otimes (x \rightarrow (x \wedge y))) \ll x \wedge y \wedge z$$

Then by (WHRL3), $(x \rightarrow (x \wedge y)) \ll (x \wedge z) \rightarrow (x \wedge y \wedge z)$. Hence

$$\begin{aligned} (x \wedge y) \sim x &= (x \wedge y) \leftrightarrow x = (((x \wedge y) \rightarrow x)) \wedge (x \rightarrow (x \wedge y)) \\ &\ll (((x \wedge y \wedge z) \rightarrow (x \wedge z)) \wedge ((x \wedge z) \rightarrow (x \wedge y \wedge z))) \\ &= (x \wedge y \wedge z) \leftrightarrow (x \wedge z) \\ &= (x \wedge y \wedge z) \sim (x \wedge z) \end{aligned}$$

(HEQ7): By Proposition 5.9(iii) and (iv),

$$x \otimes y \leq x \wedge y \ll (x \rightarrow y) \wedge (y \rightarrow x) = x \leftrightarrow y = x \sim y.$$

Therefore, $\mathcal{L}$ is a hyper EQ - algebra.

Let $1 \in x \sim y = x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x)$, then $1 \in x \rightarrow y$ and $1 \in y \rightarrow x$. By (WHRL2) and Proposition 5.9(v), $x \in x \otimes 1 \ll x \otimes (x \rightarrow y) \ll y$, then $x \leq y$. By the same way we have $y \leq x$, thus $x = y$. Therefore, $\mathcal{L}$ is separated hyper EQ -algebra. ■

6. Conclusions and future works

In this paper, we introduce the concept of hyper EQ -algebras, which is a generalization of the concept of EQ -algebras and we give some properties and related results and relation between hyper EQ -algebra and hyper BCK -algebra and hyper (weak)residuated lattice. The good hyper EQ -algebra, some result on quotient structure and filter theory, could be topics for my next task.

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ANTIPODAL BIPOLAR FUZZY GRAPHS

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Abstract. The concept of an antipodal bipolar fuzzy graph of a given bipolar fuzzy graph is introduced. Characterizations of antipodal bipolar fuzzy graphs are presented when the bipolar fuzzy graph is complete or strong. Some isomorphic properties of antipodal bipolar fuzzy graph are discussed. The notion of self median bipolar fuzzy graphs of a given bipolar fuzzy graph is also introduced.

Keywords and phrases: antipodal bipolar fuzzy graphs, median bipolar fuzzy graphs, self median bipolar fuzzy graphs.

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1. Introduction

Concepts of graph theory have applications in many areas of computer science (such as data mining, image segmentation, clustering, image capturing, networking etc.). For examples, a data structure can be designed in the form of trees, modeling of network topologies can be done using graph concepts. The most important

concept of graph coloring is utilized in resource allocation and scheduling. The concepts of paths, walks and circuits in graph theory are used in traveling salesman problem, database design concepts, and resource networking. This leads to the development of new algorithms and new theorems that can be used in tremendous applications.

A notion having certain influence on graph theory is fuzzy set, which is introduced by Zadeh [17] in 1965; actually, the theory of fuzzy sets has already become a vigorous research area which intersects with many research areas, such as medical and life sciences, management sciences, social sciences, engineering, statistics, graph theory, artificial intelligence, signal processing, multi-agent systems, pattern recognition, robotics, computer networks, expert systems, decision making and automata theory, etc. A fuzzy set on a given set X is just a mapping $A : X \rightarrow [0, 1]$. Based on the same idea, Zhang [20] defined the notion of bipolar fuzzy set on a given set X in 1994, which is just a mapping $A : X \rightarrow [-1, 1]$, where the membership degree 0 of an element x means that the element x is irrelevant to the corresponding property, the membership degree in $(0, 1]$ of an element x indicates that the element somewhat satisfies the property, and the membership degree in $[-1, 0)$ of an element x indicates that the element somewhat satisfies the implicit counter-property.

In 1975, Rosenfeld [13] discussed the concept of fuzzy graph whose basic idea was introduced by Kauffmann [11] in 1973. By considering fuzzy relations between fuzzy sets and developing structure of fuzzy graphs, Rosenfeld obtained analogs of several graph theoretical concepts. Bhattacharya [9] gave some remarks on fuzzy graphs. The complement of a fuzzy graph was defined by Mordeson [12] and further studied by Sunitha and Vijayakumar [15]. Ahmed and Gani discussed the concepts of perfect fuzzy graph and self median fuzzy graph in [7]. Based on the notion of intuitionistic fuzzy set [5], Atanassov [5] introduced the concepts of intuitionistic fuzzy relation and intuitionistic fuzzy graphs. Recently, the bipolar fuzzy graphs have been defined and discussed in [1-3] based on the notion of bipolar fuzzy set. The present paper continues to study bipolar fuzzy graphs. We introduce the concepts of antipodal bipolar fuzzy graph and self median bipolar fuzzy graph of a given bipolar fuzzy graph, and prove several characterizations theorems of antipodal bipolar fuzzy graphs whose bipolar fuzzy graph are complete or strong. We also discuss isomorphic properties of antipodal bipolar fuzzy graph.

2. Preliminaries

In this section, we review some elementary concepts whose understanding is necessary fully benefit from this paper.

By a graph $G^* = (V, E)$, we mean a non-trivial, finite, connected and undirected graph without loops or multiple edges. Formally, given a graph $G^* = (V, E)$, two vertices $x, y \in V$ are said to be *neighbors*, or *adjacent nodes*, if $\{x, y\} \in E$. The *antipodal graph* of a graph G , denoted by $A(G^*)$, has the same vertex set as G^* with an edge joining vertices u and v if $d(u, v)$ is equal to the

diameter of G^* . For a graph G^* of order p , the antipodal graph $A(G^*) = G^*$ if and only if $G^* = K_p$. If G^* is a non-complete graph of order p , then $A(G^*) \subset \overline{G^*}$. For a graph G^* , the antipodal graph $A(G^*) = \overline{G^*}$ if and only if (a) G^* is of diameter 2 or (b) G^* is disconnected and the components of G^* are complete graphs. A graph G^* is an antipodal graph if and only if it is the antipodal graph of its complement. A graph G^* is an antipodal graph if and only if (i) $\text{diam}(\overline{G^*}) = 2$ or (ii) $\overline{G^*}$ is disconnected and the components of $\overline{G^*}$ are complete graphs. The self median fuzzy graphs were introduced by Ahmed and Gani in [7]. The median of a graph G^* is the set of all vertices v of G^* for which the value $d_G^*(v)$ is minimized. A graph is distance-balanced (also called self-median) if its median is the whole vertex set. Thus, a graph G^* is self-median if and only if the value $d_G^*(v)$ is constant over all vertices v of G^* . The status, or distance sum, of a given vertex v in a graph is defined by $s(v) = \sum_{u \neq v} d(u, v)$, where $d(u, v)$ is the distance from a vertex u to v . In

other words, a self median graph G^* is one in which all the nodes have the same status $s(v)$. The graphs C_n , $K_{n,n}$ and K_n are self median. The status of a vertex v_i is denoted by $S(v_i)$ and is defined as $S(v_i) = \sum_{\forall v_j \in V} \delta(v_i, v_j)$. The total status of a fuzzy graph G^* is denoted by $t[S(G^*)]$ and is defined as $t[S(G^*)] = \sum_{\forall v_i \in V} S(v_i)$.

The median of a fuzzy graph G^* , denoted, is the set of nodes with minimum status. A fuzzy graph G^* is said to be self-median if all the vertices have the same status. Every self-median fuzzy graph is a self centered fuzzy graph. Every cube Q_n is self-median fuzzy graph.

Definition 2.1 [17, 18] A *fuzzy subset* μ on a set X is a map $\mu : X \rightarrow [0, 1]$. A *fuzzy binary relation* on X is a fuzzy subset μ on $X \times X$. By a fuzzy relation we mean a fuzzy binary relation given by $\mu : X \times X \rightarrow [0, 1]$.

Definition 2.2 [20] Let X be a nonempty set. A *bipolar fuzzy set* B in X is an object having the form

$$B = \{(x, \mu_B^P(x), \mu_B^N(x)) \mid x \in X\}$$

where $\mu_B^P : X \rightarrow [0, 1]$ and $\mu_B^N : X \rightarrow [-1, 0]$ are mappings.

We use the positive membership degree $\mu_B^P(x)$ to denote the satisfaction degree of an element x to the property corresponding to a bipolar fuzzy set B , and the negative membership degree $\mu_B^N(x)$ to denote the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar fuzzy set B . If $\mu_B^P(x) \neq 0$ and $\mu_B^N(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for B . If $\mu_B^P(x) = 0$ and $\mu_B^N(x) \neq 0$, it is the situation that x does not satisfy the property of B but somewhat satisfies the counter property of B . It is possible for an element x to be such that $\mu_B^P(x) \neq 0$ and $\mu_B^N(x) \neq 0$ when the membership function of the property overlaps that of its counter property over some portion of X .

For the sake of simplicity, we shall use the symbol $B = (\mu_B^P, \mu_B^N)$ for the bipolar fuzzy set

$$B = \{(x, \mu_B^P(x), \mu_B^N(x)) \mid x \in X\}.$$

A nice application of bipolar fuzzy concept is a political acceptance (map to $[0, 1]$) and non-acceptation (map to $[-1, 0]$).

Definition 2.3 [20] Let X be a nonempty set. Then, we call a mapping $A = (\mu_A^P, \mu_A^N) : X \times X \rightarrow [0, 1] \times [-1, 0]$ a *bipolar fuzzy relation* on X such that $\mu_A^P(x, y) \in [0, 1]$ and $\mu_A^N(x, y) \in [-1, 0]$.

Definition 2.4 [1] Let $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$ be bipolar fuzzy sets on a set X . If $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy relation on a set X , then $A = (\mu_A^P, \mu_A^N)$ is called a *bipolar fuzzy relation* on $B = (\mu_B^P, \mu_B^N)$ if $\mu_A^P(x, y) \leq \min(\mu_B^P(x), \mu_B^P(y))$ and $\mu_A^N(x, y) \geq \max(\mu_B^N(x), \mu_B^N(y))$ for all $x, y \in X$. A bipolar fuzzy relation A on X is called *symmetric* if $\mu_A^P(x, y) = \mu_A^P(y, x)$ and $\mu_A^N(x, y) = \mu_A^N(y, x)$ for all $x, y \in X$.

Definition 2.5 [3] Let G be a connected bipolar fuzzy graph. The μ^P -distance, $\delta_{\mu^P}(v_i, v_j)$, is the smallest μ^P -length of any $v_i - v_j$ path P in G , where $v_i, v_j \in V$. That is, $\delta_{\mu^P}(v_i, v_j) = \min(l_{\mu^P}(P))$. The μ^N -distance, $\delta_{\mu^N}(v_i, v_j)$, is the largest μ^N -length of any $v_i - v_j$ path P in G , where $v_i, v_j \in V$. That is, $\delta_{\mu^N}(v_i, v_j) = \max(l_{\mu^N}(P))$. The distance, $\delta(v_i, v_j)$, is defined as $\delta(v_i, v_j) = (\delta_{\mu^P}(v_i, v_j), \delta_{\mu^N}(v_i, v_j))$. For each $v_i \in V$, the μ^P -eccentricity of v_i , denoted by $e_{\mu^P}(v_i)$ and is defined as $e_{\mu^P}(v_i) = \max\{\delta_{\mu^P}(v_i, v_j) : v_i \in V, v_i \neq v_j\}$. For each $v_i \in V$, the μ^N -eccentricity of v_i , denoted by $e_{\mu^N}(v_i)$ and is defined as $e_{\mu^N}(v_i) = \min\{\delta_{\mu^N}(v_i, v_j) : v_i \in V, v_i \neq v_j\}$. For each $v_i \in V$, the eccentricity of v_i , denoted by $e(v_i)$ and is defined as $e(v_i) = (e_{\mu^P}(v_i), e_{\mu^N}(v_i))$. The μ^P -radius of G is denoted by $r_{\mu^P}(G)$ and is defined as $r_{\mu^P}(G) = \min\{e_{\mu^P}(v_i) : v_i \in V\}$. The μ^N -radius of G is denoted by $r_{\mu^N}(G)$ and is defined as $r_{\mu^N}(G) = \max\{e_{\mu^N}(v_i) : v_i \in V\}$. The radius of G is denoted by $r(G)$ and is defined as $r(G) = (r_{\mu^P}(G), r_{\mu^N}(G))$. The μ^P -diameter of G is denoted by $d_{\mu^P}(G)$ and is defined as $d_{\mu^P}(G) = \max\{e_{\mu^P}(v_i) : v_i \in V\}$. The μ^N -diameter of G is denoted by $d_{\mu^N}(G)$ and is defined as $d_{\mu^N}(G) = \min\{e_{\mu^N}(v_i) : v_i \in V\}$. The diameter of G is denoted by $d(G)$ and is defined as $d(G) = (d_{\mu^P}(G), d_{\mu^N}(G))$. A connected bipolar fuzzy graph G is a self centered graph, if every vertex of G is a central vertex, that is $r_{\mu^P}(G) = e_{\mu^P}(v_i)$ and $r_{\mu^N}(G) = e_{\mu^N}(v_i)$, $\forall v_i \in V$.

3. Antipodal bipolar fuzzy graphs

Definition 3.1 Let $G = (A, B)$ be a bipolar fuzzy graph. An *antipodal bipolar fuzzy graph* $A(G) = (E, F)$ is a bipolar fuzzy graph $G = (A, B)$ in which:

- (a) An bipolar fuzzy vertex set of G is taken as bipolar fuzzy vertex set of $A(G)$, that is, $\mu_E^P(x) = \mu_A^P(x)$ and $\mu_E^N(x) = \mu_A^N(x)$ for all $x \in V$,

(b) If $\delta(x, y) = d(G)$, then

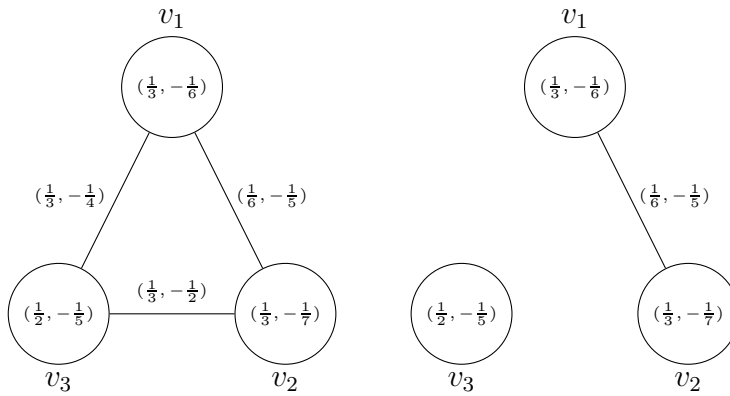
$$\mu_F^P(xy) = \begin{cases} \mu_B^P(xy) & \text{if } x \text{ and } y \text{ are neighbors in } G, \\ \min(\mu_A^P(x), \mu_A^P(y)) & \text{if } x \text{ and } y \text{ are not neighbors in } G, \end{cases}$$

$$\mu_F^N(xy) = \begin{cases} \mu_B^N(xy) & \text{if } x \text{ and } y \text{ are neighbors in } G, \\ \max(\mu_A^N(x), \mu_A^N(y)) & \text{if } x \text{ and } y \text{ are not neighbors in } G. \end{cases}$$

That is, two vertices in $A(G)$ are made as neighborhood if the $\mu^P\mu^N$ -distance between them is diameter of G .

Example 3.2 Consider a bipolar fuzzy graph G such that

$$A = \{v_1, v_2, v_3\}, \quad B = \{v_1v_2, v_1v_3, v_2v_3\}.$$



Bipolar Fuzzy Graph Antipodal Bipolar Fuzzy Graph

Figure 1

By routine calculations, we have,

$$\begin{aligned} \delta_{\mu^P}(v_1, v_2) &= 6, & \delta_{\mu^P}(v_1, v_3) &= 3, & \delta_{\mu^P}(v_2, v_3) &= 3, \\ \delta_{\mu^N}(v_1, v_2) &= -6, & \delta_{\mu^N}(v_1, v_3) &= -7, & \delta_{\mu^N}(v_2, v_3) &= -9, \\ e_{\mu^P}(v_1) &= 6, & e_{\mu^P}(v_2) &= 6, & e_{\mu^P}(v_3) &= 3, \\ e_{\mu^N}(v_1) &= -6, & e_{\mu^N}(v_2) &= -6, & e_{\mu^N}(v_3) &= -7, \\ d(G) &= (6, -6), & \delta(v_1, v_2) &= (6, -6) = d(G). \end{aligned}$$

Hence $A(G) = (E, F)$, such that $E = \{v_1, v_2, v_3\}$ and $F = \{v_1v_2\}$.

Theorem 3.3 Let $G = (A, B)$ be a complete bipolar fuzzy graph where (μ_A^P, μ_A^N) is constant function then G is isomorphic to $A(G)$.

Proof. Given that $G = (A, B)$ be a complete bipolar fuzzy graph with $(\mu_1^P, \mu_1^N) = (k_1, k_2)$, where k_1 and k_2 are constants, which implies that $\delta(v_i, v_j) = (l_1, l_2)$, $\forall v_i, v_j \in V$. Therefore, eccentricity $e(v_i) = (l_1, l_2)$, $\forall v_i \in V$, which implies that $d(G) = (l_1, l_2)$. Hence $\delta(v_i, v_j) = (l_1, l_2) = d(G)$, $\forall v_i, v_j \in V$. Hence every pair of vertices are made as neighbors in $A(G)$ such that

- (a) An bipolar fuzzy vertex set of G is taken as bipolar fuzzy vertex set of $A(G)$, that is, $\mu_E^P(v_i) = \mu_A^P(v_i)$ and $\mu_E^N(v_i) = \mu_A^N(v_i)$ for all $v_i \in V$,
- (b) $\mu_F^P(v_i v_j) = \mu_B^P(v_i v_j)$, since v_i and v_j are neighbors in G
 $\mu_F^N(v_i v_j) = \mu_B^N(v_i v_j)$, since v_i and v_j are neighbors in G .

It has same number of vertices, edges and it preserves degrees of the vertices. Hence $G \cong A(G)$. ■

Theorem 3.4 *Let $G : (A, B)$ is a connected bipolar fuzzy graph. Every antipodal bipolar fuzzy graph is spanning subgraph of G .*

Proof. By the definition of an antipodal bipolar fuzzy graph, $A(G)$ contains all the vertices of G . That is,

- (a) $\mu_E^P(x) = \mu_A^P(x)$ and $\mu_E^N(x) = \mu_A^N(x)$ for all $x \in V$, and
- (b) If $\delta(x, y) = d(G)$, then

$$\mu_F^P(xy) = \begin{cases} \mu_B^P(xy) & \text{if } x \text{ and } y \text{ are neighbors in } G, \\ \min(\mu_A^P(x), \mu_A^P(y)) & \text{if } x \text{ and } y \text{ are not neighbors in } G, \end{cases}$$

$$\mu_F^N(xy) = \begin{cases} \mu_B^N(xy) & \text{if } x \text{ and } y \text{ are neighbors in } G, \\ \max(\mu_A^N(x), \mu_A^N(y)) & \text{if } x \text{ and } y \text{ are not neighbors in } G. \end{cases}$$

Hence $A(G)$ is spanning subgraph of G . ■

Theorem 3.5 *Let G be a bipolar fuzzy graph, where crisp graph G is an even or odd cycle. If alternate edges have same membership values and non-membership values, then G is self centered bipolar fuzzy graph.*

Theorem 3.6 *Let G be a bipolar fuzzy graph, where crisp graph G^* is an even or odd cycle. If alternate edges have same positive and negative values, then $A_{\mu^P}(G)$ and $A_{\mu^N}(G)$ is the edge induced bipolar fuzzy subgraph of $\bar{G}$, whose end vertices of $A_{\mu^P}(G)$ and $A_{\mu^N}(G)$ are with maximum μ^P -eccentricity and minimum μ^N -eccentricity in G .*

Proof. If alternate edges have same positive and negative values, then μ^P -distance between non-adjacent vertices is greater than the adjacent vertices and μ^N -distance between non-adjacent vertices is lesser than the adjacent vertices. Let $\mu_B^P(v_i, v_j)$ be the least among all other edges, then

$$\delta_{\mu^P}(v_i, v_j) = \frac{1}{\mu_B^P(v_i, v_j)}.$$

Claim (i): Neighbors in G are not neighbors in $A(G)$.

Consider an arbitrary path connecting v_k, v_t such that

$$(1) \quad (v_k, v_t) \notin \mu_{B^*}^P$$

If P is a path of length 2 between v_k, v_t , then

$$(2) \quad l_{\mu^P}(P) \geq \frac{1}{\mu_B^P(v_i, v_j)}$$

Hence

$$\delta(v_k, v_t) \geq \frac{1}{\mu_B^P(v_i, v_j)},$$

since by equation (1) and (2), which implies that $\delta(v_i, v_j) < \delta(v_k, v_t) \leq d(G)$, where $\delta(v_k, v_t) \notin \mu_{B^*}^P$ and $(v_i, v_j) \in \mu_{B^*}^P$. That is, $\delta(v_i, v_j) < d(G)$, if $(v_i, v_j) \in \mu_{B^*}^P$. Therefore, if $(v_i, v_j) \in \mu_{B^*}^P$, then v_i and v_j are not neighbors in $A(G)$.

Claim (ii): Edges in $A(G)$ are edges in $\bar{G}$.

If $(v_m, v_n) \in \mu_{F^*}^P$, then by Claim (i), $(v_i, v_j) \notin \mu_{B^*}^P$. So,

$$\mu_F^P(v_m, v_n) = \min(\mu_A^P(v_m), \mu_A^P(v_n)),$$

since by the definition of $A(G)$, which implies that edges in $A(G)$ are edges in $\bar{G}$. Hence $A(G)$ is a bipolar fuzzy subgraph of $\bar{G}$, induced by the edges of $\bar{G}$, whose end vertices are with maximum μ^P eccentricity in G .

Let $(v_i, v_j) \in E$, then $\delta_{\mu^N}(v_i, v_j) = k$.

Claim (i): Neighbors in G are not neighbors in $A(G)$.

Consider an arbitrary path connecting v_k, v_t such that

$$(3) \quad (v_k, v_t) \notin \mu_{B^*}^N$$

If Q is a path of length 2 between v_k, v_t , then

$$(4) \quad \delta_{\mu^N}(P) \leq k$$

Hence

$$\delta_{\mu^N}(v_k, v_t) \leq \delta_{\mu^N}(v_i, v_j),$$

since by equation (3) and (4), which implies that $\delta_{\mu^N}(v_k, v_t) \leq d(G)$, where $(v_k, v_t) \notin \mu_{B^*}^N$ and $(v_i, v_j) \in \mu_{B^*}^N$. That is, $\delta_{\mu^N}(v_i, v_j) \geq d(G)$, if $(v_i, v_j) \in \mu_{B^*}^N$. Therefore, if $(v_i, v_j) \in \mu_{B^*}^N$, then v_i and v_j are not neighbors in $A(G)$.

Claim (ii): Edges in $A(G)$ are edges in $\bar{G}$.

If $(v_m, v_n) \in \mu_{F^*}^N$, then by Claim (i), $(v_i, v_j) \notin \mu_{B^*}^N$. So,

$$\mu_F^N(v_m, v_n) = \max(\mu_A^N(v_m), \mu_A^N(v_n)),$$

since by the definition of $A(G)$, which implies that edges in $A_{\mu^N}(G)$ are edges in $\bar{G}$. Hence $A_{\mu^N}(G)$ is a bipolar fuzzy subgraph of $\bar{G}$, induced by the edges of $\bar{G}$, whose end vertices are with minimum μ^N eccentricity in G . ■

Theorem 3.7 *Let G be a bipolar fuzzy graph, where crisp graph G^* is an even or odd cycle. If alternate edges have same positive and negative values, then $A(G)$ is a bipartite bipolar fuzzy graph.*

Theorem 3.8 *Let $G : (A, B)$ be a connected strong bipolar fuzzy graph, where crisp graph G^* is an even or odd cycle, such that $(\mu_A^P, \mu_A^N)(v_i) = (k_1, k_2)$, $\forall v_i \in \mu_A^P$ and $\forall v_i \in \mu_A^N$. Then $A(G)$ is the spanning bipolar fuzzy graph subgraph of $\bar{G}$, induced by the edges of $\bar{G}$, whose end vertices are maximum μ^P -eccentricity and minimum μ^N -eccentricity in G .*

Proof. Let $(v_i, v_j) \in \mu_{B^*}^P$, $\delta_{\mu^P}(v_i, v_j) = \frac{1}{k_1}$. But for any $(v_i, v_j) \notin \mu_{B^*}^P$, $\delta(v_k, v_m) \geq \frac{2}{k_1}$. That is $\delta_{\mu^P}(v_i, v_j) = \frac{1}{k_1} < \frac{2}{k_1} \leq \delta_{\mu^P}(v_k, v_m)$, where $(v_i, v_j) \notin \mu_{B^*}^P$, which implies that v_i, v_j are vertices in $A(G)$, but are not neighbors in $A(G)$. Now, let $(v_i, v_j) \in \mu_{B^*}^N$, $\delta_{\mu^N}(v_i, v_j) > \frac{1}{k_1}$. But for any $(v_i, v_j) \notin \mu_{B^*}^N$, $\delta_{\mu^N}(v_i, v_j) \geq \delta_{\mu^N}(v_k, v_m)$, where $(v_i, v_j) \notin \mu_{B^*}^N$, which implies that v_i, v_j are vertices in $A(G)$, but are not neighbors in $A(G)$. The remaining proof is similar to claim (ii) of above Theorem and hence we omit it. ■

Theorem 3.9 *If G_1 and G_2 are isomorphic to each other, then $A(G_1)$ and $A(G_2)$ are also isomorphic.*

Proof. As G_1 and G_2 are isomorphic, the isomorphism h , between them preserves the edge weights, so the $\mu^P \mu^N$ -length and $\mu^P \mu^N$ -distance will also be preserved. Hence, if the vertex v_i has the maximum μ^P -eccentricity and minimum μ^N -eccentricity, in G_1 , then $h(v_i)$ has the maximum μ^P -eccentricity and minimum μ^N -eccentricity, in G_2 . So, G_1 and G_2 will have the same diameter. If the $\mu^P \mu^N$ -distance between v_i and v_j is (k_1, k_2) in G_1 , then $h(v_i)$ and $h(v_j)$ will also have their $\mu^P \mu^N$ -distance as (k_1, k_2) . The same mapping h itself is a bijection between $A(G_1)$ and $A(G_2)$ satisfying the isomorphism condition.

- (i) $\mu_{E_1}^P(v_i) = \mu_{A_1}^P(v_i) = \mu_{A_2}^P(h(v_i)) = \mu_{E_2}^P(h(v_i)), \forall v_i \in G_1$
- (ii) $\mu_{E_1}^N(v_i) = \mu_{A_1}^N(v_i) = \mu_{A_2}^N(h(v_i)) = \mu_{E_2}^N(h(v_i)), \forall v_i \in G_1$
- (iii) $\mu_{F_1}^P(v_i, v_j) = \mu_{B_1}^P(v_i, v_j)$, if v_i and v_j are neighbors in G_1
 $\mu_{F_1}^P(v_i, v_j) = \min(\mu_{E_1}^P(v_i), \mu_{E_1}^P(v_j))$, if v_i and v_j are not neighbors in G_1
 and
- (iv) $\mu_{F_1}^N(v_i, v_j) = \mu_{B_1}^N(v_i, v_j)$, if v_i and v_j are neighbors in G_1
 $\mu_{F_1}^N(v_i, v_j) = \max(\mu_{E_1}^N(v_i), \mu_{E_1}^N(v_j))$, if v_i and v_j are not neighbors in G_1

As $h : G_1 \rightarrow G_2$ is an isomorphism,

$$\begin{aligned} \mu_{F_1}^N(v_i, v_j) &= \mu_{B_2}^N(h(v_i), h(v_j)), \text{ if } v_i \text{ and } v_j \text{ are neighbors in } G_1 \\ \mu_{F_1}^N(v_i, v_j) &= \max(\mu_{B_2}^N(v_i), \mu_{B_2}^N(v_j)), \text{ if } v_i \text{ and } v_j \text{ are not neighbors in } G_1 \\ \mu_{F_1}^N(v_i, v_j) &= \mu_{B_2}^N(h(v_i), h(v_j)), \text{ if } v_i \text{ and } v_j \text{ are neighbors in } G_1 \\ \mu_{F_1}^N(v_i, v_j) &= \max(\mu_{B_2}^N(v_i), \mu_{B_2}^N(v_j)), \text{ if } v_i \text{ and } v_j \text{ are not neighbors in } G_1 \end{aligned}$$

Hence $\mu_{F_1}^P(v_i, v_j) = \mu_{F_2}^P(h(v_i), h(v_j))$ and $\mu_{F_1}^N(v_i, v_j) = \mu_{F_2}^N(h(v_i), h(v_j))$. So, the same h is an isomorphism between $A(G_1)$ and $A(G_2)$. ■

Theorem 3.10 *If G_1 and G_2 are complete bipolar fuzzy graph such that G_1 is co-weak isomorphic to G_2 then $A(G_1)$ is co-weak isomorphic to $A(G_2)$.*

Proof. As G_1 is co-weak isomorphic to G_2 , there exists a bijection $h : G_1 \rightarrow G_2$ satisfying, $\mu_A^P(v_i) \leq \mu_A^P(h(v_i))$, $\mu_B^P(v_i, v_j) = \mu_B^P(h(v_i), h(v_j))$, $\forall v_i, v_j \in V_1$. If G_1 has n vertices, arrange the vertices of G_1 in such a way that $\mu_A^P(v_1) \leq \mu_A^P(v_2) \leq \mu_A^P(v_3) \dots \mu_A^P(v_n)$. As G_1 and G_2 are complete, co-weak isomorphic bipolar fuzzy graph, $\mu_B^P(v_i, v_j) = \mu_B^P(h(v_i), h(v_j))$, $\forall v_i, v_j \in V_1$. By Theorem 3.9 and the definition of antipodal bipolar fuzzy graph, we have $A(G_i)$ contains all the vertices of G , where $i = 1, 2$. That is, $\mu_E^P(x) = \mu_A^P(x)$ and $\mu_E^N(x) = \mu_A^N(x)$ for all $x \in V$ and $\mu_F^P(v_i, v_j) = \mu_F^P(h(v_i), h(v_j))$, $\forall v_i, v_j \in V_1$. So, the same bijection h is a co-weak isomorphism between $A(G_1)$ and $A(G_2)$. ■

We state the following Theorem without its proof.

Theorem 3.11 *If G_1 and G_2 are complete bipolar fuzzy graph such that G_1 is co-weak isomorphic to G_2 then $A(G_1)$ is homomorphic to $A(G_2)$.*

Remark 1 If G is a self complementary bipolar fuzzy graph, then its antipodal bipolar fuzzy graph may not be self complementary.

Example 3.12 Consider a bipolar fuzzy graph G

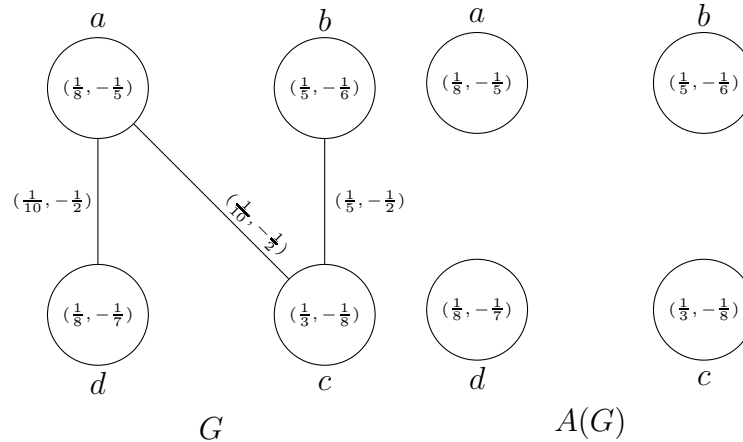


Figure 2

By routine calculations, we have

$$\begin{aligned} \delta(a, b) &= (15, -4), & \delta(a, c) &= (10, -2), & \delta(a, d) &= (10, -2), & \delta(b, c) &= (5, -2), \\ \delta(b, d) &= (25, -6), & \delta(c, d) &= (20, -4), & e(a) &= (15, -2), & e(b) &= (15, -2), \\ e(c) &= (20, -2), & e(d) &= (25, -2), & d(G) &= (25, -2). \end{aligned}$$

Since $d(G) \neq \delta(x, y)$ for all $x, y \in V$. Hence $A(G)$ is an antipodal bipolar fuzzy graph of G having same vertices as in G only, and no two vertices in $A(G)$ are made as neighborhood since their $\mu^P \mu^N$ -distance between them is not equal to the diameter of G .

Consider a bipolar fuzzy graph $\overline{G}$

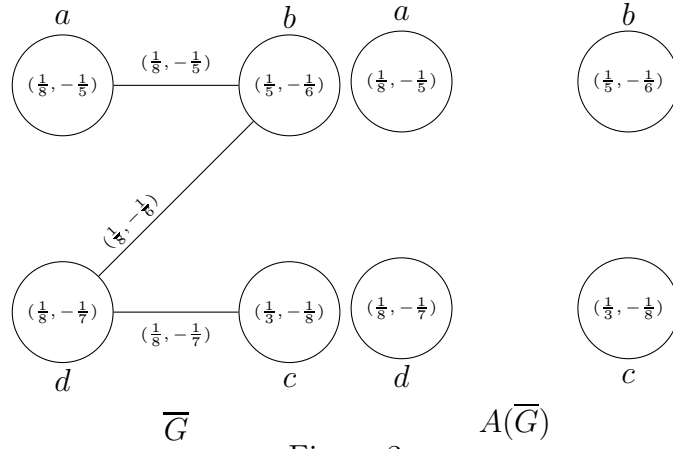


Figure 3

By routine calculations, we have

$$\begin{aligned} \delta(a, b) &= (8, -5), \quad \delta(a, c) = (24, -18), \quad \delta(a, d) = (16, -11), \quad \delta(b, c) = (16, -13), \\ \delta(b, d) &= (8, -6), \quad \delta(c, d) = (8, -7), \quad e(a) = (24, -5), \quad e(b) = (16, -5), \\ e(c) &= (24, -7), \quad e(d) = (16, -6), \quad d(\overline{G}) = (24, -5). \end{aligned}$$

Since $d(\overline{G}) \neq \delta(x, y)$ for all $x, y \in \overline{V}$. Hence $A(\overline{G})$ is an antipodal bipolar fuzzy graph of $\overline{G}$ having same vertices as in $\overline{G}$ only, and no two vertices in $A(\overline{G})$ are made as neighborhood since their $\mu^P \mu^N$ -distance between them is not equal to the diameter of $\overline{G}$.

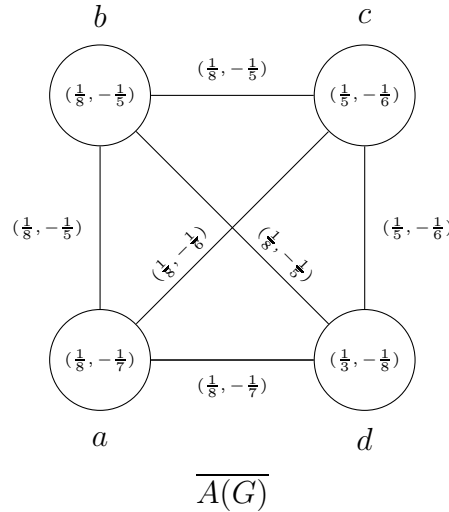


Figure 4

Clearly, $A(G)$ is not isomorphic to $\overline{A(G)}$ Hence G is self complementary, but its antipodal bipolar fuzzy graph $A(G)$ is not a self complementary bipolar fuzzy graph.

Example 3.13 Consider a bipolar fuzzy graph G

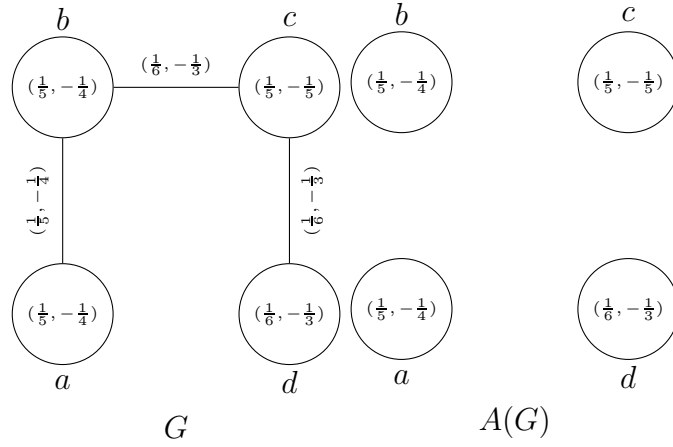


Figure 5

By routine calculations, we have

$$\begin{aligned} \delta(a, b) &= (5, -4), & \delta(a, c) &= (11, -7), & \delta(a, d) &= (17, -10), & \delta(b, c) &= (6, -3), \\ \delta(b, d) &= (12, -6), & \delta(c, d) &= (6, -3), & e(a) &= (17, -4), & e(b) &= (12, -3), \\ e(c) &= (11, -3), & e(d) &= (17, -3), & d(G) &= (17, -3). \end{aligned}$$

Since $d(G) \neq \delta(x, y)$ for all $x, y \in V$. Hence $A(G)$ is an antipodal bipolar fuzzy graph of G having same vertices as in G only, and no two vertices in $A(G)$ are made as neighborhood since their $\mu^P \mu^N$ -distance between them is not equal to the diameter of G .

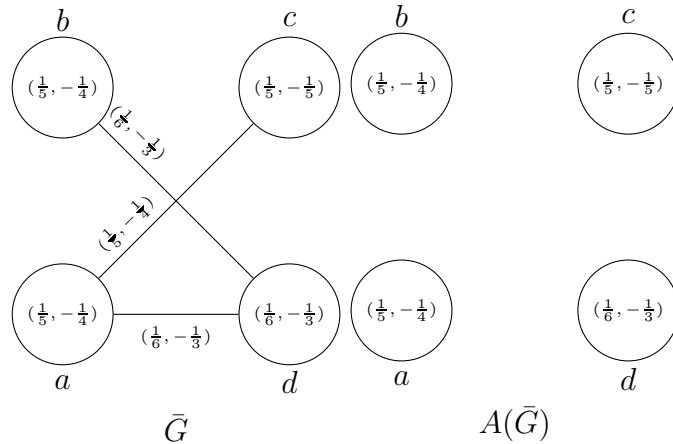


Figure 6

By routine calculations, we have

$$\begin{aligned} \delta(a, b) &= (12, -6), & \delta(a, c) &= (5, -4), & \delta(a, d) &= (6, -3), & \delta(b, c) &= (17, -10), \\ \delta(b, d) &= (6, -3), & \delta(c, d) &= (11, -7), & e(a) &= (12, -3), & e(b) &= (17, -3), \\ e(c) &= (17, -4), & e(d) &= (11, -3), & d(\bar{G}) &= (17, -3). \end{aligned}$$

Since $d(\overline{G}) \neq \delta(x, y)$ for all $x, y \in \overline{V}$. Hence $A(\overline{G})$ is an antipodal bipolar fuzzy graph of $\overline{G}$ having same vertices as in $\overline{G}$ only, and no two vertices in $A(\overline{G})$ are made as neighborhood since their $\mu^P \mu^N$ -distance between them is not equal to the diameter of $\overline{G}$.

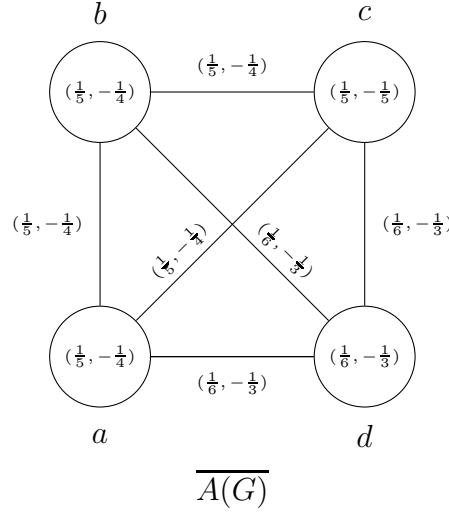


Figure 7

Clearly, $A(G)$ is not isomorphic to $\overline{A(G)}$, though $A(G)$ is isomorphic to $A(\overline{G})$. Hence G is self complementary bipolar fuzzy graph but $A(G)$ is not self complementary bipolar fuzzy graph.

We now present the concept of median bipolar fuzzy graphs.

Definition 3.14 Let G be a connected bipolar fuzzy graph. The μ^P -status of a vertex v_i is denoted by $S_{\mu^P}(v_i)$ and is defined as $S_{\mu^P}(v_i) = \sum_{\forall v_j \in V} \delta_{\mu^P}(v_i, v_j)$.

The μ^N -status of a vertex v_i is denoted by $S_{\mu^N}(v_i)$ and is defined as $S_{\mu^N}(v_i) = \sum_{\forall v_j \in V} \delta_{\mu^N}(v_i, v_j)$. The minimum μ^P -status of G is denoted by $m[S_{\mu^P}(G)]$ and

is defined as $m[S_{\mu^P}(G)] = \min(S_{\mu^P}(v_i), \forall v_i \in V)$. The minimum μ^N -status of G is denoted by $m[S_{\mu^N}(G)]$ and is defined as $m[S_{\mu^N}(G)] = \min(S_{\mu^N}(v_i), \forall v_i \in V)$. The minimum $\mu^P \mu^N$ status of G is denoted by $m[S_{\mu^P \mu^N}(G)]$ and is defined as $m[S_{\mu^P \mu^N}(G)] = (m[S_{\mu^P}(G)], m[S_{\mu^N}(G)])$. The maximum μ^P -status of G is denoted by $M[S_{\mu^P}(G)]$ and is defined as $M[S_{\mu^P}(G)] = \max(S_{\mu^P}(v_i), \forall v_i \in V)$. The maximum μ^N -status of G is denoted by $M[S_{\mu^N}(G)]$ and is defined as $M[S_{\mu^N}(G)] = \max(S_{\mu^N}(v_i), \forall v_i \in V)$. The maximum $\mu^P \mu^N$ status of G is denoted by $M[S_{\mu^P \mu^N}(G)]$ and is defined as $M[S_{\mu^P \mu^N}(G)] = (M[S_{\mu^P}(G)], M[S_{\mu^N}(G)])$. The total μ^P -status of a bipolar fuzzy graph G is denoted by $t[S_{\mu^P}(G)]$ and is defined as $t[S_{\mu^P}(G)] = \sum_{\forall v_i \in V} S_{\mu^P}(v_i)$. The total μ^N -status of a bipolar fuzzy graph

G is denoted by $t[S_{\mu^N}(G)]$ and is defined as $t[S_{\mu^N}(G)] = \sum_{\forall v_i \in V} S_{\mu^N}(v_i)$. The

total μ^N -status of a bipolar fuzzy graph G is denoted by $t[S_{\mu^P \mu^N}(G)]$ and is defined as $t[S_{\mu^P \mu^N}(G)] = (t[S_{\mu^P}(G)], t[S_{\mu^N}(G)])$. The median of a bipolar fuzzy

graph G is denoted by $M(G)$ and is defined as the set of nodes with minimum $\mu^P \mu^N$ status. An bipolar fuzzy graph G is said to be *self-median* if all the vertices have the same status. In other words, G is self-median if and only if $m[S_{\mu^P \mu^N}(G)] = M[S_{\mu^P \mu^N}(G)]$.

Example 3.15

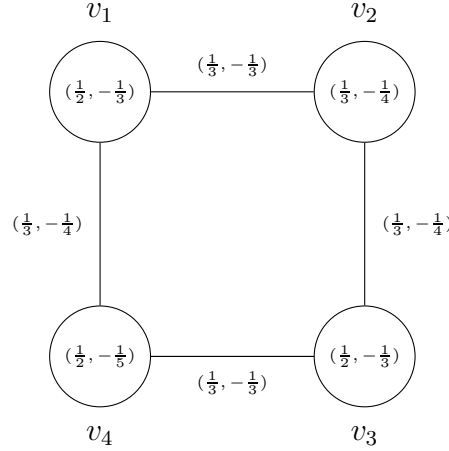


Figure 8. Selfmedianbipolarfuzzygraph.

By routine calculations, we have,

$$\begin{aligned} \delta_{\mu^P}(v_1, v_2) &= 3, & \delta_{\mu^P}(v_1, v_3) &= 6, & \delta_{\mu^P}(v_1, v_4) &= 3, \\ \delta_{\mu^P}(v_2, v_3) &= 3, & \delta_{\mu^P}(v_2, v_4) &= 6, & \delta_{\mu^P}(v_3, v_4) &= 3, \\ \delta_{\mu^N}(v_1, v_2) &= -11, & \delta_{\mu^N}(v_1, v_3) &= -7, & \delta_{\mu^N}(v_1, v_4) &= -10, \\ \delta_{\mu^N}(v_2, v_3) &= -10, & \delta_{\mu^N}(v_2, v_4) &= -7, & \delta_{\mu^N}(v_3, v_4) &= -11, \\ S_{\mu^P}(v_1) &= 12, & S_{\mu^P}(v_2) &= 12, & S_{\mu^P}(v_3) &= 12, & S_{\mu^P}(v_4) &= 12, \\ S_{\mu^N}(v_1) &= -28, & S_{\mu^N}(v_2) &= -28, & S_{\mu^N}(v_3) &= -28, & S_{\mu^N}(v_4) &= -28. \end{aligned}$$

Therefore, $S_{\mu^P \mu^N}(v_1) = (12, -28)$, $S_{\mu^P \mu^N}(v_2) = (12, -28)$, $S_{\mu^P \mu^N}(v_3) = (12, -28)$, $S_{\mu^P \mu^N}(v_4) = (12, -28)$ and $t[S_{\mu^P \mu^N}(G)] = (48, -112)$. Here, $S_{\mu^P \mu^N}(v_i) = (12, -28)$, $\forall v_i \in V$. Hence G is self median bipolar fuzzy graph.

Theorem 3.15 *Let G be a bipolar fuzzy graph, where crisp graph G^* is an even cycle. If alternate edges have same positive values and negative values, then G is self median bipolar fuzzy graph.*

Proof. Given that G is a bipolar fuzzy graph. Since crisp graph G^* is an even cycle. Also, alternate edges of G have same positive values and negative values, we have, $\delta(v_1, v_2) = \delta(v_3, v_4) = \delta(v_1, v_2) = \dots = \delta(v_{n-1}, v_n)$ and, similarly, $\delta(v_2, v_3) = \delta(v_4, v_5) = \dots = \delta(v_n, v_1)$, $\delta(v_1, v_3) = \delta(v_2, v_4) = \delta(v_3, v_5) = \dots = l$, so on. Hence $S_{\mu^P}(v_i) = k$ and $S_{\mu^N}(v_i) = m$, $\forall v_i \in V$. Hence G is a self median bipolar fuzzy graph. ■

Remark 2 Let G be a bipolar fuzzy graph, where crisp graph G^* is an odd cycle. If alternate edges have same positive and negative values, then G may not be self median bipolar fuzzy graph.

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COMPACTIFICATION OF A SOFT FUZZY PRODUCT C-SPACE

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Abstract. In this paper, the soft fuzzy product C-structure is introduced and some of the relevant properties of the associated product map on soft fuzzy product C-space are studied. Moreover, compactifying the soft fuzzy product C-space through the soft fuzzy product generalized topological space on $\mathcal{Q}(X_1 \times X_2)$ is established.

Keywords: Soft fuzzy product C-structure, functor, associated product map, weakly induced soft fuzzy product $C^\#$ -space, soft fuzzy $C^\#$ -quotient product map, soft fuzzy product dense set, soft fuzzy product strong generalized topological space, soft fuzzy $\mathfrak{G}$ -compact space.

2000 Mathematics Subject Classification: 54A40, 03E72.

1. Introduction

The fuzzy concept has penetrated almost all branches of Mathematics since the introduction of the concept of fuzzy set by Zadeh [8]. Fuzzy sets have applications in many fields such as information [4] and control [5]. The theory of fuzzy topological spaces was introduced and developed by C.L.Chang [2]. Several properties on fuzzy product topological spaces were discussed by K.K. Azad [1].

The notions of Soft fuzzy set over a poset I and soft fuzzy topological space was introduced by Ismail U. Tiriyaki [6]. The notion of C -set in general topology was introduced by E. Hatir, T. Noiri and S. Yuksel [3]. The concept of a soft fuzzy C -open set in a soft fuzzy topological space is introduced by T. Yogalakshmi, E. Roja, M.K. Uma [7].

In this paper, a new structure, called soft fuzzy product C-structure on the soft fuzzy product space is introduced. An associated product map is defined and some of its properties are studied. Moreover, a compactification of the soft fuzzy product C-space through the soft fuzzy product strong generalized topological space is established.

2. Preliminaries

Definition 2.1. [7] Let X be a nonempty set. Let μ be a fuzzy subset of X such that $\mu : X \rightarrow [0, 1]$ and M be any crisp subset of X . Then, the ordered pair (μ, M) is called as a *soft fuzzy set* in X . The family of all soft fuzzy subsets of X , will be denoted by $\mathbf{SF}(X)$.

Definition 2.2. [7] Let X be a non-empty set. Then, the *complement* of a soft fuzzy set (μ, M) is defined as $(\mu, M)' = (1 - \mu, X \setminus M)$

Definition 2.3. [7] Let X be a non-empty set and the soft fuzzy sets A and B be in the form,

$$\begin{aligned} A &= \{(\mu, M) : \mu(x) \in I^X, \forall x \in X, M \subseteq X\} \\ B &= \{(\lambda, N) : \lambda(x) \in I^X, \forall x \in X, N \subseteq X\} \end{aligned}$$

Then,

- (1) $A \subseteq B \Leftrightarrow \mu(x) \leq \lambda(x), \forall x \in X, M \subseteq N$.
- (2) $A = B \Leftrightarrow \mu(x) = \lambda(x), \forall x \in X, M = N$.
- (3) $A \cap B \Leftrightarrow \mu(x) \wedge \lambda(x), \forall x \in X, M \cap N$.
- (4) $A \sqcup B \Leftrightarrow \mu(x) \vee \lambda(x), \forall x \in X, M \cup N$.

Definition 2.4. [7] A soft fuzzy topology on a non-empty set X is a family τ of soft fuzzy sets in X satisfying the following axioms:

- (1) $(0, \phi), (1, X) \in \tau$.
- (2) For any family of soft fuzzy sets $(\lambda_j, N_j) \in \tau, j \in J, \Rightarrow \sqcup_{j \in J} (\lambda_j, N_j) \in \tau$.
- (3) For any finite number of soft fuzzy sets $(\lambda_j, N_j) \in \tau, j = 1, 2, 3, \dots, n, \Rightarrow \sqcap_{j=1}^n (\lambda_j, N_j) \in \tau$.

Then, the pair (X, τ) is called as a *soft fuzzy topological space*. (in short, **SFTS**)

Any soft fuzzy set in τ is said to be a *soft fuzzy open set* (in short, **SFOS**) in X .

The complement of SFOS in a SFTS (X, τ) is called as a *soft fuzzy closed set*, denoted **SFCS** in X .

Definition 2.5. [1] The *product* $\lambda \times \mu$ of a fuzzy set λ of X and a fuzzy set μ of Y is a fuzzy set of $X \times Y$, defined by $(\lambda \times \mu) \langle x, y \rangle = \min(\lambda(x), \mu(y))$, for each $\langle x, y \rangle \in X \times Y$.

Definition 2.6. [1] The *product* $f_1 \times f_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ of mappings $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$, defined by $(f_1 \times f_2) \langle x_1, x_2 \rangle = (f_1(x_1), f_2(x_2))$, for each $\langle x_1, x_2 \rangle \in X_1 \times X_2$.

3. On soft fuzzy product C-space

Definition 3.1. Let $\langle x_1, x_2 \rangle \in X_1 \times X_2$ and $\lambda : X_1 \times X_2 \rightarrow [0, 1]$. Define,

$$\langle x_1, x_2 \rangle_\lambda \langle y_1, y_2 \rangle = \begin{cases} \lambda & (0 < \lambda \leq 1), \text{ if } \langle x_1, x_2 \rangle = \langle y_1, y_2 \rangle \\ 0, & \text{otherwise} \end{cases}$$

Then, the soft fuzzy set $(\langle x_1, x_2 \rangle_\lambda, \{\langle x_1, x_2 \rangle\})$ is called as the *soft fuzzy point* (in short, **SFP**) in $SF(X_1 \times X_2)$, with *support*, $\langle x_1, x_2 \rangle$ and *value*, λ .

Definition 3.2. Let (λ_1, N_1) and (λ_2, N_2) be any two soft fuzzy sets. Then, the *soft fuzzy product set* is defined as $(\lambda_1, N_1) \times (\lambda_2, N_2) = (\lambda_1 \times \lambda_2, N_1 \times N_2)$

Definition 3.3. Let (X_1, τ_1) and (X_2, τ_2) be any two soft fuzzy topological spaces. The collection $\mathcal{B} = \{(\lambda_1 \times \lambda_2, N_1 \times N_2) : (\lambda, N) \in \tau_1, (\mu, M) \in \tau_2 \text{ \& } N \times M \subseteq X_1 \times X_2\}$ forms an *open base* of a soft fuzzy topology in $X_1 \times X_2$.

The soft fuzzy topology in $X_1 \times X_2$, induced by $\mathcal{B}$ is called as the *soft fuzzy product topology* of τ_1 and τ_2 , denoted by $\tau_1 \times \tau_2$.

The ordered pair $(X_1 \times X_2, \tau_1 \times \tau_2)$, which means the product of (X_1, τ_1) and (X_2, τ_2) , is called the *soft fuzzy product topological space* (in short, **SFPTS**).

Moreover, the member of a soft fuzzy product topology is called as a *soft fuzzy product open set*.

Definition 3.4. Let $(X_1 \times X_2, \tau_1 \times \tau_2)$ be a SFPTS and (λ, N) be a soft fuzzy product set in $X_1 \times X_2$. Then, the *soft fuzzy product interior* and *soft fuzzy product closure* of (λ, N) are defined by,

$$\begin{aligned} cl(\lambda, N) &= \sqcap \{(\mu, M) : (\mu, M) \text{ is a soft fuzzy closed set in } X_1 \times X_2 \\ &\quad \text{and } (\lambda, N) \sqsubseteq (\mu, M)\} \\ int(\lambda, N) &= \sqcup \{(\gamma, L) : (\gamma, L) \text{ is a soft fuzzy open set in } X_1 \times X_2 \\ &\quad \text{and } (\lambda, N) \sqsupseteq (\gamma, L)\} \end{aligned}$$

Definition 3.5. Let $(X_1 \times X_2, \tau_1 \times \tau_2)$ be a SFPTS. A soft fuzzy product set (λ, N) is said to be *soft fuzzy product α^* -open*, if $int(\lambda, N) = int(cl(int(\lambda, N)))$.

Definition 3.6. Let $(X_1 \times X_2, \tau_1 \times \tau_2)$ be a SFPTS. A soft fuzzy product set (λ, N) is said to be *soft fuzzy product C-open* (in short, **SFPcOS**), if

$$(\lambda, N) = (\mu, M) \sqcap (\gamma, K)$$

where, (μ, M) is a soft fuzzy product open set and (γ, K) is a soft fuzzy product α^* -open set.

The complement of soft fuzzy product C-open set is called as a *soft fuzzy product C-closed set* (in short, *SFPcCS*).

Definition 3.7. A *soft fuzzy product C-structure* on a non-empty set $X_1 \times X_2$ is a family $\mathfrak{st}(\tau_1 \times \tau_2)$ of soft fuzzy product C-open sets in $X_1 \times X_2$ satisfying the following axioms:

- (1) $(0, \phi), (1, X) \in \tau$.
- (2) For any finite number of soft fuzzy C-open sets $(\lambda_j, N_j) \in \tau, j = 1, 2, 3, \dots, n$,
 $\Rightarrow \bigcap_{j=1}^n (\lambda_j, N_j) \in \tau$.

Then, the pair $(X_1 \times X_2, \mathfrak{st}(\tau_1 \times \tau_2))$ is called as a *soft fuzzy product C-space*. (in short, *SFPCst*($X_1 \times X_2$))

Any soft fuzzy product set (λ, N) in $\mathfrak{st}(\tau_1 \times \tau_2)$ is said to be a *soft fuzzy product C-open set* in $X_1 \times X_2$.

The complement of a soft fuzzy product C-open set in *SFPCst*($X_1 \times X_2$) is called as a *soft fuzzy product C-closed set*.

Definition 3.8. Let $(X_1 \times X_2, \mathfrak{st}(\tau_1 \times \tau_2))$ and $(Y_1 \times Y_2, \mathfrak{st}(\sigma_1 \times \sigma_2))$ be any soft fuzzy product C-spaces. A function $f: X \rightarrow Y$ is said to be *soft fuzzy $C^\#$ -continuous product map*, if the inverse image of every soft fuzzy product C-open set in $(Y_1 \times Y_2, \mathfrak{st}(\sigma_1 \times \sigma_2))$ is a soft fuzzy product C-open set in $(X_1 \times X_2, \mathfrak{st}(\tau_1 \times \tau_2))$.

Definition 3.9. Let $(X_1 \times X_2, \mathfrak{st}(\tau_1 \times \tau_2))$ and $(Y_1 \times Y_2, \mathfrak{st}(\sigma_1 \times \sigma_2))$ be any soft fuzzy product topological spaces. A surjective function $f: X \rightarrow Y$ is said to be *soft fuzzy $C^\#$ -quotient product map*, if the inverse image of every soft fuzzy product C-open set in $(Y_1 \times Y_2, \mathfrak{st}(\sigma_1 \times \sigma_2))$ is a soft fuzzy product C-open set in $(X_1 \times X_2, \mathfrak{st}(\tau_1 \times \tau_2))$.

Definition 3.10. Let $(X_1 \times X_2, T_1 \times T_2)$ be a product topological space and $I = [0, 1]$ equipped with the usual topology, a *lower semi $C^\#$ -continuous pair* (μ, M) , where $\mu: (X_1 \times X_2, T_1 \times T_2) \rightarrow I$ with a C-open set $\mu^{-1}((\alpha, 1])$ and $M \subseteq X_1 \times X_2$ is also a C-open set in $X_1 \times X_2$, for all $\alpha \in [0, 1]$.

Definition 3.11. A soft fuzzy product C-space $(X_1 \times X_2, \mathfrak{st}(\tau_1 \times \tau_2))$ is said to be a *weakly induced soft fuzzy product $C^\#$ -space*, which is the soft fuzzy product C-space induced by a topological space $(X_1 \times X_2, T_1 \times T_2)$ if the following conditions hold :

- (a) $T_1 \times T_2 = \{A \subset X_1 \times X_2 \text{ is a product C-open set} \mid (\chi_A, A) \in St(\tau_1 \times \tau_2)\}$
- (b) Every $(\mu, M) \in \mathfrak{st}(\tau_1 \times \tau_2)$ is a lower semi $C^\#$ -continuous pair.

Definition 3.12. Let *PrTop* be the category of all the product topological spaces and the continuous product maps. Let *SFPrCst* be the category of all the soft fuzzy product C-space and SF $C^\#$ -continuous product maps. Define a *functor*, $\omega:$

$PrTop \rightarrow SFPrCst$, which associates to any product topological space $(X_1 \times X_2, \tau_1 \times \tau_2)$, the soft fuzzy product C-space $(X_1 \times X_2, \omega(\tau_1 \times \tau_2))$, where $\omega(\tau_1 \times \tau_2)$ is the totality of all lower semi $C^\#$ -continuous pair. Then, $(X_1 \times X_2, \omega(\tau_1 \times \tau_2))$ is called as the weakly induced soft fuzzy product $C^\#$ -space by $(X_1 \times X_2, \tau_1 \times \tau_2)$.

Proposition 3.1. *For mappings $f_i : X_i \rightarrow Y_i$ and soft fuzzy sets (λ_i, N_i) of Y_i , $(i = 1, 2)$; we have $(f_1 \times f_2)^{-1}(\lambda_1 \times \lambda_2, N_1 \times N_2) = f_1^{-1}(\lambda_1, N_1) \times f_2^{-1}(\lambda_2, N_2)$.*

Proof. The proof is clear.

Proposition 3.2. *For mappings $f_i : X_i \rightarrow Y_i$ and soft fuzzy sets (λ_i, N_i) of Y_i , $(i = 1, 2)$; we have $(f_1 \times f_2)(\lambda_1 \times \lambda_2, N_1 \times N_2) \sqsubseteq f_1(\lambda_1, N_1) \times f_2(\lambda_2, N_2)$.*

Proof. The proof is clear.

4. Properties of the associated product map on soft fuzzy product C-space

Definition 4.1. Let $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$ be any two maps. Let $X_1 \times X_2$ and $Y_1 \times Y_2$ be two product sets and $f_1 \times f_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ be a product map. Then, define the product associated map $f_1 \times f_2$ as $f_1 \times f_2(\langle x_1, x_2 \rangle_\lambda, \{\langle x_1, x_2 \rangle\}) = f_1 \times f_2(\langle x_1, x_2 \rangle_\lambda, \{\langle x_1, x_2 \rangle\})$, for each soft fuzzy point $(\langle x_1, x_2 \rangle_\lambda, \{\langle x_1, x_2 \rangle\})$ in $SFPCst(X_1 \times X_2)$.

Proposition 4.1. *Let $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$ be any two onto maps. Let $X_1 \times X_2$ and $Y_1 \times Y_2$ be two product sets. If $f_1 \times f_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is a product onto map, then for each soft fuzzy point $(\langle x_1, x_2 \rangle_\lambda, \{\langle x_1, x_2 \rangle\})$ in $SFPCst(X_1 \times X_2)$, $f_1 \times f_2(\langle x_1, x_2 \rangle_\lambda, \{\langle x_1, x_2 \rangle\})$ is the soft fuzzy point in $SFPCst(Y_1 \times Y_2)$ that takes the value λ in $f_1 \times f_2 \langle x_1, x_2 \rangle$.*

Proof. For $0 < \lambda \leq 1$,

$$\begin{aligned} \widetilde{f_1 \times f_2(\langle x_1, x_2 \rangle_\lambda, \{\langle x_1, x_2 \rangle\})} &= f_1 \times f_2(\langle x_1, x_2 \rangle_\lambda, \{\langle x_1, x_2 \rangle\}) \\ &= (f_1 \times f_2 \langle x_1, x_2 \rangle_\lambda, f_1 \times f_2(\{\langle x_1, x_2 \rangle\})) \\ &= (f_1 \times f_2(\langle x_1, x_2 \rangle_\lambda), \{\langle f_1(x_1), f_2(x_2) \rangle\}) \end{aligned}$$

where, $f_1 \times f_2(\langle x_1, x_2 \rangle_\lambda)(\langle y_1, y_2 \rangle)$

$$= \begin{cases} \sup_{\langle x, y \rangle \in (f_1 \times f_2)^{-1}(\langle y_1, y_2 \rangle)} \langle x_1, x_2 \rangle_\lambda(\langle x, y \rangle), & \text{if } (f_1 \times f_2)^{-1}(\langle y_1, y_2 \rangle) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \lambda, & \text{if } (f_1 \times f_2)^{-1}(\langle y_1, y_2 \rangle) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

Then, $\widetilde{f_1 \times f_2(\langle x_1, x_2 \rangle_\lambda, \{\langle x_1, x_2 \rangle\})}$ is the soft fuzzy point in $SFPCst(Y_1 \times Y_2)$ that takes the value λ in $f_1 \times f_2(\langle x_1, x_2 \rangle)$. ■

Proposition 4.2. Let $f_1 : X_1 \rightarrow Y_1$, $f_2 : X_2 \rightarrow Y_2$, $g_1 : Y_1 \rightarrow Z_1$ and $g_2 : Y_2 \rightarrow Z_2$ be any maps. Let $f_1 \times f_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ and $g_1 \times g_2 : Y_1 \times Y_2 \rightarrow Z_1 \times Z_2$ be the two product onto maps. Then, $(g_1 \times g_2) \circ (f_1 \times f_2) = (g_1 \times g_2) \circ (f_1 \times f_2)$.

Proof. By using the above Property 4.1, we have for each soft fuzzy point $(\langle x_1, x_2 \rangle_\lambda, \{\langle x_1, x_2 \rangle\})$ in $SFPCst(X_1 \times X_2)$

$$\begin{aligned} & \widetilde{(g_1 \times g_2) \circ (f_1 \times f_2)}(\langle x_1, x_2 \rangle_\lambda, \{\langle x_1, x_2 \rangle\}) \\ &= (g_1 \times g_2) \circ (f_1 \times f_2)(\langle x_1, x_2 \rangle_\lambda, \{\langle x_1, x_2 \rangle\}) \\ &= (g_1 \times g_2)((f_1 \times f_2)(\langle x_1, x_2 \rangle_\lambda, \{\langle x_1, x_2 \rangle\})) \\ &= (g_1 \times g_2)((f_1 \times f_2)(\langle x_1, x_2 \rangle_\lambda, \{\langle x_1, x_2 \rangle\})) \\ &= (g_1 \times g_2)((f_1 \times f_2)(\langle x_1, x_2 \rangle_\lambda, \{\langle x_1, x_2 \rangle\})) \\ &= (g_1 \times g_2)((f_1 \times f_2)(\langle x_1, x_2 \rangle_\lambda, \{\langle x_1, x_2 \rangle\})) \\ &= (g_1 \times g_2) \circ (f_1 \times f_2)(\langle x_1, x_2 \rangle_\lambda, \{\langle x_1, x_2 \rangle\}) \end{aligned}$$

Thus, $\widetilde{(g_1 \times g_2) \circ (f_1 \times f_2)} = \widetilde{(g_1 \times g_2)} \circ \widetilde{(f_1 \times f_2)}$. ■

Proposition 4.3. Let $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$ be any two onto maps. Let $X_1 \times X_2$ and $Y_1 \times Y_2$ be two product sets. Let $f_1 \times f_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is a product onto map. If $(f_1 \times f_2)$ is the identity map, then $f_1 \times f_2$ is also the identity map.

Proof. Since $\widetilde{(f_1 \times f_2)}$ is the identity map, $\widetilde{(f_1 \times f_2)}(\langle x_1, x_2 \rangle_\lambda, \{\langle x_1, x_2 \rangle\}) = \langle (x_1, x_2)_\lambda, \{\langle x_1, x_2 \rangle\} \rangle$, for each soft fuzzy point $(\langle x_1, x_2 \rangle_\lambda, \{\langle x_1, x_2 \rangle\})$ in $SFPCst(X_1 \times X_2)$. This implies that, $(f_1 \times f_2)(\langle x_1, x_2 \rangle_\lambda, \{\langle x_1, x_2 \rangle\}) = \langle (x_1, x_2)_\lambda, \{\langle x_1, x_2 \rangle\} \rangle$, for each soft fuzzy point $(\langle x_1, x_2 \rangle_\lambda, \{\langle x_1, x_2 \rangle\})$. Now, by the definition of the associated product map, $(f_1 \times f_2)(\langle x_1, x_2 \rangle_\lambda, \{\langle x_1, x_2 \rangle\}) = (f_1 \times f_2)(\langle x_1, x_2 \rangle_\lambda, \{\langle x_1, x_2 \rangle\}) = \langle (f_1 \times f_2)(\langle x_1, x_2 \rangle_\lambda), \{(f_1(x_1), f_2(x_2))\} \rangle$. This implies that, $((f_1 \times f_2)(\langle x_1, x_2 \rangle_\lambda), \{(f_1(x_1), f_2(x_2))\})$ is the soft fuzzy point which takes the value λ in $(f_1 \times f_2) \langle x_1, x_2 \rangle$ and $(\langle x_1, x_2 \rangle_\lambda, \{\langle x_1, x_2 \rangle\})$ is also the soft fuzzy point which takes the value λ in $\langle x_1, x_2 \rangle$. Thus, $(f_1 \times f_2)(\langle x_1, x_2 \rangle) = \langle x_1, x_2 \rangle$, for each $\langle x_1, x_2 \rangle \in X_1 \times X_2$. Hence, $f_1 \times f_2$ is the identity map. ■

Proposition 4.4. Let $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$ be any two maps.

- (1) If $f_1 \times f_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is a product onto map, then $\widetilde{(f_1 \times f_2)}$ is also the product onto map.
- (2) If $f_1 \times f_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is a product one- to-one map, then $\widetilde{(f_1 \times f_2)}$ is also the product one-to-one map.

Proof. (1) For each $(\langle y_1, y_2 \rangle_\alpha, \{\langle y_1, y_2 \rangle\})$ soft fuzzy point in $SFPCst(Y_1 \times Y_2)$, we have $\langle y_1, y_2 \rangle \in Y_1 \times Y_2$, then there exists at least $\langle x_1, x_2 \rangle \in X_1 \times X_2$ such that $(f_1 \times f_2) \langle x_1, x_2 \rangle = \langle y_1, y_2 \rangle$. Now, $(f_1 \times f_2)(\langle x_1, x_2 \rangle_\alpha, \{\langle x_1, x_2 \rangle\}) = (f_1 \times f_2)(\langle x_1, x_2 \rangle_\alpha, \{\langle x_1, x_2 \rangle\}) = ((f_1 \times f_2)(\langle x_1, x_2 \rangle_\alpha), (f_1 \times f_2)(\{\langle x_1, x_2 \rangle\}))$ which takes the value α in $(f_1 \times f_2) \langle x_1, x_2 \rangle$ and since $(f_1 \times f_2) \langle x_1, x_2 \rangle = \langle y_1, y_2 \rangle$, this

shows that $((f_1 \times f_2)(\langle x_1, x_2 \rangle_\alpha), (f_1 \times f_2)(\{\langle x_1, x_2 \rangle\})) = (\langle y_1, y_2 \rangle_\alpha, \{\langle y_1, y_2 \rangle\})$. Therefore, $(f_1 \times f_2)(\langle x_1, x_2 \rangle_\alpha, \{\langle x_1, x_2 \rangle\}) = (\langle y_1, y_2 \rangle_\alpha, \{\langle y_1, y_2 \rangle\})$. Thus, $f_1 \times f_2$ is the product onto map.

(2) If $(\langle x_1, x_2 \rangle_\alpha, \{\langle x_1, x_2 \rangle\}), (\langle x'_1, x'_2 \rangle_\beta, \{\langle x'_1, x'_2 \rangle\})$ are the two soft fuzzy points in $SFPCst(X_1 \times X_2)$ such that $(f_1 \times f_2)(\langle x_1, x_2 \rangle_\alpha, \{\langle x_1, x_2 \rangle\}) = (f_1 \times f_2)(\langle x'_1, x'_2 \rangle_\beta, \{\langle x'_1, x'_2 \rangle\})$. This implies that $(f_1 \times f_2)(\langle x_1, x_2 \rangle_\alpha, \{\langle x_1, x_2 \rangle\}) = (f_1 \times f_2)(\langle x'_1, x'_2 \rangle_\beta, \{\langle x'_1, x'_2 \rangle\})$. This shows $(f_1 \times f_2)(\langle x_1, x_2 \rangle) = (f_1 \times f_2)(\langle x'_1, x'_2 \rangle)$ and $\alpha = \beta$. Since $(f_1 \times f_2)$ is a one-to-one map, $\langle x_1, x_2 \rangle = \langle x'_1, x'_2 \rangle$ and $\alpha = \beta$, it follows

$$(\langle x_1, x_2 \rangle_\alpha, \{\langle x_1, x_2 \rangle\}) = (\langle x'_1, x'_2 \rangle_\beta, \{\langle x'_1, x'_2 \rangle\}).$$

Thus, $f_1 \times f_2$ is one-to-one. ■

Proposition 4.5. *Let $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$ be any two maps. If $f_1 \times f_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is a product one-to-one map, then $(f_1 \times f_2)^{-1} = (f_1 \times f_2)^{-1}$.*

Proof. For each soft fuzzy point $(\langle y_1, y_2 \rangle_\alpha, \{\langle y_1, y_2 \rangle\})$ in $SFPCst(Y_1 \times Y_2)$ and by the hypothesis, there exists a unique $\langle x_1, x_2 \rangle \in X_1 \times X_2$ such that $(f_1 \times f_2)(\langle x_1, x_2 \rangle) = \langle y_1, y_2 \rangle$. It must be shown that $(f_1 \times f_2)^{-1}$ is well-defined. It is enough to show that $(f_1 \times f_2)^{-1}(\langle y_1, y_2 \rangle_\alpha, \{\langle y_1, y_2 \rangle\}) = (\langle x_1, x_2 \rangle_\alpha, \{\langle x_1, x_2 \rangle\})$. Otherwise, let $(f_1 \times f_2)^{-1}(\langle y_1, y_2 \rangle_\alpha, \{\langle y_1, y_2 \rangle\}) = (\langle x_1, x_2 \rangle_\lambda, \{\langle x_1, x_2 \rangle\})$ and $\alpha \neq \lambda$. Then, $(f_1 \times f_2)(\langle x_1, x_2 \rangle_\lambda, \{\langle x_1, x_2 \rangle\}) = (f_1 \times f_2)((f_1 \times f_2)^{-1}(\langle y_1, y_2 \rangle_\alpha, \{\langle y_1, y_2 \rangle\})) = (f_1 \times f_2)(\langle x_1, x_2 \rangle_\alpha, \{\langle x_1, x_2 \rangle\})$. Since $(f_1 \times f_2)$ is a one-to-one map, $(f_1 \times f_2)$ is one-to-one. Thus, $(\langle x_1, x_2 \rangle_\alpha, \{\langle x_1, x_2 \rangle\}) = (\langle x_1, x_2 \rangle_\lambda, \{\langle x_1, x_2 \rangle\})$. But, $\lambda \neq \alpha$. Thus, it leads to a contradiction. Therefore, $(f_1 \times f_2)^{-1}(\langle y_1, y_2 \rangle_\alpha, \{\langle y_1, y_2 \rangle\})$ is uniquely the soft fuzzy point in $SFPCst(X_1 \times X_2)$ which takes the value α in $(f_1 \times f_2)^{-1} < y_1, y_2 >$. Thus, it is well defined.

Next, $(f_1 \times f_2)^{-1}(\langle y_1, y_2 \rangle_\alpha, \{\langle y_1, y_2 \rangle\}) = (f_1 \times f_2)^{-1}(\langle y_1, y_2 \rangle_\alpha, \{\langle y_1, y_2 \rangle\}) = (f_1 \times f_2)^{-1}(\langle y_1, y_2 \rangle_\alpha, \{\langle y_1, y_2 \rangle\})$. Hence, $(f_1 \times f_2)^{-1} = (f_1 \times f_2)^{-1}$. ■

Proposition 4.6. *Let $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$ be any two maps. Let $f_1 \times f_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ be a product map.*

(a) *If $f_1 \times f_2$ is onto, then $(f_1 \times f_2)$ is also onto.*

(b) *If $f_1 \times f_2$ is one-to-one, then $(f_1 \times f_2)$ is also one-to-one.*

Proof. (a) For each $\langle y_1, y_2 \rangle \in Y_1 \times Y_2$, let $(\langle y_1, y_2 \rangle_1, \{\langle y_1, y_2 \rangle\})$ be the soft fuzzy point in $SFPCst(Y_1 \times Y_2)$ which takes the value 1 in $\langle y_1, y_2 \rangle$. By hypothesis, there exists a soft fuzzy point $(\langle x_1, x_2 \rangle_\alpha, \{\langle x_1, x_2 \rangle\})$ in $SFPCst(X_1 \times X_2)$ such that $f_1 \times f_2(\langle x_1, x_2 \rangle_\alpha, \{\langle x_1, x_2 \rangle\}) = (\langle y_1, y_2 \rangle_1, \{\langle y_1, y_2 \rangle\})$. Then,

$$f_1 \times f_2(\langle x_1, x_2 \rangle_\alpha, \{\langle x_1, x_2 \rangle\}) = (\langle y_1, y_2 \rangle_1, \{\langle y_1, y_2 \rangle\}) \text{ and } (f_1 \times f_2)^{-1}(\langle y_1, y_2 \rangle) \neq \phi.$$

Hence, $(f_1 \times f_2)$ is a onto map.

(b) Let $\langle x_1, x_2 \rangle, \langle x'_1, x'_2 \rangle \in X_1 \times X_2$ with $(f_1 \times f_2)(\langle x_1, x_2 \rangle) = (f_1 \times f_2)(\langle x'_1, x'_2 \rangle)$.
Now, $(f_1 \times f_2)(\langle x_1, x_2 \rangle_1, \{\langle x_1, x_2 \rangle\}) = (f_1 \times f_2)(\langle x_1, x_2 \rangle_1, \{\langle x_1, x_2 \rangle\}) =$
 $\langle (f_1 \times f_2)(\langle x_1, x_2 \rangle)_1, (f_1 \times f_2)(\{\langle x_1, x_2 \rangle\}) \rangle$, where, for $\lambda=1$

$$\begin{aligned} & (f_1 \times f_2)(\langle x_1, x_2 \rangle_1)(\langle y_1, y_2 \rangle) \\ &= \begin{cases} \sup_{\langle x, y \rangle \in (f_1 \times f_2)^{-1}(\langle y_1, y_2 \rangle)} \langle x_1, x_2 \rangle_\lambda (< x, y >), & \text{if } (f_1 \times f_2)^{-1}(\langle y_1, y_2 \rangle) \neq \phi \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 1, & \text{if } \langle x_1, x_2 \rangle = \langle x, y \rangle \text{ and } (f_1 \times f_2)^{-1}(\langle y_1, y_2 \rangle) \neq \phi \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 1, & \text{if } (f_1 \times f_2)(\langle x'_1, x'_2 \rangle) = \langle y_1, y_2 \rangle \\ 0, & \text{otherwise} \end{cases} \\ &= (f_1 \times f_2)(\langle x'_1, x'_2 \rangle_1)(\langle y_1, y_2 \rangle). \end{aligned}$$

This implies that, $\widetilde{(f_1 \times f_2)(\langle x_1, x_2 \rangle_1, \{\langle x_1, x_2 \rangle\})} = \widetilde{(f_1 \times f_2)(\langle x'_1, x'_2 \rangle_1, \{\langle x'_1, x'_2 \rangle\})}$.
Since $(f_1 \times f_2)$ is a one-to-one map, $(\langle x'_1, x'_2 \rangle_1, \{\langle x'_1, x'_2 \rangle\}) = (\langle x_1, x_2 \rangle_1, \{\langle x_1, x_2 \rangle\})$.
This implies that, $\langle x_1, x_2 \rangle = \langle x'_1, x'_2 \rangle$. Thus, $(f_1 \times f_2)$ is a one-to-one map. ■

Definition 4.2. Let $(X_1 \times X_2, T_1 \times T_2)$ and $(Y_1 \times Y_2, S_1 \times S_2)$ be any two product topological spaces. A function $f: X \rightarrow Y$ is said to be a *C-irresolute product map*, if the inverse image of every product C-open set in $(Y_1 \times Y_2, S_1 \times S_2)$ is a product C-open set in $(X_1 \times X_2, T_1 \times T_2)$.

Proposition 4.7. Let $f_1: X_1 \rightarrow Y_1$ and $f_2: X_2 \rightarrow Y_2$ be any two maps. Let $(X_1 \times X_2, \tau_1 \times \tau_2)$ and $(Y_1 \times Y_2, \sigma_1 \times \sigma_2)$ be any two product topological spaces. If $f_1 \times f_2: (X_1 \times X_2, \tau_1 \times \tau_2) \rightarrow (Y_1 \times Y_2, \sigma_1 \times \sigma_2)$ is a C-irresolute product map iff $\widetilde{f_1 \times f_2}: (X_1 \times X_2, \omega(\tau_1 \times \tau_2)) \rightarrow (Y_1 \times Y_2, \omega(\sigma_1 \times \sigma_2))$ is a soft fuzzy $C^\#$ -continuous product map.

Proof. For each soft fuzzy C-open set (μ, M) in $(Y_1 \times Y_2, \omega(\sigma_1 \times \sigma_2))$, we have $\mu^{-1}((\alpha, 1])$ is a C-open set in $\sigma_1 \times \sigma_2$ for all $\alpha \in [0, 1]$ and by hypothesis $(f_1 \times f_2)^{-1}(\mu^{-1}(\alpha, 1])$ is a C-open set in $\tau_1 \times \tau_2$. Then, $(\mu \circ (f_1 \times f_2))^{-1}(\alpha, 1]$ is a C-open set in $\tau_1 \times \tau_2$, and also $(f_1 \times f_2)^{-1}(M) \subseteq X_1 \times X_2$ is a C-open in $X_1 \times X_2$. Therefore, $< (\mu \circ (f_1 \times f_2)), (f_1 \times f_2)^{-1}(M) >$ is a soft fuzzy C-open set in $(X_1 \times X_2, \omega(\tau_1 \times \tau_2))$. Now,

$$\begin{aligned} \widetilde{(f_1 \times f_2)}^{-1}(\mu, M) &= (f_1 \times f_2)^{-1}(\mu, M) \\ &= ((f_1 \times f_2)^{-1}(\mu), (f_1 \times f_2)^{-1}(M)) \\ &= ((\mu \circ (f_1 \times f_2)), (f_1 \times f_2)^{-1}(M)) \end{aligned}$$

Thus, $\widetilde{(f_1 \times f_2)}^{-1}(\mu, M)$ is a soft fuzzy C-open set in $(X_1 \times X_2, \omega(\tau_1 \times \tau_2))$. Hence, $(f_1 \times f_2)$ is a soft fuzzy $C^\#$ -continuous product map.

The converse part is clear by using the definition of the weakly induced soft fuzzy product $C^\#$ -space. ■

Definition 4.3. Let $(X_1 \times X_2, T_1 \times T_2)$ and $(Y_1 \times Y_2, S_1 \times S_2)$ be any two product topological spaces. A surjective function $f : X \rightarrow Y$ is said to be *C-quotient product map*, if the inverse image of every product C-open set in $(Y_1 \times Y_2, S_1 \times S_2)$ is a product C-open set in $(X_1 \times X_2, T_1 \times T_2)$.

Proposition 4.8. Let $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$ be any two maps. Let $(X_1 \times X_2, \tau_1 \times \tau_2)$ and $(Y_1 \times Y_2, \sigma_1 \times \sigma_2)$ be any two product topological spaces. Then, $f_1 \times f_2 : (X_1 \times X_2, \tau_1 \times \tau_2) \rightarrow (Y_1 \times Y_2, \sigma_1 \times \sigma_2)$ is a C-quotient product map iff $\widetilde{f_1 \times f_2} : (X_1 \times X_2, \omega(\tau_1 \times \tau_2)) \rightarrow (Y_1 \times Y_2, \omega(\sigma_1 \times \sigma_2))$ is also a soft fuzzy $C^\#$ -quotient product map.

Proof. By the definition of a weakly induced soft fuzzy product C-space, A is a C-open set in $(\sigma_1 \times \sigma_2)$ iff (χ_A, A) is a soft fuzzy C-open set in $((Y_1 \times Y_2), \omega(\sigma_1 \times \sigma_2))$. Now,

$$\begin{aligned} \widetilde{(f_1 \times f_2)}^{-1}(\chi_A, A) &= (f_1 \times f_2)^{-1}(\chi_A, A) \\ &= ((f_1 \times f_2)^{-1}(\chi_A), (f_1 \times f_2)^{-1}(A)) \\ &= (\chi_A \circ (f_1 \times f_2), (f_1 \times f_2)^{-1}(A)) \\ &= (\chi_{(f_1 \times f_2)^{-1}(A)}, (f_1 \times f_2)^{-1}(A)) \end{aligned}$$

That is, $\widetilde{(f_1 \times f_2)}^{-1}(\chi_A, A) = (\chi_{(f_1 \times f_2)^{-1}(A)}, (f_1 \times f_2)^{-1}(A))$ is the soft fuzzy C-open set in $(X_1 \times X_2, \omega(\tau_1 \times \tau_2))$. Hence, $(f_1 \times f_2)^{-1}(A)$ is a C-open set in $\tau_1 \times \tau_2$. Therefore, $f_1 \times f_2$ is a C-quotient product map.

Conversely, let (μ, M) be a soft fuzzy C-open set in $(Y_1 \times Y_2, \omega(\sigma_1 \times \sigma_2))$ iff $\mu^{-1}(\alpha, 1]$ is a C-open set in $\sigma_1 \times \sigma_2$, and $M \subseteq Y_1 \times Y_2$ is a C-open set in $\sigma_1 \times \sigma_2$, for all $\alpha \in [0, 1]$. By the hypothesis, $(f_1 \times f_2)^{-1}(\mu^{-1}(\alpha, 1])$ is a C-open set in $\tau_1 \times \tau_2$ and $(f_1 \times f_2)^{-1}(M) \subseteq X_1 \times X_2$ is a C-open set in $\tau_1 \times \tau_2$, for each $\alpha \in [0, 1]$. That is, $((\widetilde{(f_1 \times f_2)}^{-1}(\mu), \widetilde{(f_1 \times f_2)}^{-1}(M))) = \widetilde{(f_1 \times f_2)}^{-1}(\mu, M)$ is a soft fuzzy C-open set in $(X_1 \times X_2, \omega(\tau_1 \times \tau_2))$. Hence, $\widetilde{f_1 \times f_2}$ is also a soft fuzzy $C^\#$ -quotient product map. ■

Definition 4.4. Let $(X_1 \times X_2, \omega(\tau_1 \times \tau_2))$ and $(Y_1 \times Y_2, \omega(\sigma_1 \times \sigma_2))$ be any two soft fuzzy product C-spaces and $f_1 \times f_2$ be a soft fuzzy product map from $X_1 \times X_2$ to $Y_1 \times Y_2$. Then, $f_1 \times f_2$ is called as a *soft fuzzy $C^\#$ -homeomorphism* from $X_1 \times X_2$ to $Y_1 \times Y_2$, if

- (i) $f_1 \times f_2$ is a soft fuzzy $C^\#$ -continuous product function.
- (ii) $f_1 \times f_2$ is a soft fuzzy bijective product function.
- (iii) Inverse of $f_1 \times f_2$ is also soft fuzzy $C^\#$ -continuous product function.

Proposition 4.9. Let $(X_1 \times X_2, \omega(\tau_1 \times \tau_2))$ and $(Y_1 \times Y_2, \omega(\sigma_1 \times \sigma_2))$ be two weakly induced soft fuzzy product C-spaces, and $(f_1 \times f_2)$ be a soft fuzzy $C^\#$ -continuous map from $(X_1 \times X_2, \omega(\tau_1 \times \tau_2))$ onto $(Y_1 \times Y_2, \omega(\sigma_1 \times \sigma_2))$. If there exists a soft fuzzy $C^\#$ -continuous map $(g_1 \times g_2)$ from $(Y_1 \times Y_2, \omega(\sigma_1 \times \sigma_2))$ to $(X_1 \times X_2, \omega(\tau_1 \times \tau_2))$ such that $(f_1 \times f_2) \circ (g_1 \times g_2) = 1_{Y_1 \times Y_2}$, then $(Y_1 \times Y_2, \omega(\sigma_1 \times \sigma_2))$ is soft fuzzy $C^\#$ -homeomorphic with $(X_1 \times X_2) \mid R$, where R is the equivalence relation.

Proof. Since $(f_1 \times f_2) \circ (g_1 \times g_2) = 1_{Y_1 \times Y_2}$, then by using all the above propositions, we have $(f_1 \times f_2) \circ (g_1 \times g_2) = 1_{Y_1 \times Y_2}$. Then, the map $h_1 \times h_2 : (X_1 \times X_2) \mid R \rightarrow Y_1 \times Y_2$ induced by $f_1 \times f_2$ is a $C^\#$ -homeomorphism. Finally, by the above all propositions, $h_1 \times h_2$ is clearly a soft fuzzy $C^\#$ -homeomorphism. ■

5. Compactification of $SFPCst(X_1 \times X_2)$

Definition 5.1. Let $X_1 \times X_2$ be a product space. Let $(X_1 \times X_2) \mid R$ be a quotient set on $X_1 \times X_2$ with R , an equivalence relation. Then, the collection of all quotient sets on $X_1 \times X_2$, denoted by $\mathcal{Q}(X_1 \times X_2)$.

Definition 5.2. Let R_x be an equivalence relation. Then, $X_1 \times X_2 \mid R_{\langle x_1, x_2 \rangle} = \{[\langle x_1, x_2 \rangle], [\langle y_1, y_2 \rangle] : \langle x_1, x_2 \rangle R \langle z_1, z_2 \rangle, \langle y_1, y_2 \rangle R \langle z_1, z_2 \rangle, \forall \langle z_1, z_2 \rangle \in X_1 \times X_2\}$ is also a quotient product set on $X_1 \times X_2$.

Definition 5.3. Let $(X_1 \times X_2, st(\tau_1 \times \tau_2))$ be a soft fuzzy product C-space and A be a subset of $X_1 \times X_2$. If χ_A is a characteristic function of A in $X_1 \times X_2$, then

$$st(\tau_1 \times \tau_2)_A = \{(\lambda, N) \sqcap (\chi_A, A) : (\lambda, N) \in st(\tau_1 \times \tau_2)\}$$

is called as a soft fuzzy product C-substructure. Now, the pair $(A, st(\tau_1 \times \tau_2)_A)$ is called as a soft fuzzy product C-subspace.

Let $(X_1 \times X_2, st(\tau_1 \times \tau_2))$ be a non compact soft fuzzy product C-space. Associated with each $(\mu, M) \in st(\tau_1 \times \tau_2)$, we define $(\mu, M)^* = (\mu^*, M^*) \in SF(\mathcal{Q}(X_1 \times X_2))$. For each $(X_1 \times X_2) \mid R \in \mathcal{Q}(X_1 \times X_2)$

$$\mu^*((X_1 \times X_2) \mid R) = \begin{cases} \mu(\langle x_1, x_2 \rangle), & \text{if } \exists \langle x_1, x_2 \rangle \in X_1 \times X_2 \\ & \text{such that } X_1 \times X_2 \mid R = X_1 \times X_2 \mid R_{\langle x_1, x_2 \rangle}; \\ \bigvee_{[\langle x_1, x_2 \rangle] \in X_1 \times X_2 \mid R} \mu(\langle x_1, x_2 \rangle), & \text{otherwise.} \end{cases}$$

$$M^* = \begin{cases} \phi, & \text{if } M = \phi; \\ \mathcal{Q}(X_1 \times X_2), & \text{if } M = X_1 \times X_2; \\ \{(X_1 \times X_2) \mid R_{\langle x_1, x_2 \rangle_i}\}, & \text{if } \langle x_1, x_2 \rangle_i \in M \subset X_1 \times X_2, i \in I. \end{cases}$$

Proposition 5.1. Under the previous conditions the following identities hold.

- (i) $(0, \phi)^* = (0, \phi)$.
- (ii) $(1_{X_1 \times X_2}, X_1 \times X_2)^* = (1_{\mathcal{Q}(X_1 \times X_2)}, \mathcal{Q}(X_1 \times X_2))$.

Definition 5.4. A soft fuzzy product strong generalized topology on a non-empty set X is a family $\mathfrak{G}$ of soft fuzzy product sets in X satisfying the following axioms:

- (1) $(0, \phi), (1, X) \in \tau$.
- (2) For any family of soft fuzzy sets $(\lambda_j, N_j) \in \tau, j \in J$, $\Rightarrow \sqcup_{j \in J} (\lambda_j, N_j) \in \tau$.

Then, the pair $(X, \mathfrak{G})$ is called as a *soft fuzzy product strong generalized topological space*. (in short, **SFPsGTS**)

Any soft fuzzy product set in $\mathfrak{G}$ is said to be a *soft fuzzy product $\mathfrak{G}$ -open set* (in short, **SFPGOS**) in X .

The complement of SFPGOS in a SFPsGTS (X, τ) is called as a *soft fuzzy product $\mathfrak{G}$ -closed set*, denoted **SFPGCS** in X .

Proposition 5.2. Under the previous conditions the collection

$$\mathfrak{B}^* = \{(\mu, M)^* : (\mu, M) \in \mathfrak{st}(\tau_1 \times \tau_2)\}$$

is a base for some soft fuzzy product strong generalized topology on $\mathcal{Q}(X_1 \times X_2)$.

Proof.

- (i) For $(\mu_1, M_1), (\mu_2, M_2) \in \mathfrak{st}(\tau_1 \times \tau_2)$ and $X_1 \times X_2 \mid R \in \mathcal{Q}(X_1 \times X_2)$, we have
 $(\sqcup_{i \in I} (\mu_i, M_i))^* = (\bigvee_{i \in I} \mu_i, \bigcup_{i \in I} M_i)^* = ((\bigvee_{i \in I} \mu_i)^*, (\bigcap_{i \in I} M_i)^*)$

$$\begin{aligned} & (\bigvee_{i \in I} \mu_i)^* ((X_1 \times X_2) \mid R) \\ &= \begin{cases} (\bigvee \mu_i)(\langle x_1, x_2 \rangle), & \text{if } \exists \langle x_1, x_2 \rangle \in X_1 \times X_2 \\ & \text{such that } (X_1 \times X_2) \mid R = (X_1 \times X_2) \mid R_{\langle x_1, x_2 \rangle}; \\ \bigvee_{[\langle x_1, x_2 \rangle] \in (X_1 \times X_2) \mid R} (\bigvee_{i \in I} \mu_i)(\langle x_1, x_2 \rangle), & \text{otherwise.} \end{cases} \\ &= \begin{cases} \bigvee_{i \in I} \mu_i(\langle x_1, x_2 \rangle), & \text{if } \exists \langle x_1, x_2 \rangle \in X_1 \times X_2 \\ & \text{such that } (X_1 \times X_2) \mid R = (X_1 \times X_2) \mid R_{\langle x_1, x_2 \rangle}; \\ \bigvee_{i \in I} \bigvee_{[\langle x_1, x_2 \rangle] \in (X_1 \times X_2) \mid R} \mu_i(\langle x_1, x_2 \rangle), & \text{otherwise.} \end{cases} \\ &= \bigvee_{i \in I} \mu_i^* (X_1 \times X_2) \mid R). \end{aligned}$$

Thus $(\bigvee_{i \in I} \mu_i)^* = \bigvee_{i \in I} \mu_i^*$.

Now,

$$(M_1 \cap M_2)^* = \begin{cases} \phi, & \text{if } M_1 \cap M_2 = \phi; \\ \mathcal{Q}(X_1 \times X_2), & \text{if } M_1 \cap M_2 = X_1 \times X_2; \\ X_1 \times X_2 \mid R_{\langle x_1, x_2 \rangle}, & \text{if } \langle x_1, x_2 \rangle \in M_1 \cap M_2 \subset X_1 \times X_2. \end{cases}$$

$$\begin{aligned}
&= \begin{cases} \phi, & \text{if } M_1 = \phi \text{ and } M_2 = \phi \text{ or } M_1 \neq \phi \text{ and } M_2 \neq \phi; \\ \mathcal{Q}(X_1 \times X_2), & \text{if } M_1 = X_1 \times X_2 \text{ and } M_2 = X_1 \times X_2; \\ X_1 \times X_2 \mid R_{\langle x_1, x_2 \rangle}, & \text{if } \langle x_1, x_2 \rangle \in M_1 \subset X_1 \times X_2 \\ & \text{and } \langle x_1, x_2 \rangle \in M_2 \subset X_1 \times X_2. \end{cases} \\
&= M_1^* \cap M_2^*
\end{aligned}$$

Therefore $((\mu_1, M_1) \cap (\mu_2, M_2))^* = (\mu_1, M_1)^* \cap (\mu_2, M_2)^*$.

Thus, $\mathfrak{b}^*$ forms a base for $\mathcal{Q}(X_1 \times X_2)$. ■

Definition 5.5. The soft fuzzy product C-space, generated by the base $\mathfrak{b}^*$ of soft fuzzy product C-open sets, is denoted by $(\tau_1 \times \tau_2)^* = \tau_1^* \times \tau_2^*$.

Definition 5.6. Let $\mathfrak{q} : X_1 \times X_2 \rightarrow \mathcal{Q}(X_1 \times X_2)$ defined by

$$\mathfrak{q}(\langle x_1, x_2 \rangle) = X_1 \times X_2 \mid R_{\langle x_1, x_2 \rangle}$$

for each $\langle x_1, x_2 \rangle \in X_1 \times X_2$.

Proposition 5.3. Under the previous conditions, $\mathfrak{q}(X_1 \times X_2)$ is soft fuzzy Cst-dense in $(\mathcal{Q}(X_1 \times X_2), (\tau_1 \times \tau_2)^*)$, that is $C-cl_{(\tau_1 \times \tau_2)^*}(\mathfrak{q}(1_{X_1 \times X_2}), X_1 \times X_2) = (1_{\mathcal{Q}(X_1 \times X_2)}, \mathcal{Q}(X_1 \times X_2))$.

Proof. Given $(\mu, M) \in SFPCst(X_1 \times X_2)$, we have $\mathfrak{q}(\mu, M) \in SF(\mathcal{Q}(X_1 \times X_2))$. Then for each $(\mu, M) \in SFPCst(X_1 \times X_2)$. Now $\mathfrak{q}(\mu, M) = (\mathfrak{q}(\mu), \mathfrak{q}(M))$

$$\begin{aligned}
q(\mu)(X_1 \times X_2 \mid R) &= \begin{cases} \sup_{\langle x_1, x_2 \rangle \in \mathfrak{q}^{-1}(X_1 \times X_2 \mid R)} \mu(\langle x_1, x_2 \rangle), & \text{if } \mathfrak{q}^{-1}(X_1 \times X_2 \mid R) \neq \phi; \\ 0, & \text{if } \mathfrak{q}^{-1}(X_1 \times X_2 \mid R) = \phi. \end{cases} \\
&= \begin{cases} \mu(\langle x_1, x_2 \rangle), & \text{if } \exists \langle x_1, x_2 \rangle \in X_1 \times X_2 \text{ such that} \\ & X_1 \times X_2 \mid R = X_1 \times X_2 \mid R_{\langle x_1, x_2 \rangle}; \\ 0, & \text{if } \forall \langle x_1, x_2 \rangle \in X_1 \times X_2, X_1 \times X_2 \mid R \neq X_1 \times X_2 \mid R_{\langle x_1, x_2 \rangle}. \end{cases}
\end{aligned}$$

$$\mathfrak{q}(M) = \{\mathfrak{q}(\langle x_1, x_2 \rangle), \forall \langle x_1, x_2 \rangle \in M\}.$$

Now $C-cl_{(\tau_1 \times \tau_2)^*}(\mathfrak{q}(1_{X_1 \times X_2}), X_1 \times X_2)$

$$= \begin{cases} C-cl_{(\tau_1 \times \tau_2)^*}(1_{\mathcal{Q}(X_1 \times X_2)}, \mathcal{Q}(X_1 \times X_2)), & \text{if } \exists \langle x_1, x_2 \rangle \in X_1 \times X_2 \text{ such that} \\ & X_1 \times X_2 \mid R = X_1 \times X_2 \mid R_{\langle x_1, x_2 \rangle}; \\ C-cl_{(\tau_1 \times \tau_2)^*}(0, \phi), & \text{if } \forall \langle x_1, x_2 \rangle \in X_1 \times X_2, X_1 \times X_2 \mid R \neq X_1 \times X_2 \mid R_{\langle x_1, x_2 \rangle}. \end{cases}$$

Let $(\delta, L)^* = \bigcap_{j \in J} (\mu_j, M_j)^* = C-cl_{(\tau_1 \times \tau_2)^*}(\mathfrak{q}(1_{X_1 \times X_2}), X_1 \times X_2)$.

Since $\mathfrak{q}(1_{X_1 \times X_2}, X_1 \times X_2) \sqsubseteq (\delta, L)^*$, we have for each $\langle x_1, x_2 \rangle \in X_1 \times X_2$, $\delta^*((X_1 \times X_2) \mid R_{\langle x_1, x_2 \rangle}) \geq \mathfrak{q}(1_{X_1 \times X_2})((X_1 \times X_2) \mid R_{\langle x_1, x_2 \rangle}) = 1$ and $L^* \supseteq \mathfrak{q}(X_1 \times X_2)$. This implies that, $\delta^*((X_1 \times X_2) \mid R_{\langle x_1, x_2 \rangle}) = 1$ and $L^* = \mathcal{Q}(X_1 \times X_2)$. Now, for

each $\langle x_1, x_2 \rangle \in X_1 \times X_2$ and $j \in J$, $\wedge \mu_j^*((X_1 \times X_2) \mid R_{\langle x_1, x_2 \rangle}) = 1$ and $\cap M_j^* = \mathcal{Q}(X_1 \times X_2)$. It implies $\mu_j^*((X_1 \times X_2) \mid R_{\langle x_1, x_2 \rangle}) = 1$ and $M_j^* = \mathcal{Q}(X_1 \times X_2)$. Thus, for each $\langle x_1, x_2 \rangle \in X_1 \times X_2$ and $j \in J$, $\mu_j(\langle x_1, x_2 \rangle) = 1$ and $M_j = X_1 \times X_2$. That is, for each $j \in J$, $(\mu_j, M_j) = (1_{X_1 \times X_2}, X_1 \times X_2)$. Now, we conclude that, $(\delta^*, L^*) = \prod_{j \in J} (\mu_j, M_j)^* = \prod_{j \in J} (1_{X_1 \times X_2}, X_1 \times X_2)^* = (1_{\mathcal{Q}(X_1 \times X_2)}, \mathcal{Q}(X_1 \times X_2))$. This implies that, $\mathbf{q}(1_{X_1 \times X_2}, X_1 \times X_2)$ is a soft fuzzy Cst-dense set in $(\mathcal{Q}(X_1 \times X_2), (\tau_1 \times \tau_2)^*)$. ■

Proposition 5.4. *The function $\mathbf{q}$ is a soft fuzzy Cst-embedding of $X_1 \times X_2$ into $\mathcal{Q}(X_1 \times X_2)$.*

Proof.

(i) $\mathbf{q}$ is a one to one function:

If $\langle x_1, x_2 \rangle \neq \langle y_1, y_2 \rangle$, we have $R_{\langle x_1, x_2 \rangle} \neq R_{\langle y_1, y_2 \rangle}$. Let

$(\langle x_1, x_2 \rangle)_\alpha, \{\langle x_1, x_2 \rangle\} \neq (\langle y_1, y_2 \rangle)_\beta, \{\langle y_1, y_2 \rangle\}$ be two soft fuzzy points.

(a) If $\langle x_1, x_2 \rangle \neq \langle y_1, y_2 \rangle$ for each $X_1 \times X_2 \mid R \in \mathcal{Q}(X_1 \times X_2)$. We have

$$\mathbf{q}((\langle x_1, x_2 \rangle)_\alpha, \{\langle x_1, x_2 \rangle\}) = (X_1 \times X_2 \mid R_{\langle x_1, x_2 \rangle}_\alpha, \{X_1 \times X_2 \mid R_{\langle x_1, x_2 \rangle}\}).$$

Similarly, $\mathbf{q}((\langle y_1, y_2 \rangle)_\beta, \{\langle y_1, y_2 \rangle\})$ and it is clear that

$$\mathbf{q}((\langle x_1, x_2 \rangle)_\alpha, \{\langle x_1, x_2 \rangle\}) \neq \mathbf{q}((\langle y_1, y_2 \rangle)_\beta, \{\langle y_1, y_2 \rangle\}).$$

(b) If $\langle x_1, x_2 \rangle = \langle y_1, y_2 \rangle$, then $\alpha \neq \beta$ and therefore, clearly

$$\mathbf{q}((\langle x_1, x_2 \rangle)_\alpha, \{\langle x_1, x_2 \rangle\}) \neq \mathbf{q}((\langle y_1, y_2 \rangle)_\beta, \{\langle y_1, y_2 \rangle\}).$$

Hence $\mathbf{q}$ is one to one.

(ii) $\mathbf{q}$ is soft fuzzy product Cst-continuous:

For each $(\mu, M)^* \in \mathfrak{b}^*$ and $\langle x_1, x_2 \rangle \in X_1 \times X_2$, we have

$$\begin{aligned} \mathbf{q}^{-1}(\mu, M)^* &= \mathbf{q}^{-1}(\mu^*, M^*) \\ &= (\mathbf{q}^{-1}(\mu^*), \mathbf{q}^{-1}(M^*)) \\ &= (\mu^* \circ \mathbf{q}, \mathbf{q}^{-1}(M^*)) \end{aligned}$$

where

$$\begin{aligned} \mu^* \circ \mathbf{q}(\langle x_1, x_2 \rangle) &= \mu^*(\mathbf{q}(\langle x_1, x_2 \rangle)) \\ &= \mu^*(X_1 \times X_2 \mid R_{\langle x_1, x_2 \rangle}) \\ &= \mu(\langle x_1, x_2 \rangle) \end{aligned}$$

and $\mathbf{q}^{-1}(M^*) = M$. Thus $\mathbf{q}^{-1}(\mu, M)^* = (\mu, M) \in St(\tau_1 \times \tau_2)$. Hence $\mathbf{q}$ is soft fuzzy product Cst-continuous.

(iii) $\mathbf{q}$ is a soft fuzzy product Cst-open function on $\mathcal{Q}(X_1 \times X_2)$:

For each $X_1 \times X_2 \mid R \in \mathcal{Q}(X_1 \times X_2)$ and $(\mu, M) \in St(\tau_1 \times \tau_2)$.

$$\begin{aligned} (\mu^*, M^*) \sqcap (\chi_{\mathbf{q}(X_1 \times X_2)}, \{\mathbf{q}(X_1 \times X_2)\}) &= (\mu^* \wedge \chi_{\mathbf{q}(X_1 \times X_2)}, M^* \cap \{\mathbf{q}(X_1 \times X_2)\}) \\ &= (\mathbf{q}(\mu), \mathbf{q}(M)) = \mathbf{q}(\mu, M) \in (\tau_1 \times \tau_2)^* \end{aligned}$$

Thus $\mathbf{q}$ is a soft fuzzy product Cst-open function. Hence $\mathbf{q}$ is a soft fuzzy Cst-embedding of $X_1 \times X_2$ into $\mathcal{Q}(X_1 \times X_2)$. ■

Proposition 5.5. *The soft fuzzy quotient product space $(\mathcal{Q}(X_1 \times X_2), (\tau_1 \times \tau_2)^*)$ is soft fuzzy product Cst-compact.*

Proof. Let $\mathfrak{F} = \{(\lambda_i^*, N_i^*) \in \tau_1^* \times \tau_2^* : (\lambda_i, N_i) \in St(\tau_1 \times \tau_2) \text{ for } i \in J\}$ be a soft fuzzy product Cst-open cover of $\mathcal{Q}(X_1 \times X_2)$. That is,

$$\sqcup_{i \in J} (\lambda_i^*, N_i^*) = (1_{\mathcal{Q}(X_1 \times X_2)}, \mathcal{Q}(X_1 \times X_2)).$$

By definition of (λ_i^*, N_i^*) , $\sqcup_{i \in F} (\lambda_i^*, N_i^*) \sqsubseteq (1_{\mathcal{Q}(X_1 \times X_2)}, \mathcal{Q}(X_1 \times X_2))$, for some finite subfamily F of J. Thus $\mathfrak{F}$ has a soft fuzzy product finite subcover.

Hence, $(\mathcal{Q}(X_1 \times X_2), (\tau_1 \times \tau_2)^*)$ is soft fuzzy product Cst-compact. ■

Conclusion. Cst-compactification of the category of all product topological space can be done, using section- 5.

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DESCRIBING GREEN'S RELATIONS IN ORDERED Γ -GROUPOIDS USING A NEW CONCEPT: FUZZY SUBSETS¹

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Abstract. This work extends the idea of the Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{I}$ and the usual relation $\mathcal{N}$ of ordered groupoids to ordered Γ -groupoids in terms of fuzzy subsets, the concept of the Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{I}$ and the usual relation $\mathcal{N}$ of ordered Γ -groupoids is introduced and investigated, which is an interesting for ordered Γ -groupoids and some interesting characterizations of the Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{I}$ and the usual relation $\mathcal{N}$ of ordered Γ -groupoids are obtained. Also, we define relations $\mathcal{R}^F, \mathcal{L}^F, \mathcal{I}^F$ and $\mathcal{N}^F$ of ordered Γ -groupoids in terms of fuzzy subsets and we prove that $\mathcal{R}^F = \mathcal{R}$, $\mathcal{L}^F = \mathcal{L}$, $\mathcal{I}^F = \mathcal{I}$ and $\mathcal{N}^F = \mathcal{N}$ in the same way as of the results of Kehayopulu and Tsingelis [8].

Keywords: Green's relations; ordered Γ -groupoid; fuzzy subset.

Mathematics Subject Classification: 20M10.

1. Introduction and prerequisites

The notion of the Green's relations of semigroups introduced by Green [5] in 1951, several researches have characterized the many type of the Green's relations on the algebraic structures such as: In 2002, Petro and Pasku [14] studied the

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Green-Kehayopulu relation $\mathcal{H}$ in le -semigroups mimics the definition of the usual Green relation $\mathcal{H}$ in semigroups. They proved that certain properties of $\mathcal{H}$ -classes essentially differ from those of $\mathcal{H}$ -classes need not constitute a subsemigroup. In 2007, Kehayopulu and Tsingelis [8] characterized the Green's relations $\mathcal{R}, \mathcal{L}$ and $\mathcal{I}$ of ordered groupoids in terms of fuzzy subsets. For an ordered groupoid S , they defined relations $\mathcal{R}^F, \mathcal{L}^F$ and $\mathcal{I}^F$ in terms of fuzzy subsets and showed that each of them coincides with the corresponding Green's relations on S . In 2008, Chinram and Siammai [2] studied the Green's relations for Γ -semigroups and reductive Γ -semigroups. In this year, Siripitukdet and Iampan [21] introduced the concept of the Green-Kehayopulu relations in le - Γ -semigroups mimics the definition of the Green-Kehayopulu relations in le -semigroups. They proved that an $\mathcal{H}_\gamma$ -class of an le - Γ -semigroup M satisfies Green's condition if and only if it contains a γ -idempotent and an $\mathcal{H}_\gamma$ -class of an le - Γ -semigroup M is a subgroup of $\langle M_\gamma, \circ \rangle$ if and only if it consists of a single idempotent.

A fuzzy subset f of a set S is a function from S to a closed interval $[0, 1]$. The concept of a fuzzy subset of a set was first considered by Zadeh [22] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere.

After the introduction of the concept of fuzzy sets by Zadeh [22], several researches were conducted on the generalizations of the notion of fuzzy set and application to many algebraic structures such as: In 1971, Rosenfeld [15] was the first who studied fuzzy sets in the structure of groups. Fuzzy semigroups have been first considered by Kuroki [9], [10], [11], [12], and fuzzy ordered groupoids and ordered semigroups, by Kehayopulu and Tsingelis [6], [7]. In 2007, Kehayopulu and Tsingelis [8] characterized the Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{I}$ of ordered groupoids in terms of fuzzy subsets. In 2008, Shabir and Khan [19] defined fuzzy bi-ideal subsets and fuzzy bi-filters in ordered semigroups and characterized ordered semigroups in terms of fuzzy bi-ideal subsets and fuzzy bi-filters. In 2009, Majumder and Sardar [13] studied fuzzy ideals and fuzzy ideal extensions in ordered semigroups. In 2010, Chinram and Saelee [1] studied fuzzy ternary subsemigroups (left ideals, right ideals, lateral ideals, ideals) and fuzzy left filters (right filters, lateral filters, filters) of ordered ternary semigroups. In 2010, Shah and Rehman [20] introduced Γ -ideals and Γ -bi-ideals of Γ -AG-groupoids which are in fact a generalization of ideals and bi-ideals of AG-groupoids and studied some characteristics of Γ -ideals and Γ -bi-ideals of Γ -AG-groupoids. In 2010, Iampan [3] characterized the relationship between the fuzzy ordered ideals (fuzzy ordered filters) and the characteristic mappings of fuzzy ordered ideals (fuzzy ordered filters) in ordered Γ -semigroups. In 2011, Saelee and Chinram [16] studied rough, fuzzy and rough fuzzy bi-ideals of ternary semigroups.

As we know, Γ -groupoids are a generalization of groupoids. This paper is a sequel to our study [7] of fuzzification in ordered groupoids. The Green's relations defined in terms of ordered right ideals, ordered left ideals, ordered ideals and the concept of fuzzy sets play an important role in studying the structure of ordered groupoids. Now, we characterize the Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{I}$ and the

usual relation $\mathcal{N}$ defined in terms of ordered filters on ordered Γ -groupoids in terms of fuzzy subsets. To present the main theorems we discuss some elementary definitions that we use later.

Definition 1.1. Let M be a set. A *fuzzy subset* of M is an arbitrary mapping $f: M \rightarrow [0, 1]$ where $[0, 1]$ is the unit segment of the real line.

Definition 1.2. Let M be a set and $A \subseteq M$. The *characteristic mapping* $f_A: M \rightarrow [0, 1]$ defined via

$$x \mapsto f_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

By the definition of characteristic mapping, f_A is a mapping of M into $\{0, 1\} \subset [0, 1]$. Hence f_A is a fuzzy subset of M .

Definition 1.3. Let M and Γ be any two nonempty sets. Then (M, Γ) is called a Γ -*groupoid* [18] if there exists a mappings $M \times \Gamma \times M \rightarrow M$, written as $(a, \gamma, b) \mapsto a\gamma b$. A nonempty subset K of M is called a Γ -*subgroupoid* of M if $K\Gamma K \subseteq K$.

Definition 1.4. A partially ordered Γ -groupoid $(M, \Gamma, \leq)$ is called an *ordered Γ -groupoid* [17] if for any $a, b, c \in M$ and $\gamma \in \Gamma$, $a \leq b$ implies $a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$.

Definition 1.5. Let $(M, \Gamma, \leq)$ be an ordered Γ -groupoid. For each nonempty subset H of M , we define

$$(H] = \{t \in M \mid t \leq h \text{ for some } h \in H\}.$$

Definition 1.6. Let $(M, \Gamma, \leq)$ be an ordered Γ -groupoid. A nonempty subset A of M is called an *ordered left ideal* of M if

$$\text{L1. } M\Gamma A \subseteq A.$$

$$\text{L2. For any } b \in M \text{ and } a \in A, b \leq a \text{ implies } b \in A.$$

A nonempty subset A of M is called an *ordered right ideal* of M if

$$\text{R1. } A\Gamma M \subseteq A.$$

$$\text{R2. For any } b \in M \text{ and } a \in A, b \leq a \text{ implies } b \in A.$$

A nonempty subset A of M is called an *ordered ideal* of M if it is both an ordered left ideal and an ordered right ideal of M . That is,

$$\text{I1. } M\Gamma A \subseteq A \text{ and } A\Gamma M \subseteq A.$$

$$\text{I2. For any } b \in M \text{ and } a \in A, b \leq a \text{ implies } b \in A.$$

We denote by $R(x)$, $L(x)$, $I(x)$ the ordered right ideal, ordered left ideal, ordered ideal of M , respectively, generated by x .

Definition 1.7. The Greens relations $\mathcal{R}, \mathcal{L}$ and $\mathcal{I}$ are the equivalence relations on ordered Γ -groupoid M , defined as follows:

$$\begin{aligned}\mathcal{R} &:= \{(x, y) \mid R(x) = R(y)\}, \\ \mathcal{L} &:= \{(x, y) \mid L(x) = L(y)\}, \\ \mathcal{I} &:= \{(x, y) \mid I(x) = I(y)\}.\end{aligned}$$

Definition 1.8. Let $(M, \Gamma, \leq)$ be an ordered Γ -groupoid. A Γ -subgroupoid F of M is called an *ordered filter* of M if

- F1. For any $a, b \in M$ and $\gamma \in \Gamma$, $a\gamma b \in F$ implies $a, b \in F$.
- F2. For any $b \in M$ and $a \in F$, $a \leq b$ implies $b \in F$.

For an element $x \in M$, we denote by $N(x)$ the ordered filter of M generated by x , and by $\mathcal{N}$ the relation on M defined by

$$\mathcal{N} := \{(x, y) \mid N(x) = N(y)\}.$$

We can easily prove that for two subsets A and B of ordered Γ -groupoid M , we have $A \subseteq B$ if and only if $f_A \subseteq f_B$ and $A = B$ if and only if $f_A = f_B$. We denote by $F(M)$ the set of all fuzzy subsets of M . We define an order relation “ $\subseteq$ ” on $F(M)$ as follows: For $f, g \in F(M)$, we define $f \subseteq g$ if and only if $f(x) \leq g(x)$ for all $x \in M$. Clearly, $(F(M), \subseteq)$ is an ordered set. Let $0_F, 1_F$ be the fuzzy subsets of M defined by: $0_F: M \rightarrow [0, 1] \mid x \mapsto 0_F(x) := 0$ and $1_F: M \rightarrow [0, 1] \mid x \mapsto 1_F(x) := 1$. It is clear that the mapping 0_F is the least element of $F(M)$ and the mapping 1_F is the greatest element $F(M)$.

Definition 1.9. Let $(M, \Gamma, \leq)$ be an ordered Γ -groupoid. A fuzzy subset f of M is called a *fuzzy ordered left ideal* of M if

- FL1. For any $a, b \in M$, $a \leq b$ implies $f(a) \geq f(b)$.
- FL2. $f(a\gamma b) \geq f(b)$ for all $a, b \in M$ and $\gamma \in \Gamma$.

A fuzzy subset f of M is called a *fuzzy ordered right ideal* of M if

- FR1. For any $a, b \in M$, $a \leq b$ implies $f(a) \geq f(b)$.
- FR2. $f(a\gamma b) \geq f(a)$ for all $a, b \in M$ and $\gamma \in \Gamma$.

A fuzzy subset f of M is called a *fuzzy ordered ideal* of M if it is both a fuzzy ordered left and a fuzzy ordered right ideal of M . That is,

- FI1. For any $a, b \in M$, $a \leq b$ implies $f(a) \geq f(b)$.
- FI2. $f(a\gamma b) \geq f(b)$ and $f(a\gamma b) \geq f(a)$ for all $a, b \in M$ and $\gamma \in \Gamma$.

Definition 1.10. Let $(M, \Gamma, \leq)$ be an ordered Γ -groupoid. A fuzzy subset f of M is called a *fuzzy ordered filter* of M if

FF1. For any $a, b \in M$, $a \leq b$ implies $f(a) \leq f(b)$.

FF2. $f(a\gamma b) = \min\{f(a), f(b)\}$ for all $a, b \in M$ and $\gamma \in \Gamma$.

Definition 1.11. Let (M, Γ) be a Γ -groupoid. A fuzzy subset f of M is called a *fuzzy Γ -subgroupoid* of M if $f(a\gamma b) \geq \min\{f(a), f(b)\}$ for all $a, b \in M$ and $\gamma \in \Gamma$.

Definition 1.12. Let (M, Γ) be a Γ -groupoid and f a fuzzy subset of M . The mapping

$$f': M \rightarrow [0, 1] \text{ defined via } f'(x) = 1 - f(x)$$

is a fuzzy subset of M called the *complement* of f in S .

Definition 1.13. Let $(M, \Gamma, \leq)$ be an ordered Γ -groupoid. A fuzzy subset f of M is called *prime* if $f(a\gamma b) \leq \max\{f(a), f(b)\}$ for all $a, b \in M$ and $\gamma \in \Gamma$.

2. The Green's relations in terms of Fuzzy Subsets

In this section, we give some interesting properties of fuzzy subsets and study the Green's Relations of ordered Γ -groupoids in terms of fuzzy subsets.

Proposition 2.1. Let $(M, \Gamma, \leq)$ be an ordered Γ -groupoid and $\{f_i \mid i \in I\}$ a nonempty family of fuzzy subsets of M . Then

$$\bigwedge_{i \in I} f_i \in F(M) \text{ and } \bigvee_{i \in I} f_i \in F(M).$$

Proof. Let $x \in M$. The set $\{f_i(x) \mid i \in I\}$ is a nonempty lower bounded subset of $\mathbb{R}$, so there exists the $\inf\{f_i(x) \mid i \in I\}$ in $\mathbb{R}$. Since $0 \leq f_i(x) \leq 1$ for each $i \in I$, we have $0 \leq \inf\{f_i(x) \mid i \in I\} \leq 1$. If $x, y \in M$ is such that $x = y$, then clearly $(\bigwedge_{i \in I} f_i)(x) = (\bigwedge_{i \in I} f_i)(y)$. Hence $\bigwedge_{i \in I} f_i \in F(M)$. Similarly, $\bigvee_{i \in I} f_i \in F(M)$. ■

Remark 2.2. By Proposition 2.1, we have $(F(M), \subseteq)$ is a complete lattice.

Remark 2.3. The following two statements hold true:

$$(i) \bigwedge_{i \in I} f_i \subseteq f_j \text{ for all } j \in I,$$

$$(ii) f_j \subseteq \bigvee_{i \in I} f_i \text{ for all } j \in I.$$

Proposition 2.4. *Let $(M, \Gamma, \leq)$ be an ordered Γ -groupoid and $\{f_i \mid i \in I\}$ a nonempty family of fuzzy subsets of M . Then we have the following:*

- (i) $\bigwedge_{i \in I} f_i = \inf\{f_i \mid i \in I\},$
- (ii) $\bigvee_{i \in I} f_i = \sup\{f_i \mid i \in I\}.$

Proof. (i) By Proposition 2.1, $\bigwedge_{i \in I} f_i \in F(M)$ and by Remark 2.3, $\bigwedge_{i \in I} f_i \subseteq f_i$ for all $i \in I$. Thus $\bigwedge_{i \in I} f_i$ is a lower bound of $\{f_i \mid i \in I\}$. Let $g \in F(M)$ be such that $g \subseteq f_i$ for all $i \in I$. Then $g(x) \leq f_i(x)$ for all $x \in M$ and $i \in I$, so $g(x)$ is a lower bound of $\{f_i(x) \mid i \in I\}$ for all $x \in M$. By the proof of Proposition 2.1, we have

$$g(x) \leq \inf\{f_i(x) \mid i \in I\} := (\bigwedge_{i \in I} f_i)(x) \text{ for all } x \in M.$$

Thus $g \subseteq \bigwedge_{i \in I} f_i$, so $\bigwedge_{i \in I} f_i = \inf\{f_i \mid i \in I\}$. The proof of (ii) is similar. $\blacksquare$

Proposition 2.5. *Let $(M, \Gamma, \leq)$ be an ordered Γ -groupoid and $\{f_i \mid i \in I\}$ a nonempty family of fuzzy subsets of M . Then we have the following:*

- (i) *If f_i is a fuzzy ordered right ideal of M for all $i \in I$, then $\bigwedge_{i \in I} f_i$ and $\bigvee_{i \in I} f_i$ are fuzzy ordered right ideals of M .*
- (ii) *If f_i is a fuzzy ordered left ideal of M for all $i \in I$, then $\bigwedge_{i \in I} f_i$ and $\bigvee_{i \in I} f_i$ are fuzzy ordered left ideals of M .*
- (iii) *If f_i is a fuzzy ordered ideal of M for all $i \in I$, then $\bigwedge_{i \in I} f_i$ and $\bigvee_{i \in I} f_i$ are fuzzy ordered ideals of M .*
- (iv) *If f_i is a fuzzy Γ -subgroupoid of M for all $i \in I$, then $\bigwedge_{i \in I} f_i$ is a fuzzy Γ -subgroupoid of M .*
- (v) *If f_i is a fuzzy ordered filter of M for all $i \in I$, then $\bigwedge_{i \in I} f_i$ is a fuzzy ordered filter of M .*

Proof. (i) By Proposition 2.1, $\bigwedge_{i \in I} f_i$ is a fuzzy subset of M . Let $x, y \in M$ and $\gamma \in \Gamma$. Since f_i is a fuzzy ordered right ideal of M , we have $f_i(x\gamma y) \geq f_i(x)$ for all $i \in I$. Then

$$(\bigwedge_{i \in I} f_i)(x\gamma y) := \inf\{f_i(x\gamma y) \mid i \in I\} \geq \inf\{f_i(x) \mid i \in I\} := (\bigwedge_{i \in I} f_i)(x).$$

Let $x, y \in M$ be such that $x \leq y$. Since f_i is a fuzzy ordered right ideal of M , we have $f_i(x) \geq f_i(y)$ for all $i \in I$. Then

$$(\bigwedge_{i \in I} f_i)(x) := \inf\{f_i(x) \mid i \in I\} \geq \inf\{f_i(y) \mid i \in I\} := (\bigwedge_{i \in I} f_i)(y).$$

Hence $\bigwedge_{i \in I} f_i$ is a fuzzy ordered right ideals of M . Similarly, $\bigvee_{i \in I} f_i$ is a fuzzy ordered right ideals of M .

(ii) It is proved similarly.

(iii) It follows from (i) and (ii).

(iv) Let $x, y \in M$ and $\gamma \in \Gamma$. By Proposition 2.1, $\bigwedge_{i \in I} f_i$ is a fuzzy subset of M . Since f_i is a fuzzy Γ -subgroupoid of M , we have $f_i(x\gamma y) \geq \min\{f_i(x), f_i(y)\}$ for all $i \in I$. Then $f_i(x\gamma y) \geq f_i(x)$ or $f_i(x\gamma y) \geq f_i(y)$ for all $i \in I$. Since $f_i(x) \geq \inf\{f_i(x) \mid i \in I\}$ and $f_i(y) \geq \inf\{f_i(y) \mid i \in I\}$, we have $f_i(x\gamma y) \geq \inf\{f_i(x) \mid i \in I\} := (\bigwedge_{i \in I} f_i)(x)$ or $f_i(x\gamma y) \geq \inf\{f_i(y) \mid i \in I\} := (\bigwedge_{i \in I} f_i)(y)$ for all $i \in I$. Thus $f_i(x\gamma y) \geq \min\{(\bigwedge_{i \in I} f_i)(x), (\bigwedge_{i \in I} f_i)(y)\}$ for all $i \in I$. Hence

$$(\bigwedge_{i \in I} f_i)(x\gamma y) := \inf\{f_i(x\gamma y) \mid i \in I\} \geq \min\{(\bigwedge_{i \in I} f_i)(x), (\bigwedge_{i \in I} f_i)(y)\}.$$

Therefore $\bigwedge_{i \in I} f_i$ is a fuzzy Γ -subgroupoid of M .

(v) Let $x, y \in M$ and $\gamma \in \Gamma$. By Proposition 2.1, $\bigwedge_{i \in I} f_i$ is a fuzzy subset of M . Since f_i is a fuzzy ordered filter of M , we have $f_i(x\gamma y) = \min\{f_i(x), f_i(y)\}$ for all $i \in I$. Then $f_i(x\gamma y) = \min\{f_i(x), f_i(y)\} \leq f_i(x), f_i(y)$ for all $i \in I$. Thus $(\bigwedge_{i \in I} f_i)(x\gamma y) := \inf\{f_i(x\gamma y) \mid i \in I\} \leq \inf\{f_i(x) \mid i \in I\} := (\bigwedge_{i \in I} f_i)(x)$ and $(\bigwedge_{i \in I} f_i)(x\gamma y) := \inf\{f_i(x\gamma y) \mid i \in I\} \leq \inf\{f_i(y) \mid i \in I\} := (\bigwedge_{i \in I} f_i)(y)$. Hence

$$(\bigwedge_{i \in I} f_i)(x\gamma y) \leq \min\{(\bigwedge_{i \in I} f_i)(x), (\bigwedge_{i \in I} f_i)(y)\}.$$

On the other hand, by (iv), the mapping $\bigwedge_{i \in I} f_i$ is a fuzzy Γ -subgroupoid of M , which means that $(\bigwedge_{i \in I} f_i)(x\gamma y) \geq \min\{(\bigwedge_{i \in I} f_i)(x), (\bigwedge_{i \in I} f_i)(y)\}$. Thus

$$(\bigwedge_{i \in I} f_i)(x\gamma y) = \min\{(\bigwedge_{i \in I} f_i)(x), (\bigwedge_{i \in I} f_i)(y)\}.$$

Let $x, y \in M$ be such that $x \leq y$. Since f_i is a fuzzy ordered filter of M , we have $f_i(x) \leq f_i(y)$ for all $i \in I$. Then

$$\left(\bigwedge_{i \in I} f_i\right)(x) := \inf\{f_i(x) \mid i \in I\} \leq \inf\{f_i(y) \mid i \in I\} := \left(\bigwedge_{i \in I} f_i\right)(y).$$

Therefore $\bigwedge_{i \in I} f_i$ is a fuzzy ordered filter of M . ■

Notation 2.6. For an ordered Γ -groupoid $(M, \Gamma, \leq)$ and a fuzzy subset f of M , we denote

$$\begin{aligned} R_f &:= \{g \mid g \text{ fuzzy ordered right ideal of } M \text{ and } g \supseteq f\}, \\ L_f &:= \{g \mid g \text{ fuzzy ordered left ideal of } M \text{ and } g \supseteq f\}, \\ I_f &:= \{g \mid g \text{ fuzzy ordered ideal of } M \text{ and } g \supseteq f\}, \\ N_f &:= \{g \mid g \text{ fuzzy ordered filter of } M \text{ and } g \supseteq f\}. \end{aligned}$$

Remark 2.7. Clearly, $1_F \in R_f, 1_F \in L_f, 1_F \in I_f$ and $1_F \in N_f$. Since R_f, L_f, I_f and N_f are nonempty subsets of $F(M)$, by Proposition 2.1, the elements $\bigwedge_{g \in R_f} g$,

$\bigwedge_{g \in L_f} g$, $\bigwedge_{g \in I_f} g$ and $\bigwedge_{g \in N_f} g$ exist in $F(M)$. Moreover, by Proposition 2.5, we have the following:

- (i) $\bigwedge_{g \in R_f} g$ is a fuzzy ordered right ideal of M ,
- (ii) $\bigwedge_{g \in L_f} g$ is a fuzzy ordered left ideal of M ,
- (iii) $\bigwedge_{g \in I_f} g$ is a fuzzy ordered ideal of M ,
- (iv) $\bigwedge_{g \in N_f} g$ is a fuzzy ordered filter of M .

Proposition 2.8. Let $(M, \Gamma, \leq)$ be an ordered Γ -groupoid and f a fuzzy subsets of M . Then we have the following:

- (i) $\bigwedge_{g \in R_f} g \in R_f$ (resp. $\bigwedge_{g \in L_f} g \in L_f$, $\bigwedge_{g \in I_f} g \in I_f$ and $\bigwedge_{g \in N_f} g \in N_f$),
- (ii) $\bigwedge_{g \in R_f} g \subseteq h$ (resp. $\bigwedge_{g \in L_f} g \subseteq h$, $\bigwedge_{g \in I_f} g \subseteq h$ and $\bigwedge_{g \in N_f} g \subseteq h$) for all $h \in R_f$ (resp. $h \in L_f, I_f, N_f$).

Proof. (i) By Remark 2.7, the element $\bigwedge_{g \in R_f} g$ is a fuzzy ordered right ideal of M .

Let $x \in M$ and $g \in R_f$. Since $g \supseteq f$, we have $f(x) \leq g(x)$. Thus $f(x) \leq g(x)$ for all $g \in R_f$, so

$$f(x) \leq \inf\{g(x) \mid g \in R_f\} := (\bigwedge_{g \in R_f} g)(x).$$

Hence $\bigwedge_{g \in R_f} g \supseteq f$, so $\bigwedge_{g \in R_f} g \in R_f$.

(ii) It follows from Remark 2.3. ■

Remark 2.9. By Proposition 2.8, we have the following:

- (i) $\bigwedge_{g \in R_f} g$ is a fuzzy ordered right ideal of M generated by f ,
- (ii) $\bigwedge_{g \in L_f} g$ is a fuzzy ordered left ideal of M generated by f ,
- (iii) $\bigwedge_{g \in I_f} g$ is a fuzzy ordered ideal of M generated by f ,
- (iv) $\bigwedge_{g \in N_f} g$ is a fuzzy ordered filter of M generated by f .

Proposition 2.10. Let $(M, \Gamma, \leq)$ be an ordered Γ -groupoid and f a fuzzy subsets of M . Let h be a fuzzy ordered right ideal (resp. fuzzy ordered left ideal, fuzzy ordered ideal and fuzzy ordered filter) of M generated by f . Then $h = \bigwedge_{g \in R_f} g$ (resp.

$$h = \bigwedge_{g \in L_f} g, h = \bigwedge_{g \in I_f} g \text{ and } h = \bigwedge_{g \in N_f} g).$$

Proof. Since $h \in R_f$ and by Remark 2.3, we have $\bigwedge_{g \in R_f} g \subseteq h$. By Proposition 2.8, we have $\bigwedge_{g \in R_f} g \in R_f$. Since h is a fuzzy ordered right ideal of M generated by f , we have $h \subseteq \bigwedge_{g \in R_f} g$. Hence $h = \bigwedge_{g \in R_f} g$. The other cases are proved similarly. ■

Notation 2.11. If $(M, \Gamma, \leq)$ be an ordered Γ -groupoid and f a fuzzy subsets of M , we denote by $R(f)$ (resp. $L(f)$, $I(f)$ and $N(f)$) the fuzzy ordered right ideal (resp. fuzzy ordered left ideal, fuzzy ordered ideal and fuzzy ordered filter) of M generated by f . That is,

$$R(f) := \bigwedge_{g \in R_f} g \text{ (resp. } L(f) := \bigwedge_{g \in L_f} g, I(f) := \bigwedge_{g \in I_f} g \text{ and } N(f) := \bigwedge_{g \in N_f} g).$$

Remark 2.12. Clearly, $1_F \in R_x, 1_F \in L_x, 1_F \in I_x$ and $1_F \in N_x$. Since R_x, L_x, I_x and N_x are nonempty subsets of $F(M)$, by Proposition 2.1, the elements $\bigwedge_{g \in R_x} g, \bigwedge_{g \in L_x} g, \bigwedge_{g \in I_x} g$ and $\bigwedge_{g \in N_x} g$ exist in $F(M)$. Moreover, by Proposition 2.5, we have the following:

- (i) $\bigwedge_{g \in R_x} g$ is a fuzzy ordered right ideal of M ,
- (ii) $\bigwedge_{g \in L_x} g$ is a fuzzy ordered left ideal of M ,
- (iii) $\bigwedge_{g \in I_x} g$ is a fuzzy ordered ideal of M ,
- (iv) $\bigwedge_{g \in N_x} g$ is a fuzzy ordered filter of M .

Theorem 2.13. *Let $(M, \Gamma, \leq)$ be an ordered Γ -groupoid and $x \in M$. Then we have the following:*

- (i) $R_x = R_{f_{\{x\}}}$,
- (ii) $L_x = L_{f_{\{x\}}}$,
- (iii) $I_x = I_{f_{\{x\}}}$,
- (iv) $N_x = N_{f_{\{x\}}}$.

Proof. (i) Let $f \in R_x$. Then f is a fuzzy ordered right ideal of M and $f(x) = 1$. Let $y \in M$. If $y = x$, then $1 = f(x) = f(y)$ and $f_{\{x\}}(y) = f_{\{x\}}(x) = 1$, so $f_{\{x\}}(y) = f(y)$. If $y \neq x$, then $f_{\{x\}}(y) = 0 \leq f(y)$. Since $f_{\{x\}}(y) \leq f(y)$ for all $y \in M$, we have $f \supseteq f_{\{x\}}$. Thus $f \in R_{f_{\{x\}}}$. On the other hand, let $g \in R_{f_{\{x\}}}$. Then g is a fuzzy ordered right ideal of M and $g \supseteq f_{\{x\}}$. Thus $1 = f_{\{x\}}(x) \leq g(x)$, so $g(x) = 1$. Hence $g \in R_x$. Therefore $R_x = R_{f_{\{x\}}}$. The other cases can be proved similarly. ■

Proposition 2.14. *Let $(M, \Gamma, \leq)$ be an ordered Γ -groupoid and f a fuzzy ordered right ideal (resp. fuzzy ordered left ideal, fuzzy ordered ideal and fuzzy ordered filter) of M . Then we have the following:*

- (i) $f^{-1}(1) = \emptyset$ or $f^{-1}(1)$ is an ordered right ideal (resp. ordered left ideal, ordered ideal and ordered filter) of M ,
- (ii) $f_{h^{-1}(1)} \subseteq h$ for all $h \in R_x$ (resp. for all $h \in L_x, h \in I_x$ and $h \in N_x$).

Proof. (i) Assume that $f^{-1}(1) \neq \emptyset$. Let $a \in f^{-1}(1), b \in M$ and $\gamma \in \Gamma$. Then $f(a) = 1$. Since f is a fuzzy ordered right ideal of M , we have $f(a\gamma b) \geq f(a) = 1$. Thus $f(a\gamma b) = 1$, so $a\gamma b \in f^{-1}(1)$. Let $a \in M$ and $b \in f^{-1}(1)$ be such that $a \leq b$. Then $f(b) = 1$. Since f is a fuzzy ordered right ideal of M and $a \leq b$, we have $f(a) \geq f(b) = 1$. Thus $f(a) = 1$, so $a \in f^{-1}(1)$. Hence $f^{-1}(1)$ is an ordered right ideal of M . The rest of the proof is similar.

(ii) Let $h \in R_x$ and let $x \in M$. If $x \in h^{-1}(1)$, then $f_{h^{-1}(1)}(x) = 1$ and $h(x) = 1$, so $f_{h^{-1}(1)}(x) = h(x)$. If $x \notin h^{-1}(1)$, then $f_{h^{-1}(1)}(x) = 0 \leq h(x)$. Since $f_{h^{-1}(1)}(x) \leq h(x)$ for all $x \in M$, we have $f_{h^{-1}(1)} \subseteq h$. ■

Lemma 2.15. [4] *Let $(M, \Gamma, \leq)$ be an ordered Γ -groupoid. A nonempty subset R (resp. L) of M is an ordered right ideal (resp. ordered left ideal) of M if and only if the characteristic function f_R (resp. f_L) is a fuzzy ordered right ideal (resp. fuzzy ordered left ideal) of M . A nonempty subset I of M is an ordered ideal (resp. ordered filter) of M if and only if the characteristic function f_I is a fuzzy ordered ideal (resp. fuzzy ordered filter) of M .*

Remark 2.16. If $(M, \Gamma, \leq)$ is an ordered Γ -groupoid and A an ordered right ideal (resp. ordered left ideal, ordered ideal and ordered filter) of M , then there exists a fuzzy ordered right ideal (resp. fuzzy ordered left ideal, fuzzy ordered ideal and fuzzy ordered filter) f of M such that $f^{-1}(1) = A$. Indeed: If A an ordered right ideal of M , then the characteristic function f_A is a fuzzy ordered right ideal of M . On the other hand, we have $x \in A$ if and only if $f_A(x) = 1$ if and only if $x \in f_A^{-1}(1)$. Hence $f_A^{-1}(1) = A$.

Theorem 2.17. *Let $(M, \Gamma, \leq)$ be an ordered Γ -groupoid and $x \in M$. Then we have the following:*

- (i) $R(f_{\{x\}}) = f_{R(x)}$,
- (ii) $L(f_{\{x\}}) = f_{L(x)}$,
- (iii) $I(f_{\{x\}}) = f_{I(x)}$,
- (iv) $N(f_{\{x\}}) = f_{N(x)}$.

Proof. (i) According to Notation 2.11, Theorem 2.13 and Remark 2.3, we have

$$(\star) \quad R(f_{\{x\}}) := \bigwedge_{h \in R_{f_{\{x\}}}} h = \bigwedge_{h \in R_x} h \subseteq h \text{ for all } h \in R_x.$$

Since $R(x)$ is an ordered right ideal of M , it follows from Lemma 2.15 that $f_{R(x)}$ is a fuzzy ordered right ideal of M . Since $x \in R(x)$, we get $f_{R(x)}(x) = 1$. Thus $f_{R(x)} \in R_x$. Then, by $(\star)$, we have $R(f_{\{x\}}) \subseteq f_{R(x)}$. On the other hand, let $h \in R_x$. Since $h(x) = 1$, we have $x \in h^{-1}(1)$. By Proposition 2.14(i), we have $h^{-1}(1)$ is an ordered right ideal of M containing x , so $R(x) \subseteq h^{-1}(1)$. By Proposition 2.14(ii), we have $f_{R(x)} \subseteq f_{h^{-1}(1)} \subseteq h$. Thus

$$(\star\star) \quad f_{R(x)} \subseteq h \text{ for all } h \in R_x.$$

By $(\star\star)$, Proposition 2.4 and $(\star)$, we have

$$f_{R(x)} \subseteq \inf\{h \mid h \in R_x\} = \bigwedge_{h \in R_x} h = R(f_{\{x\}}).$$

Hence $R(f_{\{x\}}) = f_{R(x)}$.

The other cases can be proved similarly. ■

As we have already seen, $R(f_{\{a\}})$ is the fuzzy ordered right ideal of M generated by $f_{\{a\}}$, $L(f_{\{a\}})$ is the fuzzy ordered left ideal of M generated by $f_{\{a\}}$ and $I(f_{\{a\}})$ (resp. $N(f_{\{a\}})$ is the fuzzy ordered ideal (resp. fuzzy ordered filter) of M generated by $f_{\{a\}}$.

Notation 2.18. For an element $a \in M$, we denote:

$$\begin{aligned} R^F(a) &= R(f_{\{a\}}), \\ L^F(a) &= L(f_{\{a\}}), \\ I^F(a) &= I(f_{\{a\}}), \\ N^F(a) &= N(f_{\{a\}}). \end{aligned}$$

Definition 2.19. Let $(M, \Gamma, \leq)$ be an ordered Γ -groupoid, f a fuzzy subset of M and $x \in M$. We say that f contains x if $f(x) = 1$.

Remark 2.20. According to Notation 2.18, Notation 2.11, and Theorem 2.13, for any $x \in M$, we have the following:

$$\begin{aligned} \text{(i)} \quad R^F(x) &= R(f_{\{x\}}) = \bigwedge_{g \in R_{f_{\{x\}}}} g = \bigwedge_{g \in R_x} g, \\ \text{(ii)} \quad L^F(x) &= L(f_{\{x\}}) = \bigwedge_{g \in L_{f_{\{x\}}}} g = \bigwedge_{g \in L_x} g, \\ \text{(iii)} \quad I^F(x) &= I(f_{\{x\}}) = \bigwedge_{g \in I_{f_{\{x\}}}} g = \bigwedge_{g \in I_x} g, \\ \text{(iv)} \quad N^F(x) &= N(f_{\{x\}}) = \bigwedge_{g \in N_{f_{\{x\}}}} g = \bigwedge_{g \in N_x} g. \end{aligned}$$

Proposition 2.21. Let $(M, \Gamma, \leq)$ be an ordered Γ -groupoid and $x \in M$. Then we have the following:

- (i) $R^F(x) \in R_x$ and $R^F(x) \subseteq h$ for all $h \in R_x$,
- (ii) $L^F(x) \in L_x$ and $L^F(x) \subseteq h$ for all $h \in L_x$,
- (iii) $I^F(x) \in I_x$ and $I^F(x) \subseteq h$ for all $h \in I_x$,

(iv) $N^F(x) \in N_x$ and $N^F(x) \subseteq h$ for all $h \in N_x$.

Proof. (i) Since $R^F(x) = \bigwedge_{g \in R_x} g, R_x \neq \emptyset$ and each $g \in R_x$ is a fuzzy ordered right ideal of M and by Proposition 2.5, we have $R^F(x)$ is a fuzzy ordered right ideal of M . On the other hand, $(\bigwedge_{g \in R_x} g)(x) := \inf\{g(x) \mid g \in R_x\} = 1$. According to Definition 2.19, $R^F(x)$ is a fuzzy ordered right ideal of M containing x and so $R^F(x) \in R_x$. By Remark 2.3, we have $R^F(x) = \bigwedge_{g \in R_x} g \subseteq h$ for all $h \in R_x$.

The other cases can be proved similarly. ■

Definition 2.22. Let $(M, \Gamma, \leq)$ be an ordered Γ -groupoid and $x \in M$. Then $R^F(x)$ (resp. $L^F(x), I^F(x)$ and $N^F(x)$) is called the *fuzzy ordered right ideal* (resp. fuzzy ordered left ideal, fuzzy ordered ideal and fuzzy ordered filter) of M generated by x .

Definition 2.23. Let $(M, \Gamma, \leq)$ be an ordered Γ -groupoid. We define relations $\mathcal{R}^F, \mathcal{L}^F, \mathcal{I}^F, \mathcal{N}^F$ and $\mathcal{N}^F$ on M as follows:

$$\begin{aligned} a\mathcal{R}^Fb &\Leftrightarrow R^F(a) = R^F(b), \\ a\mathcal{L}^Fb &\Leftrightarrow L^F(a) = L^F(b), \\ a\mathcal{I}^Fb &\Leftrightarrow I^F(a) = I^F(b), \\ a\mathcal{N}^Fb &\Leftrightarrow N^F(a) = N^F(b). \end{aligned}$$

Theorem 2.24. Let $(M, \Gamma, \leq)$ be an ordered Γ -groupoid. Then we have the following:

- (i) $\mathcal{R}^F = \mathcal{R}$,
- (ii) $\mathcal{L}^F = \mathcal{L}$,
- (iii) $\mathcal{I}^F = \mathcal{I}$,
- (iv) $\mathcal{N}^F = \mathcal{N}$.

Proof. (i) For all $a, b \in M$, we have

$$\begin{aligned} a\mathcal{R}^Fb &\Leftrightarrow R^F(a) = R^F(b) \\ &\Leftrightarrow R(f_{\{a\}}) = R(f_{\{b\}}) \\ &\Leftrightarrow f_{R(a)} = f_{R(b)} \text{ (by Theorem 2.17)} \\ &\Leftrightarrow R(a) = R(b) \\ &\Leftrightarrow a\mathcal{R}b. \end{aligned}$$

Hence $\mathcal{R}^F = \mathcal{R}$. The other cases can be proved similarly. ■

Proposition 2.25. *Let $(M, \Gamma, \leq)$ be an ordered Γ -groupoid and $\{A_i \mid i \in I\}$ a nonempty family of fuzzy subsets of M . Then*

$$\bigwedge_{i \in I} f_{A_i} = f_{\bigcap_{i \in I} A_i}.$$

Proof. Since f_{A_i} is a fuzzy subset of M for all $i \in I$ and by Proposition 2.1, we have $\bigwedge_{i \in I} f_{A_i}$ is a fuzzy subset of M . Let $x \in M$. Then

$$(\bigwedge_{i \in I} f_{A_i})(x) := \inf\{f_{A_i}(x) \mid i \in I\}.$$

If $x \in A_i$ for all $i \in I$, then $f_{A_i}(x) = 1$ for all $i \in I$ and $(f_{\bigcap_{i \in I} A_i})(x) = 1$. Thus

$$(\bigwedge_{i \in I} f_{A_i})(x) := \inf\{f_{A_i}(x) \mid i \in I\} = 1 = (f_{\bigcap_{i \in I} A_i})(x).$$

If $x \notin A_j$ for some $j \in I$, then $x \notin \bigcap_{i \in I} A_i$. Thus

$$(f_{\bigcap_{i \in I} A_i})(x) = 0.$$

Since $x \notin A_j$, we have

$$f_{A_j}(x) = 0.$$

Since $0 \leq f_{A_i}(x)$ for all $i \in I$, we have

$$0 \leq \inf\{f_{A_i}(x) \mid i \in I\} \leq f_{A_j}(x) = 0.$$

Thus

$$(\bigwedge_{i \in I} f_{A_i})(x) := \inf\{f_{A_i}(x) \mid i \in I\} = 0 = (f_{\bigcap_{i \in I} A_i})(x).$$

Hence $\bigwedge_{i \in I} f_{A_i} = f_{\bigcap_{i \in I} A_i}$. ■

In a similar way we prove the following.

Proposition 2.26. *Let $(M, \Gamma, \leq)$ be an ordered Γ -groupoid and $\{A_i \mid i \in I\}$ a nonempty family of fuzzy subsets of M . Then*

$$\bigvee_{i \in I} f_{A_i} = f_{\bigcup_{i \in I} A_i}.$$

Remark 2.27. For an ordered Γ -groupoid $(M, \Gamma, \leq)$, we have

$$R(a) = \bigcap\{R \mid R \text{ ordered right ideal of } M \text{ and } a \in R\}.$$

Thus, by Proposition 2.25, we have

$$f_{R(a)} = \bigwedge\{f_R \mid R \text{ ordered right ideal of } M \text{ and } a \in R\}.$$

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PROPERTIES OF BIPOLAR FUZZY HYPERGRAPHS

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Abstract. In this article, we apply the concept of bipolar fuzzy sets to hypergraphs and investigate some properties of bipolar fuzzy hypergraphs. We introduce the notion of A -tempered bipolar fuzzy hypergraphs and present some of their properties. We also present application examples of bipolar fuzzy hypergraphs.

Keywords: Bipolar fuzzy hypergraph, bipolar fuzzy partition, dual bipolar fuzzy hypergraph, A -tempered bipolar fuzzy hypergraphs, clustering problem.

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1. Introduction

In 1994, Zhang [27] initiated the concept of bipolar fuzzy sets as a generalization of fuzzy sets [25]. Bipolar fuzzy sets are an extension of fuzzy sets whose membership degree range is $[-1, 1]$. In a bipolar fuzzy set, the membership degree 0 of an

element means that the element is irrelevant to the corresponding property, the membership degree $(0, 1]$ of an element indicates that the element somewhat satisfies the property, and the membership degree $[-1, 0)$ of an element indicates that the element somewhat satisfies the implicit counter-property. Although bipolar fuzzy sets and intuitionistic fuzzy sets look similar to each other, they are essentially different sets [20]. In many domains, it is important to be able to deal with bipolar information. It is noted that positive information represents what is granted to be possible, while negative information represents what is considered to be impossible. This domain has recently motivated new research in several directions. In particular, fuzzy and possibilistic formalisms for bipolar information have been proposed [15], because when we deal with spatial information in image processing or in spatial reasoning applications, this bipolarity also occurs. For instance, when we assess the position of an object in a space, we may have positive information expressed as a set of possible places and negative information expressed as a set of impossible places.

At present, graph theoretical concepts are highly utilized by computer science applications. Especially in research areas of computer science including data mining, image segmentation, clustering, image capturing and networking, for example a data structure can be designed in the form of tree which in turn utilized vertices and edges. Similarly, modeling of network topologies can be done using graph concepts. In the same way the most important concept of graph coloring is utilized in resource allocation, scheduling. Also, paths, walks and circuits in graph theory are used in tremendous applications say traveling salesman problem, database design concepts, resource networking. This leads to the development of new algorithms and new theorems that can be used in tremendous applications. Hypergraphs are the generalization of graphs (cf. [10]) in case of set of multiarity relations. It means the expansion of graph models for the modeling complex systems. In case of modeling systems with fuzzy binary and multiarity relations between objects, transition to fuzzy hypergraphs, which combine advantages both fuzzy and graph models, is more natural. It allows to realize formal optimization and logical procedures. However, using of the fuzzy graphs and hypergraphs as the models of various systems (social, economic systems, communication networks and others) leads to difficulties. The graph isomorphic transformations are reduced to redefinition of vertices and edges. This redefinition does not change properties the graph determined by an adjacent and an incidence of its vertices and edges. Fuzzy independent set, domination fuzzy set, fuzzy chromatic set are invariants concerning the isomorphism transformations of the fuzzy graphs and fuzzy hypergraph and allow make theirs structural analysis [11]. Lee-kwang et al. [21] generalized and redefined the concept of fuzzy hypergraphs whose basic idea was given by Kaufmann [18]. Further, the concept of fuzzy hypergraphs was discussed in [17]. Chen [14] introduced the concept of interval-valued fuzzy hypergraphs. Parvathi *et al.* [23] introduced the concept of intuitionistic fuzzy hypergraphs. Samanta and Pal [20] introduced the concept of a bipolar fuzzy hypergraph and studied some of its elementary properties. In this article, we first investigate some interesting properties of bipolar fuzzy hypergraphs. We introduce the regularity of bipolar

fuzzy hypergraphs. We then introduce the notion of A -tempered bipolar fuzzy hypergraphs and present some of their properties. Finally, we present an example of a bipolar fuzzy partition on the digital image processing.

We used standard definitions and terminologies in this paper. For notations, terminologies and applications are not mentioned in the paper, the readers are referred to [1]-[9].

2. Preliminaries

A *hypergraph* is a pair $H^* = (V, E^*)$, where V is a finite set of nodes (vertices) and E^* is a set of edges (or hyperedges) which are arbitrary nonempty subsets of V such that $\bigcup_j E_j^* = V$. A hypergraph is a generalization of an ordinary undirected graph, such that an edge need not contain exactly two nodes, but can instead contain an arbitrary nonzero number of vertices. An ordinary undirected graph (without self-loops) is, of course, a hypergraph where every edge has exactly two nodes (vertices). A hypergraph is *simple* if there are no repeated edges and no edge properly contains another. Hypergraphs are often defined by an incidence matrix with columns indexed by the edge set and rows indexed by the vertex set. The *rank* $r(H)$ of a hypergraph is defined as the maximum number of nodes in one edge, $r(H) = \max_j(|E_j|)$, and the *anti-rank* $s(H)$ is defined likewise, i.e., $s(H) = \min_j(|E_j|)$. We say that a hypergraph is *uniform* if $r(H) = s(H)$. A uniform hypergraph of rank k is called k -uniform hypergraph. Hence a simple graph is a 2-uniform hypergraph, and thus all simple graphs are also hypergraphs. A hypergraph is *vertex* (resp. *hyperedge*) *symmetric* if for any two vertices (resp. hyperedges) v_i and v_j (resp. e_i and e_j), there is an automorphism of the hypergraph that maps v_i to v_j (resp. e_i to e_j). The *dual of a hypergraph* $H^* = (V, E^*)$ with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and hyperedge set $E^* = \{e_1^*, e_2^*, \dots, e_m^*\}$ is a hypergraph $H^d = (V^d, E^{*d})$ with vertex set $V^d = \{v_1^d, v_2^d, \dots, v_m^d\}$ and hyperedge set $E^d = \{(e_1^*)^d, (e_2^*)^d, \dots, (e_n^*)^d\}$ such that v_j^d corresponds to e_j^* with hyperedges $(e_i^*)^d = \{v_j^d \mid v_i \in e_j^* \text{ and } e_j^* \in E^*\}$. In other words, H^d is obtained from H^* by interchanging of vertices and hyperedges in H^* . The incidence matrix of H^d is the transpose of the incidence matrix of H^* . Thus, $(H^d)^d = H^*$.

Definition 2.1. [25, 26] A *fuzzy set* μ on a nonempty set X is a map $\mu : X \rightarrow [0, 1]$. In the clustering, the fuzzy set μ , is called a *fuzzy class*. We define the *support* of μ by $\text{supp}(\mu) = \{x \in X \mid \mu(x) \neq 0\}$ and say μ is nontrivial if $\text{supp}(\mu)$ is nonempty. The *height* of μ is $h(\mu) = \max\{\mu(x) \mid x \in X\}$. We say μ is *normal* if $h(\mu) = 1$. A map $\nu : X \times X \rightarrow [0, 1]$ is called a *fuzzy relation* on X if $\nu(x, y) \leq \min(\mu(x), \mu(y))$ for all $x, y \in X$. A *fuzzy partition* of a set X is a family of nontrivial fuzzy sets $\{\mu_1, \mu_2, \mu_3, \dots, \mu_m\}$ such that

$$(1) \bigcup_i \text{supp}(\mu_i) = X, \quad i = 1, 2, \dots, m$$

$$(2) \sum_{i=1}^m \mu_i(x) = 1 \text{ for all } x \in X.$$

We call a family $\{\mu_1, \mu_2, \mu_3, \dots, \mu_m\}$ a *fuzzy covering* of X if it verifies only the above conditions (1) and (2).

Definition 2.2. [20] Let V be a finite set and let E be a finite family of nontrivial fuzzy sets on V such that $V = \bigcup_j \text{supp}(\mu_j)$, where μ_j is membership function defined on $E_j \in E$. Then the pair $H = (V, E)$ is a *fuzzy hypergraph* on V , E is the family of fuzzy edges of H and V is the (crisp) vertex set of H .

Definition 2.3. [19, 27] Let X be a nonempty set. A *bipolar fuzzy set* B in X is an object having the form

$$B = \{(x, \mu^P(x), \mu^N(x)) \mid x \in X\}$$

where $\mu^P : X \rightarrow [0, 1]$ and $\mu^N : X \rightarrow [-1, 0]$ are mappings.

We use the positive membership degree $\mu^P(x)$ to denote the satisfaction degree of an element x to the property corresponding to a bipolar fuzzy set B , and the negative membership degree $\mu^N(x)$ to denote the satisfaction degree of an element x to some explicit or implicit property corresponding to a bipolar fuzzy set B . If $\mu^P(x) \neq 0$ and $\mu^N(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for B . If $\mu^P(x) = 0$ and $\mu^N(x) \neq 0$, it is the situation that x does not satisfy the property of B but somewhat satisfies the counter property of B . It is possible for an element x to be such that $\mu^P(x) \neq 0$ and $\mu^N(x) \neq 0$ when the membership function of the property overlaps that of its counter property over some portion of X .

For the sake of simplicity, we shall use the symbol $B = (\mu^P, \mu^N)$ for the bipolar fuzzy set $B = \{(x, \mu^P(x), \mu^N(x)) \mid x \in X\}$.

Definition 2.4. [27] Let X be a nonempty set. Then we call a mapping $A = (\mu_A^P, \mu_A^N) : X \times X \rightarrow [0, 1] \times [-1, 0]$ a *bipolar fuzzy relation* on X such that $\mu_A^P(x, y) \in [0, 1]$ and $\mu_A^N(x, y) \in [-1, 0]$.

Definition 2.5. [19] The *support* of a bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$, denoted by $\text{supp}(A)$, is defined by

$$\begin{aligned} \text{supp}(A) &= \text{supp}^P(A) \cup \text{supp}^N(A), \quad \text{supp}^P(A) = \{x \mid \mu_A^P(x) > 0\}, \\ \text{supp}^N(A) &= \{x \mid \mu_A^N(x) < 0\}. \end{aligned}$$

We call $\text{supp}^P(A)$ as positive support and $\text{supp}^N(A)$ as negative support.

Definition 2.6. [19] Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set on X and let $\alpha \in [0, 1]$. α -cut A_α of A can be defined as

$$A_\alpha = A_\alpha^P \cup A_\alpha^N, \quad A_\alpha^P = \{x \mid \mu_A^P(x) \geq \alpha\}, \quad A_\alpha^N = \{x \mid \mu_A^N(x) \leq -\alpha\}.$$

We call A_α^P as positive α -cut and A_α^N as negative α -cut.

Definition 2.7. The *height* of a bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ is defined as $h(A) = \max\{\mu_A^P(x) \mid x \in X\}$. The *depth* of a bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ is defined as $d(A) = \min\{\mu_A^N(x) \mid x \in X\}$. We shall say that bipolar fuzzy set A is *normal*, if there is at least one $x \in X$ such that $\mu_A^P(x) = 1$ or $\mu_A^N(x) = -1$.

3. Bipolar fuzzy hypergraphs

Definition 3.1. [24] Let V be a finite set and let $E = \{E_1, E_2, \dots, E_m\}$ be a finite family of nontrivial bipolar fuzzy subsets of V such that

$$V = \bigcup_j \text{supp}(\mu_j^P, \mu_j^N), \quad j = 1, 2, \dots, m,$$

where μ_j^P and μ_j^N are positive and negative membership functions defined on $E_j \in E$. Then the pair $H = (V, E)$ is a *bipolar fuzzy hypergraph* on V , E is the family of bipolar fuzzy edges of H and V is the (crisp) vertex set of H . The order of H (number of vertices) is denoted by $|V|$ and the number of edges is denoted by $|E|$.

Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy subset of V and let E be a collection of bipolar fuzzy subsets of V such that for each $B = (\mu_B^P, \mu_B^N) \in E$ and $x \in V$, $\mu_B^P(x) \leq \mu_A^P(x)$, $\mu_B^N(x) \geq \mu_A^N(x)$. Then the pair (A, B) is a bipolar fuzzy hypergraph on the bipolar fuzzy set A . The bipolar fuzzy hypergraph (A, B) is also a bipolar fuzzy hypergraph on $V = \text{supp}(A)$, the bipolar fuzzy set A defines a condition for positive membership and negative membership in the edge set E . This condition can be stated separately, so without loss of generality we restrict attention to bipolar fuzzy hypergraphs on crisp vertex sets.

Example 3.2. Consider a bipolar fuzzy hypergraph $H = (V, E)$ such that $V = \{a, b, c, d\}$ and $E = \{E_1, E_2, E_3\}$, where

$$\begin{aligned} E_1 &= \left\{ \frac{a}{(0.2, -0.3)}, \frac{b}{(0.4, -0.5)} \right\}, \\ E_2 &= \left\{ \frac{b}{(0.4, -0.5)}, \frac{c}{(0.5, -0.2)} \right\}, \\ E_3 &= \left\{ \frac{a}{(0.2, -0.3)}, \frac{d}{(0.2, -0.4)} \right\}. \end{aligned}$$

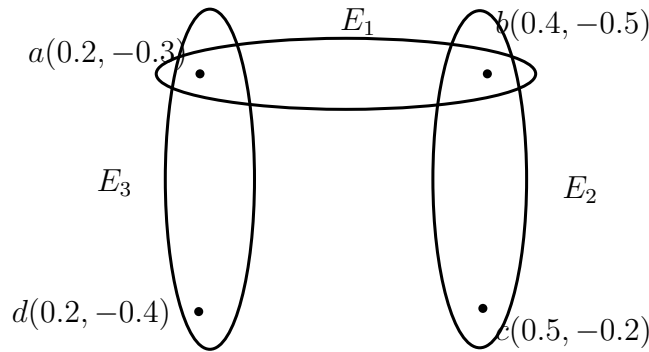


Figure 1: Bipolar fuzzy hypergraph

Table 1: The corresponding incidence matrix is given below:

M_H	E_1	E_2	E_3
a	(0.2, -0.3)	(0, 0)	(0.2, -0.3)
b	(0.4, -0.5)	(0.4, -0.5)	(0, 0)
c	(0, 0)	(0.5, -0.2)	(0, 0)
d	(0, 0)	(0, 0)	(0.2, -0.4)

Definition 3.3. A bipolar fuzzy set $A = (\mu_A^P, \mu_A^N) : X \rightarrow [0, 1] \times [-1, 0]$ is an *elementary bipolar fuzzy set* if A is single valued on $\text{supp}(A)$. An elementary bipolar fuzzy hypergraph $H = (V, E)$ is a bipolar fuzzy hypergraph whose edges are elementary.

We explore the sense in which a bipolar fuzzy graph is a bipolar fuzzy hypergraph.

Proposition 3.4. *Bipolar fuzzy graphs are special cases of the bipolar fuzzy hypergraphs.*

A bipolar fuzzy multigraph is a multivalued symmetric mapping $D = (\mu_D^P, \mu_D^N) : V \times V \rightarrow [0, 1] \times [-1, 0]$. A bipolar fuzzy multigraph can be considered to be the “disjoint union” or “disjoint sum” of a collection of simple bipolar fuzzy graphs, as is done with crisp multigraphs. The same holds for multidigraphs. Therefore, these structures can be considered as “disjoint unions” or “disjoint sums” of bipolar fuzzy hypergraphs.

Definition 3.5. A bipolar fuzzy hypergraph $H = (V, E)$ is *simple* if $A = (\mu_A^P, \mu_A^N)$, $B = (\mu_B^P, \mu_B^N) \in E$ and $\mu_A^P \leq \mu_B^P$, $\mu_A^N \geq \mu_B^N$ imply that $\mu_A^P = \mu_B^P$, $\mu_A^N = \mu_B^N$. In particular, a (crisp) hypergraph $H^* = (V, E^*)$ is simple if $X, Y \in E^*$ and $X \subseteq Y$ imply that $X = Y$. A bipolar fuzzy hypergraph $H = (V, E)$ is *support simple* if $A = (\mu_A^P, \mu_A^N)$, $B = (\mu_B^P, \mu_B^N) \in E$, $\text{supp}(A) = \text{supp}(B)$, and $\mu_A^P \leq \mu_B^P$, $\mu_A^N \geq \mu_B^N$ imply that $\mu_A^P = \mu_B^P$, $\mu_A^N = \mu_B^N$. A bipolar fuzzy hypergraph $H = (V, E)$ is *strongly support simple* if $A = (\mu_A^P, \mu_A^N)$, $B = (\mu_B^P, \mu_B^N) \in E$ and $\text{supp}(A) = \text{supp}(B)$ imply that $A = B$.

Remark. The definition 3.5 reduces to familiar definitions in the special case where H is a crisp hypergraph. The bipolar fuzzy definition of simple is identical to the crisp definition of simple. A crisp hypergraph is support simple and strongly support simple if and only if it has no multiple edges. For bipolar fuzzy hypergraphs all three concepts imply no multiple edges. Simple bipolar fuzzy hypergraphs are support simple and strongly support simple bipolar fuzzy hypergraphs are support simple. Simple and strongly support simple are independent concepts.

Definition 3.6. Let $H = (V, E)$ be a bipolar fuzzy hypergraph. Suppose that $\alpha \in [0, 1]$, $\beta \in [-1, 0]$. Let

- $E_{(\alpha,\beta)} = \{A_{(\alpha,\beta)} \mid \text{where } A \text{ is positive and negative membership function defined on } E_j \in E\}, A_{(\alpha,\beta)} = \{x \mid \mu_A^P(x) \geq \alpha \text{ or } \mu_A^N(x) \leq \beta\}, \text{ and}$
- $V_{(\alpha,\beta)} = \bigcup_{A \in E} A_{(\alpha,\beta)}.$

If $E_{(\alpha,\beta)} \neq \emptyset$, then the crisp hypergraph $H_{(\alpha,\beta)} = (V_{(\alpha,\beta)}, E_{(\alpha,\beta)})$ is the (α, β) -level hypergraph of H .

Clearly, it is possible that $A_{(\alpha,\beta)} = B_{(\alpha,\beta)}$ for $A \neq B$, by using distinct markers to identity the various members of E a distinction between $A_{(\alpha,\beta)}$ and $B_{(\alpha,\beta)}$ to represent multiple edges in $H_{(\alpha,\beta)}$. However, we do not take this approach unless otherwise stated, we will always regard $H_{(\alpha,\beta)}$ as having no repeated edges.

The families of crisp sets (hypergraphs) produced by the (α, β) -cuts of a bipolar fuzzy hypergraph share an important relationship with each other, as expressed below: suppose $\mathbb{X}$ and $\mathbb{Y}$ are two families of sets such that for each set X belonging to $\mathbb{X}$ there is at least one set Y belonging to $\mathbb{Y}$ which contains X . In this case we say that $\mathbb{Y}$ *absorbs* $\mathbb{X}$ and symbolically write $\mathbb{X} \sqsubseteq \mathbb{Y}$ to express this relationship between $\mathbb{X}$ and $\mathbb{Y}$. Since it is possible for $\mathbb{X} \sqsubseteq \mathbb{Y}$ while $\mathbb{X} \cap \mathbb{Y} = \emptyset$, we have that $\mathbb{X} \subseteq \mathbb{Y} \Rightarrow \mathbb{X} \sqsubseteq \mathbb{Y}$, whereas the converse is generally false. If $\mathbb{X} \sqsubseteq \mathbb{Y}$ and $\mathbb{X} \neq \mathbb{Y}$, then we write $\mathbb{X} \sqsubset \mathbb{Y}$.

Definition 3.7. Let $H = (V, E)$ be a bipolar fuzzy hypergraph. Let $H_{(s,t)}$ be the (s, t) -level hypergraph of H . The sequence of real numbers

$$\{(s_1, r_1), (s_2, r_2), \dots, (s_n, r_n)\}, \quad 0 < s_1 < s_2 < \dots < s_n \text{ and } 0 > r_1 > r_2 > \dots > r_n,$$

where $(s_n, r_n) = h(H)$,

which satisfies the properties:

- if $(s_{i-1}, r_{i-1}) < (u, v) \leq (s_i, r_i)$, then $E_{(u,v)} = E_{(s_i, r_i)}$, and
- $E_{(s_i, r_i)} \sqsubset E_{(s_{i+1}, r_{i+1})}$,

is called the *fundamental sequence* of H , and is denoted by $F(H)$ and the set of (s_i, r_i) -level hypergraphs $\{H_{(s_1, r_1)}, H_{(s_2, r_2)}, \dots, H_{(s_n, r_n)}\}$ is called the *set of core hypergraphs* of H or, simply, the *core set* of H , and is denoted by $C(H)$.

Definition 3.8. Suppose $H = (V, E)$ is a bipolar fuzzy hypergraph with

$$F(H) = \{(s_1, r_1), (s_2, r_2), \dots, (s_n, r_n)\},$$

and $s_{n+1} = 0, r_{n+1} = 0$, then H is called *sectionally elementary* if for each edge $A = (\mu_A^P, \mu_A^N) \in E$, each $i = \{1, 2, \dots, n\}$, and $(s_i, r_i) \in F(H)$, $A_{(s,t)} = A_{(s_i, r_i)}$ for all $(s, t) \in ((s_{i-1}, r_{i-1}), (s_i, r_i)]$.

Clearly H is sectionally elementary if and only if $A(x) = (\mu_A^P(x), \mu_A^N(x)) \in F(H)$ for each $A \in E$ and each $x \in X$.

Definition 3.9. A sequence of crisp hypergraphs $H_i = (V_i, E_i^*)$, $1 \leq i \leq n$, is said to be *ordered* if $H_1 \subset H_2 \subset \dots \subset H_n$. The sequence $\{H_i \mid 1 \leq i \leq n\}$ is *simply ordered* if it is ordered and if whenever $E^* \in E_{i+1}^* - E_i^*$, then $E^* \not\subseteq V_i$.

Definition 3.10. A bipolar fuzzy hypergraph H is *ordered* if the H induced fundamental sequence of hypergraphs is ordered. The bipolar fuzzy hypergraph H is *simply ordered* if the H induced fundamental sequence of hypergraphs is simply ordered.

Example 3.11. Consider the bipolar fuzzy hypergraph $H = (V, E)$, where $V = \{a, b, c, d\}$ and $E = \{E_1, E_2, E_3, E_4, E_5\}$ which is represented by the following incidence matrix:

Table 2: Incidence matrix of H

H	E_1	E_2	E_3	E_4	E_5
a	$(0.7, -0.2)$	$(0.9, -0.2)$	$(0, 0)$	$(0, 0)$	$(0.4, -0.3)$
b	$(0.7, -0.2)$	$(0.9, -0.2)$	$(0.9, -0.2)$	$(0.7, -0.2)$	$(0, 0)$
c	$(0, 0)$	$(0, 0)$	$(0.9, -0.2)$	$(0.7, -0.2)$	$(0.4, -0.3)$
d	$(0, 0)$	$(0.4, -0.3)$	$(0, 0)$	$(0.4, -0.3)$	$(0.4, -0.3)$

Clearly, $h(H) = (0.9, -0.1)$.

Now

$$E_{(0.9, -0.1)} = \{\{a, b\}, \{b, c\}\}$$

$$E_{(0.7, -0.2)} = \{\{a, b\}, \{b, c\}\}$$

$$E_{(0.4, -0.3)} = \{\{a, b\}, \{a, b, d\}, \{b, c\}, \{b, c, d\}, \{a, c, d\}\}.$$

Thus for $0.4 < s \leq 0.9$ and $-0.1 > t \geq -0.3$, $E_{(s,t)} = \{\{a, b\}, \{b, c\}\}$, and for $0 < s \leq 0.4$ and $-1 < t \geq -0.3$,

$$E_{(s,t)} = \{\{a, b\}, \{a, b, d\}, \{b, c\}, \{b, c, d\}, \{a, c, d\}\}.$$

We note that $E_{(0.9, -0.1)} \subseteq E_{(0.4, -0.3)}$. The fundamental sequence is

$$F(H) = \{(s_1, r_1) = (0.9, -0.1), (s_2, r_2) = (0.4, -0.3)\}$$

and the set of core hypergraph is

$$C(H) = \{H_1 = (V_1, E_1) = H_{(0.9, -0.1)}, H_2 = (V_2, E_2) = H_{(0.4, -0.3)}\},$$

where

$$V_1 = \{a, b, c\}, \quad E_1 = \{\{a, b\}, \{b, c\}\}$$

$$V_2 = \{a, b, c, d\}, \quad E_2 = \{\{a, b\}, \{a, b, d\}, \{b, c\}, \{b, c, d\}, \{a, c, d\}\}.$$

H is support simple, but not simple. H is not sectionally elementary since $E_{1(s,t)} \neq E_{1(0.9, -0.1)}$ for $s = 0.7$, $t = -0.2$. Clearly, bipolar fuzzy hypergraph H is simply ordered.

Proposition 3.12. *Let $H = (V, E)$ be an elementary bipolar fuzzy hypergraph. Then H is support simple if and only if H is strongly support simple.*

Proof. Suppose that H is elementary, support simple and that $\text{supp}(A) = \text{supp}(B)$. We assume without loss of generality that $h(A) \leq h(B)$. Since H is elementary, it follows that $\mu_A^P \leq \mu_B^P$, $\mu_A^N \geq \mu_B^N$ and since H is support simple that $\mu_A^P = \mu_B^P$, $\mu_A^N = \mu_B^N$. Therefore, H is strongly support simple. The proof of converse part is obvious. ■

The complexity of a bipolar fuzzy hypergraph depends in part on how many edges it has. The natural question arises: is there an upper bound on the number of edges of a bipolar fuzzy hypergraph of order n ?

Proposition 3.13. *Let $H = (V, E)$ be a simple bipolar fuzzy hypergraph of order n . Then there is no upper bound on $|E|$.*

Proof. Let $V = \{x, y\}$, and define $E_N = \{A_i = (\mu_{A_i}^P, \mu_{A_i}^N) \mid i = 1, 2, \dots, N\}$, where

$$\begin{aligned} \mu_{A_i}^P(x) &= \frac{1}{i+1}, & \mu_{A_i}^N(x) &= -1 + \frac{1}{i+1}, \\ \mu_{A_i}^P(y) &= \frac{1}{i+1}, & \mu_{A_i}^N(y) &= -\frac{i}{i+1}. \end{aligned}$$

Then $H_N = (V, E_N)$ is a simple bipolar fuzzy hypergraph with N edges. This ends the proof. ■

Proposition 3.14. *Let $H = (V, E)$ be a support simple bipolar fuzzy hypergraph of order n . Then there is no upper bound on $|E|$.*

Proof. The class of support simple bipolar fuzzy hypergraphs contains the class of simple bipolar fuzzy hypergraphs, thus the result follows from Proposition 3.13. ■

Proposition 3.15. *Let $H = (V, E)$ be an elementary simple bipolar fuzzy hypergraph of order n . Then there is no upper bound on $|E| \leq 2^n - 1$ if and only if $\{\text{supp}(A) \mid A \in E\} = P(V) - \emptyset$.*

Proof. Since H is elementary and simple, each nontrivial $W \subseteq V$ can be the support of at most one $A = (\mu_A^P, \mu_A^N) \in E$. Therefore, $|E| \leq 2^n - 1$. To show there exists an elementary, simple H with $|E| = 2^n - 1$, let $E = \{A = (\mu_A^P, \mu_A^N) \mid W \subseteq V\}$ be the set of functions defined by

$$\begin{aligned} \mu_A^P(x) &= \frac{1}{|W|}, & \text{if } x \in W, & & \mu_A^P(x) &= 0, & \text{if } x \notin W, \\ \mu_A^N(x) &= -1 + \frac{1}{|W|}, & \text{if } x \in W, & & \mu_A^N(x) &= -1, & \text{if } x \notin W. \end{aligned}$$

Then each one element has height $(1, -1)$, each two elements has height $(0.5, -0.5)$, and so on. Hence H is an elementary and simple, and $|E| = 2^n - 1$. ■

We state the following proposition without proof.

Proposition 3.16. (a) If $H = (V, E)$ is an elementary bipolar fuzzy hypergraph, then H is ordered.

(b) If H is an ordered bipolar fuzzy hypergraph with simple support hypergraph, then H is elementary.

Definition 3.17. The dual of a bipolar fuzzy hypergraph $H = (V, E)$ is a bipolar fuzzy hypergraph $H^D = (E^D, V^D)$ whose vertex set is the edge set of H and with edges $V^D : E^D \rightarrow [0, 1] \times [-1, 0]$ by $V^D(A^D) = (\mu_A^D(x), \nu_A^D(x))$. H^D is a bipolar fuzzy hypergraph whose incidence matrix is the transpose of the incidence matrix of H , thus $H^{DD} = H$.

Example 3.18. Consider a bipolar fuzzy hypergraph $H = (V, E)$ such that

$$V = \{x_1, x_2, x_3, x_4\}, \quad E = \{E_1, E_2, E_3, E_4\},$$

where

$$E_1 = \left\{ \frac{x_1}{(0.5, -0.3)}, \frac{x_2}{(0.4, -0.2)} \right\}, \quad E_2 = \left\{ \frac{x_2}{(0.4, -0.2)}, \frac{x_3}{(0.3, -0.6)} \right\},$$

$$E_3 = \left\{ \frac{x_3}{(0.3, -0.6)}, \frac{x_4}{(0.5, -0.1)} \right\}, \quad E_4 = \left\{ \frac{x_4}{(0.5, -0.1)}, \frac{x_1}{(0.5, -0.3)} \right\}.$$

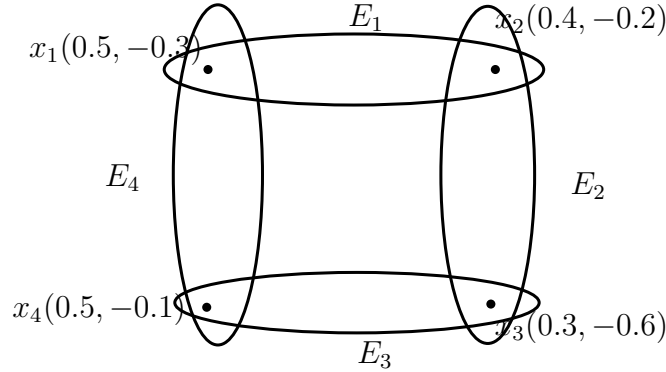


Figure 2: Bipolar fuzzy hypergraph

Table 3: The corresponding incidence matrix of H is given below:

M_H	E_1	E_2	E_3	E_4
x_1	$(0.5, -0.3)$	$(0, 0)$	$(0, 0)$	$(0.5, -0.3)$
x_2	$(0.4, -0.2)$	$(0.4, -0.2)$	$(0, 0)$	$(0, 0)$
x_3	$(0, 0)$	$(0.3, -0.6)$	$(0.3, -0.6)$	$(0, 0)$
x_4	$(0, 0)$	$(0, 0)$	$(0.5, -0.1)$	$(0.5, -0.1)$

Consider the dual bipolar fuzzy hypergraph $H^D = (E^D, V^D)$ of H such that

$$E^D = \{e_1, e_2, e_3, e_4\}, \quad V^D = \{A, B, C, D\}$$

where

$$A = \left\{ \frac{e_1}{(0.5, -0.3)}, \frac{e_4}{(0.5, -0.3)} \right\}, \quad B = \left\{ \frac{e_1}{(0.4, -0.2)}, \frac{e_2}{(0.4, -0.2)} \right\},$$

$$C = \left\{ \frac{e_2}{(0.3, -0.6)}, \frac{e_3}{(0.3, -0.6)} \right\}, \quad D = \left\{ \frac{e_3}{(0.5, -0.1)}, \frac{e_4}{(0.5, -0.1)} \right\}.$$

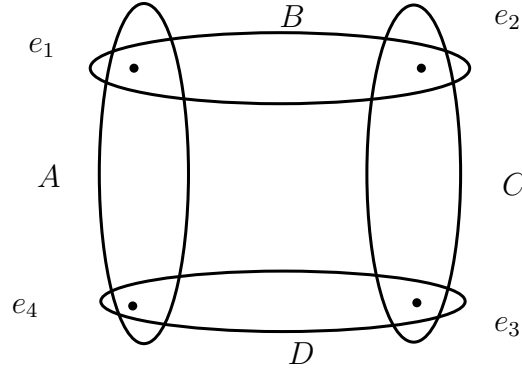


Figure 3: Dual bipolar fuzzy hypergraph

Table 4: The corresponding incidence matrix of H^D is given below:

M_{H^D}	A	B	C	D
e_1	$(0.5, -0.3)$	$(0.4, -0.2)$	$(0, 0)$	$(0, 0)$
e_2	$(0, 0)$	$(0.4, -0.2)$	$(0.3, -0.6)$	$(0, 0)$
e_3	$(0, 0)$	$(0, 0)$	$(0.3, -0.6)$	$(0.5, -0.1)$
e_4	$(0.5, -0.3)$	$(0, 0)$	$(0, 0)$	$(0.5, -0.1)$

We see that some edges contain only vertices having high positive membership degree and high negative membership degree. We define here the concept of strength of an edge.

Definition 3.19. The *strength* η of an edge E is the maximum positive membership $\mu^P(x)$ of vertices and maximum negative membership $\mu^N(x)$ of vertices in the edge E . That is, $\eta(E_j) = \{\max(\mu_j^P(x) \mid \mu_j^P(x) > 0), \max(\mu_j^N(x) \mid \mu_j^N(x) < 0)\}$.

Its interpretation is that the edge E_j groups elements having participation degree at least $\eta(E_j)$ in the hypergraph.

Example 3.20. Consider a bipolar fuzzy hypergraph $H = (V, E)$ such that $V = \{a, b, c, d\}$, $E = \{E_1, E_2, E_3, E_4\}$.

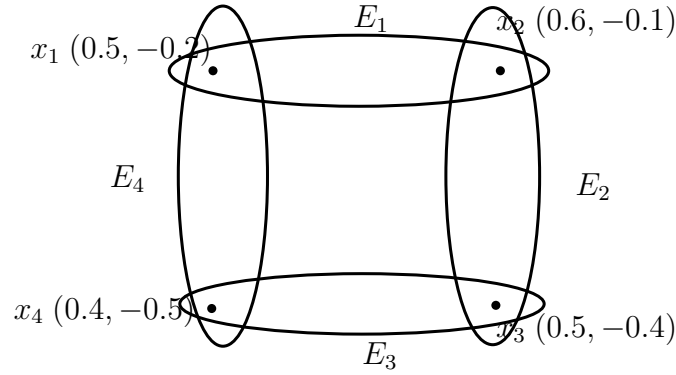


Figure 4: Bipolar fuzzy hypergraph

It is easy to see that E_1 is strong than E_3 , and E_2 is strong than E_4 . We call the edges with high strength the strong edges because the cohesion in them is strong.

Definition 3.21. The (α, β) -cut of a bipolar fuzzy hypergraph H , denoted by $H_{(\alpha, \beta)}$, is defined as an ordered pair

$$H_{(\alpha, \beta)} = (V_{(\alpha, \beta)}, E_{(\alpha, \beta)}),$$

where

- (i) $V_{(\alpha, \beta)} = \{x_1, x_2, \dots, x_n\} = V$,
- (ii) $E_{j(\alpha, \beta)} = \{x_i \mid \mu_j^P(x_i) \geq \alpha \text{ and } \mu_j^N(x_i) \leq \beta, j = 1, 2, 3, \dots, m\}$,
- (iii) $E_{m+1(\alpha, \beta)} = \{x_i \mid \mu_j^P(x_i) < \alpha \text{ and } \mu_j^N(x_i) > \beta, \forall j\}$.

The edge $E_{m+1(\alpha, \beta)}$ is added to the group of elements which are not contained in any edge $E_{j(\alpha, \beta)}$ of $H_{(\alpha, \beta)}$. The edges in the (α, β) -cut hypergraph are now crisp sets.

Example 3.22. Consider the bipolar fuzzy hypergraph $H = (V, E)$, where $V = \{x, y, z\}$ and $E = \{E_1, E_2\}$, which is represented by the following incidence matrix:

Table 5: Incidence matrix of H

H	E_1	E_2
x	(0.4, -0.2)	(0, 0)
y	(0.5, -0.3)	(0.6, -0.2)
z	(0, 0)	(0.2, -0.05)

From this matrix we understand that, for example, $E_1 = (\mu_1^P, \mu_1^N) : V \rightarrow [0, 1] \times [-1, 0]$ satisfies:

$$\mu_1^P(x) = 0.4, \mu_1^N(x) = -0.2; \mu_1^P(y) = 0.5, \mu_1^N(y) = -0.3; \mu_1^P(z) = 0, \mu_1^N(z) = 0.$$

$(0.3, -0.1)$ -cut of bipolar fuzzy hypergraph H is

$$E_{1(0.3, -0.1)} = \{x, y\}, \quad E_{2(0.3, -0.1)} = \{y\}, \quad E_{3(0.3, -0.1)} = \{z\}.$$

The incidence matrix of $H_{(0.3, -0.1)}$ is given below.

Table 6: Incidence matrix of $H_{(0.3, -0.1)}$

$H_{(0.3, -0.1)}$	$E_{1(0.3, -0.1)}$	$E_{2(0.3, -0.1)}$	$E_{3(0.3, -0.1)}$
x	1	0	0
y	1	1	0
z	0	0	1

Definition 3.23. A bipolar fuzzy hypergraph $H = (V, E)$ is called a $A = (\mu_A^P, \mu_A^N)$ -tempered bipolar fuzzy hypergraph of $H = (V, E)$ if there is a crisp hypergraph $H^* = (V, E^*)$ and a bipolar fuzzy set $A = (\mu_A^P, \mu_A^N) : V \rightarrow [0, 1] \times [-1, 0]$ such that $E = \{B_F = (\mu_{B_F}^P, \mu_{B_F}^N) \mid F \in E^*\}$, where

$$\mu_{B_F}^P(x) = \begin{cases} \min(\mu_A^P(y) \mid y \in F) & \text{if } x \in F, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu_{B_F}^N(x) = \begin{cases} \max(\mu_A^N(y) \mid y \in F) & \text{if } x \in F, \\ -1 & \text{otherwise.} \end{cases}$$

Let $A \otimes H$ denote the A -tempered bipolar fuzzy hypergraph of H determined by the crisp hypergraph $H = (V, E^*)$ and the bipolar fuzzy set $A : V \rightarrow [0, 1] \times [-1, 0]$.

Example 3.24. Consider the bipolar fuzzy hypergraph $H = (V, E)$, where $V = \{a, b, c, d\}$ and $E = \{E_1, E_2, E_3, E_4\}$ which is represented by the following incidence matrix:

Table 7: Incidence matrix of H

H	E_1	E_2	E_3	E_4
a	$(0.2, -0.7)$	$(0, 0)$	$(0, 0)$	$(0.2, -0.7)$
b	$(0.2, -0.7)$	$(0.3, -0.4)$	$(0.0, -0.9)$	$(0, 0)$
c	$(0, 0)$	$(0, 0)$	$(0, -0.9)$	$(0.2, -0.7)$
d	$(0, 0)$	$(0.3, -0.4)$	$(0, 0)$	$(0, 0)$

Then

$$\begin{aligned} E_{(0, -0.1)} &= \{\{b, c\}\}, \\ E_{(0.2, -0.7)} &= \{\{a, b\}, \{a, c\}, \{b, c\}\}. \\ E_{(0.3, -0.1)} &= \{\{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}\}. \end{aligned}$$

Define $A = (\mu_A^P, \mu_A^N) : V \rightarrow [0, 1] \times [-1, 0]$ by

$$\begin{aligned} \mu_A^P(a) = 0.2, \quad \mu_A^P(b) = \mu_A^P(c) = 0.0, \quad \mu_A^P(d) = 0.3, \quad \mu_A^N(a) = -0.7, \\ \mu_A^N(b) = \mu_A^N(c) = -0.9, \quad \mu_A^N(d) = -0.4. \end{aligned}$$

Note that

$$\begin{aligned}\mu_{B_{\{a,b\}}}^P(a) &= \min(\mu_A^P(a), \mu_A^P(b)) = 0.0, & \mu_{B_{\{a,b\}}}^P(b) &= \min(\mu_A^P(a), \\ \mu_A^P(b)) &= 0.0, & \mu_{B_{\{a,b\}}}^P(c) &= 0.0, \quad \mu_{B_{\{a,b\}}}^P(d) = 0.0, \\ \mu_{B_{\{a,b\}}}^N(a) &= \max(\mu_A^N(a), \mu_A^N(b)) = -0.9, & \mu_{B_{\{a,b\}}}^N(b) &= \max(\mu_A^N(a), \\ \mu_A^N(b)) &= -0.9, & \mu_{B_{\{a,b\}}}^N(c) &= -1, \quad \mu_{B_{\{a,b\}}}^N(d) = -1.\end{aligned}$$

Thus

$$\begin{aligned}E_1 &= (\mu_{B_{\{a,b\}}}^P, \mu_{B_{\{a,b\}}}^P), & E_2 &= (\mu_{B_{\{b,d\}}}^P, \mu_{B_{\{b,d\}}}^N), \\ E_3 &= (\mu_{B_{\{b,c\}}}^P, \mu_{B_{\{b,c\}}}^N), & E_4 &= (\mu_{B_{\{a,c\}}}^P, \mu_{B_{\{a,c\}}}^N).\end{aligned}$$

Hence H is A -tempered hypergraph.

Theorem 3.25. *A bipolar fuzzy hypergraph H is a $A = (\mu_A^P, \mu_A^N)$ -tempered bipolar fuzzy hypergraph of some crisp hypergraph H^* if and only if H is elementary, support simple and simply ordered.*

Proof. Suppose that $H = (V, E)$ is a A -tempered bipolar fuzzy hypergraph of some crisp hypergraph H^* . Clearly, H is elementary and support simple. We show that H is simply ordered. Let

$$C(H) = \{(H_1^*)^{r_1} = (V_1, E_1^*), (H_2^*)^{r_2} = (V_2, E_2^*), \dots, (H_n^*)^{r_n} = (V_n, E_n^*)\}.$$

Since H is elementary, it follows from Proposition 3.16 that H is ordered. To show that H is simply ordered, suppose that there exists $F \in E_{i+1}^* \setminus E_i^*$. Then there exists $x^* \in F$ such that $\mu_A^P(x^*) = r_{i+1}$, $\mu_A^N(x^*) = r'_{i+1}$. Since $\mu_A^P(x^*) = r_{i+1} < r_i$ and $\mu_A^N(x^*) = r'_{i+1} < r'_i$, it follows that $x^* \notin V_i$ and $F \not\subseteq V_i$, hence H is simply ordered.

Conversely, suppose $H = (V, E)$ is elementary, support simple and simply ordered. Let

$$C(H) = \{(H_1^*)^{r_1} = (V_1, E_1^*), (H_2^*)^{r_2} = (V_2, E_2^*), \dots, (H_n^*)^{r_n} = (V_n, E_n^*)\}$$

where $D(H) = \{r_1, r_2, \dots, r_n\}$ with $0 < r_n < \dots < r_1$. Since $(H^*)^{r_n} = H_n^* = (V_n, E_n^*)$ and define $A = (\mu_A^P, \mu_A^N) : V_n \rightarrow [0, 1] \times [-1, 0]$ by

$$\begin{aligned}\mu_A^P(x) &= \begin{cases} r_1 & \text{if } x \in V_1, \\ r_i & \text{if } x \in V_i \setminus V_{i-1}, i = 1, 2, \dots, n \end{cases} \\ \mu_A^N(x) &= \begin{cases} s_1 & \text{if } x \in V_1, \\ s_i & \text{if } x \in V_i \setminus V_{i-1}, i = 1, 2, \dots, n \end{cases}\end{aligned}$$

We show that $E = \{B_F = (\mu_{B_F}^P, \mu_{B_F}^N) \mid F \in E^*\}$, where

$$\begin{aligned}\mu_{B_F}^P(x) &= \begin{cases} \min(\mu_A^P(y) \mid y \in F) & \text{if } x \in F, \\ 0 & \text{otherwise,} \end{cases} \\ \mu_{B_F}^N(x) &= \begin{cases} \max(\mu_A^N(y) \mid y \in F) & \text{if } x \in F, \\ -1 & \text{otherwise.} \end{cases}\end{aligned}$$

Let $F \in E_n^*$. Since H is elementary and support simple, there is a unique bipolar fuzzy edge $C_F = (\mu_{C_F}^P, \mu_{C_F}^N)$ in E having support E^* . Indeed, distinct edges in E must have distinct supports that lie in E_n^* . Thus, to show that $E = \{B_F = (\mu_{B_F}^P, \mu_{B_F}^N) \mid F \in E_n^*\}$, it suffices to show that for each $F \in E_n^*$, $\mu_{C_F}^P = \mu_{B_F}^P$ and $\mu_{C_F}^N = \mu_{B_F}^N$. As all edges are elementary and different edges have different supports, it follows from the definition of fundamental sequence that $h(C_F)$ is equal to some number r_i of $D(H)$. Consequently, $E^* \subseteq V_i$. Moreover, if $i > 1$, then $F \in E^* \setminus E_{i-1}^*$. Since $F \subseteq V_i$, it follows from the definition of $A = (\mu_A^P, \mu_A^N)$ that for each $x \in F$, $\mu_A^P(x) \geq r_i$ and $\mu_A^N(x) \leq s_i$. We claim that $\mu_A^P(x) = r_i$ and $\mu_A^N(x) = s_i$, for some $x \in F$. If not, then by definition of $A = (\mu_A^P, \mu_A^N)$, $\mu_A^P(x) > r_i$ and $\mu_A^N(x) < s_i$ for all $x \in F$ which implies that $F \subseteq V_{i-1}$ and so $F \in E_{i-1}^*$ and since H is simply ordered $F \subsetneq V_{i-1}$, a contradiction. Thus it follows from the definition of B_F that $B_F = C_F$. This completes the proof. ■

As a consequence of the above theorem we obtain.

Proposition 3.26. *Suppose that H is a simply ordered bipolar fuzzy hypergraph and $F(H) = \{r_1, r_2, \dots, r_n\}$. If H^{r_n} is a simple hypergraph, then there is a partial bipolar fuzzy hypergraph $\dot{H}$ of H such that the following assertions hold:*

- (1) $\dot{H}$ is a $A = (\mu_A^P, \mu_A^N)$ -tempered bipolar fuzzy hypergraph of H_n .
- (2) $E \sqsubseteq \dot{E}$.
- (3) $F(\dot{H}) = F(H)$ and $C(\dot{H}) = C(H)$.

4. Application examples of bipolar fuzzy hypergraphs

Definition 4.1. Let X be a reference set. Then, a family of nontrivial bipolar fuzzy sets $\{A_1, A_2, A_3, \dots, A_m\}$ where $A_i = (\mu_i^P, \mu_i^N)$ is a *bipolar fuzzy partition* if

- (1) $\bigcup_i \text{supp}(A_i) = X, \quad i = 1, 2, \dots, m,$
- (2) $\sum_{i=1}^m \mu_i^P(x) = 1$ for all $x \in X$,
- (3) $\sum_{i=1}^m \mu_i^N(x) = -1$ for all $x \in X$.

Note that this definition generalizes fuzzy partitions because the definition is equivalent to a fuzzy partition when for all x , $\mu_i(x) = 0$. We call a family $\{A_1, A_2, A_3, \dots, A_m\}$ a *bipolar fuzzy covering* of X if it satisfies above conditions (1) – (3).

A bipolar fuzzy partition can be represented by a bipolar fuzzy matrix $[a_{ij}]$ where a_{ij} , is the positive membership degree and negative membership degree of

element x_i in class j . We see that the matrix is the same as the incidence matrix in bipolar fuzzy hypergraph. Then we can represent a bipolar fuzzy partition by a bipolar fuzzy hypergraph $H = (V, E)$ such that

- (1) V : a set of elements $x_i, i = 1, \dots, n$,
- (2) $E = \{E_1, E_2, \dots, E_m\}$ = a set of nontrivial bipolar fuzzy classes,
- (3) $V = \bigcup_j \text{supp}(E_j), j = 1, 2, \dots, m$,
- (4) $\sum_{i=1}^m \mu_i^P(x) = 1$ for all $x \in X$,
- (5) $\sum_{i=1}^m \mu_i^N(x) = -1$ for all $x \in X$.

Note that conditions (4)–(5) are added to the bipolar fuzzy hypergraph for bipolar fuzzy partition. If these conditions are added, the bipolar fuzzy hypergraph can represent a bipolar fuzzy covering. Naturally, we can apply the (α, β) -cut to the bipolar fuzzy partition.

Example 4.2. [Radio coverage network] In telecommunications, the coverage of a radio station is the geographic area where the station can communicate.

Let V be a finite set of radio receivers (vertices); perhaps a set of representative locations at the centroid of a geographic region. For each of m radio transmitters we define the bipolar fuzzy set “listening area of station j ” where $A_j(x) = (\mu_{A_j}^P(x), \mu_{A_j}^N(x))$ represents the “quality of reception of station j at location x .” Positive membership value near to 1, could signify “very clear reception on a very sensitive radio” while negative membership value near to -1 , could signify “very poor reception on a very poor radio”. Since graphy affects signal strength, each “listening area” is an bipolar fuzzy set. Also, for a fixed radio the reception will vary between different stations. Thus this model uses the full definition of an bipolar fuzzy hypergraph. The model could be used to determine station programming or marketing strategies or to establish an emergency broadcast network. Further variables could relate signal strength to changes in time of day, weather and other conditions.

Example 4.3. [Clustering problem] We consider here the clustering problem, which is a typical example of a bipolar fuzzy partition on the digital image processing.

There are five objects and they are classified into two classes: tank and house. To cluster the elements x_1, x_2, x_3, x_4, x_5 into A_t (tank) and B_h (house), a bipolar fuzzy partition matrix is given as the form of incidence matrix of bipolar fuzzy hypergraph.

We can apply the (α, β) -cut to the hypergraph and obtain a hypergraph $H_{(\alpha, \beta)}$ which is not bipolar fuzzy hypergraph. We denote the edge (class) in (α, β) -cut hypergraph $H_{(\alpha, \beta)}$ as $E_{j(\alpha, \beta)}$. This hypergraph H , represents generally the covering

Table 8: Bipolar fuzzy partition matrix

H	A_t	B_h
x_1	(0.96, -0.04)	(0.04, -0.96)
x_2	(0.95, -0.5)	(0.5, -0.95)
x_3	(0.61, -0.39)	(0.39, -0.61)
x_4	(0.05, -0.95)	(0.95, -0.05)
x_5	(0.03, -0.97)	(0.97, -0.03)

because the conditions: (4) $\sum_{i=1}^m \mu_i^P(x) = 1$ for all $x \in X$, (5) $\sum_{i=1}^m \mu_i^N(x) = -1$ for all $x \in X$, is not always guaranteed. The hypergraph $H_{(0.61, -0.03)}$ is shown in Table 8.

 Table 9: Hypergraph $H_{(0.61, -0.03)}$

$H_{(0.61, -0.03)}$	$A_{t(0.61, -0.03)}$	$B_{h(0.61, -0.03)}$
x_1	1	0
x_2	1	0
x_3	1	0
x_4	0	1
x_5	0	1

We obtain dual bipolar fuzzy hypergraph $H_{(0.61, -0.03)}^D$ of $H_{(0.61, -0.03)}$ which is given in Table 9.

Table 10: Dual bipolar fuzzy hypergraph

$H^D(0.61, -0.03)$	X_1	X_2	X_3	X_4	X_5
A_t	1	1	1	0	0
B_h	0	0	0	1	1

We consider the strength of edge (class) $E_{j(\alpha, \beta)}$, or in the (α, β) -cut hypergraph $H_{(\alpha, \beta)}$. It is necessary to apply Definition 3.19 to obtain the strength of edge $E_{j(\alpha, \beta)}$ in $H_{(\alpha, \beta)}$.

The possible interpretations of $\eta(E_{j(\alpha, \beta)})$ are:

- the edge (class) in the hypergraph (partition) $H_{(\alpha, \beta)}$, groups elements having at least η positive and negative memberships,
- the strength (cohesion) of edge (class) $E_{j(\alpha, \beta)}$ in $H_{(\alpha, \beta)}$ is η .

Thus we can use the strength as a measure of the cohesion or strength of a class in a partition. For example, the strengths of classes $A_t(0.61, -0.03)$ and $B_h(0.61, -0.03)$ at $s=0.61$, $t=-0.03$ are $\eta(A_t(0.61, -0.03))=(0.96, -0.04)$, $\eta(B_h(0.61, -0.03))=(0.97, -0.03)$. Thus we say that the class $\eta(B_h(0.61, -0.03))$ is stronger than $\eta(A_t(0.61, -0.03))$ because $\eta(B_h(0.61, -0.03)) > \eta(A_t(0.61, -0.03))$.

From the above discussion on the hypergraph $H_{(0.61, -0.03)}$ and $H_{(0.61, -0.03)}^D$, we can state that:

- The bipolar fuzzy hypergraph can represent the fuzzy partition visually. The (α, β) -cut hypergraph also represents the (α, β) -cut partition.
- The dual hypergraph $H_{(0.61, -0.03)}^D$ can represent elements X_i , which can be grouped into a class $E_{j(\alpha, \beta)}$. For example, the edges X_1, X_2, X_3 of the dual hypergraph in Table 9 represent that the elements x_1, x_2, x_3 that can be grouped into A_t at level $(0.61, -0.03)$.
- At $(\alpha, \beta) = (0.61, -0.03)$ level, the strength of class $B_h(0.61, -0.03)$ is the highest $(0.95, -0.05)$, so it is the strongest class. It means that this class can be grouped independently from the other parts. Thus we can eliminate the class B_h from the others and continue clustering. Therefore, the discrimination of strong classes from the others can allow us to decompose a clustering problem into smaller ones. This strategy allows us to work with the reduced data in a clustering problem.

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VARIATION OF PARAMETERS METHOD FOR THIN FILM FLOW OF A THIRD GRADE FLUID DOWN AN INCLINED PLANE

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Abstract. Thin film Flow of a third grade fluid down an inclined plane is considered. Non-linear differential equation which governs the flow model is obtained by using laws of conservation along with suitable similarity transforms. Variation of Parameters Method (VPM) is then employed to solve aforementioned differential equation. Analytical solution is backed by RK-4 numerical solution and both show excellent agreement. Effects of flow parameters β and m on velocity field are demonstrated graphically with comprehensive discussions.

Keywords: non-Newtonian fluid, thin film flow, nonlinear equation, inclined plane, Variation of Parameters Method (VPM).

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1. Introduction

Recently, the importance of non-Newtonian fluids has become prominent with developments in industries like polymer, petroleum, pulp etc. Many industrial materials fall into this category, such as solutions, melts of polymers, soap, biological solutions, paints, tars, asphalts and glues. Due to complex nature of non-Newtonian fluids it is hard to establish one mathematical model that can describe characteristics of all non-Newtonian fluids. So many mathematical models are used to discuss flow of non-Newtonian flows, third grade fluids lie in one of such class and are popular among researchers due to their simpler mathematical simulation. Denson [1], Waters and Keeley [2] studied some problems of third grade fluids. Siddiqui et al. [3] discussed the solution for thin film flow of third grade fluid down an inclined plane using perturbation method and Homotopy Perturbation Method (HPM). Hayat et al. [4] obtained exact solutions to same problem under certain assumptions. Sajid et al. [5] used Homotopy Analysis Method (HAM) to solve thin film flow of third grade fluid down an inclined plane and proved that HAM works better as compared to HPM.

Most of the physical problems involving non-Newtonian fluids are in form of nonlinearities, so exact solutions may not be possible. Therefore, over the years different analytical techniques have been developed. Variation of Parameters Method (VPM) is one of these techniques which do not require any linearization, discretization or existence of small and large parameters for expansion. Ma et al. [6]-[8] used VPM for solving non-homogeneous partial differential equations. Mohyud-Din et al. [9, 10, 11]-[11] further applied this technique for solving initial and boundary value problems of diversified physical nature.

Inspired and motivated by the ongoing research in this direction, we apply VPM to study thin film flow of a third grade fluid down an inclined plane. This technique is comparatively novel in fluid mechanics and has a good scope in further studies.

2. Governing equation

The basic equations for the flow of an incompressible third grade fluid flow down an inclined plane in absence of body force are

$$(1) \quad \nabla \cdot \mathbf{V} = 0$$

$$(2) \quad \rho \frac{D\mathbf{V}}{Dt} = -\nabla p + \text{div} \boldsymbol{\tau}$$

Where $\mathbf{V}$, ρ , p , $\boldsymbol{\tau}$ denote velocity vector, constant density, pressure and stress tensor respectively, further $\frac{D}{Dt}$ is the material derivative and defined as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{V} \cdot \nabla).$$

The stress tensor defining third grade fluid is given by

$$(3) \quad \tau = S_1 + S_2 + S_3,$$

where

$$(4) \quad S_1 = \mu A_1$$

$$(5) \quad S_2 = \alpha_1 A_2 + \alpha_2 A_1^2$$

$$(6) \quad S_3 = \beta_1 A_3 + \beta_2 (A_1 A_2 + A_2 A_1) + \beta_3 (\text{tr} A_2) A_1.$$

Here μ is the coefficient of viscosity and $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3$, are material constants. The Rivlin-Ericksen tensor, A_n , are define by $A_0 = I$, the identity tensor, and

$$(7) \quad A_n = \frac{DA_{n-1}}{Dt} + A_{n-1}(\nabla V) + (\nabla V)^T A_{n-1}, \quad n \geq 1$$

3. Problem formulation

Thin film flow of an incompressible third grade fluid down an inclined plane is considered. Ambient air is assumed to be stationary and flow is due to gravity only. It is assumed that the surface tension of the fluid is negligible and film is of uniform thickness δ . We seek a one-dimensional steady velocity field of the form

$$(8) \quad V = (u(y), 0, 0)$$

Using equation (3) and (8) in equations (1) and (2), we obtain the momentum equation with boundary conditions as

$$(9) \quad \mu \frac{d^2 u}{dy^2} + 6(\beta_2 + \beta_3) \left(\frac{du}{dy} \right)^2 \frac{d^2 u}{dy^2} + \rho g \sin \alpha = 0$$

$$(10) \quad u(y) = 0, \quad \text{at} \quad y = 0,$$

$$(11) \quad \frac{du}{dy} = 0, \quad \text{at} \quad y = \delta,$$

where, g is the gravity of the fluid.

Introduce the suitable dimensionless variables as,

$$(12) \quad u^* = \frac{u}{\nu/\delta}, \quad y^* = \frac{y}{\delta}, \quad \beta^* = \frac{\beta}{\rho\delta^4/\nu}$$

and dropping asterisks for simplicity, dimensionless form of the momentum equation (9) with boundary conditions (10) and (11) become

$$(13) \quad \frac{d^2 u}{dy^2} + 6\beta \left(\frac{du}{dy} \right)^2 \frac{d^2 u}{dy^2} + m = 0$$

$$(14) \quad u(0) = 0, \quad \frac{du}{dy} = 0, \quad \text{at} \quad y = 1$$

where $m = \frac{\delta^3 g \sin \alpha}{\nu^2}$.

4. Variation of Parameter Method

To illustrate the basic concept of the variation of parameter method for differential equations, we consider the general differential equation in operator form

$$(15) \quad Lu(y) + Nu(y) + Ru(y) = g(y),$$

where L is a higher order linear operator, R is a linear operator of order less than L , N is a nonlinear operator, and g is a source term. We have following general solution of equation

$$(16) \quad u(y) = \sum_{i=0}^{n-1} \frac{A_i y^i}{i!} + \int_0^y \lambda(y, s) (-Nu(s) - Ru(s) + g(s)) ds,$$

where n is the order of given differential equation and A_i 's are unknowns which can be further determined by initial/boundary conditions. Here, $\lambda(y, s)$ is a multiplier which can be obtained with the help of Wronskian technique as

$$(17) \quad \lambda(y, s) = y - s.$$

Hence, we have the following iterative scheme,

$$(18) \quad u_{k+1}(y) = \sum_{i=0}^{n-1} \frac{A_i y^i}{i!} + \int_0^y (y - s) (-Nu(s) - Ru(s) + g(s)) ds,$$

where $k = 0, 1, 2, \dots$

It is observed that the fix value of initial guess in each iteration provide the better approximation.

5. Solution of the problem

Using Equation (13) along (14) in (18), we have the following iterative scheme

$$u_{k+1}(y) = A - \int_0^y (y - s) \left(6\beta \left(\frac{du}{ds} \right)^2 \frac{d^2u}{ds^2} + m \right) ds,$$

where $A = \frac{du}{dy}(0)$, can be determined by using boundary conditions (14).

Few iterations of the solution are given below

$$\begin{aligned} u_0 &= Ay \\ u_1 &= Ay - \frac{1}{2}my^2 \\ u_2 &= \frac{1}{2}\beta m^3 y^4 - 2\beta Am^2 y^3 + \left(3\beta A^2 m - \frac{1}{2}m \right) y^2 + Ay \end{aligned}$$

$$\begin{aligned}
 u_3 = & Ay + \left(3\beta A^2 m - \frac{1}{2}m - 18\beta^2 A^4 m \right) y^2 \\
 & + \left(-2\beta Am^2 - 72\beta^3 A^5 m^2 + 36\beta^2 A^3 m^2 \right) y^3 \\
 & + \left(-108\beta^4 A^6 m^3 + 162\beta^3 A^4 m^3 - 30\beta^2 A^2 m^3 + \frac{1}{2}\beta m^3 \right) y^4 \\
 & + \left(12\beta^2 Am^4 - \frac{792}{5}\beta^3 A^3 m^4 + \frac{1296}{5}\beta^4 A^5 m^4 \right) y^5 \\
 & + \left(-288\beta^4 m^5 A^4 - 2\beta^2 m^5 + 84\beta^3 m^5 A^2 \right) y^6 \\
 & + \left(-24\beta^3 m^6 A + \frac{1296}{7}\beta^4 m^6 A^3 \right) y^7 \\
 & + \left(-72\beta^4 A^2 m^7 + 3\beta^3 m^7 \right) y^8 + 16\beta^4 Am^8 y^9 - \frac{8}{5}\beta^4 m^9 y^{10},
 \end{aligned}$$

6. Results and discussion

Fig. 1 depicts the variations in velocity profile for different values of non-Newtonian parameter β and fixed the value of m . It can be observed that as the value of β increases, velocity decreases. For $\beta = 0$ solution reduces to the case of Newtonian fluid, which is represented by solid line in the graph. From Fig. 2 it can be observed that for fixed value of β velocity increases by increasing m . By using RK-4 method and setting tolerance of 0.000001 same problem is solved numerically and results are compared with VPM solution in Fig. 3 and Table 1. It can be seen that both results have excellent agreement.

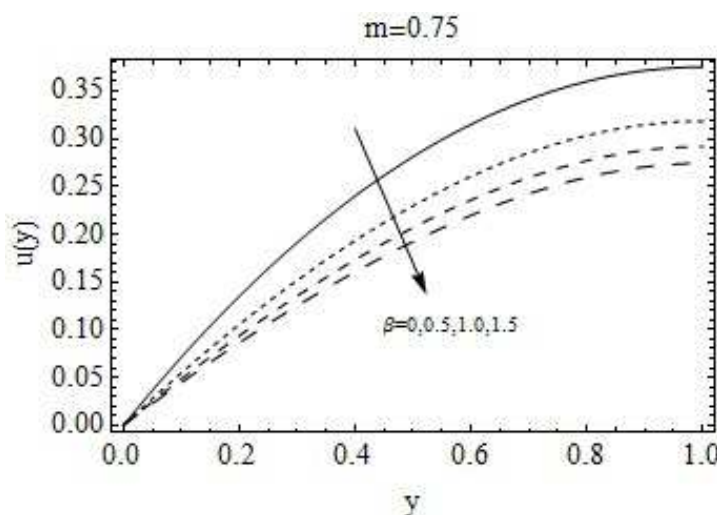


Figure 1. Effects of β on velocity profile

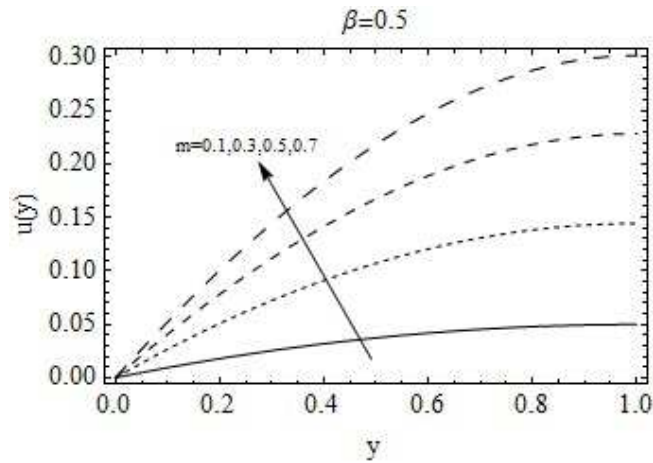


Figure 2. Effects of m on velocity profile

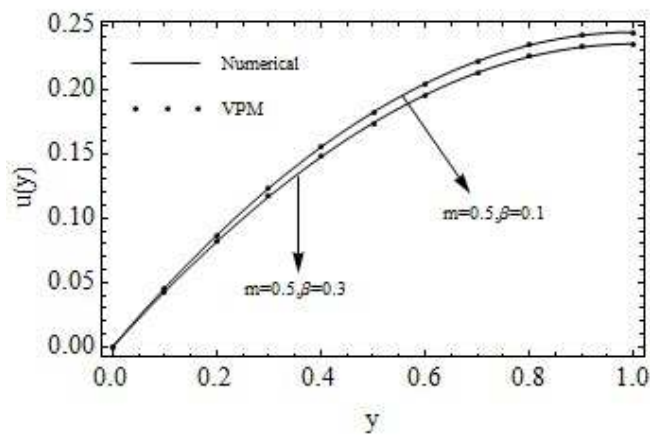


Figure 3. Comparison of VPM and numerical solution

Table 1 presents a comparison of VPM and numerical solutions which is graphically represented in Fig. 3. From Fig. 3 it can be analyzed in even in first glance that VPM has good agreement with numerical results.

y	$\beta = 0.1$		$\beta = 0.3$	
	VPM	Numerical	VPM	Numerical
0.1	0.045599	0.045599	0.044069	0.044069
0.2	0.086705	0.086705	0.084013	0.084013
0.3	0.123228	0.123228	0.11969	0.11969
0.4	0.155078	0.155078	0.150961	0.150961
0.5	0.182177	0.182177	0.177691	0.177691
0.6	0.204453	0.204453	0.199756	0.199756
0.7	0.221846	0.221846	0.217045	0.217045
0.8	0.234305	0.234305	0.229466	0.229466
0.9	0.241796	0.241796	0.236947	0.236947
1.0	0.244296	0.244296	0.239446	0.239446

Table.1: Comparison between VPM and numerical solutions

7. Conclusions

In the present work, VPM is applied to solve nonlinear equation governing thin film flow of a third grade fluid down an inclined plane. Effects of viscoelastic parameter β and dimensionless inclination parameter m on velocity profile are recorded. Velocity is observed to be decreasing for increasing β while it increases with increasing inclination parameter m . Numerical solution affirms the results obtained by VPM and it can be concluded that VPM can successfully be applied to problems in fluid mechanics.

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A NOVEL APPROACH IN HOMOTOPY PERTURBATION METHOD FOR THE EXPANSION OF NON-LINEAR TERMS

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Abstract. Homotopy Perturbation Method (HPM) is widely used in approximate calculation. The complexity of the method is to expand the nonlinear terms in series with higher powers. To make the calculation of nonlinear terms easier and friendly, we have used new algorithm based on derivatives. The same approach can be introduced to expand the higher power nonlinear terms in place of He's polynomial in HPM.

Keywords: Nonlinear, Partial differential equations, Homotopy perturbation method, Zaidian modification.

1. Introduction

Recently, the interest of scientists and engineers in finding the analytic solution to nonlinear problems has increased, for this purpose many new techniques and methods have been developed. Traditional perturbation methods have their own limitations like, presence of a very large or very small parameter inside the problem is essential, so that the solution of the problem may be expressed as a series expansion in terms of that small parameter. Choosing the small parameter is not an easy task and requires special skills. A proper and good choice of small

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parameter will make results more accurate, while on the other hand a wrong choice may lead to inaccurate results.

In 1998, J.H. He [1]–[3] introduced the Homotopy Perturbation Method (HPM) with the advantage that it absorbs all the positive traits, in which, the presence of small parameter was not necessary. J.H. He used the idea of Homotopy along with traditional perturbation method to develop this technique. Many mathematicians and engineers used this method to solve highly nonlinear differential equations, Integral equations and boundary value problem. The solution is assumed to be the summation of infinite series in HPM. Noor and Mohyud-Din [4], [5] applied HPM to fifth, sixth order boundary value problems and many nonlinear problems to show the efficiency of the method. Also, there are analytical techniques to deal with local fractional differential equations [6], [7], such as Local fractional variational iteration method [8], local fractional Fourier series method [9], Yang-Fourier and Yang-Laplace transform method [10]. Recently by introducing He's polynomial in homotopy perturbation method, especially for the nonlinear terms Ghorbani et al. [11], [12] proved the compatibility of He's polynomials with Adomian's polynomials which are easy to calculate. He's polynomials and correction functional of the VIM was combined with HPM by Noor and Mohyud-Din [13]–[14] and used to solve many real world problems as an application by VIMHP. A new method was exploited by E. Babolian [15] to calculate the Adomian's polynomials which was the easier way to handle the higher power nonlinear terms. He has utilized this method only to find the Adomian's polynomials.

It has been observed that in HPM the complex and difficult task is to split and expand the nonlinear terms like $(v_0 + pv_1 + p^2v_2 + \dots)^i$, $i = 2, 3, 4, \dots$ with conventional Homotopy Perturbation Method or by using He's polynomials. Manually it is always difficult to expand the fourth and above power of the series having more than four terms or product of two or more terms. In the present study, we have widened the horizon of the method by introducing approach based on derivative [15] in Homotopy Perturbation Method for the highly nonlinear terms. Hence, there was a dire need to develop some new algorithm to tackle the nonlinearities. Inspired from the wide range of applications of He's polynomials in various analytical techniques which not only include Homotopy Perturbation Method but also many other well-known solution techniques like Variation Iteration Method (VIM), Variation of Parameters Method (VPM) and Adomian's Decomposition Method (ADM). The basic inspiration of this paper is that we have split nonlinear terms in two parts, i.e., $N(u) = Z_0 + Z_k$, $k = 1, 2, 3, \dots$, then applied action of T on $N(u)$, which distributes the highly nonlinear terms in the coefficient of $p_0, p_1, p_2, \dots$. We introduce a new scheme to cope with the complex nonlinear terms. Our proposed technique is easier to implement and the results so obtained are fully compatible with the traditional versions.

2. Homotopy Perturbation Method

To explain the basic idea of Homotopy Perturbation Method for solving nonlinear differential equations, we consider the following nonlinear differential equation

$$(1) \quad A(u) - f(\eta) = 0, \quad \eta \in \Omega^*,$$

with boundary conditions

$$(2) \quad B' \left(u, \frac{\partial u}{\partial \zeta} \right) = 0, \quad \eta \in \partial\Omega^*$$

where A is a general non-linear operator, B' is a boundary operator, $f(\eta)$ is known as analytic function, $\partial\Omega^*$ is the boundary domain and $\frac{\partial u}{\partial \zeta}$ is the directional derivative. The non-linear operator A can further be divided into two parts, linear L and non-linear N , so equation (1) can be expressed as

$$(3) \quad L(u) + N(u) - f(\eta) = 0.$$

He constructed a homotopy as

$$(4) \quad v(\eta, r) : \Omega^* \times [0, 1] \rightarrow R,$$

that satisfies the equation

$$(1 - r) [L(v) - L(u_0)] + r [A(v) - f(\eta)] = 0,$$

Where, $r \in [0, 1]$ is an embedding parameter, u_0 is the initial approximation. Therefore, equation (1) can be written as

$$(5) \quad H[\eta, r] = L(v) - L(u_0) + rL[u_0] + r[N(v) - f(\eta)],$$

$$(6) \quad H[\eta, r] = L(v) - L(u_0) + rL[u_0] + r[Z_0 + Z_k - f(\eta)], \quad k \geq 1$$

On setting $r = 0$ in equation (5), we get

$$H(v, 0) = L[v] - L[u_0] = 0,$$

which provides us initial guess u_0 . Similarly by setting $r = 1$ in equation (5), we have

$$\begin{aligned} H(v, 1) &= L[v] - L[u_0] + L[u_0] + Z_0 + Z_k - f(\eta), \quad k \geq 1 \\ &= L[v] + Z_0 + Z_k - f(\eta) \\ &= A[v] - f(\eta), \end{aligned}$$

where $Z_0 = N(v_0)$, for $Z_k, k \geq 1$, applying action of T on $Z_0, Z_1, \dots, Z_{n+1}$ and, by using the result of Lemma [15], it can be prove that $Z_{n+1} = \frac{1}{n+1} T(Z_n)$

$$\begin{aligned}
Z_1 &= T(Z_0) = v_1 N^1(v_0), \\
Z_2 &= \frac{1}{2} T(Z_1) = \frac{1}{2} T(v_1 N^1(v_0)) = \frac{1}{2} (T(v_1) N^1(v_0) + v_1 T(N^1(v_0))) \\
&= v_2 N^1(v_0) + \frac{1}{2} v_1^2 N^2(v_0), \\
Z_3 &= \frac{1}{3} T(Z_2) = \frac{1}{3} T\left(v_2 N^{(1)}(v_0) + \frac{1}{2} v_1^2 N^{(2)}(v_0)\right) \\
&= \frac{1}{3} (T(v_2) N^{(1)}(v_0) + v_2 T(N^{(1)}(v_0)) \\
&\quad + \frac{1}{2} (T(v_1^2) N^{(2)}(v_0) + v_1^2 T(N^{(2)}(v_0)))) \\
&= v_3 N^1(v_0) + v_2 v_1 N^{(2)}(v_0) + \frac{1}{3!} v_1^3 N^{(3)}(v_0), \\
&\vdots
\end{aligned}$$

and so on we can find $Z_4, Z_5, \dots$

Thus, the changing process of r from zero to unity is essentially the change in of $v(\eta, r)$ from $u_o(\eta)$ to $u(\eta)$. In topology, this is called deformation. We expand $v(\eta, r)$ in terms of a series in the form

$$v(\eta, r) = v_0 + r v_1 + r^2 v_2 + \dots,$$

and then taking limit $r \rightarrow 1$, equation (??) yields

$$u(\eta) = \lim_{r \rightarrow 1} v(\eta, r) = v_0 + v_1 + v_2 + \dots,$$

The question of convergence of series has been discussed extensively by J.H. He [2].

3. Properties of the T operator

From [15], consider the operator T with the following properties:

$$\begin{aligned}
T((w_{i_1} w_{i_2} \dots w_{i_l}) G^{(k)}(u_0)) &= T(w_{i_1} w_{i_2} \dots w_{i_l}) G^{(k)}(u_0) + (w_{i_1} w_{i_2} \dots w_{i_l}) T(G^{(k)}(u_0)), \\
&\quad (k \in \mathbf{N} \cup \{0\}, l \in \mathbf{N}, l \geq 2),
\end{aligned}$$

$$T(w_i) = (i+1)(w_{i+1}),$$

$$T(G^{(k)}(u_0)) = u_1 G^{(k+1)}(u_0) \quad (k \in \mathbf{N} \cup \{0\}),$$

$$T(w_{i_1} w_{i_2} \dots w_{i_j}) = T(w_{i_1})(w_{i_2} \dots w_{i_j}) + w_{i_1} T(w_{i_2} \dots w_{i_j}), \quad (j \in \mathbf{N}, j \geq 2),$$

$$\begin{aligned}
T(\alpha w_{i_1} w_{i_2} \dots w_{i_l} G^{(k)}(u_0) + \beta w_{j_1} w_{j_2} \dots w_{j_m} G^{(k')}(u_0)) \\
= \alpha T(w_{i_1} w_{i_2} \dots w_{i_l} G^{(k)}(u_0)) + \beta T(w_{j_1} w_{j_2} \dots w_{j_m} G^{(k')}(u_0)), \\
\quad (k, k' \in \mathbf{N} \cup \{0\}, l, m \in \mathbf{N}, m \geq 2)
\end{aligned}$$

where $w_i \in S$, $i \in \mathbf{N} \cup \{0\}$. Define $u(\lambda) = \sum_{j=0}^{\infty} \mathbf{u}_j (\lambda^j)$ the i -th derivative of $u(\lambda)$ is $u^{(i)}(\lambda) = \sum_{j=1}^{\infty} j \alpha_j \mathbf{u}_j \lambda^{j-1}$, where $\alpha_j = (j-1) \dots (j-i+1)$. Putting $\lambda = 0$ gives $u_i = (u^{(i)}(0) / i!)$.

4. Applications

Example 5.1. $N(u) = u^2 u_x$

Expansion of nonlinear term by using conventional HPM.

$$\begin{aligned} N(u) &= (u_0 + p^1 u_1 + p^2 u_2 + p^3 u_3 \dots)^2 (u_{0x} + p^1 u_{1x} + p^2 u_{2x} + p^3 u_{3x} \dots) \\ &\quad \left(u_0^2 + (p^1 u_1 + p^2 u_2 + p^3 u_3 \dots)^2 + 2u_0 (p^1 u_1 + p^2 u_2 + p^3 u_3 \dots) \right) \\ &\quad (u_{0x} + p^1 u_{1x} + p^2 u_{2x} + p^3 u_{3x} \dots) \\ &= p^0 (u_0^2 u_{0x}) + p^1 (u_0^2 u_{1x} + 2u_0 u_1 u_{0x}) \\ &\quad + p^2 (2u_0 u_1 u_{1x} + u_0^2 u_{2x} + u_1^2 u_{0x} + 2u_0 u_2 u_{0x}) \dots \end{aligned}$$

Expansion by using the new approach we get like powers of $p_0, p_1, p_2, \dots$ without expanding the series.

$$\begin{aligned} N(u) &= N(u_0) + T(Z_0 + Z_1 + Z_2 + Z_3 + Z_4 + \dots), \\ &= N(u_0) + T(Z_0) + \frac{1}{2} T(Z_1) + \frac{1}{3} T(Z_2) + \frac{1}{4} T(Z_3) + \dots \\ &= u_0^2 u_{0x} + T(u_0^2 u_{0x}) + \frac{1}{2} T(2u_0 u_1 u_{0x} + u_0^2 u_{1x}) \\ &\quad + \frac{1}{3} T(2u_0 u_2 u_{0x} + u_1^2 u_{0x} + 2u_0 u_1 u_{1x} + u_0^2 u_{2x}) + \dots \\ &= \underbrace{u_0^2 u_{0x}}_{p_0} + \underbrace{2u_0 u_1 u_{0x} + u_0^2 u_{1x}}_{p_1} + \underbrace{2u_0 u_2 u_{0x} + u_1^2 u_{0x} + 2u_0 u_1 u_{1x} + u_0^2 u_{2x}}_{p_2} + \dots \end{aligned}$$

Example 5.2. $N(u) = u u_x u_{xx}^2$

Expanding the nonlinear term by using series expansion method in HPM, we get,

$$\begin{aligned} N(u) &= (u_0 + p^1 u_1 + p^2 u_2 + p^3 u_3 \dots) (u_{0x} + p^1 u_{1x} + p^2 u_{2x} + p^3 u_{3x} \dots) \\ &\quad (u_{0xx} + p^1 u_{1xx} + p^2 u_{2xx} + p^3 u_{3xx} \dots)^2 \\ &= (u_0 (u_{0x} + p^1 u_{1x} + p^2 u_{2x} + p^3 u_{3x} \dots) + p^1 u_1 (u_{0x} + p^1 u_{1x} + p^2 u_{2x} + p^3 u_{3x} \dots) + \dots) \\ &\quad (u_{0xx}^2 + t(p^1 u_{1xx} + p^2 u_{2xx} + p^3 u_{3xx} \dots)^2 + 2u_{0xx} (p^1 u_{1xx} + p^2 u_{2xx} + p^3 u_{3xx} \dots)) \\ &= p^0 (u_0 u_{0x} u_{0xx}^2) + p^1 (2u_0 u_{0x} u_{0xx} u_{1xx} + u_1 u_{0x} u_{0xx}^2 + u_0 u_{1x} u_{0xx}^2 t) \\ &\quad + p^2 (2u_0 u_{0x} u_{0xx} u_{2xx} + u_1 u_{1x} u_{0xx}^2 + u_0 u_{0x} u_{1xx}^2 + 2u_0 u_{1x} u_{0xx} u_{1xx} \\ &\quad + u_0 u_{2x} u_{0xx}^2 + 2u_1 u_{0x} u_{0xx} u_{1xx} + u_2 u_{0x} u_{0xx}^2) \end{aligned}$$

Expansion of $N(u)$ by using the new approach we get $p_0, p_1, p_2, \dots$ without expanding the series

$$\begin{aligned}
N(u) &= N(u_0) + T(Z_0 + Z_1 + Z_2 + Z_3 + Z_4 + \dots) \\
&= N(u_0) + T(Z_0) + \frac{1}{2}T(Z_1) + \frac{1}{3}T(Z_2) + \frac{1}{4}T(Z_3) + \dots \\
&= u_0 u_{0x} u_{0xx}^2 + T(u_0 u_{0x} u_{0xx}^2) \\
&\quad + \frac{1}{2}T(2u_0 u_{0x} u_{0xx} u_{1xx} + u_0 u_{1x} u_{0xx}^2 + u_1 u_{0x} u_{0xx}^2) \\
&\quad + \frac{1}{3}T(2u_0 u_{0x} u_{0xx} u_{2xx} + u_0 u_{0xx} u_{1xx}^2 + 2u_0 u_{1x} u_{0xx} u_{1xx} + u_2 u_{0x} u_{0xx}^2) + \dots \\
&= \underbrace{u_0 u_{0x} u_{0xx}^2}_{p_0} + \underbrace{2u_0 u_{0x} u_{0xx} u_{1xx} + u_0 u_{1x} u_{0xx}^2 + u_1 u_{0x} u_{0xx}^2}_{p_1} + \underbrace{2u_0 u_{0x} u_{0xx} u_{2xx}}_{p_2} \\
&\quad + \underbrace{u_0 u_{0xx} u_{1xx}^2 + 2u_0 u_{1x} u_{0xx} u_{1xx} + u_2 u_{0x} u_{0xx}^2}_{p_2} + \dots
\end{aligned}$$

Example 5.3. $N(u) = u^4 + uu_x$.

Expansion of nonlinear term by using conventional HPM

$$\begin{aligned}
N(u) &= (u_0 + p^1 u_1 + p^2 u_2 + p^3 u_3 \dots)^4 + (u_0 + p^1 u_1 + p^2 u_2 + p^3 u_3 \dots) \\
&\quad (u_{0x} + p^1 u_{1x} + p^2 u_{2x} + p^3 u_{3x} \dots) \\
N(u) &= (u_0 + p^1 u_1 + p^2 u_2 + p^3 u_3 \dots)^2 + (u_0 + p^1 u_1 + p^2 u_2 + p^3 u_3 \dots)^2 \\
&\quad + u_0 (u_{0x} + p^1 u_{1x} + p^2 u_{2x} + p^3 u_{3x} \dots) + p^1 u_1 u_{0x} + p^1 u_{1x} + p^2 u_{2x} \\
&\quad + p^3 u_{3x} \dots + \dots \\
&= p^0 (u_0^4 + u_0 u_{0x}) + p^1 (u_0 u_{1x} + u_1 u_{0x} + 4u_0^3 u_{1x}) \\
&\quad + p^2 (u_1 u_{1x} + 6u_0^2 u_1^2 + u_2 u_{0x} + u_0 u_{2x} + 4u_0^3 u_2) + \dots
\end{aligned}$$

Expansion of given nonlinear term by using the new approach we get $p_0, p_1, p_2, \dots$ without expanding the series

$$\begin{aligned}
N(u) &= N(u_0) + T(Z_0 + Z_1 + Z_2 + Z_3 + Z_4 + \dots) \\
&= N(u_0) + T(Z_0) + \frac{1}{2}T(Z_1) + \frac{1}{3}T(Z_2) + \frac{1}{4}T(Z_3) + \dots \\
&= u_0^4 + u_0 u_{0x} + T(u_0^4 + u_0 u_{0x}) + \frac{1}{2}T(4u_0^3 u_1 + u_1 u_{0x} + u_0 u_{1x}) + \dots \\
&= \underbrace{u_0^4 + u_0 u_{0x}}_{p_0} + \underbrace{4u_0^3 u_1 + u_1 u_{0x} + u_0 u_{1x}}_{p_1} + \underbrace{4u_0^3 u_2 + 6u_0^2 u_1^2 + u_2 u_{0x}}_{p_2} \\
&\quad + \underbrace{u_1 u_{1x} + u_0 u_{2x}}_{p_2} + \dots
\end{aligned}$$

5. Conclusions

The Homotopy Perturbation Method is very powerful and efficient method to solve the linear and nonlinear equations. For the physical problems it provides more realistic series solution but the complexity is to expand the terms with higher powers. The *Zaidian* approach presented in this paper to handle such situation made the application of HPM much easier and friendly. Moreover, application of this paper is very wide in a way that it can be used to expand nonlinear terms in Variation Iteration Method, Variation of Parameters Method, perturbation method and series solution method.

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STRONGLY PARACOMPACT MAPPINGS

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Abstract. In this paper, we define and study strongly paracompact mappings which are the fibrewise topological analogues of strongly paracompact spaces. Several characterizations and properties of strongly paractompact mappings are obtained.

Keywords. fiberwise, mapping, paracompact, perfect, strongly paracompact.

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1. Preliminaries

Unless otherwise stated, Y is a fixed topological space with topology τ , collection of all neighborhoods ($nbd(s)$) of $y \in Y$ is denoted by $N(y)$. If $f : X \rightarrow Y$ is a mapping and $A \subseteq X$ and $B \subseteq X$, then $[A]_B$ means the closure of A in B . For continuous mappings $f : X \rightarrow Y$ and $g : Z \rightarrow Y$, a continuous mapping $\lambda : X \rightarrow Z$ such t etc. if λ is hat $f = g \lambda$ is called a morphism of f into g and is denoted by $\lambda : f \rightarrow g$. λ is called surjective, closed, perfect surjective, closed, perfect etc., respectively. Open covers will be denoted by $\hat{U}, \hat{V}, \hat{W}, \dots$

To proceed we need the following definitions and results. For more details one can consult [3] and [5].

Definition 1.1 A mapping $f : X \rightarrow Y$ is called a T_2 -mapping, if there exist disjoint nbd s in X for every $x, x^* \in f^{-1}y$ such that $x \neq x^*$.

Definition 1.2 If A and B are subsets of X , then we say that A and B are

1. nbd separated in $U \subseteq X$.
2. Functionally separated in $U \subseteq X$.

if, respectively, the sets $A \cap U$ and $B \cap U$.

1. Have disjoint *nbd*s in U .
2. There exists a continuous function $f : U \rightarrow [0, 1]$ such that $A \cap U \subseteq f^{-1}(0)$ and $B \cap U \subseteq f^{-1}(1)$.

Definition 1.3 A mapping $f : X \rightarrow Y$ is said to be completely regular (regular) if for every $x \in X$ and every closed set F in X such that $x \notin F$ there exists a neighborhood $O \in N(fx)$ such that $\{x\}$ and F are functionally separated (*nbd* separated) in $f^{-1}O$.

A completely regular (regular) T_0 -mapping is called a Tychonoff or $T_{3\frac{1}{2}}$ -mapping (regular or T_3 -mapping).

Definition 1.4 A mapping $f : X \rightarrow Y$ is called functionally prenormal (prenormal) if for every $y \in Y$ and every disjoint closed (in X) sets F and H there exists a neighborhood O of y such that F and H are functionally separated (*nbd* separated) in $f^{-1}O$. If for every open subset O of Y the mapping $f|_{f^{-1}O} : f^{-1}O \rightarrow O$ is functionally prenormal (prenormal), then f is called functionally normal (normal). A normal T_3 -mapping is called T_4 -mapping.

A mapping $g : A \rightarrow B$ is said to be a (closed, open, dense, etc.) submapping of the mapping $f : X \rightarrow Y$ if g is the restriction of f on the (closed, open, dense, etc.) subset A of the space X and $g(A) = f(A) \subseteq B \subseteq Y$. A mapping $f : X \rightarrow Y$ is said to be compact if and only if f is perfect. If $f : X \rightarrow Y$ is a compact T_2 -mapping and $g : A \rightarrow B$ is a submapping of f where B is a closed subset of Y , then g is compact.

Definition 1.5 [3] A mapping $f : X \rightarrow Y$ is called paracompact if for every $y \in Y$ and every open (in X) cover $\hat{U} = \{U_\alpha; \alpha \in \Delta\}$ of $f^{-1}y$ there exists $O_y \in N(y)$ such that $f^{-1}y$ is covered by $\hat{U}$ and $(f^{-1}O_y \wedge \hat{U})$ has an open (in X) y -locally finite refinement in $f^{-1}O_y$.

Theorem 1.6 [3] If $f : X \rightarrow Y$ is a regular mapping, then the following conditions are equivalent.

- 1) f is a paracompact T_2 -mapping.
- 2) For every $y \in Y$ and every open (in X) cover $\hat{U}$ of $f^{-1}y$ there exists $O_y \in N(y)$ such that $f^{-1}y$ is covered by $\hat{U}$ and $(f^{-1}O_y \wedge \hat{U})$ has an open (in X) σ -locally finite refinement $\hat{V}$ in $f^{-1}O_y$; that is $\hat{V} = \bigcup_{i < \omega} \hat{V}_i$, where $\hat{V}_i$ is locally finite in $f^{-1}O_y$ for every $i < \omega$.

Definition 1.7 [1] Let $f : X \rightarrow Y$ be a mapping. Then f is called countably paracompact if for every $y \in Y$ and every countable open (in X) cover $\hat{U}$ of $f^{-1}y$ there exists $O_y \in N(y)$ such that $f^{-1}y$ is covered by $\hat{U}$ and $f^{-1}O_y \wedge \hat{U}$ has an open (in X) y -locally finite refinement in $f^{-1}O_y$.

Theorem 1.8 [1] *Let $f : X \rightarrow Y$ be a mapping. Then the following conditions are equivalent.*

- (i) *f is normal and countably paracompact.*
- (ii) *For every $y \in Y$ and every countable open (in X) cover $\hat{U} = \{U_i; i \in \mathbb{N}\}$ of $f^{-1}y$ there exists a neighborhood $O_y \in N(y)$ such that $f^{-1}O_y$ is covered by $\hat{U}$; furthermore, for every $i = 1, 2, \dots$ there exists $O_{i(y)} \in N(y)$, where $O_{i(y)} \subseteq O_y$, and a closed (in $f^{-1}O_{i(y)}$) subset $F_i \subseteq f^{-1}O_{i(y)} \cap U_i$ such that $f^{-1}O_y = \bigcup_{i=1}^{\infty} F_i$.*

2. Strongly paracompact mappings

Definition 2.1 Let $f : X \rightarrow Y$ be a mapping. For every $y \in Y$ the open (in X) cover $\hat{U} = \{U_\alpha\}_{\alpha \in \Delta}$ of $f^{-1}y$ is said to be y -star-finite if for every $x \in f^{-1}y$ there exists $U_\alpha \in \hat{U}$ such that $x \in U_\alpha$ and the family $S(U_\alpha, \hat{U}) = \{U_\beta \in \hat{U}; U_\beta \cap U_\alpha \neq \emptyset\}$ is finite.

Definition 2.2 Let $f : X \rightarrow Y$ be a mapping. Then f is said to be strongly paracompact if for every $y \in Y$ and every open (in X) cover $\hat{U} = \{U_\alpha\}_{\alpha \in \Delta}$ of $f^{-1}y$ there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$ and $\hat{U} \wedge f^{-1}O_y$ has a y -star-finite open (in X) refinement in $f^{-1}O_y$.

The following results can be derived easily from Definitions 2.1 and 2.2.

Theorem 2.3 *If $f : X \rightarrow Y$ is a strongly paracompact mapping, L a subset of Y and A is a closed subset of X , then*

- 1) *f is closed.*
- 2) *For every $y \in Y$ and every open (in X) cover $\hat{U} = \{U_\alpha\}_{\alpha \in \Delta}$ of $f^{-1}y$ there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$ and $\hat{U} \wedge f^{-1}O_y$ has a star-finite open (in X) refinement $\hat{V} = \{V_\beta\}_{\beta \in \Lambda}$ in $f^{-1}O_y$; that is, V_β intersects finitely many elements of $\hat{V}$ for every $\beta \in \Lambda$.*
- 3) *f is paracompact.*
- 4) *If f is, also, Hausdorff, then it is normal.*
- 5) *$f|_A : A \rightarrow Y$ is strongly paracompact.*
- 6) *$f_L : f^{-1}L \rightarrow L$ is strongly paracompact.*
- 7) *Each fibre of f is a strongly paracompact space.*

If the fibres of a mapping f are strongly paracompact spaces, then f is not necessarily a strongly paracompact mapping, even if f is closed and Tychonoff.

Example 2.4 Let L be the Niemytzki plane and let L_1 be the line $y = 0$. Then L_1 is closed in L and so the quotient mapping $q : L \rightarrow L/L_1$ is closed. Since L is Tychonof space, q is Tychonof. Since each fibre is discrete, the fibres of f are strongly paracompact space. But f is still not strongly paracompact because it is not paracompact. See Buhagiar [1997].

We know that if $\{V_\alpha; \alpha \in \Delta\}$ is a locally finite open cover of a space X , then the cover $\{[V_\alpha]_X; \alpha \in \Delta\}$ is locally finite in X . The same holds for star-finite families; more precisely

Lemma 2.5 *If $\{V_\alpha; \alpha \in \Delta\}$ is a star-finite cover of the space X , then the corresponding cover $\{St(V_\alpha); \alpha \in \Delta\}$ such that $St(V_\alpha) = \bigcup \{V_\gamma; V_\gamma \cap V_\alpha \neq \emptyset, \gamma \in \Delta\}$ is a star-finite cover of X .*

Corollary 2.6 *If $\{V_\alpha; \alpha \in \Delta\}$ is a star-finite open cover of the space X , then the corresponding cover $\{[V_\alpha]_X; \alpha \in \Delta\}$ is a star-finite cover of X .*

Now, we shall prove the following characterization of strongly paracompact mappings in the presence of regular mappings.

Theorem 2.7 *If $f : X \rightarrow Y$ is a regular mapping, then the following conditions are equivalent.*

- 1) f is strongly paracompact.
- 2) *For every $y \in Y$ and every open (in X) cover $\hat{U} = \{U_\alpha\}_{\alpha \in \Delta}$ of $f^{-1}y$ there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$ and $\hat{U} \wedge f^{-1}O_y$ has y -closed star-finite locally finite refinement in $f^{-1}O_y$.*

Proof. (1) $\rightarrow$ (2). Let $\hat{U} = \{U_\alpha\}_{\alpha \in \Delta}$ be an open (in X) cover of $f^{-1}y$. For every $x \in f^{-1}y$ there exists W_x (open neighborhood of x) and $O_{y(x)} \in N(y)$ such that $[W_x]_{f^{-1}O_{y(x)}} \subseteq U_{\alpha(x)} \cap f^{-1}O_{y(x)}$, where $W_x \subseteq U_{\alpha(x)}$. Let $\hat{W} = \{W_x; x \in f^{-1}y\}$. Then $\hat{W}$ is an open cover of $f^{-1}y$, so that there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup \hat{W}$ and $\hat{W} \wedge f^{-1}O_y$ has a star-finite (and so locally finite) open (in X) refinement in $f^{-1}O_y$, say $\hat{V} = \{V_\beta; \beta \in B\}$. Since $\hat{V}$ is star-finite and open (in $f^{-1}O_y$), we have (by the previous lemma) the family $\{[V_\beta]_{f^{-1}O_y}; \beta \in B\}$ is star-finite (and so locally finite) closed refinement of $\hat{W} \wedge f^{-1}O_y$ in $f^{-1}O_y$. For every $x \in X$ let $O_{y(x)}^* = O_y \cap O_{y(x)}$. But for every $\beta \in B$ there exists $x \in X$ such that $V_\beta \subseteq W_x \cap f^{-1}O_y$; let $V_\beta^* = [V_\beta]_{f^{-1}O_{y(x)}^*}$. Since $V_\beta \subseteq W_x \cap f^{-1}O_y$, we've $V_\beta^* \subseteq [W_x \cap f^{-1}O_y]_{f^{-1}O_{y(x)}^*}$. Note that

$$\begin{aligned}
[W_x \cap f^{-1}O_y]_{f^{-1}O_{y(x)}^*} &= [W_x \cap f^{-1}O_y]_X \cap f^{-1}O_{y(x)}^* \\
&\subseteq [W_x]_X \cap [f^{-1}O_y]_X \cap f^{-1}O_{y(x)}^* \\
&\subseteq [W_x]_X \cap f^{-1}O_{y(x)}^* \\
&\subseteq [W_x]_X \cap f^{-1}O_{y(x)} \\
&\subseteq U_{\alpha(x)} \cap f^{-1}O_{y(x)}
\end{aligned}$$

for some $\alpha(x) \in \Delta$; hence $V_\beta^* \subseteq U_{\alpha(x)} \cap f^{-1}O_{y(x)}$ for some $\alpha(x) \in \Delta$. Let $\hat{V}_\beta^* = \{V_\beta^{**}; \beta \in B\}$ such that $V_\beta^{**} = V_\beta^* \cup V_\beta$. It is clear that V_β^{**} is y -closed for every $\beta \in B$. Since $V_\beta^* \subseteq f^{-1}O_{y(x)}^* \subseteq f^{-1}O_y$, $V_\beta^* \subseteq U_{\alpha(x)} \cap f^{-1}O_y$ and $V_\beta^{**} = V_\beta^* \cup V_\beta \subseteq U_{\alpha(x)} \cap f^{-1}O_y$; this implies that $\hat{V}^*$ is a refinement of $\hat{U} \wedge f^{-1}O_y$. One can readily prove that $V_\beta^{**} \subseteq [V_\beta]_{f^{-1}O_y}$ for every $\beta \in B$. Since $\{[V_\beta]_{f^{-1}O_y}\}$ is star-finite and locally finite in $f^{-1}O_y$, we have $\hat{V}^*$ is star-finite locally finite in $f^{-1}O_y$. This completes the proof of (1) $\rightarrow$ (2).

Now, we shall show that (2) $\rightarrow$ (1). If $\hat{U}$ is an open cover of $f^{-1}y$, then there is $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup \hat{U}$ and $\hat{U} \wedge f^{-1}O_y$ has a star-finite locally finite y -closed refinement in $f^{-1}O_y$, say $\{P_\alpha; \alpha \in \Delta\}$. For every $\alpha \in \Delta$ let $\hat{Z}_\alpha = \{P_{\alpha^*}; P_{\alpha^*} \cap P_\alpha = \Phi\}$ and $Z_\alpha = \bigcup \{P_{\alpha^*}; P_{\alpha^*} \cap P_\alpha = \Phi\}$. Now, for every $x \in f^{-1}y - Z_\alpha$ there exists an open (in $f^{-1}O_y$) neighborhood V_x of x such that V_x intersects finitely many elements of $\{P_\alpha; \alpha \in \Delta\}$, so that V_x intersects finitely many elements of $\hat{Z}_\alpha$, say $P_{\alpha_1}, P_{\alpha_2}, \dots, P_{\alpha_n}$. But $\bigcup_{i=1}^n P_{\alpha_i}$ is closed in $f^{-1}O_y^*$ for some $O_y^* \in N(y)$. Set $G_x = V_x \cap (f^{-1}O_y^* - \bigcup_{i=1}^n P_{\alpha_i})$. Then G_x is an open (in X) neighborhood of x and disjoint from Z_α . Let $\hat{W} = \{W_\alpha; \alpha \in \Delta\}$ where $W_\alpha = \bigcup \{G_x; x \in f^{-1}y - Z_\alpha\}$ for every $\alpha \in \Delta$. Then $\hat{W}$ is an open (in X) cover of $f^{-1}y$ and W_α is contained in $St(P_\alpha) = \bigcup \{P_{\alpha^*}; P_{\alpha^*} \cap P_\alpha \neq \Phi\}$ for each $\alpha \in \Delta$. But the family $\{St(P_\alpha); \alpha \in \Delta\}$ is star-finite, by Lemma 2.1, so that $\hat{W}$ is a star-finite open (in X) cover of $f^{-1}y$. For every $\alpha \in \Delta$ there exists $U_\alpha \in \hat{U}$ such that P_α is contained in U_α . Let $W_\alpha^* = W_\alpha \cap U_\alpha$ and let $\hat{W}^* = \{W_\alpha^*; \alpha \in \Delta\}$. Then $\hat{W}^*$ is a star-finite open (in X) refinement of $\hat{U} \wedge f^{-1}O_y$. To complete the proof note that *ii*) implies that f is closed, so that there exists $O_y^* \in N(y)$ such that $f^{-1}O_y^* \subseteq \bigcup \hat{W}^*$; hence the cover $\hat{W}^* \wedge f^{-1}O_y^*$ is a star-finite open (in X) refinement of $\hat{U} \wedge f^{-1}O_y$ in $f^{-1}O_y^*$. This completes the proof. ■

For the sake of a later application we shall prove the following theorem.

Theorem 2.8 *If $f : X \rightarrow Y$ is a regular mapping, then the following conditions are equivalent.*

- 1) *For every $y \in Y$ and every open (in X) cover $\hat{U} = \{U_\alpha\}_{\alpha \in \Delta}$ of $f^{-1}y$ there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$ and $\hat{U} \wedge f^{-1}O_y$ has a y -closed star-countable locally finite refinement in $f^{-1}O_y$.*

- 2) For every $y \in Y$ and every open (in X) cover $\hat{U} = \{U_\alpha\}_{\alpha \in \Delta}$ of $f^{-1}y$ there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$ and $\hat{U} \wedge f^{-1}O_y$ has a star-countable open (in X) refinement in $f^{-1}O_y$.

Proof. (1) $\rightarrow$ (2). If $\hat{U}$ is an open (in X) cover of $f^{-1}y$, then there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup \hat{U}$ and $\hat{U} \wedge f^{-1}O_y$ has a locally finite star-countable y -closed refinement in $f^{-1}O_y$, say $\hat{V}$. We can write $\hat{V} = \bigcup_{t \in T} \hat{V}_t$ such that $\{\hat{V}_t; t \in T\}$ is the set of all components of $\hat{V}$ and $\hat{V}_t = \{V_{t,i}; i \in N\}$ for every $t \in T$. Set $C_t = \bigcup_{i \in N} V_{t,i}$. The family $\{C_t; t \in T\}$ covers $f^{-1}O_y$. We shall show that, C_t is closed and open in $f^{-1}O_y$ for every $t \in T$. Since $\{C_t; t \in T\}$ is a pairwise disjoint family, it is sufficient to show that C_t is open in $f^{-1}O_y$ for each $t \in T$. Let $x \in C_t$. Then, by locally finiteness, there exists V_x (open neighborhood of x) such that V_x intersects finitely many elements of $\hat{V} - \hat{V}_t$, say $P_{\alpha_1}, P_{\alpha_2}, \dots, P_{\alpha_n}$. There exists $O_y^* \in N(y)$ such that $\bigcup_{i=1}^n P_{\alpha_i}$ is closed in $f^{-1}O_y^*$. If $G_x = (f^{-1}O_y^* - \bigcup_{i=1}^n P_{\alpha_i})$, then G_x is an open neighborhood of x contained in C_t ; this implies that C_t is open in $f^{-1}O_y$ and so closed and open in $f^{-1}O_y$. For every $t \in T$ and every natural number i let $U(t, i)$ in $\hat{U}$ such that $V_{t,i} \subseteq U(t, i)$ and $\hat{V}^* = \{C_t \cap U(t, i); t \in T, i \in N\}$. It is clear that $\hat{V}^*$ is an open (in X) star-countable refinement of $f^{-1}O_y \wedge \hat{U}$ in $f^{-1}O_y$. This completes the proof of (1) $\rightarrow$ (2).

Now, we prove (2) $\rightarrow$ (1). First, we shall show that f is paracompact. Let $\hat{U}$ be an open (in X) cover of $f^{-1}y$. Then there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup \hat{U}$ and $\hat{U} \wedge f^{-1}O_y$ has a starcountable open refinement $\hat{V}$ in $f^{-1}O_y$. Since $\hat{V}$ is star-countable, we have $\hat{V} = \bigcup_{t \in T} \hat{V}_t$ such that $\{\hat{V}_t; t \in T\}$ is the set of all components of $\hat{V}$ and $\hat{V}_t = \{V_{t,i}; i \in N\}$ for every $t \in T$. Let $\hat{V}_i = \{V_{t,i}; t \in T\}$. $\hat{V}_i$ is locally finite in $f^{-1}O_y$, so that $\hat{V} = \bigcup_{t \in T} \hat{V}_t = \bigcup_{i \in N} \hat{V}_i$ is a σ -locally finite open (in X) refinement of $\hat{U}$ in $f^{-1}O_y$. But f is regular; hence (by Theorem 1.6) f is paracompact.

(2) $\rightarrow$ (1). Let $\hat{U}$ be an open cover of $f^{-1}y$. For every $x \in f^{-1}y$ there exists W_x open (in X) neighborhood of x and $O_{y(x)} \in N(y)$ such that $[W_x]_{f^{-1}O_{y(x)}} \subseteq U_{(x)} \cap f^{-1}O_{y(x)}$, where $W_x \subseteq U_{(x)}$ for some $U_{(x)} \in \hat{U}$. Let $\hat{W} = \{W_x; x \in f^{-1}y\}$. $\hat{W}$ is an open cover of $f^{-1}y$, so that there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup \hat{W}$ and $\hat{W} \wedge f^{-1}O_y$ has a star-countable open (in X) refinement in $f^{-1}O_y$, say $\hat{V} = \{V_\alpha; \alpha \in \Delta\}$. Since $\hat{V}$ is an open cover of $f^{-1}y$ and f is a paracompact mapping, there exists $O_y^* \in N(y)$ such that $O_y^* \subseteq O_y$, $f^{-1}O_y \subseteq \bigcup \hat{V}$ and $\hat{V} \wedge f^{-1}O_y$ has a locally finite open (in X) refinement in $f^{-1}O_y^*$, say $\hat{V}^*$. If $V \in \hat{V}^*$, then there exists $\alpha(V) \in \Delta$ such that $V \subseteq V_{\alpha(V)}$. For every $V \in \hat{V}^*$ fix such an $\alpha(V)$ and let $M_\alpha = \bigcup \{V; \alpha(V) = \alpha \text{ and } V \in \hat{V}^*\}$. It is clear that $M_\alpha \subseteq V_\alpha$ for every $\alpha \in \Delta$; hence $\{M_\alpha; \alpha \in \Delta\}$ is an open (in X) cover of $f^{-1}O_y^*$. Since $\hat{V} = \{V_\alpha; \alpha \in \Delta\}$ is star-countable and $M_\alpha \subseteq V_\alpha$, the family $\{M_\alpha; \alpha \in \Delta\}$ is star-countable, and since $\hat{V}^*$ is locally finite in $f^{-1}O_y^*$, there exist U_x (open subset of $f^{-1}O_y^*$) such that U_x intersects finitely many elements of $\hat{V}^*$ for every $x \in f^{-1}O_y^*$ (and so finitely many elements of $\{M_\alpha; \alpha \in \Delta\}$); hence the family $\{M_\alpha; \alpha \in \Delta\}$ is a star-countable

locally finite open (in X) refinement of $\hat{V} \wedge f^{-1}O_y^*$ in $f^{-1}O_y^*$, so that the family $\{[M_\alpha]_{f^{-1}O_y^*}\}_{\alpha \in \Delta}$ is a star-countable locally finite (in X) refinement of $\hat{V} \wedge f^{-1}O_y^*$ in $f^{-1}O_y^*$. For every $x \in X$ let $O_{y(x)}^* = O_y^* \cap O_{y(x)}$, and for every $\alpha \in \Delta$ let $M_\alpha^* = [M_\alpha]_X \cap f^{-1}O_{y(x)}^*$ and $M_\alpha^{**} = M_\alpha^* \cup M_\alpha$. Then, one readily proves that $\hat{M} = \{M_\alpha^{**}\}_{\alpha \in \Delta}$ is a star-countable locally finite refinement of $\hat{U} \wedge f^{-1}O_y^*$ in $f^{-1}O_y^*$ consisting of y -closed subsets. This completes the proof. ■

The next two lemmas will be used to show that the conditions in Theorems 2.7 and 2.8 are equivalent in the realm of functionally normal mappings. First, we shall define functionally open sets for mappings.

Definition 2.9 Let $f : X \rightarrow Y$ be any mapping. For a subset A of X we say that A is y -functionally open if there exists $O_y \in N(y)$ and a continuous real valued function $g : f^{-1}O_y \rightarrow I$ such that $g^{-1}((0, 1]) = A \cap f^{-1}O_y$.

Lemma 2.10 If $f : X \rightarrow Y$ is a continuous mapping and $\{U_i^*; i \in N\}$ is an open cover of $f^{-1}y$ such that each U_i^* is y -functionally open, then there exists a star-finite open cover of $f^{-1}y$ and refines $\{U_i^*; i \in N\}$.

Proof. For every $i \in N$ there exists $O_i \in N(y)$ and a continuous real valued function $\alpha_i : f^{-1}O_i \rightarrow I$ such that $\alpha_i^{-1}((0, 1]) = U_i^* \cap f^{-1}O_i$. Let $U_i = U_i^* \cap f^{-1}O_i$. It is clear that $\{U_i; i \in N\}$ is an open cover of $f^{-1}y$ refinement of $\{U_i^*; i \in N\}$. For every $i \in N$ let $f_i : X \rightarrow I$ be defined such that

$$f_i(x) = \begin{cases} \alpha_i(x); & x \in f^{-1}O_i \\ 0 & x \in X - f^{-1}O_i \end{cases}$$

Note that f_i is not necessarily a continuous function. Let $g : X \rightarrow I$ be defined such that $g(x) = \sum_{i=1}^{\infty} \frac{f_i(x)}{2^i}$. g is not necessarily continuous. For every $k \in N$ set $V_k = g^{-1}((\frac{1}{k}, 1])$ and $F_k = g^{-1}([\frac{1}{k}, 1])$. Note that V_k is not necessarily an open subset of X , and F_k is not necessarily a closed subset of X . For every $i \leq k$ let $U_{i,k} = U_i \cap (V_{k+1} - F_{k-1})$. We shall show that the family $\hat{U} = \{Int(U_{i,k}); i, k \in N, i \leq k\}$ is a star-finite open cover of $f^{-1}y$ and refines $\{U_i^*; i \in N\}$. First, one can readily prove that $\{U_{i,k}; i, k \in N\}$ is a star-finite refinement of $\{U_i^*; i \in N\}$, so that the family $\hat{U} = \{Int(U_{i,k}); i, k \in N \text{ and } i \leq k\}$ is a star-finite refinement of $\{U_i^*; i \in N\}$. It remains to show that the family $\hat{U} = \{Int(U_{i,k}); i, k \in N \text{ and } i \leq k\}$ covers $f^{-1}y$. If $x \in f^{-1}y$, then $x \in U_i$ for some natural number i ; which implies that $f_i(x) \neq 0$ and so $g(x) \neq 0$, so that there exists a natural number m such that $g(x) > \frac{1}{m}$, i.e., $x \in V_m \subseteq F_m$. Let k be the smallest natural number such that $x \in F_k$. Since $F_k \subseteq V_{k+1}$, we have $x \in V_{k+1}$ and $x \notin F_{k-1}$. It is clear that $F_k \subseteq \bigcup_{i \leq k} U_i$, so that there exists $j \leq k$ such that $x \in U_j$; which implies that $x \in U_{j,k}$. Now, we shall show that $x \in int(U_{j,k})$. Since $x \in V_{k+1}$, we have $g(x) = \sum_{i=1}^{\infty} \frac{f_i(x)}{2^i} > \frac{1}{k+1}$, so that there exists $n_o \in N$ such that $\sum_{i=1}^{n_o} \frac{f_i(x)}{2^i} > \frac{1}{k+1}$. For every $i = 1, 2, \dots, n_o$ let $s_i = 0$ whenever $f_i(x) = 0$, and if $f_i(x) \neq 0$ let s_i be any real number such that $s_i < f_i(x)$ and $\sum_{i=1}^{\infty} \frac{s_i}{2^i} > \frac{1}{k+1}$; furthermore let

$$M_i = \begin{cases} f_i^{-1}O_i & \text{if } f_i(x) = 0 \\ f_i^{-1}((s_i, 1]) = \alpha_i^{-1}((s_i, 1]) & \text{if } f_i(x) \neq 0 \end{cases}$$

Set $H_x = \cap_{i=1}^{n_o} M_i$. It is clear that H_x is an open subset of X and contains x . We shall show that $H_x \subseteq V_{k+1}$. But if $t \in H_x$, then $t \in f_i^{-1}([s_i, 1])$ for every $i = 1, 2, \dots, n_o$, so that $f_i(t) \geq s_i$ and $g(t) = \sum_{i=1}^{\infty} \frac{f_i(t)}{2^i} \geq \sum_{i=1}^{n_o} \frac{f_i(t)}{2^i} \geq \sum_{i=1}^{n_o} \frac{s_i}{2^i} > \frac{1}{k+1}$; hence $t \in g^{-1}((\frac{1}{k+1}, 1]) = V_{k+1}$, which implies that $H_x \subseteq V_{k+1}$. On the other hand, since $x \notin F_{k-1}$, we've $g(x) < \frac{1}{k-1}$. Let $s \in (g(x), \frac{1}{k-1})$ and $n_o \in N$ such that $\sum_{i=n_o+1}^{\infty} \frac{1}{2^i} < \frac{1}{k-1} - s$. For every $i \leq n_o$ let $s_i = 1$ whenever $f_i(x) = 1$ and let $s_i \in (f_i(x), 1)$ such that $\sum_{i=1}^{n_o} \frac{s_i}{2^i} < s$ whenever $f_i(x) \neq 1$; furthermore let

$$A_i = \begin{cases} f_i^{-1}O_i = \alpha_i^{-1}([0, 1]) & \text{if } f_i(x) = 1, \\ \alpha_i^{-1}([0, s_i]) & \text{if } f_i(x) \neq 1. \end{cases}$$

Set $H_x^* = \cap_{i=1}^{n_o} A_i$. It is clear that H_x^* is open in X and contains x . We shall show that $H_x^* \cap F_{k-1} = \emptyset$. If $t \in H_x^*$, then $t \in A_i$ for every $i = 1, 2, \dots, n_o$, so that $f_i(t) \leq s_i$ for every s_i , which implies that

$$\begin{aligned} g(t) &= \sum_{i=1}^{\infty} \frac{f_i(t)}{2^i} = \sum_{i=1}^{n_o} \frac{f_i(t)}{2^i} + \sum_{i=n_o+1}^{\infty} \frac{f_i(t)}{2^i} \\ &\leq \sum_{i=1}^{n_o} \frac{f_i(t)}{2^i} + \sum_{i=n_o+1}^{\infty} \frac{1}{2^i} \\ &< \sum_{i=1}^{n_o} \frac{s_i}{2^i} + \frac{1}{k-1} - s \\ &< s + \frac{1}{k-1} - s = \frac{1}{k-1} \end{aligned}$$

So that $t \notin g^{-1}([\frac{1}{k-1}, 1]) = F_{k-1}$ and $H_x^* \cap F_{k-1} = \emptyset$. Let $H_x^{**} = H_x^* \cap H_x \cap U_j$. H_x^{**} is an open (in X) neighborhood of x contained in $U_{j,k}$; more precisely $x \in \text{Int}(U_{j,k})$, so that the family $\hat{U} = \{\text{Int}(U_{i,k}); i, k \in N, i \leq k\}$ covers $f^{-1}y$. This completes the proof. ■

Lemma 2.11 *If $f : X \rightarrow Y$ is a countably paracompact functionally normal mapping, then for every open (in X) cover $\hat{U} = \{U_i; i \in N\}$ of $f^{-1}y$ there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup \hat{U}$ and $\hat{U} \wedge f^{-1}O_y$ has a star-finite open (in X) refinement in $f^{-1}O_y$.*

Proof. By Theorem 1.8, there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup \hat{U}$ and $\hat{U} \wedge f^{-1}O_y$ has a refinement $\{F_i; i \in N\}$ in $f^{-1}O_y$ such that $F_i \subseteq U_i$ for every i , and there exists $O_i \in N(y)$ for every i such that $O_i \subseteq O_y$, F_i is a closed subset of $f^{-1}O_i$ and $\bigcup_{i=1}^{\infty} F_i = f^{-1}O_y$. Since f is functionally normal, the mapping $f_{O_i} : f^{-1}O_i \rightarrow O_i$ is functionally normal for every i , so that there exists $O_i^* \in N(y)$ such that $O_i^* \subseteq O_i$ and there exists a continuous function $\alpha_i : f^{-1}O_i^* \rightarrow I$ such that $\alpha_i(F_i \cap f^{-1}O_i^*) = 1$ and $\alpha_i((f^{-1}O_i^*) - U_i) = 0$. Set $V_i = \alpha_i^{-1}((0, 1])$. Since $F_i \cap f^{-1}O_i^* \subseteq V_i^*$, we have the family $\hat{V} = \{V_i; i \in N\}$ is an open cover of $f^{-1}y$ consisting of y -functionally open subsets. By Lemma 2.10, there exists a star-finite open (in X) cover $\hat{W}$ of $f^{-1}y$ refines $\hat{V} = \{V_i; i \in N\}$. Since f is closed,

there exists $O_y^* \in N(y)$ such that $f^{-1}O_y^* \subseteq \bigcup \hat{W}$. It is clear that $f^{-1}O_y^* \wedge \hat{W}$ is a star-finite open (in X) refinement of $f^{-1}O_y^* \wedge \hat{U}$ in $f^{-1}O_y^*$. This completes the proof. ■

Question 2.12 *Is Lemma 2.11 true if the functional normality is replaced by normality?*

The following theorem is a characterization of strongly paracompact mappings.

Theorem 2.13 *If $f : X \rightarrow Y$ is a functionally normal mapping, then the following conditions are equivalent.*

- 1) f is strongly paracompact.
- 2) For every $y \in Y$ and every open (in X) cover $\hat{U} = \{U_\alpha\}_{\alpha \in \Delta}$ of $f^{-1}y$ there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$ and $\hat{U} \wedge f^{-1}O_y$ has a y -closed star-finite locally finite refinement in $f^{-1}O_y$.
- 3) For every $y \in Y$ and every open (in X) cover $\hat{U} = \{U_\alpha\}_{\alpha \in \Delta}$ of $f^{-1}y$ there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$ and $\hat{U} \wedge f^{-1}O_y$ has a star-countable y -closed locally finite refinement in $f^{-1}O_y$.
- 4) For every $y \in Y$ and every open (in X) cover $\hat{U} = \{U_\alpha\}_{\alpha \in \Delta}$ of $f^{-1}y$ there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$ and $\hat{U} \wedge f^{-1}O_y$ has a star-countable open (in X) refinement in $f^{-1}O_y$.

Proof. By Theorems 2.7 and 2.8 we have 1) is equivalent to 2) and 3) is equivalent to 4). It is clear that 2) implies 3). It suffices to show that 4) implies 1).

Assume 4). First, note that f is paracompact (see the proof of Theorem 2.7). Now, we shall show that f is a strongly paracompact mapping.

Let $\hat{U}$ be an open (in X) cover of $f^{-1}y$. Then there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup \hat{U}$ and $\hat{U} \wedge f^{-1}O_y$ has a starcountable open refinement $\hat{V}$ in $f^{-1}O_y$. We can write $\hat{V} = \bigcup_{t \in T} \hat{V}_t$ such that $\{\hat{V}_t \in T\}$ is the set of all components of $\hat{V}$. But for every $t \in T$ we have $\hat{V}_t = \{V_{t,i}; i \in N\}$. Set $C_t = \bigcup_{i \in N} V_{t,i}$. Then C_t is a closed and open subset of $f^{-1}O_y$ for every $t \in T$. Since f is a paracompact functionally normal mapping, we have $f_{O_y} : f^{-1}O_y \rightarrow O_y$ is a paracompact functionally normal mapping, and since C_t is a closed subset of $f^{-1}O_y$, we have $f_t = f|_{C_t} : C_t \rightarrow O_y$ is a paracompact (and so a countably paracompact) functionally normal mapping for every $t \in T$. But $\hat{V}_t$ is an open cover of C_t so it is an open cover of $f_t^{-1}y$ and (by Lemma 2.11) there exists $O_t \in N(y)$ such that $f_t^{-1}O_t \subseteq \bigcup_{i=1}^{\infty} V_{t,i}$ and $\hat{V}_t \wedge f_t^{-1}O_t$ has a star-finite open (in X) refinement $\hat{V}_t^*$ in $f_t^{-1}O_t$. It is clear that $\hat{V}^* = \bigcup_{t \in T} \hat{V}_t^*$ is an open (in X) cover of $f^{-1}y$. Since the family $\hat{C} = \{C_t; t \in T\}$ is pairwise

disjoint and $\hat{V}_t^*$ is star-finite for every $t \in T$, we have $\hat{V}^* = \bigcup_{t \in T} \hat{V}_t^*$ is star-finite, and since f is closed, there exists $O_y^* \in N(y)$ such that $f^{-1}O_y^* \subseteq \bigcup \hat{V}^*$, so that $f^{-1}O_y^* \wedge \hat{V}^*$ is a star-finite open (in X) refinement of $f^{-1}O_y^* \wedge \hat{U}$ in $f^{-1}O_y^*$. This completes the proof. ■

Corollary 2.14 *Every Lindelöf functionally normal mapping is strongly paracompact.*

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A PRACTICAL METHOD FOR MATRIX INVERSION

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Abstract. This report presents a matrix inversion method that has the following features: general applicability for any non-singular $(n \times n)$ matrix $\mathbf{A}$ with either real or complex elements a_{ij} , $i, j = 1, 2, \dots, n$; univocally defined matrix operations; analytical representation of the sought-for inverse matrix A^{-1} as a product of three uniquely specified non-singular triangular matrices ($A^{-1} = \mathbf{P}\mathbf{G}\mathbf{V}$, $\mathbf{P}$ and $\mathbf{V}$ lower triangular, $\mathbf{G}$ upper triangular); and convenient (minimal) number, n^3 , of required multiplication/division operations.

The inversion procedure is carried out in two stages: I) transformation of matrix $\mathbf{A}$ into an upper triangular matrix $\mathbf{T}$ having unit diagonal elements; II) transformation of matrix $\mathbf{T}$ into the $(n \times n)$ unit matrix $\mathbf{U}^{(n)}$ having elements $u_{ij}^{(n)} = 0$ if $i \neq j$, and $= 1$ if $i = j$.

The first stage conforms with a Gaussian elimination procedure that can be carried out in the natural order, through n consecutive ordered steps, since at each step univocally prescribed non-zero diagonal elements (leading pivots) are made available. A sequence of transformed non-singular $(n \times n)$ matrices $\{\mathbf{A}^{(k)}\}$, $k = 1, 2, \dots, n$, having elements $a_{ij}^{(k)}$, is obtained with pivots $a_{kk}^{(k)} = 1$. The viability of the procedure is assured by the ordered use of univocally defined, very simple non-singular lower triangular $(n \times n)$ operational matrices $\mathbf{P}^{(k)}$ which post-multiply $\mathbf{A}^{(k-1)}$ ($\mathbf{A}^{(0)}$ coinciding with $\mathbf{A}$) and transform diagonal element $a_{kk}^{(k-1)}$ into a leading pivot having a positive non-zero value given by the sum of the absolute values of all the elements in its row.

A simple numerical example is detailed for illustrating practical aspects.

The present matrix inversion method is free from any operational ambiguity. The simplicity and univocal definiteness of its transformations are expected to provide operational advantages for the development of related numerical algorithms both for finding matrix inverses and for solving systems of linear algebraic equations. Further useful features are related with the final triangular matrix factorization achieved which, in particular, allows an immediate computation of the determinants of matrices $\mathbf{A}$ and $\mathbf{A}^{-1}$.

General presentation

The determination of the unique inverse matrix $\mathbf{A}^{-1}$ of a given non-singular $(n \times n)$ matrix $\mathbf{A}$, having n^2 elements a_{ij} , $i, j = 1, 2, \dots, n$, that are real or complex numbers, represents a basic problem in linear algebra with implications for many theories, and applications in various fields.

Gaussian elimination has been recognized as the most widely used method for inverting matrices [1] and it is known that it may need suitable matrix transformations such as row (column) interchanges or combinations, for achieving adequate non-zero diagonal elements (pivots). The focus of the present inversion method consists in prescribing all required transformations at the outset, in view of avoiding any operational ambiguity and potentially critical situations. With such a formulation:

- i) A numerical algorithm based on the present method is of general use with no operational ambiguity, exploits a very simple logic, and can be applied through consecutive steps taken in the natural order.
- ii) In the successive ordered matrix transformations which achieve the triangular factorization of $\mathbf{A}$, a leading pivot has a positive non-zero value given by the sum of the absolute values of all the elements in its row, which tends to diminish the possible negative effects of round-off errors.
- iii) The sought-for inverse matrix $\mathbf{A}^{-1}$ has an analytical representation, of great practical interest in some contexts, as a product of three uniquely specified non-singular triangular matrices, $\mathbf{A}^{-1} = \mathbf{P}\mathbf{G}\mathbf{V}$, $\mathbf{P}$ and $\mathbf{V}$ lower triangular, $\mathbf{G}$ upper triangular. In particular, an immediate computation of the determinants of matrices $\mathbf{A}$ and $\mathbf{A}^{-1}$ is made available.
- iv) The inversion of a matrix $\mathbf{A}$ having complex elements is obtained in a straightforward way in analogy with the case of real elements.

In general, for solving a given problem, preference is given to the numerical method which requires a minimum number of operations and is logically simpler and, thus, is realized more rapidly on a computer [2]. As for the number of operations, a specific count for the present inversion method is detailed in Appendix 1 where the minimal count, n^3 , for all required multiplication/division operations is obtained [1]. The development of different numerical methods for solving specific problems should turn out to be a useful endeavour in many cases, particularly if various difficulties and complexities are at work.

The present inversion method is described for the case of a matrix $\mathbf{A}$ having real elements a_{ij} . The case of complex elements can be handled by introducing simple variations that are detailed below.

The inverse matrix $\mathbf{A}^{-1}$ is achieved through two consecutive transformations: $\tau 1$, which transforms $\mathbf{A}$ into an upper triangular matrix $\mathbf{T}$ having all diagonal elements $t_{ii} = 1$; and $\tau 2$, which transforms $\mathbf{T}$ into the $(n \times n)$ unit matrix $\mathbf{U}^{(n)}$ having elements $u_{ij}^{(n)} = 0$ if $i \neq j$, and $= 1$ if $i = j$.

Transformation $\tau 1$ conforms with a Gaussian elimination procedure that can be carried out in the natural order, through n consecutive ordered steps, since at each step univocally prescribed non-zero diagonal elements (leading pivots)

are made available. A sequence of transformed non-singular $(n \times n)$ matrices $\mathbf{A}^{(k)}$, $k = 1, 2, \dots, n$, having elements $a_{ij}^{(k)}$, is obtained with pivots $a_{kk}^{(k)} = 1$. The feasibility of the procedure is assured by the ordered use of univocally defined, very simple non-singular lower triangular $(n \times n)$ operational matrices $\mathbf{P}^{(k)}$ which post-multiply $\mathbf{A}^{(k-1)}$ ($\mathbf{A}^{(0)}$ coinciding with $\mathbf{A}$) and transform diagonal element $a_{kk}^{(k-1)}$ into the positive sum $\sum_{q=k}^n |a_{kq}^{(k-1)}|$.

A matrix $\mathbf{P}^{(k)}$ differs (or may differ) from $\mathbf{U}^{(n)}$ only in the diagonal and lower elements of the k -th column, prescribed by the relations: $p_{ik}^{(k)} = \sigma(a_{ki}^{(k-1)})$, $i = k, k+1, \dots, n$, where symbol $\sigma(r)$ indicates, for any real number r , the values: -1 if $r \leq 0$, and 1 if $r > 0$. A matrix $\mathbf{P}^{(k)}$ has thus a very simple structure with only two possible absolute values, 0 or 1 , for its elements. This structural simplicity provides a convenient matrix multiplication property: the matrix product $\mathbf{P}^{(k)} \mathbf{P}^{(s)}$, with $s > k$, is a matrix having the same elements as the unit matrix $\mathbf{U}^{(n)}$ except for the k -th and s -th columns which equal, respectively, the k -th column of $\mathbf{P}^{(k)}$ and the s -th column of $\mathbf{P}^{(s)}$.

Transformation $\tau 1$

This transformation requires n consecutive ordered steps. At the k -th step, $k = 1, 2, \dots, n-1$, matrix $\mathbf{A}$ is transformed into a matrix $\mathbf{A}^{(k)}$ that admits the following four-block representation:

$$(1.0) \quad \mathbf{A}^{(k)} = \begin{bmatrix} \mathbf{Z}^{(k)} & \mathbf{W}^{(k)} \\ \mathbf{R}^{(k)} & \mathbf{S}^{(k)} \end{bmatrix};$$

where $\mathbf{Z}^{(k)}$ is an upper triangular square sub-matrix of order k with all diagonal elements $z_{ii}^{(k)} = 1$; $\mathbf{R}^{(k)}$ is an $(n-k) \times k$ null matrix; $\mathbf{S}^{(k)}$ is a non-singular square sub-matrix of order $(n-k)$; and $\mathbf{W}^{(k)}$ is a $(k \times (n-k))$ matrix. At the last step, $k = n$, matrix $\mathbf{A}^{(n)} = \mathbf{T}$ is obtained by transforming element $a_{nn}^{(n-1)}$ into unity.

At the k -th step, $k = 1, 2, \dots, n$, the following transformations apply:

$$(1.1) \quad \mathbf{A}^{(k-1)} \mathbf{P}^{(k)} = \mathbf{A}^{(k-1)\oplus};$$

$$(1.2) \quad \mathbf{V}^{(k)} \mathbf{A}^{(k-1)\oplus} = \mathbf{A}^{(k)};$$

where:

- $\mathbf{A}^{(k-1)}$ with elements $a_{ij}^{(k-1)}$, $\mathbf{A}^{(k-1)\oplus}$ with elements $a_{ij}^{(k-1)\oplus}$, and $\mathbf{A}^{(k)}$ with elements $a_{ij}^{(k)}$, are $(n \times n)$ non-singular matrices and $\mathbf{A}^{(0)} = \mathbf{A}$ so that elements $a_{ij}^{(0)} = a_{ij}$;
- $\mathbf{P}^{(k)}$ and $\mathbf{V}^{(k)}$ are non singular $(n \times n)$ lower triangular operational matrices which differ (or may differ) from $\mathbf{U}^{(n)}$ only in the diagonal and lower elements of the k -th column. For $\mathbf{P}^{(k)}$, these elements are: $p_{ik}^{(k)} = \sigma(a_{ki}^{(k-1)})$; $i = k, k+1, \dots, n$; while for $\mathbf{V}^{(k)}$, they are: $v_{kk}^{(k)} = (a_{kk}^{(k-1)\oplus})^{-1}$; $v_{ik}^{(k)} = -a_{ik}^{(k-1)\oplus} (a_{kk}^{(k-1)\oplus})^{-1}$; $i = k+1, k+2, \dots, n$.

Operational matrices $\mathbf{P}^{(k)}$ and $\mathbf{V}^{(k)}$ play key roles and allow transformation $\tau 1$ to be carried out through all its consecutive steps taken in the natural order, so that the sequence of transformed matrices $\{\mathbf{A}^{(k)}\}$ achieves the sought-for triangular factorization of $\mathbf{A}$.

The univocal algebraic relations among matrix elements in equations (1.1) and (1.2) may be visualized with the help of the following matrix representations:

$$\mathbf{A}^{(k-1)} =$$

$$\begin{bmatrix} 1 & a_{12}^{(k-1)} & a_{13}^{(k-1)} & \cdot & a_{1k-1}^{(k-1)} & a_{1k}^{(k-1)} & a_{1k+1}^{(k-1)} & a_{1k+2}^{(k-1)} & a_{1k+3}^{(k-1)} & \cdot & a_{1n-1}^{(k-1)} & a_{1n}^{(k-1)} \\ 0 & 1 & a_{23}^{(k-1)} & \cdot & a_{2k-1}^{(k-1)} & a_{2k}^{(k-1)} & a_{2k+1}^{(k-1)} & a_{2k+2}^{(k-1)} & a_{2k+3}^{(k-1)} & \cdot & a_{2n-1}^{(k-1)} & a_{2n}^{(k-1)} \\ 0 & 0 & 1 & \cdot & a_{3k-1}^{(k-1)} & a_{3k}^{(k-1)} & a_{3k+1}^{(k-1)} & a_{3k+2}^{(k-1)} & a_{3k+3}^{(k-1)} & \cdot & a_{3n-1}^{(k-1)} & a_{3n}^{(k-1)} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdot & 1 & a_{k-1k}^{(k-1)} & a_{k-1k+1}^{(k-1)} & a_{k-1k+2}^{(k-1)} & a_{k-1k+3}^{(k-1)} & \cdot & a_{k-1n-1}^{(k-1)} & a_{k-1n}^{(k-1)} \\ 0 & 0 & 0 & \cdot & 0 & a_{kk}^{(k-1)} & a_{kk+1}^{(k-1)} & a_{kk+2}^{(k-1)} & a_{kk+3}^{(k-1)} & \cdot & a_{kn-1}^{(k-1)} & a_{kn}^{(k-1)} \\ 0 & 0 & 0 & \cdot & 0 & a_{k+1k}^{(k-1)} & a_{k+1k+1}^{(k-1)} & a_{k+1k+2}^{(k-1)} & a_{k+1k+3}^{(k-1)} & \cdot & a_{k+1n-1}^{(k-1)} & a_{k+1n}^{(k-1)} \\ 0 & 0 & 0 & \cdot & 0 & a_{k+2k}^{(k-1)} & a_{k+2k+1}^{(k-1)} & a_{k+2k+2}^{(k-1)} & a_{k+2k+3}^{(k-1)} & \cdot & a_{k+2n-1}^{(k-1)} & a_{k+2n}^{(k-1)} \\ 0 & 0 & 0 & \cdot & 0 & a_{k+3k}^{(k-1)} & a_{k+3k+1}^{(k-1)} & a_{k+3k+2}^{(k-1)} & a_{k+3k+3}^{(k-1)} & \cdot & a_{k+3n-1}^{(k-1)} & a_{k+3n}^{(k-1)} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdot & 0 & a_{n-1k}^{(k-1)} & a_{n-1k+1}^{(k-1)} & a_{n-1k+2}^{(k-1)} & a_{n-1k+3}^{(k-1)} & \cdot & a_{n-1n-1}^{(k-1)} & a_{n-1n}^{(k-1)} \\ 0 & 0 & 0 & \cdot & 0 & a_{nk}^{(k-1)} & a_{nk+1}^{(k-1)} & a_{nk+2}^{(k-1)} & a_{nk+3}^{(k-1)} & \cdot & a_{nn-1}^{(k-1)} & a_{nn}^{(k-1)} \end{bmatrix}$$

$$\mathbf{P}^{(k)} = \begin{bmatrix} 1 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 1 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 1 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdot & 1 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & \sigma\left(a_{kk}^{(k-1)}\right) & 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & \sigma\left(a_{kk+1}^{(k-1)}\right) & 1 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & \sigma\left(a_{kk+2}^{(k-1)}\right) & 0 & 1 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & \sigma\left(a_{kk+3}^{(k-1)}\right) & 0 & 0 & 1 & \cdot & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdot & 0 & \sigma\left(a_{kn-1}^{(k-1)}\right) & 0 & 0 & 0 & \cdot & 1 & 0 \\ 0 & 0 & 0 & \cdot & 0 & \sigma\left(a_{kn}^{(k-1)}\right) & 0 & 0 & 0 & \cdot & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}^{((k-1)\oplus)} =$$

$$\begin{bmatrix} 1 & a_{12}^{(k-1)} & a_{13}^{(k-1)} \cdot a_{1k-1}^{(k-1)} & \sum_{q=k}^n (a_{1q}^{(k-1)} p_{qk}^{(k)}) & a_{1k+1}^{(k-1)} & a_{1k+2}^{(k-1)} & a_{1k+3}^{(k-1)} & \cdot & a_{1n-1}^{(k-1)} & a_{1n}^{(k-1)} \\ 0 & 1 & a_{23}^{(k-1)} \cdot a_{2k-1}^{(k-1)} & \sum_{q=k}^n (a_{2q}^{(k-1)} p_{qk}^{(k)}) & a_{2k+1}^{(k-1)} & a_{2k+2}^{(k-1)} & a_{2k+3}^{(k-1)} & \cdot & a_{2n-1}^{(k-1)} & a_{2n}^{(k-1)} \\ 0 & 0 & 1 & \cdot a_{3k-1}^{(k-1)} \sum_{q=k}^n (a_{3q}^{(k-1)} p_{qk}^{(k)}) & a_{3k+1}^{(k-1)} & a_{3k+2}^{(k-1)} & a_{3k+3}^{(k-1)} & \cdot & a_{3n-1}^{(k-1)} & a_{3n}^{(k-1)} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdot 1 & \sum_{q=k}^n (a_{k-1q}^{(k-1)} p_{qk}^{(k)}) & a_{k-1k+1}^{(k-1)} & a_{k-1k+2}^{(k-1)} & a_{k-1k+3}^{(k-1)} & \cdot a_{k-1n-1}^{(k-1)} & a_{k-1n}^{(k-1)} \\ 0 & 0 & 0 & \cdot 0 & \sum_{q=k}^n |a_{kq}^{(k-1)}| & a_{kk+1}^{(k-1)} & a_{kk+2}^{(k-1)} & a_{kk+3}^{(k-1)} & \cdot a_{kn-1}^{(k-1)} & a_{kn}^{(k-1)} \\ 0 & 0 & 0 & \cdot 0 & \sum_{q=k}^n (a_{k+1q}^{(k-1)} p_{qk}^{(k)}) & a_{k+1k+1}^{(k-1)} & a_{k+1k+2}^{(k-1)} & a_{k+1k+3}^{(k-1)} & \cdot a_{k+1n-1}^{(k-1)} & a_{k+1n}^{(k-1)} \\ 0 & 0 & 0 & \cdot 0 & \sum_{q=k}^n (a_{k+2q}^{(k-1)} p_{qk}^{(k)}) & a_{k+2k+1}^{(k-1)} & a_{k+2k+2}^{(k-1)} & a_{k+2k+3}^{(k-1)} & \cdot a_{k+2n-1}^{(k-1)} & a_{k+2n}^{(k-1)} \\ 0 & 0 & 0 & \cdot 0 & \sum_{q=k}^n (a_{k+3q}^{(k-1)} p_{qk}^{(k)}) & a_{k+3k+1}^{(k-1)} & a_{k+3k+2}^{(k-1)} & a_{k+3k+3}^{(k-1)} & \cdot a_{k+3n-1}^{(k-1)} & a_{k+3n}^{(k-1)} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdot 0 & \sum_{q=k}^n (a_{n-1q}^{(k-1)} p_{qk}^{(k)}) & a_{n-1k+1}^{(k-1)} & a_{n-1k+2}^{(k-1)} & a_{n-1k+3}^{(k-1)} & \cdot a_{n-1n-1}^{(k-1)} & a_{n-1n}^{(k-1)} \\ 0 & 0 & 0 & \cdot 0 & \sum_{q=k}^n (a_{nq}^{(k-1)} p_{qk}^{(k)}) & a_{nk+1}^{(k-1)} & a_{nk+2}^{(k-1)} & a_{nk+3}^{(k-1)} & \cdot a_{nn-1}^{(k-1)} & a_{nn}^{(k-1)} \end{bmatrix}$$

$$\mathbf{V}^{(k)} = \begin{bmatrix} 1 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 1 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 1 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdot & 1 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & (a_{kk}^{(k-1)\oplus})^{-1} & 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & -a_{k+1k}^{(k-1)\oplus} \left(a_{kk}^{(k-1)\oplus} \right)^{-1} & 1 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & -a_{k+2k}^{(k-1)\oplus} \left(a_{kk}^{(k-1)\oplus} \right)^{-1} & 0 & 1 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & -a_{k+3k}^{(k-1)\oplus} \left(a_{kk}^{(k-1)\oplus} \right)^{-1} & 0 & 0 & 1 & \cdot & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdot & 0 & -a_{n-1k}^{(k-1)\oplus} \left(a_{kk}^{(k-1)\oplus} \right)^{-1} & 0 & 0 & 0 & \cdot & 1 & 0 \\ 0 & 0 & 0 & \cdot & 0 & -a_{nk}^{(k-1)\oplus} \left(a_{kk}^{(k-1)\oplus} \right)^{-1} & 0 & 0 & 0 & \cdot & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}^{((k))} = \begin{bmatrix} 1 & a_{12}^{(k-1)\oplus} & a_{13}^{(k-1)\oplus} & \cdot & a_{1k-1}^{(k-1)\oplus} & a_{1k}^{(k-1)\oplus} & a_{1k+1}^{(k-1)\oplus} & a_{1k+2}^{(k-1)\oplus} & a_{1k+3}^{(k-1)\oplus} & 0 & \cdot & a_{1n-1}^{(k-1)\oplus} & a_{1n}^{(k-1)\oplus} \\ 0 & 1 & a_{23}^{(k-1)\oplus} & \cdot & a_{2k-1}^{(k-1)\oplus} & a_{2k}^{(k-1)\oplus} & a_{2k+1}^{(k-1)\oplus} & a_{2k+2}^{(k-1)\oplus} & a_{2k+3}^{(k-1)\oplus} & \cdot & a_{2n-1}^{(k-1)\oplus} & a_{2n}^{(k-1)\oplus} \\ 0 & 0 & 1 & \cdot & a_{3k-1}^{(k-1)\oplus} & a_{3k}^{(k-1)\oplus} & a_{3k+1}^{(k-1)\oplus} & a_{3k+2}^{(k-1)\oplus} & a_{3k+3}^{(k-1)\oplus} & \cdot & a_{3n-1}^{(k-1)\oplus} & a_{3n}^{(k-1)\oplus} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdot & 1 & a_{k-1k}^{(k-1)\oplus} & a_{k-1k+1}^{(k-1)\oplus} & a_{k-1k+2}^{(k-1)\oplus} & a_{k-1k+3}^{(k-1)\oplus} & \cdot & a_{k-1n-1}^{(k-1)\oplus} & a_{k-1n}^{(k-1)\oplus} \\ 0 & 0 & 0 & \cdot & 0 & 1 & a_{kk+1}^{(k)} & a_{kk+2}^{(k)} & a_{kk+3}^{(k)} & \cdot & a_{kn-1}^{(k)} & a_{kn}^{(k)} \\ 0 & 0 & 0 & \cdot & 0 & 0 & a_{k+1k+1}^{(k)} & a_{k+1k+2}^{(k)} & a_{k+1k+3}^{(k)} & \cdot & a_{k+1n-1}^{(k)} & a_{k+1n}^{(k)} \\ 0 & 0 & 0 & \cdot & 0 & 0 & a_{k+2k+1}^{(k)} & a_{k+2k+2}^{(k)} & a_{k+2k+3}^{(k)} & \cdot & a_{k+2n-1}^{(k)} & a_{k+2n}^{(k)} \\ 0 & 0 & 0 & \cdot & 0 & 0 & a_{k+3k+1}^{(k)} & a_{k+3k+2}^{(k)} & a_{k+3k+3}^{(k)} & \cdot & a_{k+3n-1}^{(k)} & a_{k+3n}^{(k)} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdot & 0 & 0 & a_{n-1k+1}^{(k)} & a_{n-1k+2}^{(k)} & a_{n-1k+3}^{(k)} & \cdot & a_{n-1n-1}^{(k)} & a_{n-1n}^{(k)} \\ 0 & 0 & 0 & \cdot & 0 & 0 & a_{nk+1}^{(k)} & a_{nk+2}^{(k)} & a_{nk+3}^{(k)} & \cdot & a_{nn-1}^{(k)} & a_{nn}^{(k)} \end{bmatrix}$$

In the representation of $\mathbf{A}^{(k)}$, transformed elements $a_{ij}^{(k)}$ are given by the positions:

$$\begin{aligned} a_{kj}^{(k)} &= v_{kk}^{(k)} a_{kj}^{(k-1)\oplus}; & j &= k+1, k+2, \dots, n. \\ a_{ij}^{(k)} &= a_{ij}^{(k-1)\oplus} + v_{ik}^{(k)} a_{kj}^{(k-1)\oplus}; & i, j &= k+1, k+2, \dots, n. \end{aligned}$$

It is important to notice that in the representation of $\mathbf{A}^{(k-1)\oplus}$, the key pivotal element $a_{kk}^{(k-1)\oplus}$ is non-zero since it is given by the expression:

$$a_{kk}^{(k-1)\oplus} = \sum_{q=k}^n |a_{kq}^{(k-1)}|.$$

At the end of the n -th step, the transformation of matrix $\mathbf{A}$ is expressed by the following equivalences:

$$(2.0) \quad \mathbf{VAP} = \mathbf{T};$$

$$(2.1) \quad \mathbf{V} = \prod_{q=1}^n \mathbf{V}^{(n+1-q)};$$

$$(2.2) \quad \mathbf{P} = \prod_{q=1}^n \mathbf{P}^{(q)};$$

$$(2.3) \quad \mathbf{T} = \mathbf{A}^{(n)};$$

so that the sought-for inverse $\mathbf{A}^{-1}$ may be represented by the formula:

$$(3.0) \quad \mathbf{A}^{-1} = \mathbf{PT}^{-1}\mathbf{V}.$$

Transformation τ_2

Matrix $\mathbf{T}$ is transformed into the unit matrix $\mathbf{U}^{(n)}$ through $(n - 1)$ consecutive ordered steps in such a way that at the k -th step, $k = 1, 2, \dots, n - 1$, it is transformed into an upper triangular matrix $\mathbf{T}^{(k)}$ that admits the following four-block representation:

$$(4.0) \quad \mathbf{T}^{(k)} = \begin{bmatrix} \mathbf{U}^{(k)} & \mathbf{D}^{(k)} \\ \mathbf{Q}^{(k)} & \mathbf{H}^{(k)} \end{bmatrix};$$

where $\mathbf{D}^{(k)}$ is a $(k \times (n - k))$ null matrix; $\mathbf{Q}^{(k)}$ is an $((n - k) \times k)$ null matrix; and $\mathbf{H}^{(k)}$ is a square upper triangular sub-matrix of order $(n - k)$ with all its diagonal elements $h_{ii}^{(k)} = 1$.

At the k -th step, $k = 1, 2, \dots, n - 1$, the following transformation applies:

$$(4.1) \quad \mathbf{T}^{(k-1)} \mathbf{G}^{(k)} = \mathbf{T}^{(k)};$$

where:

- $\mathbf{T}^{(0)} = \mathbf{T}$, so that elements $t_{ij}^{(0)} = t_{ij}$;
- $\mathbf{G}^{(k)}$ is a non-singular $(n \times n)$ upper triangular operational matrix which differs (or may differ) from $\mathbf{U}^{(n)}$ only in the k -th row for the elements $g_{kj}^{(k)}$, with $j > k$, given by the expressions: $g_{kj}^{(k)} = -t_{kj}^{(k-1)}$, $j = k + 1, k + 2, \dots, n$.

The following matrix representations help in visualizing the operations required by (4.1)

$$\mathbf{T}^{(k-1)} = \begin{bmatrix} 1 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 1 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 1 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdot & 1 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 1 & t_{kk+1}^{(k-1)} & t_{kk+2}^{(k-1)} & t_{kk+3}^{(k-1)} & \cdot & t_{kn-1}^{(k-1)} & t_{kn}^{(k-1)} \\ 0 & 0 & 0 & \cdot & 0 & 0 & 1 & t_{k+1k+2}^{(k-1)} & t_{k+1k+3}^{(k-1)} & \cdot & t_{k+1n-1}^{(k-1)} & t_{k+1n}^{(k-1)} \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 1 & t_{k+2k+3}^{(k-1)} & \cdot & t_{k+2n-1}^{(k-1)} & t_{k+2n}^{(k-1)} \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 1 & \cdot & t_{k+3n-1}^{(k-1)} & t_{k+3n}^{(k-1)} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 1 & t_{n-1n}^{(k-1)} \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 1 \end{bmatrix}$$

$$\mathbf{G}^{(k)} = \begin{bmatrix} 1 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 1 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 1 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ \hline 0 & 0 & 0 & \cdot & 1 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 1 & -t_{kk+1}^{(k-1)} & -t_{kk+2}^{(k-1)} & -t_{kk+3}^{(k-1)} & \cdot & -t_{kn-1}^{(k-1)} & -t_{kn}^{(k-1)} \\ 0 & 0 & 0 & \cdot & 0 & 0 & 1 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 1 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 1 & \cdot & 0 & 0 \\ \hline 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 1 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}^{(k)} = \begin{bmatrix} 1 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 1 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 1 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ \hline 0 & 0 & 0 & \cdot & 1 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 1 & 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 1 & t_{k+1k+2}^{(k-1)} & t_{k+1k+3}^{(k-1)} & \cdot & t_{k+1n-1}^{(k-1)} & t_{k+1n}^{(k-1)} \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 1 & t_{k+2k+3}^{(k-1)} & \cdot & t_{k+2n-1}^{(k-1)} & t_{k+2n}^{(k-1)} \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 1 & \cdot & t_{k+3n-1}^{(k-1)} & t_{k+3n}^{(k-1)} \\ \hline 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 1 & t_{n-1n}^{(k-1)} \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 1 \end{bmatrix}$$

At the end of the $(n-1)$ -th step, the transformation of matrix $\mathbf{T}$ is expressed by the following equivalences:

$$(5.0) \quad \mathbf{T}\mathbf{G} = \mathbf{U}^{(n)};$$

$$(5.1) \quad \mathbf{G} = \prod_{q=1}^{n-1} \mathbf{G}^{(q)}.$$

Since $\mathbf{G}$ is, according to (5.0), the inverse of $\mathbf{T}$, expression (3.0) for $\mathbf{A}^{-1}$ is rewritten as follows:

$$(6.0) \quad \mathbf{A}^{-1} = \mathbf{P}\mathbf{G}\mathbf{V}.$$

A numerical example

The practical application of the present method for matrix inversion is illustrated through an elementary example for which, it is to be noticed, the simplest Gaussian elimination procedure cannot be carried out in the natural order.

It is required to find the inverse $\mathbf{A}^{-1}$ of the matrix:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & -1 \end{bmatrix}$$

Transformation $\tau 1$ **Step 1:**

$$\mathbf{A}^{(0)}\mathbf{P}^{(1)} = \mathbf{A}^{(0)\oplus}; \quad \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 3 & 1 & -1 & -2 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & -1 \end{bmatrix}$$

$$\mathbf{V}^{(1)}\mathbf{A}^{(0)\oplus} = \mathbf{A}^{(1)}; \quad \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \frac{-3}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 & 0 \\ 3 & 1 & -1 & -2 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{-1}{2} & 0 \\ 0 & \frac{-1}{2} & \frac{1}{2} & -2 \\ 0 & 1 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix}$$

Step 2:

$$\mathbf{A}^{(1)}\mathbf{P}^{(2)} = \mathbf{A}^{(1)\oplus}; \quad \begin{bmatrix} 1 & \frac{1}{2} & \frac{-1}{2} & 0 \\ 0 & \frac{-1}{2} & \frac{1}{2} & -2 \\ 0 & 1 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & \frac{-1}{2} & 0 \\ 0 & 3 & \frac{1}{2} & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -1 \end{bmatrix}$$

$$\mathbf{V}^{(2)}\mathbf{A}^{(1)\oplus} = \mathbf{A}^{(2)}; \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{-1}{3} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & \frac{-1}{2} & 0 \\ 0 & 3 & \frac{1}{2} & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{1}{6} & \frac{-2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{-1}{3} \end{bmatrix}$$

Step 3:

$$\mathbf{A}^{(2)}\mathbf{P}^{(3)} = \mathbf{A}^{(2)\oplus}; \quad \begin{bmatrix} 1 & -1 & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{1}{6} & \frac{-2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{-1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{5}{6} & \frac{-2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{2}{3} & \frac{-1}{3} \end{bmatrix}$$

$$\mathbf{V}^{(3)}\mathbf{A}^{(2)\oplus} = \mathbf{A}^{(3)}; \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{-2}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{5}{6} & \frac{-2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{2}{3} & \frac{-1}{3} \end{bmatrix} = \begin{bmatrix} 1 & -1 & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{5}{6} & \frac{-2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{-1}{3} \end{bmatrix}$$

Step 4:

$$\mathbf{A}^{(3)}\mathbf{P}^{(4)} = \mathbf{A}^{(3)\oplus}; \quad \begin{bmatrix} 1 & -1 & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{5}{6} & \frac{-2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{-1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{5}{6} & \frac{2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$\mathbf{V}^{(4)}\mathbf{A}^{(3)\oplus} = \mathbf{A}^{(4)} = \mathbf{T}; \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{5}{6} & \frac{2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & -1 & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{5}{6} & \frac{2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformation τ_2

Step 1:

$$\mathbf{T}^{(0)}\mathbf{G}^{(1)} = \mathbf{T}^{(1)}; \quad \begin{bmatrix} 1 & -1 & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{5}{6} & \frac{2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{5}{6} & \frac{2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 2:

$$\mathbf{T}^{(1)}\mathbf{G}^{(2)} = \mathbf{T}^{(2)}; \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{5}{6} & \frac{2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{-5}{6} & \frac{-2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 3:

$$\mathbf{T}^{(2)}\mathbf{G}^{(3)} = \mathbf{T}^{(3)} = \mathbf{U}^{(4)};$$

Since $\mathbf{T}^{(2)} = \mathbf{U}^{(4)}$, also $\mathbf{G}^{(3)} = \mathbf{U}^{(4)}$ and $\mathbf{T}^{(3)} = \mathbf{U}^{(4)}$.

$$\mathbf{A}^{-1} = \mathbf{P}\mathbf{G}\mathbf{V}.$$

$$\mathbf{P} = \prod_{q=1}^4 \mathbf{P}^{(q)}; \quad \mathbf{G} = \prod_{q=1}^3 \mathbf{G}^{(q)}; \quad \mathbf{V} = \prod_{q=1}^4 \mathbf{V}^{(5-q)}.$$

$$\begin{aligned} \mathbf{P} &= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 \end{bmatrix} \end{aligned}$$

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{-5}{6} & \frac{-2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & \frac{-1}{3} & \frac{-2}{3} \\ 0 & 1 & \frac{-5}{6} & \frac{-2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \mathbf{V} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{-2}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{-1}{3} & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \frac{-3}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \frac{-1}{2} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & -1 & -2 & 3 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
\mathbf{A}^{-1} = \mathbf{P}\mathbf{G}\mathbf{V} &= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & \frac{-1}{3} & \frac{-2}{3} \\ 0 & 1 & \frac{-5}{6} & \frac{-2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \frac{-1}{2} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & -1 & -2 & 3 \end{bmatrix} \\
&= \begin{bmatrix} -1 & -1 & \frac{1}{3} & \frac{2}{3} \\ 1 & 0 & \frac{1}{2} & 0 \\ -1 & 0 & \frac{1}{2} & 0 \\ -1 & -2 & \frac{1}{6} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \frac{-1}{2} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & -1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 & 2 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{-1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{3}{2} & -1 & \frac{-1}{2} & 1 \end{bmatrix}
\end{aligned}$$

Complex elements

If the elements of matrix $\mathbf{A}$ are complex numbers: $a_{ij} = b_{ij} + \mathbf{i}c_{ij}$, $\mathbf{i}$ being the imaginary unit such that $\mathbf{i}^2 = -1$, in transformation $\tau 1$ matrices $\mathbf{A}^{(k-1)}$ (with $\mathbf{A}^{(0)} = \mathbf{A}$), $k = 1, 2, \dots, n$, have complex elements $a_{ij}^{(k-1)} = b_{ij}^{(k-1)} + \mathbf{i}c_{ij}^{(k-1)}$, so that the specification of operational matrix $\mathbf{P}^{(k)}$ must be such as to achieve a suitable non-zero leading pivot $a_{kk}^{(k-1)\oplus}$.

Specification of $\mathbf{P}^{(k)}$, $k = 1, 2, \dots, n$

$\mathbf{P}^{(k)}$ is a non singular $(n \times n)$ lower triangular matrix which differs (or may differ) from $\mathbf{U}^{(n)}$ only in the diagonal and lower elements of the k -th column. These elements are:

$$p_{ik}^{(k)} = \sigma(b_{ki}^{(k-1)}) - \mathbf{i}\sigma(c_{ki}^{(k-1)}); \quad i = k, k+1, \dots, n;$$

Specification of $a_{kk}^{(k-1)\oplus}$, $k = 1, 2, \dots, n$

$$a_{kk}^{(k-1)\oplus} = \sum_{q=k}^n \left(\left| b_{kq}^{(k-1)} \right| + \left| c_{kq}^{(k-1)} \right| \right) + \mathbf{i} \sum_{q=k}^n \left(\sigma \left(b_{kq}^{(k-1)} \right) c_{kq}^{(k-1)} - \sigma \left(c_{kq}^{(k-1)} \right) b_{kq}^{(k-1)} \right)$$

Inversion by partitioning

A given square $(n \times n)$ matrix $\mathbf{A}$ may be partitioned into four blocks $\mathbf{A}_q$, $q = 1, 2, 3, 4$, in view of operating with matrices of reduced dimensions [3]:

$$(7.0) \quad \mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix},$$

where $\mathbf{A}_1$ is a square sub-matrix of order s , $s = 1, 2, \dots, n-1$; $\mathbf{A}_2$ is an $(s \times (n-s))$ sub-matrix; $\mathbf{A}_3$ is an $((n-s) \times s)$ sub-matrix; and $\mathbf{A}_4$ is a square sub-matrix of order $n-s$.

If $\mathbf{A}$ is non-singular, the possibility of inverting it by operating on its blocks is, however, not granted since it is necessary that all required operations be feasible. In this connection, it is to be noticed that the present $\tau 1$ transformations provide feasibility.

In the four-block representation of the matrix $\mathbf{A}^{(k)}$:

$$(1.0) \quad \mathbf{A}^{(k)} = \begin{bmatrix} \mathbf{Z}^{(k)} & \mathbf{W}^{(k)} \\ \mathbf{R}^{(k)} & \mathbf{S}^{(k)} \end{bmatrix},$$

the matrix $\mathbf{Z}^{(k)}$ is non-singular so that $\mathbf{A}^{(k)}$ can be easily inverted once the inverses $(\mathbf{Z}^{(k)})^{-1}$ and $(\mathbf{S}^{(k)})^{-1}$ have been computed. Inverse $(\mathbf{A}^{(k)})^{-1}$ is provided by the following expression:

$$(8.0) \quad (\mathbf{A}^{(k)})^{-1} = \begin{bmatrix} (\mathbf{Z}^{(k)})^{-1} & -(\mathbf{Z}^{(k)})^{-1}\mathbf{W}^{(k)}(\mathbf{S}^{(k)})^{-1} \\ \mathbf{R}^{(k)} & (\mathbf{S}^{(k)})^{-1} \end{bmatrix}$$

In view of the relationship between $\mathbf{A}$ and $\mathbf{A}^{(k)}$:

$$(8.1) \quad \mathbf{V}^{\{k\}} \mathbf{A} \mathbf{P}^{[k]} = \mathbf{A}^{(k)};$$

where

$$(8.2) \quad \mathbf{V}^{\{k\}} = \prod_{q=1}^k \mathbf{V}^{(k+1-q)};$$

$$(8.3) \quad \mathbf{P}^{[k]} = \prod_{q=1}^k \mathbf{P}^{(q)};$$

the required inverse $\mathbf{A}^{-1}$ is represented by the expression:

$$(9.0) \quad \mathbf{A}^{-1} = \mathbf{P}^{[k]} (\mathbf{A}^{(k)})^{-1} \mathbf{V}^{\{k\}}.$$

Appendix 1

Operational counts

Logical simplicity and number of required arithmetic operations are important features of numerical methods.

If operations of multiplication and division are considered equivalent and operations of subtraction and addition are neglected, for the present matrix inversion method the number of required multiplications/divisions turns out to be n^3 , i.e., the best result available [1].

The overall number of operations, N_0 , required for achieving $\mathbf{A}^{-1}$, is expressed by the sum:

$$(10.0) \quad N_0 = N_{\tau 1} + N_{\tau 2} + N_{\nu} + N_p + N_g + N_{gv} + N_{p(gv)};$$

where $N_{\tau 1}$, $N_{\tau 2}$, N_ν , N_p , N_g , N_{gv} and $N_{p(gv)}$ represent, respectively, the maximal numbers of operations required for carrying out transformations $\tau 1$ and $\tau 2$; for computing $\mathbf{V}$ according to (2.1); $\mathbf{P}$ according to (2.2); $\mathbf{G}$ according to (5.1); $\mathbf{GV}$; and $\mathbf{P}(\mathbf{GV})$.

Evaluation of $N_{\tau 1}$

$$(10.1) \quad N_{\tau 1} = \sum_{k=1}^n N_{1k},$$

where N_{1k} represents the maximal number of operations required for carrying out the k -th step in transformation $\tau 1$.

Evaluation of N_{1k} , $k = 1, 2, \dots, n$

- Operations for $\mathbf{P}^{(k)} : 0$;
- Operations for $\mathbf{A}^{(k-1)}\mathbf{P}^{(k)} : 0$;
- Operations for $\mathbf{V}^{(k)} : n - k + 1$;
- Operations for $\mathbf{V}^{(k)}\mathbf{A}^{(k-1)\oplus} : (n - k + 1)(n - k)$;

$$N_{1k} = (n + 1)^2 - 2(n + 1)k + k^2.$$

$N_{\tau 1}$ is evaluated by taking into account the formulae:

$$\begin{aligned} \sum_{k=1}^n k &= 2^{-1}n(n + 1); & \sum_{k=1}^n k^2 &= 6^{-1}n(n + 1)(2n + 1). \\ N_{\tau 1} &= \sum_{k=1}^n N_{1k} = \sum_{k=1}^n ((n + 1)^2 - 2(n + 1)k + k^2) \\ &= n(n + 1)^2 - n(n + 1)^2 + 3^{-1}n^3 + 2^{-1}n^2 + 6^{-1}n \\ &= 3^{-1}n^3 + 2^{-1}n^2 + 6^{-1}n. \end{aligned}$$

Evaluation of $N_{\tau 2}$

$$(10.2) \quad N_{\tau 2} = \sum_{k=1}^{n-1} N_{2k},$$

where N_{2k} represents the maximal number of operations required for carrying out the k -th step in transformation $\tau 2$.

In view of the specifications of the elements of upper triangular matrices $\mathbf{T}^{(k-1)}$ and $\mathbf{G}^{(k)}$ in (4.1), one finds: $N_{2k} = 0$, and consequently $N_{\tau 2} = 0$.

Evaluation of N_v

By introducing the matrix $\mathbf{V}^{[s]}$, $s = 1, 2, \dots, n-1$, through the position:

$$(10.3) \quad \mathbf{V}^{[s]} = \prod_{q=1}^s \mathbf{V}^{(n+1-q)},$$

N_v may be expressed by the sum:

$$(10.4) \quad N_v = \sum_{s=1}^{n-1} N_{vs},$$

where N_{vs} represents the maximal number of operations for $\mathbf{V}^{[s]}\mathbf{V}^{(n-s)}$.

Evaluation of N_{vs} , $s = 1, 2, \dots, n-1$.

$$N_{vs} = \sum_{q=1}^s q = 2^{-1}s(s+1).$$

The value of N_v is found to be:

$$\begin{aligned} N_v &= \sum_{s=1}^{n-1} 2^{-1}s(s+1) = 4^{-1}(n-1)n + 6^{-1}n^3 - 4^{-1}n^2 + 12^{-1}n \\ &= 6^{-1}n^3 - 6^{-1}n. \end{aligned}$$

Evaluation of N_p

By introducing matrix $\mathbf{P}^{[s]}$, $s = 1, 2, \dots, n-1$, through the position:

$$(10.5) \quad \mathbf{P}^{[s]} = \prod_{q=1}^s \mathbf{P}^{(q)}.$$

N_p may be expressed by the sum:

$$(10.6) \quad N_p = \sum_{s=1}^{n-1} N_{ps},$$

where N_{ps} represents the maximal number of operations for $\mathbf{P}^{[s]}\mathbf{P}^{(s+1)}$.

In view of the specifications of the elements of lower triangular matrices $\mathbf{P}^{(k)}$ in (1.1), one finds: $N_{ps} = 0$, and consequently $N_p = 0$.

Evaluation of N_g

By introducing the matrix $\mathbf{G}^{[s]}$, $s = 1, 2, \dots, n-2$, through the position:

$$(10.7) \quad \mathbf{G}^{[s]} = \prod_{q=1}^s G^{(q)},$$

N_g may be expressed by the sum:

$$(10.8) \quad N_g = \sum_{s=1}^{n-2} N_{gs},$$

where N_{gs} represents the maximal number of operations for $\mathbf{G}^{[s]} \mathbf{G}^{(s+1)}$.

Evaluation of N_{gs} , $s = 1, 2, \dots, n-2$

$$N_{gs} = s(n-s-1)$$

The value of N_g is found to be:

$$\begin{aligned} N_g &= \sum_{s=1}^{n-2} s(n-s-1) = 2^{-1}(n-2)(n-1)^2 - 6^{-1}(n-2)(n-1)(2n-4+1) \\ &= 2^{-1}(n^3 - 4n^2 + 5n - 2) - 6^{-1}(2n^3 - 9n^2 + 13n - 6) \\ &= 6^{-1}n^3 - 2^{-1}n^2 + 3^{-1}n. \end{aligned}$$

Evaluation of N_{gv}

N_{gv} is expressed by the sum:

$$(10.9) \quad N_{gv} = \sum_{s=1}^{n-1} N_{gvs},$$

where N_{gvs} represents the maximal number of operations required for multiplying the s -th row, $s = 1, 2, \dots, n-1$, of $\mathbf{G}$ with $\mathbf{V}$.

$$\begin{aligned} N_{gvs} &= s(n-s) + \sum_{q=1}^{n-s} q = s(n-s) + 2^{-1}(n-s)(n-s+1) \\ &= ns - s^2 + 2^{-1}(n^2 - 2ns + n + s^2 - s) = 2^{-1}n^2 + 2^{-1}n - 2^{-1}s^2 - 2^{-1}s. \end{aligned}$$

The value of N_{gv} is found to be:

$$\begin{aligned} N_{gv} &= 2^{-1}n^2(n-1) + 2^{-1}n((n-1) - 2^{-1}(3^{-1}n^3 - 2^{-1}n^2 + 6^{-1}n) - 2^{-2}(n-1)n) \\ &= 3^{-1}n^3 - 3^{-1}n. \end{aligned}$$

Evaluation of $N_{p(gv)}$

Since the lower triangular matrix $\mathbf{P}$ has elements having absolute values either 0 or 1, the evaluation of $N_{p(gv)}$ is immediate:

$$N_{p(gv)} = 0.$$

Evaluation of N_0

The value of N_0 , as defined by (10.0), is found to be:

$$N_0 = 3^{-1}n^3 + 2^{-1}n^2 + 6^{-1}n + 6^{-1}n^3 - 6^{-1}n + 6^{-1}n^3 - 2^{-1}n^2 + 3^{-1}n + 3^{-1}n^3 - 3^{-1}n = n^3.$$

Acknowledgements. I dedicate this work to the memory of my father, Odino Reali, who could see me through a university education in difficult times. At the international colloquium on applications of mathematics in memoriam Lothar Collatz held in Hamburg, 4 and 5 July 1997, and for which no proceedings were published, I suggested the possibility of inverting $\mathbf{A}$ by representing it as a sum $\mathbf{B} + \mathbf{C}$ ($\mathbf{B}$ and $\mathbf{C}$ being $(n \times n)$ non-singular matrices respectively lower and upper triangular), and by applying Scarborough's inversion method [4] to the auxiliary $(2n \times 2n)$ matrix

$$\begin{bmatrix} \mathbf{B} & -\mathbf{U}^{(n)} \\ \mathbf{C} & \mathbf{U}^{(n)} \end{bmatrix}$$

having the inverse

$$\begin{bmatrix} \mathbf{A}^{-1} & \mathbf{A}^{-1} \\ -\mathbf{CA}^{-1} & \mathbf{BA}^{-1} \end{bmatrix}.$$

The diagonal elements b_{ii} of $\mathbf{B}$ played the role of free parameters to be suitably specified in view of avoiding the occurrence of null diagonal elements. A specific application for supporting a block solution of a linear algebraic system was envisaged.

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FAIL-STOP DESIGNATED RECIPIENT SIGNATURE SCHEME BASED ON ELLIPTIC CURVE DISCRETE LOGARITHM PROBLEM

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Abstract. This paper describes a new digital signature called fail-stop designated recipient signature scheme based on elliptic curve discrete logarithm (ECDLP). The scheme allows a signer and an intended recipient to co-operatively provide a proof of forgery if an attacker can successfully forge a signature on a message m . The scheme also provides that the intended recipient is the only entity to verify the resulting signature and capable to prove the validity of signature to any third party. With this property, we show that our new signature scheme is the best alternative to solve certain problems concerning the protection of confidential documents especially those which are personally sensitive (to the owner) and is also applicable for group of recipients to verify a signature jointly (shared verification) in a group oriented environment. The scheme utilizes the inherent advantage of elliptic curve cryptosystem in terms of smaller key size and lower computational overhead to its counterpart public cryptosystems such as RSA and AlGamal. Compared with the Ismail and Yahya scheme [1], we show that our scheme is more efficient in terms of both computation and communication complexity.

Keywords: Cryptography, Digital signature, Elliptic curve discrete logarithm problem, Fail-stop designated recipient signature and zero knowledge Protocol.

1. Introduction

In all classical digital signatures if a forger successfully obtains a forge signature that pass the verification procedure then the scheme is completely insecure and the signer would be blamed. This is the main disadvantage of the digital signature scheme. However, fail-stop signature completely avoids this type of attack. If an

attacker successfully forger has happened. This concept was first introduced by Waidner and Pfitzmann [13] and then was formally defined in Pfitzmann and Waidner (1990). Theoretically, fail-stop signature are known to exist if claw-free pairs of permutations (not necessarily with trapdoor) exist, see [2] and [19] for description and Pfitzmann and Waidner [13] for proof. For complete definition of claw-free permutations, see Goldwasser [3]. In particular, [2] and [5] show that fail-stop signature only exist if factoring large integer or discrete logarithm problem is hard. The first construction of fail-stop signature [13] uses a one-time signature scheme and requires message to be signed bit although the tree-authentication [12] is used. This general construction is not efficient. There is an efficient construction for fail-stop signature that can be used to protect clients unconditionally secure in online payment system [19]. However in this scheme all signatures by one client must have the same recipient like the bank in a payment system. Furthermore, signing is a 3-round protocol between the signer and the recipient. Pfitzmann [18] also presented an efficient scheme with single recipient. First fail-stop signature scheme based on factorization and discrete logarithm problems have been proposed respectively in Susilo et. al [23] and Heyst and Pedersen [4]. The former uses modulus factor as a proof of forgery whereas the latter uses a private key of the system as a proof of forgery. In Pedersen and Pfitzmann [16], a formal definition of fail-stop signature is given and a general construction using bundling homomorphism is proposed. See also Pfitzmann [17]. To provide a proof of forgery, they show that under bundling homomorphism, two signatures (respectively generated by signer and forger) are shown to collide. In 2007, Ismail and Yahya[1] fail-stop recipient designated signature scheme based on RSA have been proposed. The elliptic curve cryptosystems ECC was initiated by Koblitz [6] and Miller [14], where the security was established on the discrete logarithm problem over the points on an elliptic curve, called ECDLP. The basic operations are the execution of integer points on the elliptic curve over finite fields, including addition and multiplication. The operations associated with the ECC are more efficient than those associated with other cryptosystems, like the RSA and the DSA security solutions. Owing to the fact that the ECC has a smaller key size and faster computation, for this reason, in this paper we had the motivation to design a new type from fail-stop designated recipient signature based on ECDLP.

The paper is organized as follows.

In the next section, we present the basic concept and definition of fail-stop designated recipient signature (FDRS) and introduce the notations that are used throughout the rest of the paper.

In Section 3, we propose our new scheme.

In Sections 4 and 5, we analyze security and efficiency performance of the proposed scheme.

Next, we present a small example of our scheme and this can be found in Section 6.

Finally, Section 7 concludes the paper.

2. Preliminaries

2.1. The model of fail-stop designated recipient signature

FDRS consists of five algorithms. The algorithms are as in original fail-stop signature, but we put some modifications to suit our desired applications.

Definition 2.1 A fail-stop designated recipient signature scheme consists of five algorithms (GEN, SIGN, VER, PVER, PTEST).

- GEN, SIGN and VER are respectively for generating keys, signing messages and verifying signatures.
- PVER for recipient to prove validation of signature to a third party without revealing any secret information
- PTEST for both signer and recipient to provide a proof of forgery if an adversary successfully forges a signature.

A secure FDRS must satisfy the following properties:

1. If the signer signs a message, then the recipient can verify the signature and accept it as genuine
2. A polynomial bounded (limited) forger cannot create forged signatures that successfully pass the verification test.
3. When a forger with unlimited computational power succeeds in forging a signature that passes the VER, the signer and recipient are able to construct a proof of forgery and convince an interested third party that a forgery has occurred.

A polynomial bounded (limited) signer or recipient cannot create a signature that he can later prove to be a forgery.

To achieve the above properties, for each public key, there must exist many matching secret keys. In other words, there must exist secret keys, SK, a set consisting of secret keys and each of which fits with the signer's public key, PK. Different secret keys from SK must create different signatures on the same message, m . The real signer knows only one of the secret keys and thus can only construct one possible signature on a given message. An enemy with unlimited computational power knows all the secret keys in SK, thus can generate all the signatures but he is unable to determine which one will be used by the signer. If the enemy presents a forged signature and claiming that the signature is signed by a signer, the signer then is able to provide a proof of forgery by generating his own signature on the message and use this signature with the one presented by enemy to break the underlying assumption of the scheme.

To show security of FDRS, it suffices to prove the following properties:

- There exists a probabilistic polynomial time algorithm PVER that takes a pair of secret and public key, a message and a forged signature for that message, and outputs a proof of forgery.
- An enemy with unlimited computing power, who knows the public key of the signer and his or her signature on a message, cannot find the secret key of the signer. Thus, he or she would not be able to construct signer's signature on a new message
- A polynomially bounded signer cannot construct valid signature on a message, and later prove that it is a forgery.

Generally, FDRS can be categorized into two ways: (1) FDRS with a trusted dealer (TD) who is trusted by all other entities and the one who is responsible in initializing the system and (2) FDRS without TD, and in this case the role of TD is given to a recipient. Our proposed scheme is based on the former.

2.2. Elliptic curve

Elliptic curve cryptosystem have attracted much attention in recent years because of the relatively small size of keys they require. An elliptic over $GF(p)$ is the set of points $(x, y) \in GF(p)$ satisfying the equation

$$y^2 = x^3 + ax + b$$

together with a special element denoted $\mathcal{O}$ and called the point at infinity. Addition operation on the points of an elliptic curve can be defined that makes it into an abelian group.

For more precise definition operation, we refer the reader to [11].

Let $E_p(a, b)$ denotes an elliptic curve of the form

$$y^2 = (x^3 + ax + b)(\text{mod } p),$$

where p is a prime number, $x^3 + ax + b(\text{mod } p)$ does not have multiple roots, and

$$4a^3 + 27b^2 \neq 0(\text{mod } p)$$

$|E_p(a, b)|$ denotes the order of $E_p(a, b)$ and can be calculated in polynomial time using algorithm such as Schoof's algorithm [8, 20, 24], and combination of Schoof's algorithm with Shank's baby-step giant-step algorithm [11].

Definition 2.2 An elliptic curve discrete logarithm problem (ECDL) is defined as follows: Let $\alpha \in E_p(a, b)$ be point of order q , and let $\beta = d\alpha$. Given α and β , determine the unique integer d , where $0 < d \leq q$

3. The proposed scheme

We now present our fail-stop designated recipient signature scheme based on the elliptic curve discrete logarithm problems.

3.1. The system setup

The fail-stop designated recipient signature scheme requires TD to choose and generate the system parameters for the scheme. TD is implemented according to the following steps:

- TD chooses an elliptic curve such that $q = |E_p(a, b)|$ is also prime.
- TD randomly chooses a point $\alpha \in E_p(a, b)$ and a number $d \in GF(q)$.
- calculates $\beta = d\alpha$ over and discards d .

Finally he publishes $(E_p(a, b), q, \alpha, \beta)$.

A simple algorithm to find such a curve is to randomly select a curve, find its order, discard the curve if the order is non-prime and repeat the process. It is conjectured that this type of curve can be obtained in $\mathcal{O}(1/\log p)$ [7].

The signer then identifies the intended recipient who wishes to accept a resulting signature. The signer and intended recipient then agree on a common secret key $\lambda \in GF(q)$. Next the signer passes securely the integer β to the intended recipient who then does the following:

- Selects at random private key $v \in GF(q)$ and computes and securely sends $\gamma = v\beta$ over $E_p(a, b)$ to the signer.

The signer then proceeds by selecting secret keys and computing public keys as shown below:

- Chooses four secret keys $k_1, k_2, k_3, k_4 \in GF(q)$
- Computes the following public keys that are related to the intended recipient:

$$\begin{aligned}\beta_1 &= (k_4\alpha + k_3\gamma) \text{ (over } E_p(a, b)) \\ \alpha_1 &= (k_3\alpha + k_1\beta_1) \text{ (over } E_p(a, b)) \\ \alpha_2 &= (k_4\alpha + k_2\beta_1) \text{ (over } E_p(a, b))\end{aligned}$$

The four secret keys chosen by signer must be used once (choose different secret keys for different messages). This is because if two different messages are signed using the same four secret keys then these secret keys will be obtained easily.

3.2. Signing and verifying

To generate the signature for a message $m \in GF(q)$, the signer computes

$$\begin{aligned}y_1 &= (k_1m + k_2\lambda)(\text{mod } q) \\ y_2 &= (k_3m + k_4\lambda)(\text{mod } q)\end{aligned}$$

and publishes (y_1, y_2) as his signature on m .

The recipient accepts the signature as genuine if and only if the following check is correct:

$$(3.1) \quad y_2\alpha + y_1\beta_1 = (m\alpha_1 + \lambda\alpha_2) \text{ (over } E_p(a, b))$$

Note that only the intended recipient, who knows λ can verify the signature. In some cases, the recipient needs to prove the validity of the signature to any third party. To do this the recipient and the signer must cooperate where the signer computes and sends recipient $w_1 = k_1\beta_1$ and $w_2 = k_3\alpha$ secretly. The recipient then sends third party $w_3 = \lambda\alpha_2$ who next confirms that

$$y_2\alpha + y_1\beta_1 - m\alpha_1 = w_3$$

Finally, the recipient proves that $m(w_1 + w_2) = m_1$ in a Zero-knowledge technique for example using the protocol based on the elliptic curve discrete logarithm [10].

Now, assume that Adv successfully obtains a forged signature $(\bar{y}_1, \bar{y}_2)$ that passes VER, the signer and the recipient now can jointly prove that a forgery has happened by running the following procedure:

3.3. Algorithm proof of forgery (PTEST)

- Construct his own signature on message m as (y_1, y_2)
- Signer then calculates $(y_2 - \bar{y}_2)$ and $(\bar{y}_1 - y_1)$
- Signer and recipient jointly compute $d = \frac{(y_2 - \bar{y}_2) - (\bar{y}_1 - y_1)k_4}{(\bar{y}_1 - y_1)k_3v} \pmod{q}$ and use this as the proof of forgery.

Theorem 3.1 *If GEN and SIGN are run smoothly the validation in VER is correct.*

Proof. Let a signature on a message m is given by (y_1, y_2) . Then it is easy to show that the following congruence holds:

$$\begin{aligned} y_2\alpha + y_1\beta_1 &= (k_3m + k_4\lambda)\alpha + (k_1m + k_2\lambda)\beta_1 \\ &= (\alpha k_3m + \alpha k_4\lambda + \beta_1 k_1m + \beta_1 k_2\lambda) \\ &= (\alpha k_3 + \beta_1 k_1)m + (\alpha k_4 + \beta_1 k_2)\lambda \\ &= m\alpha_1 + \lambda\alpha_2 \end{aligned} \quad \blacksquare$$

4. Security and efficiency performance

4.1. Security

An enemy with unlimited computational power has a non-negligible probability to forge a signature and if the enemy presents the forge signature, we can provide a proof that a forgery has happened. In our scheme, t provides such proof; we need collaboration between a signer and a designated recipient. To show that our scheme is secure, we have to prove the following lemmas and theorems as suggested in Section 2. We begin by proving the following lemma.

Lemma 4.1 *There are q^2 equally likely secret keys that mach with the signer related public key.*

Proof. The public key of scheme $(\alpha_1, \alpha_2, \beta_1)$ gives the following three equations:

$$\begin{aligned}\beta_1 &= k_4\alpha + k_3\gamma(\text{over } E_p(a, b)) \\ \alpha_1 &= k_3\alpha + k_1\beta_1(\text{over } E_p(a, b)) \\ \alpha_2 &= k_3\alpha + k_2\beta_1(\text{over } E_p(a, b))\end{aligned}$$

Since $\gamma = v\beta = vd\alpha$, we have:

$$\beta_1 = k_4\alpha + k_3vd\alpha = (k_4 + k_3vd)\alpha.$$

Solving the ECDLP problem we have:

$$c = k_4 + k_3vd, \text{ where } c \in GF(q).$$

We also have:

$$\alpha_1 = k_3\alpha + k_1(k_4 + k_3vd)\alpha = (k_3 + k_1c)\alpha$$

and

$$\alpha_2 = k_4\alpha + k_2(k_4 + k_3vd)\alpha = (k_4 + k_2c)\alpha.$$

Solving the ECDLP problem, we have:

$$\begin{aligned}k_3 + k_1c &= c_1 \pmod{q} \\ k_4 + k_2c &= c_2 \pmod{q}\end{aligned}$$

where $c_1, c_2 \in GF(q)$. Equivalently,

$$\begin{pmatrix} c & 0 & 1 & 0 \\ 0 & c & 0 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \pmod{q}$$

This is a set of 2 linear equations in 4 unknowns where the rank of the coefficient matrix is equal to 2. Hence, there are q^2 solutions corresponding to assigning arbitrary values $\pmod{q}$ to k_3 and k_4 (q^2 possibilities), and calculating the values of k_1 and k_2 .

Theorem 4.2 *If the signer receives a forged signature $(\bar{y}_1, \bar{y}_2)$ not equal to his valid signature (y_1, y_2) on a message m and the forge signature passes the verification test then he is able to solve ECDL problem.*

Proof. Since both signatures (y_1, y_2) and $(\bar{y}_1, \bar{y}_2)$ pass the verification test, then, according to (3.1), we must have that:

$$\begin{aligned}y_2\alpha + y_1\beta_1 &= \bar{y}_2\alpha + \bar{y}_1\beta_1(\text{over } E_p(a, b)) \\ y_2\alpha - \bar{y}_2\alpha &= \bar{y}_1\beta_1 - y_1\beta_1(\text{over } E_p(a, b)) \\ (y_2 - \bar{y}_2)\alpha &= (\bar{y}_1 - y_1)\beta_1(\text{over } E_p(a, b))\end{aligned}$$

$$(4.1) \quad (y_2 - \bar{y}_2)\alpha = (\bar{y}_1 - y_1)(k_4 + k_3vd)\alpha \text{ (over } E_p(a, b))$$

$$(4.2) \quad (y_2 - \bar{y}_2) = (\bar{y}_1 - y_1)k_4 + (\bar{y}_1 - y_1)k_3vd \pmod{q}$$

$$(4.3) \quad (y_2 - \bar{y}_2) - (\bar{y}_1 - y_1)k_4 = (\bar{y}_1 - y_1)k_3vd \pmod{q}$$

$$(4.4) \quad d = \frac{(y_2 - \bar{y}_2) - (\bar{y}_1 - y_1)k_4}{(\bar{y}_1 - y_1)k_3v}$$

The correctness of driving equation (4.2) from (4.1) is ensured by the following lemmas.

Lemma 4.3 [11], [22], [8] *Any elliptic curve $E_p(a, b)$ where $q = |E_p(a, b)|$ is also prime, forms a cyclic group, which is isomorphic to $GF(q)$. Thus, any point other than the point at infinity is a generator of $E_p(a, b)$.*

Lemma 4.4 *If there is an equation of the form, $c\mu = a\mu + b\mu$ over a curve $E_p(a, b)$, where $a, b, c \in GF(q)$, $\mu \in E_p(a, b)$, and $q = |E_p(a, b)|$ is prime, then we have*

$$h = (a + b) \pmod{q}$$

Proof. According to Lemma 4.2, every point on $E_p(a, b)$, including μ , is a generator of the group and has order q , where $q = |E_p(a, b)|$.

Theorem 4.5 *Knowing the public key together with the signature for a message m , an enemy unlimited computational power can calculate q possible secret keys that could have been used for signing the message.*

Proof. With the public key $(\alpha_1, \alpha_2, \beta_1)$ and the signature (y_1, y_2) on m , an unlimited computational power enemy can solve the ECDL problem and rewrite these equations as follows:

$$\begin{aligned} c_1 &= (k_3 + k_1c) \pmod{q} \\ c_2 &= (k_4 + k_2c) \pmod{q} \\ \bar{y}_1 &= (k_1m + k_2) \pmod{q} \\ \bar{y}_2 &= (k_3m + k_4) \pmod{q} \end{aligned}$$

where $c = (k_4 + k_3vd)$, $c_1, c_2 \in GF(q)$ and the last two equations are an acceptable signature on m . These equations next can be formed as

$$\begin{pmatrix} c & 0 & 1 & 0 \\ 0 & c & 0 & 1 \\ m & 1 & 0 & 0 \\ 0 & 0 & m & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} \pmod{q}$$

It is easy to see that this matrix has rank 3 (this is true because $cr_3 - r_2 - mr_1 + r_4 = 0$, where r_i is the i^{th} row of the matrix and noting that the submatrix consisting of the first 3 columns has 3 independent rows), and so there are exactly q solutions to this equation.

Theorem 4.6 *The signer can prove a forgery with probability equal $\frac{q-1}{q}$.*

Proof. Given a forged signature that passes the verification test, the presumed signer can generate a different signature which passes the verification test with probability $\frac{q-1}{q}$, where q is the number of possible signatures on a message (this can be seen from previous Theorem 4.4).

Corollary 4.7 *A computationally bounded signer cannot make signatures which he can later prove to be forgeries.*

Proof. In order to deny a signature, given $(\alpha, \alpha_1, \alpha_2, \beta_1)$ dishonest signer must find four secret keys $\bar{k}_1, \bar{k}_2, \bar{k}_3, \bar{k}_4$ such as

$$\begin{aligned}\alpha_1 &= (\bar{k}_3\alpha_1 + \bar{k}_1\beta_1)(\text{mod } q) \text{ and} \\ \alpha_2 &= (\bar{k}_4\alpha + \bar{k}_2\beta_1)(\text{mod } q)\end{aligned}$$

However, finding those secrete keys is hard to solve because it is difficult to solve elliptic curve discrete logarithm problems.

Lemma 4.8 *Different secret keys that match with the public key and pass the verification test for a message m create different $\bar{m} \neq m$.*

Proof. As shown in Theorem (3), an enemy with unlimited computational power can obtain q secret keys. Say, there is another signature which also passes the verification test, then we have

$$\begin{aligned}\bar{y}_1 &= (k_1\bar{m} + k_2)(\text{mod } q) \\ \bar{y}_2 &= (k_3\bar{m} + k_4)(\text{mod } q)\end{aligned}$$

and obtain the following equation

$$\begin{pmatrix} c & 0 & 1 & 0 \\ 0 & c & 0 & 1 \\ m & 1 & 0 & 0 \\ 0 & 0 & m & 1 \\ \bar{m} & 1 & 0 & 0 \\ 0 & 0 & \bar{m} & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ y_1 \\ y_2 \\ \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} (\text{mod } q)$$

The coefficient matrix has rank 4 and the equations contain 4 variables. Thus there is only one unique solution. Theorems 4.4, 4.5 and Lemma 4.7 show that the proposed scheme satisfies all requirements of fail-stop signature mentioned in Section 2.

5. Efficiency performance

The performance of our scheme is described in terms of number of keys, computational complexity. We use the following notations to analyze the performance of our scheme.

Table 1: The comparison among our scheme and Esmail and Yahya scheme

Items	scheme by Eddie and Yahya		Our scheme	
	Time complexity	Complexity in T_{mul}	Time Complexity	Complexity in T_{mul}
Key generation	$8T_{exp} + 3T_{mul}$	$1923T_{mul}$	$8T_{ec-mul} + 3T_{ec-add}$	$232.36T_{mul}$
Signature generation	$4T_{mul}$	$4T_{mul}$	$4T_{ec-mul} + 2T_{ec-add}$	$32.24T_{mul}$
Signature verification	$4T_{exp} + 2T_{mul}$	$962T_{mul}$	$4T_{ec-mul} + 2T_{ec-add}$	$232.36T_{mul}$

- T_{mul} is the time complexity for executing the modular multiplication,
- T_{exp} is the time complexity for executing the modular exponentiation,
- T_{ec-add} is the time complexity for executing the addition of two points in elliptic curve,
- T_{ec-mul} is the time complexity for executing the multiplication of a number and an elliptic curve,
- T_h is the time complexity for performing a one-way hash function

We ignore the negligible time performing for modular addition. The computational complexity for the key generation, signing generation and verification is given in Table 1 and the last column converts various operation units to T_{mul} where $T_{exp} \approx 40T_{mul}$, $T_{ec-mul} \approx 29T_{mul}$ and $T_{ec-add} \approx 0.12T_{mul}$ given by Koblitiz, Menezes and Vanstone [9].

Ismail and Yahya proposed fail-stop designated signature scheme. In the scheme, the authors proved that fail-stop designated signature scheme proposed by them is superior in performance to other RSA-like schemes, but our scheme can be instantiated using elliptic curve arithmetic over small prime fields. This is important for many applications. We can seen in Table 1 that our scheme is mush more efficient than the Ismail and Yahya scheme [1] for key generation, signature and verification.

6. Numerical simulation of the scheme

Let say, a recipient Bob wishes to obtain a signature on his message $m = 603$. The scheme's setup is done by a trusted dealer, TD who generates the following parameters:

$$\begin{aligned}
y^2 &= (x^3 - 3x + 33)(\text{mod } 1091) \\
p &= 1091, \quad q = 1051, \\
G &= [299, 62] \text{ is a base with order } q \\
d &= 233, \quad \alpha = 195G = [401, 993] \\
\beta &= d\alpha = 233(195G) = 242G = [4, 1080]
\end{aligned}$$

and TD sends β to signer Alice via secure channel but broadcasts the pair (α, q) . To communicate, Bob and Alice must agree on a common secret key, say they choose $\lambda = 721$ at random. Alice then passes securely β to Bob and next Bob selects his random private key $v = 1021$ and computes and securely sends Alice

$$\gamma = v\beta = 1021(242G) = 97G = [795, 484]$$

Alice then proceeds by selecting his four secret keys in $GF(q)$. Let the keys be the following:

$$k_1 = 171, k_2 = 341, k_3 = 521 \text{ and } k_4 = 622$$

These keys must be used only once. This means to sign different message, Alice should select another keys because if the same keys are used twice, the keys will be recovered easily. Now Alice will generate three public keys related to Bob. The keys are given by

$$\begin{aligned}
\beta_1 &= 622(195G) + 521(97G) = 514G = [27, 759] \\
\alpha_1 &= 521(195G) + 171(514G) = 309G = [60, 933] \\
\beta_2 &= 622(195G) + 341(514G) = 182G = [759, 198]
\end{aligned}$$

and will be used by Bob to validate the produced signature. Now, to sign $m = 603$, Alice computes the following:

$$\begin{aligned}
y_1 &= k_1m + k_2\lambda = 171(603) + 341(721) = 42 \\
y_2 &= k_3m + k_4\lambda = 521(603) + 622(721) = 650
\end{aligned}$$

and these values are the signature of $m = 603$. To validate, only Bob has a right to do so. He checks that $y_2\alpha + y_1\beta_1 = m\alpha_1 + \lambda\alpha_2$ (*over* $E_p(a, b)$) holds. Since

$$650(195G) + 42(514G) = 147G = 603(309G) + 721(182G)$$

Bob accepts the signature. It is very important for Bob to prove to any third party say, Johnson that the signature is valid otherwise Bob and the signature cannot be trusted. This is done via zero-knowledge techniques.

Now, say an enemy, Song claims to Johnson that $(\bar{y}_1, \bar{y}_2) = (1023, 53)$ is also a valid signature of 603 produced by Alice. Unfortunately, this is true since

$$53(195G) + 1023(514G) = 147G = 603(309G) + 721(182G)$$

and Johnson may bring this to court because he feels that Alice and Bob is trying to cheat him. To prove that they are not guilty, Alice and Bob must show that the private keys have been broken. To do this, Alice first generates a signature on the message 603 and this is given by (42, 650). He then calculates the two numbers as below:

$$\begin{aligned} y_2 - \bar{y}_2 &= 650 - 53 = 597 \\ \bar{y}_1 - y_1 &= 1023 - 42 = 981 \end{aligned}$$

and next Alice cooperates with Bob to prove that a forgery has happened by compute the following :

$$\begin{aligned} d &= \frac{(y_2 - \bar{y}_2) - (\bar{y}_1 - y_1)k_4}{(\bar{y}_1 - y_1)k_3v} \pmod{q} \\ &= \frac{597 - 981(622)}{981(521)1021} \equiv 1046(584) \equiv 233 \pmod{1051} \end{aligned}$$

and use this as the proof of forgery.

7. Conclusion

In this article, we presented a new fail-stop designated recipient signature scheme based on elliptic curve discrete logarithm problems. with two properties: (1) only an intended recipient can validate the resulting signature and prove it to any requested third party and (2) if there was a forgery then both the signer and recipient can co-operate to provide a proof that a forgery has happened. The security of Ismail and Yahya fail-stop designated recipient signature scheme is constructed on the integer factorization problem while the security of the proposed scheme is based on the difficulty of solving the elliptic curve discrete logarithm problem. According to [15], the elliptic curve discrete logarithm problem is significantly more difficult than the integer factorization problem. For the most part, the well-known RSA system [21] must use 1024 bit keys, only then can it attain computationally reasonable security; the ECC needs only 160 bit keys. So, at the same level of security, the speed of ECC is several times faster than RSA system; it can also save on key storage space. Therefore, we showed that our scheme is much more efficient than the Ismail and Yahya scheme. Since implemented on the elliptic curve cryptosystem, the proposed scheme enables to reach the best trade-off between security and efficiency.

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PROPERTIES OF (γ, γ') -SEMIOPEN SETS

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Abstract. In this paper, we introduce and study the concept of (γ, γ') - θ -semiconnectedness and (γ, γ') - θ -semicomponents using (γ, γ') - θ -semiopen sets.

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1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real Analysis concerns the various modified forms of continuity, separation axioms etc. by utilizing generalized open sets. Kasahara [3] defined the concept of an operation on topological spaces. Umehara et. al. [9] introduced the notion of $\tau_{(\gamma, \gamma')}$ which is the collection of all (γ, γ') -open sets

in a topological space (X, τ) . Recently, G.S.S. Krishnan and K. Balachandran (see [4], [6], [5] and [8]) studied in this field. In [1] the authors, introduced the notion of (γ, γ') -semiopeness and investigated its fundamental properties. In this paper, we introduce and study the concept of (γ, γ') - θ -semiconnectedness and (γ, γ') - θ -semicomponents using (γ, γ') - θ -semiopen sets.

2. Preliminaries

Definition 2.1 [3] Let (X, τ) be a topological space. An operation γ on the topology τ is function from τ on to power set $\mathcal{P}(X)$ of X such that $V \subset V^\gamma$ for each $V \in \tau$, where V^γ denotes the value of γ at V . It is denoted by $\gamma : \tau \rightarrow \mathcal{P}(X)$.

Definition 2.2 [9] A subset A of a topological space (X, τ) is said to be (γ, γ') -open set if, for each $x \in A$, there exist open neighborhoods U and V of x such that $U^\gamma \cup V^{\gamma'} \subset A$. The complement of (γ, γ') -open set is called (γ, γ') -closed. $\tau_{(\gamma, \gamma')}$ denotes set of all (γ, γ') -open sets in (X, τ) .

Remark 2.3 It is easily to see that $\tau_{(\gamma, \gamma')} = \tau_\gamma \cap \tau_{\gamma'}$, where τ_γ denotes set of all γ -open sets in (X, τ) .

Definition 2.4 [2] Let A be a subset of a space X . A point $x \in A$ is said to be (γ, γ') -interior point of A , if there exist open neighborhoods U and V of x such that $U^\gamma \cup V^{\gamma'} \subset A$ and we denote the set of all such points by $\text{Int}_{(\gamma, \gamma')}(A)$. Thus $\text{Int}_{(\gamma, \gamma')}(A) = \{x \in A : x \in U \in \tau, V \in \tau \text{ and } U^\gamma \cup V^{\gamma'} \subset A\}$. Note that A is (γ, γ') -open if and only if $A = \text{Int}_{(\gamma, \gamma')}(A)$. A set A is called (γ, γ') -closed if and only if $X \setminus A$ is (γ, γ') -open.

Definition 2.5 [9] A point $x \in X$ is called a (γ, γ') -closure point of $A \subset X$, if $(U^\gamma \cup V^{\gamma'}) \cap A \neq \emptyset$, for any open neighborhoods U and V of x . The set of all (γ, γ') -closure points of A is called (γ, γ') -closure of A and is denoted by $\text{cl}_{(\gamma, \gamma')}(A)$. A subset A of X is called (γ, γ') -closed, if $\text{cl}_{(\gamma, \gamma')}(A) \subset A$. Note that $\text{cl}_{(\gamma, \gamma')}(A)$ is contained in every (γ, γ') -closed superset of A .

Definition 2.6 [1] A subset A of a space (X, τ) is said to be (γ, γ') -semiopen set, if there exists a (γ, γ') -open set O such that $O \subset A \subset \text{cl}_{(\gamma, \gamma')}(O)$. The set of all (γ, γ') -semiopen sets is denoted by $SO_{(\gamma, \gamma')}(X)$. A is (γ, γ') -semiclosed if and only if $X \setminus A$ is (γ, γ') -semiopen in X . Note that A is (γ, γ') -semiclosed if and only if $\text{Int}_{(\gamma, \gamma')}(\text{cl}_{(\gamma, \gamma')}(A)) \subset A$.

Remark 2.7 Observe that if $\{A_i : i \in I\}$ is a collection of (γ, γ') -open set, then $\bigcup_{i \in I} A_i$ is (γ, γ') -open set. From this, it follows that $\text{cl}_{(\gamma, \gamma')}(A)$ is a (γ, γ') -closed set. Moreover, if we define $\beta : \tau \rightarrow \mathcal{P}(X)$ as $\beta(U) = \text{cl}_{(\gamma, \gamma')}(U)$, then β is a monotone operation on τ .

Definition 2.8 [1] Let A be a subset of a space X . The intersection of all (γ, γ') -semiclosed sets containing A is called (γ, γ') -semiclosure of A and is denoted by $\text{scl}_{(\gamma, \gamma')}(A)$. Note that A is (γ, γ') -semiclosed if and only if $\text{scl}_{(\gamma, \gamma')}(A) = A$.

Definition 2.9 [1] Let A be a subset of a space X . The union of (γ, γ') -semiopen subsets of A is called (γ, γ') -semiinterior of A and is denoted by $s\text{Int}_{(\gamma, \gamma')}(A)$.

Definition 2.10 [1] A point $x \in X$ is said to be (γ, γ') -semi- θ -adherent point of a subset A of X , if $scl_{(\gamma, \gamma')}(U) \cap A \neq \emptyset$, for every $U \in SO_{(\gamma, \gamma')}(X, x)$. The set of all (γ, γ') -semi- θ -adherent points of A is called the (γ, γ') -semi- θ -closure of A and is denoted by $s_{(\gamma, \gamma')}cl_{\theta}(A)$. A subset A is called (γ, γ') -semi- θ -closed, if $s_{(\gamma, \gamma')}cl_{\theta}(A) = A$. A subset A is called (γ, γ') -semi- θ -open, if and only if $X \setminus A$ is (γ, γ') -semi- θ -closed.

Definition 2.11 [1] A subset A of a space X is said to be (γ, γ') -semiregular, if it is both (γ, γ') -semiopen and (γ, γ') -semiclosed. The class of all (γ, γ') -semiregular sets of X is denoted by $SR_{(\gamma, \gamma')}(A)$.

3. (γ, γ') -semi- θ -closed sets

Theorem 3.1 For a subset A of a topological space X , $s_{(\gamma, \gamma')}cl_{\theta}(A) = \cap \{V : A \subset V \text{ and } V \text{ is } SR_{(\gamma, \gamma')}(X)\}$.

Proof. Let $x \notin s_{(\gamma, \gamma')}cl_{\theta}(A)$. Then there exists a (γ, γ') -semiopen set U containing x such that $scl_{(\gamma, \gamma')}(U) \cap A = \emptyset$. Then $A \subset X \setminus scl_{(\gamma, \gamma')}(U) = V$ (say). Thus $V \in SR_{(\gamma, \gamma')}(X)$ such that $x \notin V$. Hence $x \notin \cap \{V : A \subset V \text{ and } V \text{ is } SR_{(\gamma, \gamma')}(X)\}$. Again, if $x \notin \cap \{V : A \subset V \text{ and } V \text{ is } SR_{(\gamma, \gamma')}(X)\}$, then there exists $V \in SR_{(\gamma, \gamma')}(X)$ containing A such that $x \notin V$. Then $(X \setminus V) (= U, \text{ say})$ is a (γ, γ') -semiopen set containing x such that $scl_{(\gamma, \gamma')}(U) \cap V = \emptyset$. This shows that $scl_{(\gamma, \gamma')}(U) \cap A = \emptyset$ so that $x \notin s_{(\gamma, \gamma')}cl_{\theta}(A)$. ■

Corollary 3.2 A subset A of X is (γ, γ') -semi- θ -closed if and only if $A = \cap \{V : A \subset V \in SR_{(\gamma, \gamma')}(X)\}$.

Theorem 3.3 For any subset A of X , $s_{(\gamma, \gamma')}cl_{\theta}(s_{(\gamma, \gamma')}cl_{\theta}(s_{(\gamma, \gamma')}cl_{\theta}(A))) = s_{(\gamma, \gamma')}cl_{\theta}(A)$.

Proof. Obviously, $s_{(\gamma, \gamma')}cl_{\theta}(A) \subset s_{(\gamma, \gamma')}cl_{\theta}(s_{(\gamma, \gamma')}cl_{\theta}(s_{(\gamma, \gamma')}cl_{\theta}(A)))$. Now, let

$$x \in s_{(\gamma, \gamma')}cl_{\theta}(s_{(\gamma, \gamma')}cl_{\theta}(s_{(\gamma, \gamma')}cl_{\theta}(A))) \text{ and } U \in SO_{(\gamma, \gamma')}(X, x).$$

Then $scl_{(\gamma, \gamma')}(U) \cap s_{(\gamma, \gamma')}cl_{\theta}(A) \neq \emptyset$. Let $y \in scl_{(\gamma, \gamma')}(U) \cap s_{(\gamma, \gamma')}cl_{\theta}(A)$. Since $scl_{(\gamma, \gamma')}(U) \in SO_{(\gamma, \gamma')}(X, y)$, $s_{(\gamma, \gamma')}cl(s_{(\gamma, \gamma')}cl(U)) \cap A \neq \emptyset$, that is, $s_{(\gamma, \gamma')}cl(U) \cap A \neq \emptyset$. Thus $x \in s_{(\gamma, \gamma')}cl_{\theta}(A)$. ■

Corollary 3.4 $s_{(\gamma, \gamma')}cl_{\theta}(A)$ is (γ, γ') -semi- θ -closed for any $A \subset X$.

Theorem 3.5 Intersection of arbitrary collection of (γ, γ') -semi- θ -closed sets in X is (γ, γ') -semi- θ -closed.

Proof. Let $\{A_\alpha : \alpha \in \Lambda\}$ be any collection of (γ, γ') -semi- θ -closed sets in a topological space (X, τ) and $A = \bigcap_{\alpha \in \Lambda} A_\alpha$. Now, using Definition 2.10, $x \in s_{(\gamma, \gamma')} \text{cl}_\theta(A)$, in consequence, $x \in s_{(\gamma, \gamma')} \text{cl}_\theta(A_\alpha)$ for all $\alpha \in \Lambda$. Follows that $x \in A_\alpha$ for all $\alpha \in \Lambda$. Therefore, $x \in A$. Thus $A = s_{(\gamma, \gamma')} \text{cl}_\theta(A)$. ■

Corollary 3.6 *For any $A \subset X$, $s_{(\gamma, \gamma')} \text{cl}_\theta(A)$ is the intersection of all (γ, γ') -semi- θ -closed sets each containing A .*

Remark 3.7 The following example shows that the union of (γ, γ') -semi- θ -closed sets may fail to be (γ, γ') -semi- θ -closed. Thus (γ, γ') -semi- θ -closure operator is not a Kuratowski's closure operator.

Example 3.8 Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Let $\gamma : \tau \rightarrow \mathcal{P}(X)$ and $\gamma' : \tau \rightarrow \mathcal{P}(X)$ be operators defined as follows: for every $A \in \tau$,

$$\gamma(A) = \text{Int}(\text{cl}(A))$$

$$\gamma'(A) = \begin{cases} X & \text{if } A = \{a, b\}, \\ A & \text{if } A \neq \{a, b\}. \end{cases}$$

Then the subsets $\{a\}$ and $\{b\}$ are (γ, γ') -semi- θ -closed, but their union $\{a, b\} = A \cup B$ is not (γ, γ') -semi- θ -closed.

Example 3.9 Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Let $\gamma : \tau \rightarrow \mathcal{P}(X)$ and $\gamma' : \tau \rightarrow \mathcal{P}(X)$ be operators defined as follows: for every $A \in \tau$,

$$\gamma(A) = \begin{cases} X & \text{if } A = \{a, b\}, \{c\} \\ A & \text{if } A \neq \{a, b\}, \{c\}. \end{cases}$$

$$\gamma'(A) = \begin{cases} X & \text{if } A = \{a, b\}, \{c\}, \{a, c\} \\ A & \text{if } A \neq \{a, b\}, \{c\}, \{a, c\}. \end{cases}$$

$\{c\}$ is (γ, γ') -semi- θ -closed but not (γ, γ') -semiregular.

Remark 3.10 It is proved by Carpintero et. al. [1] that a (γ, γ') -semiregular set is (γ, γ') -semi- θ -closed set. In the above example, $\{c\}$ is (γ, γ') -semi- θ -closed but not (γ, γ') -semiregular. Again, for a subset A , we always have $A \subset s_{(\gamma, \gamma')} \text{cl}(A) \subset s_{(\gamma, \gamma')} \text{cl}_\theta(A)$. Therefore, every (γ, γ') -semi- θ -open set is (γ, γ') -semiopen. The following example shows that the converse is not true in general.

Example 3.11 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ and $\gamma : \tau \rightarrow \mathcal{P}(X)$ and $\gamma' : \tau \rightarrow \mathcal{P}(X)$ be operators defined as follows: for every $A \in \tau$,

$$\gamma(A) = \begin{cases} \text{cl}(A) & \text{if } A = \{a\}, \\ A & \text{if } A \neq \{a\}, \end{cases}$$

$$\gamma'(A) = \begin{cases} \text{cl}(A) & \text{if } A = \{b\}, \\ A & \text{if } A \neq \{b\}. \end{cases}$$

Then $\{a, b\}$ is a (γ, γ') -semiopen set but not a (γ, γ') -semi- θ -open set.

Remark 3.12 The notions (γ, γ') -openness and (γ, γ') -semi- θ -openness are independent. In Example 3.9, $\{a, b\}$ is a (γ, γ') -semi- θ -open set but not a (γ, γ') -open set, whereas in Example 3.11, $\{b\}$ is a (γ, γ') -open set but not a (γ, γ') -semi- θ -open set.

4. (γ, γ') -semi- θ -connectedness

Definition 4.1 A pair (A, B) of nonempty subsets of a topological space (X, τ) is said to be a (γ, γ') -semi- θ -separation relative to X , if $s_{(\gamma, \gamma')}cl_{\theta}(A) \cap B = A \cap s_{(\gamma, \gamma')}cl_{\theta}(B) = \emptyset$.

Definition 4.2 A subset N of a topological space (X, τ) is said to be (γ, γ') -semi- θ -connected relative to X (or, simply (γ, γ') -semi- θ -connected), if there exists no (γ, γ') -semi- θ -separation (A, B) relative to X such that $N = A \cup B$. If N is not (γ, γ') -semi- θ -connected, then it is called (γ, γ') -semi- θ -disconnected.

Lemma 4.3 If (A, B) is a (γ, γ') -semi- θ -separation relative to X and P, Q are nonempty subsets of A and B respectively, then (P, Q) is also a (γ, γ') -semi- θ -separation relative to X .

Proof. The proof is obvious. ■

Theorem 4.4 For a subset A of X , the following are equivalent:

1. A is (γ, γ') -semi- θ -connected.
2. If (L, M) is a (γ, γ') -semi- θ -separation relative to X and $A \subset L \cup M$, then either $A \subset L$ or $A \subset M$.

Theorem 4.5 Let A be a (γ, γ') -semi- θ -closed set in a topological space (X, τ) . If (P, Q) is a (γ, γ') -semi- θ -separation relative to X such that $A = P \cup Q$, then P and Q are (γ, γ') -semi- θ -closed sets in X .

Proof. We have $A = P \cup Q$, where $s_{(\gamma, \gamma')}cl_{\theta}(P) \cap Q = P \cap s_{(\gamma, \gamma')}cl_{\theta}(Q) = \emptyset$. Now, $A \cap s_{(\gamma, \gamma')}cl_{\theta}(P) = (P \cup Q) \cap s_{(\gamma, \gamma')}cl_{\theta}(P) = (P \cap s_{(\gamma, \gamma')}cl_{\theta}(P)) \cup (Q \cap s_{(\gamma, \gamma')}cl_{\theta}(P)) = P$. In view of Corollary 3.4 and Theorem 3.5, P is (γ, γ') -semi- θ -closed. Similarly, Q is (γ, γ') -semi- θ -closed. ■

Theorem 4.6 If C is a (γ, γ') -semi- θ -connected set relative to a topological space X and $C \subset T \subset s_{(\gamma, \gamma')}cl_{\theta}(C)$, then T is also (γ, γ') -semi- θ -connected.

Corollary 4.7 The (γ, γ') -semi- θ -closure of a (γ, γ') -semi- θ -connected set is (γ, γ') -semi- θ -connected.

Remark 4.8 Since for any subset A of X , $A \subset s_{(\gamma, \gamma')}cl(A) \subset s_{(\gamma, \gamma')}cl_{\theta}(A)$, the (γ, γ') -semiclosure of a (γ, γ') -semi- θ -connected set is (γ, γ') -semi- θ -connected set.

Theorem 4.9 *A set E in a topological space X is (γ, γ') -semi- θ -connected if and only if any two points of E are contained in some (γ, γ') -semi- θ -connected set contained in E .*

Theorem 4.10 *The union of any collection of (γ, γ') -semi- θ -connected sets, no two of which are (γ, γ') -semi- θ -separated relative to X is a (γ, γ') -semi- θ -connected set.*

Corollary 4.11 *The union of any family of (γ, γ') -semi- θ -connected sets, every pair of which has an element in common is (γ, γ') -semi- θ -connected.*

Theorem 4.12 *For a topological space (X, τ) the following statements are equivalent:*

1. X is (γ, γ') -semi- θ -connected.
2. There do not exist disjoint nonempty (γ, γ') -semi- θ -clopen sets A, B in X such that $X = A \cup B$.
3. The only subset of X which are both (γ, γ') -semi- θ -open and (γ, γ') -semi- θ -closed are X and $\emptyset$.

Definition 4.13 [7] A topological space (X, τ) is said to be a (γ, γ') -connected, if there does not exist a pair A, B of nonempty disjoint (γ, γ') -open subsets of X such that $X = A \cup B$, otherwise X is called (γ, γ') -disconnected.

The relation between (γ, γ') -semi- θ -connectedness and (γ, γ') -connectedness, is as follows.

Theorem 4.14 *Every (γ, γ') -semi- θ -connected space X is (γ, γ') -connected.*

Proof. If X is (γ, γ') -disconnected, there exist nonempty disjoint (γ, γ') -closed sets U and V such that $X = U \cup V$. Then U and V are (γ, γ') -semiregular sets and hence each is (γ, γ') -semi- θ -closed. Thus X is not (γ, γ') -semi- θ -connected. ■

Remark 4.15 The converse of the above theorem is false. In fact, in Example 3.9 the topological space (X, τ) is (γ, γ') -connected but not (γ, γ') -semi- θ -connected.

5. (γ, γ') -semi- θ -components and (γ, γ') -semi- θ -quasi components

Definition 5.1 Let (X, τ) be a topological space. A maximal (γ, γ') -semi- θ -connected subset relative to X , that is, an (γ, γ') -semi- θ -connected set in X which is not properly contained in any larger (γ, γ') -semi- θ -connected set of X , is called a (γ, γ') -semi- θ -component of the space X .

Theorem 5.2 *For a topological space (X, τ) , the following statements are true:*

1. Each point of X is contained in exactly one (γ, γ') -semi- θ -component of X .
2. X is equal to the union of its (γ, γ') -semi- θ -components.
3. Each (γ, γ') -semi- θ -connected subset of X is contained in an (γ, γ') -semi- θ -component of X .
4. Each (γ, γ') -semi- θ -component of X is (γ, γ') -semi- θ -closed.
5. Two different (γ, γ') -semi- θ -components of X are disjoint.

Proof. (1). Let $x \in X$. Consider the class $\{C_\alpha : \alpha \in \Lambda\}$ of all (γ, γ') -semi- θ -connected sets of X , each of which contains x . The class is nonempty, since $\{x\}$ belongs to the class. By Corollary 4.11, $C = \bigcup_{\alpha \in \Lambda} C_\alpha$ is a maximal (γ, γ') -semi- θ -connected set containing x that is, a (γ, γ') -semi- θ -component of X . Uniqueness of the (γ, γ') -semi- θ -component (containing x) is obvious. (2). Follows from (1). (3). By construction in (1), any (γ, γ') -semi- θ -connected set is contained in the (γ, γ') -semi- θ -component which contains any one of its points. (4). Follows from Corollary 4.7. (5). If C_1 and C_2 are two (γ, γ') -semi- θ -components and $C_1 \cap C_2 \neq \emptyset$, then by Corollary 4.11, $C_1 \cup C_2$ is a (γ, γ') -semi- θ -connected set in X . By the maximality of C_1 and C_2 , $C_1 = C_1 \cup C_2 = C_2$. ■

Definition 5.3 Let A be a nonempty subset of X . Two elements x, y of A are said to be (γ, γ') -semi- θ -equivalent, written $x \sim y$, if whenever (L, M) is a (γ, γ') -semi- θ -separation relative to X and $A = L \cup M$, we have either $x, y \in L$ or $x, y \in M$.

The relation " $\sim$ " is an equivalence relation on A . The equivalence classes are called (γ, γ') -semi- θ -quasi components of A . We shall denote by $A(x)$ the (γ, γ') -semi- θ -quasi component of A containing x . For a nonempty subset A of X , let us call a subset P of A a (γ, γ') -semi- θ -component of A if P is (γ, γ') -semi- θ -connected and there does not exist any subset Q of A such that $P \subsetneq Q$ and Q is (γ, γ') -semi- θ -connected.

Theorem 5.4 Let (X, τ) be a topological space and A a nonempty subset of X . Then $A(x)$ is an (γ, γ') -semi- θ -component of A for each $x \in A$ for which $A(x)$ is (γ, γ') -semi- θ -connected.

Proof. Let $x \in A$ for which $A(x)$ is (γ, γ') -semi- θ -connected and $B \subset A$ be a (γ, γ') -semi- θ -connected set and also assume that $A(x) \subset B$. If $Y \in B$ and $A = L \cup M$, where (L, M) is a (γ, γ') -semi- θ -separation relative to X , then either $B \subset L$ or $B \subset M$. Thus either $x, y \in L$ or $x, y \in M$. Hence $y \in A(x)$ and consequently, $A(x) = B$. Thus $A(x)$ is a (γ, γ') -semi- θ -component of A . ■

Definition 5.5 Let A be a nonempty subset of X and $x \in A$. We define $L_x(A) = \{L \subset X : \text{for some } M \subset X, (L, M) \text{ is a } (\gamma, \gamma')\text{-semi-}\theta\text{-separation relative to } X \text{ satisfying } A = L \cup M \text{ and } x \in L\}$.

Theorem 5.6 *Let A be a nonempty subset of a topological space (X, τ) and $x \in A$. Then a (γ, γ') -semi- θ -quasi component $A(x)$ of A is the intersection of the elements of $L_x(A)$.*

Proof. Let $y \in A(x)$ and L_x be any member of $L_x(A)$. Then for some $M_x \subset X$, (L_x, M_x) is a (γ, γ') -semi- θ -separation relative to X satisfying $A = L_x \cup M_x$ and $x \in L_x$. Since $x \sim y$, we have $x, y \in L_x$. Thus $y \in \cap L_x(A)$; hence $A(x) \subset \cap L_x(A)$. On the other hand, if $y \in \cap L_x(A)$ then obviously, $y \in A(x)$. ■

Theorem 5.7 *Each (γ, γ') -semi- θ -quasi component Q in the topological space (X, τ) is the intersection of all (γ, γ') -semi- θ -clopen sets containing a given point belonging to Q .*

Proof. Straightforward. ■

Corollary 5.8 *Every (γ, γ') -semi- θ -quasi component of a topological space (X, τ) is (γ, γ') -semi- θ -closed in X .*

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OPERATION VIA-REGULAR OPEN SETS

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Abstract. The aim of this paper is to introduce and study the concepts of γ -regular open sets and their related notions in topological spaces.

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1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, separation axioms etc. by utilizing generalized open sets. Kasahara [3] defined the concept of an operation on topological spaces and introduce the concept of γ -closed graphs of a function. Ogata [6] introduced the notion γ -open

sets in a topological space (X, τ) . In this paper, we have introduced and study the notion of γ -regular open sets by using the operation γ on a topological space (X, τ) . We also introduce the almost (γ, β) -continuous functions and investigate some of its important properties.

2. Preliminaries

Definition 2.1 Let (X, τ) be a topological space. An operation γ [3] on the topology τ is a function from τ onto a power set $\mathcal{P}(X)$ of X such that $V \subset V^\gamma$ for each $V \in \tau$, where V^γ denotes the value of γ at V . It is denoted by $\gamma : \tau \rightarrow \mathcal{P}(X)$.

Definition 2.2 A subset A of a topological space (X, τ) is called γ -open [6] set if, for each $x \in A$, there exists an open set U such that $x \in U$ and $U^\gamma \subset A$. τ_γ denotes the set of all γ -open sets in (X, τ) . The complement of a γ -open set is called γ -closed.

Definition 2.3 Let A be subset of a topological space (X, τ) and γ be an operation on τ . Then

- (i) the τ_γ -closure of A is defined as the intersection of all γ -closed sets containing A . That is, $\tau_\gamma\text{-cl}(A) = \bigcap \{F : F \text{ is } \gamma\text{-closed and } A \subset F\}$.
- (ii) the τ_γ -interior of A is defined as the union of all γ -open sets contained in A . That is, $\tau_\gamma\text{-Int}(A) = \bigcup \{U : U \text{ is } \gamma\text{-open and } U \subset A\}$.

Definition 2.4 A subset A of a topological space (X, τ) is said to be

- (i) γ -semiopen [5] if $A \subset \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A))$.
- (ii) γ -preopen [4] if $A \subset \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))$.
- (iii) γ - α -open [2] if $A \subset \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A)))$.
- (iv) γ - β -open [1] if $A \subset \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A)))$.

The complement of a γ -semiopen (resp. γ -preopen, γ - α -open, γ - β -open) set is called a γ -semiclosed (resp. γ -preclosed, γ - α -closed, γ - β -closed) set. The family of all γ -semiopen (resp. γ -preopen, γ - α -open, γ - β -open) sets of (X, τ) is denoted by $\gamma SO(X)$ (resp. $\gamma PO(X)$, $\gamma \alpha(X)$, $\gamma \beta(X)$)

Definition 2.5 A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be:

- (i) (γ, β) -continuous [6] at a point $x \in X$ if, for each β -open subset V in Y containing $f(x)$, there exists a γ -open subset U of X containing x such that $f(U) \subset V$;
- (ii) (γ, β) -continuous [6] if it has this property at each point of X .

3. γ -Regular open sets

Throughout this paper, the operator γ is defined on (X, τ) and the operator β is defined on (Y, σ) .

Definition 3.1 A subset A of a topological space (X, τ) is said to be γ -regular open set, if $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A)) = A$. We call a subset A of X is γ -regular closed, if its complement is γ -regular open.

The family of all γ -regular open (resp. γ -regular closed) sets of (X, τ) is denoted by $\gamma RO(X)$ (resp. $\gamma RC(X)$).

Lemma 3.2 For a topological space (X, τ) , we have $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))$.

Proof. It is obvious that $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A)))) \subseteq \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))$. Conversely, $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A)) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))) \subseteq \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))))$. ■

Definition 3.3 An operation γ on τ is said to be regular, if for any open neighborhoods U, V of $x \in X$, there exists an open neighborhood W of x such that $W^\gamma \subseteq U^\gamma \cap V^\gamma$.

Lemma 3.4 If A and B are two γ -open subsets and γ is a regular operator. then $A \cap B$ is a γ -open set.

Remark 3.5 The condition in the above Lemma that γ is a regular operator can not be omitted as we see in the following example

Example 3.6 Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Let $\gamma : \tau \rightarrow \mathcal{P}(X)$ be an operation defined as follows: for every $A \in \tau$,

$$A^\gamma = \begin{cases} A & \text{if } b \in A, \\ \text{cl}(A) & \text{if } b \notin A, \end{cases}$$

Then the sets $\{a, b\}$ and $\{a, c\}$ are γ -open sets but their intersection $\{a\}$ is not a γ -open set.

Lemma 3.7 Let A and B be subsets of a topological space (X, τ) . Then the following properties hold:

- (i) $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))$ is γ -regular open.
- (ii) If A and B are γ -regular open, then $A \cap B$ is also γ -regular open.

Proof. (i). Follows from Lemma 3.2. (ii). Let A and B be γ -regular open sets of X . Then using Lemma 3.4, we have $A \cap B = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A)) \cap \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(B)) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A) \cap \tau_\gamma\text{-cl}(B)) \supset \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A \cap B)) \supset \tau_\gamma\text{-Int}(A \cap B) = A \cap B$. Therefore, we obtain $A \cap B = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A \cap B))$. This shows that $A \cap B$ is γ -regular open. ■

Remark 3.8 The following example shows that the union of any two γ -regular open sets need not be γ -regular open.

Example 3.9 [6] Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Let $\gamma : \tau \rightarrow \mathcal{P}(X)$ be an operation defined as follows: for every $A \in \tau$,

$$A^\gamma = \begin{cases} A & \text{if } c \notin A, \\ \text{cl}(A) & \text{if } c \in A, \end{cases}$$

Then the sets $\{a\}$ and $\{b\}$ are γ -regular open sets but their union $\{a, b\}$ is not a γ -regular open set.

Theorem 3.10 *The following statements are true:*

- (i) A γ -open set A is γ -regular open if and only if $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A)) \subseteq A$.
- (ii) For every γ -closed set A , the set $\tau_\gamma\text{-Int}(A)$ is γ -regular open.

Proof. (i). It suffices to prove that every γ -open set A satisfying $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A)) \subset A$ is γ -regular open. Since $A \subset \tau_\gamma\text{-cl}(A)$ holds, then $A = \tau_\gamma\text{-Int}(A) \subset \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))$ is true, so we have $A = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))$. (ii). If A is γ -closed, then the following holds: $\tau_\gamma\text{-Int}(A) \subset \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A)) \subset \tau_\gamma\text{-cl}(A) = A$. Hence $\tau_\gamma\text{-Int}(A) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A))) \subset \tau_\gamma\text{-Int}(A)$. So, $\tau_\gamma\text{-Int}(A) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A)))$ holds. That is, $\tau_\gamma\text{-Int}(A)$ is γ -regular open. ■

Theorem 3.11 *For a subset A of X , the following properties are equivalent:*

- (i) A is γ -preopen.
- (ii) there exists a γ -regular open subset $G \subset X$ such that $A \subset G$ and $\tau_\gamma\text{-cl}(A) = \tau_\gamma\text{-cl}(G)$.
- (iii) $A = G \cap D$, where G is γ -regular open and D is γ -dense.
- (iv) $A = G \cap D$, where G is γ -open and D is γ -dense.

Proof. (i) $\Rightarrow$ (ii): Let A be γ -preopen. We have $A \subset \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A)) \subset \tau_\gamma\text{-cl}(A)$ which implies that $\tau_\gamma\text{-cl}(A) \subset \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))) \subset \tau_\gamma\text{-cl}(A)$ and so $\tau_\gamma\text{-cl}(A) = \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A)))$. Let $G = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))$. Then $A \subset G$ and $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(G)) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A)))) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A)) = G$ which implies that G is γ -regular open. Also $\tau_\gamma\text{-cl}(G) = \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(G))) = \tau_\gamma\text{-cl}(A)$. (ii) $\Rightarrow$ (iii): Let G be a γ -regular open set such that $A \subset G$ and $\tau_\gamma\text{-cl}(A) = \tau_\gamma\text{-cl}(G)$. Let $D = A \cup (X \setminus G)$. Then $A = G \cap D$ where G is γ -regular open. Now, $\tau_\gamma\text{-cl}(D) = \tau_\gamma\text{-cl}(A \cup (X \setminus G)) = \tau_\gamma\text{-cl}((A) \cup \tau_\gamma\text{-cl}(X \setminus G)) = \tau_\gamma\text{-cl}(G) \cup \tau_\gamma\text{-cl}(X \setminus G) = \tau_\gamma\text{-cl}((G \cup (X \setminus G))) = \tau_\gamma\text{-cl}(X) = X$. Hence D is γ -dense. (iii) $\Rightarrow$ (iv): The proof follows from the fact that every γ -regular open set is γ -open. (iv) $\Rightarrow$ (i): Suppose $A = G \cap D$ where G is γ -open and D is γ -dense. Now $G = G \cap X = G \cap \tau_\gamma\text{-cl}(D) \subset \tau_\gamma\text{-cl}(G \cap D)$ and so $G = \tau_\gamma\text{-Int}(G) \subset \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(G \cap D)) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))$ which implies that $A \subset \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))$. Hence A is γ -preopen. ■

Theorem 3.12 *For a subset A of X , the following are equivalent:*

- (i) A is γ -regular closed.
- (ii) A is γ -preclosed and γ -semiopen.
- (iii) A is γ - α -closed and γ - β -open.

Proof. (i) $\Rightarrow$ (ii): Let A be γ -regular closed. Then $A = \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A))$ and A is γ -preclosed and γ -semiopen. (ii) $\Rightarrow$ (iii): Let A be γ -preclosed and γ -semiopen. Then $A \subset \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A))$ and $\tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A)) \subset A$. Therefore, we have $\tau_\gamma\text{-cl}(A) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))$ and hence $\tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A)))) = \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A)) \subset A$. This shows that A is γ - α -closed. Since every γ -semiopen set is γ - β -open set, it is obvious that A is γ - β -open. (iii) $\Rightarrow$ (i): Let A be γ - α -closed and γ - β -open. Then $A = \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A)))$ and hence $\tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A)) = \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A)))) = A$. Therefore, A is γ -regular closed. **Proof.**

Definition 3.13 A topological space (X, τ) is said to be τ_γ -extremally disconnected, if $\tau_\gamma\text{-cl}(A) \in \tau_\gamma$, for every $A \in \tau_\gamma$.

Theorem 3.14 *For a topological space (X, τ) , the following properties are equivalent:*

- (i) X is τ_γ -extremally disconnected.
- (ii) Every γ -regular open subset of X is γ -closed.
- (iii) Every γ -regular closed subset of X is γ -open.

Proof. (i) $\rightarrow$ (ii): Let X be τ_γ -extremally disconnected. Let A be a γ -regular open subset of X . Then $A = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))$. Since A is a γ -open set, then $\tau_\gamma\text{-cl}(A) \in \tau_\gamma$. Thus, $A = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A)) = \tau_\gamma\text{-cl}(A)$; hence A is γ -closed. (ii) $\rightarrow$ (iii): Suppose that every γ -regular open subset of X is γ -closed in X . Let $A \in \tau_\gamma$. Since $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))$ is γ -regular open, then it is γ -closed in X . This implies that $\tau_\gamma\text{-cl}(A) \subset \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))$ since $A \subset \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))$. Thus, $\tau_\gamma\text{-cl}(A) \in \tau_\gamma$; hence X is τ_γ -extremally disconnected. (ii) $\Leftrightarrow$ (iii): Obvious. ■

Theorem 3.15 *For a topological space (X, τ) , the following properties are equivalent:*

- (i) X is τ_γ -extremally disconnected.
- (ii) γ -regular open sets coincide with γ -regular closed sets.

Proof. (i) $\Rightarrow$ (ii): Suppose A is a γ -regular open subset of X . Since γ -regular open sets are γ -open, by (i), $A = \tau_\gamma\text{-cl}(A) = \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A))$ and so A is γ -regular closed. If A is γ -regular closed, then $A = \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A)) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A))) = \tau_\gamma\text{-Int}(A)$ so A is γ -open. Also, $A = \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A)) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A))) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))$. Hence A is γ -regular open. (ii) $\Rightarrow$ (i):

Let A be a γ -open subset of X . Then $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))$ is γ -regular open and so it is γ -regular closed, by (ii). Hence $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}((A)))) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))$ which implies that $\tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))$. Therefore, $\tau_\gamma\text{-cl}(A) = \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A)) \subset \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))$ and so $\tau_\gamma\text{-cl}(A) = \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A))$. Hence $\tau_\gamma\text{-cl}(A)$ is γ -open. This shows that X is τ_γ -extremally disconnected. ■

Theorem 3.16 *For a topological space (X, τ) , the following properties are equivalent:*

- (i) X is τ_γ -extremally disconnected.
- (ii) Every γ -regular closed set is γ -preopen.

Proof. (i) $\Rightarrow$ (ii): The proof follows from the fact that every γ -regular open sets is γ -preopen set. (ii) $\Rightarrow$ (i): If A is γ -open, then $\tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A))$ is γ -regular closed and so it is γ -preopen. Therefore, $\tau_\gamma\text{-cl}(A) = \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A)) \subset \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(\tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A)))) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A))) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))$. Thus, $\tau_\gamma\text{-cl}(A) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))$ which implies that $\tau_\gamma\text{-cl}(A)$ is γ -open. Hence X is τ_γ -extremally disconnected. ■

Theorem 3.17 *If (X, τ) is a τ_γ -extremally disconnected, then the following properties hold:*

- (i) $A \cap B$ is γ -regular closed for all γ -regular closed subsets of A and B of X .
- (ii) If γ is regular open. Then $A \cap B$ is γ -regular open for all γ -regular open subsets A and B of X .

Proof. (i). Let X be τ_γ -extremally disconnected. Let A and B be γ -regular closed subsets of X . Since A and B are γ -closed, $\tau_\gamma\text{-Int}(A)$ and $\tau_\gamma\text{-Int}(B)$ are γ -closed. This implies that $A \cap B = \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A)) \cap \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(B)) = \tau_\gamma\text{-Int}(A) \cap \tau_\gamma\text{-Int}(B) = \tau_\gamma\text{-Int}(A \cap B) \subset \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A \cap B))$. On the other hand, we have $\tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A \cap B)) = \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A) \cap \tau_\gamma\text{-Int}(B)) \subset \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A)) \cap \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(B)) = A \cap B$. Thus, $A \cap B$ is γ -regular closed. (ii). It follows from (i). ■

Theorem 3.18 *For a topological space (X, τ) , the following properties are equivalent:*

- (i) X is τ_γ -extremally disconnected.
- (ii) $\tau_\gamma\text{-cl}(A) \in \tau_\gamma$ for every $A \in \gamma SO(X)$.
- (iii) $\tau_\gamma\text{-cl}(A) \in \tau_\gamma$ for every $A \in \gamma PO(X)$.
- (iv) $\tau_\gamma\text{-cl}(A) \in \tau_\gamma$ for every $A \in \gamma RO(X)$.

Proof. (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii): Let A be a γ -semiopen (γ -preopen) set. Then A is γ - β -open; hence $\tau_\gamma\text{-cl}(A) \in \tau_\gamma$. (ii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (iv): $A \in \gamma RO(X)$. Then $A \in \gamma SO(X)$ and $A \in \gamma PO(X)$ and hence $\tau_\gamma\text{-cl}(A) \in \tau_\gamma$. (iv) $\Rightarrow$ (i). Suppose that the γ -closure of every γ -semi open subset of X is γ -open. Let $A \subset X$ be a γ -open set. This implies that $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))$ is a γ -open set. Then $\tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A)))$ is γ -open. We have $\tau_\gamma\text{-cl}(A) \subset (\tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A)))) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))$. Thus, $\tau_\gamma\text{-cl} \in \tau_\gamma$; hence X is γ -extremally disconnected. ■

Definition 3.19 Let (X, τ) be a topological space, $S \subset X$ and $x \in X$. Then

- (i) x is called γ - δ -cluster point of S , if $S \cap \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(U)) \neq \emptyset$, for each τ_γ -open set U containing x .
- (ii) The family of all γ - δ -cluster point of S is called the γ - δ -closure of S and is denoted by $\tau_\gamma\text{-cl}_\delta(S)$.
- (iii) A subset S is said to be γ - δ -closed, if $\tau_\gamma\text{-cl}_\delta(S) = S$. The complement of an γ - δ -closed set is said to be an γ - δ -open set.

Lemma 3.20 Let A and B be subsets of a topological space (X, τ) . Then the following properties hold:

- (i) $A \subset \tau_\gamma\text{-cl}_\delta(A)$.
- (ii) If $A \subset B$, then $\tau_\gamma\text{-cl}_\delta(A) \subset \tau_\gamma\text{-cl}_\delta(B)$.
- (iii) $\tau_\gamma\text{-cl}_\delta(A) = \cap \{F \subset X : A \subset F \text{ and } F \text{ is } \gamma\text{-}\delta\text{-closed}\}$.
- (iv) If A is an γ - δ -closed set of X for each $\alpha \in \Delta$, then $\cap \{A_\alpha : \alpha \in \Delta\}$ is γ - δ -closed.
- (v) $\tau_\gamma\text{-cl}_\delta(A)$ is γ - δ -closed.

Proof. (i). For any $x \in A$ and any γ -open set V containing x , we have $\emptyset \neq A \cap V \subset A \cap \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(V))$ and hence $x \in \tau_\gamma\text{-cl}_\delta(A)$. This shows that $A \subset \tau_\gamma\text{-cl}_\delta(A)$. (ii). Suppose that $x \notin \tau_\gamma\text{-cl}_\delta(B)$. Then there exists a γ -open set V containing x such that $\emptyset = \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(V)) \cap B$. Hence $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(V)) \cap A = \emptyset$. Therefore, we have $x \notin \tau_\gamma\text{-cl}_\delta(A)$. (iii). Suppose that $x \in \tau_\gamma\text{-cl}_\delta(A)$. For any γ -open set V containing x and any γ - δ -closed set F containing A , we have $\emptyset \neq A \cap \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(V)) \subset F \cap \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(V))$ and hence $x \in \tau_\gamma\text{-cl}_\delta(F) = F$. This shows that $x \in \cap \{F \subset X : A \subset F \text{ and } F \text{ is } \gamma\text{-}\delta\text{-closed}\}$. Conversely, suppose that $x \notin \tau_\gamma\text{-cl}_\delta(A)$. Then there exists a γ -open set V containing x such that $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(V)) \cap A = \emptyset$. Then $X \setminus \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(V))$ is a γ - δ -closed set which contains A and does not contain x . Therefore, we have $x \notin \cap \{F \subset X : A \subset F \text{ and } F \text{ is } \gamma\text{-}\delta\text{-closed}\}$. (iv). For each $\alpha \in \Delta$, $\tau_\gamma\text{-cl}_\delta(\cap_{\alpha \in \Delta} A_\alpha) \subset \tau_\gamma\text{-cl}_\delta(A_\alpha) = A_\alpha$ and hence $\tau_\gamma\text{-cl}_\delta(\cap_{\alpha \in \Delta} A_\alpha) \subset (\cap_{\alpha \in \Delta} A_\alpha)$. By (i) we obtain $\tau_\gamma\text{-cl}_\delta(\cap_{\alpha \in \Delta} A_\alpha) = (\cap_{\alpha \in \Delta} A_\alpha)$. This shows that $\cap_{\alpha \in \Delta} A_\alpha$ is γ - δ -closed. (v). This follows immediately from (iii) and (iv). ■

Proposition 3.21 *Let A and B be subsets of a topological space (X, τ) . Then the following properties hold:*

- (i) *If A is γ -regular open, then it is γ - δ -open.*
- (ii) *Every γ - δ -open set is the union of a family of γ -regular open sets.*

Proof. (i). Let A be any γ -regular open set. For each $x \in A$, $(X \setminus A) \cap A = \emptyset$ and A is γ -regular open. Hence $x \notin \tau_\gamma\text{-cl}_\delta(X \setminus A)$ for each $x \in A$. This shows that $x \notin (X \setminus A) \Rightarrow x \notin \tau_\gamma\text{-cl}_\delta(X \setminus A)$. Therefore we have $\tau_\gamma\text{-cl}_\delta(X \setminus A) \subset (X \setminus A)$. Since in general, for any subset S of X , $S \subseteq \tau_\gamma\text{-cl}_\delta(S)$, $\tau_\gamma\text{-cl}_\delta(X \setminus A) = (X \setminus A)$; A is γ - δ -open. (ii). Let A be a γ - δ -open set. Then $(X \setminus A)$ is γ - δ -closed and hence $(X \setminus A) = \tau_\gamma\text{-cl}_\delta(X \setminus A)$. For each $x \in A$, $x \notin \tau_\gamma\text{-cl}_\delta(X \setminus A)$ and there exists a γ -open set V_x such that $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(V_x)) \cap (X \setminus A) = \emptyset$. Therefore, $x \in V_x \subset \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(V_x)) \subset A$ and hence $A = \cup \{ \tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(V_x)) : x \in A \}$. By (i) $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(V_x))$ is γ -regular open for each $x \in A$. ■

Theorem 3.22 *Let (X, τ) be topological space and $\tau_{\gamma\delta} = \{A \subset X : A \text{ is a } \gamma\text{-}\delta\text{-open set of } (X, \tau)\}$.*

Proof. (i). It is obvious that $\emptyset, X \in \tau_{\gamma\delta}$. (ii). Let $V_\alpha \in \tau_{\gamma\delta}$ for each $\alpha \in \Delta$. Then $X \setminus V_\alpha$ is γ - δ -closed, for each $\alpha \in \Delta$. By Proposition 3.21, $\bigcap_{\alpha \in \Delta} (X \setminus V_\alpha)$ is γ - δ -closed and $\bigcap_{\alpha \in \Delta} (X \setminus V_\alpha) = X \setminus \bigcup_{\alpha \in \Delta} V_\alpha$. Hence $\bigcup_{\alpha \in \Delta} V_\alpha$ is γ - δ -open. (iii) Let $A, B \in \tau_{\gamma\delta}$. Then $A = \bigcup_{\kappa \in \Delta_1} A_\kappa$ and $B = \bigcup_{\omega \in \Delta_2} B_\omega$, where A_κ and B_ω are γ -regular open sets for each $\kappa \in \Delta_1$ and $\omega \in \Delta_2$. Thus, $A \cap B = \cup \{A_\kappa \cap B_\omega : \kappa \in \Delta_1, \omega \in \Delta_2\}$. Since $A_\kappa \cap B_\omega$ is γ -regular open, $A \cap B$ is a γ - δ -open set. ■

Remark 3.23 It is clear that γ - δ -open sets in (X, τ) form a topology $\tau_{\gamma\delta}$ on X weaker than τ_γ for which the γ -regular open sets of X form a base.

Lemma 3.24 *If A is a γ - β -open set in a topological space (X, τ) , then $\tau_\gamma\text{-cl}(A) = \tau_\gamma\text{-cl}_\delta(A)$.*

Proof. Let A be a γ - β -open set. Suppose that $x \notin \tau_\gamma\text{-cl}(A)$. Then there exists an γ -open set U containing x such that $U \cap A = \emptyset$. We have $U \cap \tau_\gamma\text{-cl}(A) = \emptyset$. This implies that $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(U)) \cap \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(A))) = \emptyset$. Since A is a γ - β -open set, then $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(U)) \cap A = \emptyset$. Thus, $x \notin \tau_\gamma\text{-cl}_\delta(A)$ and $\tau_\gamma\text{-cl}(A) \supset \tau_\gamma\text{-cl}_\delta(A)$. On the other hand, we have $\tau_\gamma\text{-cl}(A) \subset \tau_\gamma\text{-cl}_\delta(A)$. Hence, we obtain $\tau_\gamma\text{-cl}(A) = \tau_\gamma\text{-cl}_\delta(A)$. ■

Lemma 3.25 *If A is a γ -semiopen set in a topological space (X, τ) , then $\tau_\gamma\text{-cl}(A) = \tau_\gamma\text{-cl}_\delta(A)$.*

Proof. Let A be a γ -semiopen set. We have $\tau_\gamma\text{-cl}(A) \subset \tau_\gamma\text{-cl}_\delta(A)$. Suppose that $x \notin \tau_\gamma\text{-cl}(A)$. Then there exists a γ -open set U containing x such that $U \cap A = \emptyset$. We have $U \cap \tau_\gamma\text{-Int}(A) = \emptyset$. This implies that $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(U)) \cap \tau_\gamma\text{-cl}(\tau_\gamma\text{-Int}(A)) = \emptyset$. Since A is a γ -semiopen set, then $\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(U)) \cap A = \emptyset$. Thus, $x \notin \tau_\gamma\text{-cl}_\delta(A)$ and $\tau_\gamma\text{-cl}(A) \supset \tau_\gamma\text{-cl}_\delta(A)$. Hence $\tau_\gamma\text{-cl}(A) = \tau_\gamma\text{-cl}_\delta(A)$. ■

Theorem 3.26 *For a topological space (X, τ) , the following properties are equivalent:*

- (i) X is τ_γ -extremally disconnected.
- (ii) $\tau_\gamma\text{-cl}(A) \in \tau_\gamma$ for every $A \in \gamma SO(X)$.
- (iii) $\tau_\gamma\text{-cl}(A) \in \tau_\gamma$ for every $A \in \gamma PO(X)$.
- (iv) $\tau_\gamma\text{-cl}(A) \in \tau_\gamma$ for every $A \in \gamma RO(X)$.

Theorem 3.27 *For a topological space (X, τ) , the following properties are equivalent:*

- (i) X is τ_γ -extremally disconnected.
- (ii) $\tau_\gamma\text{-cl}_\delta(A) \in \tau_\gamma$ for every $A \in \gamma SO(X)$.
- (iii) $\tau_\gamma\text{-cl}_\delta(A) \in \tau_\gamma$ for every $A \in \gamma PO(X)$.
- (iv) $\tau_\gamma\text{-cl}_\delta(A) \in \tau_\gamma$ for every $A \in \gamma RO(X)$.

Proof. The proof follows from Theorems 3.22, 3.26 and Lemmas 3.24, 3.25. ■

4. Almost (γ, β) -continuous functions

Definition 4.1 A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be:

- (i) almost (γ, β) -continuous at a point $x \in X$ if, for each β -open subset V in Y containing $f(x)$, there exists a γ -open set U of X containing x such that $f(U) \subset \tau_\beta\text{-Int}(\tau_\beta\text{-cl}(V))$;
- (ii) almost (γ, β) -continuous, if it has this property at each point of X .

Remark 4.2 Almost (γ, β) -continuity implies (γ, β) -continuity. But the converse is not true in general as the following examples shows.

Example 4.3 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $\sigma = \{\emptyset, \{a\}, \{a, b\}, X\}$. Define a function $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = b$, $f(b) = c$ and $f(c) = a$. Then f is almost (id, id) -continuous but not (id, id) -continuous, where "id" denotes the identity operator.

Theorem 4.4 *For a function $f : (X, \tau) \rightarrow (Y, \sigma)$, the following statements are equivalent:*

- (i) f is almost (γ, β) -continuous at $x \in X$;
- (ii) $x \in \tau_\gamma\text{-Int}(f^{-1}(\tau_\gamma\text{-Int}(\tau_\gamma\text{-cl}(V))))$ for every β -open set V of Y containing $f(x)$;

- (iii) $x \in \tau_\gamma\text{-Int}(f^{-1}(V))$ for every β -regular open set V of Y containing $f(x)$;
- (iv) For any β -regular open set V containing $f(x)$, there exists a γ -open set U containing x such that $f(U) \subset V$.

Proof. (i) $\Rightarrow$ (ii): Let V be any β -open set V of Y containing $f(x)$. By (i), there exists a γ -open set U of X containing x such that $f(U) \subset \tau_\beta\text{-Int}(\tau_\beta\text{-cl}(V))$. Since $x \in U \subset f^{-1}(\tau_\beta\text{-Int}(\tau_\beta\text{-cl}(V)))$, we have $x \in \tau_\gamma\text{-Int}(f^{-1}(\tau_\beta\text{-Int}(\tau_\beta\text{-cl}(V))))$. (ii) $\Rightarrow$ (iii): Let V be any β -regular open set V of Y containing $f(x)$. Then since $V = \tau_\beta\text{-Int}(\tau_\beta\text{-cl}(V))$, by (ii), we have $x \in \tau_\gamma\text{-Int}(f^{-1}(V))$. (iii) $\Rightarrow$ (iv): Let V be any β -regular open set of Y containing $f(x)$. From (iii), there exists a γ -open set U containing x such that $U \subset f^{-1}(V)$. Hence we have (iv). (iv) $\Rightarrow$ (i): Let V be any β -open set V of Y containing $f(x)$. Then $f(x) \in V \subset \tau_\beta\text{-Int}(\tau_\beta\text{-cl}(V))$. Since $\tau_\beta\text{-Int}(\tau_\beta\text{-cl}(V))$ is β -regular open, by (iv), there exists a γ -open set U containing x such that $f(U) \subset \tau_\beta\text{-Int}(\tau_\beta\text{-cl}(V))$. Hence f is almost (γ, β) -continuous at $x \in X$. ■

Theorem 4.5 For a function $f : (X, \tau) \rightarrow (Y, \sigma)$, the following statements are equivalent:

- (i) f is almost (γ, β) -continuous;
- (ii) $f^{-1}(V)$ is γ -open for every β -regular open set V of Y ;
- (iii) $f^{-1}(V)$ is γ -closed for every β -regular closed set V of Y ;
- (iv) $f(\tau_\gamma\text{-cl}(A)) \subset \tau_\gamma\text{-cl}_\delta(f(A))$ for every subset A of X ;
- (v) $\sigma_\beta\text{-cl}(f^{-1}(B)) \subset f^{-1}(\tau_\gamma\text{-cl}_\delta(B))$ for every subset B of Y ;
- (vi) $f^{-1}(F)$ is γ -open for every β - δ -open set F of Y ;
- (vii) $f^{-1}(V)$ is γ -closed for every β - δ -closed set V of Y .

Proof. (i) $\Rightarrow$ (ii): Clear. (ii) $\Rightarrow$ (iii): Let $F \in \beta RC(Y)$. Then $Y \setminus F \in \beta RO(Y)$. Take $x \in f^{-1}(Y \setminus F)$, then $f(x) \in Y \setminus F$ and since f is almost (γ, β) -continuous, there exists a β open set W_x of X such that $x \in W_x$ and $f(W_x) \subset Y \setminus F$. Then $x \in W_x \subset f^{-1}(Y \setminus F)$ so that $f^{-1}(Y \setminus F) = \bigcup_{x \in f^{-1}(Y \setminus F)} W_x$. Since any union of γ -open sets is γ -open, $f^{-1}(Y \setminus F)$ is γ -open in X and hence $f^{-1}(F) \in \beta RC(X)$. (iii) $\Rightarrow$ (iv): Let A be a subset of X . Since $\sigma_\beta\text{-cl}_\delta(f(A))$ is β - δ -closed in Y , it is equal to $\bigcap \{F_\alpha : F_\alpha \text{ is } \beta\text{-regular closed in } Y, \alpha \in \Lambda\}$, where Λ is an index set. From (iii), we have $A \subset f^{-1}(\tau_\gamma\text{-cl}_\delta(f(A))) = \bigcap \{f^{-1}(F_\alpha) : \alpha \in \Lambda\} \in \gamma RC(X)$ and hence $\tau_\gamma\text{-cl}(A) \subset f^{-1}(\sigma_\beta\text{-cl}_\delta(f(A)))$. Therefore, we obtain $f(\tau_\gamma\text{-cl}(A)) \subset \sigma_\beta\text{-cl}_\delta(f(A))$. (iv) $\Rightarrow$ (v): Set $A = f^{-1}(B)$ in (iv), then $f(\tau_\gamma\text{-cl}(f^{-1}(B))) \subset \sigma_\beta\text{-cl}_\delta(f(f^{-1}(B))) \subset \sigma_\beta\text{-cl}_\delta(B)$ and hence $\tau_\gamma\text{-cl}(f^{-1}(B)) \subset f^{-1}(\sigma_\beta\text{-cl}_\delta(B))$. (v) $\Rightarrow$ (vi): Let F be β - δ -closed set of Y , then $\tau_\gamma\text{-cl } f^{-1}(F) \subset f^{-1}(F)$ so $f^{-1}(F) \in \gamma RC(X)$. (vi) $\Rightarrow$ (vii): Let V be β - δ -open set of Y , then $Y \setminus V$ is β - δ -closed set in Y . This gives $f^{-1}(Y \setminus V) \in \gamma RC(X)$ and hence $f^{-1}(V) \in \beta O(X)$. (viii) $\Rightarrow$ (i): Let V be any β -regular open set of Y . Since V is β - δ -open in Y , then $f^{-1}(V) \in \tau_\gamma$ and hence from $f(f^{-1}(V)) \subset V = \tau_\beta\text{-Int}(\tau_\beta\text{-cl}(V))$. Therefore, f is almost (γ, β) -continuous. ■

Theorem 4.6 *For a function $f : (X, \tau) \rightarrow (Y, \sigma)$, the following statements are equivalent:*

- (i) f is almost (γ, β) -continuous;
- (ii) $f^{-1}(V) \subset \tau_\gamma\text{-Int}(f^{-1}(\tau_\beta\text{-Int}(\tau_\beta\text{-cl}(V))))$ for every β -open set V in Y ;
- (iii) $\tau_\gamma\text{-cl}(f^{-1}(\tau_\beta\text{-cl}(\tau_\beta\text{-Int}(F)))) \subset f^{-1}(F)$ for every β -closed set F in Y ;
- (iv) $\tau_\gamma\text{-cl}(f^{-1}(\tau_\beta\text{-cl}(\tau_\beta\text{-Int}(\tau_\beta\text{-cl}(B)))) \subset f^{-1}(\tau_\beta\text{-cl}(B))$ for every subset B in Y ;
- (v) $f^{-1}(\tau_\beta\text{-Int}(B)) \subset \tau_\gamma\text{-Int}(f^{-1}(\tau_\beta\text{-Int}(\tau_\beta\text{-cl}(\tau_\beta\text{-Int}(B))))$ for every subset B in Y .

Proof. (i) $\Rightarrow$ (ii): Let V be any β -open set in Y and $x \in f^{-1}(V)$. Then there exists a γ -open set U containing x such that $f(U) \subset \tau_\beta\text{-Int}(\tau_\beta\text{-cl}(V))$. This implies $x \in \tau_\gamma\text{-Int}(f^{-1}(\tau_\beta\text{-Int}(\tau_\beta\text{-cl}(V))))$. Hence $f^{-1}(V) \subset \tau_\gamma\text{-Int}(f^{-1}(\tau_\beta\text{-Int}(\tau_\beta\text{-cl}(V))))$. (ii) $\Rightarrow$ (iii): Let F be any β -closed set in Y . Then $f^{-1}(Y \setminus F) \subset \tau_\gamma\text{-Int}(f^{-1}(\tau_\beta\text{-Int}(\tau_\beta\text{-cl}(Y \setminus F)))) = X \setminus \tau_\gamma\text{-cl}(f^{-1}(\tau_\beta\text{-cl}(\tau_\beta\text{-Int}(F))))$. Hence $\tau_\gamma\text{-cl}(f^{-1}(\tau_\beta\text{-cl}(\tau_\beta\text{-Int}(F)))) \subset f^{-1}(F)$. (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (v): It is obvious. (v) $\Rightarrow$ (i): Let V be any β -regular open set in Y . Since $\tau_\beta\text{-Int}(\tau_\beta\text{-cl}(\tau_\beta\text{-Int}(V))) = V$, from (v), it follows $f^{-1}(V) \subset \tau_\beta\text{-Int}(f^{-1}(V))$ and so $f^{-1}(V) = \tau_\beta\text{-Int}(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is γ -open in X . By Theorem 4.5 (ii), f is almost (γ, β) -continuous. ■

Theorem 4.7 *For a function $f : (X, \tau) \rightarrow (Y, \sigma)$, the following statements are equivalent:*

- (i) f is almost (γ, β) -continuous;
- (ii) $\tau_\gamma\text{-cl}(f^{-1}(G)) \subset f^{-1}(\tau_\beta\text{-cl}(G))$ for every β - β -open set G of Y ;
- (iii) $\tau_\gamma\text{-cl}(f^{-1}(G)) \subset f^{-1}(\tau_\beta\text{-cl}(G))$ for every β -semiopen set G of Y ;
- (iv) $\tau_\gamma\text{-cl}(f^{-1}(G)) \subset f^{-1}(\tau_\beta\text{-cl}(G))$ for every β -preopen set G of Y .

Proof. (i) $\Rightarrow$ (ii): Let G be any β - β -open set of Y . Since $\tau_\beta\text{-cl}(G)$ is β -regular closed, $\tau_\gamma\text{-cl}(f^{-1}(\tau_\beta\text{-cl}(G))) = f^{-1}(\tau_\beta\text{-cl}(G))$. Thus, $\tau_\gamma\text{-cl}(f^{-1}(G)) \subset \tau_\gamma\text{-cl}(f^{-1}(\tau_\beta\text{-cl}(G))) = f^{-1}(\tau_\beta\text{-cl}(G))$. (ii) $\Rightarrow$ (iii): It is obvious since every β -semiopen set is β - β -open. (iii) $\Rightarrow$ (i): Let F be any β -regular closed set of Y ; then since F is β -semiopen, we have $\tau_\gamma\text{-cl}(f^{-1}(G)) \subset f^{-1}(\tau_\beta\text{-cl}(G)) = f^{-1}(F)$. Thus, from Theorem 4.5 (iii), f is almost (γ, β) -continuous. (i) $\Rightarrow$ (iv): Let V be any β -preopen set of Y ; then $V \subset \tau_\beta\text{-Int}(\tau_\beta\text{-cl}(V))$ and $\tau_\beta\text{-Int}(\tau_\beta\text{-cl}(V))$ is β -regular open. By Theorem 4.5 (ii), $f^{-1}(\tau_\beta\text{-Int}(\tau_\beta\text{-cl}(V))) = \tau_\gamma\text{-Int}(f^{-1}(\tau_\beta\text{-Int}(\tau_\beta\text{-cl}(V))))$. Thus, we have $f^{-1}(V) \subset f^{-1}(\tau_\beta\text{-Int}(\tau_\beta\text{-cl}(V))) = \tau_\gamma\text{-Int}(f^{-1}(\tau_\beta\text{-Int}(\tau_\beta\text{-cl}(V))))$. (iv) $\Rightarrow$ (i): Let V be any β -regular open set of Y ; then since V is β -preopen and $f^{-1}(V) \subset \tau_\gamma\text{-cl}(f^{-1}(\tau_\beta\text{-Int}(\tau_\beta\text{-cl}(V)))) = \tau_\gamma\text{-Int}(f^{-1}(V))$. Hence by Theorem 4.5 (ii), f is almost (γ, β) -continuous. ■

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LIE-ADMISSIBLE HYPERALGEBRAS

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Abstract. We review Albert's axioms of Lie-admissible algebras, Santilli's geno- and hyper-mathematics with a Lie-admissible structure, and Vougiouklis H_v -hyperstructures; we then introduce Lie-admissible hyperstructures; and point out their expected relevance for biological structures and other fields.

Keywords: hyperstructures, H_v -structures, hopes, ∂ -hopes.

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1. Introduction

In 1948, A.A. Albert [1] defined a generally non associative algebra U with elements $a, b, c, \dots$, and product ab over a field of characteristic zero as being *Lie-admissible* (*Jordan-admissible*) when the attached antisymmetric algebra U^- (attached symmetric algebra U^+) which is the same vector space as U equipped with the product $[a, b] = ab - ba$ (the product $\{a, b\} = ab + ba$) verifies all axioms of a Lie algebras (Jordan algebra). In the same paper [1], Albert identified the *flexible algebras* with product

$$(1.1) \quad (A, B) = \lambda AB - (1 - \lambda)BA,$$

(where $A, B, \dots$ are Hermitean n -dimensional matrices, λ is a scalar and AB is the conventional associative product) as a realization of Lie-admissible and Jordan-admissible algebras.

In 1967, R.M. Santilli [2] noted that the algebras with product (1.1) do not admit Lie algebras in their classification and, consequently, cannot be used for the construction of covering theories. Consequently, Santilli introduced the product

$$(1.2) \quad \begin{aligned} (A, B) &= \lambda AB - \mu BA = \alpha(AB - BA) + \beta(AB + BA), \\ \lambda &= \alpha + \beta, \mu = \beta - \alpha, \end{aligned}$$

(where $\lambda, \mu, \lambda \pm \mu$ are non-null scalars) which is jointly Lie-admissible and Jordan-admissible while admitting Lie algebras in their classification.

By recalling that the theories based on Lie algebras are reversible over time because of the anti-Hermiticity of the Lie product $[A, B] = -[A, B]^\dagger$, Santilli introduced product Lie-admissible product (1.2) to break the anti-Hermiticity of the Lie product for the intent of initiating quantitative studies on processes that are irreversible over time via a covering Lie-admissible theory admitting reversible Lie processes as particular cases. For that scope, Santilli [3], [4] introduced in 1968 the following infinitesimal and finite generalizations of Heisenberg equations

$$(1.3a) \quad i \frac{dA}{dt} = (A, H) = \lambda AH - \mu HA,$$

$$(1.3b) \quad \begin{aligned} A(t) &= U(t)A(0)V(t)^\dagger = e^{Ht\mu i} A(0)e^{-i\lambda tH}, \\ U &= e^{Ht\mu i}, \quad V = e^{i\lambda tH}UV^\dagger \neq I, \end{aligned}$$

where H is the Hamiltonian.

By noting that equation (1.3b) is non-unitary, thus not invariant under the time evolution, Santilli introduced in 1978 [5], [6] the following most general known realization of products that are jointly lie-admissible and Jordan-admissible

$$\begin{aligned} (A, B) &= ARB - BSA = (ATB - BTA) + \{AWB + BWA\} = \\ &= [A, B]^* + \{A, B\}^* = (ATH - HTA) + \{AWH + HWA\}, \\ R &= T + W, S = W - T, \end{aligned} \quad (1.4)$$

where $R, S, R \pm S$ are now non singular operators. In this broader realization, the attaches product $[A, B]^* = ATB - BTA$ and $\{A, B\}^* = AWB + BWA$ still verify the Lie and Jordan axioms, respectively, but characterize broader algebras called *Lie-Santilli and Jordan-Santilli isoalgebras*, where the prefix "iso" is interpreted in the Greek sense of preserving Lie axioms.

In the same papers [5], [6], Santilli introduced the most general possible broadening of Heisenberg's equations in their infinitesimal and finite form with a Lie-isotopic and Jordan-isotopic structure defined over a field of characteristic zero

$$(1.5a) \quad i \frac{dA}{dt} = (A, H) = ARH - BSA,$$

$$(1.5b) \quad \begin{aligned} A(t) &= U(t)A(0)V(t)^\dagger = e^{HSti}A(0)e^{-itRH}, \\ U &= e^{HSti}, \quad V = e^{HRti}, \quad UV^\dagger \neq I, \end{aligned}$$

and assumed them as the foundation of a Lie-admissible covering of quantum mechanics for irreversible processes proposed under the name of *hadronic mechanics* [6].

In the 1990s, Santilli noted that equation (1.5) are indeed the most general possible equations with a Lie-isotopic-admissible and Jordan-isotopic-admissible structure that persist under arbitrary (non-singular) non-unitary transforms. However, the operators R and S are not invariant under time evolution (1.5b). But said operators represent the irreversible component of the systems considered. Therefore, as proposed in 1978, generalized dynamical equations (1.5) do not admit consistent physical applications.

In order to achieve the invariance over time of the R and S operators, Santilli had to build a new mathematics, today known as *Santilli genomathematics* (where the prefix "geno" is now intended in the Greek sense of inducing new axioms), which can be defined as a mathematics admitting the generalized units $1^> = 1/S$ ($<1 = 1/R$) and corresponding ordered multiplications $A > B = ASB$ ($A < B = ARB$) at all levels, including numeric field, functional analysis, differential calculus, etc, for the representation of motion forward (backward) in time and basic rules

$$1^> = 1/S, A > B = ASB, \quad 1^> > A = A > 1^> = A, \quad (1.6a)$$

$$<1 = 1/R, A < B = ARB, \quad <1 < A = A < <1 = A, \quad (1.6b)$$

Real, complex and quaternionic numbers, when equipped with the above forward and, separately, backward generalized units and products, were shown in Ref. [7] of 1993 to verify the abstract axioms of a numeric field and are now called *Santilli forward and backward genoreal, genocomplex and genoquaternionic numbers*. Corresponding forward and backward generalization of functional analysis, differential calculus, and other aspects of applied mathematics were introduced by Santilli in memoir [8] of 1996, The invariance of the R and S operator under a Lie-admissible and Jordan-admissible time evolution (1.5) was first proved in Ref. [9] of 1997.

Santilli additionally noted that genomathematics is indeed effective for the initiation of quantitative studies on irreversible processes while admitting, for the first time, a direct connection to thermodynamics [10-13], but said forward and backward genomathematics are insufficient for quantitative studies of biological structure because they are *single-valued* (in the sense that each ordered products $A > B$ and $A < B$ yield one single result). To initiate quantitative studies on complex biological structures such as the DNA, Santilli introduced in 1996 [8] the following ordered *multi-valued* realization of genomathematics today known as *Santilli multi-valued, forward and backward hypermathematics*

$$\hat{1}^> = \{1_1^>, 1_2^>, 1_3^>, \dots, 1_n^>\} = \{1/S_1, 1/S_2, \dots, 1/S_n\}$$

$$\begin{aligned}
(1.7a) \quad & 1_k^> = 1/S_k < 0, k = 1, 2, \dots, n \\
(1.7b) \quad & A \hat{>} B = AS_1B + AS_2B + AS_3B + \dots + AS_nB, \\
(1.7c) \quad & <\hat{1} = \{<\hat{1}_1, <\hat{1}_2, <\hat{1}_3, \dots, <\hat{1}_n\} = \{R_1, R_2, R_3, \dots, R_n\}, \\
(1.7d) \quad & A \hat{<} B = AR_1B + AR_2B + AR_3B + \dots + AR_nB,
\end{aligned}$$

whose multi-valued character is evident.

It should be indicate that Santilli forward and backward hypermathematics *are different* than hyperstructures (see, e.g., Refs. [14]–[19]) for numerous reasons, such as: the former are based on classical operations while the latter are characterized by hyper operations; the former are formulated over rings verifying the axioms of a numeric field, while the latter are not; etc.

Jointly, T. Vougiouklis [20]–[30] introduced the most general known formulation of hyperstructures, today known as *Vougiouklis H_v hyperstructures*, which are relevant for the study of irreversible processes, such as biological structure, since Santilli forward and backward genomathematics are expected to show limitations due to their formulations via classical operations. Therefore, in this paper, we shall reformulated Santilli hypermathematics in terms of Vougiouklis.

2. The hyperstructures

In this section we present an introduction on the theory of hyperstructures.

The largest class of hyperstructures were introduced in 1990 [24] and are called H_v -structures. They satisfy the weak axioms where the non-empty intersection replaces the equality. Some basic definitions are the following:

In a set H equipped with a hyperoperation (abbreviation as *hope*)

$$\cdot : H \times H \rightarrow P(H) - \{\emptyset\},$$

we have the following properties in abbreviated notation

WASS, the *weak associativity*: $(xy)z \cap x(yz) \neq \emptyset, \forall x, y, z \in H$ and

COW, the *weak commutativity*: $xy \cap yx \neq \emptyset, \forall x, y \in H$.

The hyperstructure $(H, \cdot)$ is called H_v -semigroup if it is *WASS*, and it is called H_v -group if it is a reproductive H_v -semigroup, i.e.,

$$xH = Hx = H, \forall x \in H.$$

As it is well known, in the classical theory, F. Marty stated in 1934 that the quotient of a group with respect to any subgroup is a hypergroup. Also, the quotient of a group with respect to any partition (or any equivalence relation) is an H_v -group illustrating the motivation to introduce the H_v -structures [24], [26].

In an H_v -semigroup the powers of an element $h \in H$ are defined as follows: $h^1 = \{h\}$, $h^2 = h \cdot h$, ..., $h^n = h \circ h \circ \dots \circ h$, where $(\circ)$ denotes the *n-ary circle hope*, i.e., take the union of hyperproducts, n times, with all possible patterns of parentheses put on them. An H_v -semigroup $(H, \cdot)$ is called *cyclic of period s* , if there exists an element h , called *generator*, and a natural number s , the minimum

one, such that $H = h^1 \cup h^2 \dots \cup h^s$. We define analogously the cyclicity for the infinite period [13]. If there is an element h and a natural number s , the minimum one, such that $H = h^s$, then $(H, \cdot)$ is called *single-power cyclic of period s* .

More complicated hyperstructures can be defined: in a similar way, such as $(R, +, \cdot)$ is called *H_v -ring* if $(+)$ and $(\cdot)$ are WASS, the reproduction axiom is valid for $(+)$, and $(\cdot)$ is *weak distributive* with respect to $(+)$, i.e.,

$$x(y + z) \cap (xy + xz) \neq \emptyset, (x + y)z \cap (xz + yz) \neq \emptyset, \forall x, y, z \in R.$$

Let $(R, +, \cdot)$ be an H_v -ring, $(M, +)$ be a COW H_v -group and there exists an external hope

$$\cdot : R \times M \rightarrow P(M) : (a, x) \rightarrow ax$$

such that $\forall a, b \in R$ and $\forall x, y \in M$ we have

$$a(x + y) \cap (ax + ay) \neq \emptyset, (a + b)x \cap (ax + bx) \neq \emptyset, (ab)x \cap a(bx) \neq \emptyset,$$

then M is called an *H_v -module* over F . In the case of an H_v -field F , which is defined later, instead of an H_v -ring R , then the *H_v -vector space* is defined.

The main tool to study hyperstructures is the fundamental relation. In 1970 M. Koscas defined in hypergroups the relation β and its transitive closure β^* . This relation connects the hyperstructures with the corresponding classical structures and is defined in H_v -groups as well. T. Vougiouklis [26, 28] introduced the γ^* and ϵ^* relations, which are defined in H_v -rings and H_v -vector spaces, respectively. Vougiouklis also named all these relations β^* , γ^* and ϵ^* *Fundamental Relations* because they play a very important role to the study of hyperstructures especially in their representation theory.

Definition 2.1. The fundamental relations β^* , γ^* and ϵ^* , are defined in H_v -groups, H_v -rings and H_v -vector space, respectively, as the smallest equivalences so that the quotient would be group, ring and vector space, respectively. ([23], [24], [26], [28])

To specifying the above motivation, we note the following: Let $(G, \cdot)$ be a group and R be an equivalence relation (or a partition) in G , then $(G/R, \cdot)$ is an H_v -group, therefore we have the quotient $(G/R, \cdot)/\beta^*$ which is a group, the fundamental one. Remark that the classes of the fundamental group $(G/R, \cdot)/\beta^*$ are a union of some of the R -classes. Otherwise, the $(G/R, \cdot)/\beta^*$ has elements classes of G where they form a partition which classes are larger than the classes of the original partition R .

The way to find the fundamental classes is given by the following:

Theorem 2.2. Let $(H, \cdot)$ be an H_v -group and denote by U the set of all finite products of elements of H . We define the relation β in H by setting $x\beta y$ iff $\{x, y\} \subset u$ where $u \in U$. Then β^* is the transitive closure of β .

Analogous to the above theorem, in the case of an H_v -ring [26], is the following:

Theorem 2.3. *Let $(R, +, \cdot)$ be an H_v -ring. Denote by U the set of all finite polynomials of elements of $\mathbf{R}$. We define the relation γ in $\mathbf{R}$ as follows:*

$$x\gamma y \text{ iff } \{x, y\} \subset u \text{ where } u \in U.$$

Then the relation γ^* is the transitive closure of the relation γ .

An element is called *single* if its fundamental class is singleton [26].

Fundamental relations are used for general definitions. Thus, an H_v -ring $(R, +, \cdot)$ is called **H_v -field** if R/γ^* is a field. The elements of an H_v -field are called **H_v -numbers** or hyper-numbers

Let $(H, \cdot), (H, *)$ be H_v -semigroups defined on the same set H . The hope $(\cdot)$ is called *smaller* than the hope $(*)$, and $(*)$ *greater* than $(\cdot)$, iff there exists an

$$f \in \text{Aut}(H, *) \text{ such that } xy \subset f(x * y), \forall x, y \in H.$$

Then we write $\cdot \leq *$ and we say that $(H, *)$ contains $(H, \cdot)$. If $(H, \cdot)$ is a structure then it is called *basic structure* and $(H, *)$ is called H_b -structure.

Theorem 2.4. (The Little Theorem). *Greater hopes than the ones which are WASS or COW, are also WASS or COW, respectively.*

This Theorem leads to a partial order on H_v -structures and mainly to a correspondence between hyperstructures and posets. Therefore we can obtain an extreme large number of H_v -structures just putting more elements on any result. Using the partial ordering with the fundamental relations, one can give several definitions to obtain constructions used in several applications [28]:

Let $(H, \cdot)$ be hypergroupoid. We *remove* $h \in H$, if we consider the restriction of $(\cdot)$ in the set $H - \{h\}$. $\underline{h} \in H$ *absorbs* $h \in H$ if we replace h by $\underline{h}$ and h does not appear in the structure. $\underline{h} \in H$ *merges* with $h \in H$, if we take as product of any $x \in H$ by $\underline{h}$, the union of the results of x with both $h, \underline{h}$, and consider h and $\underline{h}$ as one class with representative $\underline{h}$.

A class of H_v -structures is the following:

An H_v -structure is called *very thin* iff all hopes are operations except one, which has all hyperproducts singletons except only one, which is a subset of cardinality more than one. Therefore, in a very thin H_v -structure in a set H there exists a hope $(\cdot)$ and a pair $(a, b) \in H^2$ for which $ab = A$, with $\text{card}A > 1$, and all the other products, with respect to any other hopes (so they are operations), are singletons.

A large class of H_v -structures is the following [29]:

Let $(G, \cdot)$ be groupoid (resp., hypergroupoid) and $f : G \rightarrow G$ be a map. We define a hope (∂) , called *theta-hope*, we write ∂ -hope, on G as follows

$$x\partial y = \{f(x) \cdot y, x \cdot f(y)\}, \forall x, y \in G. \text{ (resp. } x\partial y = (f(x) \cdot y) \cup (x \cdot f(y)), \forall x, y \in G)$$

If $(\cdot)$ is commutative then ∂ is commutative. If $(\cdot)$ is COW, then ∂ is COW.

Let $(G, \cdot)$ be a groupoid (or hypergroupoid) and $f : G \rightarrow P(G) - \{\emptyset\}$ be any multivalued map. We define the (∂) , on G as follows

$$x\partial y = (f(x) \cdot y) \cup (x \cdot f(y)), \quad \forall x, y \in G$$

Let $(G, \cdot)$ be a groupoid, $f_i : G \rightarrow G, i \in I$, be a set of maps on G . The

$$f_{\cup} : G \rightarrow \mathbf{P}(G) : f_{\cup}(x) = \{f_i(x) | i \in I, \}$$

is the union of $f_i(x)$. We have the union ∂ -hope (∂) , on G if we take $f_{\cup}(x)$. If $\underline{f} \equiv f \cup (id)$, then we have the $b - \partial - hope$.

Motivation for the definition of the theta-hope is the map *derivative* where only the multiplication of functions can be used. The basic property is that if $(G, \cdot)$ is a semigroup then for every f , the (∂) is WASS.

Another well known and large class of hopes is given as follows [22], [26]:

Let $(G, \cdot)$ be a groupoid then for every $P \subset G, P \neq \emptyset$, we define the following hopes called P -hopes: for all $x, y \in G$

$$\underline{P} : x\underline{P}y = (xP)y \cup x(Py), \quad \underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP), \quad \underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy).$$

The $(G, \underline{P}), (G, \underline{P}_r)$ and $(G, \underline{P}_l)$ are called P -hyperstructures. The most usual case is if $(G, \cdot)$ is semigroup, then $x\underline{P}y = (xP)y \cup x(Py) = xPy$ and $(G, \underline{P})$ is a semihypergroup but we do not know about $(G, \underline{P}_r)$ and $(G, \underline{P}_l)$. In some cases, depending on the choice of P , the $(G, \underline{P}_r)$ and $(G, \underline{P}_l)$ can be associative or WASS.

A generalization of P -hopes, introduced by Davvaz, Santilli, Vougiouklis in [17], [18] is the following:

Construction 2.5. Let $(G, \cdot)$ be an abelian group and P any subset of G with more than one elements. We define the hope $\times P$ as follows:

$$x \times_p y = \begin{cases} x \times_P y = x \cdot P \cdot y = \{x \cdot h \cdot y | h \in P\} & \text{if } x \neq e \text{ and } y \neq e \\ x \cdot y & \text{if } x = e \text{ and } y = e \end{cases}$$

we call this hope P_e -hope. The hyperstructure $(G, \times_p)$ is an abelian H_v -group.

H_v -structures are used in Representation Theory of H_v -groups which can be achieved either by generalized permutations or by H_v -matrices [25], [26]. Representations by generalized permutations can be faced by translations. H_v -matrix is called a matrix if has entries from an H_v -ring. The hyperproduct of H_v -matrices is defined in a usual manner. The problem of the H_v -matrix representations is the following:

Definition 2.6. Let $(H, \cdot)$ be H_v -group, find an H_v -ring R , a set

$$M_R = \{(a_{ij}) \mid a_{ij} \in R\}$$

and a map

$$T : H \rightarrow M_R : h \mapsto T(h) \text{ such that } T(h_1 h_2) \cap T(h_1)T(h_2) \neq \emptyset, \forall h_1, h_2 \in H.$$

Then, the map T is called H_v -matrix representation.

If the $T(h_1 h_2) \subset T(h_1)(h_2), \forall h_1, h_2 \in H$ is valid, then T is called *inclusion representation*.

If $T(h_1 h_2) = T(h_1)(h_2) = \{T(h) | h \in h_1 h_2\}, \forall h_1, h_2 \in H$, then T is called *good representation* and then an induced representation T^* for the hypergroup algebra is obtained.

If T is one to one and good then it is a *faithful representation*.

In the representations of H_v -groups there are two difficulties: To find an H_v -ring or an H_v -field and an appropriate set of H_v -matrices. Hopes on any type of ordinary matrices can be defined [17], they are called *helix hopes*.

Using several classes of H_v -structures one can face several representations. Some of those classes are as follows:

Definition 2.7. Let $\mathbf{M} = \mathbf{M}_{m \times n}$ be a module of $m \times n$ matrices over a ring $\mathbf{R}$ and $\mathbf{P} = \{P_i : i \in I\} \subseteq \mathbf{M}$. We define, a kind of, a $\mathbf{P}$ -hope $\underline{\mathbf{P}}$ on $\mathbf{M}$ as follows

$$\underline{\mathbf{P}} : \mathbf{M} \times \mathbf{M} \rightarrow \mathbf{P}(\mathbf{M}) : (A, B) \rightarrow \underline{APB} = \{AP_i^t B : i \in I\} \subseteq \mathbf{M}$$

where P^t denotes the transpose of the matrix P .

The hope $\underline{\mathbf{P}}$, which is a bilinear map, is a generalization of Rees' operation where, instead of one sandwich matrix, a set of sandwich matrices is used. The hope $\underline{\mathbf{P}}$ is strong associative and the inclusion distributivity with respect to addition of matrices

$$\underline{AP}(B + C) \subseteq \underline{APB} + \underline{APC} \text{ for all } A, B, C \text{ in } \mathbf{M}$$

is valid. Therefore, $(\mathbf{M}, +, \underline{\mathbf{P}})$ defines a multiplicative hyperring on non-square matrices. Multiplicative hyperring means that only the multiplication is a hope.

Definition 2.8. Let $\mathbf{M} = \mathbf{M}_{m \times n}$ be a module of $m \times n$ matrices over $\mathbf{R}$ and let us take sets $\mathbf{S} = \{s_k : k \in K\} \subseteq R, \mathbf{Q} = \{Q_j : j \in J\} \subseteq \mathbf{M}, \mathbf{P} = \{P_i : i \in I\} \subseteq \mathbf{M}$. Define three hopes as follows

$$\underline{\mathbf{S}} : R \times \mathbf{M} \rightarrow \mathbf{P}(\mathbf{M}) : (r, A) \rightarrow r\underline{\mathbf{S}}A = \{(rs_k)A : k \in K\} \subseteq \mathbf{M}$$

$$\underline{\mathbf{Q}}_+ : \mathbf{M} \times \mathbf{M} \rightarrow \mathbf{P}(\mathbf{M}) : (A, B) \rightarrow \underline{AQ}_+ B = \{A + Q_j + B : j \in J\} \subseteq \mathbf{M}$$

$$\underline{\mathbf{P}} : \mathbf{M} \times \mathbf{M} \rightarrow \mathbf{P}(\mathbf{M}) : (A, B) \rightarrow \underline{APB} = \{AP_i^t B : i \in I\} \subseteq \mathbf{M}$$

Then $(\mathbf{M}, \underline{\mathbf{S}}, \underline{\mathbf{Q}}_+, \underline{\mathbf{P}})$ is a hyperalgebra over $\mathbf{R}$ called *general matrix $\mathbf{P}$ -hyperalgebra*.

In a similar way, a generalization of this hyperalgebra can be defined if one considers an H_v -ring or an H_v -field instead of a ring and using H_v -matrices.

3. Lie-hyperalgebras

Since the algebras are defined on vector spaces, we now present the analogous to the above Theorem 2.3, on H_v -vector spaces. The proof is similar.

Theorem 3.1. *Let $(V, +)$ be an H_v -vector space over the H_v -field F . Denote by U the set of all expressions consisting of finite hopes either on F and V or the external hope applied on finite sets of elements of F and V . We define the relation ϵ in V as follows:*

$$x\epsilon y \text{ iff } \{x, y\} \subset u \text{ where } u \in U.$$

Then the relation ϵ^* is the transitive closure of the relation ϵ .

Proof. Let $\underline{\epsilon}$ be the transitive closure of ϵ , and denote by $\underline{\epsilon}(x)$ the class of the element x . First we prove that the quotient set $\mathbf{M}/\underline{\epsilon}$ is a module over $\mathbf{R}/\gamma^*$.

In $\mathbf{M}/\underline{\epsilon}$ the sum $(\oplus)$ and the product $(\otimes)$ using the γ^* classes in $\mathbf{R}$, are defined in the usual manner:

$$\begin{aligned} \underline{\epsilon}(x) \oplus \underline{\epsilon}(y) &= \{\underline{\epsilon}(z) : z \in \underline{\epsilon}(x) + \underline{\epsilon}(y)\}, \\ \gamma^*(a) \otimes \underline{\epsilon}(x) &= \{\underline{\epsilon}(z) : z \in \gamma^*(a) \cdot \underline{\epsilon}(x)\} \forall a \in R, x, y \in \mathbf{M}. \end{aligned}$$

Take $x' \in \underline{\epsilon}(x)$, $y' \in \underline{\epsilon}(y)$. Then we have $x' \underline{\epsilon} x$ iff $\exists x_1, \dots, x_{m+1}$ with $x_1 = x'$, $x_{m+1} = x$ and $u_1, \dots, u_m \in U$ such that $\{x_i, x_{i+1}\} \subset u_i$, $i = 1, \dots, m$, and $y' \underline{\epsilon} y$ iff $\exists y_1, \dots, y_{n+1}$ with $y_1 = y'$, $y_{n+1} = y$ and $v_1, \dots, v_n \in U$ such that $\{y_j, y_{j+1}\} \subset v_j$, $j = 1, \dots, n$. From the above, we obtain

$$\{x_i, x_{i+1}\} + y_1 \subset u_i + v_1, \quad i = 1, \dots, m-1,$$

$$x_{m+1} + \{y_j, y_{j+1}\} \subset u_m + v_j, \quad j = 1, \dots, n.$$

The sums

$$u_i + v_1 = t_i, \quad i = 1, \dots, m-1 \text{ and } u_m + v_j = t_{m+j-1}, \quad j = 1, \dots, n$$

are also elements of $\mathbf{U}$, therefore $t_k \in U$ for all $k \in \{1, \dots, m+n-1\}$.

Now, pick up elements $z_1, \dots, z_{m+n}$ such that

$$z_i \in x_i + y_1, \quad i = 1, \dots, n \text{ and } z_{m+j} \in x_{m+1} + y_{j+1}, \quad j = 1, \dots, n.$$

Therefore, using the above relations, we obtain $\{z_k, z_{k+1}\} \subset t_k$, $k = 1, \dots, m+n-1$. Thus, every element $z_1 \in x_1 + y_1 = x' + y'$ is $\underline{\epsilon}$ equivalent to every element $z_{m+n} \in x_{m+1} + y_{n+1} = a + b$. Thus, $\underline{\epsilon}(x) \oplus \underline{\epsilon}(y)$ is a singleton, so we can write

$$\underline{\epsilon}(x) \oplus \underline{\epsilon}(y) = \underline{\epsilon}(z) \text{ for all } z \in \underline{\epsilon}(x) + \underline{\epsilon}(y)$$

In a similar way, using the properties of γ^* in $\mathbf{R}$, one can prove that

$$\gamma^*(a) \otimes \underline{\epsilon}(x) = \underline{\epsilon}(z) \text{ for all } z \in \gamma^*(a) \cdot \underline{\epsilon}(x).$$

The WASS and the weak distributivity on $\mathbf{R}$ and $\mathbf{M}$ guarantee that the associativity and the distributivity are valid for the quotient $\mathbf{M}/\underline{\epsilon}$ over $\mathbf{R}/\gamma^*$. Therefore $\mathbf{M}/\underline{\epsilon}$ is a module over $\mathbf{R}/\gamma^*$.

Now, let σ be an equivalence relation in $\mathbf{M}$ such that $\mathbf{M}/\sigma$ is a module over $\mathbf{R}/\gamma^*$. Denote $\sigma(x)$ the class of x . Then $\sigma(x) \oplus \sigma(y)$ and $\gamma^*(a) \otimes \sigma(x)$ are singletons for all $a \in \mathbf{R}$ and $x, y \in \mathbf{M}$, i.e.,

$$\begin{aligned}\sigma(x) \oplus \sigma(y) &= \sigma(z) \quad \text{for all } z \in \sigma(x) + \sigma(y), \\ \gamma^*(a) \otimes \sigma(x) &= \sigma(z) \quad \text{for all } z \in \gamma^*(a) \cdot \sigma(x).\end{aligned}$$

Thus, we can write, for every $a \in R, x, y \in M$ and $\mathbf{A} \subset \gamma^*(a), \mathbf{X} \subset \sigma(x), \mathbf{Y} \subset \sigma(y)$,

$$\begin{aligned}\sigma(x) \oplus \sigma(y) &= \sigma(x + y) = \sigma(\mathbf{X} + \mathbf{Y}), \\ \gamma^*(a) \otimes \sigma(x) &= \sigma(ax) = \sigma(\mathbf{A} \cdot \mathbf{B})\end{aligned}$$

By induction, we extend these relations on finite sums and products. Thus, for every $u \in U$, we have the relation $\sigma(x) = \sigma(u)$ for all $x \in u$. Consequently,

$$x \in \epsilon(x) \text{ implies } x \in \sigma(x) \text{ for every } x \in \mathbf{M}.$$

But σ is transitively closed, so we obtain:

$$x' \in \underline{\epsilon}(x) \text{ implies } x' \in \sigma(x).$$

That means that $\underline{\epsilon}$ is the smallest equivalence relation on $\mathbf{M}$ such that $\mathbf{M}/\underline{\epsilon}$ is a module over $\mathbf{R}/\gamma^*$, i.e., $\underline{\epsilon} = \epsilon^*$. ■

The general definition of an H_v -Lie algebra was given in [30] as follows:

Definition 3.2. Let $(L, +)$ be an H_v -vector space over the H_v -field $(F, +, \cdot)$, $\phi : F \rightarrow F/\gamma^*$ the canonical map and $\omega_F = \{x \in F : \phi(x) = 0\}$, where 0 is the zero of the fundamental field F/γ . Similarly, let ω_L be the core of the canonical map $\phi' : L \rightarrow L/\epsilon^*$ and denote by the same symbol 0 the zero of L/ϵ^* . Consider the *bracket (commutator) hope*:

$$[,] : L \times L \rightarrow P(L) : (x, y) \rightarrow [x, y].$$

Then $\mathbf{L}$ is an H_v -Lie algebra over F if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e.,

$$\begin{aligned}[\lambda_1 x_1 + \lambda_2 x_2] \cap (\lambda_1 [x_1, y] + \lambda_2 [x_2, y]) &\neq \emptyset \\ [x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1 [x, y_1] + \lambda_2 [x, y_2]) &\neq \emptyset, \\ \forall x, x_1, x_2, y, y_1, y_2 \in L, \quad \lambda_1, \lambda_2 \in F\end{aligned}$$

(L2) $[x, x] \cap \omega_L \neq \emptyset, \quad \forall x \in L$

(L3) $([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \emptyset, \quad \forall x, y \in L$

This is a general definition thus one can use special cases in order to face problems in applied sciences.

Now, we can see theta hopes in H_v -vector spaces and H_v -Lie algebras:

Theorem 3.3. *Let $(\mathbf{A}, +, \cdot)$ be an algebra over the field F . Take any map $f : A \rightarrow A$, then the ∂ -hope on the Lie bracket $[x, y] = xy - yx$, is defined as follows*

$$x\partial y = \{f(x)y - f(y)x, f(x)y - yf(x), xf(y) - f(y)x, xf(y) - yf(x)\}.$$

Then $(\mathbf{A}, +, \partial)$ is an H_v -algebra over F , with respect to the ∂ -hopes on Lie bracket, where the weak anti-commutativity and the inclusion linearity is valid.

Remark that if we take the identity map $f(x) = x, \forall x \in A$, then $x\partial y = \{xy - yx\}$, thus we have not a hope and remains the same operation.

4. An application

During last decades hyperstructures have a variety of applications in other branches of mathematics and in many other sciences. These applications range from biomathematics -conchology, inheritance- and hadronic physics or on leptons to mention but a few. The hyperstructures theory is closely related to fuzzy theory; consequently, hyperstructures can now be widely applicable in industry and production, too. In several books and review papers [8], [10], [11], [13], [14], [15], [21], [26], [31] one can find numerous applications.

The Lie-Santilli theory on *isotopies* was born in 1970's to solve Hadronic Mechanics problems. Santilli proposed a "lifting" of the n -dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n -dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The *isofields* needed in this theory correspond into the hyperstructures were introduced by Santilli and Vougiouklis in 1996 [20] and they are called *e-hyperfields*. The H_v -fields can give e-hyperfields which can be used in the isotopy theory in applications as in physics or biology. We present in the following the main definitions and results restricted in the H_v -structures.

Definition 4.1. A hyperstructure $(H, \cdot)$ which contain a unique scalar unit e , is called e-hyperstructure. In an e-hyperstructure, we assume that for every element x , there exists an inverse x^{-1} , i.e. $e \in x \cdot x^{-1} \cap x^{-1} \cdot x$. Remark that the inverses are not necessarily unique.

Definition 4.2. A hyperstructure $(F, +, \cdot)$, where $(+)$ is an operation and $(\cdot)$ is a hope, is called e-hyperfield if the following axioms are valid:

1. $(F, +)$ is an abelian group with the additive unit 0,
2. $(\cdot)$ is WASS,
3. $(\cdot)$ is weak distributive with respect to $(+)$,

4. 0 is absorbing element: $0 \cdot x = x \cdot 0 = 0, \forall x \in F$,
5. exist a multiplicative scalar unit 1, i.e. $1 \cdot x = x \cdot 1 = x, \forall x \in F$,
6. for every $x \in F$ there exists a unique inverse x^{-1} , such that $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$.

The elements of an e-hyperfield are called *e-hypernumbers*. In the case that the relation $1 = x \cdot x^{-1} = x^{-1} \cdot x$ is valid, we say that we have a *strong e-hyperfield*.

Now, we present a general construction which is based on the partial ordering of the H_v -structures and on the Little Theorem.

Definition 4.3. *The Main e-Construction.* Given a group $(G, \cdot)$, where e is the unit, then we define in G , a large number of hopes $(\otimes)$ as follows:

$$x \otimes y = \{xy, g_1, g_2, \dots\}, \forall x, y \in G - \{e\}, \text{ and } g_1, g_2, \dots \in G - \{e\}$$

$g_1, g_2, \dots$ are not necessarily the same for each pair (x, y) . Then $(G, \otimes)$ becomes an H_v -group, actually is an H_b -group which contains the $(G, \cdot)$. The H_v -group $(G, \otimes)$ is an e-hypergroup. Moreover, if for each x, y such that $xy = e$, so we have $x \otimes y = xy$, then $(G, \otimes)$ becomes a strong e-hypergroup

The proof is immediate since we enlarge the results of the group by putting elements from G and applying the Little Theorem. Moreover one can see that the unit e is a unique scalar and for each x in G , there exists a unique inverse x^{-1} , such that $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$ and if this condition is valid then we have $1 = x \cdot x^{-1} = x^{-1} \cdot x$. So the hyperstructure $(G, \otimes)$ is a strong e-hypergroup.

The above main e-construction gives an extremely large number of e-hopes. These e-hopes can be used in the several more complicate hyperstructures to obtain appropriate e-hyperstructures.

One can see that we can have more strict hopes. The reason we gave the above example is to see that there is a large variety of e-hyperstructures we can construct from given classical structures. One can see that some classes of e-hyperstructures and their properties and results connected them with the classical theory. The representation theory and the Lie algebras as well as in hypermatrix theory large classes of e-hyperstructures appear and can offer to Lie-Santilli algebraic theory models to represent their theory

5. The Santilli's Lie-admissibility in hyperstructures

Now, we present the Santilli's admissibility in hyperstructures in a general form:

Definition 5.1. Let $(L, +)$ be an H_v -vector space H_v -field $(F, +, \cdot)$, $\phi : F \rightarrow F/\gamma^*$ the canonical map and $\omega_F = \{x \in F : \phi(x) = 0\}$, where 0 is the zero of the fundamental field F/γ . Similarly, let ω_L be the core of the canonical map $\phi' : L \rightarrow L/\epsilon^*$ and denote by the same symbol 0 the zero of L/ϵ^* . Consider the H_v -Lie algebra over $\mathbf{F}$ taking the bracket(commulator) hope:

$$[,] : \mathbf{L} \times \mathbf{L} \rightarrow \mathbf{P}(\mathbf{L}) : (x, y) \rightarrow [x, y]$$

Take any two subsets $\mathbf{R}, \mathbf{S} \subseteq \mathbf{L}$ then the **general Santilli's Lie-admissible hyperalgebra** is obtained by reformulating the Lie bracket into the hope:

$$[,]_{RS} : \mathbf{L} \times \mathbf{L} \rightarrow \mathbf{P}(\mathbf{L}) : [x, y]_{RS} = x\mathbf{R}y - y\mathbf{S}x.$$

Notice that $[x, y]_{RS}$ is not a single element but a set

$$[x, y]_{RS} = x\mathbf{R}y - y\mathbf{S}x = \{xry - ysx / r \in \mathbf{R} \text{ and } s \in \mathbf{S}\}.$$

Remark that this definition is, in some way, a generalization of the P -hopes.

Special cases, but not degenerate, are the "small" and "strict" ones:

(a) $\mathbf{R} = \mathbf{e}$. Then $[x, y]_{RS} = xy - y\mathbf{S}x = \{xy - ysx / s \in \mathbf{S}\}$

(b) $\mathbf{S} = \mathbf{e}$. Then $[x, y]_{RS} = x\mathbf{R}y - yx = \{xry - yx / r \in \mathbf{R}\}$

(c) $\mathbf{R} = \{r_1, r_2\}$ and $\mathbf{S} = \{s_1, s_2\}$. Then

$$[x, y]_{RS} = x\mathbf{R}y - y\mathbf{S}x = \{xr_1y - ys_1x, xr_1y - ys_2x, xr_2y - ys_1x, xr_2y - ys_2x\}$$

Since the above is the most general definition one can take special cases in order to obtain applications. Therefore if one take e-hyperstructures used in Lie-Santilli theory then the admissibility is transferred in an obvious reason. Finally using the fundamental structures the classical algebraic structures and the hyperstructures are connected.

Now, we can transfer the Santilli's admissibility problem, presented in the Introduction, into the hyperstructure theory. The Santilli's admissibility in the in the hyperstructure theory can be achieved in the following ways: (a) The use of an H_v -field instead of an ordinary field. (b) The replacement, or enlargement, of the single valued external or internal operations on vectors by multivalued ones. (c) The replacement of the selected elements $\mathbf{R}$ and $\mathbf{S}$ by sets of elements.

Therefore, (i) In equation (1.1), the hyperstructure form can be faced by using the elements λ from an H_v -field or the external operation λA could be a hope or, of course, both the above generalizations can be used.

(ii) In the realization presented by equation (1.4), the hyperstructure form can be achieved by using an H_v -field or by replace R and S (consequently, T and W) by sets of elements. In the later case, since they are P -hopes, one can use also the Construction 2.5 as well.

(iii) The case of equation (1.7) is already a generalization into the multivalued case. In fact, this generalization is an Hb -hope since it is an extension of an single valued operation. However, if we replace $S_1, S_2, \dots, S_n$ and $R_1, R_2, \dots, R_n$ by sets of elements, then we obtain enlarged hopes.

6. Concluding remarks

As it is well known, the correlation of two or more atoms of a DNA can yield entire organs during the growth of a biological structure with a very large number

of atoms. But biological correlations can be mathematically represented with the multiplication that, for the DNA is expected to be multi-valued. Therefore, Santilli [8], [13] suggested hypermathematics with ordered basic rules (1.7) in which the product of two quantities can yield a large number of ordered results. However, rules (1.7) are classical and, as such, they cannot provide the most general possible mathematics as expected for the DNA in view of its complexity. This insufficiency has been resolved in this paper via Vougiouklis H_v formulation of Santilli's classical Lie-admissible hyperstructures (1.7). Specific initial applications to the DNA are in progress for reporting in a subsequent paper.

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FINITE p -GROUPS DETERMINED BY AN INEQUALITY OF THE ORDER OF ANY TWO-ELEMENTS GENERATED SUBGROUPS

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Abstract. In this note, we study a class of finite p -group determined by an inequality of the order of any two-elements generated subgroups. If $|\langle x, y \rangle| \leq p^i \max\{|x|, |y|\}$ for all $x, y \in G$, then G is called a M_i -group. If $|\langle x, y \rangle| \leq p^i |x|$ for all $x, y \in G$, then G is called a P_i -group. Such groups relate to a problem posed by Berkovich and Janko (Groups of prime order, Walter de Gruyter, Berlin, vol. 1, 2008) (Problem 461 and 237). In this paper, we mainly get the nilpotent class of P_i (or M_i)-groups, the exponent of derived subgroup of P_i (or M_i)-groups and $G^{p^2} \leq Z(G)$.

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1. Introduction

In this paper, only finite p -groups are considered, where p is a prime.

Finite p -groups are often determined by their subgroup structure, especially determined by their two-elements generated subgroups. For example, see [1], [5], and [6]. As in [1], authors have studied the finite p -groups with small subgroups generated by two conjugate elements and define K_1 -group. Recall a finite p -group is called a K_1 -group if $|\langle x, x^y \rangle : \langle x \rangle| \leq p$ for all $x, y \in G$. Generally, a finite p -group is called a K_i -group if $|\langle x, x^y \rangle : \langle x \rangle| \leq p^i$ for all $x, y \in G$. We denote this class by K_i .

In this paper, we consider finite p -groups in which any two-elements generated subgroups with large cyclic subgroup, that is, $|\langle x, y \rangle| \leq p^i \max\{|x|, |y|\}$ for all $x, y \in G$. We define that a group G is called a M_i -group if $|\langle x, y \rangle| \leq p^i \max\{|x|, |y|\}$ for all $x, y \in G$. This problem is posed by Berkovich and Janko in [2] problem 461 and 237. We denote this class by M_i .

In addition, we also consider the stronger case. We define that a group G is called a P_i -group if $|\langle x, y \rangle| \leq p^i |x|$ for all $x, y \in G$. We denote this class by P_i .

Clearly, we have $P_i \subseteq M_i \subseteq K_i$. It is obvious that $K_i \not\subseteq M_i$ and $K_i \not\subseteq P_i$. For example, $Q_{2^4} \in K_1$, but $Q_{2^4} \notin M_1$ and $Q_{2^4} \notin P_1$. In addition, we have $D_{2^4} \in M_2$, but $D_{2^4} \notin P_2$.

In Section 2, we mainly study P_2 -group. We get the following results:

- (1) If $p \geq 3$, then $G^p \leq Z(G)$ and $\exp(G') \leq p$. Moreover, if $p \geq 5$, then $Cl(G) \leq 2$; If $p = 3$, then $Cl(G) \leq 3$.
- (2) If $p = 2$, then $G^4 \leq Z(G)$, $\exp(G') \leq 4$ and $Cl(G) \leq 3$.

Moreover, in Section 3, we study the group M_i -groups. Firstly, since $M_1 \subseteq K_1$, we apply another method to get more precise results: $Cl(G) \leq 2$ and $G^p \leq Z(G)$, if $G \in M_1$. Secondly, we prove that G^{p^i} is cyclic and $G^{p^2} \leq Z(G)$, if $G \in M_2$. Furthermore, if $G \in M_2$, we have $Cl(G) \leq 4$, where $p \neq 2$ and $p \neq 5$ and the exponent of derived subgroup of G .

We use standard notation. Throughout this paper, G denotes a finite p -group for some prime p . $H \triangleleft G$ means that H is a normal subgroup of G . $\Omega_n(G) = \langle x \in G \mid x^{p^n} = 1 \rangle$, $G^{p^n} = \langle x^{p^n} \mid x \in G \rangle$. $Cl(G)$ is the nilpotent class of G . $\exp(G)$ is the exponent of G . $\gamma_i(G)$, $Z_i(G)$ is the low central series, the upper central series of G , respectively. $Z(G) = Z_1(G)$. $G' = \gamma_2(G)$.

2. P_2 -groups

Lemma 2.1 *Let G be a nonabelian p -group of order p^{n+1} with cyclic subgroup $A = \langle a \rangle$ of index p . Then G is isomorphic to one of the following groups:*

- (a) $M_{p^{n+1}} = \langle a, b \mid a^{p^n} = b^p = 1, b^{-1}ab = a^{1+p^{n-1}} \rangle$, where $n \geq 3$ if $p = 2$.
- (b) $p = 2$, $D_{2^{n+1}} = \langle a, b \mid a^{2^n} = b^2 = 1, bab = a^{-1} \rangle$.
- (c) $p = 2$, $Q_{2^{n+1}} = \langle a, b \mid a^{2^n} = 1, b^2 = a^{2^{n-1}}, b^{-1}ab = a^{-1} \rangle$.
- (d) $p = 2$, $SD_{2^{n+1}} = \langle a, b \mid a^{2^n} = b^2 = 1, bab = a^{-1+2^{n-1}} \rangle$, where $n > 2$.

Proof. See [2, Theorem 1.2]. ■

Lemma 2.2 *Let $G \in P_2$ and $x \in G$ of order p . Then, for each $y \in G$, let $H = \langle x, y \rangle$, we have that H is either abelian or $Cl(H) = 2$; and $|H| \leq p^3$ and $\exp(H) = p$, when $p \geq 3$; $H \cong D_8$, when $p = 2$.*

Proof. By $G \in P_2$, we have $|H : \langle x \rangle| \leq p^2$, then $|H| \leq p^3$. If H is nonabelian, when $p \geq 3$, then $|H| = p^3$ and $\exp(H) = p$; when $p = 2$, then $H \cong D_8$. ■

Theorem 2.3 *Let $G = \langle x, y \rangle$ is a nonabelian p -group. If $G \in P_2$, then*

- (1) *when $p > 2$, then $|G| \leq p^4$ and G is a minimal nonabelian p -group;*
- (2) *when $p = 2$, then $|G| \leq 2^4$; and when $|G| = 2^4$, we have $G \cong Q_{2^4}$; when $|G| = 2^3$, we have $G \cong D_{2^3}$.*

Proof. Since $G \in P_2$, we have $|\langle x, y \rangle : \langle x \rangle| \leq p^2$. Thus there exists subgroup H such that $|H : \langle x \rangle| \leq p$. Therefore, H has a cyclic subgroup $\langle x \rangle$ of index p . If $p > 2$, by Lemma 2.1, we have $H = \langle x, g \rangle$ and $|g| = p$. Since $H \in P_2$, we get $|H| \leq p^2|g| = p^3$ and $|G| \leq p^4$. If $p = 2$, we have $H = \langle x, g \rangle$ and $|g| \leq p^2$. Therefore, $|H| \leq p^2|g| = p^4$ and $|G| \leq p^5$.

If $p > 2$ and $|G| = p^4$, then G is not a maximal class group. Otherwise, we can choose generators such that $G = \langle x, g \rangle$ and $|g| = p$. We have $|G| \leq p^3$, which is a contradiction to $|G| = p^4$. Therefore, $Cl(G) \leq 2$. If $|G'| = p^2$, then $|\langle h, G' \rangle| = p^3$ for $h \notin G'$ and $\langle h, G' \rangle$ is abelian maximal subgroup of G . We get G is a maximal class group, which is a contradiction. Thus $|G'| = p$, that is, G is a minimal nonabelian p -group.

If $p = 2$ and $|G| = 2^4$, then $|G'| \neq 2$. Otherwise, we have G is a minimal nonabelian 2-group, that is, Q_8 , which is a contradiction to $|G| = 2^4$. Therefore, $|G'| = 2^2$ and G is a maximal class 2-group. Since D_{2^4} and SD_{2^4} not in P_2 , we have $G \cong Q_{2^4}$.

If $|G| = 2^5$, we have G is a maximal class 2-group when $|G'| = 2^3$. By Lemma 2.1, we get $|G| \leq 2^4$, which is a contradiction to $|G| = 2^5$. When $|G'| = 2$, G is Q_8 , which is a contradiction to $|G| = 2^5$. Therefore, $|G'| = 2^2$. It is obvious that $|x| \geq 2^3$ and $|y| \geq 2^3$. Let $H = \langle x^p, y \rangle$, then we have $|H| = 2^4$ and $G' < H < G$. Therefore, $H \cong Q_{2^4}$ and $H = \langle x^p, g \rangle$, where $|g| = 2^2$. Thus, $G = \langle x, g \rangle$ and $|G| \leq 2^4$, which is a contradiction to $|G| = 2^5$.

In addition, it is obvious that D_{2^3} is in P_2 , when $|G| = 2^3$. ■

Theorem 2.4 *Let $G \in P_2$.*

- (1) *If $p \geq 3$, then $G^p \leq Z(G)$ and $\exp(G') \leq p$. Moreover, If $p \geq 5$, then $Cl(G) \leq 2$; If $p = 3$, then $Cl(G) \leq 3$.*
- (2) *If $p = 2$, then $G^4 \leq Z(G)$, $\exp(G') \leq 4$ and $Cl(G) \leq 3$.*

Proof. Let $x, y \in G$ and $H = \langle x, y \rangle$.

If $p \geq 3$, by Theorem 2.3, we get that H is either a abelian or minimal nonabelian p -group. Therefore $H^p \in Z(H)$ and $|H'| \leq p$, that is, $G^p \leq Z(G)$ and $\exp(G') \leq 4$. Since $Cl(H) \leq 2$, we get G is a 2-Engle group. Therefore, the conclusions are obvious.

If $p = 2$, by Theorem 2.3, we have $H \cong Q_{2^4}$ or $H \cong Q_{2^3}$. Therefore, $H^4 \in Z(H)$ and $|H'| \leq 4$, that is, $G^4 \leq Z(G)$ and $\exp(G') \leq 4$. Let $\bar{G} = G/G^4$, then $\exp(\bar{G}) \leq 4$. Since the quotient group $\bar{G}$ is also G_2 -group, by Theorem 2.3, we get $\bar{H} \cong Q_{2^3}$. Thus $\bar{G}$ is a 2-Engle group. Therefore, $Cl(\bar{G}) \leq 2$. Note that $G^4 \leq Z(G)$, we have $Cl(G) \leq 3$. ■

3. M_2 -groups

Theorem 3.1 *Let $G \in M_i$ ($i \in N$). Then G^{p^i} is cyclic.*

Proof. If G^{p^i} is noncyclic, let $x \in G$ such that $|x| = \exp(G)$. Then there exists $y \in G$, such that $y^{p^i} \notin \langle x^{p^i} \rangle$. Since $\langle x \rangle \langle y \rangle \subseteq \langle x, y \rangle$ and $\langle x \rangle \cap \langle y \rangle \subseteq \langle y^{p^{i+1}} \rangle$, we have $|\langle x, y \rangle| \geq |\langle x \rangle \langle y \rangle| = \frac{|x||y|}{|\langle x \rangle \cap \langle y \rangle|} \geq \frac{|x||y|}{|y^{p^{i+1}}|} \geq p^{i+1}|x|$, which contradicts $G \in M_i$. ■

Remark. Clearly, we have $P_i \subseteq M_i$. Therefore, if $G \in P_i$, then G^{p^i} is cyclic.

Proposition 3.2 *Let $G = \langle a, b \rangle$ nonabelian. Then $G \in M_1$ if and only if G is (a) or (c) in Lemma 2.1, where $G = Q_{2^3}$ if G is (c).*

Proof. Let $|a| = p^n \geq |b|$. Since $G \in M_1$, we have $|\langle a, b \rangle| \leq p|a|$, that is, G be a nonabelian p -group of order p^{n+1} with cyclic subgroup $A = \langle a \rangle$ of index p . Then G is isomorphic to one of groups in Lemma 2.1.

Case 1. G is (b). Since $(ab)^2 = aa^b = 1$ and $G = \langle a, b \rangle = \langle b, ab \rangle$, then $|G| = |\langle b, ab \rangle| \leq 2^2$, which contradicts the fact that G is a nonabelian.

Case 2. G is (c). Since $(ab)^4 = ab^2(b^{-1}ab)ab^2(b^{-1}ab) = b^4 = 1$ and $G = \langle a, b \rangle = \langle b, ab \rangle$, then $|G| = |\langle b, ab \rangle| \leq 2^3$, that is, $G = Q_{2^3}$.

Case 3. G is (d). Since $(ab)^2 = abab = aa^b = a^{2^{n-1}}$, then $|ab| = 4$. By $G \in M_1$ and $G = \langle a, b \rangle = \langle b, ab \rangle$, we have $|G| = |\langle b, ab \rangle| \leq 2^3$, which contradicts $n > 2$. ■

Theorem 3.3 *Let $G \in M_1$. Then $Cl(G) \leq 2$ and $G^p \leq Z(G)$.*

Proof. Let $H = \langle x, y \rangle$ for $x, y \in G$.

(1) If $p > 3$, then H is (a) by proposition 3.2, that is, $Cl(H) \leq 2$. Thus G is 2-Engle group, by [7, Theorem 3.3.2], we have $Cl(G) \leq 2$.

If $p = 3$, then $Cl(H) \leq 2$ by proposition 3.2 and $|H'| = p$. Thus H is a regular 3-group with cyclic derived subgroup. It follows that G satisfies that every two-elements generate subgroups have cyclic derived subgroups. By [7, Theorem 5.2.11], we have G is a regular 3-group. Let $G = \langle x, y_1, \dots, y_k \rangle$, where $|x| = \exp(G)$.

If $[x, y_i] \neq 1$ for every $y_i \in G (i = 1, \dots, k)$, by Proposition 3.2, we can choose $y_i \in G (i = 1, \dots, k)$ such that $|y_i| = p$. so $G = \langle x \rangle \Omega_1(G)$, where $\exp(\Omega_1(G)) = p$. Since $G \in P_1$, then $\Omega_1(G)$ is elementary abelian. Therefore, $Cl(G) \leq 1 + 1 = 2$.

If there exists $y_i \in G$ such that $[x, y_i] = 1$, then G has $y_j \in G$ such that $[x, y_j] \neq 1$ and $|y_j| = p$ (Otherwise G is abelian, conclusion is correct). Since G is a regular 3-group, then we get $|y_i y_j| = |y_i|$ and $[x, y_i y_j] \neq 1$. So, we can substitute $y_i y_j$ for y_i in $G = \langle x, y_1, \dots, y_k \rangle$ such that $|y_i| = p$. Therefore, $Cl(G) \leq 2$ by proof in above paragraph.

Since $Cl(G) \leq 2$ and $|H'| = p$, we have $[x^p, y] = [x, y]^p = 1$, that is, $G^p \leq Z(G)$.

(2) If $p = 2$, by Proposition 3.2, we also have $Cl(H) \leq 2$. Then, $Cl(G) \leq 2$ and $G^2 \leq Z(G)$. ■

Remark. Clearly, $M_1 \subseteq K_1$. In [1], if $G \in K_1$, authors mainly get the nilpotent class of G and $G^p \leq Z(G)$ (or $G^4 \leq Z(G)$). In Theorem 3.3, we get more concise conclusions, if $G \in M_1$.

Lemma 3.4 *Let G is a maximal class 2-group. If $G \in M_2$, Then $|G| \leq 2^4$.*

Proof. Since G is a maximal class 2-group, by Lemma 2.1, then G is one of (b),(c) or (d).

If G is (b), since $G = \langle a, b \rangle = \langle b, ab \rangle$ and $(ab)^2 = 1$, then $|G| = |\langle b, ab \rangle| \leq 2^3$.

If G is (c), since $G = \langle a, b \rangle = \langle b, ab \rangle$ and $(ab)^4 = 1$, then $|G| = |\langle b, ab \rangle| \leq 2^4$.

If G is (d), since $G = \langle a, b \rangle = \langle b, ab \rangle$ and $|ab| = 4$, we have $|G| = |\langle b, ab \rangle| \leq 2^4$. ■

Theorem 3.5 *If $G \in M_2$, then $G^{p^2} \leq Z(G)$.*

Proof. Let $x \in G$ such that $|x| = \exp(G)$. By Theorem 3.1, $G^{p^2} = \langle x^{p^2} \rangle$. If $|x| \leq p^3$, by $|G^{p^2}| \leq p$ and $G^{p^2} \triangleleft G$, we have $G^{p^2} \leq Z(G)$. Now, suppose that $|x| = p^n \geq p^4$. Let $y \in G$ and let subgroup $H = \langle x, y \rangle$. We will prove that $H^{p^2} = \langle x^{p^2} \rangle \leq Z(H)$.

Since H is also a P_2 group, then $|H| \leq p^{n+2}$. If $|H| = p^{n+1}$, then $|H : \langle x \rangle| = p$. By Lemma 2.1, we get $H = \langle x, y_0 \rangle$, where $|y_0| \leq p^2$. Since $|y_0| = p^2$ if and only if $p = 2$ and $H \cong Q_{2^{n+1}}$, it means that $H = \langle xy_0, y_0 \rangle$, where $|xy_0| = |y_0| = 2^2$, a contradiction. Hence we have $y_0^p = 1$. By Lemma 2.1 and 3.4, it is easy to get that $Cl(H) \leq 2$ and $H^p \leq Z(H)$.

Now consider the case that $|H| = p^{n+2}$. Let H_1 be the maximal subgroup of H such that $x \in H_1$. Then we have $H_1 \triangleleft H$ and $Cl(H_1) \leq 2$. It follows Lemma 2.1 we have $H_1 = \langle x, x_1 \rangle$, where $x_1^p = 1, x^{x_1} = x^{1+p^{n-1}}$ or $x^{x_1} = x$. We see that $\Omega_1(H_1) = \langle x^{p^{n-1}}, x_1 \rangle$ and $\Omega_1(H_1) \leq \Phi(H)$. Let the quotient group $\bar{H} = H/\Omega_1(H_1)$ and $\bar{x} = x\Omega_1(H_1)$. Then $|\bar{H} : \langle \bar{x} \rangle| = p$. If $p = 2$, by $|\bar{x}| \geq p^3$, we get $\bar{H}$ is a 2-group of maximal class. Hence there is an element $y_1 \in H$ such that $\bar{H} = \langle \bar{x}, \bar{y}_1 \rangle$, where $|\bar{y}_1| \leq 4$. Then we have $\bar{H} = \langle x\bar{y}_1, \bar{y}_1 \rangle$, where $|x\bar{y}_1| = |\bar{y}_1| \leq 4$. Since $\Omega_1(H_1) \leq \Phi(H)$, we have $H = \langle xy_1, y_1 \rangle$ and $|xy_1| = |y_1| \leq 8$. Since $H \in M_2$, we have $|H| \leq 2^5$. By $2^n \geq 2^4$, which contradicts with $|H| = 2^{n+2} > 2^5$. Hence by Lemma 2.1 we get that $Cl(\bar{H}) \leq 2$. It follows that $\bar{H} = \langle \bar{x}, \bar{y}_1 \rangle$, where $|\bar{y}_1| = p$

and $\bar{x}^{\bar{y}_1} = \bar{x}$ or $\bar{x}^{\bar{y}_1} = \bar{x}^{1+p^{n-2}}$. Then we have $H = \langle x, y_1 \rangle$ and $x^{y_1} = xx_1^k$ or $x^{y_1} = x^{1+p^{n-2}}x_1^k$, where $0 \leq k \leq p-1$.

Note that H_1' is a subgroup of order p . If $x^{y_1} = xx_1^k$, by Hall-Petrescu formula, we have

$$(x^{p^2})^{y_1} = (xx_1^k)^{p^2} = x^{p^2}(x_1^k)^{p^2} = x^{p^2},$$

which means that $x^{p^2} \leq Z(H)$. If $x^{y_1} = x^{1+p^{n-2}}x_1^k$, similarly, by Hall-Petrescu formula, we have $(x^{p^2})^{y_1} = (x^{1+p^{n-2}}x_1^k)^{p^2} = x^{p^2}$, and then $x^{p^2} \leq Z(H)$. So, we get that $G^{p^2} \leq Z(G)$. ■

Lemma 3.6 Suppose $G = \langle x, y \rangle$ is a p -group. If $G \in M_2$ and $p \geq 3$, then $|\Omega_1(G)| \leq p^3$; If $G \in M_3$ and $p \geq 5$, then $|\Omega_1(G)| \leq p^4$.

Proof. Let $H = \langle a, b \rangle$ for every $a, b \in G$. If $|a| = |b| = p$ and $G \in M_2$, then $|H| \leq p^3$. Therefore, $|H'| \leq p$. Since $p \geq 3$, then $\Omega_1(H)$ is regular, that is, $\Omega_1(G)$ is. Without loss of generality, we assume $|x| \geq |y|$. Hence, we have $p^2|x| \geq |G| \geq |\langle x, \Omega_1(G) \rangle| = \frac{|x||\Omega_1(G)|}{|\langle x \rangle \cap \Omega_1(G)|} = \frac{|x||\Omega_1(G)|}{p}$. Thus $|\Omega_1(G)| \leq p^3$.

If $G \in M_3$ and $p \geq 5$, then $|H| \leq p^4$. Therefore, H is regular. Thus $\Omega_1(G)$ is regular. Similarly, we have $p^3|x| \geq |G| \geq |\langle x, \Omega_1(G) \rangle| = \frac{|x||\Omega_1(G)|}{|\langle x \rangle \cap \Omega_1(G)|} = \frac{|x||\Omega_1(G)|}{p}$. Thus $|\Omega_1(G)| \leq p^4$. ■

Theorem 3.7 If $G \in M_2$, then $Cl(G) \leq 4$, where $p \neq 2$ and $p \neq 5$.

Proof. Let $H = \langle a, b \rangle$ for every $a, b \in G$. By Lemma 3.6, we get $|\Omega_1(H)| \leq p^3$.

If $|\Omega_1(H)| = p^3$, since $p^2|a| \geq |H| \geq |\langle a, \Omega_1(H) \rangle| = \frac{|a||\Omega_1(H)|}{p} = p^2|a|$, we have $H = \langle a, \Omega_1(H) \rangle$. Therefore, $H/\Omega_1(H)$ is cyclic. Then, $H' = [H, \Omega_1(H)] < \Omega_1(H)$ and $|H'| \leq p^2$.

If $|\Omega_1(H)| = p^2$, since $\frac{|a|}{p} \leq |H/\Omega_1(H)| \leq |a|$, then $H/\Omega_1(H)$ is cyclic (where $|H/\Omega_1(H)| = \frac{|a|}{p}$) or has a cyclic subgroup of index p (where $|H/\Omega_1(H)| = |a|$).

(1) If $H/\Omega_1(H)$ is cyclic, we get $H' \leq \Omega_1(H)$, that is, $|H'| \leq |\Omega_1(H)| \leq p^2$.

(2) If $H/\Omega_1(H)$ has a cyclic subgroup of index p , when $|H/\Omega_1(H)| = |a| = p^2$, then $|H'| \leq |\Omega_1(H)| \leq p^2$. Without loss of generality, we assume $|a| = p^n \geq p^3$. By Lemma 2.1, there exists $b_1 \in H$ such that $H = \langle a, b_1 \rangle$, where $|b_1| = p^2$.

If $\langle a \rangle \cap \langle b_1 \rangle = 1$, since $p^2|a| \geq |H| = |\langle a, b_1 \rangle| \geq |\langle a \rangle||\langle b_1 \rangle| = p^2|a|$, then $H = \langle a \rangle \langle b_1 \rangle$, that is, H is metacyclic. Therefore $H/\langle b_1 \rangle^H$ is cyclic. Thus, we have $H' \leq \langle b_1 \rangle^H$. By H is metacyclic, we get H is regular. Then, $\exp(\langle b_1 \rangle^H) \leq |b_1| \leq p^2$. We have $|H'| \leq p^2$.

If $\langle a \rangle \cap \langle b_1 \rangle \neq 1$, let $H_1 = \Omega_2(H) = \langle a^{p^{n-2}}, b_1 \rangle$. Since $\langle a^p \rangle \trianglelefteq H$, then $\langle a^{p^{n-2}} \rangle \trianglelefteq H$. Therefore, $H_1 = \langle a^{p^{n-2}} \rangle \langle b_1 \rangle$ and $|H_1| = p^3$. We have $Cl(H_1) \leq 2$. Since we have $(a^{p^{n-2}}b_1^{-1})^p = a^{p^{n-1}}(b_1^{-1})^p = 1$ ($a^{p^{n-1}} = b_1^p$), let $b_2 = a^{p^{n-2}}b_1^{-1}$, then we get $H = \langle a, b_1 \rangle = \langle a, b_2 \rangle$, where $|b_2| = p$. Therefore, $H/\Omega_1(H) = \langle a\Omega_1(H), b_2\Omega_1(H) \rangle = \langle a\Omega_1(H) \rangle$, that is, $H/\Omega_1(H)$ is cyclic. Then, we have $|H'| \leq |\Omega_1(H)| \leq p^2$.

Since every two-elements generated subgroup H of G has $|H'| \leq p^2$ and $Cl(H) \leq 3$, then we get G is a 3-Engle group. By [8], we have $Cl(G) \leq 4$, where $p \neq 2$ and $p \neq 5$. ■

Theorem 3.8 *If $G \in M_2$, then:*

- (1) *if $\exp(G) = p$, then, if and only if G is $Cl(G) \leq 3$, where $p = 3$; or $Cl(G) = 2$, where $p > 3$.*
- (2) *if $\exp(G) = p^2$ and $p \geq 3$, then $\exp(G') \leq p$.*
- (3) *if $\exp(G) > p^2$ and $p \geq 3$, $\exp(G') \leq p^2$.*

Proof. (1) Let $H = \langle a, b \rangle$, for every $a, b \in G$. Since $|H| \leq p^3$, by Theorem 3.3 in [9], (1) holds.

(2) Let $H = \langle a, b \rangle$, for every $a, b \in G$. If $|a| = |b| = p$, then $|H'| \leq p$. Therefore, let $|a| = p^2 \geq |b|$. Since $\langle a^p \rangle = H^p \trianglelefteq G$ and $\exp(G) = p^2$, then $a^p \in Z(G)$. By Theorem 3.7, we have $|H'| \leq p^2$ and $Cl(H) \leq 3$. Therefore, $1 = [a^p, b] = [a, b]^p [a, b, a]^{\frac{p(p-1)}{2}}$. Since $\gamma_3(H) < \gamma_2(H) = H'$, then $[a, b]^p = 1$. We have $G' = \langle [a, b] | a, b \in G \rangle \leq \Omega_1(G)$, by Lemma 3.6, G' is regular and $\exp(G') \leq p$.

(3) Let $H = \langle a, b \rangle$, for every $a, b \in G$. If $|b| \leq |a| \leq p^2$, by (2), then we have $[a, b]^p = 1$. Without loss of generality, we assume $|a| > p^2$ and $|a| \geq |b|$. By Theorem 3.7, we have $|H'| \leq p^2$ and $Cl(H) \leq 3$. Since $\gamma_3(H) < \gamma_2(H) = H'$, we have $[a^p, b] = [a, b]^p [a, b, a]^{\frac{p(p-1)}{2}} = [a, b]^p$. By Theorem 3.5, we get $1 = [a^{p^2}, b] = [a, b]^{p^2}$. Therefore, $G' \leq \Omega_2(G)$.

For every $x, y \in \Omega_2(G)$ and $|y| \leq |x| \leq p^2$, we have $|\langle x, y \rangle| \leq p^4$. If $\langle x, y \rangle$ is a maximal class group, then $|xy| \leq p^2$. If $\langle x, y \rangle$ is not a maximal class group, then $Cl(\langle x, y \rangle) \leq 2$ and $|xy| \leq p^2$. Therefore $\exp(G') \leq p^2$. ■

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FINITE GROUPS WITH THEIR AUTOMORPHISM GROUPS HAVING ORDERS $4pq^2$ ($2 < p < q$)

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Abstract. The authors find all finite nilpotent groups with automorphism groups having orders $4pq^2$ and prove that there exists no finite non-nilpotent group such that $|\text{Aut}(G)| = 4pq^2$ ($2 < p < q$, $p \nmid q^2 - 1$). So, the authors classify finite groups with their automorphism groups having orders $4pq^2$ ($2 < p < q$, $p \nmid q^2 - 1$).

Keywords: a finite group; automorphism group; Sylow subgroup; order; classification.

1. Introduction

Given a positive integer n , to find out all finite groups in equation $|\text{Aut}(G)| = n$ is a very difficult and interesting topic. Iyer proved in [8] that there are at most finite numbers of groups satisfy $|\text{Aut}(G)| = n$. Flannery and Machale solved equations $|\text{Aut}(G)| = p^n$ ($1 \leq n \leq 4$) or pq and proved that $|\text{Aut}(G)| = p^n$ ($5 \leq n \leq 7$) has no solution in [5, 14]. Curran proved that $|\text{Aut}(G)| = p^n$ ($1 \leq n \leq 5$) has no solution in [3]. Flym and Machale found all finite groups with $|\text{Aut}(G)| = 2^n$ ($1 \leq n \leq 7$) in [6]. Chen solved the equation $|\text{Aut}(G)| = p_1 p_2 \cdots p_n$ or pq^2 in [2]. Li found all finite groups such that $|\text{Aut}(G)| = 2^3 p$, $p^3 q$ or $p^2 q^2$ in [10, 11, 13]. Huang solved the equation $|\text{Aut}(G)| = prq^2$ in [7]. Xia and Chen classified finite groups with their automorphism groups having orders $4p_1 p_2 \cdots p_n$ in [17].

In this paper, we find all finite nilpotent groups with $|\text{Aut}(G)| = 4pq^2$ and prove that there exists no finite non-nilpotent group such that $|\text{Aut}(G)| = 4pq^2$ ($2 < p < q$, $p \nmid q^2 - 1$). That is to say, we get the classification of finite groups G with $|\text{Aut}(G)| = 4pq^2$ ($2 < p < q$, $p \nmid q^2 - 1$).

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All groups considered are finite. We use C_n to denote cyclic group of order n , $\text{Cen}(G)$ the group of central automorphisms of G , and $[A]B$ the semi-direct product of groups A and B with A normal in $[A]B$. All other notations are standard.

2. Preliminaries

In this section, we list some known results in this topic and establish some lemmas needed.

Lemma 2.1. [4] *Let P be a non-cyclic p -group such that $|P| \geq p^3$. Suppose that $|P/Z(P)| \leq p^4$. Then $|P| \mid |\text{Aut}(P)|$.*

Lemma 2.2. [9] *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where $p_1, p_2, \dots, p_r$ are distinct primes. Set $W(n) = \sum_{i=1}^r \alpha_i$. Then there exists no finite group G such that $|\text{Aut}(G)|$ is odd and $W(|\text{Aut}(G)|) \leq 4$.*

Lemma 2.3. [17] *Let G be a nilpotent group of order m , where $m = 2^\alpha p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ and $p_1, p_2, \dots, p_n$ are distinct primes such that $p_1 < p_2 < \cdots < p_n$. If Sylow p_i -subgroup of G is cyclic, Sylow 2-subgroup of G is cyclic or is isomorphic to $C_2 \times C_2$, then $2^n \mid |\text{Aut}(G)|$ and $2^{\alpha+n-1} \mid |\text{Aut}(G)|$ when $\alpha \geq 1$.*

Lemma 2.4. [16] *Let G be a group without nontrivial abelian direct factor, and p a prime. If $p \mid (|G/G'|, |Z(G)|)$, then G has a central automorphism of order p .*

Lemma 2.5. [16] *Let $\pi(G) = \{p_1, p_2, \dots, p_r\}$. Suppose that the Sylow decomposition of G/G' and $Z(G)$ respectively are*

$$\begin{aligned} G/G' &= \overline{G_{p_1}} \times \overline{G_{p_2}} \times \cdots \times \overline{G_{p_r}}, \quad \text{and} \\ Z(G) &= Z_{p_1} \times \cdots \times Z_{p_r}, \end{aligned}$$

where the groups on the right side may be 1. If G does not contain nontrivial abelian direct factor, then $|\text{Cen}(G)| = \prod_{i=1}^r \prod_{j=1}^{k_i} |Z_{p_i,j}|^{r_{i,j}}$, where $Z_{p_i,j} = \{z \in Z_{p_i} \mid z^{p_i^j} = 1\}$, $p_i^{k_i}$ is the order of $\overline{G_{p_i}}$ and $r_{i,j}$ is the number of direct factors of order p_i^j in the direct composition of $\overline{G_{p_i}}$.

Lemma 2.6. [20] *If all Sylow subgroups of a finite non-abelian group G are cyclic. Then G is a metacyclic group such that $G = \langle a, b \mid a^m = b^n = 1, b^{-1}ab = a^r, ((r-1)n, m) = 1, r^n \equiv 1 \pmod{m} \rangle$.*

Lemma 2.7. [12] *For every prime p , suppose that $|\text{Aut}(G)|$ can not be divided by p^3 . Then G has a Sylow Tower.*

Lemma 2.8. [11] *Suppose $G = [A]B$ and A is abelian, k is an interger such that $(k, |A|) = 1$, then the map*

$$\sigma : ab \mapsto a^k b, \quad a \in A, \quad b \in B,$$

is an automorphism of G .

Lemma 2.9. [17] *Suppose*

$$G = \langle a, b \mid a^m = b^n = 1, b^{-1}ab = a^r, ((r-1)n, m) = 1, r^n \equiv 1 \pmod{m} \rangle.$$

Then one of the following holds:

- (1) *If $Z(G) = 1$, then $|Aut(G)| = m\varphi(m)$.*
- (2) *If $|Z(G)| = 2$, then $|Aut(G)| = 2m\varphi(m)$.*

Lemma 2.10. *Let G be a group of order $2q^2$, where q is an odd prime. Then G is supersolvable.*

Proof. It follows from the definition relations of finite groups with order $2q^2$ in [21]. ■

Lemma 2.11. [20] *Suppose G/N and G/K are both nilpotent, then $G/(N \cap K)$ is nilpotent.*

Lemma 2.12. [13] *Let G be a group of even order such that $G = [Q]R$, where $R \in Syl_2(G)$, $Q \in Syl_q(G)$. Suppose that $Q \cong C_q \times C_q$ and $C_R(Q) = 1 = Z(G)$, where q is an odd prime. Then $2^3 \mid |Aut(G)|$.*

Lemma 2.13. *Let G be a group of even order such that $G = [P \times Q]R$, where $R \in Syl_2(G)$, $P \in Syl_p(G)$ and $Q \in Syl_q(G)$. Suppose that $P \cong C_p$, $Q \cong C_q \times C_q$ and $C_R(PQ) = 1 = Z(G)$, where p and q are distinct odd primes. Then $2^3 \mid |Aut(G)|$.*

Proof. If $|R| \geq 2^3$, consider the inner automorphisms induced by elements of R , since $Z(G) = 1$, we get $2^3 \mid |Aut(G)|$. If $|R| \leq 4$, since $PQ \trianglelefteq G$, we investigate the automorphism I_a of PQ induced by any $a \in R$. Let $\widehat{R} = \{I_a \mid a \in R\}$. Since $C_R(PQ) = 1$, we get $\widehat{R} \cong R$. ■

If $|R| = 4$, we consider $Aut(PQ)$. Since $PQ = P \times Q$, then $GL(2, q) \cong Aut(Q) \leq Aut(PQ)$. Because $2^3 \mid |GL(2, q)|$, we can find a 2-element $\sigma \in Aut(PQ) \setminus \widehat{R}$ which normalizes $\widehat{R}$. Then $\sigma^{-1}I_a\sigma = I_{a'}$ holds for every $a \in R$. Because of $C_R(PQ) = 1$, a' mentioned above is unique. Since $(I_{ab})^\sigma = (I_a I_b)^\sigma = I_{a'b'}$ holds for arbitrary $a, b \in R$, we have

$$(ab)' = a'b', \quad a, b \in R.$$

Define a map ρ as

$$(ax)^\rho = a'x^\sigma, \quad a \in R, \quad x \in PQ.$$

Then, for every $a, b \in R$ and $x, y \in PQ$, we have

$$(axy)^\rho = (ab)'(b^{-1}xy)^\sigma = a'b'x^{I_b\sigma}y^\sigma = a'b'x^{\sigma I_{b'}}y^\sigma = a'x^\sigma b'y^\sigma = (ax)^\rho(by)^\rho$$

This implies that ρ is an automorphism of G . By the definition of a' , we see that there exists some $a \in R$ such that $a \neq a'$, hence $\rho \neq 1$. Obviously, $|\rho| \mid |\sigma|$. If $\rho \in \text{Inn}(G)$, then there exists a 2-element of G , for example $y^{-1}by$ ($y \in PQ$, $b \in R$), such that $xy^{-1}by = x^\sigma$ for any $x \in PQ$. Consequently, $x^b = x^\sigma$, which implies that $\sigma = I_b \in \widehat{R}$, a contradiction. Therefore $2^3 \mid |\text{Aut}(G)|$.

If $|R| = 2$, then $|G| = 2pq^2$. But $P, Q \leq G$, it follows from Lemma 2.10 and Lemma 2.11 that G is supersolvable. Let $R = \langle a \rangle$ and $P = \langle z \rangle$. Because G is supersolvable and $C_Q(R) = 1$, we can take x and y such that $Q = \langle x, y \rangle$ and $x^a = x^{-1}$, $y^a = y^{-1}$. Then $I_a \in Z(\text{Aut}(Q))$ and thus $\alpha^{-1}I_a\alpha = I_a$ holds for every $\alpha \in \text{Aut}(Q)$. Every element σ of $\text{Aut}(Q)$ can be extended to an element ρ of $\text{Aut}(G)$, through the following definition of ρ :

$$(a'z'w')^\rho = a'z'(w')^\sigma, \quad a' \in R, \quad z' \in R, \quad w' \in Q.$$

Thus $\text{Aut}(Q)$ is a subgroup of $\text{Aut}(G)$ and then $2^3 \mid |\text{Aut}(G)|$. ■

Lemma 2.14. [13] *Let p be an odd prime. Suppose H is a p' -group acting on a p -group V and A is an H -invariant maximal abelian subgroup of V . Then H acts trivially on V only if H acts trivially on A .*

Lemma 2.15. [20] *Suppose H is a π' -group acting on an abelian π -group G , A is an H -invariant subgroup and a direct factor of G . Then there exists an H -invariant subgroup K of G such that $G = A \times K$.*

3. Main results

In this section, G is a finite group such that $|\text{Aut}(G)| = 4pq^2$, where p and q are odd primes such that $2 < p < q$.

Theorem 3.1. *Suppose G is nilpotent. Then G is isomorphic to one of following groups:*

- $G_1 = C_{p_1^2} \times C_{p_2}$, where $p_1 = q$, $p_1 = 2p + 1$ and $p_2 = 2q + 1$.
- $G_2 = C_2 \times C_{p_1^2} \times C_{p_2}$, where $p_1 = q$, $p_1 = 2p + 1$ and $p_2 = 2q + 1$.
- $G_3 = C_{p_1} \times C_{p_2}$, where $p_1 = 2q + 1$ and $p_2 = 2pq + 1$.
- $G_4 = C_2 \times C_{p_1} \times C_{p_2}$, where $p_1 = 2q + 1$ and $p_2 = 2pq + 1$.
- $G_5 = C_{p_1} \times C_{p_2^3}$, where $p_1 = 3$ and $p_2 = 2p + 1 = q$.
- $G_6 = C_2 \times C_{p_1} \times C_{p_2^3}$, where $p_1 = 3$ and $p_2 = 2p + 1 = q$.
- $G_7 = C_4 \times C_{p_1}$, where $p_1 = 2pq + 1$.
- $G_8 = C_4 \times C_{p_1^3}$, where $p_1 = 2p + 1 = q$.

Proof. Since G is nilpotent, we may assume that $G = R \times P_1 \times P_2 \times \cdots \times P_m$, where $R \in \text{Syl}_2(G)$, $P_1 \in \text{Syl}_{p_1}(G)$, $\dots$, $P_m \in \text{Syl}_{p_m}(G)$, $p_i (1 \leq i \leq m)$ are odd primes such that $p_1 < p_2 < \cdots < p_m$. Then

$$4pq^2 = |\text{Aut}(G)| = \left(\prod_{i=1}^m |\text{Aut}(P_i)| \right) \times |\text{Aut}(R)|.$$

We divide the proof into 5 steps.

- (1) P_i is cyclic and $|P_i| \mid p_i^2$, where $1 \leq i \leq m$.

If $|P_i| \geq p_i^3$. By $|P_i/Z(P_i)| \mid |P_i/P_i \cap Z(G)|$, we have

$$|P_i/P_i \cap Z(G)| \mid |P_i Z(G)/Z(G)| \mid |G/Z(G)| \mid |\text{Aut}(G)|.$$

This implies $|P_i/Z(P_i)| \leq p_i^2$. Then $|P_i| \mid |\text{Aut}(P_i)|$ by Lemma 2.1.

But $|\text{Aut}(P_i)| \mid |\text{Aut}(G)|$, it follows that $p_i^3 \mid 4pq^2$, a contradiction. Therefore $|P_i| \mid p_i^2$. If P_i is not cyclic, then $P_i \cong C_{p_i} \times C_{p_i}$ and thus $8 \mid |\text{Aut}(P_i)| = 4pq^2$, a contradiction too.

- (2) R is cyclic of order ≤ 8 or $R \cong C_2 \times C_2$.

Otherwise R is non-cyclic and $|R| \geq 2^3$. Similar to step (1), we can prove that $|R/Z(R)| \leq 4$. Then $|R| \mid |\text{Aut}(R)|$ by Lemma 2.1.

But $|\text{Aut}(R)| \mid |\text{Aut}(G)|$, so $8 \mid 4pq^2$, a contradiction. Hence R is cyclic or $|R| \leq 4$.

Now if R is cyclic, let $|R| = 2^s (s > 1)$, then $2^{s-1} \mid |\text{Aut}(R)| \mid |\text{Aut}(G)|$, so $2^{s-1} \leq 4$, which yields $s \leq 3$. Hence (2) follows.

- (3) While $|R| = 8$ one has that $G = R \cong C_8$ and $|\text{Aut}(G)| = 4$, a contradiction.

In the following steps, we need to notice the fact that $2q^2 + 1$ is not a prime when q is an odd prime such that $q > 3$.

- (4) While $|R| = 1$ or 2 , by Lemma 2.3, we may assume that $G = RP_1P_2$ ($P_1 \neq 1 \neq P_2$).

If $R = 1$. Then one of the following occurs:

If $G \cong C_{p_1^2} \times C_{p_2}$, then $|\text{Aut}(G)| = p_1(p_1 - 1)(p_2 - 1)$. Hence $|\text{Aut}(G)| = 4pq^2 (2 < p < q)$ if only if $p_1 = q$, $p_1 = 2p + 1$ and $p_2 = 2q + 1$.

If $G \cong C_{p_1^2} \times C_{p_2^2}$, then $|\text{Aut}(G)| = p_1p_2(p_1 - 1)(p_2 - 1)$, it is impossible.

If $G \cong C_{p_1^2} \times C_{p_2^3}$, then $|\text{Aut}(G)| = p_1p_2(p_1 - 1)(p_2 - 1)$, which concludes $p_1 - 1 = p_2 - 1 = 2$, a contradiction.

If $G \cong C_{p_1} \times C_{p_2}$, then $|\text{Aut}(G)| = (p_1 - 1)(p_2 - 1)$. Hence $|\text{Aut}(G)| = 4pq^2 (2 < p < q)$ if only if $p_1 = 2q + 1$ and $p_2 = 2pq + 1$.

If $G \cong C_{p_1} \times C_{p_2^2}$, then $|\text{Aut}(G)| = p_2(p_1 - 1)(p_2 - 1)$, which is impossible.

If $G \cong C_{p_1} \times C_{p_2^3}$, then $|\text{Aut}(G)| = p_2^2(p_1 - 1)(p_2 - 1)$. Hence $|\text{Aut}(G)| = 4pq^2 (2 < p < q)$ if only if $p_1 = 3$ and $p_2 = 2p + 1 = q$.

Therefore G is isomorphic to G_1 , G_3 or G_5 .

If $|R| = 2$, then G is isomorphic to G_2 , G_4 or G_6 according to the conclusion above.

(5) While $|R| = 4$, we can only have that $G = RP_1$ by Lemma 2.3.

If $G \cong C_4 \times C_{p_1}$, then $|\text{Aut}(G)| = 2(p_1 - 1)$. So $|\text{Aut}(G)| = 4pq^2$ ($2 < p < q$) if only if $p_1 = 2pq + 1$.

If $G \cong C_4 \times C_{p_1^2}$, then $|\text{Aut}(G)| = 2p_1(p_1 - 1)$, which is impossible.

If $G \cong C_4 \times C_{p_1^3}$, then $|\text{Aut}(G)| = 2p_1^2(p_1 - 1)$. Hence $|\text{Aut}(G)| = 4pq^2$ ($2 < p < q$) if only if $p_1 = 2p + 1 = q$.

If $G \cong C_2 \times C_2 \times C_{p_1}$, then $|\text{Aut}(G)| = 6(p_1 - 1)$. So $|\text{Aut}(G)| = 4pq^2$ ($2 < p < q$) if only if $p_1 = 2q^2 + 1$, $p = 3$, a contradiction.

If $G \cong C_2 \times C_2 \times C_{p_1^2}$, then $|\text{Aut}(G)| = 6p_1(p_1 - 1)$, which is impossible.

If $G \cong C_2 \times C_2 \times C_{p_1^3}$, then $|\text{Aut}(G)| = 6p_1^2(p_1 - 1)$, which is impossible.

Therefore G is isomorphic to G_7 or G_8 . The proof is completed. ■

Now we begin to study the non-nilpotent groups. We will prove that no non-nilpotent group G exists if $p \nmid q^2 - 1$. To make the proof clear, we give the following lemma.

Lemma 3.2. *Suppose that G is a non-nilpotent group such that $|\text{Aut}(G)| = 4pq^2$, where $p \nmid q^2 - 1$. Then G has a nontrivial abelian direct factor.*

Proof. Suppose that G has no nontrivial abelian direct factor.

Obviously, we may assume that $G = RPQ$, where $R \in \text{Syl}_2(G)$, $P \in \text{Syl}_p(G)$ and $Q \in \text{Syl}_q(G)$. Because $|\text{Aut}(G)| = 4pq^2$, G obtains a Sylow tower by Lemma 2.7, which implies that G has a normal series $Q \trianglelefteq PQ \trianglelefteq G$ and $P \trianglelefteq RP$.

(a) $|R \cap Z(G)| \leq 2$, meanwhile $|R| \mid 4$ if $R \cap Z(G) = 1$, and $R \cong C_4$ if $|R \cap Z(G)| = 2$.

If $R \cap Z(G) = 1$, then $|R| \mid 4$ since $|R| = |RZ(G)/Z(G)|$.

If $R \cap Z(G) \neq 1$, let $H = PQ$. We prove $|R \cap Z(G)| = 2$. Since $|G/HZ(G)| \mid |G/Z(G)|_2$, $G/HZ(G)$ is abelian and thus $G' \leq HZ(G)$. We assert that $G \neq HZ(G)$. Otherwise R is a nontrivial abelian direct factor of G , which contradicts to our assumption. Hence $2 \mid (|G/G'|, |Z(G)|)$, then $2 \mid |\text{Cen}(G)|$ and thus $|\text{Cen}(G)|_2 = 2$ or 4 by Lemma 2.4.

If $|\text{Cen}(G)|_2 = 4$, then every Sylow 2-subgroup of $G/Z(G)$ centralizes $G/Z(G)$, which means

$$G/Z(G) = RZ(G)/Z(G) \times HZ(G)/Z(G).$$

This implies that R acts trivially on $H/H \cap Z(G)$ and $H \cap Z(G)$, and then acts trivially on H . Thus $G = R \times H$. Hence by our assumption, R is

non-abelian and then $|R| \geq 8$. However, $|R/Z(R)| \mid |R/R \cap Z(G)| \mid 4$, then $8 \mid |\text{Aut}(R)|$ by Lemma 2.1. It follows that $8 \mid |\text{Aut}(G)|$, a contradiction.

If $|\text{Cen}(G)|_2 = 2$. By Lemma 2.5, we have that $|R \cap Z(G)| = 2$ and every Sylow 2-subgroup of G/G' is cyclic. This implies $R/R \cap G' \cong RG'/G'$ is cyclic. Since $G/H(R \cap Z(G)) \cong R/R \cap Z(G)$ is abelian, $G' \leq H(R \cap Z(G))$. Then $R \cap G' \leq R \cap H(R \cap Z(G)) = R \cap Z(G)$ and thus $R/R \cap Z(G)$ is cyclic, which implies R is abelian. Hence $G/H \cong R$ is abelian and $G' \leq H$, therefore $R \cap G' = 1$ and R is cyclic. Since $|R \cap Z(G)| = 2$, one has that $|R| \mid 8$. If $|R| = 2$, then $R \leq Z(G)$ and hence R is a nontrivial abelian direct factor of G , a contradiction. If $|R| = 8$, then $|RZ(G)/Z(G)| = |R/R \cap Z(G)| = 4$ and thus $|\text{Inn}(G)|_2 = 4$. Hence $|Z(\text{Inn}(G))|_2 \neq 1$ since $|\text{Cen}(G)|_2 = 2$ and $4 \nmid |\text{Aut}(G)|$. Let I_u be an element of order 2 in $Z(\text{Inn}(G))$, where I_u is an inner automorphism induced by $u \in R$. Hence $\langle u \rangle$ acts trivially on $H/H \cap Z(G)$ and $H \cap Z(G)$, which forces $[\langle u \rangle, H] = 1$. Because R is abelian, $u \in Z(G)$, a contradiction. Thus $|R \cap Z(G)| = 2$ and $R \cong C_4$.

(b) $P \cap Z(G) = 1$ and $|P| \mid p$.

We may assume $P \neq 1$. Because G is solvable and has a normal series $Q \trianglelefteq PQ \trianglelefteq G$, G must have a $\{2, p\}$ -Hall subgroup L . Moreover $L \cap PQ \in \text{Syl}_p(G)$, which implies that L has a normal p -Sylow subgroup. Without loss of generality, we assume that $L = RP$ and $P \trianglelefteq L$. Since $|P/Z(P)| \mid p$, we have that P is abelian and thus $P = C_P(R) \times [R, P]$. Because $P \cap Z(G) \leq C_P(R)$ and $|P/P \cap Z(G)| \mid p$, $|P/C_P(R)| \mid p$ and $|[R, P]| = 1$ or p .

Assume $|[R, P]| = 1$. Let $B = RQ$, then B is normal in G , and $G' \leq B$ for G/B is abelian. This implies $p \mid |G/G'|$. Moreover if $P \cap Z(G) \neq 1$, then $p \mid |\text{Cen}(G)|$. We have $p \mid |\text{Inn}(G)|_p$ for there exists no nontrivial abelian direct factor of G . However $|\text{Aut}(G)|_p = p$, then $p \mid |Z(\text{Inn}(G))|$. Take $I_v \in Z(\text{Inn}(G))$ such that $|I_v| = p$, where I_v is an inner automorphism induced by $v \in P$. Then $\langle v \rangle$ acts trivially on $B/B \cap Z(G)$ and $B \cap Z(G)$. Because P is abelian, $[\langle v \rangle, P] = 1$ and thus $v \in Z(G)$ which contradicts $|I_v| = p$. Hence $P \cap Z(G) = 1$, $|P| \mid p$.

If $|[R, P]| = p$, then it follows by $|P/C_P(R)| \mid |P/P \cap Z(G)| = p$, that $P \cap Z(G) = C_P(R)$. This implies $G = (P \cap Z(G)) \times [R, P]B$. Since there exists no nontrivial abelian direct factor of G , we get that $P \cap Z(G) = 1$ and $|P| \mid p$.

(c) $G = [P \times Q]R$.

Since $|P| \mid p$ and $p \nmid q^2 - 1$, P acts trivially on $Q/Q \cap Z(G)$ and $Q \cap Z(G)$ by conjugation. This gives $[P, Q] = 1$ and then $G = [P \times Q]R$.

In the following, we divide our arguments into several parts by considering the order of $G/Z(G)$.

(1) There exists no finite group G such that $|\text{Aut}(G)| = 4pq^2$ and $|G/Z(G)|_q \mid q$.

At first, we assert that $Q \cap Z(G) = 1$ and $|Q| \mid q$. Since $|Q/Z(Q)| \mid |G/Z(G)|_q \mid q$, Q is abelian and thus $Q = C_Q(RP) \times [RP, Q]$. Clearly, $G = RP[RP, Q] \times C_Q(RP)$. Because there is no nontrivial abelian direct factor of G , $C_Q(RP) = 1$. Therefore $Q \cap Z(G) \leq C_Q(RP) = 1$, which gives $|Q| = |Q/Q \cap Z(G)| \mid q$.

Now, we continue the proof by investigating R .

If R is cyclic, then every Sylow subgroup of G is cyclic. It follows from Lemma 2.6 that $G = \langle a, b \mid a^m = b^t = 1, b^{-1}ab = a^r, ((r-1)t, m) = 1, r^t \equiv 1 \pmod{m} \rangle$. If $Z(G) = 1$, then $m\varphi(m) = |\text{Aut}(G)| = 4pq^2$ by Lemma 2.9. If $q \mid m$, then $q^3 \mid m\varphi(m)$ or $q \parallel m\varphi(m)$, a contradiction. If $q^2 \mid \varphi(m)$, then $m \mid 4p$, again a contradiction. If $|Z(G)| = 2$, then $2m\varphi(m) = |\text{Aut}(G)| = 4pq^2$ by Lemma 2.9, which leads to a contradiction similarly.

If $R \cong C_2 \times C_2$. When P and Q are both trivial, we have that $|\text{Aut}(G)| = 6$, a contradiction. If only one of P and Q is trivial, we may assume $Q \neq 1$ and $P = 1$. In this case $Z(G) = 1$ by (a), (b) and $Q \cap Z(G) = 1$. But $C_R(Q) = 1$, so $\text{Aut}(Q)$ contains a subgroup which is isomorphic to $C_2 \times C_2$, a contradiction. If $P \neq 1$ and $Q \neq 1$, then

$$G = \langle a, b, x, y \mid a^2 = b^2 = x^p = y^q = 1, [a, b] = [x, y], x^a = x^{-1}, y^a = y, x^b = x, y^b = y^{-1} \rangle.$$

It is easy to check that $|\text{Aut}(G)| = pq(p-1)(q-1)$, which contradicts $|\text{Aut}(G)| = 4pq^2$.

- (2) There exists no finite group G such that $|\text{Aut}(G)| = 4pq^2$ and $|G/Z(G)| = q^2$ or pq^2 .

Since $G = [P \times Q]R$, it follows that G is nilpotent by Lemma 2.10 and 2.11, a contradiction.

- (3) There exists no finite group G such that $|\text{Aut}(G)| = 4pq^2$ and $|G/Z(G)| = 2q^2$.

In this case, R is cyclic, $|R| \mid 4$ and $P=1$ by (a) and (b).

Now, we show that Q is non-abelian. Suppose that Q is abelian, then $Q = C_Q(R) \times [R, Q]$ and $G = R[R, Q] \times C_Q(R)$. Because G contains no nontrivial abelian direct factor, $C_Q(R) = 1$ holds. This gives $Q \cap Z(G) \leq C_Q(R) = 1$ and then $|Q| = q^2$. If Q is cyclic, then any automorphism of Q can be extended to an automorphism of G , which forces that $\text{Aut}(G)$ contains an outer automorphism of order q , a contradiction to $|G/Z(G)| = 2q^2$. Then $Q \cong C_q \times C_q$. If we can prove $Z(G) = 1$, and then by Lemma 2.12 we come to a contradiction: $2^3 \mid |\text{Aut}(G)|$.

To prove $Z(G) = 1$, it suffices to prove that $R \cong C_2$. Otherwise, $R \cong C_4$. According to Lemma 2.10, G is supersolvable since $|G/Z(G)| = 2q^2$. By Lemma 2.15 we can choose x and y such that $Q = \langle x \rangle \times \langle y \rangle$, $\langle x \rangle \trianglelefteq G$ and $\langle y \rangle \trianglelefteq G$. Let $G = [\langle x \rangle]K$, where $K = R\langle y \rangle$. Define

$$\alpha: x^t b \mapsto x^{-t} b, b \in K, t \in \mathbb{Z},$$

then α is an automorphism of G of order 2. Let $R = \langle a \rangle$ and I_a be an inner automorphism of G induced by a , then $|a| = 4$ and $|I_a| = 2$. Since $C_Q(R) = 1$, it is easy to show that

$$x^{I_a} = x^{-1}, y^{I_a} = y^{-1} \quad \text{and} \quad \alpha I_a = I_a \alpha.$$

Therefore, $\langle I_a, \alpha \rangle = \langle I_a \rangle \times \langle \alpha \rangle$ makes itself a Sylow 2-subgroup of $\text{Aut}(G)$. On the other hand, by Lemma 2.4, there exists a central automorphism σ ($\in \text{Aut}(G)$) of order 2 since $|R \cap Z(G)| = 2$. Replacing σ by its suitable conjugate, we may assume $\sigma \in \langle I_a, \alpha \rangle$. But neither I_a nor α is a central automorphism of G , which forces $\sigma = \alpha I_a$. Then

$$yZ(G) = (yZ(G))^\sigma = (yZ(G))^{\alpha I_a} = y^{-1}Z(G).$$

This implies $y^2 \in Z(G)$, which contradicts $Q \cap Z(G) = 1$. Thus Q is non-abelian.

Now, we show that G does not exist.

Since Q is non-abelian and $|Q/Z(Q)| \mid |Q/Q \cap Z(G)| = q^2$, $Z(Q) = Q \cap Z(G) > 1$ and $Q/Z(Q) \cong C_q \times C_q$. And since $|G/Z(G)| = 2q^2$, $G/Z(G)$ is supersolvable, which means that there exists $w \in Q$ such that $\langle wZ(G) \rangle \trianglelefteq G/Z(G)$. For $a \in R$, there exists an integer k such that $w^k w^a \in Q \cap Z(G) = Z(Q)$, which implies that $\langle wZ(Q) \rangle$ is R -invariant. Let $A/Z(Q) = \langle wZ(Q) \rangle$, then by Lemma 2.15, it follows that

$$Q/Z(Q) = A/Z(Q) \times B/Z(Q),$$

where both of $A/Z(Q)$ and $B/Z(Q)$ are R -invariant subgroups of $Q/Z(Q)$ with order q . This implies that A and B are R -invariant abelian subgroups of Q . By Lemma 2.14, it follows that $[R, A] \neq 1 \neq [R, B]$ (Otherwise $[R, Q] = 1$, which contradicts the fact that there exists no nontrivial abelian direct factor of G). Let $[R, A] = \langle x \rangle$ and $[R, B] = \langle y \rangle$, where x and y are elements of order q . And let $G = [A]K$, where $K = R\langle y \rangle$. Notice that A is an abelian q -group. Then for every m such that $(|A|, m) = 1$, there is an automorphism α_m of G by Lemma 2.8, defined as $(au)^{\alpha_m} = a^m u$, $a \in A$, $u \in K$. If A contains an element of order q^2 , then we can take m such that $|\alpha_m| = q$. Obviously α_m is an outer automorphism of G . But $|\alpha_m \text{Inn}(G)| \mid (q, 2p)$, by $|\text{Aut}(G)| = 4pq^2$, one has that $\alpha_m \in \text{TMInn}(G)$, a contradiction. Therefore A an elementary abelian q -group. At this moment we can take an automorphism α_m of G , such that $|\alpha_m| = q - 1$. Clearly α_m acts untrivially on $Q \cap Z(G)$, which means that α_m is an outer automorphism of G too. But $|\alpha_m \text{Inn}(G)| \mid (q - 1, 2p)$, by $|\text{Aut}(G)| = 4pq^2$, it follows that $(q - 1, 2p) = 2$ by $p \nmid q^2 - 1$. Therefore $|\alpha_m \text{Inn}(G)| = 2$ and $|\alpha_m^2| = \frac{q-1}{2}$ since α_m is an outer automorphism. Thus $\frac{q-1}{2} \mid 2q^2$ and so $\frac{q-1}{2} \mid 2$, which contradicts $q > p > 2$ and $p \nmid q^2 - 1$.

(4) There exists no finite group G such that $|\text{Aut}(G)| = 4pq^2$ and $|G/Z(G)| = 4q^2$.

In this case $R \cap Z(G) = 1$, $|R| = 4$ and $P = 1$ by (a) and (b). This implies $C_R(Q) = 1$.

By Lemma 2.3 of [12], one has that $Q = [A]B$, where A is abelian, B is cyclic, and $A = Q$ if Q is abelian. It is easy to take an automorphism α of Q having order 2, defined as

$$\alpha : ab \mapsto a^{-1}b, \quad a \in A, \quad b \in B.$$

If $|Q| \geq q^3$, then $Z = Q \cap Z(G) \cap A \neq 1$ and $\alpha|_Z \neq 1$. Let $X = \{\sigma \in \text{Aut}(Q) \mid Z^\sigma = Z\}$ and $\text{Aut}_Z(Q) = \{\sigma \in \text{Aut}(Q) \mid \sigma|_Z = 1\}$, then both of X and $\text{Aut}_Z(Q)$ are groups such that $\text{Aut}_Z(Q) \trianglelefteq X$. Since $C_R(Q) = 1$, one has that $R \leq \text{Aut}_Z(Q)$. Obviously, $2 \mid |X/\text{Aut}_Z(Q)|$ for $\alpha \in X \setminus \text{Aut}_Z(Q)$ and $|\alpha| = 2$. Thus $8 \mid |X|$ and R is not a Sylow 2-subgroup of X . Then there exists $\beta \in X \setminus R$ such that $R^\beta = R$ and β is a 2-element. Clearly β can be extended to an outer automorphism of G , which contradicts $|G/Z(G)| = 4q^2$ for $|\beta|$ is a power of 2. This forces $|Q| = q^2$. If Q is cyclic, then every automorphism of Q can be extended to an automorphism of G , which means that there is an outer automorphism of G with order q , a contradiction. Therefore $Q = C_q \times C_q$ and $Q \cap Z(G) = 1$, which implies $Z(G) = 1$. And then by Lemma 2.12, we come to a contradiction: $2^3 \mid |\text{Aut}(G)|$.

(5) There exists no finite group G such that $|\text{Aut}(G)| = 4pq^2$ and $|G/Z(G)| = 2pq^2$.

In this case R is cyclic, $|R| \mid 4$ and $P \cong C_p$. Since $G = [P \times Q]R$, we may define $\sigma_i \in \text{Aut}(G)$ as:

$$\sigma_i : xyz \mapsto xy^i z, \quad x \in R, \quad y \in P, \quad z \in Q, \quad (i, p) = 1.$$

It is easy to show that $\{\sigma_i \mid (i, p) = 1, i \in \mathbb{Z}\}$ forms a cyclic group of $p - 1$. For any $z \in Q$ and any $i \in \mathbb{Z}$ satisfying $(i, p) = 1$, we have $I_z \sigma_i = \sigma_i I_z$. Thus $|\langle \sigma_i, I_z \mid (i, p) = 1, i \in \mathbb{Z}, z \in Q \rangle| = (p - 1)q^2$, which implies that $(p - 1)q^2 \mid 4pq^2$. Noticing $p \nmid q^2 - 1$, we get that $p = 5$.

Now, we show that Q is non-abelian. Otherwise, if Q is abelian, then $Q = C_Q(RP) \times [RP, Q]$ and $G = RP[RP, Q] \times C_Q(RP)$. Because G contains no nontrivial abelian direct factor, $C_Q(RP) = 1$. This gives that $Q \cap Z(G) \leq C_Q(RP) = 1$ and then $|Q| = q^2$. If Q is cyclic, then any automorphism of Q can be extended to an automorphism of G , which forces that $\text{Aut}(G)$ contains an outer automorphism of order q , a contradiction. Hence $Q \cong C_q \times C_q$. If we can prove that $Z(G) = 1$, and then by Lemma 2.13 we come to a contradiction: $2^3 \mid |\text{Aut}(G)|$.

To prove $Z(G) = 1$, it suffices to prove that $R \cong C_2$. Otherwise, $R \cong C_4$. By Lemma 2.10 and Lemma 2.11, G is supersolvable since $G = [P \times Q]R$, $|G/Z(G)/PZ(G)/Z(G)| = 2q^2$ and $|G/Z(G)/QZ(G)/Z(G)| = 2p$. By Lemma 2.15, we can take x and y such that $Q = \langle x \rangle \times \langle y \rangle$, $\langle x \rangle \trianglelefteq G$ and $\langle y \rangle \trianglelefteq G$. Let $R = \langle a \rangle$ and $P = \langle z \rangle$. Define α by

$$\alpha : a^i z^j x^k y^l \mapsto a^i z^j x^{-k} y^l, \quad i, j, k, l \in \mathbb{Z},$$

then α is an automorphism of G having order 2. Let I_a be an inner automorphism of G induced by a , then $|I_a| = 2$ and $C_Q(R) = 1$ for $|R \cap Z(G)| = 2$ and $[P, Q] = 1 = C_Q(RP)$. Therefore,

$$x^{I_a} = x^{-1}, \quad y^{I_a} = y^{-1}, \quad z^{I_a} = z^{-1} \text{ or } z.$$

Clearly $\alpha I_a = I_a \alpha$, thus $\langle I_a, \alpha \rangle = \langle I_a \rangle \times \langle \alpha \rangle$ is a Sylow 2-subgroup of $\text{Aut}(G)$. By Lemma 2.4, there exists a central automorphism $\sigma (\in \text{Aut}(G))$ of order 2 since $|R \cap Z(G)| = 2$. Replacing σ by its suitable conjugate, we may assume $\sigma \in \langle I_a, \alpha \rangle$. But neither I_a nor α is a central automorphism of G , which forces $\sigma = \alpha I_a$. Then

$$yZ(G) = (yZ(G))^\sigma = (yZ(G))^{I_a \alpha} = y^{-1}Z(G),$$

which implies $y^2 \in Z(G)$, a contradiction to $Q \cap Z(G) = 1$. Thus Q is non-abelian.

Now, we show that G doesn't exist.

Because Q is non-abelian and $|Q/Z(Q)| \mid |Q/Q \cap Z(G)| = q^2$, one has that $Z(Q) = Q \cap Z(G) > 1$ and $Q/Z(Q) \cong C_q \times C_q$. Since

$$G/Z(G) = [PZ(G)/Z(G) \times QZ(G)/Z(G)]RZ(G)/Z(G),$$

it follows by Lemma 2.10 and Lemma 2.11 that $G/Z(G)$ is supersolvable, which implies that there exists $w \in Q$ such that $\langle wZ(G) \rangle \leq QZ(G)/Z(G)$ and $\langle wZ(G) \rangle \trianglelefteq G/Z(G)$. For $a \in RP$, there exists an integer k such that $w^k w^a \in Q \cap Z(G) = Z(Q)$, consequently $\langle wZ(Q) \rangle$ is RP -invariant. Let $A/Z(Q) = \langle wZ(Q) \rangle$, then by Lemma 2.15, one has that

$$Q/Z(Q) = A/Z(Q) \times B/Z(Q),$$

where both of $A/Z(Q)$ and $B/Z(Q)$ are RP -invariant subgroups of $Q/Z(Q)$ of order q . This implies A and B are RP -invariant abelian subgroups of Q .

If $[RP, Q] = 1$, then $G = RP \times Q$ and $[R, P] \neq 1$ for G has no nontrivial abelian direct factor. Because $|G/Z(G)| = 2pq^2$, we have $RP \cong \langle a, b \mid a^2 = b^5 = 1, b^a = b^{-1} \rangle$ or $\langle a, b \mid a^4 = b^5 = 1, b^a = b^{-1} \rangle$. Clearly, $|\text{Aut}(RP)| = 20$ or 40 . But $|\text{Aut}(RP)||\text{Aut}(Q)| \mid 20q^2$, we have that $|\text{Aut}(Q)| \mid q^2$. By Lemma 2.2, we come to a contradiction $|Q| \leq 2$.

If $[RP, Q] \neq 1$, then $[RP, A] \neq 1 \neq [RP, B]$ by Lemma 2.14. Let $[RP, A] = \langle x \rangle$, $[RP, B] = \langle y \rangle$, where x and y are two elements of order q . And let $G = [A]K$, where $K = RP\langle y \rangle$. Notice that A is an abelian q -group. Then for every m such that $(|A|, m) = 1$, there is an automorphism α_m of G by Lemma 2.8, defined by $(au)^{\alpha_m} = a^m u$, $a \in A$, $u \in K$. If A contains an element of order q^2 , then there exists an integer m such that $|\alpha_m| = q$. Obviously α_m is an outer automorphism of G . On the other hand, $|\alpha_m \text{Inn}(G)| \mid (q, 2)$, it forces that $\alpha_m \in \text{Inn}(G)$, a contradiction. Hence A is an elementary abelian q -group. By the same way we can take an automorphism α_m of G such that $|\alpha_m| = q - 1$. Clearly α_m acts untrivially on $Q \cap Z(G)$, which means that α_m is an outer automorphism of G . But $|\alpha_m \text{Inn}(G)| \mid (q - 1, 2)$, it follows that $(q - 1, 2) = 2$. Consequently $|\alpha_m \text{Inn}(G)| = 2$ and $|\alpha_m^2| = \frac{q-1}{2}$. Therefore $\frac{q-1}{2} \mid 2$. But $\frac{q-1}{2} \mid 2pq^2$ and $p \nmid q^2 - 1$, which contradicts the assumptions $q > p > 2$ and $p \nmid q^2 - 1$.

(6) There exists no finite group G such that $|\text{Aut}(G)| = 4pq^2$ and $|G/Z(G)| = 4pq^2$.

In this case, $R \cap Z(G) = 1$, $|R| = 4$ and $P \cong C_p$. This implies $C_R(PQ) = 1$. We show that G doesn't exist.

Since $G = [P \times Q]R$, we can define $\sigma_i \in \text{Aut}(G)$ as:

$$\sigma_i : xyz \mapsto xy^i z, \quad x \in R, \quad y \in P, \quad z \in Q, \quad (i, p) = 1.$$

It is easy to show that $\{\sigma_i \mid (i, p) = 1, i \in \mathbb{Z}\}$ forms a cyclic group of $p - 1$. For any $z \in Q$ and any $i \in \mathbb{Z}$ satisfying $(i, p) = 1$, we have $I_z \sigma_i = \sigma_i I_z$. Thus $|\langle \sigma_i, I_z \mid (i, p) = 1, i \in \mathbb{Z}, z \in Q \rangle| = (p - 1)q^2$. This implies that $(p - 1)q^2 \mid 4pq^2$. But $p \nmid q^2 - 1$, we get that $p = 5$. Because $4 \parallel |\text{Aut}(G)|$, $R \cap Z(G) = 1$ and $|R| = 4$, we have $\langle \sigma_i \mid (i, 5) = 1 \rangle$ consists of inner automorphisms of G . That is to say, replacing R by its suitable conjugate, we can get the subgroup of inner automorphism group $RZ(G)/Z(G) = \langle \sigma_i \mid (i, 5) = 1 \rangle$, then $[R, Q] = 1$, $[RP, Q] = 1$ and $[R, P] \neq 1$. Hence $G = RP \times Q$ ($R \cong C_4$, $P \cong C_5$). Therefore, RP is isomorphic to $\langle a, b \mid a^4 = b^5 = 1, b^a = b^2 \rangle$ or $\langle a, b \mid a^4 = b^5 = 1, b^a = b^{-2} \rangle$. We can easily see that $|\text{Aut}(RP)| = 40$, which contradicts to $|\text{Aut}(RP)||\text{Aut}(Q)| \mid 20q^2$.

This is the end of the proof. ■

Theorem 3.3. *Suppose that G is a non-nilpotent group such that $|\text{Aut}(G)| = 4pq^2$ and $p \nmid q^2 - 1$. Then G does not exist.*

Proof. Assume that the theorem is not true, then there exists a non-nilpotent group G such that $|\text{Aut}(G)| = 4pq^2$.

By Lemma 3.2, we have $G = H \times A$, where H is a non-nilpotent group without nontrivial abelian direct factor and A is a nontrivial abelian group. Hence

$$|\text{Aut}(H)||\text{Aut}(A)| \mid 4pq^2.$$

We get that $|\text{Aut}(H)|$ cannot be $4pq^2$ or $4q^2$ by Lemma 3.2 and [13, Theorem 3.2]. Since q is an odd prime, by Lemma 2.2, neither $|\text{Aut}(H)|$ nor $|\text{Aut}(A)|$ is equal to any one of q , q^2 , p , pq and pq^2 . Hence,

$$|\text{Aut}(H)| \in \{2p, 2q, 2pq, 2q^2, 2pq^2\}.$$

- (1) If $|\text{Aut}(H)| = 2p$ or $2q$, then H is nilpotent by [2, Theorem 1], a contradiction.
- (2) If $|\text{Aut}(H)| = 2pq$, then it follows by [2, Theorem 1] that $H \cong \langle a, b \mid a^m = b^q = 1, a^{-1}ba = b^t, t^m \equiv 1 \pmod{q}, m \mid 2p \rangle$, $q = 2p + 1$, a contradiction to $p \nmid q^2 - 1$.
- (3) If $|\text{Aut}(H)| = 2pq^2$, then H doesn't exist by [7, Lemma 2.2], so G does not exist.
- (4) If $|\text{Aut}(H)| = 2q^2$, then H doesn't exist by [2, Theorem 1], and G does not exist either.

Thus there doesn't exist a finite non-nilpotent group G such that $|\text{Aut}(G)| = 4pq^2$, where $p \nmid q^2 - 1$. ■

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FINITE GROUPS WITH $6PQ$ ELEMENTS OF THE LARGEST ORDER¹

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Abstract. It is an interesting topic to determine the structure of a finite group with a given number of elements of the largest order. In this article, it is proved that finite groups with $6pq$ elements of the largest order, where p, q are primes and $13 < p < q$, are solvable.

Keywords: finite group; number of elements of the largest order; solvable group; Thompson's Problem.

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1. Introduction

In this paper, all groups considered are finite. Let $\pi_e(G)$ be a set of orders of elements of a group G , k the largest order of elements of G and n the number of cyclic subgroups of order k . If t is a positive integer, then $\pi(t)$ denotes the set of prime divisors of t , especially, $\pi(G) = \pi(|G|)$. $M_t(G)$ denotes the set of elements of order t of G , especially, $M(G) = M_k(G)$. $N_t(G)$ denotes the subgroups generated by elements of order t , e.g., $N_t(G) = \langle a | a \in M_t(G) \rangle$, especially, $N(G)$ denotes the one generated by elements of the largest order. $\varphi(x)$ the Euler function of x . The other notations and terminology are standard (see [4]).

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Definition. ([9]) Two groups G_1 and G_2 are said to be conformal groups if $\pi_e(G_1) = \pi_e(G_2)$ and $|M_t(G)| = |M_t(S)|$ for all $t \in \pi_e(G_1)$.

Two conformal groups are also called Grassmann equivalent. An open problem was proposed by Thompson (see [9]): *Let G be a conformal group of a finite solvable group. Is G solvable or not?*

It is an interesting topic to determine the structure of finite groups by the number of elements of the largest order. Because it may give useful information to the open problem on the solvability of a group proposed by Thompson. Many authors had investigated the groups with some kinds of restriction to the number of elements of the largest order, they proved that these groups are solvable (see [2], [3], [6]-[8], [12]-[14]). In this paper we will continue this work and obtain the following theorem:

Main Theorem. *Let G be a finite group. If $|M(G)| = 6pq$, where p, q are primes and $13 < p < q$, then G is solvable.*

2. Preliminary results

Lemma 2.1. ([14, Lemma 2.2]) *Suppose that G has m cyclic subgroups of order l . Then $|M_l(G)| = m\varphi(l)$, particularly, $|M(G)| = n\varphi(k)$, where k is the largest order of elements of G .*

Corollary 2.1. *Let G be a group with k and n mentioned as above. If $|M(G)| = 6pq$, where p, q are primes, then all the possibilities for n, k and $\varphi(k)$ are as in Table 1.*

Table 1

n	$\varphi(k)$	k
1	$6pq$	k
3	$2pq$	(1) $r, 2r$, where $r = 2pq + 1$ is a prime; (2) q^2 , where $q = 2p + 1$ is a prime
p	$6q$	$r, 2r$, where $r = 6q + 1$ is a prime
$3p$	$2q$	$r, 2r$, where $r = 2q + 1$ is a prime
q	$6p$	$r, 2r$, where $r = 6p + 1$ is a prime
$3q$	$2p$	$r, 2r$, where $r = 2p + 1$ is a prime
pq	6	7, 14, 9, 18
$3pq$	2	3, 4, 6
$6pq$	1	2

Lemma 2.2. ([14, Theorem 1.1]) *Suppose k is the largest order of elements of G . If $|M(G)| = \varphi(k)$, then G is supersolvable.*

Let $n \in \mathbb{N}$. We say that a finite non-abelian simple group G is a simple K_n -group if $|\pi(G)| = n$.

Lemma 2.3. ([5]) *Let G be a simple K_3 -group. Then G is isomorphic to one of following simple groups: $A_5(2^2 \cdot 3 \cdot 5)$, $A_6(2^3 \cdot 3^2 \cdot 5)$, $L_2(7)(2^3 \cdot 3 \cdot 7)$, $L_2(8)(2^3 \cdot 3^2 \cdot 7)$, $L_2(17)(2^4 \cdot 3^2 \cdot 17)$, $L_3(3)(2^4 \cdot 3^3 \cdot 13)$, $U_3(3)(2^5 \cdot 3^3 \cdot 7)$, $U_4(2)(2^6 \cdot 3^4 \cdot 5)$.*

Lemma 2.4. ([10, Theorem 2]) *Let G be a simple K_4 -group. Then G is isomorphic to one of following simple groups:*

- (1) $A_7(2^3 \cdot 3^2 \cdot 5 \cdot 7)$, $A_8(2^6 \cdot 3^2 \cdot 5 \cdot 7)$, $A_9(2^6 \cdot 3^4 \cdot 5 \cdot 7)$, $A_{10}(2^7 \cdot 3^4 \cdot 5^2 \cdot 7)$.
- (2) $M_{11}(2^4 \cdot 3^2 \cdot 5 \cdot 11)$, $M_{12}(2^6 \cdot 3^3 \cdot 5 \cdot 11)$, $J_2(2^7 \cdot 3^3 \cdot 5^2 \cdot 7)$.
- (3) $L_2(16)(2^4 \cdot 3 \cdot 5 \cdot 17)$, $L_2(25)(2^3 \cdot 3 \cdot 5^2 \cdot 13)$, $L_2(49)(2^4 \cdot 3 \cdot 5^2 \cdot 7^2)$, $L_2(81)(2^4 \cdot 3^4 \cdot 5 \cdot 41)$, $L_3(4)(2^6 \cdot 3^2 \cdot 5 \cdot 7)$, $L_3(5)(2^5 \cdot 3 \cdot 5^3 \cdot 31)$, $L_3(7)(2^5 \cdot 3^2 \cdot 7^3 \cdot 19)$, $L_3(8)(2^9 \cdot 3^2 \cdot 7^2 \cdot 73)$, $L_3(17)(2^9 \cdot 3^2 \cdot 17^3 \cdot 307)$, $L_4(3)(2^7 \cdot 3^6 \cdot 5 \cdot 13)$, $S_4(4)(2^8 \cdot 3^2 \cdot 5^2 \cdot 17)$, $S_4(5)(2^6 \cdot 3^2 \cdot 5^4 \cdot 13)$, $S_4(7)(2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4)$, $S_4(9)(2^8 \cdot 3^8 \cdot 5^2 \cdot 41)$, $S_6(2)(2^9 \cdot 3^4 \cdot 5 \cdot 7)$, $O_8^+(2)(2^{12} \cdot 3^5 \cdot 5^2 \cdot 7)$, $G_2(3)(2^6 \cdot 3^6 \cdot 7 \cdot 13)$, $U_3(4)(2^6 \cdot 3 \cdot 5^2 \cdot 13)$, $U_3(5)(2^4 \cdot 3^2 \cdot 5^3 \cdot 7)$, $U_3(7)(2^7 \cdot 3 \cdot 7^3 \cdot 43)$, $U_3(8)(2^9 \cdot 3^4 \cdot 7 \cdot 19)$, $U_3(9)(2^5 \cdot 3^6 \cdot 5^2 \cdot 73)$, $U_4(3)(2^7 \cdot 3^6 \cdot 5 \cdot 7)$, $U_5(2)(2^{10} \cdot 3^5 \cdot 5 \cdot 11)$, $Sz(8)(2^6 \cdot 5 \cdot 7 \cdot 13)$, $Sz(32)(2^{10} \cdot 5^2 \cdot 31 \cdot 41)$, ${}^3D_4(2)(2^{12} \cdot 3^4 \cdot 7^2 \cdot 13)$, ${}^2F_4(2)'(2^{11} \cdot 3^3 \cdot 5^2 \cdot 13)$.
- (4) $L_2(t)$, where t is a prime satisfying the equation $t^2 - 1 = 2^a \cdot 3^b \cdot u^c$ for some $a, b, c \geq 1$ and a prime $u > 3$.
- (5) $L_2(2^m)$, satisfies the equations $2^m - 1 = u$ and $2^m + 1 = 3t^b$ for some $t > 3$, $b \geq 1$ and primes u, t .
- (6) $L_2(3^m)$, satisfies the equations $3^m - 1 = 2u^b$ and $3^m + 1 = 4t$ or $3^m - 1 = 2u$ and $3^m + 1 = 4t^b$, where u and t are odd primes and $b \geq 1$.

Lemma 2.5. ([10, Theorem 1]) *Let G be a simple K_4 -group with $3 \notin \pi(G)$. Then $G \cong Sz(8)(2^6 \cdot 5 \cdot 7 \cdot 13)$ or $G \cong Sz(32)(2^{10} \cdot 5^2 \cdot 31 \cdot 41)$.*

Lemma 2.6. ([14, Theorem 2.2]) *Suppose that $a \in M_r(G)$. If $\pi_e(C_G(a)) \subseteq \pi_e(\langle a \rangle)$, then $C_G(a) \leq N_r(G)$.*

Lemma 2.7. *If $|M(G)| = 2m$, where m is an odd positive integer, then:*

- (1) $k = 4$, q^r or $2q^r$, where q is an odd prime and $r \in \mathbb{N}$.
- (2) If G is non-solvable, then $k = 2q^r$ for some odd prime q and $2 \parallel \varphi(k) = (q - 1)q^{r-1}$. Moreover, any Sylow 2-subgroup of G contains a maximal subgroup which is elementary abelian.
- (3) If $k = 14$, $|G| = 2^\alpha \cdot 3 \cdot 7^\beta$ and G is non-solvable, then $G \cong E \times L_2(7)$, where E is an elementary abelian 2-subgroup.

Proof. By [6, Lemma 2.7], (1)-(2) is true. (3) is hold by the proof of [6, Lemma 3.4(2)]. ■

Lemma 2.8. ([11, p. 11]) *Suppose that H is a proper subgroup of a group G . Let $H_1, H_2, \dots, H_n$ be all conjugate classes of H in G . Then $\langle H_1, H_2, \dots, H_n \rangle = H_1 H_2 \dots H_n$.*

Lemma 2.9. *Suppose that G has n cyclic subgroups A_i of order k , where $i = 1, 2, \dots, n$. Let $\{A_1, A_2, \dots, A_s\}$ be a complete representative system of the conjugate classes of n cyclic subgroups of order k , and let n_i be the length of the conjugate classes containing A_i . Then the following statements hold:*

- (1) $n_i = |G : N_G(A_i)|$, $n = \sum_{i=1}^n n_i$, $\pi(n_i) \cup \pi(A_i) = \pi(n_1) \cup \pi(A_1)$,
where $i = 1, 2, \dots, s$.
- (2) $\pi(C_G(A_i)) = \pi(A_i)$, $|N_G(A_i) : C_G(A_i)| \mid \varphi(k)$ and $|G| = |G : N_G(A_i)| |N_G(A_i) : C_G(A_i)| |C_G(A_i)|$, where $i = 1, 2, \dots, s$.
- (3) Let $A = \langle a \rangle$, $|a| = k$. If $i = 1$, $|M(G)| = 2m$, where m is an odd positive integer, then G is solvable.

Proof. By [3, Lemma 2.6], (1), (2) are true. Now, we prove (3). Since $|a| = k$ is an element of the largest order, we have $\pi_e(C_G(a)) \subseteq \pi_e(\langle a \rangle)$. By Lemma 2.6, $C_G(A) \leq N(G)$. If $i = 1$, then the all cyclic subgroups A_i of order k are conjugate in G , so $N(G) = A^G = AA^{g_1} \dots A^{g_{n-1}}$ by Lemma 2.8. Since $|M(G)| = \varphi(k) = 2m$, then by Lemma 2.7, $k = 4$, q^r or $2q^r$, where q is an odd positive integer, thus $\pi(N(G)) \leq 2$, so $N(G)$ is solvable. Since $N(G) \triangleleft G$, we get $|G/N(G)| \mid |G : C_G(A)| = n \cdot |N_G(A) : C_G(A)| \mid |M(G)|$, so $G/N(G)$ is solvable, hence G is solvable. ■

Lemma 2.10. ([11]) *Let $G = L_2(q)$. Then G has a subgroup which is isomorphic to the dihedral group containing the cyclic subgroup of order $\frac{q \pm 1}{2}$.*

3. Proof of Main Theorem

Proof of Main Theorem. We will prove the theorem step by step according to the possible values of $\varphi(k)$.

Case 1. If $n = 1$ and $\varphi(k) = 6pq$, it can be shown that G is supersolvable from Lemma 2.1 and Lemma 2.2.

Case 2. If $n = 3$ and $\varphi(k) = 2pq$, then by Corollary 2.1, (1) $k = r$ or $2r$, where $r = 2pq + 1$ is a prime; (2) $k = q$ or q^2 where $q = 2p + 1$ is a prime. Let a be an element with largest order k of G and let $A = \langle a \rangle$. Then it is a well-known fact that

$$|G| = |G : N_G(A)| \cdot |N_G(A) : C_G(A)| \cdot |C_G(A)|. \quad (*)$$

Suppose that G is non-solvable, by Lemma 2.7 (2), we get $k = 2r$. Let $|C_G(A)| = 2^\alpha \cdot r^\beta$, where $\alpha, \beta \in \mathbb{N}$. Clearly, $C_G(A)$ is solvable. The fact $N_G(A)/C_G(A) \leq \text{Aut}(A)$ implies that $|N_G(A)/C_G(A)| \mid \varphi(k)$, thus $N_G(A)/C_G(A)$ is solvable, hence $N_G(A)$ is solvable. Let η be the permutation representation of G on right cosets of $N_G(A)$ in G . Then $G/\ker \eta \lesssim S_3$. Since $\ker \eta \leq N_G(A)$ is solvable, we get G is solvable, a contradiction.

Case 3. If $n = p$ and $\varphi(k) = 6q$, then by Corollary 2.1, $k = r$ or $2r$, where $r = 6q + 1$ is a prime. Suppose that G is non-solvable, by Lemma 2.7 (2), we get $k = 2r$. Using the same argument as *Case 2*, we get $N_G(A)$ is solvable. By Lemma 2.9 (2) and (3), we have $i \geq 2$, so there exists a i such that $p \nmid n_i$, thus G is a $\{2, 3, q, r\}$ -group from (*). Let $|C_G(A)| = 2^\alpha \cdot r^\beta$, where $\alpha, \beta \in \mathbb{N}$. Then $C_G(A)$ has no elements of order 2^2 and r^2 . If $\beta \geq 2$, then $C_G(A)$ has at least $r^2 - 1$ elements of order $2r$. By $q > p$, we get $r^2 - 1 = 6q(6q + 2) > 6pq$, a contradiction. Thus $\beta = 1$. Let R be a Sylow r -subgroup of $C_G(A)$. If $R \ntriangleleft C_G(A)$, then $C_G(A)$ has at least $r + 1$ Sylow r -subgroups, so $C_G(A)$ has at least $(r + 1) \cdot (r - 1) = r^2 - 1$ elements of order $2r$, a contradiction. So $R \triangleleft C_G(A)$, thus $R \text{ Char } C_G(A) \triangleleft N_G(A)$, hence $N_G(A) \leq N_G(R)$. Since $|G : N_G(A)| = |G : N_G(R)| \cdot |N_G(R) : N_G(A)|$ and $|G : N_G(A)| < p$. By Sylow Theorem, $|G : N_G(R)| = 1$, so $R \triangleleft G$. Thus G/R is a $\{2, 3, q\}$ -group. By Lemma 2.3 and $q > p > 13$, G/R is solvable, so is G , a contradiction.

Case 4. If $n = 3p$ and $\varphi(k) = 2q$, then by Corollary 2.1, $k = r$ or $2r$, where $r = 2q + 1$ is a prime. Suppose that G is non-solvable, by Lemma 2.7 (2), we get $k = 2r$. Using the same argument as *Case 2*, we get $N_G(A)$ is solvable and $\pi(G) \subseteq \{2, 3, p, q, r\}$. By Lemma 2.9 (2) and (3), we have $i \geq 2$. If $3 \nmid (n_1, n_2, \dots, n_s)$, then there exists a i such that $3 \nmid n_i$. By $|G| = n_i \cdot |N_G(A) : C_G(A)| \cdot |C_G(A)|$, we get G is a $\{2, p, q, r\}$ -group. Since p, q are primes and $q > p > 13$, we have G is solvable by Lemma 2.3 and Lemma 2.5, a contradiction. If $3 \mid (n_1, n_2, \dots, n_s)$, then $3 \parallel (n_1, n_2, \dots, n_s)$ by $n_1 + n_2 + \dots + n_s = 3p$. Let $n_i = 2^{\alpha_i} \cdot 3^{\beta_i} \cdot p^{\gamma_i} \cdot q^{\theta_i} \cdot r^{\lambda_i}$, where $\alpha_i, \beta_i, \gamma_i, \theta_i, \lambda_i \in \mathbb{N}$. By $|G| = n_i \cdot |N_G(A_i) : C_G(A_i)| \cdot |C_G(A_i)|$ and $n_i < 3p$, there exists a i such that $\beta_i = 1, \gamma_i = \theta_i = \lambda_i = 0$, so G is a $\{2, 3, q, r\}$ -group and $3 \parallel |G|$. Since G is non-solvable and $q > p > 13$, we get no section of G which is isomorphic to the one of simple K_3 -groups, so G has a section W which is isomorphic to the one of simple K_4 -groups. By Lemma 2.4, W is isomorphic to one of following simple K_4 -groups:

(II): Assume that $W \cong L_2(t)$, where t is a prime satisfying the equation $t^2 - 1 = 2^a \cdot 3^b \cdot u^c$ for some $a, b, c \geq 1$ and a prime $u > 3$.

Now, $t = r$, $u = q$. Then $r^2 - 1 = 2^a \cdot 3^b \cdot q^c$. Clearly, $b = 1$, so $2q(r + 1) = 2^a \cdot 3 \cdot q^c$, thus $2q + 2 = r + 1 = 2^{a-1} \cdot 3 \cdot q^{c-1}$. If $c \geq 2$, then $q \mid 2$, a contradiction. Hence $c = 1$. By Lemma 2.10, $L_2(r)$ has a subgroup which is isomorphic to the dihedral group containing the cyclic subgroup of order $\frac{r+1}{2}$, then it has a cyclic subgroup of order $\frac{r+1}{2} = 2^{a-2} \cdot 3$, so it has the element of order 2^{a-2} . If $a \geq 5$, then G has an element of order 8, contrary to the Lemma 2.7 (2), thus $a \leq 4$, hence $q + 1 = 2^{a-2} \cdot 3 \leq 12$, a contradiction. Therefore, $W \not\cong L_2(r)$.

(III): Assume that $W \cong L_2(2^m)$, satisfies the equations $2^m - 1 = u$ and $2^m + 1 = 3t^b$ for some $t > 3$, $b \geq 1$ and primes u, t .

Now, if $u = q$, $t = r$, then $2 = 3r^b - q = 3(2q + 1)^b - q \geq 5q + 3 > 3$, a contradiction. If $u = r$, $t = q$, then $2 = 3q^b - r = 3q^b - (2q + 1)$, so $3q^b - 3 = 2q$, thus $3 \mid 2q$, hence $3 \mid q$, a contradiction. Therefore, $W \not\cong L_2(2^m)$.

(IIII): Assume that $W \cong L_2(3^m)$, satisfies the equations $3^m - 1 = 2u^b$ and $3^m + 1 = 4t$ or $3^m - 1 = 2u$ and $3^m + 1 = 4t^b$, where u and t are odd primes and $b \geq 1$.

Since $|L_2(3^m)| = \frac{3^m \cdot (3^m + 1) \cdot (3^m - 1)}{2}$ and $|L_2(3^m)| \mid |G|$, we have $m = 1$, it is impossible. Therefore, $W \not\cong L_2(3^m)$.

Case 5. If $n = q$ and $\varphi(k) = 6p$, then by Corollary 2.1, $k = r$ or $2r$, where $r = 6p + 1$ is a prime. Suppose that G is non-solvable, by Lemma 2.7 (2), we get $k = 2r$. Using the same argument as *Case 2*, we get $N_G(A)$ is solvable and $\pi(G) \subseteq \{2, 3, p, q, r\}$. By Lemma 2.9 (2) and (3), we have $i \geq 2$. The same discussion as *Case 4*, we get G is a $\{2, 3, p, r\}$ -group and $3 \parallel |G|$. Since G is non-solvable and $q > p > 13$, by Lemma 2.3, if there is a section Y of G which is isomorphic to the simple K_3 -groups, then $Y \cong L_2(17)$, so $3^2 \mid |G|$, a contradiction. Thus G has a section W which is isomorphic to the one of simple K_4 -groups. By Lemma 2.4, W is isomorphic to one of following simple K_4 -groups:

(I): Assume that $W \cong L_2(t)$, where t is a prime satisfying the equation $t^2 - 1 = 2^a \cdot 3^b \cdot u^c$ for some $a, b, c \geq 1$ and a prime $u > 3$.

Now, $t = r$, $u = p$. Then $r^2 - 1 = 2^a \cdot 3^b \cdot p^c$. Clearly, $b = 1$, so $6p(r + 1) = 2^a \cdot 3 \cdot p^c$, $6p + 2 = r + 1 = 2^{a-1} \cdot p^{c-1}$. If $c \geq 2$, then $p \mid 2$, a contradiction. Hence $c = 1$. By Lemma 2.10, $L_2(r)$ has a subgroup which is isomorphic to the dihedral group containing the cyclic subgroup of order $\frac{r+1}{2}$, then it has a cyclic subgroup of order $\frac{r+1}{2} = 2^{a-2}$, so it has the element of order 2^{a-2} . If $a \geq 5$, then G has an element of order 8, contrary to the Lemma 2.7 (2), thus $a \leq 4$, hence $3p + 1 = 2^{a-2} \leq 4$, a contradiction. Therefore, $W \not\cong L_2(r)$.

(III): Assume that $W \cong L_2(2^m)$, satisfies the equations $2^m - 1 = u$ and $2^m + 1 = 3t^b$ for some $t > 3$, $b \geq 1$ and primes u, t .

Now, if $u = p$, $t = r$, then $2 = 3r^b - q = 3(6p + 1)^b - p \geq 17p + 3 > 3$, a contradiction. If $u = r$, $t = p$, then $2 = 3p^b - r = 3p^b - (6p + 1)$, so $p^b - 2p = 1$, thus $p \mid 1$, a contradiction. Therefore, $W \not\cong L_2(2^m)$.

(IIII): Assume that $W \cong L_2(3^m)$, satisfies the equations $3^m - 1 = 2u^b$ and $3^m + 1 = 4t$ or $3^m - 1 = 2u$ and $3^m + 1 = 4t^b$, where u and t are odd primes and $b \geq 1$.

Since $|L_2(3^m)| = \frac{3^m \cdot (3^m + 1) \cdot (3^m - 1)}{2}$ and $|L_2(3^m)| \mid |G|$, we have $m = 1$, it is impossible. Therefore, $W \not\cong L_2(3^m)$.

Case 6. If $n = 3q$ and $\varphi(k) = 2p$, then by Corollary 2.1, $k = r$ or $2r$, where $r = 2p + 1$ is a prime. Suppose that G is non-solvable. Using the same argument as *Case 5*, we get a contradiction.

Case 7. If $n = pq$ and $\varphi(k) = 6$, then by Corollary 2.1, $k = 7, 14, 9, 18$. Suppose that G is non-solvable, by Lemma 2.7 (2), we get $k = 14, 18$. Assume that $k = 14$. By $n_1 + n_2 + \cdots + n_s = pq$, there exists a $i \in \mathbb{N}$ such that $3 \nmid n_i$, so $3^2 \nmid |G|$ by (*). Since p, q are primes and $q > p > 13$, we get G is a $\{2, 3, 7\}$ -group. Then by Lemma 2.7 (3), $G \cong A \times L_2(7)$, where A is an elementary abelian 2-subgroup, so G has $(2^a - 1) \times 48$ elements of order 14. Clearly, $(2^a - 1) \times 48 \neq 6pq$, a

contradiction. Assume that $k = 18$. By Lemma 2.9 (2) and (3), we have $i \geq 2$. The same discussion as *Case 4*, we get G is a $\{2, 3, p\}$ -group or $\{2, 3, q\}$ -group. Since G is non-solvable and $q > p > 13$, by Lemma 2.3, we get G is a $\{2, 3, p\}$ -group and $p = 17$. Now, G has a section W which is isomorphic to $L_2(17)$, then $8 \in \pi(L_2(17))$, contrary to the Lemma 2.7 (2).

Case 8. If $n = 3pq$ and $\varphi(k) = 2$, then by Corollary 2.1, $k = 3, 4, 6$. Let a be an element with largest order k of G . Then $C_G(\langle a \rangle)$ is a $\{2\}$ -group or $\{2, 3\}$ -group. Since $|N_G(\langle a \rangle) : C_G(\langle a \rangle)| \mid \varphi(k) = 2$, $k \leq 6$ and $q > p > 13$. The same discussion as *Case 4*, we get G is a $\{2\}$ -group or $\{2, 3\}$ -group. Hence G is solvable.

Case 9. If $n = 6pq$ and $\varphi(k) = 1$, then G is an elementary abelian 2-group for $k = 2$, the number of maximal order elements in G is $2^t - 1$ for some positive integer t , a contradiction. ■

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**THE BLACK BOX OF ECONOMIC INTERDEPENDENCE
IN THE PROCESS OF STRUCTURAL CHANGE.
EU AND EA ON THE STAGE**

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Abstract. The search of a hierarchy in economic systems, using triangulation method, has a long tradition in economics. This study, after revisiting Authors' optimization algorithm, based on "path restricted unimodularity", tries to cast new light into the black box of the rules governing the transformation of the economic structure in growth process. Inter-temporal, cross-sectional and simulated comparisons have been made on European Community, Union and Euro Area input-output tables (1965-2007) aiming at verifying above all whether the asymmetrical hierarchy prevailed or not. The main results prove that the agents (branches) work in competition rather than in cooperation, thus the dominance criterion rules inducing asymmetry. Contrary to expectations, the richest Euro Area presents a more hierarchical structure than European Union and is more reactive to any fluctuation of final demand. Also the classical Agribusiness complex, important for investigating new comers, presents a highly inductive triangular arrangement and plays an intermediate function.

Keywords: linear ordering, input-output table, economic structures, triangularization, Agribusiness.

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Introduction

Scientific works, published in the context of input-output analysis, had a suddenly rise in the turn of XXI century. This increase has been made possible by the availability of updated input-output tables put forth by the main international (and national) statistical offices (OECD, UN, Eurostat, US-BL etc.) so helping scholars to apply this model mainly developed by Leontief (Leontief, 1986). These organizations recognize that intersectoral input-output matrixes are a fundamental tool to study the interrelations of an economic structure. Since the pioneering work of Chenery-Watanabe (1958), their use to establish a hierarchy in economic systems by triangulation method has a long tradition even if finding an exact algorithm is technically very complex since the problem is NP-hard and computationally heavy for matrixes of large dimension (over 50)². According to Korte and Oberhofer (1971, pp. 512-513), it is an important analytical tool in economics since it performs, among others, the following tasks: i) It sheds light on the functioning of the economy from a structural point of view; ii) allows comparison between economies of different countries and periods; iii) can be used to influence optimally cycles and growth; iv) is useful for forecasting and economic planning.

This study tries to cast new light into the black box of the rules governing the transformation of the economic structure in growth process. Inter-temporal, cross-sectional and simulated comparisons have been made on European Community, Union and Euro Area input-output tables (1965-2007) aiming at verifying above all whether the asymmetrical hierarchy prevailed. Moreover, the validity of some scholars' statements about the connections between the level of hierarchical arrangement and the dimension, richness and technological standards have been investigated comparing whole European Union (27 countries) and Euro Area (17 countries) economies. Finally the classical case of Agribusiness complex in its evolution has been deeply studied in view of the development unbalances of new comers (mainly countries from Eastern Europe).

1. The database

In order to examine the structure of economic systems, it needs a complete set of macro-level data that only the intersectoral input-output tables can offer. The use of these tables implies the acceptance of input-output scientific framework (Leontief, 1986).

For this instance, we have used Eurostat five yearly symmetric input-output tables (1965, 1970, 1975 and 1980) pertaining to various aggregates of the European Economic Community in 44 sectors and annual tables of the European Union and of the Euro Area (2000-2007) in 59 branches (Eurostat 1976; 1978a; 1983; 1986; 2008a; 2012). The first series was compiled using the old system of national accounts ESA-70 (Eurostat, 1978b) and the NACE-CLIO classification (Eurostat, various years), while the second is applying the ESA-95 (European Commission,

²An exact description of the problem and a synthesis of its development, inclusive of authors' approach is contained in Section 2.

1996) and the CPA, NACE-Rev. 1 and NACE-Rev. 2 classifications (Eurostat, 1996; 2008b). In particular, the old series 1965-1980 consists of 12 aggregate domestic and total symmetric input-output matrixes at d.u. prices relative to EUR6, 7 and 9.

The new set of annual domestic and total product by product symmetric input-output tables 2000-07 is composed by 32 matrixes at basic prices of the 27 countries European Union and of the 17 countries the Euro Area.

These two series are only partially comparable due to changes in the European accounting system and in its classifications (Eurostat, 1978b; 2008a).

To apply the triangularization algorithms described below, the well-known domestic and total (imports included) vertical coefficients are calculated by dividing the transactions (domestic and total) by the total production at départ-usine or at basic prices ($a_{ij} = x_{ij}/p_j$) in accordance with the time series considered.

2. The triangularization problem and its methods

Triangularization problem consists in exchanging rows and columns of a matrix so as to minimize non-zero coefficients above the main diagonal. Many authors proposed heuristic techniques, founded on a suitable (but not exhaustive) sequence of rearrangements. Simpson-Tsukui (1965) make comparisons between sequences of symmetrical elements and interchange of a single branch at each step. Implementation is easy and the method is very fast and useful to initialize other heuristics. Korte-Oberhofer (1971) test all possible interchanges of two sequences of adjoining branches, and choose the best terminating when no interchange improves the solution (local maximum). Fukui (1986) applies the same principle of Korte-Oberhofer method, but three sequences of branches are involved. Other works implement heuristic techniques (e.g., Chiarini, 2004, p. vi), but heuristics, even sophisticated, cannot ensure optimality because the function to be optimized is not convex, so that these algorithms may end at a local maximum and even if absolute optimality is attained, it cannot be detected³. As in every heuristic, examples of bad performance can be constructed (Howe, 1991). Branch and bound methods can solve this drawback, but, for large matrixes, they become inefficient and achieve only suboptimal results with estimation of errors (Haltia, 1992; Östblom, 1997). Other studies implement different exact algorithms using linear and integer programming to find the absolute optimal solution without any threshold, initialization with heuristics, or ring shift permutations (Grötschel et al., 1984; Dietzenbacher, 1996; Kondo, 2010; Piccinini and Chang, 1997; 2007; Chiarini, 2004; Charon and Hundry, 2007).

Because of the complexity of their method these algorithms have been disregarded by most economic studies in favor of those more user-friendly, such as Simpson-Tsukui and Fukui. The method developed and used by the authors has a very simple code, and, in its enhanced version, even when it does not succeed

³In our experimental work we found cases of Fukui's algorithm success (unlike other simpler algorithms).

in solving the optimality problem it gives very strict bounds from below and from above for the optimal solution. In the present study the exact solution was reached in 42 cases out of 44, and in the remaining two cases the confidence interval was less than 10^{-4} .

The exact terms of the problem can be so described. Let M be a square non-negative matrix. By P a permutation of its rows and columns (the same for both) is denoted; this operation is called rearrangement of the matrix. Let $T(M)$ denote the total sum of the elements outside the diagonal. To each rearrangement P the sum $S(M, P)$ of all the terms that lie below the diagonal is associated. The *linearity coefficient of M under rearrangement P* is the ratio

$$L(M, P) = S(M, P)/T(M)$$

By the name of *linear ordering problem or triangularization* of the matrix M it is meant the search of the rearrangement that maximizes the linearity coefficient among all rearrangements. Since the number of arrangements is finite (even if big) that is an actual maximum, and it is assumed at least by one rearrangement. The *optimal linearity degree* $L^\circ(M)$ is thus defined as

$$L^\circ(M) = L(M, P^\circ) = \max\{L(M, P), P \text{ rearrangement of } M\},$$

where P° is an optimal rearrangement.

Of course, the value of $L^\circ(M)$ ranges from 0.5 (random matrix) to 1 (total order matrix)⁴. This is a combinatorial problem of dimension $n!$, and is an NP -hard problem, hence in the case of large dimensions (usually over 50 for input-output) it can be computationally heavy.

The problem can be formulated in a different form using graphs. A *graph* is a set of nodes connected two by two by an arc. When the direction is relevant, as it happens in input-output study, the graph is oriented⁵. When a positive number is associated to each arc the graph is called *weighted* graph (0 means a non-existing arc). Therefore the numerical data of a positive matrix M can be associated to a weighted oriented graph, that does not depend on the rearrangement. It will still be denoted by M . A graph that contains no cycle is preordered, so that it generates one or more linear order structures, corresponding to the optimal rearrangement(s). The presence of cycles does not allow the construction of any order relation: hence it is necessary to cut a sufficient, but not redundant, number of arcs, so that all cycles vanish. An elementary way of performing the task is the following *Procedure E*:

- a) Initially the ordered list is empty.
- b) A node for which the incoming arcs start only from the ordered list is chosen. If no such node exists, the node for which the sum of incoming arcs starting outside the list reaches the minimum is chosen.

⁴It may be remarked that actual permutations can give also values of $L(M, P) < 0.5$, since when a rearrangement P is inverted its linearity is $1 - L(M, P)$.

⁵Sometimes it is called digraph.

- c) The node is added to the ordered list.
- d) If the length of the list is less than the number of nodes (= dimension of the matrix) return to step b).

Should a certain number of cuts have already been performed by any technique, the above algorithm allows to find out a total order, if any, or otherwise to construct a suboptimal order consistent with the given system of cuts.

After the rearrangement of the matrix the arcs that are cut appear in the upper part of the matrix. Let W be the total weight of the cut arcs, and P be the associate rearrangement: it holds

$$L(M, P) = (T(M) - W)/T(M), \quad \text{where } L(M, P) \leq L^\circ(M).$$

Thus, the system of cuts that reduces the oriented weighted graph into a pre-ordered graph paying the *least penalty* for the cuts (i.e., minimize W) gives the optimal linearity degree and allows an optimal rearrangement through procedure E . This is an integer programming problem, that can be formalized as follows. The non-zero arcs of the graph are listed (from 1 to N); the weight of each arc is recorded in the vector $\mathbf{w}$, and the number of cuts performed on each arc is stored in the vector $\mathbf{n}$. We list all the cycles that exist in the graph, from 1 to K . Each cycle is built by a certain number of arcs (no matter the order of the list), hence the cycles can be described by a matrix $A \equiv \{a_{ij}, i = 1, \dots, K; j = 1, \dots, N\}$, where value 1 means that the j -th arc belongs to the i -th cycle, 0 that it does not belong. Vector $\mathbf{1}$ denotes vector $(1, \dots, 1)$. Using these notations the condition that all cycles have been cut becomes

$$(1) \quad A\mathbf{n} \geq \mathbf{1}, \quad n_j \text{ non-negative integer}$$

while the objective function to be minimized (total weight of cuts), is

$$(2) \quad \min \langle \mathbf{w}, \mathbf{n} \rangle$$

The problem of integer programming may be computationally heavy. A simpler problem is the released problem of *fractional cut*, where each cycle is requested to have partial cuts x_j of non-negative weight such that the total weight of cuts along each cycle is at least 1.

$$(3) \quad A\mathbf{x} \geq \mathbf{1}, \quad x_j \geq 0$$

$$(4) \quad \min \langle \mathbf{w}, \mathbf{x} \rangle.$$

This is a standard linear programming problem. Of course, the fractional cut problem in some cases has a total weight less the integer cut problem, as is classically shown by Moebius's structure (compare Grötschel and Piccinini-Chang 97). Denoting by $FC^\circ(M)$ the minimum of the fractional cut problem divided by $T(M)$, in general it holds

$$L^\circ(M) \leq (1 - FC^\circ(M)).$$

Strict equality holds when the optimal system of fractional cuts happens to be already a system of integer cuts. In the actual Input-Output tables this situation is almost always true. The dual of the fractional cut problem is the maximal circulation problem along the graph; u_i represents the flow along the i -th cycle. The problem is to optimize (6) under the constraints (5):

$$(5) \quad A^t \mathbf{u} \leq \mathbf{w}, \quad u_i \geq 0,$$

$$(6) \quad \max \langle \mathbf{1}, \mathbf{u} \rangle.$$

For each admissible (i.e., satisfying (5)) system of flows $\mathbf{u}$, let $C(M, \mathbf{u})$ be the circulation $\langle \mathbf{1}, \mathbf{u} \rangle$ divided by $T(M)$; thus, the *degree of circularity* $C^\circ(M)$ of a matrix is defined as

$$C^\circ(M) = C(M, \mathbf{u}^\circ) = \max\{C(M, \mathbf{u}), \mathbf{u} \text{ admissible circulation on } M\}.$$

By duality, $C^\circ(M)$ coincides with the minimum of the fractional cut problem, so that $C^\circ(M) = FC^\circ(M)$. Since, in general, it holds $L^\circ(M) \leq 1 - C^\circ(M)$, for any rearrangement P and for any admissible system of flows $\mathbf{u}$ it holds

$$(7) \quad L(M, P) \leq L^\circ(M) \leq 1 - C^\circ(M) \leq 1 - C(M, \mathbf{u}).$$

The sequence of inequalities (7) is fundamental, since it gives an upper estimate, that together with the left estimate, allows to fix a reference window where $L^\circ(M)$ must lie. It can be observed that (7) holds however P and $\mathbf{u}$ are found, even by different techniques and by different authors.

In the integer case, when $L^\circ(M) + FC^\circ(M) = L^\circ(M) + C^\circ(M) = 1$, the solution of the optimal circulation allows to find the optimal rearrangement. In fact, in view of duality, the knowledge of the optimal circulation allows to detect the optimal structure of cuts and procedure E allows to find immediately the optimal rearrangement. The contrary does not hold, since there is no fast way of constructing the optimal system of flows starting from the linear order⁶.

Should the exact solution be unknown, experimental P^e and $\mathbf{u}^e$ can be found out with any procedure and by any author, so that the confidence window⁷

$$(8) \quad W^e(M) = [1 - C(M, \mathbf{u}^e)] - L(M, P^e)$$

can be improved by different methods and also by heuristics.

In particular, it was known from the very beginning of the search for linear ordering of matrixes that an easy heuristic for the system of flows was to consider only the binary cycles, that in the matrix, whatever the rearrangement, correspond to simultaneous non-zero values in cell (i, j) and in its symmetric (j, i) . Let $\mathbf{u}^2$ denote this system. For each cycle the maximum allowed flow is the minimum

⁶Anyhow the knowledge of an optimal or suboptimal arrangement allows to find some heuristic very powerful for a good choice of "promising" cycles, that help to accelerate the simplex method for finding optimal circulation. The knowledge of a suboptimal circulation allows the construction of a suboptimal order, hence can improve the left side bound.

⁷The arguments about released problem, duality and confidence windows were thoroughly discussed in Piccinini and Chang, 2007.

between the two values. Hence a simple (but rather loose) estimate is easily reached

$$(9) \quad L^\circ(M) \leq 1 - C(M, \mathbf{u}2)$$

The value $1 - C(M, \mathbf{u}2)$ received the somehow misleading name of *theoretical linearity* of the matrix, and was denoted by the symbol l^* . This number means that there exists a matrix M^* with the same couples of symmetric values, but endowed with a special structure in which the lower term of the couple always lies above the diagonal⁸, so that actually $L^\circ(M^*) = l^*$. Any other ordering of the symmetric couples leads to a lower value for $L^\circ(M)$.

A sketch of authors' method is now given (compare Chang-Piccinini 1997, 2007 for the mathematical and procedural full details). The core of the method is the search of an integer solution of the dual problem (5)–(6). The finding of $C^\circ(M)$ or of a suitable suboptimal estimate is performed by the simplex method. The problem of the huge number of cycles is overcome by working on "smart" selections of cycles found out using Floyd's algorithm (Floyd, 1962). A *path restricted unimodularity* is searched, that is only unitary pivots are chosen, hence only unitary (provisional) cuts are determined. When the procedure stops, either for optimality (case O) or for impossibility of finding a unit pivot (case S), Floyd's algorithm is used for a new battery of cycles containing no cut. Should no more cycle be available, if the last exit was (O) the optimal solution is found and integrity of solutions is ensured. In the case (S), the estimate of the confidence window according to (8) is expected.

The enhanced version of the method tested in the present paper improved the (S) terminal phase. The use of procedure E allows to find suboptimal rearrangements. One possibility is to start from the (S) solution, but this is not very efficient when the lack of optimality is large. It may be more advisable to use a simplified matrix A , where data below a certain threshold are set to zero, and to look for the exact solution. Usually, a better linearity coefficient is achieved. Whatever the suboptimal order is, there is the possibility of *sectional optimization*, namely a square submatrix along the diagonal is abstracted, and the algorithm is used to find its optimal rearrangement.⁹ In small dimension problems (20–25 rows), usually path restricted unimodularity is preserved, hence the partial rearrangement P^e improves $L(M, P^e)$. An expert system (or the user in the supervised case) repeats the procedure on the sequence of new re-ordered matrix until no further improvement is obtained. A final check is to reorder the whole matrix starting from the current suboptimal order; actually often the choice of cycles is simpler and allows path-restricted unimodularity until optimality is certified.

An attractive conjecture is that path restricted unimodularity holds whenever an integer solution of problem (3)–(4) can be found.

⁸The property is stable under any arrangement of the matrix, hence the property cannot be achieved by a generic matrix M through arrangement.

⁹Actually only the columns and the rows containing the submatrix are changed, and the parts lying above the submatrix, rearranged, remain in the upper triangle, so the parts lying below. Similarly for the rows: the left/right of the submatrix lies below/over the diagonal and, rearranged, still stays below/over.

3. Discussion of the main results: the hierarchical structure of EC, EU and EA (1965-2007)

In the structural analysis, some basic concepts such as those of dependence, independence, hierarchy and circularity or multi-regional interdependence have been widely accepted (Leontief, 1986, p. 166). Following Leontief, these concepts derive from a particular typology of the economic structure: a) circular or mutually interdependent (symmetrical dependence); b) independent; c) hierarchical or asymmetrical dependent; and d) multi-regional interdependent that in some cases takes a block-diagonal form.

A natural order structure of the economy can be made triangular with suitable arrangements. From past studies, it may be argued that the economic growth can produce the forming of more or less compact blocks, sustained at the base by a nucleus of intermediate productive branches (Simpson and Tsukui, 1965). It can also favor the concatenation among branches by means of their progressive specialization and differentiation (Leontief, 1986; Chenery and Watanabe, 1958). Any dissolution of blocks involves a re-composition of the economic structure in a block-diagonal or triangular form. During the whole transformation process, the formation of circular relationships among branches takes place. Due to the modification in the composition of aggregates (notably the countries of European Community, Union and Euro Area), in the number of branches, in classification and accounting systems, it is not possible to establish where the fundamental structure of the economy is going, whether toward greater asymmetry and reactivity to any changes happening inside or the contrary. This is the dilemma to which even the Nobel prize Leontief was not able to give a definitive answer and till now scholars are discordant. The attempt of this study is *inter alia* to cast a flash into this black box.

European aggregates ^a	Euristic Degree of linearity		Optimal Degree of Linearity (L ^o)	Degree of Circularity (C ^o)	Theoretical Linearity (I [*])
	Simpson - Tsukui	Fukui			
EC6-1965D	0.85759117	0.85757940	0.8579079	0.14209210	0.8852777
EC6-1970D	0.85401789	0.85505873	0.8550611	0.14493890	0.8873708
EC9-1970D	0.84936883	0.84936251	0.8494286	0.15057140	0.8817202
EC9-1975D	0.86540901	0.86540901	0.8654094	0.13459060	0.8908107
EC7-1980D	0.85739044	0.85739044	0.8573904	0.14260960	0.8823065
EC9-1980D	0.85770894	0.85770894	0.8587776	0.14122240	0.8838542
EC6-1965T	0.86241903	0.86322235	0.8632224	0.13677760	0.8897137
EC6-1970T	0.85959278	0.85959278	0.8595929	0.14040710	0.8909524
EC9-1970T	0.85473716	0.85473143	0.8547941	0.14520590	0.8847106
EC9-1975T	0.86627481	0.86627481	0.8662748	0.13372520	0.8914440
EC7-1980T	0.85892011	0.85892011	0.8591983	0.14080170	0.8845728
EC9-1980T	0.86094468	0.86094469	0.8609453	0.13905470	0.8864342

^a 6,7,9 = n° EC countries; D = Domestic Matrix; T = Total Matrix.

Source: our triangularization of Eurostat matrixes

Table 1: Degree of linearity and circularity (1965-1980) of European Community (EC) economies (44 branches)

European aggregates	Euristic Degree of linearity		Optimal Degree of Linearity (L^*)	Degree of Circularity (C^*)	Theoretical Linearity (I^*)
	Simpson - Tsukui	Fukui			
EU 2000 domestic	0.82415258	0.82597481	0.8276916	0.1723084	0.8537313
EU 2001 Domestic	0.81992401	0.82299486	0.8292979	0.1707021	0.8582740
EU 2002 Domestic	0.82442576	0.82695645	0.8283759*	0.1715835 *	0.8534707
EU 2003 Domestic	0.82423946	0.82795434	0.8329239	0.1670761	0.8616793
EU 2004 Domestic	0.82666418	0.82963492	0.8318227	0.1681773	0.8570203
EU 2005 Domestic	0.82694386	0.82734970	0.8290128	0.1709872	0.8538818
EU 2006 Domestic	0.83108726	0.83294258	0.8345255	0.1654745	0.8581509
EU 2007 Domestic	0.83234251	0.83299220	0.8331445	0.1668555	0.8574879
EU 2000 Total	0.82342160	0.82391489	0.8239150	0.1760850	0.8504848
EU 2001 Total	0.81864795	0.81913605	0.8248758	0.1751242	0.8542230
EU 2002 Total	0.82039237	0.82266706	0.8251293	0.1748707	0.8503877
EU 2003 Total	0.82116401	0.82458580	0.8287056	0.1712944	0.8573315
EU 2004 Total	0.82174980	0.82570788	0.8277029	0.1722971	0.8529726
EU 2005 Total	0.82352681	0.82217678	0.8255847	0.1744153	0.8509302
EU 2006 Total	0.82570056	0.83058190	0.8309914	0.1690086	0.8544440
EU 2007 Total	0.82370579	0.82719076	0.8295527	0.1704473	0.8533781
EA 2000 Domestic	0.83248052	0.83315126	0.8334054*	0.1665522 *	0.8609319
EA 2001 Domestic	0.83247432	0.83286189	0.8335599	0.1664401	0.8618690
EA 2002 Domestic	0.83109094	0.83143653	0.8361254	0.1638746	0.8640377
EA 2003 Domestic	0.83231587	0.83606146	0.8382770	0.1617230	0.8698823
EA 2004 Domestic	0.83219498	0.83614236	0.8372842	0.1627158	0.8680130
EA 2005 Domestic	0.83578652	0.83721130	0.8372113	0.1627887	0.8702890
EA 2006 Domestic	0.83755797	0.83836031	0.8395422	0.1604578	0.8695757
EA 2007 Domestic	0.83784407	0.83843213	0.8396743	0.1603257	0.8633359
EA 2000 Total	0.82917690	0.82966452	0.8296646	0.1703354	0.8571389
EA 2001 Total	0.82695573	0.82960332	0.8297118	0.1702882	0.8574511
EA 2002 Total	0.82959834	0.82995714	0.8308473	0.1691527	0.8593709
EA 2003 Total	0.82930840	0.83092646	0.8328405	0.1671595	0.8640419
EA 2004 Total	0.82755983	0.83129776	0.8319883	0.1680117	0.8626096
EA 2005 Total	0.82932317	0.83248670	0.8324867	0.1675133	0.8648868
EA 2006 Total	0.83429986	0.83736837	0.8380381	0.1619619	0.8660557
EA 2007 Total	0.83167136	0.83227068	0.8333391	0.1666609	0.8572904

* Confidence window $w^* 0.00004-5$; the value is the left bound of formula (7) hence may be compared with Fukui estimate.
Source: our triangularization of Eurostat matrixes

Table 2: Degree of linearity and circularity (2000-2007) of European Union (EU) and Euro Area (EA) economies (59 branches)

This essay purports to find the optimal triangular arrangement of all the 44 matrixes (1965-2007) of European Community (EC) and Union (EU) as well as Euro Area (EA) economies using the Authors' exact algorithm in the framework of the above points a) and c). The comparison of the results has allowed to draw some conclusions about the dominant typology of those structures and their fundamental structural transformations as the result of the economic development process. Preliminary, it must be pointed out that the application of the Authors' algorithm has achieved the optimum degree of linearity (L^*) in 42 out of 44 cases (64 out of 66 adding the 22 inverses) and that in the remaining two the confidence window was negligible. Also in those two last cases an improvement over Simpson and Tsukui and Fukui heuristic algorithms has been obtained (Tabb. 1-2).

The main object of this section is also to verify the validity of statements made by some scholars (Korte and Oberhofer, 1971, p. 512-13) and questioned in an interesting work (Grötschel et al., 1984, p. 292) to which any final evidence has not yet been given.

As above explained, the degree of linearity is the ratio between the sum of coefficients below the main diagonal and that of all coefficients out of the diagonal. In this framework it can be interpreted as a measure of permeability of the economy as a whole. The statements (to be verified) regard the connections between the degree of linearity and: i) country GDP per capita (statement 1) and its trend (statement 2); ii) the dimension of the country (statement 3); iii) the technological standards (statement 4). A relation between the level of development and the sector ranking is also asserted (statement 5).

From the analysis of the input-output tables 1965-2007 performed using Authors' and Simpson and Tsukui and Fukui triangularization algorithms (in particular authors' exact one) it was possible to deduce what follows:

- the EC, EU and EA economic structures present an essentially triangular order for all matrixes, thus proving the existence of a dominant hierarchical arrangement characterized by asymmetric dependence between branches described above in c) (Tab. 1-2);
- the degree of linearity (Lo) of EU-27 countries is lower than that of EA-17 countries. This result is therefore a confirmation of statement 3 according to which "large countries have a lower degree of linearity than small ones". Moreover, in EU an higher number of large countries is present compared to EA and the former (EU) registers in fact a less hierarchical economic structure (Tab. 2);
- since Euro Area GDP in PPS (Purchase Power Standard) per inhabitant is always greater than that of European Union, this performance seems to disprove the above statement 1 which maintains that "Highly developed countries have a lower degree of linearity than less developed ones";
- during the period examined of nearly 50 years, trends were substantially univocal. In the first period (1965-1980), the tendency in the European Community was to maintain constant both the degree of asymmetry and that of mutual interdependence or circularity already acquired (Tab. 1). Actually, the degree of linearity, deriving from the triangulation of total coefficient matrixes, fluctuated but conserved its level (0.863 in 1965 and 0.861 in 1980) and that of domestic coefficients did the same before returning little over the baseline levels. In the second period (2000-2007), the trend for EU was similar notwithstanding a slight increase of asymmetry (the L° of EU total coefficient matrixes varied with fluctuations from 0.824 to 0.830 and those of domestic coefficients from 0.828 to 0.833) and respectively a specular decrease of symmetry or circularity (the degree of circularity – C° – of EU total coefficient matrixes has decreased from 0.176 to 0.170 and those of domestic coefficients from 0.172 to 0.167). In synthesis for EC and EU economies, during the last half century, an "essential" constancy of the level

of asymmetry with a tendency to a slight increase in the last decade was registered. This result seems in broad sense to give substance to the validity of the above statement 4 whereby "the degree of linearity is a structural constant that can be used to compare the technological standards of different countries";

- in Euro Area, it happened the extraordinary fact of the introduction of the Euro. Its impact affected EA economic structure in the experimental period. From 2000 to 2003, the Degree of Linearity of EA total coefficient matrixes rose from 0.830 up to 0.833 and those of domestic coefficients did the same, rising from 0.833 to 0.838, while the Degree of Circularity of EA total coefficient matrixes decreased from 0.170 to 0.167 and those of domestic coefficients dropped from 0.167 to 0.160. It is well known that, in this period, the world was characterized by a high level of uncertainty from geopolitical, economic and especially financial point of view. This uncertainty, coupled with Euro introduction, contributed to a significant and sustained fall in equity markets and to an increase of volatility (BCE, 2013, p. 61). The result for Euro Area was a strengthening of the hierarchical arrangement or asymmetry and a weakening of the mutual interdependence or circularity. After on (2004-2007), there was, both for the close and the open EA economic structure, an ulterior reduction of the degree of circularity or mutual interdependence and the attempt to recover the levels achieved before the monetary revolution failed (Tab. 2). The effect was a further deepening of the asymmetry in EA hierarchical economy favored also by the increasing of US Dollar to Euro exchange rate. This performance contradicts the above statement 2 owing which "for highly developed countries the degree of linearity is almost constant or slightly decreasing over time"; in the EA case on the contrary the hierarchical arrangement of the economy was increased;
- for the European Community (1965-80), the degree of linearity/respectively circularity after the triangularization of the total coefficients matrixes is always higher/respectively lower than that obtained for domestic coefficient matrixes. That performance reversed itself both for European Union and Euro Area (2000-07). These contrasting trends may be attributed to the relatively lower level of globalization with financial protectionism in the period 1965-80 versus the higher level of globalization without financial protectionism in the period 2000-07¹⁰.

Those performances demonstrate that in the former European Community the international division of labor favored an accentuation of the asymmetry in the structure open to foreign trade respect to the domestic, while in the EU and EA, the rationalization of domestic productive and integration processes among countries prevailed so that the economic hierarchy of the domestic structure became stronger.

That means that in the European economic systems of the last fifty years is persisting a hierarchical arrangement in which goods flows more probably from

¹⁰A due acknowledgement to F. Bortot for this important suggestion.

the top to the bottom of the triangular structure and not in the opposite direction (Korte and Oberhofer, 1971). This property of economic triangularity supports the hypothesis that, in these systems, the agents (branches) do not work in a context of cooperation but in one of competition in which the dominance criterion is prevailing (Perroux 1964, Chang 1994).

It is opinion of some qualified researchers that, during the growth process, the structure of economic systems, starting from a block-diagonality arrangement (Simpson-Tsukui), moves towards a triangular one and afterwards gradually to mutual interdependence (Leontief), but this last statement has not yet been proved. The results of this study seem to demonstrate the maintenance of certain level of asymmetry although both the degree of linearity and the economic system triangularity have been slightly increasing.

The above results must be ascribed to a real structural change in the economic systems and, more specifically, to the process of service expansion. In post-industrial, tertiary, quaternary and quinary services absorb the main part of employment and GDP of the economy while industry is dropping. Mature economies are characterized by the specialization of functions most of which originally belonged to industry, and at present are pertinence mainly of services, but to some degree also of satellite industries. Therefore as the growth process induced economic branches to develop circular interrelations at the same time specialization operated in the opposite direction transferring these functions to service or other industrial branches. These contrasting trends lead to a substantial stability of the asymmetric structure of economic systems which is lasting over time (Piccinini and Chang, 2007).

3.1. A new method to evaluate the similarity in the economic structures

Frequently Spearman ρ rank correlation coefficient is used to establish whether two hierarchical arrangements are similar (Spearman, 1904, pp. 72-101). As a former work asserted, that correlation measure "... can be an arbitrarily poor measure of 'similarity' between two (optimum) linear ordering" especially when the real input-output matrixes contain a great number of zero-entries (Grötschel et al., 1984, p. 288) and a plurality of optimal arrangements may rise. In the current real matrices, if all coefficients are considered (no threshold), zeros are infrequent. To filter data by thresholds generates zero-entries proliferation and the consequent enhancement of equivalent optimal rankings.

Moreover, the Spearman coefficient is too sensitive to branch or block interchanges. When blocks are independent, it can be high even if the change of linearity degree is negligible. Also, the Kendall τ correlation coefficient creates the same drawback (Yule and Kendall, 1950, p. 268). Furthermore, these indexes take into account only the ranking and not the weight of the various branches on the assumption that order change of a branch drawn at random is equivalent to that of any other. Corrections are possible, but still remain arbitrary.

To overcome these drawbacks and in accordance with Leontief (1986; p. 169), the Authors have developed a method, which consists in placing one matrix of coef-

ficients into the optimal arrangement of a reference matrix and then to recalculate the degree of linearity in a context of sub-optimality. The results are placed in a "simulated linearity" table which reports on the column headers the reference matrix name and on the rows that of matrix under test. For each couple, the difference between optimal linearity and "simulated linearity" is always positive. But, this difference respects neither the rule of symmetry nor the triangular inequality, hence the half-sum of the two corresponding elements of the distance matrix has been calculated and then applied the Floyd algorithm to obtain a distance function that satisfies both rules (Floyd, 1962). This algorithm was applied to the complete dataset of 32 matrices.

Triangulated Coefficient Matrices ^{a)}	COT 2000 EU	COT 2005 EU	COT 2007 EU	COD 2000 EU	COD 2005 EU	COD 2007 EU	COT 2000 EA	COT 2005 EA	COT 2007 EA	COD 2000 EA	COD 2005 EA	COD 2007 EA
COT2000EU	-											
COT2005EU	0.0013	-										
COT2007EU	0.0014	0.0006	-									
COD2000EU	0.0002	0.0014	0.0015	-								
COD2005EU	0.0016	0.0002	0.0007	0.0015	-							
COD2007EU	0.0020	0.0010	0.0015	0.0020	0.0008	-						
COT2000EA	0.0026	0.0039	0.0040	0.0028	0.0041	0.0046	-					
COT2005EA	0.0039	0.0026	0.0026	0.0039	0.0026	0.0034	0.0026	-				
COT2007EA	0.0020	0.0007	0.0007	0.0020	0.0007	0.0015	0.0034	0.0019	-			
COD2000EA	0.0048	0.0046	0.0051	0.0048	0.0044	0.0036	0.0045	0.0044	0.0051	-		
COD2005EA	0.0049	0.0039	0.0043	0.0049	0.0037	0.0036	0.0047	0.0044	0.0044	0.0034	-	
COD2007EA	0.0030	0.0020	0.0024	0.0029	0.0017	0.0009	0.0050	0.0044	0.0025	0.0026	0.0026	-

^{a)} COT = Total coefficient matrix; COD = Domestic coefficient matrix; EU = EU-27 countries; EA = EA-17 countries.

Source: Authors' elaboration on Eurostat data.

Table 3: Distances between the rankings in European Union and Euro Area economic structures (2000-07).

The symmetrical distances derived by this algorithm can be in principle smaller than those which could result with the application of a more limited dataset. Therefore, the synthesis presented in Tab. 3, although of only 12 rows and 12 columns, has been derived from 32 rows and 32 columns dataset. These measures are substantially related to the phenomenon studied, while Spearman and Kendall correlation coefficients are "general purpose" linked only to the rank and independent from the assessment of what has generated it and from its sensitivity. The range of distance values relative to the new measure exposed in this study is much smaller than that of the above indexes even if the distortion in the case of EU and EA matrixes is weak due to few zero-entries. The distances obtained comparing various EU and EA hierarchical structures (2000-07) prove that the Degree of Linearity differences obtained using the symmetric Floyd distance are negligible, mostly in the order of thousandths (Tab. 3). That performance gives support to the above statement five which argues that "for highly developed countries the ranking of sector in different countries is similar".

4. The triangularized structure of European economy and the role of Agribusiness

In the context of the input-output vertical model called also "demand-driven", the optimal order of the European (EC, EU, EA) matrixes (1965-2007) has been achieved through the application of Authors' algorithm. Examining the results, it was possible to reveal the basic hierarchical structure of the EC, EU and EA economy. For lack of space, only the results emerging from the EU domestic most recent matrix (2007) are discussed in order to establish what are the main suppliers of the whole economy.

The salient features of the EU-27 triangulated structure, with the reconstruction of the purchase links, are as follows¹¹ (Fig. 1):

- straddling the middle of the hierarchical structure (from 27th to 32th place) is placed the first main block called "Agribusiness" (examined widely afterwards);
- in the second half of the triangular structure (from 33rd to 46th place), is positioned the second main block of the system called "Metals and machinery". Its components result to be well integrated and are: "Collected and purified water, distribution services of water", "Sewage and refuse disposal services, sanitation and similar services", "Trade, maintenance and repair services of motor vehicles and motorcycles, retail sale of automotive fuel", "Motor vehicles, trailers and semi-trailers", "Coke. refined petroleum products and nuclear fuels", "Crude petroleum and natural gas, services incidental to oil and gas extraction", "Construction work", "Other non-metallic mineral products", "Machinery and equipment nec", "Electrical machinery and apparatus nec", "Rubber and plastic products", "Fabricated metal products", "Basic metals", "Chemicals, chemical products and man-made fibers";
- at the base of the triangular structure (from 47th to 59th place) are firmly positioned the principal suppliers of the economic system and of the majority of the branches; this third main block is called "Services" and is composed as following: "Retail trade services", "Renting services of machinery and equipment", "Insurance and pension funding services", "Services auxiliary to financial intermediation", "Electrical energy, gas, steam and hot water", "Wholesale trade and commission trade services", "Supporting and auxiliary transport services, travel agency services", "Land transport, transport via pipeline services", "Financial intermediation services", "Post and telecommunication services", "Computer and related services", "Real estate services" and the "Other business services". The last branch appears to be the main supplier of the EU economy;
- at the top of the triangular structure, are located 12 branches (from 1st to 12th place). oriented primarily to satisfy the final demand, that are variously related to other three main blocks (Agribusiness, Metals and machinery and Services);

¹¹After first quotation, branch name is shortened.

4.1. Position and Composition of Agribusiness

It is well known that Davis and Goldberg (1957) of the Harvard Business School proved, using input-output technique, the presence in the US economy of a complex of sectors called "Agribusiness" (AB). They define it as "the sum total of all operations involved in the manufacture and distribution of farm supplies; production operations on the farm; and the storage, processing and distribution of farm commodities and items made from them". Their conception involves substantially all activities linked upstream and downstream to agriculture: a very innovative approach in a scientific world dominated by sectorial analysis.

Subsequently, numerous works demonstrated the presence in various countries of a complex of branches, strongly interrelated, whose first and second main suppliers were respectively agriculture and food industry (Chenery and Watanabe, 1958; Simpson and Tsukui, 1965; Chang, 1985). But in agricultural economics studies a dichotomy arose among scholars between those who continued to center their attention on agriculture (Agribusiness approach) and those who revolved their focus on food industry (French School approach; Malassis, 1973). The role of primary production versus food function was exalted in their respective studies (Corsani, 1986; see also Bortot, 1988 for comparison of the two approaches).

In this subsection, the aim is to verify the position of Agribusiness complex (Chang, 1981;1985) and that of its sub-systems, linked to both branches, and to identify the structural changes involved in the last 50 years.

In Graph. 1, the block of Agribusiness (AB) is integrating with its main buyers and, only for representation, the coefficients greater than/or equal to $1/2n$ (n is the number of branches) have been highlighted.

Order Number		1	5	9	18	19	21	25	27	28	29	30	31	32
		3	10	12	30	13	11	57	38	9	15	14	2	1
Order	Number Group of Products	FISH	TOBAC	WEAR	FURNIT	LEATHER	TEXTILE	RECREAT	HOTEL	FOOD	PAPER	WOOD	FOREST	AGRIC
1	3 FISH	X												
5	10 TOBAC		X											
9	12 WEAR			X		T								
18	30 FURNIT				X									
19	13 LEATHER			X	T	X								
21	11 TEXTILE	X		X	X	X	X							
25	57 RECREAT							/						
27	38 HOTEL							X	X					
28	9 FOOD	X				X		X	X	X				X
29	15 PAPER		X		X					X	X			
30	14 WOOD				X						X	X		
31	2 FOREST										X	X	X	
32	1 AGRIC		X				X		X	X			X	X

Source: Authors' elaboration on Eurostat Data

Graph 1. EU-27 (2007) Agribusiness complex main domestic (X) and total linkages ($T = 1/2n$)

Agribusiness has a nearly perfect triangular form and is composed, from vertex to base, by the following branches:

- "Recreational, cultural and sporting services";
- "Hotel and restaurant services";
- "Food products and beverages";
- "Pulp, paper and paper products";
- "Wood and products of wood and cork";
- "Products of forestry, logging and related services";
- "Products of agriculture, hunting and related services".

The AB main supplier is Agriculture, that presides the triangle base, and establishes three significant sale relations with Hotel, Food and Products of forestry as well as with itself (intra-industry relationship).

Inside of AB complex operate the following two sub-systems:

- A) the Food System which is located at the vertex of the AB triangle and consists of the following branches:
 - Recreational and Hotels that buy one another and as well from themselves;
 - Food which is currently the main supplier of this system and the second of AB with its deliveries to Recreational and to Hotel. Moreover Food is connected, on the purchase side, with itself and with Agriculture and is also one of the main suppliers of the very Agriculture. Actually the existence of a structural purchase/sale or circular relationship between them has been long recognized, since livestock production buys feed for animals from Food industry as well as sells to the latter primary agricultural products to be processed;
 - Food System also acts as a supplier of the branches above the AB block, i.e., those closest to the final demand such as: "Fish and other fishing products" (to supply feed), Membership organization, Water transport, Air transport, "Health and social work services" and "Leather and leather products" through the delivery of skins coming from the slaughtering industry. Regarding the branches underlying the AB block, the Food System is among the main suppliers of Chemicals, Wholesale trade and Retail trade.
- B) The Forest System which is situated at the base of AB triangular structure and it is composed by these branches:
 - "Pulp, paper and paper products" which buys from itself and from both "Wood and products of Wood" and "Products of forestry, logging and related services";
 - Wood which purchases, within the block, only from Products of forestry as well as from itself;
 - "Products of forestry" that buys from Agriculture and from itself.

About relationships with the branches above it, Forest system results to provide meaningfully with Wood and with Products of forestry the following branches: "Uranium and thorium ores", Pulp and paper, Agriculture and Tobacco. Pulp and paper are also a leading provider of: "Secondary raw materials", Wood and "Coal and lignite". Last "Furniture, other manufactured goods" has as its main suppliers the branches Pulp and paper, Wood and "Wearing apparel; furs".

Regard to Forest system sale relations with the branches ranking downstream, Pulp and paper branch (inside it) is one of the major provider of "Rubber and plastic products" and of Chemicals, while the branch of Wood plays the same role with its deliveries to "Crude petroleum and natural gas".

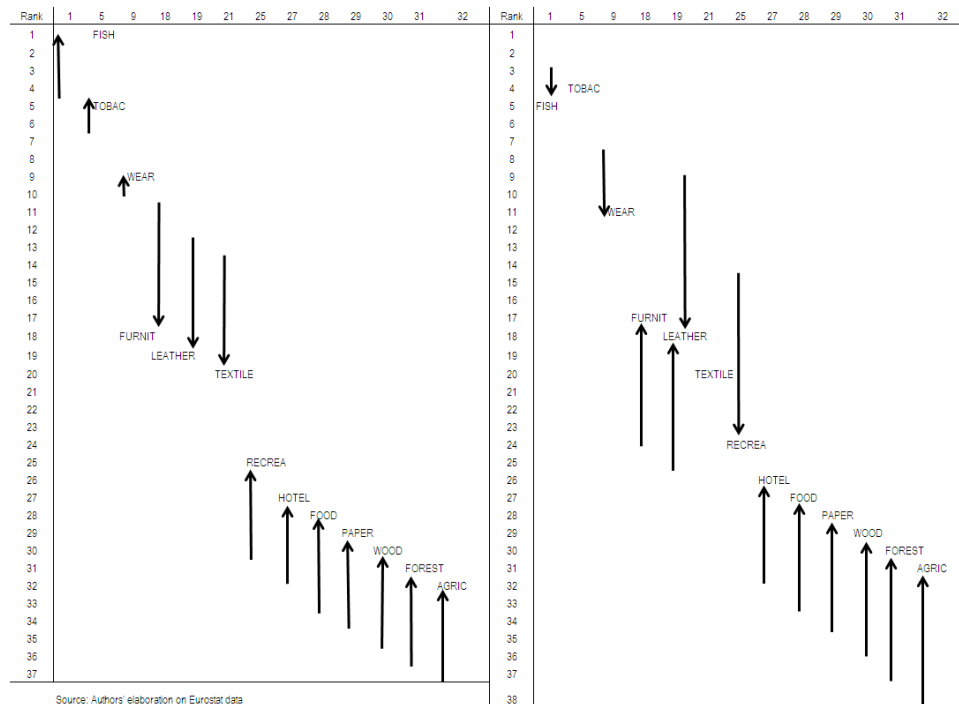
Remark that Agriculture, with its deliveries, creates a *trait-d'union* inside the AB between both the subsystems. Furthermore the branch of Recreational, with its purchases and sales from/towards the "Printed matter and recorded media" branch, generates, inside of the AB, a first round indirect linkage with Pulp and paper. Finally Agriculture gives rise to a second indirect round linkage with Wood, so ensuring the integration of Printing and publishing with the AB and especially with the Forest system.

Considering the industries originally component of the AB complex (Chang, 1981), it was possible to reconstruct the so called "Textile system", since EU, not being endowed by enough natural fibers, has to resort to imports. Therefore it is already positive that Wearing branch reveals a significant purchase link with European domestic agriculture. Wearing, typical final production, is located up to the 9th place of the triangular structure and establishes purchase relationships with Leather and with Textiles, as well as with other branches at the base of the economy triangle. It is thus detectable a link between the Textile system and the Food system through the linkage that Leather forms with Food since Leather purchases skins from the slaughter industry. The branch of Textiles in turn is placed at the 21st place and is not entertaining other significant purchase relationships with the subsequent branches until Agriculture (32th place). Textiles branch is also connected with the second main block of the economy "Metals and machinery" and in particular with some of its main suppliers that are "Machinery and equipment", Rubber, Chemicals, as well as with some service sectors standing at the base of the third block.

Considering the hierarchical order of the 2007 coefficient total matrix (import of goods and services included), the results confirm (for AB) the structure described above for the domestic matrix with a maximum slip down of 1 or 2 positions. This means that, due to imports, the AB block is somewhat more intermediate.

From the inter-temporal 2000-07 Migration graph (Graph 2, left side), of the AB branches in EU-27 countries, calculated on the optimal arrangements of the total matrix, the following trends are found:

- AB complex stands in 2007 within the 32nd place of the 59 position ranking (in 2000 stood within the 37th place);
- "Tobacco products" tends, among all branches, to settle itself at the vertex of the economy triangle closest to final demand. Also the branch of Fish shows the same trend;
- the most final branches, like Wearing, Leather and Furniture glide down considerably towards the AB block;
- the block of Agribusiness in turn moves itself towards a higher position, confirming its tendency to become less intermediate and so approaching the AB more final branches, standing up in the asymmetrical structure of the economy.



Graph 2. - Agribusiness Complex Inter-temporal Migration Graph 2000-2007
(Left EU-27, Right EA-17 countries).

In EA (Graph 2, right side), the final sequence of the hierarchical order has become substantially the same of that of EU, except for the interchanging position of Tobacco and Fish at the top of the triangle. Hence the differences at the start vanished. This is the result of both light industry (Wearing, Furniture, Leather and Textile) compacting and the upward migration of Agribusiness Block toward its main non-food processors.

In synthesis, Agribusiness presents a highly inductive triangular arrangement in a hierarchical structure already strict. Due to its location in the center of the economy it plays an intermediate function. Eventually some of its original branches (Wearing, Leather and Textiles) moved closer to final demand ranking among the main clients of the economy. Recently there is a slight tendency both in EU and EA of the AB block and of its final branches to approach themselves. Finally the real innovation consists in the insertion of a typical post-industrial branch (Recreational, cultural and sporting services) at the top of the block. That branch generates linkages with the Food System (backward) and with the Printed matter and recorded media system (circular). This last is also linked to the Pulp and paper inside the AB so strengthening its internal cohesion.

5. Conclusions

In the last fifty years, the EU and EA economic structures observe the dominance of an asymmetrical development pattern characterized by hierarchical dependence. Lacking theoretical directions, some scholars have controversial opinions about the *sequentia temporum* of various typological structure, thus the Black box should

be revealed. The horn of the dilemma lies in determining whether, during the development process, there is an increase of circularity (mutual dependence). On the contrary, the results of this study seem to support a slight weakening of circularity in favor of a strengthening of asymmetrical dependence. Authors' hypothesis is that in post-industrial economies the rationalization of services may lead to a better division of labor, restoring a hierarchical dependence pattern. If that trend will continue, last statement could achieve a general value since the structures of mature economies have proved to be very similar.

Furthermore, the performance described above seems to provide, from a theoretical point of view, a significant empirical evidence to the hypothesis that make the hierarchy or asymmetric dependence, instead of the mutual interdependence, the hub of a different theory of structural analysis based on the dependence/dominance among the economic "agents". According to the theory of unbalanced growth, the activation of the system, caused by a variation of final demand, occurs mainly through the so-called backward linkages determined by input purchases (Hirschman, 1958). Although backward linkages are considered the development *primum mobile*, also the forward linkages, induced by sales, have their relevance. Those last strengthen the backward linkages by engendering a pincer movement able to help investment decisions. If the structure is hierarchical, the more it becomes asymmetrical, the higher is the probability that a variation of final demand produces its effects from top to bottom of the triangular structure and not in the opposite direction, inducing a cumulative development. For European Union and Euro Area, the inference of this study seems to support the interpretation that the hierarchical structure of the economy favors the permeability of the system to any variation of final demand inducing a growth in case of increase and a de-growth in case of decrease. If an asymmetrical structure is prevailing, the task of policy makers is facilitated both in the expansion and in the regression phases of economic cycle. They may attempt, in the first case, to contain inflation e.g. augmenting taxation while in the second case to incentivize final demand reducing fiscal pressure (or by monetary base reduction and respectively expansion).

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A COMMUTATIVE REGULAR MONOID ON ROUGH SETS

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Abstract. In this paper, we intend to study an algebraic approach on rough sets. We introduce the concept of rough semigroup, rough monoid and rough ideals on the set of all rough sets for the given information system together with the operation Praba Δ . We illustrate these concepts through examples.

Keyword: roughset, semigroup, monoid, regular semigroup, ideals.

1. Introduction

The concept of rough set theory was introduced by Z. Pawalak [12] in 1982. This formal tool was implemented to process incomplete information in the information systems. Rough set theory is an extension of set theory and it is defined by a pair of sets called lower and upper approximations. In the content of data analysis, this concept will be used to discover fundamental patterns in data, remove redundancies and generate decision rules. Also rough set theory

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will be applied in several field like computational intelligence such as machine learning, intelligent systems, pattern recognition, knowledge discovery, expert systems and others [14], [10], [6], [2], [4]. Zadeh[15] introduced the concept of fuzzy sets in his paper. B. Praba and R. Mohan [13] discussed the concept of rough lattice. In this paper the authors considered an information system $I = (U, A)$. A partial ordering relation was defined on $T = \{RS(X) \mid X \subseteq U\}$. The least upper bound and greatest lower bound were established. They have also defined the operation Praba Δ . N. Kuroki and P.P. Wang [9] discussed some properties of lower and upper approximations with respect to the normal subgroup. R. Biswas and S. Nanda [1] introduced the notion of rough groups and rough subgroups. Also the authors T.B. Iwinski [8] Z. Bonikowaski [3] have studied algebraic properties of rough sets. Then the concept of rough fuzzysets and fuzzy rough sets was introduced by D. Dubois, H. Parade [5] and Nick C. Fiala [11] discussed about semigroup, monoid and group models of groupoid identities in his paper. In the recent and past, rough set theory has triggered many researchers all around the world. The concept of rough set theory is the approximation space such as lower and upper approximations of a set determined by attributes. The pair of lower and upper approximation is called rough set also in rough set theory data can be represented in the form of an information system. An information system is a pair $I = (U, A)$ where U is a non empty finite set of objects, called universal set and A is a nonempty set of fuzzy attributes defined by $\mu_a : U \rightarrow [0, 1]$, $a \in A$, is a fuzzy set.

Indiscernibility is a core concept of rough set theory and it is defined as an equivalence between objects. Objects in the information system about which we have the same knowledge forms an equivalence relation.

Formally, any set $P \subseteq A$, there is an associated equivalence relation called *P-Indiscernibility* relation defined as follows,

$$IND(P) = \{(x, y) \in U^2 \mid \forall a \in P, \mu_a(x) = \mu_a(y)\}.$$

The partition induced by $IND(P)$ consists of equivalence classes defined by

$$[x]_P = \{y \in U \mid (x, y) \in IND(P)\}.$$

For any $X \subseteq U$, define the lower approximation space

$$\underline{P}(x) = \{x \in U \mid [x]_P \subseteq X\}.$$

Also, define the upper approximation space $\overline{P}(x) = \{x \in U \mid [x]_P \cap X \neq \emptyset\}$.

For every subset $X \subseteq U$, there is an associated rough set

$$RS(X) = (\underline{P}(x), \overline{P}(x)).$$

In this paper, we consider an information system $I = (U, A)$ where U is a non empty finite set of objects, called universal set and A is a nonempty set of fuzzy attributes and let $T = \{RS(X) \mid X \subseteq U\}$ be the set of all rough sets on U . A binary relation Praba Δ [13] is defined on T then we prove that (T, Δ) is a

commutative regular monoid of idempotents. Also we characterize the principal ideals generated by these idempotents. The paper is organized as follows.

In Section 2, we give the necessary definitions pertaining to rough set theory and semigroup theory.

In Section 3, we define the binary operation Δ [13] on T and prove that (T, Δ) is a commutative regular monoid of idempotents called as the rough monoid.

In Section 4, a characterization for the lower and upper approximation of the principal ideal generated by the elements of T is discussed in detail and section 5 illustrates these concepts with example.

Section 6 gives the conclusion.

2. Preliminaries

In this section, we present some preliminaries on rough sets and algebraic structures.

2.1. Rough sets

Let $I = (U, A)$ be an information system and for any subset X of U and $(\underline{P}(x), \overline{P}(x))$ are the lower and upper approximations respectively as defined in previous section.

Definition 2.1. [rough set] A rough set corresponding to X , where X is an arbitrary subset of U in the approximation space P , we mean the ordered pair $RS(X) = (\underline{P}(x), \overline{P}(x))$.

Remarks 2.1. [13] If $X \subseteq U$, then $X \subseteq \bigcup_{i=1}^r X_{\alpha_i}$ where none, one or more of the equivalence classes are contained in X . Here X_{α_i} , $i = 1, 2, \dots, r$ are the equivalence classes induced by $Ind(P)$.

Definition 2.2. [13] If $X \subseteq U$, then the number of equivalence classes (Induced by $Ind(P)$) contained in X is called as the Ind. weight of X . It is denoted by $IW(X)$.

Example 2.1. [13] Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $A = \{a_1, a_2, a_3, a_4\}$ where each a_i ($i = 1$ to 4) is a fuzzy set whose membership values are shown in Table 1.

Table 1:

A/U	a_1	a_2	a_3	a_4
x_1	0	0.1	0.3	0.2
x_2	1	0.6	0.7	0.3
x_3	0	0.1	0.3	0.2
x_4	1	0.6	0.7	0.3
x_5	0.8	0.5	0.2	0.4
x_6	1	0.6	0.7	0.3

Let $X = \{x_1, x_3, x_5, x_6\}$ and $P = A$. Then the equivalence classes induced by $IND(P)$ are given below.

$$\begin{aligned} (1) \quad [x_1]_p &= \{x_1, x_3\} \\ (2) \quad [x_2]_p &= \{x_2, x_4, x_6\} \\ (3) \quad [x_5]_p &= \{x_5\} \end{aligned}$$

Hence,

$$\begin{aligned} \underline{P}(x) &= \{x_1, x_3, x_5\} \quad \text{and} \\ \overline{P}(x) &= \{x_1, x_2, x_3, x_4, x_5, x_6\}. \end{aligned}$$

Therefore

$$RS(X) = (\{x_1, x_3, x_5\}, \{x_1, x_2, x_3, x_4, x_5, x_6\}).$$

Note that the upper approximation space consists of those objects that are possibly members of the target set X . The set $U - \overline{P}(x)$ represents the negative region containing the set of objects that can be definitely ruled out to be the members of the target set X . The boundary region given by the set difference $\underline{P}(x) - \overline{P}(x)$ consists of those objects that can neither be ruled in nor ruled out as the members of the target set X . In the previous example the negative region is an empty set and the boundary is $\underline{P}(x) - \overline{P}(x) = \{x_2, x_4, x_6\}$. Throughout this paper, we use this same Example 2.1 to illustrate our concepts.

Example 2.2. Let $U = \{x_1, x_2, \dots, x_6\}$ as in Table 1. The equivalence classes induced by $Ind(P)$ are

$$\begin{aligned} [x_1]_p &= \{x_1, x_3\} \\ [x_2]_p &= \{x_2, x_4, x_6\} \\ [x_5]_p &= \{x_5\} \end{aligned}$$

Let $X = \{x_1, x_4, x_5\} \subseteq U$ then by definition, Ind. weight of $X = IW(X) = 1$ (since there is only one equivalence class $[x_5]_p = \{x_5\}$ present in X).

Definition 2.3. [13] Let $X, Y \subseteq U$. The Praba Δ is defined as $X \Delta Y = X \cup Y$, if $IW(X \cup Y) = IW(X) + IW(Y) - IW(X \cap Y)$. If $IW(X \cup Y) > IW(X) + IW(Y) - IW(X \cap Y)$, then identify the equivalence class obtained by the union of X and Y . Then delete the elements of that class belonging to Y . Call the new set as Y . Now, obtain $X \Delta Y$. Repeat this process until

$$IW(X \cup Y) = IW(X) + IW(Y) - IW(X \cap Y).$$

Example 2.3. [13] Let $U = \{x_1, x_2, \dots, x_6\}$ as in Table 1. Let $X = \{x_2, x_4, x_5\}, Y = \{x_1, x_6\} \subseteq U$ then by definition,

$$IW(X) = 1; IW(Y) = 0; IW(X \cup Y) = 2; IW(X \cap Y) = 0.$$

Here,

$$IW(X \cup Y) > IW(X) + IW(Y) - IW(X \cap Y).$$

The new equivalence class formed in $X \cup Y$ is $[x_2]_p$. As $x_6 \in Y$ and x_6 is an element of $[x_2]_p$, delete x_6 from Y . Now the new Y is $\{x_1\}$. Now for $X = \{x_2, x_5, x_6\}$ and $Y = \{x_1\}$. Finding $IW(X \cup Y)$,

$$IW(X \cup Y) = IW(X) + IW(Y) - IW(X \cap Y).$$

Therefore, $X \Delta Y = X \cup Y = \{x_1, x_2, x_4, x_5\}$.

2.2. Algebraic structures

Definition 2.4. [Groupoid] [11], [7] A groupoid consists of a non-empty set equipped with a binary operation $*$, and it is denoted by $(S, *)$.

Definition 2.5. [Semigroup] [11], [7] A *semigroup* $(S, *)$ is a groupoid that is *associative* $(x * y) * z = x * (y * z)$ for all $x, y, z \in S$.

Definition 2.6. [Monoid] [11], [7] A *semigroup* $(S, *)$ is said to be a monoid if it contains an identity element $e \in S$ such that $e * x = x * e = x$ for all $x \in S$.

Definition 2.7. [Commutative monoid] [11], [7] A *monoid* $(S, *)$ which satisfies commutative axiom namely $x * y = y * x$ for all $x, y \in S$, is known as Commutative monoid

Definition 2.8. [Idempotent] [11], [7] An element x in a groupoid $(S, *)$ is said to be idempotent, if $x * x = x$.

Definition 2.9. [Regular semigroup] [11], [7] A semigroup $(S, *)$ is said to be regular, if there exists an element $y \in S$ such that $x = x * y * x$ for all $x \in S$.

Definition 2.10. [Right (Left) ideal] [7] A nonempty subset I of a semigroup $(S, *)$ is a right (left) ideal, if it satisfies $I * S \subseteq I$ ($S * I \subseteq I$).

Definition 2.11. [Ideal] A nonempty subset I of a semigroup $(S, *)$ is said to be an ideal, if it is both right and left ideal.

Definition 2.12. [Principal ideal] [7] Let $(S, *)$ be a semigroup and for any element $a \in S$ such that $a * S$ ($S * a$) is called as principal right (left) ideal generated by a and $S * a * S$ is called as principal ideal of S

Example 2.4. Ideals nZ of the semigroup Z are all principal and in fact all ideals of Z are principal.

In the following section, a binary operation Praba Δ [13] is defined on the set of all rough sets and its algebraic structure is studied.

3. Monoids on Rough sets

Throughout this section, we consider an information system $I = (U, A)$. Now, for any $X \subseteq U$, $RS(X) = (\underline{P}(X), \overline{P}(X))$, and let $T = \{RS(X) | X \subseteq U\}$ be the set of all rough sets on I

Definition 3.1. [Binary operation] Let T be the collection of rough sets and let $\Delta : T \times T \rightarrow T$ such that $\Delta(RS(X), RS(Y)) = RS(X \Delta Y)$

Theorem 3.1. Let $I = (U, A)$ be an information system where U be the universal (finite) set and A be the set of attributes and T be the set of all rough sets then (T, Δ) is a Monoid.

Proof.

(a) **Closure axiom:** For $X, Y \subseteq U$, and letting $X \Delta Y = Z \subseteq U$,

$$\implies RS(X \Delta Y) = RS(Z) \in T$$

(b) **Associative axiom:** For all $X, Y, Z \subseteq U$ and $RS(X), RS(Y), RS(Z) \in T$ such that

$$RS(X \Delta (Y \Delta Z)) = RS((X \Delta Y) \Delta Z),$$

i.e., to prove $RS(X) \Delta (RS(Y) \Delta RS(Z)) = (RS(X) \Delta RS(Y)) \Delta RS(Z)$,

$$(4) \implies RS(X \Delta (Y \Delta Z)) = RS((X \Delta Y) \Delta Z).$$

Claim:

$$(i) \underline{P}(X \Delta (Y \Delta Z)) = \underline{P}((X \Delta Y) \Delta Z)$$

$$(ii) \overline{P}(X \Delta (Y \Delta Z)) = \overline{P}((X \Delta Y) \Delta Z)$$

Proof for Claim (i): For $x \in \underline{P}(X \Delta (Y \Delta Z))$, we have

$$\begin{aligned} & \{x \in U | [x]_p \subseteq X \Delta (Y \Delta Z)\} \\ \implies & \text{i.e., } [x]_p \subseteq X \text{ or } [x]_p \subseteq Y \Delta Z \\ \implies & [x]_p \subseteq X \text{ or } [x]_p \subseteq Y \text{ or } [x]_p \subseteq Z \\ \implies & [x]_p \subseteq X \Delta Y \text{ or } [x]_p \subseteq Z \\ \implies & [x]_p \subseteq (X \Delta Y) \Delta Z \\ \implies & x \in \underline{P}((X \Delta Y) \Delta Z) \\ \implies & \underline{P}(X \Delta (Y \Delta Z)) \subseteq \underline{P}((X \Delta Y) \Delta Z). \end{aligned}$$

Similarly, we prove that

$$\begin{aligned} & \underline{P}((X \Delta Y) \Delta Z) \subseteq \underline{P}(X \Delta (Y \Delta Z)) \\ \therefore & \underline{P}((X \Delta Y) \Delta Z) = \underline{P}(X \Delta (Y \Delta Z)). \end{aligned}$$

Proof for claim (ii): For $x \in \overline{P}(X\Delta(Y\Delta Z))$, we have

$$\begin{aligned}
 & \{x \in U \mid [x]_P \cap (X\Delta(Y\Delta Z)) \neq \phi\} \\
 \implies & \text{ i.e., } [x]_P \cap X \neq \phi \text{ or } [x]_P \cap (Y\Delta Z) \neq \phi \\
 \implies & [x]_P \cap X \neq \phi \text{ or } [x]_P \cap Y \neq \phi \text{ or } [x]_P \cap Z \neq \phi \\
 \implies & [x]_P \cap (X\Delta Y) \neq \phi \text{ or } [x]_P \cap Z \neq \phi \\
 \implies & [x]_P \cap ((X\Delta Y)\Delta Z) \neq \phi \\
 \implies & x \in \overline{P}((X\Delta Y)\Delta Z) \\
 \implies & \overline{P}(X\Delta(Y\Delta Z)) \subseteq \overline{P}((X\Delta Y)\Delta Z).
 \end{aligned}$$

Similarly, we prove that

$$\begin{aligned}
 & \overline{P}((X\Delta Y)\Delta Z) \subseteq \overline{P}(X\Delta(Y\Delta Z)) \\
 \therefore & \overline{P}((X\Delta Y)\Delta Z) = \overline{P}(X\Delta(Y\Delta Z)).
 \end{aligned}$$

Thus $RS(X\Delta(Y\Delta Z)) = RS((X\Delta Y)\Delta Z)$. Hence (T, Δ) is a semigroup. This Semigroup is called as rough semigroup.

(c) Identity axiom: For $RS(X) \in T$ there exists $RS(\phi) \in T$ such that

$$\begin{aligned}
 & RS(X)\Delta RS(\phi) = RS(X\Delta\phi) = RS(X) \\
 \therefore & RS(X\Delta\phi) = RS(\phi\Delta X) = RS(X).
 \end{aligned}$$

Thus (T, Δ) is a monoid. This monoid is called as the rough monoid. ■

Theorem 3.2. (T, Δ) is a commutative rough monoid.

Proof. From Theorem 3.1, we have (T, Δ) is a rough monoid. Now, it is enough to prove Δ is commutative, i.e., for any $X, Y \subseteq U$ and $RS(X), RS(Y) \in T$, such that $RS(X\Delta Y) = RS(Y\Delta X)$, i.e.,

Claim:

- (i) $\underline{P}(X\Delta Y) = \underline{P}(Y\Delta X)$,
- (ii) $\overline{P}(X\Delta Y) = \overline{P}(Y\Delta X)$.

Proof of Claim (i): For $x \in \underline{P}(X\Delta Y)$, i.e., $[x]_P \subseteq (X\Delta Y)$

$$\begin{aligned}
 \implies & [x]_P \subseteq X \text{ or } [x]_P \subseteq Y \\
 \implies & [x]_P \subseteq Y \text{ or } [x]_P \subseteq X \\
 \implies & [x]_P \subseteq (Y\Delta X) \\
 \implies & \underline{P}(X\Delta Y) \subseteq \underline{P}(Y\Delta X).
 \end{aligned}$$

Similarly, we prove that

$$\begin{aligned}
 & \underline{P}(Y\Delta X) \subseteq \underline{P}(X\Delta Y) \\
 \therefore & \underline{P}(X\Delta Y) = \underline{P}(Y\Delta X)
 \end{aligned}$$

Proof of claim (ii): For $x \in \overline{P}(X\Delta Y)$, i.e., $[x]_P \cap (X\Delta Y) \neq \phi$

$$\begin{aligned} \implies & [x]_P \cap X \neq \phi \text{ or } [x]_P \cap Y \neq \phi \\ \implies & [x]_P \cap Y \neq \phi \text{ or } [x]_P \cap X \neq \phi \\ \implies & [x]_P \cap (Y\Delta X) \neq \phi \\ \implies & \overline{P}(X\Delta Y) \subseteq \overline{P}(Y\Delta X). \end{aligned}$$

Similarly, we prove that

$$\begin{aligned} \overline{P}(Y\Delta X) &\subseteq \overline{P}(X\Delta Y) \\ \therefore \overline{P}(X\Delta Y) &= \overline{P}(Y\Delta X). \end{aligned}$$

Hence, $RS(X\Delta Y) = RS(Y\Delta X)$. Thus (T, Δ) is a commutative monoid. This commutative monoid is called as commutative rough monoid. ■

Theorem 3.3. (T, Δ) is a regular rough monoid of idempotents.

Proof. To prove: (T, Δ) is a regular monoid. We need to prove that, for any $RS(X) \in T$, there exist $RS(Y) \in T$ such that

$$RS(X)\Delta RS(Y)\Delta RS(X) = RS(X).$$

For $RS(X) \in T$, take $Y = E_X = \{x \in U \mid [x]_P \subseteq X\}$.
Then $Y\Delta X = X$ and hence $X\Delta Y\Delta X = X$.

$$\therefore RS(X)\Delta RS(Y)\Delta RS(X) = RS(X\Delta Y\Delta X) = RS(X).$$

Hence

$$(T, \Delta) \text{ is a regular monoid.}$$

Now, for $RS(X) \in T$,

$$RS(X)\Delta RS(X) = RS(X\Delta X) = RS(X).$$

This implies that $RS(X)$ is an idempotent in T , i.e., all elements of T are idempotent. Hence (T, Δ) is a regular monoid of idempotents called as regular rough monoid of idempotents on T . ■

4. Rough Ideals

In this section, we discuss about the principal ideals of commutative regular monoid of idempotents (T, Δ) .

Definition 4.2. [Rough ideal] Consider the commutative regular monoid of idempotents (T, Δ) . For any $RS(X) \in T$, $RS(X)T$ is the principal ideal generated by $RS(X)$. This ideal is called as the principal rough ideal on T .

Remarks 4.1. The Rough ideal is $RS(X)T = \{RS(X)\Delta RS(Y) \mid RS(Y) \in T\}$.

Next, we prove a representation theorem for these principal ideals and for any information system $I = (U, A)$, $T = \{RS(X) : X \subseteq U\}$ and $X = E_X \cup P_X$ where E_X be the union of equivalence classes which is completely contained in X , i.e., $E_X = \{x \in U \mid [x]_P \subseteq X\}$ and $P_X = \bigcup_{i=1}^r X_{\alpha_i}$ where X_{α_i} are the proper subset of all indiscernable classes also $\overline{P_X}$ be the union of equivalence classes containing the elements of P_X .

For example, let $X = \{x_1, x_2, x_3\}$ as in Table 1. Then, $X = E_X \cup P_X$, where $E_X = \{x_1, x_3\}$, $P_X = \{x_2\}$ and $\overline{P_X} = \{x_2, x_4, x_6\}$.

Using this characterization for any subset X of U , we have the following theorem representing the lower and upper approximation of the elements of the rough ideals.

Theorem 4.1. *If $I=(U, A)$ be an information system and $T=\{RS(X) : X \subseteq U\}$, for any $RS(Y) \in T$, then $RS(X \Delta Y) = (E_X \cup E_Y, E_X \cup \overline{P_X} \cup E_Y \cup \overline{P_Y})$*

Proof. For $X \subseteq U$, $X = E_X \cup P_X$ and for $Y \subseteq U$, $Y = E_Y \cup P_Y$

$$(5) \quad RS(X \Delta Y) = (\underline{P}(X \Delta Y), \overline{P}(X \Delta Y))$$

$$\begin{aligned} \underline{P}(X \Delta Y) &= [x]_P \subseteq X \Delta Y \\ (6) \quad &= E_X \cup E_Y \end{aligned}$$

$$\begin{aligned} \overline{P}(X \Delta Y) &= [x]_P \cap X \Delta Y \neq \phi \\ (7) \quad &= E_X \cup \overline{P_X} \cup E_Y \cup \overline{P_Y} \end{aligned}$$

From (5), (6) and (7) we have,

$$(8) \quad RS(X \Delta Y) = (E_X \cup E_Y, E_X \cup \overline{P_X} \cup E_Y \cup \overline{P_Y})$$

This completes the proof. ■

In the following section the described concepts are illustrated through examples.

5. Examples

Example 5.1. Let us consider The following information system $I = (U, A)$, where $U = \{x_1, x_2, \dots, x_6\}$ as in table 1, then (T, Δ) is a Commutative Monoid.

As we have, from table 1, $|T| = 18$ [13], also the Cayley's table of (T, Δ) will be very large. So, we have illustrated through one simple example.

(Closure axiom) Let $X = \{x_1, x_2, x_3, x_4\} \subseteq U$, $Y = \{x_5, x_6\} \subseteq U$ from equations (1), (2) and (3). We have, $X \Delta Y = X \cup Y$ if $IW(X \cup Y) = IW(X) + IW(Y) - IW(X \cap Y)$ but from $X \cup Y$ we have $IW(X \cup Y) > IW(X) + IW(Y) - IW(X \cap Y)$. so by definition of *Ind.weight* deleting the elements from the new class, we get $X \Delta Y = X \cup Y = \{x_1, x_2, x_3, x_4, x_5\}$.

Therefore, $RS(X\Delta Y) = \{[x_1]_p \cup [x_5]_p, [x_1]_p \cup [x_2]_p \cup [x_5]_p\}$.
Hence, $RS(X\Delta Y) \in T$.

(Associative axiom) Let us consider from closure $X\Delta Y = \{x_1, x_2, x_3, x_4, x_5\}$ and let $Z = \{x_2, x_3\}$ then from the definition of *Ind.weight* we have $(X\Delta Y)\Delta Z = \{x_1, x_2, x_3, x_4, x_5\}$

$$(9) \quad \text{Therefore, } RS((X\Delta Y)\Delta Z) = \{[x_1]_p \cup [x_5]_p, [x_1]_p \cup [x_2]_p \cup [x_5]_p\}.$$

Also, $Y\Delta Z = Y \cup Z$ if $IW(X \cup Y) = IW(X) + IW(Y) - IW(X \cap Y)$ so, $Y\Delta Z = \{x_2, x_3, x_5, x_6\}$ and $X\Delta(Y\Delta Z) = \{x_1, x_2, x_3, x_4, x_5\}$

$$(10) \quad \text{Therefore, } RS(X\Delta(Y\Delta Z)) = \{[x_1]_p \cup [x_5]_p, [x_1]_p \cup [x_2]_p \cup [x_5]_p\}.$$

From (9) and (10), we have $RS((X\Delta Y)\Delta Z) = RS(X\Delta(Y\Delta Z))$.

(Identity axiom) For $U = \{x_1, x_2, \dots, x_6\}$ as in Table 1 and $X = \{x_1, x_2, x_3, x_4\} \subseteq U$, there exist an empty set ϕ such that

$$(11) \quad RS(X\Delta\phi) = RS(\phi\Delta X) = RS(X) = \{[x_1]_p, [x_1]_p \cup [x_2]_p\}.$$

Thus, we have (T, Δ) is a Rough monoid.

(Commutative axiom) Let us consider, $U = \{x_1, x_2, \dots, x_6\}$ as in Table 1 and let $X = \{x_1, x_2, x_3, x_4\} \subseteq U$, $Y = \{x_5, x_6\} \subseteq U$ also from equations (1), (2) and (3). We have, $X\Delta Y = X \cup Y$ if $IW(X \cup Y) = IW(X) + IW(Y) - IW(X \cap Y)$ but from $X \cup Y$ we have $IW(X \cup Y) > IW(X) + IW(Y) - IW(X \cap Y)$. so by definition of *Ind.weight* deleting the elements from the new class, we get $X\Delta Y = X \cup Y = \{x_1, x_2, x_3, x_4, x_5\} = Y \cup X = Y\Delta X$.

Therefore, $RS(X\Delta Y) = \{[x_1]_p \cup [x_5]_p, [x_1]_p \cup [x_2]_p \cup [x_5]_p\} = RS(Y\Delta X)$.

(T, Δ) is a Commutative rough monoid.

Example 5.2. (Regular Monoid) Let us consider, $U = \{x_1, x_2, \dots, x_6\}$ as in Table 1 and let $X = \{x_1, x_2, x_3\} \subseteq U$, $Y = \{x_4, x_6\} \subseteq U$ also from equations (1), (2) and (3). We have, $X\Delta Y = X \cup Y = \{x_1, x_2, x_3\}$ then

$$RS(X\Delta Y) = RS(X) = \{[x_1]_p, [x_1]_p \cup [x_2]_p\} = RS(X\Delta Y\Delta X).$$

5.1. Examples for Ideals

- The ideals for $RS(X_1)\Delta T$ are $RS(U)$, $RS(X_1)$, $RS(X_1 \cup X_2)$, $RS(X_1 \cup X_3)$, $RS(X_1 \cup \{x_2\})$, $RS(X_1 \cup \{x_2\} \cup X_3)$.
- The ideals for $RS(X_3)\Delta T$ are $RS(X_1 \cup X_3)$, $RS(X_2 \cup X_3)$, $RS(X_3)$, $RS(U)$, $RS(\{x_1\} \cup X_3)$, $RS(\{x_2\} \cup X_3)$, $RS(\{x_1\} \cup X_2 \cup X_3)$, $RS(X_1 \cup \{x_2\} \cup X_3)$, $RS(\{x_1\} \cup \{x_2\} \cup X_3)$.

- The ideals for $RS(\{x_1\})\Delta T$ are $RS(U)$, $RS(X_1)$, $RS(X_1 \cup X_2)$, $RS(X_1 \cup X_3)$, $RS(X_1 \cup \{x_2\})$, $RS(\{x_1\} \cup X_2)$, $RS(\{x_1\} \cup X_3)$, $RS(\{x_1\} \cup X_2 \cup X_3)$, $RS(X_1 \cup \{x_2\} \cup X_3)$, $RS(\{x_1\} \cup X_2 \cup X_3)$, $RS(\{x_1\})$, $RS(\{x_1\} \cup \{x_2\})$, $RS(\{x_1\} \cup \{x_2\} \cup X_3)$.

6. Conclusion

In this paper, we have introduced a new operation Praba Δ on the set of all rough sets T for a given information system $I = (U, A)$. Also we have proved (T, Δ) is a commutative regular monoid of idempotents. We also gave a characterization for the principal rough ideal in T . The future study is to investigate and explore this regular monoid.

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CONTROLLABILITY RESULTS FOR SECOND ORDER IMPULSIVE NEUTRAL FUNCTIONAL INTEGRODIFFERENTIAL INCLUSIONS WITH INFINITE DELAY

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Abstract. This paper is concerned with the controllability of a partial neutral functional integrodifferential inclusion of second order with impulse effect and infinite delay in Banach spaces. The controllability of mild solutions using a fixed point theorem for contraction multi-valued maps and without assuming compactness of the family of cosine operators.

Keywords: Controllability; Second Order Abstract Cauchy problem; Impulsive systems; Phase space; Cosine function of operators; Integrodifferential inclusions.

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1. Introduction

The controllability for functional differential and integrodifferential systems in Banach spaces has found wide applications in many branches of physics and technical sciences. The theory of impulsive integrodifferential equations is an important branch of differential equations, which has an extensively physical background [14], [16], [22]. Delay differential equations are a special type of functional differential and integrodifferential equations which is similar to the ordinary differential equation, but the evolution involves past values of the state variable. The solutions of delay differential equations therefore requires knowledge of not only the current state, but also of the state of a certain time previously. System of delay differential equations have occurred all areas of science and particularly in the biological sciences (e.g., population dynamics and epidemiology). They also often arise in either natural or technological control problems. In these systems, a controller monitors the state of the system, and makes adjustments to the system based on its observations. Since these adjustments can never be made instantaneously, a delay arises between the observation and the control action. There are different kinds of delay existing in dynamical systems. The development of the theory of functional differential equations with infinite delay heavily depends on a choice of a phase space. In fact, various phase spaces have been considered and each different phase space has required a separate development of the theory. When the delay is infinite, the selection of the state (i.e. phase space) plays an important role in the study of both qualitative and quantitative theory. A usual choice is a normed space satisfying suitable axioms, which was introduced by Hale and Kato [15]. Hernandez et al. [16] proved the existence of solutions for impulsive partial neutral functional differential equations for first and second order systems with infinite delay. Balachandran et al. [1], [2], [3] discussed the controllability first and second order functional differential and integrodifferential evolution system in a Banach space without impulse effect. Chang and Li [9] obtained the controllability result for functional integrodifferential inclusions on an unbounded domain without impulse term. In recent years, significant progress has been made in the controllability of linear and nonlinear deterministic systems [19], [20], [21]

On the other hand, researchers have been proving the controllability results using compactness assumption of semigroups and the family of cosine operators. Bing Liu [23] studied the controllability of first order impulsive neutral functional integrodifferential inclusions with infinite delay in a Banach space with the assumption of compactness of the semigroup. However, as remarked by Triggiani [24], in an infinite dimensional Banach space, the linear control system is never exactly controllable on a given interval of time, if either a bounded linear operator (from control space to state space) is compact or a semigroup is compact. According to Triggiani [24], this is a typical case for most control systems governed by parabolic partial differential equations, and hence the concept of exact controllability is very limited for many parabolic partial differential equations. Nowadays, researchers are engaged to overcome this problem, refer to [15], [16], [17], [23], [24]. Recently, Chalishajar and Gunasekar et al. [9], [6], [7], [8], [14] studied the

controllability of second order neutral functional differential inclusion, with infinite delay and impulse effect on unbounded domain, without compactness of the family of cosine operators.

In this paper, we discuss the controllability for the second order impulsive neutral functional integrodifferential inclusions, with infinite delay through the phase space defined in [17], and without compactness of the family of cosine operators. To the best of our knowledge, there is no work reported on the controllability of neutral functional integrodifferential inclusion of second order with impulse effect and infinite delay in a Phase space, and the aim of this paper is to close the gap. The result obtained in this paper are generalizations of the results given by Chalishajar [8] and Covitz and Nadler [11].

2. Preliminaries

The purpose of this paper is to study the controllability of impulsive partial neutral functional integrodifferential inclusion of second order with infinite delay. Specifically, we are concerned with the inclusion form

$$(1) \quad \begin{cases} \frac{d}{dt} [x'(t) - g(t, x_t, x'(t))] \in Ax(t) + Bu(t) \\ \quad + F\left(t, x_t, x'(t), \int_0^t e(t, s, x_s, x'(s))ds\right), \quad t \in J, \quad t \neq t_k; \\ \Delta x(t_k) = I_k^1(x(t_k), x'(t_k)); \quad k = 1, 2, \dots, m \\ \Delta x(t_k) = I_k^2(x(t_k), x'(t_k)); \quad k = 1, 2, \dots, m \\ x(0) = \phi \in \mathcal{B}_h, \quad x'(0) = x_0, \end{cases}$$

where $A : D(A) \subset X \rightarrow X$ is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in R\}$ defined on X , and $F : J \times \mathcal{B}_h \times X \times X \Rightarrow X$ is bounded, closed and convex multivalued map, $h : J \times J \times \mathcal{B}_h \times X \Rightarrow X$ is appropriate function. Let $J_0 = (-\infty, 0]$, and non-local condition $\phi \in \mathcal{B}_h$ (defined below) and $x_0 \in X$ be the given initial values. $g : J \times \mathcal{B}_h \times X \rightarrow X$ is a given function, the state the function $x(t)$ takes values in X , and the control function $u \in L^2(J, U)$, a Banach space of admissible control functions with U as a Banach space. B is a bounded linear operator U to X . X is a Banach space with the norm $|\cdot|$.

Also, $0 < t_0 < t_1 < \dots < t_p < t_{p+1} = m$ ($\rightarrow \infty$ as $t \rightarrow \infty$); $I_k^1, I_k^2 \in C(X \times X, X)$, $k = 1, 2, \dots, p$ are bounded, $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, $\Delta x'|_{t=t_k} = x'(t_k^+) - x'(t_k^-)$ and $x(t_k^-), x(t_k^+), x'(t_k^-), x'(t_k^+)$ represents the left and right limits of $x(t)$ and $x'(t)$, respectively, at $t = t_k$. Furthermore, for any continuous function x defined on the interval $J_1 = (-\infty, m]$ with values in X and for any $t \in J$. We denote by x_t an element of $C(J_0, X)$ defined by $x_t(\theta) = x(t + \theta)$, $\theta \in J_0$.

We present the abstract phase space $\mathcal{B}_h$. Assume that $h : (-\infty, 0] \rightarrow [0, \infty)$ be a continuous function with $l = \int_{-\infty}^0 h(s)ds < +\infty$. Define

$$\mathcal{B}_h := \left\{ \begin{array}{l} \phi :]-\infty, 0] \rightarrow X \text{ such that, for any } r > 0, \\ \phi(\theta) \text{ is bounded and measurable function on } [-r, 0] \\ \text{and } \int_{-\infty}^0 h(s) \sup_{s \leq \theta \leq 0} |\phi(\theta)| ds < +\infty \end{array} \right\}.$$

Here, $\mathcal{B}_h$ is endowed with the norm

$$\|\phi\|_{\mathcal{B}_h} = \int_{-\infty}^0 h(s) \sup_{s \leq \theta \leq 0} |\phi(\theta)| ds, \quad \forall \phi \in \mathcal{B}_h.$$

Then, it is easy to show that $(\mathcal{B}_h, \|\cdot\|_{\mathcal{B}_h})$ is a Banach space.

Lemma 2.1 [8] *Suppose $x \in \mathcal{B}_h$; then, for each $t \in J, x_t \in \mathcal{B}_h$. Moreover,*

$$l|x(t)| \leq \|x_t\|_{\mathcal{B}_h} \leq \sup_{s \leq \theta \leq 0} (|x(s)| + \|x_0\|_{\mathcal{B}_h}),$$

where $l := \int_{-\infty}^0 h(s)ds < +\infty$.

Next, we introduce definitions, notation and preliminary facts from multi-valued analysis, which are useful for the development of this paper. Let $C(J, X)$ be the Banach space of continuous functions from J to X with the norm

$$\|y\|_{\infty} = \sup_{t \in J} |y(t)|.$$

$B(X)$ denotes the Banach space of bounded linear operators from X to X .

Let $L^1(J, X)$ denotes the Banach space of continuous function $x : J \rightarrow X$, which are integrable and endowed with the norm

$$\|x\|_{L^1} = \int_0^m |x(t)| dt, \quad x \in L^1(J, X).$$

For a metric space (X, d) , we introduce the following notations:

$$\begin{aligned} P(X) &:= \{Y \in \mathcal{P}(X) : Y \neq \emptyset\}, \\ P_{cl}(X) &:= \{Y \in P(X) : Y \text{ is closed}\}, \\ P_b(X) &:= \{Y \in P(X) : Y \text{ is bounded}\}, \\ P_{cp}(X) &:= \{Y \in P(X) : Y \text{ is compact}\}, \\ P_{b,cl}(X) &:= \{Y \in P(X) : Y \text{ is bounded and closed}\}. \end{aligned}$$

We define a Hausdorff space $H_d : P(X) \times P(X) \rightarrow \mathbb{R}^+ \cup \{\infty\}$ by

$$H_d(A, B) := \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\},$$

where $d(A, b) := \inf_{a \in A} d(a, b)$, $d(a, B) := \inf_{b \in B} d(a, b)$.

Then $(P_{b,cl}(X), H_d)$ is a metric space, and $(P_{cl}(X), H_d)$ is a generalized (complete) metric space.

We now recall some preliminaries about multi-valued maps.

Let $(X, \|\cdot\|)$ be a Banach space. A multi-valued map $G : X \Rightarrow X$ is a convex (resp. closed) iff $G(y)$ is convex (resp. closed) in X , for all $y \in X$. The map G is bounded on bounded sets iff $G(B) = \bigcup_{y \in B} G(y)$ is bounded in X for any bounded set B on X (i.e., $\sup_{y \in B} \{\sup_{x \in G(y)} \|x\|\} < \infty$), G is called upper semi-continuous (u.s.c.) on X iff for each $y_0 \in X$ the set $G(y_0)$ is a nonempty, closed subset of X and if for each open set B of X containing $G(y_0)$ there exists an open neighbourhood A of y_0 such that $G(A) \subseteq B$. The map G is said to be completely continuous iff $G(B)$ is relatively compact for every bounded subset $B \subseteq X$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. iff G has a closed graph. That is, if $y_n \rightarrow y_0$ and $x_n \rightarrow x_0$, where $x_n \in G(y_n)$, then $x_0 \in G(y_0)$. We say that G has a fixed point iff there is $y \in X$ such that $y \in G(y)$. In the following, $BCC(X)$ denotes the set of all nonempty bounded, closed and convex subsets of X .

A multivalued map $G : J \rightarrow BCC(X)$ is said to be measurable iff for each $y \in X$, the distance function $Y : J \rightarrow \mathbb{R}$, defined by

$$Y(t) := d(y, G(t)) = \inf\{|y - z| : z \in G(t)\},$$

is measurable. For more details on multivalued maps see [12], [18].

An upper semi-continuous map $G : X \Rightarrow X$ is said to be condensing iff for any subset $B \subseteq X$, with $\alpha(B) \neq 0$ we have $\alpha(G(B)) < \alpha(B)$, where α denotes the Kuratowski measure of non-compactness. For the properties of the Kuratowski measure, we refer to Banas and Geobel [4].

We note that completely continuous multivalued map is the easiest example of a condensing map. For more details on multivalued maps, see the book of Deiming [12] and the research article of Travis and Webb [25].

We say that the family $\{C(t) : t \in \mathbb{R}\}$ of operators in $B(X)$ is a strongly continuous cosine family iff

1. $C(0) = I$, I is the identity operator on X .
2. $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $s, t \in \mathbb{R}$.
3. The map $t \mapsto C(t)y$ is strongly continuous for each $y \in X$.

The strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$, associated with the strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$, is defined by

$$S(t)y := \int_0^t C(s)y ds, y \in X, t \in \mathbb{R}.$$

The infinitesimal generator $A : X \rightarrow X$ of a cosine family $C(t) : t \in \mathbb{R}$ is defined by

$$Ay := \frac{d^2}{dt^2} C(t)y|_{t=0}, y \in D(A).$$

where $D(A) = \{y \in X : C(t)y \text{ is twice continuously differentiable}\}$.

We refer to the book of Goldstein [13] for the detailed study of the family of cosine and sine operators.

Definition 2.1 A multivalued operator $G : C \Rightarrow P_{cl}(X)$ is called

- (a) γ -Lipschitz iff there exists $\gamma > 0$ such that $H_d(G(x), G(y)) \leq \gamma d(x, y)$ for each $x, y \in X$;
- (b) a contraction iff it is γ -Lipschitz with $\gamma < 1$.

Definition 2.2 The integral formulation $x(t)$ of system (1) is given by

$$x(t) := \phi(t), \text{ if } t \in J_0,$$

$$\begin{aligned} x(t) := & C(t)\phi(0) + S(t)[x_0 - g(0, \phi, x_0)] + \int_0^t C(t-s)g(s, x_s, x'(s))ds \\ & + \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)f(s)ds \\ & + \sum_{0 < t_k < t} C(t-t_k)I_k^1(x_{t_k}, x'(t_k)) + \sum_{0 < t_k < t} S(t-t_k)I_k^2(x_{t_k}, x'(t_k)), \text{ if } t \in J, \end{aligned}$$

where $f \in S_{F,z,y}^1 := \{f \in L^1(J, X) : f(t) \in F(t, x_t, x'(t), \int_0^t e(t, s, x_s, x'(s))ds), \text{ for a.e. } t \in J\} \neq \emptyset$ and $x_0 = \phi \in B_h$ is called a mild solution of the inclusion (1) provided $\int_0^t C(t-s)g(s, x_s, x'(s))ds$ is integrable.

Definition 2.3 System (1) is said to be controllable on $J' = J \setminus \{t_1, t_2, \dots, t_p\}$ iff for every $\phi \in \mathcal{B}_h$ with $\phi(0) = D(A)x_0 \in X$, and $x_1 \in X$, there exists a control $u \in L^2(J, U)$ such that the mild solution $x(\cdot)$ of the system (1) satisfies $x(m) = x_1 \in D(A)$, $x'(m) = y_1 \in X$, the conditions $\Delta x|_{t=t_k} = I_k^1(x_{t_k}, x'(t_k))$, and $\Delta x'|_{t=t_k} = I_k^2(x_{t_k}, x'(t_k))$; $k = 1, 2, \dots, p$; and $x'(0) = x_0 \in \mathcal{B}_h$.

Lemma 2.2 ([25]) Let $\{C(t) : t \in \mathbb{R}\}$ be a strongly continuous cosine family in X with infinitesimal generator A . If $h_1 : \mathbb{R} \rightarrow X$ is continuously differentiable, with $z_0 \in D(A)$, $z_1 \in X$, and

$$w(t) = C(t)z_0 + S(t)z_1 + \int_0^t S(t-s)h(s)ds, \quad t \in \mathbb{R},$$

then $w(t) \in D(A)$ for $t \in \mathbb{R}$, w is twice continuously differentiable, and w satisfies

$$w''(t) = Aw(t) + h_1(t), \quad t \in \mathbb{R}, \quad w(0) = x_1, \quad w'(0) = y_0.$$

Conversely, if $h_1 : \mathbb{R} \rightarrow X$ is continuous, $w(t) : \mathbb{R} \rightarrow X$ is twice continuously differentiable, $w(t) \in D(A)$ for $t \in \mathbb{R}$, and w satisfies system (1). Then

$$w(t) = C(t)z_0 + S(t)z_1 + \int_0^t S(t-s)h(s)ds, \quad t \in \mathbb{R}.$$

Lemma 2.3 (Covitz and Nadler [11]) *Let (X, d) be a complete metric space. If $G : X \Rightarrow P_{cl}(X)$ is a contraction, then $\text{fix } G = \phi$ (where $\text{fix } G$ denotes set of fixed points of the multi-valued operator G).*

3. Controllability results

In this section, we prove the controllability of system (1). We adopt the following hypotheses:

(H1) An operator A is an infinitesimal generator of a strongly continuous and bounded cosine family $\{C(t) : t \in R\}$ with $M = \sup\{|C(t)| : t \in J\}$, and sine family $\{S(t) : t \in R\}$ with $\widetilde{M} = \sup\{|S(t)| : t \in J\}$.

(H2) $F : J \times \mathcal{B}_h \times X \times X \Rightarrow X : P_{cp}(X) : (\cdot, \psi, x, w) \Rightarrow F(\cdot, \psi, x, w)$ is measurable with respect to t for each $\psi \in \mathcal{B}_h$ and $x, w \in X$, and F is u.s.c. with respect to second, third and fourth variables, for each $t \in J$.

Indeed, by the Caratheodory condition of F , F has a measurable selection (see Theorem III in [5]).

(H3) The linear operator $W : L^2(J, U) \rightarrow X$ is defined by

$$Wu := \int_0^b S(t-s)Bu(s)ds,$$

W has an invertible operator W^{-1} which take the values in $L^2(J, U)|\ker W$ (since the restriction of W to the domain $L^2(J, U)|\ker W$ is invertible) and there exists positive constants M_1 such that $\|W^{-1}\| \leq M_1$.

(H4) The linear operator $\overline{W} : L^2(J, U) \rightarrow X$ is defined by

$$\overline{W}u := \int_0^b C(t-s)Bu(s)ds,$$

$\overline{W}$ has an invertible operator $\overline{W}^{-1}$ which take the values in $L^2(J, U)|\ker \overline{W}$ and there exists positive constants M_2 such that $\|\overline{W}^{-1}\| \leq M_2$.

(H5) There exist a positive constant M_3 such that $\|B\| \leq M_3$ for $t \in J$.

(H6) $H_d(F(t, \psi_1, u_1, w_1), F(t, \psi_2, u_2, w_2)) \leq l(t)(\|\psi_1 - \psi_2\| + |u_1 - u_2| + |w_1 - w_2|)$, for each $t \in J$, $\psi_1, \psi_2 \in \mathcal{B}_h$ and $u_1, u_2, w_1, w_2 \in X$; where $l \in L^1(J, R^+)$ and $d(0, F(t, 0, 0, 0)) \leq l(t)$, for a.e. $t \in J$.

(H7) The function $e : J \times J \times \mathcal{B}_h \times X \Rightarrow X$ satisfies the following conditions:

- (i) For every $(t, s) \in J \times J$ the function $h(t, s, \cdot) : \mathcal{B}_h \times X \rightarrow X$ is continuous.
- (ii) For every $(\psi, w) \in \mathcal{B}_h \times X$ the function $e(\cdot, \psi, w) : J \times J \rightarrow X$ is strongly measurable.

- (iii) $H_d(e(t, s, \psi_1, u_1), e(t, s, \psi_2, u_2)) \leq l'(t)(\|\psi_1 - \psi_2\| + |u_1 - u_2|)$, for each $t \in J$, $\psi_1, \psi_2 \in \mathcal{B}_h$ and $u_1, u_2 \in X$; where $l' \in L^1(J, R^+)$ and $d(0, e(t, s, 0, 0)) \leq l'(t)$, for a.e. $t \in J$.
- (H8) (i) $I_k^1 : \mathcal{B}_h \times X \rightarrow X$ are completely continuous and there exist constants $c_k^j, j = 1, 2$, such that $\|I_k^1(\psi, x)\|_X \leq (c_k^1(\|\psi\|_{\mathcal{B}_h} + |x|)) + c_k^2$; $k = 1, 2, \dots, p$; for every $(\psi, x) \in \mathcal{B}_h \times X$.
- (ii) $I_k^2 : \mathcal{B}_h \times X \rightarrow X$ are completely continuous and there exist constants $d_k^j, j = 1, 2$, such that $\|I_k^2(\psi, x)\|_X \leq (d_k^1(\|\psi\|_{\mathcal{B}_h} + |x|)) + d_k^2$; $k = 1, 2, \dots, p$; for every $(\psi, x) \in \mathcal{B}_h \times X$.

Theorem 3.1 Assume that the hypotheses (H1)-(H8) be satisfied. System (1) is controllable for x_0 and x_1 on J , provided $m\widetilde{M}L[1 + (1/2)M_3m(M_1\widetilde{M} + M_2M)] < 1$.

Proof. For $k = 1, 2, \dots, p$, consider $\mathcal{B}_b(J_1, X) = \{x :] - \infty, m] \rightarrow X; x(t) \text{ is continuous at } t \neq t_k, x(t_k^-) = x(t_k), \text{ and } x(t_k^+) \text{ exists}\}$, and $\mathcal{B}_b^1(J_1, X) = \{x \in \mathcal{B}_b(J_1, X); x'(t) \text{ is continuous at } t \neq t_k, x'(t_k^-) = x'(t_k), \text{ and } x'(t_k^+) \text{ exists}\}$, with the norm

$$\|x\|_{b^1} = \|x\|_{\mathcal{B}_h} + \sup_{t \in J_m} \{\|x(s)\|_b, \|x'(s)\|_{b^1}\} : 0 \leq s \leq b, x \in \mathcal{B} \text{ and } x' \in \mathcal{B}_b^1;$$

where $\|x\|_b = \sup_{t \in J_m} |x(t)|$. Here, x_k is restriction of x to $J_k = (t_k, t_{k+1}]$; such that

$$\|x_k\|_{J_k} = \sup_{s \in J_k} \|x_k(s)\|.$$

Let us define the spaces

$$Z_1 := \{\mathcal{B}_b(J_1, X) \cap C^2(J'_1, X)\}, \quad Z_2 := \{\mathcal{B}_b^1(J_1, X) \cap C^2(J'_1, X)\},$$

where $J'_1 = J_1 \setminus \{t_k, k = 1, 2, \dots, p\}$.

Using hypotheses (H3) and (H4), we defined the control formally as

$$\begin{aligned} u_y(t) = & \frac{1}{2} \left\{ W^{-1} \left[x_1 - C(m)\phi(0) - S(m)[x(0) - g(0, \phi, x_0)] \right. \right. \\ & - \int_0^m C(m-s)g(s, x_s, x'(s))ds - \int_0^m S(m-s)f(s)ds \\ & - \sum_{0 < t_k < t} C(m-t_k)I_k^1(x_{t_k}, x'(t_k)) - \sum_{0 < t_k < t} S(m-t_k)I_k^2(x_{t_k}, x'(t_k)) \Big] \\ & + \overline{W}^{-1} \left[y_1 - AS(m)\phi(0) - C(m)[x(0) - g(0, \phi, x_0)] \right. \\ & - \int_0^m AS(m-s)g(s, x_s, x'(s))ds - \int_0^m C(m-s)f(s)ds \\ & \left. \left. - \sum_{0 < t_k < t} AS(m-t_k)I_k^1(x_{t_k}, x'(t_k)) - \sum_{0 < t_k < t} C(m-t_k)I_k^2(x_{t_k}, x'(t_k)) \right] \right\}(t), \end{aligned}$$

where $v \in S_{F,z,x}^1$.

Using this control, we shall now show that the operator $G_1 : Z_1 \Rightarrow \mathcal{P}(Z_1)$ and $G_2 : Z_2 \Rightarrow \mathcal{P}(Z_2)$, defined by

$$\begin{aligned} G_1(y) := & \left\{ x \in Z_1 : x(t) = C(t)\phi(0) + S(t)[x_0 - g(0, \phi, x_0)] \right. \\ & + \int_0^t C(t-s)g(s, x_s, x'(s))ds + \int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)Bu_y(s)ds \\ & \left. + \sum_{0 < t_k < t} C(t-t_k)I_k^1(x_{t_k}, x'(t_k)) + \sum_{0 < t_k < t} S(t-t_k)I_k^2(x_{t_k}, x'(t_k)) \right\}, \end{aligned}$$

$$\begin{aligned} G_2(y') := & \left\{ x' \in Z_1 : x(t) = AS(t)\phi(0) + C(t)[x_0 - g(0, \phi, x_0)] \right. \\ & + \int_0^t AS(t-s)g(s, x_s, x'(s))ds + \int_0^t C(t-s)f(s)ds + \int_0^t C(t-s)Bu_y(s)ds \\ & \left. + \sum_{0 < t_k < t} AS(t-t_k)I_k^1(x_{t_k}, x'(t_k)) + \sum_{0 < t_k < t} C(t-t_k)I_k^2(x_{t_k}, x'(t_k)) \right\}, \end{aligned}$$

has a fixed point. This fixed point is then a mild solution of system (1).

Obviously, $x_1 \in (G_1 y)(m)$ and $x_2 \in (G_2 y')(m)$. Next, we shall show that G_1 and G_2 satisfy the assumptions of Lemma (2.3). The proof will be given in two steps.

Step 1. We show that $G_1(y) \in P_{cl}(Z_1)$, $G_2(y) \in P_{cl}(Z_2)$.

Indeed, let $\{x_n\}_{n \geq 0} \in G_1(y)$ such that $x_n \rightarrow x_*$ in Z_1 . Then $x_* \in Z_1$, and there exists $f_n \in S_{F,z,x}^1$ such that for each $t \in J$.

$$\begin{aligned} x_n(t) = & C(t)\phi(0) + S(t)[x_0 - g(0, \phi, x_0)] \\ & + \int_0^t C(t-s)g(s, x_s, x'(s))ds + \int_0^t S(t-s)f_n(s)ds \\ & + \sum_{0 < t_k < t} C(t-t_k)I_k^1(x_{t_k}, x'(t_k)) + \sum_{0 < t_k < t} S(t-t_k)I_k^2(x_{t_k}, x'(t_k)) \\ & + \frac{1}{2} \int_0^t S(t-s)B \left\{ W^{-1} \left[x_1 - C(m)x_0 - S(m)[x_1 - g(0, \phi, x_0)] \right. \right. \\ & - \int_0^m C(m-\tau)g(\tau, x_\tau, x'(\tau))ds - \int_0^m S(m-\tau)f_n(\tau)d\tau \\ & - \sum_{0 < t_k < t} C(m-t_k)I_k^1(x_{t_k}, x'(t_k)) - \sum_{0 < t_k < t} S(m-t_k)I_k^2(x_{t_k}, x'(t_k)) \left. \right] \\ & + \overline{W}^{-1} \left[y_1 - AS(m)\phi(0) - C(m)[x(0) - g(0, \phi, x_0)] \right. \\ & - \int_0^m AS(m-\tau)g(\tau, x_\tau, x'(\tau))ds - \int_0^m C(m-\tau)f_n(\tau)d\tau \\ & \left. \left. - \sum_{0 < t_k < t} AS(m-t_k)I_k^1(x_{t_k}, x'(t_k)) - \sum_{0 < t_k < t} C(m-t_k)I_k^2(x_{t_k}, x'(t_k)) \right] \right\}(s)ds \end{aligned}$$

Using the fact that F has compact values and (H6) holds, we may pass a subsequence which is necessary to find that f_n converges to f in $L^1(J, X)$; hence $f \in S_{F,z,x}^1$.

Then, for each $t \in J$,

$$\begin{aligned}
x_n(t) &\rightarrow x_*(t) = C(t)\phi(0) + S(t)[x_0 - g(0, \phi, x_0)] \\
&+ \int_0^t C(t-s)g(s, x_s, x'(s))ds + \int_0^t S(t-s)f_n(s)ds \\
&+ \sum_{0 < t_k < t} C(t-t_k)I_k^1(x_{t_k}, x'(t_k)) + \sum_{0 < t_k < t} S(t-t_k)I_k^2(x_{t_k}, x'(t_k)) \\
&+ \frac{1}{2} \int_0^t S(t-s)B \left\{ W^{-1} \left[x_1 - C(m)x_0 - S(m)[x_1 - g(0, \phi, x_0)] \right. \right. \\
&- \int_0^m C(m-\tau)g(\tau, x_\tau, x'(\tau))ds - \int_0^m S(m-\tau)f_n(\tau)d\tau \\
&- \sum_{0 < t_k < t} C(m-t_k)I_k^1(x_{t_k}, x'(t_k)) - \sum_{0 < t_k < t} S(m-t_k)I_k^2(x_{t_k}, x'(t_k)) \left. \right] \\
&+ \overline{W}^{-1} \left[y_1 - AS(m)\phi(0) - C(m)[x(0) - g(0, \phi, x_0)] \right. \\
&- \int_0^m AS(m-\tau)g(\tau, x_\tau, x'(\tau))ds - \int_0^m C(m-\tau)f_n(\tau)d\tau \\
&- \sum_{0 < t_k < t} AS(m-t_k)I_k^1(x_{t_k}, x'(t_k)) - \sum_{0 < t_k < t} C(m-t_k)I_k^2(x_{t_k}, x'(t_k)) \left. \right] \Big\} (s)ds.
\end{aligned}$$

So, $x_* \in G_1(y)$. In particular, $G_1(y) \in P_{cl}(Z_1)$. Similarly, we can show that $G_2(y') \in P_{cl}(Z_2)$.

Step 2. We shall show $G_1(y)$ and $G_2(y')$ are contractive multi-valued maps for each $x \in Z_1$ and $x' \in Z_2$.

Let $x_t, \overline{x}_t, x', \overline{x}' \in Z_1$ and $x \in G_1(y)$. then there exists $f \in S_{F,z,x}^1$ such that

$$\begin{aligned}
x(t) &= C(t)\phi(0) + S(t)[x_0 - g(0, \phi, x_0)] \\
&+ \int_0^t C(t-s)g(s, x_s, x'(s))ds + \int_0^t S(t-s)f(s)ds \\
&+ \sum_{0 < t_k < t} C(t-t_k)I_k^1(x_{t_k}, x'(t_k)) + \sum_{0 < t_k < t} S(t-t_k)I_k^2(x_{t_k}, x'(t_k)) \\
&+ \frac{1}{2} \int_0^t S(t-s)B \left\{ W^{-1} \left[x_1 - C(m)x_0 - S(m)[x(0) - g(0, \phi, x_0)] \right. \right. \\
&- \int_0^m C(m-\tau)g(\tau, x_\tau, x'(\tau))ds - \int_0^m S(m-\tau)f(\tau)d\tau \\
&- \sum_{0 < t_k < t} C(m-t_k)I_k^1(x_{t_k}, x'(t_k)) - \sum_{0 < t_k < t} S(m-t_k)I_k^2(x_{t_k}, x'(t_k)) \left. \right] \\
&+ \overline{W}^{-1} \left[y_1 - AS(m)\phi(0) - C(m)[x(0) - g(0, \phi, x_0)] \right. \\
&- \int_0^m AS(m-\tau)g(\tau, x_\tau, x'(\tau))ds - \int_0^m C(m-\tau)f(\tau)d\tau \\
&- \sum_{0 < t_k < t} AS(m-t_k)I_k^1(x_{t_k}, x'(t_k)) - \sum_{0 < t_k < t} C(m-t_k)I_k^2(x_{t_k}, x'(t_k)) \left. \right] \Big\} (s)ds
\end{aligned}$$

From (H6),(H7) it follows that, for each $t \in J$.

$$H_d(F(t, x_t, x'(t), w), F(t, \bar{x}_t, \bar{x}'(t), \bar{w})) \leq l(t) \left(\|x_t - \bar{x}(t)\| + |x'(t) - \bar{x}'(t)| + |w - \bar{w}| \right).$$

Hence, there exists $\bar{f}(t) \in F(t, \bar{x}_t, \bar{x}'(t), \bar{w})$ such that

$$|f(t, x_t, x'(t), w) - \bar{f}(t, \bar{x}_t, \bar{x}'(t), \bar{w})| \leq l(t) \left(\|x_t - \bar{x}(t)\| + |x'(t) - \bar{x}'(t)| + |w - \bar{w}| \right).$$

Consider $V : J \Rightarrow \mathcal{P}(X)$, given by

$$V(t) := \left\{ \bar{f}(t) \in X : |f(t, x_t, x'(t), w) - \bar{f}(t, \bar{x}_t, \bar{x}'(t), \bar{w})| \leq l(t) \left(\|x_t - \bar{x}(t)\| + |x'(t) - \bar{x}'(t)| + |w - \bar{w}| \right) \right\}.$$

Since the multi-valued operator $W(t) = V(t) \cap F(t, \bar{x}_t, \bar{x}'(t), \bar{w})$ is a measurable (see Proposition III.4 in [5]), there exists a function $\bar{f}(t)$ which is measurable selection for W . So, $\bar{f}(t) \in F(t, \bar{x}_t, \bar{x}'(t), \bar{w})$, and

$$|f(t) - \bar{f}(t)| \leq l(t) \left(\|x_t - \bar{x}(t)\| + |x'(t) - \bar{x}'(t)| + |w - \bar{w}| \right), \text{ for each } t \in J.$$

Furthermore, for each $t \in J$, we define

$$\begin{aligned} \bar{x}(t) = & C(t)\phi(0) + S(t)[x_0 - g(0, \phi, x_0)] \\ & + \int_0^t C(t-s)g(s, x_s, x'(s))ds + \int_0^t S(t-s)\bar{f}(s)ds \\ & + \sum_{0 < t_k < t} C(t-t_k)I_k^1(x_{t_k}, x'(t_k)) + \sum_{0 < t_k < t} S(t-t_k)I_k^2(x_{t_k}, x'(t_k)) \\ & + \frac{1}{2} \int_0^t S(t-s)B \left\{ W^{-1} \left[x_1 - C(m)x_0 - S(m)[x(0) - g(0, \phi, x_0)] \right. \right. \\ & - \int_0^m C(m-\tau)g(\tau, x_\tau, x'(\tau))ds - \int_0^m S(m-\tau)\bar{f}(\tau)d\tau \\ & - \sum_{0 < t_k < t} C(m-t_k)I_k^1(x_{t_k}, x'(t_k)) - \sum_{0 < t_k < t} S(m-t_k)I_k^2(x_{t_k}, x'(t_k)) \left. \right] \\ & + \bar{W}^{-1} \left[y_1 - AS(m)\phi(0) - C(m)[x(0) - g(0, \phi, x_0)] \right. \\ & - \int_0^m AS(m-\tau)g(\tau, x_\tau, x'(\tau))ds - \int_0^m C(m-\tau)\bar{f}(\tau)d\tau \\ & \left. \left. - \sum_{0 < t_k < t} AS(m-t_k)I_k^1(x_{t_k}, x'(t_k)) - \sum_{0 < t_k < t} C(m-t_k)I_k^2(x_{t_k}, x'(t_k)) \right] \right\} (s)ds \end{aligned}$$

Then, for each $t \in J$, we get

$$\begin{aligned}
|x(t) - \bar{x}(t)| &\leq \left| \int_0^t S(t-s)[f(s) - \bar{f}(s)]ds \right| \\
&+ \frac{1}{2} \left| \int_0^t S(t-s)B \left[W^{-1} \int_0^m S(m-\tau)[f(\tau) - \bar{f}(\tau)]d\tau \right](s)ds \right| \\
&+ \frac{1}{2} \left| \int_0^t S(t-s)B \left[\bar{W}^{-1} \int_0^m C(m-\tau)[f(\tau) - \bar{f}(\tau)]d\tau \right](s)ds \right| \\
&\leq \widetilde{M}m \left[\|x_t - \bar{x}_t\| + |x'(t) - \bar{x}'(t)| \right] \int_0^m [l(s) + l'(s)]ds \\
&+ \frac{1}{2} \int_0^t |S(t-s)|M_3M_1\widetilde{M}m \left[\|x_t - \bar{x}_t\| + |x'(t) - \bar{x}'(t)| \right] \int_0^m [l(\tau) + l'(\tau)]d\tau \\
&+ \frac{1}{2} \int_0^t |S(t-s)|M_3M_2Mm \left[\|x_t - \bar{x}_t\| + |x'(t) - \bar{x}'(t)| \right] \int_0^m [l(\tau) + l'(\tau)]d\tau \\
&\leq m\widetilde{M}L \left[\|x_t - \bar{x}_t\| + |x'(t) - \bar{x}'(t)| \right] \\
&+ \frac{1}{2}\widetilde{M}M_3m(M_1\widetilde{M}m + M_2mM)L \left[\|x_t - \bar{x}_t\| + |x'(t) - \bar{x}'(t)| \right] \\
&\leq m\widetilde{M}L[1 + (1/2)M_3(M_1\widetilde{M}m + M_2mM)](\|x_t - \bar{x}_t\| + |x'(t) - \bar{x}'(t)|),
\end{aligned}$$

where $L = \int_0^m [l(s) + l'(s)]ds$. Then

$$\|x(t) - \bar{x}(t)\| \leq m\widetilde{M}L[1 + (1/2)M_3m(M_1\widetilde{M} + M_2M)](\|x_t - \bar{x}_t\| + |x'(t) - \bar{x}'(t)|).$$

By interchanging the roles of x and $\bar{x}$, we get an analogous relation

$$H_d(G_1(y), G_2(\bar{y})) \leq m\widetilde{M}L[1 + (1/2)M_3m(M_1\widetilde{M} + M_2M)](\|x_t - \bar{x}_t\| + |x'(t) - \bar{x}'(t)|).$$

Similarly, for any x' and $\bar{x}'$, we obtain

$$H_d(G_1(y'), G_2(\bar{y}')) \leq m\widetilde{M}L[1 + (1/2)M_3m(M_1\widetilde{M} + M_2M)](\|x_t - \bar{x}_t\| + |x'(t) - \bar{x}'(t)|).$$

Thus G_1 and G_2 are contractive. As a consequence of Lemma 2.3, we deduce that G_1 and G_2 have a fixed point. Hence, system (1) is controllable on J .

4. Conclusion

In this paper, we discussed the controllability results for the second order impulsive neutral functional integrodifferential inclusions, with infinite delay through phase space defined in Section 2. We have proved the result without compactness of family of cosine operators. The result is obtained using a Hausdorff space (defined in Section 2) and a contraction mapping for multi-valued maps due to [11].

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AN IMPLICIT METHOD FOR SOLVING NONCONVEX VARIATIONAL INEQUALITIES

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Abstract. This paper presents a predictor-corrector algorithm for solving the strongly nonlinear general nonconvex variational inequality, which is a class of nonconvex variational inequalities involving three nonlinear operators. We establish the equivalence between the strongly nonlinear general nonconvex variational inequalities and the fixed point problem, and show that the convergence of the predictor-corrector method only requires pseudomonotonicity, which is a weaker condition than monotonicity. Some special cases are also discussed.

Keywords: nonconvex variational inequalities, fixed point problems, predictor-corrector methods.

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1. Introduction

Variational inequality theory, introduced by Stampacchia [1], provides simple and unified framework to study a large number of problems arising in finance, economics, transportation, network and structural analysis, elasticity and optimization. Variational inequalities have been generalized and extended in many different directions using novel and innovative techniques to study wide classes of pure and applied sciences.

The existence and iterative schemes of variational inequalities have been investigated over convex sets, and that is due to the fact that all techniques are mainly based on the properties of the projection operator over convex sets. Recently, the concept of convex sets has been generalized in many different ways, which has a considerable impact in various fields, such as optimization, economics, control theory, etc.. The uniformly prox-regular sets are an immediate consequence of the

generalization of convex sets, these sets are nonconvex and include convex sets as a particular case.

Recently, Bounkhel et al. [2], Noor [3], Moudafi [4], and Pang et al. [5] considered the variational inequality problems and equilibrium problems over these nonconvex sets. They suggested and analyzed some projection type iterative algorithms by using the prox-regular technique and auxiliary principle technique. In [6]–[10] Noor has shown that the projection technique can be extended to nonconvex variational inequalities and has established the equivalence between the nonconvex variational inequalities and fixed point problems using the projection technique. This equivalent alternative formulation has been used to investigate the existence of a solution of the nonconvex variational inequalities on one hand and to introduce some iterative methods on the other hand.

Motivated by the recent research going on in this area, we have introduced and studied a generalization of nonconvex variational inequalities, which is called the strongly nonlinear general nonconvex variational inequality. We then solved this class using the projection method and the Wiener-Hopf equations technique and showed that the convergence of the two methods requires the strong monotonicity and the Lipschitz continuity of the operators [14], [15].

In this paper we present a new implicit iterative algorithm using the predictor-corrector technique to solve this class of variational inequalities, and showed that the convergence of the new algorithm requires the pseudomonotonicity, which is a weaker condition than monotonicity, and the Lipschitz requiring continuity of the operator is not needed.

This paper is divided into four sections, the first is the introduction, the second is the preliminaries and formulation, the third is devoted to the study of the iterative algorithm, and in the fourth section the convergence is proved.

For more information about applications, numerical methods and other aspects of variational inequalities, one may refer to [1]–[18].

2. Preliminaries and formulation

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ respectively. Let K be a nonempty closed subset of H . Recall the following well-known concepts from nonlinear convex analysis and non-smooth analysis [11]–[13].

Definition 2.1. The proximal normal cone of K at $u \in H$ is given by

$$N_K^P(u) := \{w \in H : \exists \alpha > 0 \mid u \in P_K[u + \alpha w]\},$$

where

$$P_K[u] = \{u^* \in K : d_K(u) = \|u - u^*\|\}.$$

and $d_K(\cdot)$ is the usual distance function to the subset K , that is

$$d_K(u) = \inf_{v \in K} \|v - u\|$$

The proximal normal cone $N_K^P(u)$ has the following characterization.

Lemma 2.1. *Let K be a nonempty, closed subset in H . Then $w \in N_K^P(u)$, if and only if, there exists a constant $\alpha > 0$ such that*

$$\langle w, v - u \rangle \leq \alpha \|v - u\|^2, \forall v \in K.$$

Definition 2.2. ([11]) The Clarke normal cone, denoted by $N_K^C(u)$, is defined by

$$N_K^C(u) = \overline{\text{co}}[N_K^P(u)],$$

where $\overline{\text{co}}[S]$ denotes the closure of the convex hull of S .

It is true that $N_K^P(u) \subset N_K^C(u)$, while the converse is not true in general. Note that $N_K^C(u)$ is always a closed and convex cone and that $N_K^P(u)$ is always a convex cone but may be non-closed (see [11], [12]). Furthermore, if K is convex all the existing normal cones coincide with the normal cone in the sense of convex analysis $N_K(u)$ given by

$$N_K(u) := \{v \in H : \langle v, u^* - u \rangle \leq 0, \forall u^* \in K\}.$$

A new class of nonconvex sets, called *uniformly r-prox-regular sets* (see [13]), has been introduced and studied in [11]. It has been successfully used in many non-convex applications such as optimization, economic models, dynamical systems, and differential inclusions. This class seems particularly well suited to overcome the difficulties which arise due to the nonconvexity assumption on K .

Definition 2.3. ([4]) For a given $r \in (0, \infty]$, a subset K is said to be uniformly r -prox-regular if and only if every nonzero proximal normal to K can be realized by an r -ball, that is, $\forall u \in K$ and $0 \neq w \in N_K^P(u)$, one has

$$\langle w / \|w\|, v - u \rangle \leq (1/2r) \|v - u\|^2, \forall v \in K.$$

Recall that for $r = +\infty$ the uniform r -prox-regularity of K is equivalent to the convexity of K . The following lemma summarizes some important consequences of the uniform-prox-regularity and is needed in the sequel.

Lemma 2.2. *Let K be a nonempty closed subset of H , $r \in (0, \infty]$ and set $K_r = \{u \in H : d(u, K) < r\}$. If K is uniformly r -prox-regular, then the following holds:*

- (i) $\forall u \in K_r, P_K(u) \neq \emptyset$;
- (ii) $\forall r \in (0, r)$, the operator P_K is Lipschitz continuous with constant $\frac{r}{r - r'}$ on $K_{r'}$.

Now we define the strongly nonlinear general nonconvex variational inequality problem (SNGNVIP) as the problem of finding $u \in H$ such that $g(u) \in K$ and

$$(2.1) \quad \begin{aligned} \langle Tu, g(v) - g(u) \rangle + \alpha \|g(v) - g(u)\|^2 &\geq \langle A(u), g(v) - g(u) \rangle, \\ \forall v \in H : g(v) &\in K, \end{aligned}$$

where T, A, g are given nonlinear operators from H into itself, and α is a positive parameter. Problem (2.1) was studied by the author in [14], [15].

Special cases

- If $g = I$, the identity operator, then (1) is equivalent to finding $u \in K$, such that

$$(2.2) \quad \langle Tu, v - u \rangle + \alpha \|v - u\|^2 \geq \langle A(u), v - u \rangle, \forall v \in K,$$

which is known as the strongly nonlinear nonconvex variational inequality, and was studied in [10].

- If $A(u) \equiv 0$, then problem (2.1) is equivalent to finding $u \in K$, such that

$$(2.3) \quad \langle Tu, g(v) - g(u) \rangle + \alpha \|g(v) - g(u)\|^2 \geq 0, \forall v \in H : g(v) \in K,$$

which is called the general nonconvex variational inequality studied in [7].

- If $g \equiv I$, the identity operator, then problem (2.3) is equivalent to finding $u \in K$, such that

$$(2.4) \quad \langle Tu, v - u \rangle + \alpha \|v - u\|^2 \geq 0, \forall v \in K,$$

which is called the nonconvex variational inequality.

- if K is a convex set in H , then problem (2.4) is equivalent to finding $u \in K$, such that

$$(2.5) \quad \langle Tu, v - u \rangle \geq 0, \forall v \in K.$$

Problem (2.5) is known as the classical variational inequality, which was introduced and studied by Stampacchia [1].

Now, if K is a nonconvex (uniformly r -prox-regular) set, then problem (2.1) is equivalent to finding $u \in K$, such that

$$(2.6) \quad 0 \in Tu - A(u) + g(u) - g(u) + N_K^P(g(u)),$$

where $N_K^P(g(u))$ denotes the normal cone of K at $g(u)$ in the sense of nonconvex analysis. Problem (2.6) is called the nonconvex variational inclusion problem

associated with nonconvex variational inequality (2.1). This implies that the variational inequality (2.1) is equivalent to finding a zero of the sum of two monotone operators (2.6).

3. Main results

In this section, we establish the equivalence between the nonconvex variational inequality (SNGNVIP) (2.1) and the fixed point problem using the projection operator.

Lemma 3.1. ([1]) *Let P_K be the projection of H onto the uniformly r -prox-regular set K , then $u \in H : g(u) \in K$ is a solution of the strongly nonlinear nonconvex variational inequality (2.1) if and only if $u \in H : g(u) \in K$ satisfies the relation*

$$(3.1) \quad g(u) = P_K[g(u) - \rho Tu + \rho A(u)],$$

Lemma 3.1 implies that the strongly nonlinear general nonconvex variational inequality (2.1) is equivalent to the fixed point problem (3.1). This will be invoked to suggest the following algorithms for solving the (SNGNVIP) (2.1).

Algorithm 3.1. *For a given $u_0 \in K$, find the approximate solution u_{n+1} using the iterative scheme*

$$g(u_{n+1}) = P_K[g(u_n) - \rho Tu_n + \rho A(u_n)], n = 0, 1, 2, \dots$$

Algorithm 3.1 is an explicit iterative method for solving the nonconvex variational inequality (2.1).

Now we use the fixed point formulation (3.1) to suggest another iterative method for solving (2.1) and this is the motivation behind this paper.

Algorithm 3.2. *For a given $u_0 \in K$, find the approximate solution u_{n+1} using the iterative scheme*

$$(3.2) \quad g(u_{n+1}) = P_K[g(u_n) - \rho Tu_{n+1} + \rho A(u_{n+1})], n = 0, 1, 2, \dots$$

Algorithm 3.2 is an implicit iterative algorithm, which is itself difficult to execute. To overcome this drawback, we use the predictor–corrector technique to suggest a two-step iterative method for solving the nonconvex variational inequalities (2.1). We use Algorithm 3.1 as a predictor and Algorithm 3.2 as a corrector. Consequently, we have the following algorithm.

Algorithm 3.3. *For a given $u_0 \in K$, find the approximate solution u_{n+1} using the iterative scheme*

$$\begin{aligned} g(y_n) &= P_K[g(u_n) - \rho Tu_n + \rho A(u_n)], \quad n = 0, 1, 2, \dots \\ g(u_{n+1}) &= P_K[g(u_n) - \rho Ty_n + \rho A(y_n)], \quad n = 0, 1, 2, \dots \end{aligned}$$

Algorithm 3.3 is a two-step forward–backward iterative method, and is also known as an extragradient method. The extragradient methods for solving the classical variational inequalities (2.5) were introduced and considered in [7].

Remark 3.1. The predictor-corrector method represented by Algorithm 3.3 and the implicit method represented by Algorithm 3.2 are equivalent. This equivalent formulation is used to prove the convergence of the predictor-corrector method represented by Algorithm 3.3.

4. Convergence

In this section, we consider the convergence analysis of Algorithm 3.2, the following well-known definitions and results are needed for the proof of the convergence.

Definition 4.1. An operator $T : H \rightarrow H$ with respect to an arbitrary operator g is called:

(i) g -monotone if and only if

$$\langle Tu - Tv, g(u) - g(v) \rangle + \alpha \|g(v) - g(u)\|^2 \geq 0, \forall u, v \in H.$$

(ii) g -pseudomonotone if

$$\langle Tu, g(u) - g(v) \rangle + \alpha \|g(v) - g(u)\|^2 \geq 0,$$

implies

$$\langle Tv, g(u) - g(v) \rangle + \alpha \|g(v) - g(u)\|^2 \geq 0, \forall u, v \in H.$$

It is well-known that g -monotonicity implies g -pseudomonotonicity, but the converse is not true.

Definition 4.2. An operator g is called r -strongly monotone if and only if for all $u, v \in H$, there exists a constant $r > 0$, such that

$$\langle g(u) - g(v), u - v \rangle \geq r \|u - v\|^2.$$

Lemma 4.1. ([16]) Let (a_n) and (β_n) be sequences of nonnegative numbers satisfying:

$$a_{n+1} \leq (1 + \beta_n)a_n$$

if $\sum_n \beta_n < +\infty$ then (a_n) is convergent.

Now we need the following lemma for the proof of the convergence.

Lemma 4.2. Let $u \in H : g(u) \in K$ be a solution of (2.1) and let u_{n+1} be the approximate solution obtained from Algorithm 3.2. If the operator $T - A$ is g -pseudomonotone, then

$$(4.1) \quad (1 - 4\alpha\rho_n) \|g(u) - g(u_{n+1})\|^2 \leq \|g(u) - g(u_n)\|^2 - \|g(u_{n+1}) - g(u_n)\|^2.$$

where (ρ_n) is a sequence of nonnegative numbers.

Proof. Since $u \in H : g(u) \in K$ be a solution of (2.1). Then, using the g -pseudo-monotonicity of $T - A$, one has

$$(4.2) \quad \langle \rho_n T v - \rho_n A(v), g(v) - g(u) \rangle + \rho \alpha \|g(v) - g(u)\|^2 \geq 0, \forall v \in K$$

Taking $v = u_{n+1}$ in (4.2), we have

$$(4.3) \quad \langle \rho_n T u_{n+1} - \rho_n A(u_{n+1}), g(u_{n+1}) - g(u) \rangle + \rho \alpha \|g(u_{n+1}) - g(u)\|^2 \geq 0.$$

Using Lemma 3.1, equation (3.2) can be written as

$$(4.4) \quad \begin{aligned} & \langle \rho_n T u_{n+1} - \rho_n A(u_{n+1}) + g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1}) \rangle \\ & + \alpha \rho_n \|g(v) - g(u_{n+1})\|^2 \geq 0, \quad \forall v \in H : g(v) \in K \end{aligned}$$

Taking $v = u$ in (4.4), one has

$$(4.5) \quad \begin{aligned} & \langle \rho_n T u_{n+1} - \rho_n A(u_{n+1}) + g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1}) \rangle \\ & + \alpha \rho_n \|g(u) - g(u_{n+1})\|^2 \geq 0. \end{aligned}$$

From (4.3) and (4.5), it follows that

$$\langle g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1}) \rangle \geq -2\alpha \rho_n \|g(u) - g(u_{n+1})\|^2.$$

Since

$$\begin{aligned} 2 \langle g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1}) \rangle &= \|g(u) - g(u_n)\|^2 \\ &\quad - \|g(u) - g(u_{n+1})\|^2 - \|g(u_{n+1}) - g(u_n)\|^2 \end{aligned}$$

one obtains

$$(1 - 4\alpha \rho_n) \|g(u) - g(u_{n+1})\|^2 \leq \|g(u) - g(u_n)\|^2 - \|g(u_{n+1}) - g(u_n)\|^2.$$

Hence the claim. ■

Theorem 4.1. *Let $u \in H : g(u) \in K$ be a solution of (2.1) and let u_{n+1} be the approximate solution obtained from Algorithm 3.2. Let the operators T, A be continuous and $T - A$ be g -pseudomonotone. If g is r -strongly monotone, H is a finite dimensional space and $\sum \rho_n < +\infty$, then the sequence $\{u_n\}$ obtained from Algorithm 3.2 converges to a solution u of (2.1).*

Proof. Using Lemma 4.2, then inequality (4.1) holds and we have

$$\|g(u_{n+1}) - g(u)\|^2 \leq \frac{1}{1 - 4\alpha \rho_n} \|g(u) - g(u_n)\|^2 - \frac{1}{1 - 4\alpha \rho_n} \|g(u_n) - g(u_{n+1})\|^2$$

which can be written as

$$(4.6) \quad \begin{aligned} \|g(u_{n+1}) - g(u)\|^2 &\leq \left(1 + \frac{4\alpha \rho_n}{1 - 4\alpha \rho_n}\right) \|g(u) - g(u_n)\|^2 \\ &\quad - \frac{1}{1 - 4\alpha \rho_n} \|g(u_n) - g(u_{n+1})\|^2 \end{aligned}$$

Applying Lemma 4.1 to (4.6), we have that the sequence $(\|g(u_n) - g(u)\|)_{n \in N}$ is convergent, which implies that $(g(u_n))_{n \in N}$ is bounded and, by passing to the limit in (4.1), where $\rho_n \rightarrow 0$ since $\sum \rho_n < +\infty$, we obtain

$$(4.7) \quad \lim_{n \rightarrow +\infty} \|g(u_n) - g(u_{n+1})\| = 0$$

Since g is r -strongly monotone we have

$$\langle g(u_n) - g(u), u_n - u \rangle \geq r \|u_n - u\|^2,$$

i.e.,

$$\begin{aligned} \|u_n - u\|^2 &\leq r^{-1} \langle g(u_n) - g(u), u_n - u \rangle \\ &\leq r^{-1} \|g(u_n) - g(u)\| \|u_n - u\|, \end{aligned}$$

that is,

$$\|u_n - u\| \leq r^{-1} \|g(u_n) - g(u)\|,$$

this implies that (u_n) is bounded and thus admits a weak-cluster point $\hat{u}$. This is equivalent to the existence of a subsequence of (u_n) converges to $\hat{u}$.

Passing to the limit on a subsequence in (4.5) and using (4.7) and the continuity of T and A , we have

$$\langle T\hat{u} - A(\hat{u}), g(v) - g(\hat{u}) \rangle + \rho\alpha \|g(v) - g(\hat{u})\|^2 \geq 0, \forall v \in H : g(v) \in K.$$

Thus, $\hat{u} \in H : g(\hat{u}) \in K$ solves the strongly nonlinear general non convex variational inequality (2.1) and we have

$$\|g(u_{n+1}) - g(u)\|^2 \leq \left(1 + \frac{4\alpha\rho_n}{1 - 4\alpha\rho_n}\right) \|g(u) - g(u_n)\|^2.$$

Now, since $(\|g(u_n) - g(\hat{u})\|^2)$ converges, set $l(\hat{u}) = \lim_{n \rightarrow +\infty} \|g(u_n) - g(\hat{u})\|^2$ and let $\tilde{u}$ be another cluster point. We have

$$\begin{aligned} \|g(u_n) - g(\hat{u})\|^2 &= \|g(u_n) - g(\tilde{u}) + g(\tilde{u}) - g(\hat{u})\|^2 \\ &= \|g(u_n) - g(\tilde{u})\|^2 + 2 \langle g(u_n) - g(\tilde{u}), g(\tilde{u}) - g(\hat{u}) \rangle \\ &\quad + \|g(\tilde{u}) - g(\hat{u})\|^2 \end{aligned}$$

by passing to the limit of a subsequence in the last inequality, we obtain

$$(4.8) \quad l(\hat{u}) = l(\tilde{u}) + \|g(\tilde{u}) - g(\hat{u})\|^2$$

because $\lim_{n \rightarrow +\infty} \langle g(u_n) - g(\tilde{u}), g(\tilde{u}) - g(\hat{u}) \rangle = 0$.

Reversing the role of $\hat{u}$ and $\tilde{u}$, we have

$$(4.9) \quad l(\tilde{u}) = l(\hat{u}) + \|g(\hat{u}) - g(\tilde{u})\|^2$$

Adding (4.8) and (4.9), we obtain

$$\|g(\hat{u}) - g(\tilde{u})\|^2 = 0 \implies g(\hat{u}) = g(\tilde{u}).$$

The r -strong monotonicity of g gives

$$\|\hat{u} - \tilde{u}\| \leq r^{-1} \|g(\hat{u}) - g(\tilde{u})\| = 0 \implies \hat{u} = \tilde{u}.$$

and thus the whole sequence converges to a solution of (SNAGGING)(2.1), the required result. $\blacksquare$

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FIXED POINT THEOREMS ON INTUITIONISTIC FUZZY QUASI-METRIC SPACES WITH APPLICATION TO THE DOMAIN OF WORDS

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Abstract. In this paper, we prove intuitionistic fuzzy quasi-metric version of the Banach contraction principle which extend the famous Grabiec fixed point theorem. By using this result we show the existence of fixed point for contraction mappings on the domain of words and apply this approach to deduce the existence of solution for some recurrence equations associated to the analysis of Quicksort algorithms and divide and Conquer algorithms, respectively.

Key words and phrases: Intuitionistic fuzzy quasi-metric space, Common fixed point, G-complete intuitionistic fuzzy quasi-metric space, B-contraction, Domain of words.

Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

Fuzzy set theory was first introduced by Zadeh [23] in 1965 to describe the situations where data are uncertain. Thenafter the concept of fuzzy sets was generalized as intuitionistic fuzzy set by Atanassov [2, 3] in 1984, has a wide range of applications in various fields.

With the help of continuous t-norm the concept of fuzzy metric space was modified by Kramosil and Michalek [13] and George and Veeramani [7]. The concept of fuzzy quasi-metric space was introduced by Gregori and Romaguera [10] by generalizing the concept of fuzzy metric space given by Kramosil and Michalek [13].

In [16], Park generalized the notion of fuzzy metric space given by George and Veeramani [7] and introduced the notion of intuitionistic fuzzy metric space. Alaca et al. [1] defined the notion of intuitionistic fuzzy metric space as a generalization of fuzzy metric space by Kramosil and Michalek [13]. On the other hand Alaca, Turkoglu and Yildiz [1] proved intuitionistic fuzzy versions of the celebrated Banach fixed point theorem and Edelstein fixed point theorem by using the notion of intuitionistic fuzzy metric space.

The concept of intuitionistic fuzzy quasi-metric space was introduced by Tirado [21] by generalizing the notion of intuitionistic fuzzy metric space given by Alaca, Turkoglu and Yildiz [1] to the quasi-metric setting and gave intuitionistic fuzzy quasi-metric version of the Banach contraction principle. Our basic references are [5], [7], [8], [9], [10], [17], [21].

In [8], Grabiec proved fuzzy versions of celebrated Banach fixed point theorem and Edelstein fixed point theorem. Romaguera, Sapena and Tirado [17] proved the Banach fixed point theorem in fuzzy quasi-metric spaces and applied the result to the domain of words.

In this paper, we prove Banach fixed point theorem in intuitionistic fuzzy quasi-metric space. The existence of a solution for a recurrence equation which appears in the average case analysis of Quicksort algorithms is obtained as an application. We generalize the results of Romaguera, Sapena and Tirado [17] and also generalize several known results.

2. Preliminaries

Definition 2.1. ([19]) A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-norm if it satisfies the following conditions:

- (1) $*$ is commutative and associative.
- (2) $*$ is continuous.
- (3) $a * 1 = a$ for all $a \in [0, 1]$.
- (4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ with $a, b, c, d \in [0, 1]$.

Definition 2.2. ([19]) A binary operation $\diamond$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-conorm if it satisfies the following conditions:

- (1) $\diamond$ is commutative and associative.
- (2) $\diamond$ is continuous.
- (3) $a \diamond 0 = a$ for all $a \in [0, 1]$.
- (4) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ with $a, b, c, d \in [0, 1]$.

Remark 2.1. The concept of triangular norms (t-norm) and triangular conorms (t-conorm) are known as the axiomatic skeletons that we use for characterizing fuzzy intersections and unions respectively. These concepts were originally introduced by Menger [15] in his study of statistical metric spaces. Several examples for these concepts were purposed by many authors [5, 11, 12, 22].

Definition 2.3. ([21]) A 5-tuple $(X, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy quasi-metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and M and N are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions:

- (1) $M(x, y, t) + N(x, y, t) \leq 1 \ \forall x, y \in X$ and $t > 0$.
- (2) $M(x, y, 0) = 0 \ \forall x, y \in X$.
- (3) $M(x, y, t) = M(y, x, t) = 1$ iff $x = y \ \forall t > 0$.
- (4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s) \ \forall x, y, z \in X$ and $s, t > 0$.
- (5) for all $x, y \in X$, $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is left continuous.
- (6) $N(x, y, 0) = 1 \ \forall x, y \in X$.
- (7) $N(x, y, t) = N(y, x, t) = 0$ iff $x = y \ \forall t > 0$.
- (8) $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s) \ \forall x, y, z \in X$ and $s, t > 0$.
- (9) for all $x, y \in X$, $N(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is right continuous.

In this case, we say that $(M, N, *, \diamond)$ is an intuitionistic fuzzy quasi-metric (an ifqm) on X . If in addition M and N satisfy $M(x, y, t) = M(y, x, t)$ and $N(x, y, t) = N(y, x, t)$ for all $x, y \in X$ and $t > 0$ then $(M, N, *, \diamond)$ is called intuitionistic fuzzy metric on X and $(X, M, N, *, \diamond)$ is called an intuitionistic fuzzy metric space.

Example 2.1. Let (X, d) be a quasi-metric space. Define t-norm $a * b = \min\{a, b\}$ and t-conorm $a \diamond b = \max\{a, b\}$ and for all $x, y \in X$ and $t > 0$,

$$M_d(x, y, t) = \frac{t}{t + d(x, y)} \quad \text{and} \quad N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}.$$

Then $(X, M, N, *, \diamond)$ is an intuitionistic fuzzy quasi-metric space. We call this intuitionistic fuzzy quasi-metric (M, N) induced by the metric d the standard intuitionistic fuzzy-quasi metric. Furthermore it is easy to check that $(M_d)^{-1} = M_{d^{-1}}$, $(M_d)^i = M_{d^i}$, $(N_d)^{-1} = N_{d^{-1}}$, $(N_d)^i = N_{d^i}$. The topology $\mathfrak{S}_d$ generated by d coincides with the topology $\mathfrak{S}_{MN_d}$ generated by the induced intuitionistic fuzzy quasi-metric $(M, N, *, \diamond)$.

Remark 2.2. It is clear that if $(X, M, N, *, \diamond)$ is an ifqm-space then $(X, M, *)$ is a fuzzy quasi-metric space. Conversely if $(X, M, *)$ is a fuzzy quasi-metric space on X , then $(X, M, 1 - M, *, \diamond)$ is an ifqm-space where $a \diamond b = 1 - [(1 - a) * (1 - b)]$ for all $a, b \in [0, 1]$. If $(M, N, *, \diamond)$ is an ifqm on X , then $(M^{-1}, N^{-1}, *, \diamond)$ is also an ifqm on X where M^{-1} and N^{-1} are the fuzzy sets in $X \times X \times (0, \infty)$ defined by

$M^{-1}(x, y, t) = M(y, x, t)$ and $N^{-1}(x, y, t) = N(y, x, t)$. Moreover if we denote M^i and N^s , the fuzzy sets on $X^2 \times [0, \infty)$ given by $M^i(x, y, t) = \min\{M(x, y, t), M^{-1}(x, y, t)\}$ and $N^s(x, y, t) = \max\{N(x, y, t), N^{-1}(x, y, t)\}$. Then $(M^i, N^s, *, \diamond)$ is an intuitionistic fuzzy metric on X . In order to construct a suitable topology on an ifqm-space $(X, M, N, *, \diamond)$ it seems natural to consider balls $B(x, r, t)$ defined similarly to Park [16] and Alaca, Turkoglu and Yildiz [1] by $B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r, N(x, y, t) < r \text{ for all } x \in X\}$ $r \in (0, 1)$ and $t > 0$. Then one can prove as in Park [16] that the family of sets of the form $\{B(x, r, t) : x \in X, 0 < r < 1, t > 0\}$ is a base for the topology $\mathfrak{S}_{M, N}$ on X .

Definition 2.4. ([16]) Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. A sequence $\{x_n\}_n$ in X is called a Cauchy if for each $\varepsilon \in (0, 1)$ and each $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \varepsilon$, $N(x_n, x_m, t) < \varepsilon$ whenever $n, m \geq n_0$. We say that $(X, M, N, *, \diamond)$ is complete if every Cauchy sequence is convergent.

Definition 2.5. ([16]) A sequence $\{x_n\}_n$ in an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be converges to a point $x \in X$ if and only if

$$\lim_{n \rightarrow \infty} M(x, x_n, t) = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} N(x, x_n, t) = 0 \text{ for all } t > 0.$$

Definition 2.6. ([1]) Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. A sequence $\{x_n\}_n$ in X is called G-Cauchy if for each $p \in N$ and each $t > 0$, $\lim_{n \rightarrow \infty} M(x_n, x_{n+p}, t) = 1$ and $\lim_{n \rightarrow \infty} N(x_n, x_{n+p}, t) = 0$. We say that $(X, M, N, *, \diamond)$ is G-complete if every G-Cauchy sequence is convergent.

3. The Banach fixed point theorem in intuitionistic fuzzy quasi-metric space

Definition 3.1. ([1]) A B-contraction on an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is a self mapping f on X such that there is a constant $k \in (0, 1)$ satisfying

$$M(f(x), f(y), kt) \geq M(x, y, t) \text{ and } N(f(x), f(y), kt) \leq N(x, y, t)$$

for all $x, y \in X, t > 0$.

Theorem A. ([1]) Let $(X, M, N, *, \diamond)$ be a G-complete intuitionistic fuzzy metric space and $f : X \rightarrow X$ be a self-map such that

$$M(f(x), f(y), kt) \geq M(x, y, t) \text{ and } N(f(x), f(y), kt) \leq N(x, y, t),$$

for all $x, y \in X, t > 0$ with $k \in (0, 1)$. Then f has a unique fixed point.

Generalizing in a natural way the notations of G-completeness and B-contraction to intuitionistic fuzzy quasi-metric spaces we introduce the following concepts.

Definition 3.2. ([21]) A sequence $\{x_n\}_n$ in an intuitionistic fuzzy quasi-metric space $(X, M, N, *, \diamond)$ is said to be G-Cauchy if it is a G-Cauchy sequence in the intuitionistic fuzzy metric space $(X, M^i, N^s, *, \diamond)$.

Definition 3.3. ([21]) An intuitionistic fuzzy quasi-metric space $(X, M, N, *, \diamond)$ is called G-bicomplete if the intuitionistic fuzzy metric space $(X, M^i, N^s, *, \diamond)$ is G-complete. In this case we say that $(M, N, *, \diamond)$ is a fuzzy quasi-metric on X .

Definition 3.4. A B-contraction on an intuitionistic fuzzy quasi-metric space $(X, M, N, *, \diamond)$ is a self mapping f on X such that there is a constant $k \in (0, 1)$ satisfying

$$M(f(x), f(y), kt) \geq M(x, y, t) \text{ and } N(f(x), f(y), kt) \leq N(x, y, t),$$

for all $x, y \in X, t > 0$. The number k is called a contraction constant of f .

Theorem 3.1. Let $(X, M, N, *, \diamond)$ be a G-bicomplete intuitionistic fuzzy quasi-metric space such that

$$\lim_{t \rightarrow \infty} M(x, y, t) = 1 \text{ and } \lim_{t \rightarrow \infty} N(x, y, t) = 0 \text{ for all } x, y \in X.$$

Then every B-contraction on X has a unique fixed point.

Proof. Let $f : X \rightarrow X$ be a B-contraction on X with contraction constant $k \in (0, 1)$. Then

$$M(f(x), f(y), kt) \geq M(x, y, t) \text{ and } N(f(x), f(y), kt) \leq N(x, y, t),$$

for all $x, y \in X, t > 0$. It immediately follows that

$$M^i(f(x), f(y), kt) \geq M^i(x, y, t) \text{ and } N^s(f(x), f(y), kt) \leq N^s(x, y, t),$$

for all $x, y \in X, t > 0$. Hence f is a B-contraction on the G-complete fuzzy metric space $(X, M^i, N^s, *, \diamond)$ and by Theorem A, f has a unique fixed point.

4. G-bicompleteness in non-Archimedean intuitionistic fuzzy quasi-metric space

Definition 4.1. ([21]) An intuitionistic fuzzy quasi-metric space $(X, M, N, *, \diamond)$ is called a non-Archimedean intuitionistic fuzzy quasi-metric space if $(M, N, *, \diamond)$ is a non-Archimedean intuitionistic fuzzy quasi-metric on X , that is,

$$\begin{aligned} M(x, y, t) &\geq \min\{M(x, z, t), M(z, y, t)\} \\ \text{and } N(x, y, t) &\leq \max\{N(x, z, t), N(z, y, t)\}, \end{aligned}$$

for all $x, y, z \in X$ and $t > 0$.

Lemma 4.1. *Each G-Cauchy sequence in a non-Archimedean intuitionistic fuzzy quasi-metric space is a Cauchy sequence.*

Proof. Let (x_n) be a G-Cauchy sequence in the non-Archimedean intuitionistic fuzzy quasi-metric space $(X, M, N, *, \Diamond)$, then for each $t > 0$, we have

$$\lim_{n \rightarrow \infty} M^i(x_n, x_{n+1}, t) = 1 \text{ and } \lim_{n \rightarrow \infty} N^s(x_n, x_{n+1}, t) = 0,$$

which implies that, for each $\varepsilon \in (0, 1)$, there is $n_0 \in N$ such that

$$M^i(x_n, x_{n+1}, t) > 1 - \varepsilon \text{ and } N^s(x_n, x_{n+1}, t) < \varepsilon, \text{ for each } n \geq n_0,$$

Now let $m > n \geq n_0$. Then $m = n + j$, for some $j \in N$. So

$$\begin{aligned} M^i(x_n, x_m, t) &\geq \min\{M^i(x_n, x_{n+1}, t), M^i(x_{n+1}, x_{n+2}, t), \\ &\quad \dots, M^i(x_{n+j-1}, x_{n+j}, t)\} \\ &> 1 - \varepsilon \end{aligned}$$

and

$$\begin{aligned} N^s(x_n, x_m, t) &\leq \max\{N^s(x_n, x_{n+1}, t), N^s(x_{n+1}, x_{n+2}, t), \\ &\quad \dots, N^s(x_{n+j-1}, x_{n+j}, t)\} \\ &< \varepsilon. \end{aligned}$$

We conclude that (x_n) is a Cauchy sequence in $(X, M, N, *, \Diamond)$.

Theorem 4.1. *Each bicomplete non-Archimedean intuitionistic fuzzy quasi-metric space is G-bicomplete.*

Proof. Let (x_n) be a G-Cauchy sequence in the bicomplete non-Archimedean intuitionistic fuzzy quasi-metric space $(X, M, N, *, \Diamond)$. By Lemma 4.1, (x_n) is a Cauchy sequence in $(X, M, N, *, \Diamond)$. Hence there is $x \in X$ such that $\lim_{n \rightarrow \infty} M^i(x, x_n, t) = 1$ and $\lim_{n \rightarrow \infty} N^s(x, x_n, t) = 0$, for all $t > 0$. We conclude that $(X, M^i, N^s, *, \Diamond)$ is G-complete, that is, $(X, M, N, *, \Diamond)$ is G-bicomplete.

Corollary 4.2. *Each complete non-Archimedean intuitionistic fuzzy metric space is G-complete.*

5. Application to the domain of words

Let Σ be a non-empty alphabet. Let Σ^∞ be the set of all finite and infinite sequences ("words") over Σ , where we adopt the convention that the empty sequence ϕ is an element of Σ^∞ . The symbol $\sqsubseteq$ denote the prefix order on Σ^∞ , that is, $x \sqsubseteq y \iff x$ is a prefix of y . Now, for each $x \in \Sigma^\infty$ denote by $l(x)$ the length of x . Then $l(x) \in [1, \infty)$ whenever $x \neq \phi$ and $l(\phi) = 0$. For each $x, y \in \Sigma^\infty$ let $x \sqcap y$ be the common prefix of x and y . Thus the function $d_{\sqsubseteq}$ defined on $\Sigma^\infty \times \Sigma^\infty$ by

$$d_{\sqsubseteq}(x, y) = \begin{cases} 0, & \text{if } x \sqsubseteq y \\ 2^{-l(x \sqcap y)}, & \text{otherwise} \end{cases}$$

is a quasi-metric on Σ^∞ (We adopt the convention that $2^{-\infty} = 0$). Actually, $d_{\sqsubseteq}$ is a non-Archimedean quasi-metric on Σ^∞ and the non-Archimedean quasi-metric $(d_{\sqsubseteq})^s$ is the Baire metric on Σ^∞ , that is, $(d_{\sqsubseteq})^s(x, x) = 0$ and $(d_{\sqsubseteq})^s(x, y) = 2^{-l(x \sqcap y)}$ for all $x, y \in \Sigma^\infty$ such that $x \neq y$. It is well known that $(d_{\sqsubseteq})^s$ is complete. From this fact it is clear that $d_{\sqsubseteq}$ is bicomplete. The quasi-metric $d_{\sqsubseteq}$, which was introduced by Smyth [20], will be called the Baire quasi-metric. Observe that condition $d_{\sqsubseteq}(x, y) = 0$ can be used to distinguish between the case that x is a prefix of y and the remaining cases.

Example 5.1. Let $d_{\sqsubseteq}$ be a (non-Archimedean) quasi-metric on a set X and let $M_{d_{\sqsubseteq}}$ and $N_{d_{\sqsubseteq}}$ are fuzzy sets in $X \times X \times [0, \infty)$ given by

$$M_{d_{\sqsubseteq}}(x, y, t) = \frac{t}{t + d_{\sqsubseteq}(x, y)} \quad \text{and} \quad N_{d_{\sqsubseteq}}(x, y, t) = \frac{d_{\sqsubseteq}(x, y)}{t + d_{\sqsubseteq}(x, y)},$$

for all $x, y \in X$ and $t > 0$. Then $(M_{d_{\sqsubseteq}}, N_{d_{\sqsubseteq}}, \wedge, \vee)$ is a (non-Archimedean) intuitionistic fuzzy quasi-metric on X , where $\wedge$ denotes the continuous t-norm and $\vee$ denotes the continuous t-conorm given by $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. It is clear that $\mathfrak{S}_{MN_{d_{\sqsubseteq}}} = \mathfrak{S}_{d_{\sqsubseteq}}$ and that $(X, M_{d_{\sqsubseteq}}, N_{d_{\sqsubseteq}}, \wedge, \vee)$ is bicomplete if and only if $(X, d_{\sqsubseteq})$ is bicomplete.

Proposition 5.1. $(\Sigma^\infty, M_{d_{\sqsubseteq}}, N_{d_{\sqsubseteq}}, \wedge, \vee)$ is a G -bicomplete non-Archimedean intuitionistic fuzzy quasi-metric space.

Consequently, Theorem 3.1 can be applied to this useful space.

Proposition 5.2. $(\Sigma^\infty, M_{d_{\sqsubseteq 1}}, N_{d_{\sqsubseteq 0}}, \wedge, \vee)$ is a G -bicomplete non-Archimedean intuitionistic fuzzy quasi-metric space.

The intuitionistic fuzzy non-Archimedean quasi-metric $(M_{d_{\sqsubseteq 1}}, N_{d_{\sqsubseteq 0}}, \wedge, \vee)$ is given by

$$\begin{aligned} M_{d_{\sqsubseteq 1}}(x, y, 0) &= 0 \text{ and } N_{d_{\sqsubseteq 0}}(x, y, 0) = 1 \text{ for all } x, y \in \Sigma^\infty. \\ M_{d_{\sqsubseteq 1}}(x, y, t) &= 1 \text{ and } N_{d_{\sqsubseteq 0}}(x, y, t) = 0 \text{ if } x \text{ is a prefix of } y \text{ and } t > 0, \\ M_{d_{\sqsubseteq 1}}(x, y, t) &= 1 - 2^{-l(x \sqcap y)} \\ &\text{and } N_{d_{\sqsubseteq 0}}(x, y, t) = 2^{-l(x \sqcap y)} \text{ if } x \text{ is not a prefix of } y \text{ and } t \in (0, 1), \\ M_{d_{\sqsubseteq 1}}(x, y, t) &= 1 \text{ and } N_{d_{\sqsubseteq 0}}(x, y, t) = 0 \text{ if } x \text{ is not a prefix of } y \text{ and } t > 1. \end{aligned}$$

Proposition 5.2 allows us to apply any of the Proposition 5.1 and Theorem 3.1 to the complexity analysis of quicksort algorithm, to show, in direct way, the existence and uniqueness of solution for the following recurrence equation:

$$T(1) = 0 \text{ and } T(n) = \frac{2(n-1)}{n} + \frac{n+1}{n}T(n-1), \quad n \geq 2.$$

The average case analysis of Quicksort is discussed in [14] (see also [6]), where the above recurrence equation is obtained

Consider as an alphabet Σ the set of non-negative real numbers, that is, $\Sigma = [0, \infty)$. We associate to T the functional $\Phi : \Sigma^\infty \rightarrow \Sigma^\infty$ given by

$$(\Phi(x))_1 = T(1) \quad \text{and} \quad (\Phi(x))_n = \frac{2(n-1)}{n} + \frac{n+1}{n}x_{n-1}, \text{ for all } n \geq 2.$$

If $x \in \Sigma^\infty$ has length $n < \infty$, we write $x = x_1x_2x_3\dots x_n$, and if x is an infinite word we write $x = x_1x_2x_3\dots$. Next we show that Φ is a B-contraction on the G-bicomplete non-Archimedean intuitionistic fuzzy quasi-metric space $(\Sigma^\infty, M_{d_\sqsubseteq}, N_{d_\sqsubseteq}, \wedge, \vee)$ with contraction constant $\frac{1}{2}$.

To this end, we first note that, by construction, we have $l(\Phi(x)) = l(x) + 1$ for all $x \in \Sigma^\infty$ (in particular $l(\Phi(x)) = \infty$ whenever $l(x) = \infty$).

Furthermore, it is clear that

$$x \sqsubseteq y \iff \Phi(x) \sqsubseteq \Phi(y),$$

and consequently

$$\Phi(x \sqcap y) \sqsubseteq \Phi(x) \sqcap \Phi(y), \text{ for all } x, y \in \Sigma^\infty.$$

Hence

$$l(\Phi(x \sqcap y)) \leq l(\Phi(x) \sqcap \Phi(y)), \text{ for all } x, y \in \Sigma^\infty.$$

From the preceding observations we deduce that for all $x, y \in X$, if x is a prefix of y , then

$$M_{d_\sqsubseteq} \left(\Phi(x), \Phi(y), \frac{t}{2} \right) = M_{d_\sqsubseteq}(x, y, t) = 1$$

and

$$N_{d_\sqsubseteq} \left(\Phi(x), \Phi(y), \frac{t}{2} \right) = N_{d_\sqsubseteq}(x, y, t) = 0.$$

and if x is not a prefix of y , then for all $t > 0$,

$$\begin{aligned} M_{d_\sqsubseteq} \left(\Phi(x), \Phi(y), \frac{t}{2} \right) &= \frac{\frac{t}{2}}{\frac{t}{2} + 2^{-l(\Phi(x) \sqcap \Phi(y))}} \\ &\geq \frac{\frac{t}{2}}{\frac{t}{2} + 2^{-l(\Phi(x \sqcap y))}} \\ &\geq \frac{\frac{t}{2}}{\frac{t}{2} + 2^{-(l(x \sqcap y)+1)}} \\ &\geq \frac{t}{t + 2^{-l(x \sqcap y)}} \\ &\geq M_{d_\sqsubseteq}(x, y, t) \end{aligned}$$

and

$$\begin{aligned}
 N_{d_{\sqsubseteq}} \left(\Phi(x), \Phi(y), \frac{t}{2} \right) &= \frac{2^{-l(\Phi(x) \sqcap \Phi(y))}}{\frac{t}{2} + 2^{-l(\Phi(x) \sqcap \Phi(y))}} \\
 &\leq \frac{2^{-l(\Phi(x \sqcap y))}}{\frac{t}{2} + 2^{-l(\Phi(x \sqcap y))}} \\
 &\leq \frac{2^{-(l(x \sqcap y)+1)}}{\frac{t}{2} + 2^{-(l(x \sqcap y)+1)}} \\
 &\leq \frac{2^{-l(x \sqcap y)}}{t + 2^{-l(x \sqcap y)}} \\
 &\leq N_{d_{\sqsubseteq}}(x, y, t).
 \end{aligned}$$

Therefore, Φ is a B-contraction on $(\Sigma^\infty, M_{d_{\sqsubseteq}}, N_{d_{\sqsubseteq}}, \wedge, \vee)$ with contraction constant $\frac{1}{2}$. So, by Theorem 3.1, Φ has a unique fixed point $z = z_1 z_2 z_3 \dots$, which is obviously the unique solution to the recurrence equation T , that is, $z_1 = 0$ and $z_n = \frac{2(n-1)}{n} + \frac{n+1}{n} z_{n-1}$ for all $n \geq 2$.

Conclusion

We conclude the paper by applying our results to the complexity analysis of Divide and Conquer algorithm. Recall [4], [18] that Divide and Conquer algorithms solve a problem by recursively splitting it into subproblems each of which is solved separately by the same algorithm, after which the results are combined into a solution of the original problem. Thus, the complexity of a Divide and Conquer algorithm typically is the solution to the recurrence equation given by

$$T(1) = c \text{ and } T(n) = aT\left(\frac{n}{b}\right) + h(n).$$

where $a, b, c \in N$ with $a, b \geq 2$, n range over the set $\{b^p : p = 0, 1, 2, \dots\}$, and $h(n) \geq 0$ for all $n \in N$. As in the case of Quicksort algorithm, take $\Sigma = [0, \infty)$ and put $\Sigma^N = \{x \in \Sigma^\infty : l(x) = \infty\}$. Clearly Σ^N is a closed subset of $(\Sigma^\infty, (M_{d_{\sqsubseteq}})^i, (N_{d_{\sqsubseteq}})^s, \wedge, \vee)$, $(\Sigma^N, M_{d_{\sqsubseteq}}, N_{d_{\sqsubseteq}}, \wedge, \vee)$ is a non-Archimedean intuitionistic G-bicomplete fuzzy quasi-metric space by Proposition 5.1.

Now, we associate to T the functional $\Phi : \Sigma^N \rightarrow \Sigma^N$ given by

$$\begin{aligned}
 (\Phi(x))_1 &= T(1) \text{ and } (\Phi(x))_n = \frac{ax_n}{b} + h(n) \text{ if } n \in \{b^p : p = 1, 2, \dots\}, \\
 \text{and } (\Phi(x))_n &= 0 \text{ otherwise for all } x \in \Sigma^N.
 \end{aligned}$$

For our purposes here it suffices to observe that for each $x, y \in \Sigma^N$, the following inequality holds $l(\Phi(x) \sqcap \Phi(y)) \geq 1 + l(x \sqcap y)$. In fact, If $l(x \sqcap y) = 0$, then $l(\Phi(x) \sqcap \Phi(y)) \geq 1$ and if $b^p > l(x \sqcap y) \geq b^{p-1}$, $p \geq 1$, then $b^{p+1} > l(\Phi(x) \sqcap \Phi(y)) \geq b^p$.

Hence, for each $x, y \in \Sigma^N$ and $t > 0$, we obtain

$$\begin{aligned}
 M_{d_{\sqsubseteq}} \left(\Phi(x), \Phi(y), \frac{t}{2} \right) &= \frac{\frac{t}{2}}{\frac{t}{2} + 2^{-l(\Phi(x) \sqcap \Phi(y))}} \\
 &\geq \frac{\frac{t}{2}}{\frac{t}{2} + 2^{-l(\Phi(x \sqcap y))}} \\
 &\geq \frac{\frac{t}{2}}{\frac{t}{2} + 2^{-(l(x \sqcap y) + 1)}} \\
 &\geq \frac{t}{t + 2^{-l(x \sqcap y)}} \\
 &\geq M_{d_{\sqsubseteq}}(x, y, t)
 \end{aligned}$$

and

$$\begin{aligned}
 N_{d_{\sqsubseteq}} \left(\Phi(x), \Phi(y), \frac{t}{2} \right) &= \frac{2^{-l(\Phi(x) \sqcap \Phi(y))}}{\frac{t}{2} + 2^{-l(\Phi(x) \sqcap \Phi(y))}} \\
 &\leq \frac{2^{-l(\Phi(x \sqcap y))}}{\frac{t}{2} + 2^{-l(\Phi(x \sqcap y))}} \\
 &\leq \frac{2^{-(l(x \sqcap y) + 1)}}{\frac{t}{2} + 2^{-(l(x \sqcap y) + 1)}} \\
 &\leq \frac{2^{-l(x \sqcap y)}}{t + 2^{-l(x \sqcap y)}} \\
 &\leq N_{d_{\sqsubseteq}}(x, y, t).
 \end{aligned}$$

Therefore Φ is a B-contraction on $(\Sigma^N, M_{d_{\sqsubseteq}}, N_{d_{\sqsubseteq}}, \wedge, \vee)$ with contraction constant $\frac{1}{2}$. So, by Theorem 3.1, Φ has a unique fixed point $z = z_1 z_2 z_3 \dots$. Consequently, the function F defined on $\{b^p : p = 0, 1, 2, \dots\}$ by $F(b^p) = z_{b^p}$ for all $p \geq 0$, is the unique solution to the recurrence equation of the given Divide and Conquer algorithm.

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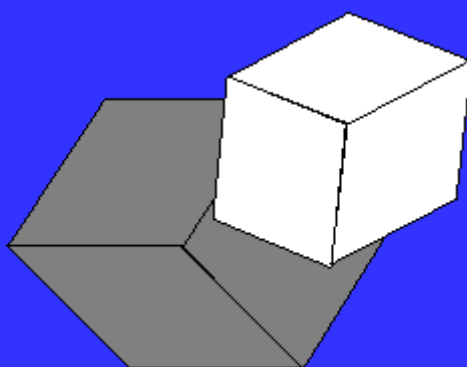
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