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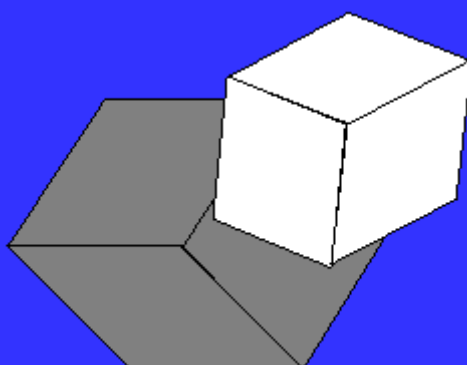
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Department of Mathematics
University of Tehran
Tehran, Iran
reza_ameri@yahoo.com

Luisa Arlotti
Department of Civil Engineering and Architecture
Via delle Scienze 206 - 33100 Udine, Italy
luisa.arlotti@dic.uniud.it

Krassimir Atanassov
Centre of Biomedical Engineering, Bulgarian Academy of Science
BL 105 Acad. G. Bontchev Str.
1113 Sofia, Bulgaria
krat@argo.bas.bg

Malvina Baica
University of Wisconsin-Whitewater
Dept. of Mathematical and Computer Sciences
Whitewater, W.I. 53190, U.S.A.
baicam@uww.edu

Federico Bartolozzi
Dipartimento di Matematica e Applicazioni
via Archirafi 34 - 90123 Palermo, Italy
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Rajabali Borzooei
Department of Mathematics
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Dipartimento di Matematica e Informatica
Via delle Scienze 206 - 33100 Udine, Italy
cecchini@dimi.uniud.it

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School of Mathematics and Statistics,
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gychen1963@sina.com

Domenico (Nico) Chillemi
Executive IT Specialist, IBM Software Group
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nicochillemi@it.ibm.com

Stephen Comer
Department of Mathematics and Computer Science
The Citadel, Charleston S. C. 29409, USA
comers@citadel.edu

Irina Cristea
Department of Civil Engineering and Architecture
Via delle Scienze 206 - 33100 Udine, Italy
irinacri@yahoo.co.uk

Mohammad Reza Darafsheh
School of Mathematics, College of Science
University of Tehran
Tehran - Iran
darafsheh@ut.ac.ir

Bal Kishan Dass
Department of Mathematics
University of Delhi, Delhi - 110007, India
dassbk@rediffmail.com

Bijan Davvaz
Department of Mathematics,
Yazd University, Yazd, Iran
bdavvaz@yahoo.com

Mario De Salvo
Dipartimento di Matematica e Informatica
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98166 Messina
desalvo@unime.it

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Via delle Scienze 206 - 33100 Udine, Italy
detoni@uniud.it

Franco Eugeni
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Università di Teramo, Italy
eugenif@tin.it

Giovanni Falcone
Dipartimento di Metodi e Modelli Matematici
viale delle Scienze Ed. 8
90128 Palermo, Italy
gfalcone@unipa.it

Yuming Feng
College of Math. and Comp. Science, Chongqing Three-Gorges University,
Wanzhou, Chongqing, 404000, P.R.China
yumingfeng25928@163.com

Antonino Giambruno
Dipartimento di Matematica e Applicazioni
via Archirafi 34 - 90123 Palermo, Italy
giambruno@math.unipa.it

Furio Honsell
Dipartimento di Matematica e Informatica
Via delle Scienze 206 - 33100 Udine, Italy
honsell@dimi.uniud.it

James Jantosciak
Department of Mathematics
Brooklyn College (CUNY)
Brooklyn, New York 11210, USA
jsjbc@cunyvm.cuny.edu

Tomas Kepka
MFF-UK
Sokolovská 83
18600 Praha 8, Czech Republic
kepka@karlin.mff.cuni.cz

David Kinderlehrer
Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, PA15213-3890, USA
davidk@andrew.cmu.edu

Andrzej Lasota
Silesian University
Institute of Mathematics
Bankowa 14
40-007 Katowice, Poland
lasota@ux2.math.us.edu.pl

Violeta Leoreanu
Faculty of Mathematics
Al. I. Cuza University
6600 Iasi, Romania
leoreanu2002@yahoo.com

Mario Marchi
Università Cattolica del Sacro Cuore
via Trieste 17, 25121 Brescia, Italy
geomei@bs.unicatt.it

Donatella Marini
Dipartimento di Matematica
Via Ferrata 1- 27100 Pavia, Italy
marini@imati.cnr.it

Angelo Marzollo
Dipartimento di Matematica e Informatica
Via delle Scienze 206 - 33100 Udine, Italy
marzollo@dimi.uniud.it

Antonio Maturo
University of Chieti-Pescara, Department of Social Sciences,
Via dei Vestini, 31
66013 Chieti, Italy
amaturo@unich.it

M. Reza Moghadam
Faculty of Mathematical Science
Ferdowsi University of Mashhad
P.O.Box 1159 - 91775 Mashhad, Iran
moghadam@math.um.ac.ir

Petr Nemeč
Czech University of Life Sciences, Kamýcka' 129
16521 Praha 6, Czech Republic
nemec@tf.czu.cz

Vasile Oproiu
Faculty of Mathematics
Al. I. Cuza University
6600 Iasi, Romania
voproui@uaic.ro

Livio C. Piccinini
Department of Civil Engineering and Architecture
Via delle Scienze 206 - 33100 Udine, Italy
piccinini@uniud.it

Goffredo Pieroni
Dipartimento di Matematica e Informatica
Via delle Scienze 206 - 33100 Udine, Italy
pieroni@dimi.uniud.it

Flavio Pressacco
Dept. of Economy and Statistics
Via Tomadini 30
33100, Udine, Italy
flavio.pressacco@uniud.it

Vito Roberto
Dipartimento di Matematica e Informatica
Via delle Scienze 206 - 33100 Udine, Italy
roberto@dimi.uniud.it

Ivo Rosenberg
Département de Mathématique et de Statistique
Université de Montréal
C.P. 6128 Succursale Centre-Ville
Montréal, Québec H3C 3J7 - Canada
rosenb@DMS.UMontreal.CA

Gaetano Russo
Department of Civil Engineering and Architecture
Via delle Scienze 206
33100 Udine, Italy
gaetano.russo@uniud.it

Paolo Salmon
Dipartimento di Matematica
Università di Bologna
Piazza di Porta S. Donato 5
40126 Bologna, Italy
salmon@dm.unibo.it

Maria Scafati Tallini
Dipartimento di Matematica
"Guido Castelnuovo"
Università La Sapienza
Piazzale Aldo Moro 2 - 00185 Roma, Italy
maria.scafati@gmail.com

Kar Ping Shum
Faculty of Science
The Chinese University of Hong Kong
Hong Kong, China (SAR)
kpshum@ynu.edu.cn

Alessandro Silva
Dipartimento di Matematica
"Guido Castelnuovo"
Università La Sapienza
Piazzale Aldo Moro 2 - 00185 Roma, Italy
silva@mat.uniroma1.it

Sergio Spagnolo
Scuola Normale Superiore
Piazza dei Cavalieri 7 - 56100 Pisa, Italy
spagnolo@dm.unipi.it

Stefanos Spartalís
Department of Production Engineering and Management,
School of Engineering
Democritus University of Thrace
V.Sofias 12, Prokat, Bdg A1, Office 308
67100 Xanthi, Greece
sspart@pme.duth.gr

Hari M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia
V8W3P4, Canada
hmsri@uwv.univ.ca

Marzio Strassoldo
Department of Statistical Sciences
Via delle Scienze 206 - 33100 Udine, Italy
m.strassoldo@gmail.com

Yves Sureau
27, rue d'Aubiere
63170 Perignat, Les Sarlieve - France
hysuroq@orange.fr

Carlo Tasso
Dipartimento di Matematica e Informatica
Via delle Scienze 206 - 33100 Udine, Italy
tasso@dimi.uniud.it

Ioan Tofan
Faculty of Mathematics
Al. I. Cuza University
6600 Iasi, Romania
tofan@uaic.ro

Aldo Ventre
Seconda Università di Napoli, Fac. Architettura, Dip. Cultura del Progetto
Via San Lorenzo s/n
81031 Aversa (NA), Italy
aldoventre@yahoo.it

Thomas Vougiouklis
Democritus University of Thrace,
School of Education,
681 00 Alexandroupolis, Greece
tvougiou@eled.duth.gr

Hans Weber
Dipartimento di Matematica e Informatica
Via delle Scienze 206 - 33100 Udine, Italy
weber@dimi.uniud.it

Yunqiang Yin
School of Mathematics and Information Sciences,
East China Institute of Technology, Fuzhou, Jiangxi
344000, P.R. China
yunqiangyin@gmail.com

Mohammad Mehdi Zahedi
Department of Mathematics, Faculty of Science
Shahid Bahonar, University of Kerman
Kerman, Iran
zahedi_mm@mail.uk.ac

Fabio Zanolin
Dipartimento di Matematica e Informatica
Via delle Scienze 206 - 33100 Udine, Italy
zanolin@dimi.uniud.it

Paolo Zellini
Dipartimento di Matematica
Università degli Studi
Tor Vergata, via Orazio Raimondo
(loc. La Romanina) - 00173 Roma, Italy
zellini@axp.mat.uniroma2.it

Jianming Zhan
Department of Mathematics, Hubei Institute for Nationalities
Enshi, Hubei Province, 445000, China
zhanjianming@hotmail.com

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127.	Mathematical Journal - Academy of Sciences - Uzbekistan	CSI
128.	Mathematical Journal of Okayama University - Okayama	J
129.	Mathematical Preprint - Dep. of Math., Computer Science, Physics – University of Amsterdam	NL
130.	Mathematical Reports - Kyushu University - Fukuoka	J
131.	Mathematics Applied in Science and Technology – Sangyo University, Kyoto	J
132.	Mathematics Reports Toyama University - Gofuku	J
133.	Mathematics for Applications - Institute of Mathematics of Brno University of Technology, Brno	CZ
134.	MAT - Prepublicacions - Universidad Austral	AR
135.	Mediterranean Journal of Mathematics – Università di Bari	I
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152.	Note di Matematica - Lecce	I
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155.	Octagon Mathematical Magazine - Braşov	RO
156.	Osaka Journal of Mathematics - Osaka	J
157.	Periodica Matematica Hungarica - Budapest	H
158.	Periodico di Matematiche - Roma	I
159.	Pliska - Sofia	BG
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165.	Proceeding of the Indian Academy of Sciences - Bangalore	IND
166.	Proceeding of the School of Science of Tokai University - Tokai University	J
167.	Proceedings - Institut Teknologi Bandung - Bandung	RI
168.	Proceedings of the Academy of Sciences Tasked – Uzbekistan	CSI
169.	Proceedings of the Mathematical and Physical Society of Egypt – University of Cairo	ET
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209.	Rivista di Matematica - Università di Parma	I
210.	Sains Malaysiana - Selangor	MAL

211.	Saitama Mathematical Journal - Saitama University	J
212.	Sankhya - Calcutta	IND
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248.	Yugoslav Journal of Operations Research – Beograd	SRB
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COCATÉGORIE ET NILPOTENCE

Mohammed El Haouari

*Université des Sciences et Technologies de Lille
U.F.R. de Mathématiques Pures et Appliquées
F-59655 Villeneuve d'Ascq Cedex
France
email: haouari@math.univ-lille1.fr*

Abstract. The Lusternik-Schnirelmann category of a topological space X , denoted $catX$, is the least integer n such that X can be covered by $n + 1$ open sets, each of them contractible in X . This is a homotopical invariant. We give dualisation of this invariant in the sense of Eckmann-Hilton and we show that the nilpotent class of the space $[G, X]$ is a lower bound.

Résumé. La catégorie de Lusternik-Schnirelmann d'un espace X , notée $catX$, est le plus petit entier naturel n tel que X soit recouvert par $n + 1$ ouverts contractiles dans X . C'est un invariant topologique. Des définitions équivalentes ont été données dans la littérature, en particulier au sens de Whitehead et de Ganea. De même plusieurs résultats donnent le lien entre la catégorie de Lusternik-Schnirelmann et d'autres invariants topologiques: Invariant de Toomer, nilpotence de la cohomologie. Nous donnons une dualisation de cet invariant au sens d'Eckmann-Hilton et nous le minorons par la classe de nilpotence de l'espace $[G, X]$ des classes d'homotopies d'applications continues de G dans X .

Mots clés: L.S.catégorie, co-H-espace.

Mathematical Subject Classification: 55P30, 55P50.

1. Introduction

Dans cet article, nous nous intéressons à des invariants de type d'homotopie: L.S. catégorie et L.S. cocatégorie des espaces topologiques simplement connexes. Rappelons que la L.S.catégorie de Lusternik-Schnirelmann d'un espace X , notée $catX$, est le plus petit entier naturel n tel que X soit recouvert par $n + 1$ ouverts contractiles dans X . T. Ganea et G.W. Whitehead ont donné d'autres descriptions de cet invariant en termes de sections de fibrations $g_n : G_n(X) \rightarrow X$ appelées fibrations de Ganea et relèvements de la diagonale $\Delta : X \rightarrow X^{n+1}$ via des applications $T^{n+1} \rightarrow X^{n+1}$ où T^n est le $n^{\text{ième}}$ bouquet garni de X .

Une dualisation des deux approches Ganea et Whitehead donnent naissance à deux invariants topologiques: les cocatégories de Ganea et Whitehead. L'égalité entre ces deux invariants n'est pas établie et reste un problème ouvert. Ce qui rend l'étude de ces invariants difficiles.

Notre contribution consiste à établir une liaison entre les cocatégories de Ganea et Whitehead et la classe de nilpotence de l'espace $[G, X]$ des classes d'homotopies d'applications continues de G dans X .

Dans la suite nous donnons les définitions des cocatégories au sens de Ganea et Whitehead, et nous démontrons les deux théorèmes suivants:

Théorème 4 *Soit X un espace 1-connexe et G un co- H -espace. Si la cocatégorie de Whitehead $\text{cocat}_W X \leq n$ alors l'espace $[G, X]$ des classes d'homotopie d'applications continues de G dans X est nilpotent de classe n .*

Théorème 7 *Soit X un espace 1-connexe et G un co- H -espace. Si la cocatégorie de Ganea $\text{cocat}_G X \leq n$ alors l'espace $[G, X]$ des classes d'homotopie d'applications continues de G dans X est nilpotent de classe n .*

1) La Cocatégorie cocat_W

Soit X un espace topologique 1-connexe. D'abord on définit, par récurrence, des espaces $W_n X$, $n \geq 0$ et des applications continues $w_n(X) : \bigvee^{n+1} X \rightarrow W_n X$:

$w_0(X) : X \rightarrow W_0 X = *$ est l'application triviale et $w_n(X) : \bigvee^{n+1} X \rightarrow W_n X$ est le cojoint des applications évidentes $\bigvee^{n+1} X \simeq (\bigvee^n X) \bigvee X \rightarrow \bigvee^n X \bigvee *$ et $\bigvee^{n+1} X \simeq (\bigvee^n X) \bigvee X \rightarrow W_{n-1} X \bigvee X$ (cette construction est fonctorielle).

Rappelons que le cojoint de 2 applications $A \rightarrow B$ et $A \rightarrow C$ est le produit fibré homotopique (homotopy pullback) de $B \rightarrow S$ et $C \rightarrow S$, où S est la somme amalgamée homotopique (homotopy pushout) de $A \rightarrow B$ et $A \rightarrow C$.

Définition 1 $\text{cocat}_W X \leq n$ si et seulement si l'application $\nabla : \bigvee^{n+1} X \rightarrow X$ se factorise, à homotopie près, à travers $w_n(X) : \bigvee^{n+1} X \rightarrow W_n X$, c'est à dire il existe une application continue $f : W_n X \rightarrow X$ telle que le diagramme suivant:

$$\begin{array}{ccc} \bigvee^{n+1} X & \xrightarrow{\nabla_{n+1}} & X \\ \downarrow w_n(X) & \nearrow f & \\ W_n X & & \end{array}$$

soit homotopiquement commutatif.

2) La Cocatégorie cocat_G

Soit X un espace topologique 1-connexe. On définit d'abord par récurrence, pour tout $n \geq 0$, des espaces $G'_n X$ et des applications continues $g_n(X) : X \rightarrow G'_n X$:

$g_0(X) : X \rightarrow G'_0 X = *$ est l'application triviale et $g_n(X) : X \rightarrow G'_n X$ par le cojoint de $g_{n-1}(X) : X \rightarrow G'_{n-1} X$ et $g_0(X) : X \rightarrow *$.

Les cofibrations de Ganea sont: $X \rightarrow G'_n X \rightarrow C_n(X)$ où $C_n(X)$ est la cofibre de $g_n(X) : X \rightarrow G'_n X$.

Notons que $G'_n X$ est la fibre homotopique de $G'_{n-1} X \rightarrow C_{n-1}(X)$

Définition 2 $\text{cocat}_G X \leq n$ si et seulement si l'application $g_n : X \rightarrow G'_n X$ admet une rétraction r_n .

Proposition 3

1) Il existe un diagramme:

$$\begin{array}{ccc} \bigvee^{n+1} X & \xrightarrow{w_n(X)} & W_n X \\ \nabla_{n+1} \downarrow & & \downarrow \\ X & \xrightarrow{g_n(X)} & G'_n(X) \end{array}$$

qui commute à homotopie près.

2) $\text{cocat}_W X \leq \text{cocat}_G X$

Preuve.

1) C'est une dualisation immédiate du Théorème 5.5 dans [3].

2) L'inégalité suit à partir des définitions. ■

Théorème 4 Soit X un espace 1-connexe et G un co- H -groupe. Si $\text{cocat}_W X \leq n$ alors l'espace $[G, X]$ des classes d'homotopie d'applications continues de G dans X est nilpotent de classe n .

a) Définition du commutateur $[\cdot, \cdot]_{\#} : G \rightarrow G \vee G$:

Posons $\mu : G \rightarrow G \vee G$ la loi du co- H -groupe G , $j : G \rightarrow G$ l'inverse et $p_k : G \vee G \rightarrow G$, $k = 1, 2$ les restrictions des projections p_k de $G \times G$ sur G .

Posons $\nabla_G : G \vee G \rightarrow G$ l'application qui envoie $(x, *)$ et $(*, x)$ sur x .

On a $p_1 \circ \mu \simeq p_2 \circ \mu \simeq 1_G$ et les applications $\nabla_G \circ (j \vee 1_G) \circ \mu$ et $\nabla_G \circ (1_G \vee j) \circ \mu$ sont homotopiquement triviales.

On définit alors $[\cdot, \cdot]_{\#}$ par la composée $\nabla_{G \vee G}(\mu \vee T \mu j)\mu$, où $T : G \vee G \rightarrow G \vee G$ est la restriction de l'application de transposition $T : G \times G \rightarrow G \times G$ définie par $T(x, y) = (y, x)$.

Lemme 5 Si k est l'inclusion de $G \vee G$ dans $G \times G$ alors $k \circ [,]_{\#}$ est homotopiquement triviale.

Preuve. D'après la propriété universelle du produit fibré homotopique, il suffit de montrer que $p_k \circ [,]_{\#}$ est triviale pour $k = 1, 2$.

Il est facile de voir que

$$p_k \nabla_{G \vee G}(f \vee g) = \nabla_G(p_k f \vee p_k g), \quad \forall f, g : G \rightarrow G \vee G.$$

Ainsi:

$$p_k \circ [,]_{\#} = p_k \nabla_{G \vee G}(\mu \vee T\mu j)\mu = \nabla_G(p_k \mu \vee p_k T\mu j)\mu = \nabla_G(1_G \vee j)\mu \simeq \tilde{e},$$

où $\tilde{e}$ est l'application triviale. ■

b) Définition des $\Phi_n : G \rightarrow \bigvee^n G$:

On définit $\Phi_n : G \rightarrow \bigvee^n G$ par induction,

$$\Phi_1 = id_G, \quad \Phi_2 = [,]_{\#} \text{ et } \Phi_{n+1} = (1_G \vee \Phi_n) \circ \Phi_2.$$

Proposition 6 Soit $w_n(G) : \bigvee^{n+1} G \rightarrow W_n G$. On a $w_n(G) \circ \Phi_{n+1} \simeq *$.

Preuve. Par récurrence:

Le cas $n = 0$ est trivial.

Le cas $n = 1$ est donné par le Lemme 5.

Supposons $w_{n-1}(G) \circ \Phi_n \simeq *$ par une homotopie H .

On peut supposer que $w_{n-1}(G) : \bigvee^n G \rightarrow W_{n-1}G$ une fibration et de là il existe une application continue $\Psi' : G \times I \rightarrow \bigvee^n G$ telle $\Psi'(\cdot, 0) = \Phi_n$ et $w_{n-1} \circ \Psi'(\cdot, 1) = H(\cdot, 1) = *$

Soit alors $\Psi : (G \vee G) \times I \rightarrow G \vee \bigvee^n G$ par $\Psi((x, *), t) = (x, *)$ et $\Psi((*, x), t) = (*, \Psi'(x, t))$.

On a alors $\Psi_0 = 1 \vee \Phi_n$ et posons $F = \Psi_1$.

On a: $\forall x \in G$: $F(x, *) = (x, *)$ et $F(*, x) = (*, \Psi'(x, 1)) = (*, \Psi'_1(x))$.

En considérant le produit fibré homotopique:

$$\begin{array}{ccc} W_n G & \xrightarrow{\quad} & * \vee \bigvee^n G \\ \downarrow pfh & & \downarrow * \vee w_{n-1}(G) \\ G \vee W_{n-1} G & \xrightarrow{* \vee 1_G} & * \vee W_{n-1} G \end{array}$$

Les applications $(*, \Psi'_1) : G \times G \rightarrow * \bigvee^n G$ et $(1, *) : G \times G \rightarrow G \bigvee W_{n-1}G$ induisent une application $\tilde{\Psi}_n : G \times G \rightarrow W_nG$ telle que le diagramme suivant

$$\begin{array}{ccccc}
 & & * \bigvee^n G & & \\
 & \nearrow^{(*, \Psi'_1)} & & \searrow^{* \bigvee w_{n-1}} & \\
 G \times G & \xrightarrow{\tilde{\Psi}_n} & W_nG & \xrightarrow{h.p.b.} & * \bigvee W_{n-1}G \\
 & \searrow_{(1, *)} & & \nearrow_{* \bigvee 1_G} & \\
 & & G \bigvee W_{n-1}G & &
 \end{array}$$

commute à homotopie près.

De même, les restrictions de $(*, \Psi'_1)$ et $(1, *)$ à $G \bigvee G$ déterminent l'application $w_n \circ F : G \bigvee G \rightarrow W_nG$ telle que le diagramme suivant

$$\begin{array}{ccccc}
 & & * \bigvee^n G & & \\
 & \nearrow^{(*, \Psi'_1)} & & \searrow^{* \bigvee w_{n-1}} & \\
 G \bigvee G & \xrightarrow{w_n \circ F} & W_nG & \xrightarrow{h.p.b.} & * \bigvee W_{n-1}G \\
 & \searrow_{(1, *)} & & \nearrow_{* \bigvee 1_G} & \\
 & & G \bigvee W_{n-1}G & &
 \end{array}$$

commute à homotopie près.

Ainsi par unicité, $w_n \circ F$ se factorise à travers $\tilde{\Psi}_n$ via l'inclusion

$$k : G \bigvee G \rightarrow G \times G.$$

D'où

$$w_n \circ \Phi_{n+1} \simeq w_n \circ (1_G \bigvee \Phi_n) \circ \Phi_2 \simeq w_n \circ F \circ \Phi_2 \simeq \tilde{\Psi}_n \circ k \circ \Phi_2 \simeq *$$

(d'après le Lemme 5). ■

Démonstration du théorème. Soit $\Gamma = [G, X]$.

On pose $Z_1(\Gamma) = \Gamma$ et $Z_{i+1}(\Gamma) = [\Gamma, Z_i(\Gamma)]$ pour $i \geq 1$.

Γ est nilpotent si et seulement s'il existe $c \geq 1$ tel que $Z_{c+1}(\Gamma) = 0$.

On pose, par induction, pour tout $\{f_i\}_i$ dans Γ :

$$[f_1, f_2, \dots, f_{q+1}] = [f_1[f_2[\dots f_{q+1}]]].$$

On a: Γ est nilpotent de classe $\leq n$ si et seulement si tous les commutateurs $[f_1, f_2, \dots, f_{n+1}]$ sont nuls.

Supposons que $\text{cocat}_W X \leq n$ et montrons que ces commutateurs sont nuls: Soit le diagramme commutatif

$$\begin{array}{ccccc}
 G & \xrightarrow{\Phi_{n+1}} & \bigvee^{n+1} G & \xrightarrow{\varphi_n} & \bigvee^{n+1} X & \xrightarrow{\nabla_{n+1}} & X \\
 & & \downarrow w_n G & & \downarrow w_n(X) & \nearrow f & \\
 & & W_n G & \xrightarrow{\tilde{\varphi}_n} & W_n X & &
 \end{array}$$

où $\varphi_n = f_1 \vee f_2 \vee \dots \vee f_{n+1}$, $\tilde{\varphi}_n$ est l'application induite par functorialité et f est la factorisation.

Le commutateur $[f_1[f_2[\dots f_{n+1}]]]$ est représenté par:

$$G \xrightarrow{\Phi_{n+1}} \bigvee^{n+1} G \xrightarrow{\varphi_n} \bigvee^{n+1} X \xrightarrow{\nabla_{n+1}} X$$

Ainsi

$$\nabla_{n+1} \circ \varphi_n \circ \Phi_{n+1} = f w_n(X) \varphi_n \Phi_{n+1} = f \tilde{\varphi}_n w_n(G) \Phi_{n+1} = * \quad \blacksquare$$

Théorème 7 Soit X un espace 1-connexe et G un co- H -groupe. Si $\text{cocat}_G X \leq n$ alors l'espace $[G, X]$ des classes d'homotopie d'applications continues de G dans X est nilpotent de classe n .

Avant de démontrer ce théorème, nous allons donner quelques résultats généraux:

Soit une fibration $F \xrightarrow{i} E \xrightarrow{p} B$. Il existe une opération d'holonomie $h : \Omega B \times F \rightarrow F$ appelée holonomie de la fibration [3], [4]. Elle est définie à homotopie près.

L'application h induit une action de $[G, \Omega B]$ sur $[G, F]$. On la notera “.”.

D'autre part, considérons l'application $q_* : [G, \Omega B] \rightarrow [G, F]$ induite par l'application q dans la suite

$$\Omega B \xrightarrow{q} F \longrightarrow E \longrightarrow B$$

L'application q est appelée l'application connectante de la fibration et est la composée de $\Omega B \rightarrow \Omega B \times F$ avec h .

En notant par $+$ les opérations du groupe dans $[G, \Omega B]$ et $[G, F]$, il est facile de voir que: $\forall a, b$ dans $[G, \Omega B]$ et $\forall x, y$ dans $[G, F]$ on a:

$$(1) \quad (a + b).(x + y) = (a.x) + (b.y).$$

Finalement, si on note par $\tilde{e}$ la classe d'homotopie de l'application constante. En appliquant l'égalité (1) à $b = \tilde{e}$ et $x = \tilde{e}$ et à $a = \tilde{e}$ et $y = \tilde{e}$, on obtient $a.y = q_*(a) + y$ et $a.y = y + q_*(a)$ et ainsi l'image de q_* est dans le centre de $[G, F]$.

Démonstration du théorème. Montrons d'abord que $\text{nil}[G, G'_k(X)] \leq k$.

Par induction sur $k = \text{cocat}_G X$:

Si $k = 0$, $\text{nil}[G, G'_0(X)] \leq 0$ car $G'_0(X) = *$.

Supposons que le résultat est vrai à l'ordre $k - 1$.

Considérons la suite

$$\Omega C_{k-1}(X) \xrightarrow{q} F'_k(X) \longrightarrow G'_{k-1}(X) \longrightarrow C_{k-1}(X)$$

où $F'_k(X)$ est la fibre homotopique de $G'_{k-1}(X) \rightarrow C_{k-1}(X)$.

On déduit la suite exacte de groupes:

$$[G, \Omega C_{k-1}(X)] \xrightarrow{q_*} [G, F'_k(X)] \longrightarrow [G, G'_{k-1}(X)] \longrightarrow [G, C_{k-1}(X)]$$

D'après ce qui précède, l'image de q_* est dans le centre de $[G, F'_k(X)]$.

Comme, par hypothèse,

$$\text{nil} [G, G'_{k-1}(X)] \leq k - 1,$$

il suit que

$$\text{nil} [G, G'_k(X)] = \text{nil} [G, F'_k(X)] \leq k.$$

Maintenant, supposons que $\text{cocat}_G X \leq n$. Il existe alors une rétraction $r_n : G'_n(X) \rightarrow X$ de l'application $g_n : X \rightarrow G'_n(X)$ et par suite l'application $(r_n)_* : [G, G'_n X] \rightarrow [G, X]$ est surjective.

Finalement, $\text{nil}[G, G'_k(X)] \leq n$ achève la démonstration du théorème. ■

Remarque. Le premier théorème est une dualisation de celui de Whitehead dans [11], et le deuxième est une dualisation d'un résultat d'Arkowitz dans [1]. Bien que le Théorème 4 entraîne le Théorème 7, nous avons trouvé qu'il est intéressant de donner deux démonstrations complètement différentes.

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LIE IDEAL AND GENERALIZED JORDAN LEFT DERIVATION ON SEMIPRIME RINGS

R.K. Sharma

B. Prajapati¹

*Department of Mathematics
Indian Institute of Technology
Delhi, Hauz Khas, New Delhi, 110016
India
e-mails: rksharma@maths.iitd.ac.in
Bprajapati82@gmail.com*

Abstract. Let R be a 2-torsion free semiprime ring in which $x^2 = 0$ implies $x = 0$. Let g be a generalized Jordan left (right) derivation associated with Jordan left (right) derivation d on R . Then g is a generalized left (right) derivation on R . It is proved that if $Q_r(S)$ is the Martindale quotient ring of S then there exists $q \in Q_r(S)$ such that $g(x) = qx + d(x)$ for all $x \in R$. (In right derivation case).

Keywords and Phrases: semiprime ring, biadditive mapping, generalized left derivation, left centralizer, martingale quotient ring, extended centroid, central closure.

2000 Mathematics Subject Classification: 16W25, 16N60.

1. In this paper, R will denote a 2-torsion free semiprime ring. A ring R is said to be semiprime if $xRx = 0$ with $x \in R$ implies $x = 0$. An additive subgroup U of R is said to be a Lie ideal of R if $[U, R] \subseteq U$, where $[x, y]$ denotes the Lie product $xy - yx$ of x and y . A mapping $A : R \times R \rightarrow R$ is said to be biadditive if it is additive in both the variables. Let C be the extended centroid of R and $S = RC$ be the central closure of R . The notion of extended centroid and central closure of semiprime ring R are given by Amitsur in [1]. It can also be found in [5]. Let $Q_r(S)$ be the Martindale quotient ring corresponding to S . An additive mapping $t : R \rightarrow R$ is said to be a left centralizer (resp. Jordan left centralizer) if $t(xy) = t(x)y$ (resp. $t(x^2) = t(x)x$) for all $x, y \in R$. Clearly, every left centralizer is a Jordan left centralizer but the converse may not be true in general. In [19], it is proved that if the ring is 2-torsion free semiprime then the converse is also true. An additive mapping $d : R \rightarrow R$ is said to be a derivation (resp. Jordan derivation) if $d(xy) = d(x)y + xd(y)$ (resp. $d(x^2) = d(x)x + xd(x)$) for all $x, y \in R$. Clearly, every derivation is a Jordan derivation but converse need not be true. A counter

¹Corresponding author.

example is a reverse derivation given by Herstein in [12]. An additive mapping $*$: $R \rightarrow R$ is said to be a reverse derivation if $(ab)^* = b^*a + ba^*$. However, if ring is 2-torsion free prime or semiprime then Jordan derivation is also a derivation [12], [8]. An additive mapping $d : R \rightarrow R$ is said to be a left derivation (resp. Jordan left derivation) if $d(xy) = xd(y) + yd(x)$ (resp. $d(x^2) = 2xd(x)$) for all $x, y \in R$. Every left derivation is a Jordan left derivation but the converse is not true. If the ring is 2-torsion free, prime then every Jordan left derivation is a left derivation [2]. An additive mapping $g : R \rightarrow R$ is said to be a generalized derivation (resp. generalized Jordan derivation) associated with the derivation d if $g(xy) = g(x)y + xd(y)$ (resp. $g(x^2) = g(x)x + xd(x)$) for all $x, y \in R$. An additive mapping $g : R \rightarrow R$ is said to be generalized left derivation (resp. generalized Jordan left derivation) associated with the Jordan left derivation $d : R \rightarrow R$ if $g(xy) = xg(y) + yd(x)$ ($g(x^2) = xg(x) + xd(x)$) for all $x, y \in R$. Every generalized left derivation is a generalized Jordan left derivation but converse is not true. An example is given by Ashraf and Ali in [2]. In [2] a question asked: "Is every generalized Jordan left derivation on semiprime ring generalized left derivation?" We show here that if a ring is 2-torsion free semiprime in which $x^2 = 0$ implies $x = 0$ the above question has affirmative answer.

2. To prove our theorems we need some results.

Lemma 1 ([3]). *Let R be a 2-torsion free ring and U be a Lie ideal of R such that $x^2 \in U$ for all $x \in U$. If $d : R \rightarrow R$ is an additive mapping satisfying $d(x^2) = 2xdx$ for all $x \in U$, then*

- (i) $d(xy + yx) = 2xdy + 2ydx$ for all $x, y \in U$.
- (ii) $d(xyx) = x^2dy + 3xydx - yxdx$ for all $x, y \in U$.
- (iii) $d(xyz + zyx) = (xz + zx)dy + 3xydz + 3zydx - yxdz - yzdx$ for all $x, y, z \in U$.
- (iv) $[x, y]xdx = x[x, y]dx$ for all $x, y \in U$.
- (v) $[x, y]A(x, y) = 0$ for all $x, y \in U$.

Here $A(x, y) = d(xy) - yd(x) - xdy$.

From Lemma 1, one can easily see that

$$A(x, y) + A(y, x) = 0 \text{ and } A(x, y) - A(y, x) = d[x, y].$$

Lemma 2. *Let R be a 2-torsion free semiprime ring in which $x^2 = 0$ implies $x = 0$. Let U be a Lie ideal of R such that $x^2 \in U$ for all $x \in U$. Let $A, B : U \times U \rightarrow U$ are biadditive mappings such that $A(x, y)B(x, y) = 0$ for all $x, y \in U$ then $A(x, y)B(u, v) = 0$ for all $x, y, u, v \in U$.*

Proof. Replace x by $x + u$ in $A(x, y)B(x, y) = 0$ to get

$$A(x, y)B(u, y) = -A(u, y)B(x, y).$$

Now, by the above relation, we have

$$\begin{aligned}\{A(x, y)B(u, y)\}^2 &= A(x, y)B(u, y)A(x, y)B(u, y) \\ &= -A(x, y)B(x, y)A(u, y)B(u, y).\end{aligned}$$

Thus, by given hypothesis, we get $\{A(x, y)B(u, y)\}^2 = 0$, for all $x, y, u \in U$. We get $A(x, y)B(u, y) = 0$. Again, replacing y by $y + v$ in $A(x, y)B(u, y) = 0$ and using the same argument, we get the required result. ■

Lemma 3. *Let R be a semiprime ring and $x, y \in R$ such that $[u, v]A(x, y) = 0$ for all $u, v \in R$, then $A(x, y) \in Z(R)$.*

Proof. Let $z, w \in R$ be arbitrary elements

$$\begin{aligned}[z, A(x, y)]w[z, A(x, y)] &= [z, A(x, y)]wzA(x, y) - [z, A(x, y)]wA(x, y)z \\ &= [z, A(x, y)wz]A(x, y) - A(x, y)[z, wz]A(x, y) \\ &\quad - [z, A(x, y)w]A(x, y)z + A(x, y)[z, w]A(x, y)z. \\ \Rightarrow [z, A(x, y)]w[z, A(x, y)] &= 0 \text{ for all } w, z \in R.\end{aligned}$$

Since R is semiprime, we get $[z, A(x, y)] = 0$ for all $z \in R$ and so $A(x, y) \in Z(R)$. ■

Theorem 1. *Let R be a 2-torsion free semiprime ring in which $x^2 = 0$ implies $x = 0$. Let U be a Lie ideal of R such that $x^2 \in U$ for all $x \in U$. If $d : R \rightarrow R$ is an additive mapping satisfying $d(x^2) = 2xdx$ for all $x \in U$, then $d(xy) = xdy + ydx$ for all $x, y \in U$.*

Proof. By Lemma 1 (v), $[x, y]A(x, y) = 0$ for all $x, y \in U$. Since both $[x, y]$ and $A(x, y)$ are biadditive mappings, so by Lemma 2, we get that $[u, v]A(x, y) = 0$ for all $x, y, u, v \in U$. Now for fixed x, y in U , $A(x, y) \in Z(U)$ by Lemma 3. And this is true for all $x, y \in U$.

Now, we have

$$\begin{aligned}2(A(x, y))^2 &= A(x, y)(A(x, y) + A(x, y)) \\ &= A(x, y)(A(x, y) - A(y, x)), \text{ since } A(x, y) = -A(y, x) \\ &= A(x, y)d[x, y], \text{ since } A(x, y) - A(y, x) = d[x, y],\end{aligned}$$

that is

$$2(A(x, y))^2 = A(x, y)d[x, y].$$

Since $[x, y]A(x, y) = 0$ and $A(x, y) \in Z(U)$, we have

$$A(x, y)[x, y] + [x, y]A(x, y) = 0$$

and, by Lemma 1(i), we get

$$2A(x, y)d[x, y] + 2[x, y]dA(x, y) = 0.$$

Since R is 2-torsion free, $A(x, y)d[x, y] = -[x, y]dA(x, y)$. Thus, we get

$$2(A(x, y))^2 + [x, y]dA(x, y) = 0.$$

Multiplying by $A(x, y)$ on left and using the fact that $A(x, y) \in Z(U)$, we get $2(A(x, y))^3 = 0$, since $[x, y]A(x, y) = 0$. Again, using the fact that R is 2-torsion free, we get $(A(x, y))^3 = 0$. Now, since we are assuming $x^2 = 0$ implies $x = 0$, so we have $A(x, y) = 0$, and thus, $d(xy) = xdy + ydx$, for all $x, y \in U$. ■

Corollary 1. *Let R be a 2-torsion free semiprime ring with unit element. Then, every Jordan left derivation is a left derivation.*

Proof. Since R is semiprime ring with unity, the assumption $x^2 = 0$ implies $x = 0$ is clearly satisfied. ■

Corollary 2. *Let R be a 2-torsion free semisimple ring in which $x^2 = 0$ implies $x = 0$ (or R has unity). Then, every Jordan left derivation is a left derivation.*

Proof. Since every semisimple ring is semiprime, the result follow from Theorem 1. ■

Lemma 4 ([19] Proposition 1.4). *Let R be a 2-torsion free semiprime ring and $T : R \rightarrow R$ an additive mapping which satisfies $T(x^2) = T(x)x$ for all $x \in R$. Then T is a left centralizer.*

Lemma 5. *An additive mapping $g : R \rightarrow R$ is a generalized left (right) derivation if and only if g is of the form $g = d + t$, where d is a left (right) derivation and t is a right (left) centralizer.*

Proof. Let g be a generalized left derivation on R i.e. $g(xy) = xg(y) + yd(x)$ where d is a left derivation on R . Suppose $t = g - d$, then

$$\begin{aligned} t(xy) &= (g - d)(xy) = xg(y) + yd(x) - xd(y) - yd(x) \\ &= x(g - d)(y) = xt(y). \end{aligned}$$

This shows that t is a right centralizer. Hence $g = d + t$.

Conversely, suppose $g = d + t$. Then,

$$\begin{aligned} g(xy) &= (d + t)(xy) = xd(y) + yd(x) + xt(y) \\ &= x(d + t)(y) + yd(x) = xg(y) + yd(x). \end{aligned}$$

Thus, g is generalized left derivation. ■

Theorem 2. *Let R be a 2-torsion free semiprime ring in which $x^2 = 0$ implies $x = 0$. Let $g : R \rightarrow R$ be a generalized Jordan left derivation associated with Jordan left derivation $d : R \rightarrow R$ then g is a generalized left derivation on R .*

Proof. Since g is a generalized Jordan left derivation, we have

$$g(x^2) = xg(x) + xd(x), \quad \forall x \in R$$

Since d is given to be a Jordan left derivation, by Theorem 1, it is a left derivation on R . Let us denote $g - d$ by t . Then, we have

$$t(x^2) = (g-d)(x^2) = g(x^2) - d(x^2) = xg(x) + xd(x) - 2xd(x) = x(g-d)(x) = xt(x),$$

which shows that t is a Jordan right centralizer. By Lemma 4, t is a right centralizer. Hence, g is of the form $g = d + t$, where d is a left derivation and t is a right centralizer. By Lemma 5, g is a generalized left derivation on R . This proves the theorem. ■

The third theorem is inspired by T.K. Lee [16].

We prove here that every generalized Jordan right derivation on semiprime ring has the form $g(x) = qx + d(x)$ for all $x \in R$ for some $q \in Q_r(S)$.

Remark 1. The concept of left derivation (left centralizer) and right derivation (right centralizer) are analogues.

Corollary 3. *Let R be a 2-torsion free semiprime ring with unity then every generalized Jordan left derivation is a generalized left derivation.*

Corollary 4. *Let R be a semisimple ring in which $x^2 = 0$ implies $x = 0$ (or R has unity) then every generalized Jordan left derivation is a generalized left derivation.*

Lemma 6 ([15] Lemma 2). *Let $f : R \rightarrow S$ be an additive map satisfying $f(xy) = f(x)y$ for all $x, y \in R$. Then there exists $q \in Q_r(S)$ such that $f(x) = qx$ for all $x \in R$.*

Theorem 3. *Let R be a 2-torsion free semiprime ring in which $x^2 = 0$ implies $x = 0$. Let $g : R \rightarrow R$ be a generalized Jordan right derivation associated with Jordan right derivation $d : R \rightarrow R$. Then there exist $q \in Q_r(S)$, the Martindale quotient ring of S , such that $g(x) = qx + d(x)$ for all $x \in R$.*

Proof. Since g is a generalized Jordan right derivation associated with the Jordan right derivation d on R , by Theorem 2, g is a generalized right derivation on R . Hence g is of the form $g = d + t$, where d is a right derivation and t is left centralizer. That is, $(g - d)(xy) = t(xy) = t(x)y$. Now, by Lemma 6, there exists $q \in Q_r(S)$ such that $(g - d)(x) = t(x) = qx$ for all $x \in R$. Thus $g(x) = qx + d(x)$ for all $x \in R$. Hence the theorem. ■

Remark 2. Corollaries 3 and 4 are also hold true for Theorem 3.

3. Example. Semiprimeness of ring R is essential in Theorem 1. For, let S be a commutative ring. Suppose

$$R = \left\{ X = \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in S \right\}.$$

R is not a semiprime ring, since

$$\begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} Y \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $Y = \begin{pmatrix} 0 & x & y & z \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & -x \\ 0 & 0 & 0 & 0 \end{pmatrix}$, but $\begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Define $d : R \rightarrow R$ as

$$d \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & c \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then, clearly, d is an additive mapping.

Now,

$$\begin{aligned} d(X^2) &= \begin{pmatrix} 0 & 0 & 0 & ab - ba \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= 2 \begin{pmatrix} 0 & 0 & 0 & ab - ba \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 2XdX, \end{aligned}$$

for all $X \in R$, since S is commutative. This means that d is a Jordan left derivation.

But

$$d(XY) = \begin{pmatrix} 0 & 0 & 0 & ay - bx \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq XdY + YdX = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

since S is commutative. That is d is not a left derivation.

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WEAKLY $\mathfrak{b}$ -I-OPEN SETS AND WEAKLY $\mathfrak{b}$ -I-CONTINUOUS FUNCTIONS

Jamal M. Mustafa

Samer Al Ghour¹

*Department of Mathematics
Al al-Bayt University
P.O. Box: 130095, Mafrq
Jordan
e-mail: jjmmrr971@yahoo.com
algore@just.edu.jo*

Khalid Al Zoubi

*Department of Mathematics
Faculty of Science
Yarmouk University
Jordan
e-mail : khalidz@yu.edu.jo*

Abstract. The notion of weakly $\mathfrak{b}$ -I-open sets is introduced and used to define the notions of weakly $\mathfrak{b}$ -I-continuous functions, weakly $\mathfrak{b}$ -I-open functions, and weakly $\mathfrak{b}$ -I-closed functions. Some characterizations and properties regarding these concepts are discussed.

Keywords and phrases: Ideal topological spaces, weakly $\mathfrak{b}$ -I-open sets, weakly $\mathfrak{b}$ -I-continuous functions, weakly $\mathfrak{b}$ -I-irresolute functions, weakly $\mathfrak{b}$ -I-open functions.

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Introduction

Let (X, τ) be a topological space and $A \subseteq X$. The complement of A in X , the closure of A , the interior of A and the power set of A will be denoted by $X - A = A^c$, $Cl(A)$, $Int(A)$ and $\mathcal{P}(A)$, respectively. The subject of ideals in topological spaces has been studied by Kuratowski [9] and Vaidyanathaswamy [13]. An ideal on a topological space (X, τ) is defined as a non-empty collection I of subsets of X satisfying the following two conditions:

- (1) If $A \in I$ and $B \subseteq A$, then $B \in I$;
- (2) If $A \in I$ and $B \in I$, then $A \cup B \in I$.

¹Samer Al Ghour did this research during the sabbatical leave which was given from Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid, Jordan.

An ideal topological space is a topological space (X, τ) with an ideal I on X and is denoted by (X, τ, I) . For a subset $A \subseteq X$, $A^*(I) = \{x \in X : U \cap A \notin I \text{ for every } U \in \tau \text{ with } x \in U\}$ is called the local function of A with respect to I and τ [9]. We simply write A^* instead of $A^*(I)$ in case there is no chance of confusion. It is well known that $Cl^*(A) = A \cup A^*$ defines a Kuratowski closure operator for $\tau^*(I)$. First, we shall recall some definitions used in the sequel.

Definition 1.1. Let A be a subset of a topological space (X, τ) . Then

- (a) A is called *semi-open* [10] if $A \subseteq Cl(Int(A))$.
- (b) A is called *pre-open* [11] if $A \subseteq Int(Cl(A))$.
- (c) A is called *b-open* [1] if $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$.
- (d) A is called *semi-closed* [2] if it is the complement of a semi-open set.
- (e) The *semi-closure* of A [2], denoted by $sCl(A)$, is the smallest semi-closed set that contains A .

Definition 1.2. A subset A of an ideal topological space (X, τ, I) is said to be

- (a) *I-open* [8] if $A \subseteq Int(A^*)$.
- (b) *semi-I-open* [7] if $A \subseteq Cl^*(Int(A))$.
- (c) *pre-I-open* [3] if $A \subseteq Int(Cl^*(A))$.
- (d) *b-I-open* [4] if $A \subseteq Cl^*(Int(A)) \cup Int(Cl^*(A))$.
- (e) *weakly semi-I-open* [5] if $A \subseteq Cl^*(Int(Cl(A)))$.
- (f) *weakly pre-I-open* [6] if $A \subseteq sCl(Int(Cl^*(A)))$.

2. Weakly b-I-open sets

Definition 2.1. A subset A of an ideal topological space (X, τ, I) is said to be *weakly b-I-open* if $A \subseteq Cl^*(Int(Cl(A))) \cup Cl(Int(Cl^*(A)))$.

The family of all weakly b-I-open sets of the space (X, τ, I) will be denoted by $WBIO(X, \tau)$.

Theorem 2.2. For a subset of an ideal topological space, the following properties hold:

- (a) Every b-I-open set is weakly b-I-open.
- (b) Every weakly semi-I-open set is weakly b-I-open.
- (c) Every weakly pre-I-open set is weakly b-I-open.

The converse of each part in the above theorem need not be true as shown in the following three examples.

Example 2.3. Let $X = \{1, 2, 3\}$, $\tau = \{\phi, X, \{1, 2\}\}$ and $I = \{\phi, \{1\}\}$. Then $A = \{1\}$ is weakly b-I-open but not b-I-open.

Example 2.4. Let $X = \{1, 2, 3\}$, $\tau = \{\phi, X, \{1\}, \{3\}, \{1, 3\}\}$ and $I = \{\phi, \{1\}\}$. Then $A = \{1, 2\}$ is weakly b-I-open but not weakly semi-I-open.

Example 2.5. Let $X = \{1, 2, 3, 4\}$, $\tau = \{\phi, X, \{1, 2\}, \{3, 4\}\}$ and $I = \mathcal{P}(X)$. Then $A = \{2, 3\}$ is weakly semi-I-open and so weakly b-I-open but not weakly pre-I-open since $sCl(Int(Cl^*(A))) = \phi$.

Lemma 2.6. [8] *Let A and B be subsets of an ideal topological space (X, τ, I) .*

- (a) *If $A \subseteq B$, then $A^* \subseteq B^*$.*
- (b) *If $U \in \tau$, then $U \cap A^* \subseteq (U \cap A)^*$.*
- (c) *A^* is closed in (X, τ) .*

Theorem 2.7. *Let (X, τ, I) be an ideal topological space and A, B subsets of X .*

- (1) *If $U_\alpha \in WBIO(X, \tau)$ for each $\alpha \in \Delta$, then $\bigcup\{U_\alpha : \alpha \in \Delta\} \in WBIO(X, \tau)$.*
- (2) *If $A \in WBIO(X, \tau)$ and $B \in \tau$, then $A \cap B \in WBIO(X, \tau)$.*

Proof. (1) Since $U_\alpha \in WBIO(X, \tau)$, we have

$$\begin{aligned} \bigcup_{\alpha \in \Delta} U_\alpha &\subseteq \bigcup_{\alpha \in \Delta} [Cl^*(Int(Cl(U_\alpha))) \cup Cl(Int(Cl^*(U_\alpha)))] \\ &\subseteq \bigcup_{\alpha \in \Delta} \{[(Int(Cl(U_\alpha))) \cup (Int(Cl(U_\alpha)))^*] \cup [Cl(Int(U_\alpha \cup U_\alpha^*))]\} \\ &\subseteq [Int(Cl(\bigcup_{\alpha \in \Delta} U_\alpha)) \cup (Int(Cl(\bigcup_{\alpha \in \Delta} U_\alpha)))^*] \cup [Cl(Int((\bigcup_{\alpha \in \Delta} U_\alpha) \cup (\bigcup_{\alpha \in \Delta} U_\alpha)^*))] \\ &= Cl^*[Int(Cl(\bigcup_{\alpha \in \Delta} U_\alpha))] \cup Cl[Int(Cl^*(\bigcup_{\alpha \in \Delta} U_\alpha))]. \end{aligned}$$

Hence $\bigcup_{\alpha \in \Delta} U_\alpha \in WBIO(X, \tau)$.

(2) Let $A \in WBIO(X, \tau)$ and $B \in \tau$. Then $A \subseteq Cl^*(Int(Cl(A))) \cup Cl(Int(Cl^*(A)))$ and so

$$\begin{aligned} A \cap B &\subseteq [Cl^*(Int(Cl(A))) \cup Cl(Int(Cl^*(A)))] \cap B \\ &= [Cl^*(Int(Cl(A))) \cap B] \cup [Cl(Int(Cl^*(A))) \cap B] \\ &= [(Int(Cl(A)) \cup (Int(Cl(A)))^*) \cap B] \cup [Cl(Int(A \cup A^*)) \cap B] \\ &\subseteq [(Int(Cl(A)) \cap B) \cup ((Int(Cl(A))) \cap B)^*] \cup [Cl(Int[(A \cap B) \cup (A^* \cap B)])] \\ &\subseteq [(Int(Cl(A)) \cap B) \cup (Int(Cl(A)) \cap B)^*] \cup [Cl(Int[(A \cap B) \cup (A \cap B)^*])] \\ &\subseteq [(Int(Cl(A \cap B))) \cup (Int(Cl(A \cap B)))^*] \cup [Cl(Int[(A \cap B) \cup (A \cap B)^*])] \\ &= Cl^*[Int(Cl(A \cap B))] \cup Cl[Int(Cl^*(A \cap B))]. \end{aligned}$$

This shows that $A \cap B \in WBIO(X, \tau)$.

The following example shows that the finite intersection of weakly b-I-open sets need not be weakly b-I-open.

Example 2.8. Let $X = \{1, 2, 3, 4\}$, $\tau = \{\phi, X, \{1, 2\}, \{1, 2, 3\}\}$ and $I = \{\phi, \{3\}, \{4\}, \{3, 4\}\}$. Then $A = \{2, 4\}$ and $B = \{1, 4\}$ are weakly b-I-open but $A \cap B = \{4\}$ is not weakly b-I-open.

Definition 2.9. A subset A of an ideal topological space (X, τ, I) is said to be *weakly b-I-closed* if its complement is weakly b-I-open.

Theorem 2.10. If a subset A of an ideal topological space (X, τ, I) is weakly b-I-closed, then $\text{Int}(\text{Cl}(\text{Int}(A))) \subseteq A$.

Proof. Since A is weakly b-I-closed, $X - A \in \text{WBIO}(X, \tau)$. Then

$$\begin{aligned} X - A &\subseteq \text{Cl}^*(\text{Int}(\text{Cl}(X - A))) \cup \text{Cl}(\text{Int}(\text{Cl}^*(X - A))) \\ &\subseteq \text{Cl}(\text{Int}(\text{Cl}(X - A))) \cup \text{Cl}(\text{Int}(\text{Cl}(X - A))) \\ &= \text{Cl}(\text{Int}(\text{Cl}(X - A))) \\ &= X - \text{Int}(\text{Cl}(\text{Int}(A))). \end{aligned}$$

Therefore, $\text{Int}(\text{Cl}(\text{Int}(A))) \subseteq A$.

3. Weakly b-I-continuous functions

Definition 3.1.

- (1) A function $f : (X, \tau, I) \rightarrow (Y, \rho)$ is called *b-I-continuous* [4] if the inverse image of each open set in Y is a b-I-open set in X .
- (2) A function $f : (X, \tau, I) \rightarrow (Y, \rho)$ is called *weakly semi-I-continuous* [5] if the inverse image of each open set in Y is a weakly semi-I-open set in X .
- (3) A function $f : (X, \tau, I) \rightarrow (Y, \rho)$ is called *weakly pre-I-continuous* [6] if the inverse image of each open set in Y is a weakly pre-I-open set in X .

Definition 3.2. A function $f : (X, \tau, I) \rightarrow (Y, \rho)$ is called *weakly b-I-continuous* if $f^{-1}(V)$ is weakly b-I-open in (X, τ, I) for every open set V in (Y, ρ) .

Remark 3.3.

- (1) Every b-I-continuous function is weakly b-I-continuous.
- (2) Every weakly semi-I-continuous function is weakly b-I-continuous.
- (3) Every weakly pre-I-continuous function is weakly b-I-continuous.

The converse in each part of the above remark need not be true as shown in the following three examples.

Example 3.4. Let (X, τ) be the real line with the indiscrete topology and (Y, ρ) the real line with the usual topology. Then the identity function $f : (X, \tau, \mathcal{P}(X)) \rightarrow (Y, \rho)$ is weakly b-I-continuous but not b-I-continuous.

Example 3.5. Let $X = Y = \{1, 2, 3\}$, $\tau = \{\phi, X, \{1\}, \{3\}, \{1, 3\}\}$, $I = \{\phi, \{1\}\}$ and $\rho = \{\phi, Y, \{1, 3\}\}$. Define a function $f : (X, \tau, I) \rightarrow (Y, \rho)$ as follows: $f(1) = 1$, $f(2) = 3$ and $f(3) = 2$. Then f is weakly b-I-continuous but not weakly semi-I-continuous.

Example 3.6. Let $X=Y=\{1, 2, 3, 4\}$, $\tau=\{\phi, X, \{1\}, \{3\}, \{1, 3\}\}$, $\rho = \{\phi, Y, \{1\}, \{3\}, \{1, 3\}\}$, and $I = \{\phi, \{3\}\}$. Define a function $f : (X, \tau, I) \rightarrow (Y, \rho)$ as follows: $f(1) = f(3) = 4$ and $f(2) = f(4) = 2$. Then f is weakly b-I-continuous but not weakly pre-I-continuous.

Theorem 3.7. For a function $f : (X, \tau, I) \rightarrow (Y, \rho)$ the following are equivalent:

- (1) f is weakly b-I-continuous.
- (2) For each $x \in X$ and each $V \in \rho$ with $f(x) \in V$, there exists $U \in WBIO(X, \tau)$ with $x \in U$ such that $f(U) \subseteq V$.
- (3) The inverse image of each closed set in Y is weakly b-I-closed in X .

Proof. Straightforward.

Definition 3.8. Let A be a subset of a space (X, τ, I) and let $x \in X$. Then A is called a *weakly b-I-neighborhood* of x , if there exists a weakly b-I-open set U containing x such that $U \subseteq A$.

Theorem 3.9. For a function $f : (X, \tau, I) \rightarrow (Y, \rho)$, the following statements are equivalent:

- (1) f is weakly b-I-continuous.
- (2) For each $x \in X$ and each open set V in Y with $f(x) \in V$, $f^{-1}(V)$ is weakly b-I-neighborhood of x .

Proof. (1) $\Rightarrow$ (2). Let $x \in X$ and let V be an open set in Y such that $f(x) \in V$. By Theorem 3.7, there exists a weakly b-I-open set U in X with $x \in U$ such that $f(U) \subseteq V$. So $x \in U \subseteq f^{-1}(V)$. Hence $f^{-1}(V)$ is a weakly b-I-neighborhood of x .

(2) $\Rightarrow$ (1). Let V be an open set in Y and let $f(x) \in V$. Then by assumption, $f^{-1}(V)$ is a weakly b-I-neighborhood of x . Thus for each $x \in f^{-1}(V)$ there exists a weakly b-I-open set U_x containing x such that $x \in U_x \subseteq f^{-1}(V)$. Hence $f^{-1}(V) = \cup\{U_x : x \in f^{-1}(V)\}$ and so $f^{-1}(V) \in WBOI(X, \tau)$.

Definition 3.10. A function $f : (X, \tau, I) \rightarrow (Y, \rho)$ is called *weakly b-I-irresolute* if $f^{-1}(V)$ is weakly b-I-open in (X, τ, I) for every weakly b-I-open set V in (Y, ρ) .

Theorem 3.11. Let $f : (X, \tau, I) \rightarrow (Y, \rho, J)$ and $g : (Y, \rho, J) \rightarrow (Z, \sigma, K)$ then

- (1) gof is weakly b - I -continuous if f is weakly b - I -continuous and g is continuous.
- (2) gof is weakly b - I -continuous if f is weakly b - I -irresolute and g is weakly b - I -continuous.

We note that Theorem 3.11(1) is not true if g is assumed to be only b - I -continuous as it is shown in the next example.

Example 3.12. Let $X = Y = Z = \{1, 2, 3, 4\}$, $\tau = \{\phi, X, \{1\}, \{3\}, \{1, 3\}\}$, $\rho = \{\phi, Y, \{1\}, \{3\}, \{1, 3\}\}$, $\sigma = \{\phi, Z, \{1, 2\}\}$, $I = \{\phi, \{3\}\}$ and $J = \{\phi, \{2\}\}$. Define a function $f : (X, \tau, I) \rightarrow (Y, \rho, J)$ by $f(1) = f(3) = 4$ and $f(2) = f(4) = 2$. Also let $g : (Y, \rho, J) \rightarrow (Z, \sigma)$ be the identity function. Then f is weakly b - I -continuous and g is b - I -continuous but gof is not weakly b - I -continuous since $(gof)^{-1}(\{1, 2\}) = \{2, 4\}$ is not weakly b - I -open.

Corollary 3.13. If $f : (X, \tau, I) \rightarrow \prod_{\alpha \in \Delta} X_\alpha$ is a weakly b - I -continuous function from a space (X, τ, I) into a product space $\prod_{\alpha \in \Delta} X_\alpha$ then $p_\alpha \circ f$ is weakly b - I -continuous for each $\alpha \in \Delta$, where p_α is the projection function from the product space $\prod_{\alpha \in \Delta} X_\alpha$ onto the space X_α for each $\alpha \in \Delta$.

If (X, τ, I) is an ideal topological space and A is a subset of X , we denote by $\tau|_A$ the relative topology on A and $I|_A = \{A \cap I : I \in I\}$ is obviously an ideal on A .

Lemma 3.14. [8] Let (X, τ, I) be an ideal topological space and A, B be subsets of X such that $B \subseteq A$. Then $B^*(\tau|_A, I|_A) = B^*(\tau, I) \cap A$.

Lemma 3.15. [5] Let (X, τ, I) be an ideal topological space, $A \subseteq X$ and $U \in \tau$. Then $Cl^*(A) \cap U = Cl_U^*(A \cap U)$.

Theorem 3.16. Let (X, τ, I) be an ideal topological space, $A \subseteq U \in \tau$. If $A \in WBIO(X, \tau)$ then $A \in WBIO(U, \tau|_U, I|_U)$.

Proof. Since $U \in \tau$ and $A \in WBIO(X, \tau)$, we have

$$\begin{aligned}
 A &= U \cap A \subseteq U \cap [Cl^*(Int(Cl(A))) \cup Cl(Int(Cl^*(A)))] \\
 &= [U \cap (Cl^*(Int(Cl(A))))] \cup [U \cap (Cl(Int(Cl^*(A))))] \\
 &\subseteq Cl^*(U \cap Int(Cl(A))) \cup Cl(U \cap Int(Cl^*(A))) \\
 &= Cl^*(Int(U \cap Cl(A))) \cup Cl(Int(U \cap Cl^*(A))) \\
 &= Cl^*(Int_U(U \cap Cl(A))) \cup Cl(Int_U(U \cap Cl^*(A)))
 \end{aligned}$$

Since $U \in \tau \subseteq \tau^*$, we obtain

$$\begin{aligned}
 A &= U \cap A \subseteq U \cap [Cl^*(Int_U(U \cap Cl(A))) \cup Cl(Int_U(U \cap Cl^*(A)))] \\
 &= [U \cap (Cl^*(Int_U(U \cap Cl(A))))] \cup [U \cap (Cl(Int_U(U \cap Cl^*(A))))] \\
 &= Cl_U^*(Int_U(U \cap Cl(A))) \cup Cl_U(Int_U(U \cap Cl^*(A))) \\
 &= Cl_U^*(Int_U(Cl_U(A))) \cup Cl_U(Int_U(Cl_U^*(A)))
 \end{aligned}$$

Then $A \in WBIO(U, \tau|_U, I|_U)$.

Corollary 3.17. *Let (X, τ, I) be an ideal topological space. $U \in \tau$ and $A \in WBIO(X, \tau)$, then $U \cap A \in WBIO(U, \tau|_U, I|_U)$.*

Proof. Since $U \in \tau$ and $A \in WBIO(X, \tau)$, by Theorem 2.7, $U \cap A \in WBIO(X, \tau)$. Since $U \in \tau$, by Theorem 3.16, $U \cap A \in WBIO(U, \tau|_U, I|_U)$.

Theorem 3.18. *Let $f : (X, \tau, I) \rightarrow (Y, \rho)$ be a weakly b-I-continuous function and $U \in \tau$. Then the restriction $f|_U : (U, \tau|_U, I|_U) \rightarrow (Y, \rho)$ is weakly b-I-continuous.*

Proof. Let V be any open set in (Y, ρ) . Since f is weakly b-I-continuous, we have $f^{-1}(V) \in WBIO(X, \tau)$. Since $U \in \tau$, by Theorem 3.16, we have $U \cap f^{-1}(V) \in WBIO(U, \tau|_U, I|_U)$. On the other hand, $(f|_U)^{-1}(V) = U \cap f^{-1}(V)$ and $(f|_U)^{-1}(V) \in WBIO(U, \tau|_U, I|_U)$. This shows that $f|_U : (U, \tau|_U, I|_U) \rightarrow (Y, \rho)$ is weakly b-I-continuous.

Theorem 3.19. *A function $f : (X, \tau, I) \rightarrow (Y, \rho)$ is weakly b-I-continuous if and only if the function $g : X \rightarrow X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, is weakly b-I-continuous.*

Proof. $\Rightarrow$) Let f be weakly b-I-continuous. Let $x \in X$ and W be an open set in $X \times Y$ with $g(x) \in W$. Then there exists a basic open set $U \times V$ such that $g(x) = (x, f(x)) \in U \times V \subseteq W$. Since f is weakly b-I-continuous, there exists a weakly b-I-open set U_0 in X with $x \in U_0$ such that $f(U_0) \subseteq V$. By Theorem 2.7, $U_0 \cap U$ is weakly b-I-open and $g(U_0 \cap U) \subseteq U \cap V \subseteq W$. This shows that g is weakly b-I-continuous.

$\Leftarrow$) Suppose that g is weakly b-I-continuous. Let $x \in X$ and V be any open set in Y with $f(x) \in V$. Then $X \times V$ is open in $X \times Y$. Since g is weakly b-I-continuous, there exists a weakly b-I-open set U in X with $x \in U$ such that $g(U) \subseteq X \times V$. Therefore, we obtain $f(U) \subseteq V$. This shows that f is weakly b-I-continuous.

Definition 3.20. An ideal topological space (X, τ, I) is said to be *weakly b-I-normal* if for each pair of non-empty disjoint closed subsets A and B of X , there exist two weakly b-I-open subsets U and V of X such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \phi$.

Theorem 3.21. *If $f : (X, \tau, I) \rightarrow (Y, \rho)$ is weakly b-I-continuous, closed injection and Y is normal, then X is weakly b-I-normal.*

Proof. Let A and B be two disjoint closed subsets of X . Since f is closed and injective, $f(A)$ and $f(B)$ are disjoint closed subsets of Y . Since Y is normal, there exist two open subsets U and V of Y such that $f(A) \subseteq U$, $f(B) \subseteq V$ and $U \cap V = \phi$. Now $f^{-1}(U)$ and $f^{-1}(V)$ are weakly b-I-open in X with $A \subseteq f^{-1}(U)$, $B \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \phi$. Thus X is weakly b-I-normal.

Definition 3.22. An ideal topological space (X, τ, I) is said to be *weakly b-I-connected* if X can't be written as a union of two disjoint weakly b-I-open subsets of X .

Theorem 3.23. *The weakly b-I-continuous image of a weakly b-I-connected space is connected.*

Proof. Let $f : (X, \tau, I) \rightarrow (Y, \rho)$ be a weakly b-I-continuous function of a weakly b-I-connected space X onto a topological space Y . Assume that Y is disconnected, then $Y = A \cup B$ where A and B are non-empty clopen with $A \cap B = \phi$. Since f is weakly b-I-continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty weakly b-I-open in X . Also, $X = f^{-1}(Y) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A) \cap f^{-1}(B) = \phi$. Hence X is not weakly b-I-connected which is a contradiction. Therefore, Y is connected.

Lemma 3.24. [12] *For any function $f : (X, \tau, I) \rightarrow (Y, \rho)$, $f(I)$ is an ideal on Y .*

Definition 3.25. [12] An ideal topological space (X, τ, I) is said to be *I-compact* if for every open cover $\{W_\alpha : \alpha \in \Delta\}$ of X , there exists a finite subset Δ_0 of Δ such that $(X - \cup\{W_\alpha : \alpha \in \Delta_0\}) \in I$.

Definition 3.26. An ideal topological space (X, τ, I) is said to be *weakly b-I-compact* if for every weakly b-I-open cover $\{W_\alpha : \alpha \in \Delta\}$ of X , there exists a finite subset Δ_0 of Δ such that $(X - \cup\{W_\alpha : \alpha \in \Delta_0\}) \in I$.

Theorem 3.27. *The image of a weakly b-I-compact space under a weakly b-I-continuous surjective function is $f(I)$ -compact.*

Proof. Let (X, τ, I) be a weakly b-I-compact space and $f : (X, \tau, I) \rightarrow (Y, \rho)$ be a weakly b-I-continuous surjection. Let $\{V_\alpha : \alpha \in \Delta\}$ be an open cover of Y . Then $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a weakly b-I-open cover of X . Since X is weakly b-I-compact, there exists a finite subset Δ_0 of Δ such that $(X - \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\}) \in I$. Therefore $(Y - \cup\{V_\alpha : \alpha \in \Delta_0\}) \in f(I)$. This shows that $(Y, \rho, f(I))$ is $f(I)$ -compact.

4. Weakly b-I-open functions

Recall that a subset F of a space (X, τ, I) is said to be *weakly semi-I-closed* [5] if its complement is weakly semi-I-open.

Definition 4.1. [5] A function $f : (X, \tau) \rightarrow (Y, \rho, J)$ is called *weakly semi-I-open* (resp., *weakly semi-I-closed*) if the image of every open (resp., closed) set in (X, τ) is weakly semi-I-open (resp., weakly semi-I-closed) in (Y, ρ, J) .

Definition 4.2. A function $f : (X, \tau) \rightarrow (Y, \rho, J)$ is called *weakly b-I-open* (resp., *weakly b-I-closed*) if the image of every open (resp., closed) set in (X, τ) is weakly b-I-open (resp., weakly b-I-closed) in (Y, ρ, J) .

Remark 4.3. Every weakly semi-I-open (resp., weakly semi-I-closed) function is weakly b-I-open (resp., weakly b-I-closed).

The converse of the above remark need not be true as shown in the following example.

Example 4.4. Let $X = \{1, 2, 3\}$ and $\tau = \{\phi, X, \{1, 3\}\}$. Also let $Y = X$, $\rho = \{\phi, Y, \{1\}, \{3\}, \{1, 3\}\}$ and $J = \{\phi, \{1\}\}$. Define a function $f : (X, \tau) \rightarrow (Y, \rho, J)$ as follows: $f(1) = 1$, $f(2) = 3$ and $f(3) = 2$. Then f is weakly b-I-open but not weakly semi-I-open.

Theorem 4.5. *A function $f : (X, \tau) \rightarrow (Y, \rho, J)$ is weakly b-I-open if and only if for each $x \in X$ and each neighborhood U of x there exists $V \in WBJO(Y, \rho)$ containing $f(x)$ such that $V \subseteq f(U)$.*

Proof. $\Rightarrow$ Suppose that f is a weakly b-I-open function. For each $x \in X$ and each neighborhood U of x , there exists $U_x \in \tau$ such that $x \in U_x \subseteq U$. Let $V = f(U_x)$. Since f is weakly b-I-open, $V \in WBJO(Y, \rho)$ and $f(x) \in V \subseteq f(U)$.

$\Leftarrow$ Let U be an open set in (X, τ) . For each $x \in U$, there exists $V_x \in WBJO(X, \tau)$ such that $f(x) \in V_x \subseteq f(U)$. Now $f(U) = \cup\{V_x : x \in U\}$ and so $f(U) \in WBJO(Y, \rho)$. This shows that f is weakly b-I-open.

Theorem 4.6. *Let $f : (X, \tau) \rightarrow (Y, \rho, J)$ be a weakly b-I-open function. If G is any subset of Y and C is a closed subset of X with $f^{-1}(G) \subseteq C$, then there exists a weakly b-I-closed subset H of Y with $G \subseteq H$ such that $f^{-1}(H) \subseteq C$.*

Proof. Suppose that f is a weakly b-I-open function. Let G be any subset of Y and C a closed subset of X with $f^{-1}(G) \subseteq C$. Then $X - C$ is open. Since f is weakly b-I-open, $f(X - C)$ is weakly b-I-open in Y . Let $H = Y - f(X - C)$. Then H is weakly b-I-closed in Y . Since $f^{-1}(G) \subseteq C$, $G \subseteq H$. Also, we obtain $f^{-1}(H) \subseteq C$.

Theorem 4.7. *Let $f : (X, \tau) \rightarrow (Y, \rho, J)$ be weakly b-I-closed. If G is any subset of Y and U is an open subset of X with $f^{-1}(G) \subseteq U$, then there exists a weakly b-I-open subset H of Y with $G \subseteq H$ such that $f^{-1}(H) \subseteq U$.*

Proof. Similar to that used in Theorem 4.6.

Theorem 4.8. *For any bijective function $f : (X, \tau) \rightarrow (Y, \rho, J)$, the following are equivalent:*

- (1) $f^{-1} : (Y, \rho, J) \rightarrow (X, \tau)$ is weakly b-I-continuous.
- (2) f is weakly b-I-open.
- (3) f is weakly b-I-closed.

Proof. It is straightforward.

Theorem 4.9. *Let $f : (X, \tau) \rightarrow (Y, \rho, J)$ and $g : (Y, \rho, J) \rightarrow (Z, \sigma, K)$.*

- (1) *$g \circ f$ is weakly b - I -open if f is open and g is a weakly b - I -open.*
- (2) *f is weakly b - I -open if $g \circ f$ is open and g is a weakly b - I -continuous injection.*

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REDEFINED GENERALIZED FUZZY R -SUBGROUPS OF NEAR-RINGS

Fen Luo

Jianming Zhan¹

*Department of Mathematics
Hubei University for Nationalities
Enshi, Hubei Province, 445000
P.R. China*

Abstract. By means of a kind of new idea, we redefine generalized fuzzy R -subgroups of a near-ring and investigate some of its related properties. Some new characterizations are also given. In particular, we introduce the concepts of strong prime (semiprime) $(\in, \in \vee q)$ -fuzzy R -subgroups of near-rings, and discuss the relationship between strong prime (resp., semiprime) $(\in, \in \vee q)$ -fuzzy R -subgroups and prime (resp., semiprime) $(\in, \in \vee q)$ -fuzzy R -subgroups of near-rings.

Keywords: near-ring; prime (semiprime) R -subgroup; $(\in, \in \vee q)$ -fuzzy R -subgroup; $(\overline{\in}, \overline{\in} \vee \overline{q})$ -fuzzy R -subgroup.

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1. Introduction

Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces and so on. This provides sufficient motivations to researchers to review various concepts and results from the realm of abstract algebra in the broader framework of fuzzy setting.

A near-ring satisfying all axioms of an associative ring, except for commutativity of addition and one of the two distributive laws. Abou-Zaid [1] introduced the concept of fuzzy subnear-ring and studied fuzzy ideals of near-rings. The concept was discussed further by many researchers, for example [3]-[7], [10], [11]. After the introduction of fuzzy sets by Zadeh, there have been a number of generalizations of this fundamental concept. A new type of fuzzy subgroup, that is, the $(\in, \in \vee q)$ -fuzzy subgroup, was introduced in an earlier paper of Bhakat

¹Corresponding author. E-mail address: zhanjianming@hotmail.com (J. Zhan).

and Das [2] by using the combined notions of “belongingness” and “quasicoincidence” of fuzzy points and fuzzy sets. In fact, the $(\in, \in \vee q)$ -fuzzy subgroup is an important generalization of Rosenfeld’s fuzzy subgroup. It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems with other algebraic structures, see [3], [4], [8], [9].

By means of a kind of new idea, we redefine generalized fuzzy R -subgroups of a near-ring and investigate some of its related properties. In particular, we introduce the concepts of strong prime (semiprime) $(\in, \in \vee q)$ -fuzzy R -subgroups of near-rings, and discuss the relationship between strong prime (resp., semiprime) $(\in, \in \vee q)$ -fuzzy R -subgroups and prime (resp., semiprime) $(\in, \in \vee q)$ -fuzzy R -subgroups of near-rings.

2. Preliminaries

A non-empty set R with two binary operation “+” and “ $\cdot$ ” is called a *near-ring* if it satisfies:

- (1) $(R, +)$ is a group,
- (2) $(R, \cdot)$ is a semigroup,
- (3) $x \cdot (y + z) = x \cdot y + x \cdot z$, for all $x, y, z \in R$.

We will use the word “near-ring” to mean “left near-ring” and denote xy instead of $x \cdot y$.

An R -subgroup H of a near-ring R is a subset of R such that

- (i) $(H, +)$ is a subgroup of $(R, +)$,
- (ii) $RH \subseteq H$,
- (iii) $HR \subseteq H$.

If H satisfies (i) and (ii), then it is called a left R -subgroup of R . If H satisfies (i) and (iii), then it is called a right R -subgroup of R .

If I and J are R -subgroups of near-ring R . An R -subgroup P of R is called prime if $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$ for all R -subgroups I and J of R . An R -subgroup P of R is called semiprime if $I^2 \subseteq P$ implies $I \subseteq P$ for all R -subgroups I of R .

We next state some fuzzy logic concepts. Recall that a fuzzy set is a function $\mu : R \rightarrow [0, 1]$. For any $A \subseteq R$, the characteristic function of A is denoted by χ_A . We define μ^{-1} by $\mu^{-1}(x) = \mu(-x)$, for all $x \in R$.

A fuzzy set μ of S of the form

$$\mu(y) = \begin{cases} t(\neq 0) & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a *fuzzy point with support x and value t* and is denoted by x_t . A fuzzy point x_t is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set μ , written as $x_t \in \mu$ (resp., $x_t q \mu$) if $\mu(x) \geq t$ (resp., $\mu(x) + t > 1$). If $x_t \in \mu$ or $x_t q \mu$, then we write $x_t \in \vee q \mu$. If $\mu(x) < t$ (resp., $\mu(x) + t \leq 1$), then we call $x_t \bar{\in} \mu$ (resp., $x_t \bar{q} \mu$). We note that the symbol $\bar{\in \vee q}$ means that $\in \vee q$ does not hold.

Definition 2.1. [1] A fuzzy set μ of R is called a *fuzzy right* (resp., *left*) *R -subgroup* of R if

- (F1a) $\mu(x + y) \geq \mu \wedge \mu(y), \forall x, y \in R$,
- (F1a') $\mu(-x) \geq \mu(x), \forall x \in R$,
- (F1b) $\mu(xy) \geq \mu(x)$ (resp., $\mu(yx) \geq \mu(x)$), $\forall x, y \in R$.

In what follows, a (fuzzy) R -subgroup means a (fuzzy) right R -subgroup and R is a near-ring unless otherwise specified.

Definition 2.2. [4] A fuzzy set μ of R is called an $(\in, \in \vee q)$ -fuzzy R -subgroup of R if for all $t, r \in (0, 1]$ and $x, y \in R$,

- (F2a) $x_t \in \mu$ and $y_r \in \mu$ imply $(x + y)_{t \wedge r} \in \vee q \mu$,
- (F2a') $x_t \in \mu$ implies $(-x)_t \in \vee q \mu$,
- (F2b) $x_t \in \mu$ implies $(xy)_t \in \vee q \mu$.

Theorem 2.3. [4] A fuzzy set μ of R is an $(\in, \in \vee q)$ -fuzzy R -subgroup of R if and only if for any $x, y, a \in R$,

- (F3a) $\mu(x + y) \geq \mu(x) \wedge \mu(y) \wedge 0.5$,
- (F3a') $\mu(-x) \geq \mu(x) \wedge 0.5$,
- (F3b) $\mu(xy) \geq \mu(x) \wedge 0.5$.

Naturally, we consider the concept of $(\bar{\in}, \bar{\in} \vee \bar{q})$ -fuzzy R -subgroup of R by means of Davvaz's way.

Definition 2.4. A fuzzy set μ of R called an $(\bar{\in}, \bar{\in} \vee \bar{q})$ -fuzzy R -subgroup of R if for all $t, r \in (0, 1]$ and for all $x, y \in R$,

- (F4a) $(x + y)_{t \wedge r} \bar{\in} \mu$ implies $x_t \bar{\in} \vee \bar{q} \mu$ or $y_r \bar{\in} \vee \bar{q} \mu$,
- (F4a') $(-x)_t \bar{\in} \mu$ implies $(-x)_t \bar{\in} \vee \bar{q} \mu$,
- (F4b) $(xy)_{t \wedge r} \bar{\in} \mu$ implies $x_t \bar{\in} \vee \bar{q} \mu$.

Example 2.5. Let $R = \{a, b, c, d\}$ be a set with two binary operations as follows:

$+$	a	b	c	d	$\cdot$	a	b	c	d
a	a	b	c	d	a	a	a	a	a
b	b	a	d	c	b	a	a	a	a
c	c	d	b	a	c	a	a	a	a
d	d	c	b	a	d	a	a	b	b

Then $(R, +, \cdot)$ is a near-ring. Define a fuzzy set μ of R by $\mu(a) = 0.9$, $\mu(b) = 0.8$, $\mu(c) = 0.4$ and $\mu(d) = 0.6$. Thus, μ is an $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy R -subgroup of R .

Theorem 2.6. *A fuzzy set μ of R is an $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy R -subgroup of R if and only if for any $x, y \in R$,*

$$(F5a) \quad \mu(x + y) \vee 0.5 \geq \mu(x) \wedge \mu(y),$$

$$(F5a') \quad \mu(-x) \vee 0.5 \geq \mu(x),$$

$$(F5b) \quad \mu(xy) \vee 0.5 \geq \mu(x).$$

Proof. We only prove $(F4a) \Leftrightarrow (F5a)$. The others are similar.

$(F4a1) \Rightarrow (F5a)$ If there exist $x, y \in R$ such that $\mu(x + y) \vee 0.5 < t = \mu(x) \wedge \mu(y)$, then $0.5 < t \leq 1$, $(x + y)_t \bar{\epsilon} \mu$, but $x_t \in \mu, y_t \in \mu$. By (F1), we have $x_t \bar{q} \mu$ or $y_t \bar{q} \mu$. Then, $(t \leq \mu(x) \text{ and } t + \mu(x) \leq 1)$ or $(t \leq \mu(y) \text{ and } t + \mu(y) \leq 1)$. Thus, $t \leq 0.5$, contradiction.

$(F5a) \Rightarrow (F4a)$ Let $(x + y)_{t \wedge r} \bar{\epsilon} \mu$, then $\mu(x + y) < t \wedge r$.

(1) If $\mu(x + y) \geq \mu(x) \wedge \mu(y)$, then $\mu(x) \wedge \mu(y) < t \wedge r$, and consequently, $\mu(x) < t$ or $\mu(y) < r$. It follows that $x_t \bar{\epsilon} \mu$ or $y_r \bar{\epsilon} \mu$. Thus, $x_t \bar{\epsilon} \vee \bar{q} \mu$ or $y_r \bar{\epsilon} \vee \bar{q} \mu$.

(2) If $\mu(x + y) < \mu(x) \wedge \mu(y)$ then by (F4), we have $0.5 \geq \mu(x) \wedge \mu(y)$. Putting $x_t \bar{\epsilon} \mu$ or $y_r \bar{\epsilon} \mu$, then $t \leq \mu(x) \leq 0.5$ or $r \leq \mu(y) \leq 0.5$. It follows that $x_t \bar{q} \mu$ or $y_r \bar{q} \mu$, and thus, $x_t \bar{\epsilon} \vee \bar{q} \mu$ or $y_r \bar{\epsilon} \vee \bar{q} \mu$. This completes the proof. ■

3. Main results

In this Section, we introduce the concepts of generalized fuzzy R -subgroups of near-rings by means of a new way, which is different with the related topic.

Remark 3.1. Let μ and ν be any two fuzzy sets of R . Then

(i) If $x_t \in \mu$ implies $x_t \in \vee q \nu$ for all $x \in R$ and $t \in (0, 1]$, then we can write $\mu \subseteq \vee q \nu$.

(ii) If $x_t \bar{\epsilon} \mu$ implies $x_t \bar{\epsilon} \vee \bar{q} \nu$ for all $x \in R$ and $t \in (0, 1]$, then we can write $\mu \supseteq \vee \bar{q} \nu$.

Proposition 3.2. *For any two fuzzy sets μ and ν of R .*

(i) $\mu \subseteq \vee q \nu$ if and only if $\nu(x) \geq \min\{\mu(x), 0.5\}, \forall x \in R$;

(ii) $\mu \supseteq \vee \bar{q} \nu$ if and only if $\max\{\mu(x), 0.5\} \geq \nu(x), \forall x \in R$.

Proof. (i) Let $\mu \subseteq \vee q \nu$. If there exists $x \in R$ such that $\nu(x) < t = \mu(x) \wedge 0.5$, then $x_t \in \mu$, but $x_t \bar{\epsilon} \vee \bar{q} \nu$, contradiction.

Conversely, let $\nu(x) \geq \mu(x) \wedge 0.5, \forall x \in R$. If $\mu \not\subseteq \vee q \nu$, then there exists $x_t \in \mu$, but $x_t \bar{\epsilon} \vee \bar{q} \nu$, and so $\mu(x) \geq t$ and $\nu(x) < t < 0.5$, contradiction.

(ii) Let $\mu \supseteq \vee \bar{q} \nu$. If there exists $x \in R$ such that $\mu(x) \vee 0.5 < t = \nu(x)$, then $x_t \in \nu$, but $x_t \notin \mu$ and $t > 0.5$. Hence $x_t \notin \vee \bar{q} \nu$, and so $x_t \bar{q} \nu$, that is, $G(x) + t \leq 1$, and so $t \leq 0.5$, contradiction.

Conversely, let $\mu(x) \vee 0.5 \geq \nu(x), \forall x \in R$. If $\mu \supseteq \vee \bar{q} \nu$, then there exists $x_t \notin \mu$, but $x_t \in \vee \bar{q} \nu$. Hence $\mu(x) < t, \nu(x) \geq t$ and $\nu(x) + t > 1$.

Case (1). $\mu(x) > 0.5$. Then $\mu(x) \geq \nu(x)$, contradiction.

Case (2). $\mu(x) \leq 0.5$. Then $0.5 \geq \nu(x)$. By $\nu(x) \geq t, 0.5 \geq \nu(x) \geq t$. But $2\nu(x) \geq \nu(x) + t > 1$, and so $\nu(x) > 0.5$, contradiction. ■

Now, we give the concepts of the product and sum of two fuzzy sets of R .

Definition 3.3. Let μ and ν be fuzzy sets of R . Then the sum of μ and ν is defined by

$$(\mu \circ \nu)(x) = \bigvee_{x=ab} (\mu(a) \wedge \nu(b))$$

and $(\mu \circ \nu)(x) = 0$ if x cannot be expressed as $x = ab$.

Definition 3.4. Let μ and ν be fuzzy sets of R . Then the sum of μ and ν is defined by

$$(\mu + \nu)(x) = \bigvee_{x=a+b} (\mu(a) \wedge \nu(b))$$

and $(\mu + \nu)(x) = 0$ if x cannot be expressed as $x = a + b$.

Now, by means of a new way, we consider another generalized fuzzy R -subgroup of near-rings, which is called a new $(\in, \in \vee q)$ -fuzzy R -subgroups.

Definition 3.5. A fuzzy set μ of R is called a *new $(\in, \in \vee q)$ -fuzzy R -subgroup* of R if it satisfies:

$$(F6a) \quad (\mu + \mu) \subseteq \vee q \mu,$$

$$(F6a') \quad \mu^{-1} \subseteq \vee q \mu,$$

$$(F6b) \quad (\mu \circ \chi_R) \subseteq \vee q \mu.$$

Theorem 3.6. A fuzzy set μ of R is a new $(\in, \in \vee q)$ -fuzzy R -subgroup of R if and only if it satisfies (F3a), (F3a') and (F3b).

Proof. Let μ be a new $(\in, \in \vee q)$ -fuzzy R -subgroup of R . If there exist $x, y \in R$ such that $\mu(x + y) < t < \mu(x) \wedge \mu(y) \wedge 0.5$, then $t < 0.5, x_t \in \mu, y_t \in \mu$, but $(x + y)_t \notin \mu$, and so $(x + y)_t \in \vee q \mu$. But, $(\mu + \mu)(x + y) = \bigvee_{x+y=a+b} (\mu(a) \wedge \mu(b)) \geq \mu(x) \wedge \mu(y) \geq t$, and so $(x + y)_t \in (\mu + \mu)$. Thus, $(x + y)_t \in \vee q \mu$, contradiction. This proves (F3a) holds.

Now, if there exists $x \in R$ such that $\mu(-x) < t < \mu(x) \wedge 0.5$, then $\mu(x) \geq t$ and $t < 0.5$, but $(-x)_t \notin \mu$. Thus, $(-x)_t \in \vee q \mu$. But $\mu^{-1}(-x) = \mu(x) \geq t$, and

so $(-x)_t \in \mu^{-1}$, which implies, $(-x)_t \in \vee q\mu$, contradiction. This proves (F3a') holds.

The proof of (F3b) is similar to the proof of (F3a).

Conversely, μ satisfies (F3a), (F3a') and (F3b). Let $x_t \in (\mu + \mu)$, but $x_t \notin \vee q\mu$. Then $\mu(x) < t$ and $\mu(x) < 0.5$. By definition, $(\mu + \mu)(x + y) = \bigvee_{x+y=a+b} (\mu(a) \wedge \mu(b))$. Since $0.5 > \mu(x) = \mu(a + b) \geq \mu(a) \wedge \mu(b) \wedge 0.5$, and so $\mu(x) \geq \mu(a) \wedge \mu(b)$. Thus, $t \leq (\mu + \mu)(x) \leq \bigvee_{x=a+b} \mu(x) = \mu(x)$, that is, $\mu(x) \geq t$, contradiction. This proves that (F6a) holds.

Now, let $x_t \in \mu^{-1}$, but $x_t \notin \vee q\mu$, then $\mu(x) < t$ and $\mu(x) < 0.5$. Thus $\mu(-x) \geq \mu(x) \wedge 0.5 = \mu(x)$, and so $\mu(-x) = \mu(x)$. Hence $t \leq \mu^{-1}(x) = \mu(-x) = \mu(x)$, contradiction.

The proof of (F6b) is similar to the proof of (F6a).

Therefore, μ is a new $(\in, \in \vee q)$ -fuzzy R -subgroup of R . ■

The following is a consequence of Theorem 2.3 and 3.6.

Corollary 3.7. *The concepts of new $(\in, \in \vee q)$ -fuzzy R -subgroups of R and $(\in, \in \vee q)$ -fuzzy R -subgroups are equivalent, respectively.*

Next, by means of a new way, we consider another a generalized fuzzy R -subgroup of near-rings, which is called a new $(\overline{\in}, \overline{\in} \vee \overline{q})$ -fuzzy R -subgroup.

Definition 3.8. A fuzzy set μ of R is called a *new $(\overline{\in}, \overline{\in} \vee \overline{q})$ -fuzzy R -subgroup of R* if it satisfies:

$$(F7a) \quad \mu \supseteq \vee \overline{q}(\mu + \mu),$$

$$(F7a') \quad \mu \supseteq \vee \overline{q}\mu^{-1},$$

$$(F7b) \quad \mu \supseteq \vee \overline{q}(\mu \circ \chi_R).$$

Theorem 3.9. *A fuzzy set μ of R is a new $(\overline{\in}, \overline{\in} \vee \overline{q})$ -fuzzy R -subgroup of R if and only if it satisfies (F5a), (F5a') and (F5b).*

Proof. Let μ be a new $(\in, \in \vee q)$ -fuzzy R -subgroup of R . If there exist $x, y \in R$ such that $\mu(x + y) \vee 0.5 < t < \mu(x) \wedge \mu(y)$, then $t > 0.5$, $x_t \in \mu$, $y_t \in \mu$, but $(x + y)_t \notin \mu$, and so $(x + y)_t \in \vee q(\mu + \mu)$. Thus,

$$(*) \quad (\mu + \mu)(x + y) < t$$

and

$$(**) \quad (\mu + \mu)(x + y) + t \leq 1$$

But, $(\mu + \mu)(x + y) = \bigvee_{x+y=a+b} (\mu(a) \wedge \mu(b)) \geq \mu(x) \wedge \mu(y) \geq t$, which implies, $(\mu + \mu)(x + y) \geq t$. By (*) and (**), we have $(\mu + \mu)(x + y) + t \leq 1$, and so $t \leq 0.5$, contradiction. This proves (F5a) holds. Similarly, we can prove (F5a') and (F5b).

Conversely, μ satisfies (F5a), (F5a') and (F5b). Let $x_t \in \mu$, but $x_t \notin \overline{\overline{\epsilon} \vee \overline{q}}(\mu + \mu)$. Then $\mu(x) < t$, but $(\mu + \mu)(x) \geq t$ and $(\mu + \mu)(x) + t > 1$, and so $(\mu + \mu)(x) > 0.5$. By definition, $(\mu + \mu)(x) = \bigvee_{x=a+b} (\mu(a) \wedge \mu(b))$. Since $0.5 \vee \mu(x) = \mu(a+b) \vee 0.5 \geq \mu(a) \wedge \mu(b)$, and so $0.5 \vee \mu(x) \geq \mu(a) \wedge \mu(b)$. Thus, $t \leq (\mu + \mu)(x) \leq \bigvee_{x=a+b} \mu(x) \vee 0.5$. Since $(\mu + \mu)(x) > 0.5$, then $\mu(x) \geq 0.5$, and so $\mu(x) \geq t$, contradiction. This proves that (F7a) holds. Similarly, we can prove (F7a') and (F7b) hold. Therefore, μ is a new $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy R -subgroup of R . ■

The following is a consequence of Theorem 2.5 and 3.9.

Corollary 3.10. *The concepts of new $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy R -subgroups of R and $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy R -subgroups are equivalent, respectively.*

4. Strong prime (semiprime) $(\epsilon, \epsilon \vee q)$ -fuzzy R -subgroups

In this Section, we introduce the concepts of strong prime (semiprime) $(\epsilon, \epsilon \vee q)$ -fuzzy R -subgroups of near-rings. In particular, we discuss the relationship between strong prime (resp., semiprime) $(\epsilon, \epsilon \vee q)$ -fuzzy R -subgroups and prime (resp., semiprime) $(\epsilon, \epsilon \vee q)$ -fuzzy R -subgroups of near-rings.

Definition 4.1. [8]

- (i) An $(\epsilon, \epsilon \vee q)$ -fuzzy R -subgroup μ of R is called prime if for all $x, y \in R$ and $t \in (0, 1]$, we have

$$(P) \quad (xy)_t \in \mu \Rightarrow x_t \in \vee q\mu \text{ or } y_t \in \vee q\mu.$$
- (ii) An $(\epsilon, \epsilon \vee q)$ -fuzzy R -subgroup μ of R is called semiprime if for all $x \in R, t \in (0, 1]$, we have

$$(SP) \quad (x^2)_t \in \mu \Rightarrow x_t \in \vee q\mu.$$

Theorem 4.2. [8]

- (i) An $(\epsilon, \epsilon \vee q)$ -fuzzy R -subgroup μ of R is prime if for all $x, y \in R$, it satisfies:

$$(P') \quad \mu(x) \vee \mu(y) \geq \mu(xy) \wedge 0.5.$$
- (ii) An $(\epsilon, \epsilon \vee q)$ -fuzzy R -subgroup μ of R is semiprime if for all $x \in R$, it satisfies:

$$(SP') \quad \mu(x) \geq \mu(x^2) \wedge 0.5.$$

For a fuzzy set μ of R and $t \in (0, 1]$, the crisp set $\mu_t = \{x \in R \mid \mu(x) \geq t\}$ is called the *level subset* of μ .

Theorem 4.3. [8] *An $(\in, \in \vee q)$ -fuzzy R -subgroup μ of R is prime (semiprime) if and only if $\mu_t (\neq \emptyset)$ is a prime (semiprime) R -subgroup of R for all $t \in (0, 0.5]$, respectively.*

Now, we give the concept of strong prime (semiprime) $(\in, \in \vee q)$ -fuzzy R -subgroups of near-rings.

Definition 4.4.

- (i) An $(\in, \in \vee q)$ -fuzzy R -subgroup ρ of R is called strong prime if for every $(\in, \in \vee q)$ -fuzzy R -subgroups μ and ν of R , it satisfies:
(P'') $\mu \circ \nu \subseteq \rho$ implies $\mu \subseteq \rho$ or $\nu \subseteq \rho$.
- (ii) An $(\in, \in \vee q)$ -fuzzy R -subgroup μ of R is called strong semiprime if for every $(\in, \in \vee q)$ -fuzzy R -subgroup μ of R , it satisfies:
(SP'') $\mu \circ \mu \subseteq \rho$ implies $\mu \subseteq \rho$.

Theorem 4.5. *Let μ be a strong prime (semiprime) $(\in, \in \vee q)$ -fuzzy R -subgroup of R . Then $\mu_t (\neq \emptyset)$ is a prime (semiprime) R -subgroup of R for all $t \in (0, 0.5]$, respectively.*

Proof. We only consider strong prime $(\in, \in \vee q)$ -fuzzy R -subgroups. The case for strong semiprimeness is similar.

Let $t \in (0, 0.5]$ be such that μ_t is non-empty. Then μ_t is an R -subgroup of R . Now we show that μ_t is prime. Let I and J be two R -subgroups of R such that $IJ \subseteq \mu_t$. Then it is easy to see that t_I and t_J are two $(\in, \in \vee q)$ -fuzzy R -subgroups of R and that $t_I \circ t_J \subseteq \mu$. In fact, let $x \in R$. If $(t_I \circ t_J)(x) = 0$, then $(t_I \circ t_J)(x) = 0 \leq \mu(x)$. Otherwise, there exist $a, b \in R$ such that $x = ab$ and $t_I(a) \wedge t_J(b) \neq 0$. This implies $a \in I$ and $b \in J$, hence $x \in IJ \subseteq \mu_t$, that is, $\mu(x) \geq t$. Hence $(t_I \circ t_J)(x) = \bigvee_{x=yz} t_I(y) \wedge t_J(z) \leq t \leq \mu(x)$. Therefore, $t_I \circ t_J \subseteq \mu$.

Since μ is a strong prime $(\in, \in \vee q)$ -fuzzy R -subgroup of R , we have $t_I \subseteq \mu$ or $t_J \subseteq \mu$, this implies $I \subseteq \mu_t$ or $J \subseteq \mu_t$. This completes the proof. ■

The following is a consequence of Theorems 4.3 and 4.5.

Theorem 4.6. *Every strong prime (semiprime) $(\in, \in \vee q)$ -fuzzy R -subgroup of a near-ring is a prime (semiprime) $(\in, \in \vee q)$ -fuzzy R -subgroup, respectively.*

Remark 4.7. The converse of Theorem 4.6 is not true in general as shown in the following example.

Example 4.8. Let $(\mathbb{Z}, +, \cdot)$ be the near-ring (it is also a ring) of all integers. Then $0.4_{(2)}$ is an $(\in, \in \vee q)$ -fuzzy R -subgroup of $\mathbb{Z}$ and non-empty subset $(0.4_{(2)})_t$

is a prime R -subgroup of R for all $t \in (0, 0.5]$. By Theorem 4.3, we know that $0.4_{(2)}$ is a prime $(\in, \in \vee q)$ -fuzzy R -subgroup of $\mathbb{Z}$, but it is not strong prime.

In fact, $0.4_{(3)} \circ 0.5_{(4)} \subseteq 0.4_{(2)}$, but $0.4_{(3)} \not\subseteq 0.4_{(2)}$ and $0.5_{(4)} \not\subseteq 0.4_{(2)}$, where both $0.4_{(3)}$ and $0.5_{(4)}$ are $(\in, \in \vee q)$ -fuzzy R -subgroups of $\mathbb{Z}$.

5. Conclusions

In study the structure of a fuzzy algebraic system, we notice that the (fuzzy) R -subgroups with special properties always play an important role. In this paper, by means of a kind of new idea, we redefine generalized fuzzy R -subgroups of a near-ring and investigate some of its related properties.

We hope that the research along this direction can be continued, and in fact, some results in this paper have already constituted a platform for further discussion concerning the future development of near-rings. In our future study of fuzzy structure of near-rings, may be the following topics should be considered:

- (1) To describe soft near-rings and its applications;
- (2) To establish an $(\in, \in \vee q)$ -fuzzy spectrum of near-rings.

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GENERALIZED FUZZY ALGEBRAIC HYPERSYSTEMS

Jianming Zhan¹

*Department of Mathematics
Hubei University for Nationalities
Enshi, Hubei Province, 445000
P.R. China*

Bijan Davvaz

*Department of Mathematics
Yazd University, Yazd
Iran*

Young Bae Jun

*Department of Mathematics Education
Gyeongsang National University
Chinju 660-701
Korea*

Abstract. The concept of quasi-coincidence of a fuzzy interval value with an interval valued fuzzy set, which is a generalization of quasi-coincidence of a fuzzy point with a fuzzy set, is introduced. Using this new idea, the notion of interval valued (α, β) -fuzzy subalgebraic hypersystems in an algebraic hypersystem, which is a generalization of a fuzzy subalgebraic system, is defined, and related properties are investigated. We also discuss entropy of interval valued (α, β) -fuzzy subalgebraic hypersystems. In particular, the study of interval valued $(\in, \in \vee q)$ -fuzzy subalgebraic hypersystems of an algebraic hypersystem is dealt with. Finally, we consider the concept of implication-based interval valued fuzzy subalgebraic hypersystems.

Keywords: Algebraic hypersystem; interval valued (α, β) -fuzzy subalgebraic hypersystem; interval valued $(\in, \in \vee q)$ -fuzzy subalgebraic hypersystem; entropy; fuzzy logic; implication operator.

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1. Introduction

The study of algebraic hyperstructures (or hypersystems) is a well established branch of classical algebraic theory. Hyperstructure theory was born in 1934 when Marty [19] defined hypergroups, began to analyse their properties and applied them to groups, rational functions and algebraic functions. Later on, people

¹Corresponding author. E-mail address: zhanjianming@hotmail.com(J. Zhan).

have observed that hyperstructures have many applications to several branches of both pure and applied sciences, for example, semihypergroups are the simplest algebraic hyperstructures which possess the properties of closure and associativity. A comprehensive review of the theory of hyperstructures appears in [5], [6], [23].

After the introduction of fuzzy sets by Zadeh [26], reconsideration of the concept of classical mathematics began. On the other hand, because of the importance of group theory in mathematics, as well as its many areas of application, the notion of fuzzy subgroups was defined by Rosenfeld [22] and its structure was investigated. Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces and so on. This provides sufficient motivations to researchers to review various concepts and results from the realm of abstract algebra in the broader framework of fuzzy setting, see [20]. In 1975, Zadeh [27] introduced the concept of interval valued fuzzy subset, where the values of the membership functions are intervals of numbers instead of the numbers. In [4], Biswas defined interval valued fuzzy subgroups of the same nature of Rosenfeld's fuzzy subgroups. A new type of fuzzy subgroup (viz, $(\in, \in \vee q)$ -fuzzy subgroup) was introduced in an earlier paper of Bhakat and Das [3] by using the combined notions of "belongingness" and "quasicoincidence" of fuzzy points and fuzzy sets, which was introduced by Pu and Liu [21]. In fact, $(\in, \in \vee q)$ -fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroups. Recently, Davvaz ([13]) introduced the concept of $(\in, \in \vee q)$ -fuzzy subnear-rings(ideals) and investigated some interesting results.

Fuzzy sets and hyperstructures introduced by Zadeh and Marty, respectively, are now used in the world both on the theoretical point of view and for their applications. The relations between fuzzy sets and algebraic hyperstructures (structures) have been already considered by Corsini, Davvaz, Leoreanu, Ameri, Vougiouklis, Zahedi, Zhan and others, for instance, see [1], [5]-[18], [23]-[25], [28]-[33]. In Section 2, we recall some basic definitions and results about algebraic hypersystems. In Section 3, we discuss entropy of interval valued (α, β) -fuzzy subalgebraic hypersystems in an algebraic hypersystem. Since the concept of interval valued $(\in, \in \vee q)$ -fuzzy subalgebraic hypersystems is an important and useful generalization of ordinary fuzzy algebraic hypersystem in an algebraic hypersystem, some fundamental aspects of interval valued $(\in, \in \vee q)$ -fuzzy subalgebraic hypersystems in an algebraic hypersystem have been discussed in section 4. Finally, in section 5, we consider the concept of implication-based interval valued fuzzy subalgebraic hypersystems.

2. Preliminaries

General algebraic hypersystems and some of their related concepts are introduced in this section. Examples of some familiar algebraic hypersystems are given.

Let H be a non-empty set and f a mapping $f : H \times H \rightarrow \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ is the set of all the non-empty subsets of H . Then f is called a binary

hyperoperation on H . In general, a mapping $f : H^n \rightarrow \mathcal{P}^*(H)$ is called an n -ary hyperoperation and n is called the order of hyperoperation.

A non-empty set and one or more n -ary hyperoperations on the set will be called an algebraic hypersystem. We shall denote an algebraic hypersystem by $\langle H, \Gamma \rangle$, where H is a non-empty set and $\Gamma = \{f_1, f_2, \dots\}$ is a set of hyperoperations on H .

The algebraic hypersystem $(H, \{f_i\}_{i \geq 1})$ induces a universal algebra

$$(\mathcal{P}^*(H), \{F_i\}_{i \geq 1})$$

with the operations

$$F_i(A_1, \dots, A_n) = \cup \{f_i(a_1, \dots, a_n) \mid a_k \in A_k, 1 \leq i \leq n\}$$

for $A_1, \dots, A_n \in \mathcal{P}^*(H)$.

If Γ is a singleton $\Gamma = \{f\}$ and f is a 2-ary hyperoperation, the algebraic hypersystem is called hypergroupoid, the hyperoperation is denoted by $\circ$ and the image of the pair (x, y) is denoted by $x \circ y$. Hypergroups, polygroups and hyperrings are algebraic hypersystems.

Let (H, Γ) be an algebraic hypersystem. A subset S of H is said to be closed under the n -ary hyperoperation f if $(x_1, \dots, x_n) \in S \times S \times \dots \times S$ (where S occurs n times) implies $f(x_1, \dots, x_n) \in \mathcal{P}^*(S)$. S is called a subalgebraic hypersystem of H , if S is closed under any hyperoperation in Γ .

Now, we give the following concept:

Definition 2.1. Let (H, Γ) be an algebraic hypersystem and F a fuzzy subset of H . Then F is called a fuzzy subalgebraic hypersystem of H , if for any n -ary hyperoperation $f \in \Gamma$ and for all $x_i \in H (i = 1, 2, \dots, n)$,

$$(HF1) \quad \inf_{y \in f(x_1, x_2, \dots, x_n)} F(y) \geq \min\{F(x_1), F(x_2), \dots, F(x_n)\}.$$

Example 2.2.

(i) Consider $H = \{e, a, b\}$ and define $\circ$ on H with the following table:

$\circ$	e	a	b
e	$\{e\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{a, b\}$	$\{e, a\}$
b	$\{b\}$	$\{e, a\}$	$\{e\}$

Define a fuzzy set $F : H \rightarrow [0, 1]$ by $F(a) \leq F(b) \leq F(e)$. Then F is a fuzzy subalgebraic hypersystem of H .

(ii) Let $(H = \{1, -1, i, -i\}, \circ)$, where $\circ$ defined on H with the following table:

$\circ$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

Note that every algebraic system is an algebraic hypersystem. Define $F : H \rightarrow [0, 1]$ by $F(1)=1, F(-1) = 0.5$ and $F(i) = F(-i) = 0$. Clearly F is a fuzzy subalgebraic system of H .

(iii) Let $(H = \{x, y, z, t\}, \{f_1, f_2\})$ be an algebraic hypersystem, where

f_1	x	y	z	t	f_2	x	y	z	t
x	$\{x\}$	$\{y\}$	$\{z\}$	$\{t\}$	x	x	y	z	t
y	$\{y\}$	$\{x, y\}$	$\{z\}$	$\{t\}$	y	x	y	z	t
z	$\{z\}$	$\{z\}$	$\{x, y, t\}$	$\{z, t\}$	z	x	y	z	t
t	$\{t\}$	$\{t\}$	$\{z, t\}$	$\{x, y, z\}$	t	x	y	z	t

We define a fuzzy set $F : H \rightarrow [0, 1]$ by $F(x) = 0.7, F(y) = 0.5$ and $F(z) = F(t) = 0.3$. Routine calculations give that F is a fuzzy subalgebraic hypersystem of H .

Let F be a fuzzy set. For every $t \in [0, 1]$, the set $U(F; t) = \{x \in H | F(x) \geq t\}$ is called the level subset of F .

Theorem 2.3. *Let H be an algebraic hypersystem and F a fuzzy set of H . Then F is a fuzzy subalgebraic hypersystem of H if and only if for any $t \in [0, 1]$, $U(F; t) (\neq \emptyset)$ is a subalgebraic hypersystem of H .*

By an interval number $\tilde{a}$ we mean (cf. [27]) an interval $[a^-, a^+]$, where $0 \leq a^- \leq a^+ \leq 1$. The set of all interval numbers is denoted by $D[0, 1]$. The interval $[a, a]$ is identified with the number $a \in [0, 1]$.

For interval numbers $\tilde{a}_i = [a_i^-, a_i^+], \tilde{b}_i = [b_i^-, b_i^+] \in D[0, 1], i \in I$, we define

$$\begin{aligned} \text{rmax}\{\tilde{a}_i, \tilde{b}_i\} &= [\max(a_i^-, b_i^-), \max(a_i^+, b_i^+)], \\ \text{rmin}\{\tilde{a}_i, \tilde{b}_i\} &= [\min(a_i^-, b_i^-), \min(a_i^+, b_i^+)], \\ \text{rinf}\tilde{a}_i &= \left[\bigwedge_{i \in I} a_i^-, \bigwedge_{i \in I} a_i^+ \right], \quad \text{rsup}\tilde{a}_i = \left[\bigvee_{i \in I} a_i^-, \bigvee_{i \in I} a_i^+ \right] \end{aligned}$$

and put

- (1) $\tilde{a}_1 \leq \tilde{a}_2 \iff a_1^- \leq a_2^- \text{ and } a_1^+ \leq a_2^+,$
- (2) $\tilde{a}_1 = \tilde{a}_2 \iff a_1^- = a_2^- \text{ and } a_1^+ = a_2^+,$
- (3) $\tilde{a}_1 < \tilde{a}_2 \iff \tilde{a}_1 \leq \tilde{a}_2 \text{ and } \tilde{a}_1 \neq \tilde{a}_2,$
- (4) $k\tilde{a} = [ka^-, ka^+], \text{ whenever } 0 \leq k \leq 1.$

It is clear that $(D[0, 1], \leq, \vee, \wedge)$ is a complete lattice with $0 = [0, 0]$ as the least element and $1 = [1, 1]$ as the greatest element.

Interval valued fuzzy sets provide a more adequate description of uncertainty than traditional fuzzy sets; it is therefore important to use interval valued fuzzy

sets in applications. One of main applications of fuzzy sets is fuzzy control, and one of the most computationally intensive part of fuzzy control is defuzzification. Since a transition to interval valued fuzzy sets usually increase the amount of computations, it is vitally important to design faster algorithms for the corresponding defuzzification.

By an interval valued fuzzy set F on X we mean the set

$$F = \{(x, [\mu_F^-(x), \mu_F^+(x)]) \mid x \in X\},$$

where μ_F^- and μ_F^+ are two fuzzy subsets of X such that $\mu_F^-(x) \leq \mu_F^+(x)$ for all $x \in X$. Putting $\widetilde{\mu}_F(x) = [\mu_F^-(x), \mu_F^+(x)]$, we see that $F = \{(x, \widetilde{\mu}_F(x)) \mid x \in X\}$, where $\widetilde{\mu}_F : X \rightarrow D[0, 1]$.

If A, B are two interval valued fuzzy sets of X , then we define

$A \subseteq B$ if and only if for all $x \in X$, $\mu_A^-(x) \leq \mu_B^-(x)$ and $\mu_A^+(x) \leq \mu_B^+(x)$,

$A = B$ if and only if for all $x \in X$, $\mu_A^-(x) = \mu_B^-(x)$ and $\mu_A^+(x) = \mu_B^+(x)$.

Also, the union, intersection and complement are defined as follows: let A, B be two interval valued fuzzy sets of X , then

$$A \cup B = \{(x, [\max\{\mu_A^-(x), \mu_B^-(x)\}, \max\{\mu_A^+(x), \mu_B^+(x)\}]) \mid x \in X\},$$

$$A \cap B = \{(x, [\min\{\mu_A^-(x), \mu_B^-(x)\}, \min\{\mu_A^+(x), \mu_B^+(x)\}]) \mid x \in X\},$$

$$A^c = \{(x, [1 - \mu_A^+(x), 1 - \mu_A^-(x)]) \mid x \in X\},$$

where the operation “ c ” is the complement of interval valued fuzzy set in X .

3. Entropy of interval valued (α, β) -fuzzy subalgebraic hypersystems

Based on [2], [3], we can extend the concept of quasi-coincidence of fuzzy point with a fuzzy set to the concept of quasi-coincidence of a fuzzy interval value with an interval valued fuzzy set as follows.

An interval valued fuzzy set $F = \{(x, \widetilde{\mu}_F(x)) \mid x \in H\}$ of an algebraic hypersystem H of the form

$$\widetilde{\mu}_F(y) = \begin{cases} \widetilde{t} (\neq [0, 0]) & \text{if } y = x \\ [0, 0] & \text{if } y \neq x \end{cases}$$

is said to be fuzzy interval value with support x and interval value $\widetilde{t}$ and is denoted by $U(x; \widetilde{t})$. A fuzzy interval value $U(x; \widetilde{t})$ is said to belong to (resp. be quasi-coincident with) an interval valued fuzzy set F , written as $U(x; \widetilde{t}) \in F$ (resp. $U(x; \widetilde{t}) q F$) if $\widetilde{\mu}_F(x) \geq \widetilde{t}$ (resp. $\widetilde{\mu}_F(x) + \widetilde{t} > [1, 1]$). If $U(x; \widetilde{t}) \in F$ or (resp. and) $U(x; \widetilde{t}) q F$, then we write $U(x; \widetilde{t}) \in \vee q F$ (resp. $\in \wedge q F$). The symbol $\overline{\in \vee q}$ means $\in \vee q$ does not hold.

In what follows, let (H, Γ) be an algebraic hypersystem, $f \in \Gamma$ any n -ary hyperoperation on H , and α and β will denote any one of $\in, q, \in \vee q$ or $\in \wedge q$ unless otherwise specified. Also we emphasis $\widetilde{\mu}_F(x) = [\mu_F^-(x), \mu_F^+(x)]$ must satisfy the following properties:

$$[\mu_F^-(x), \mu_F^+(x)] < [0.5, 0.5] \text{ or } [0.5, 0.5] \leq [\mu_F^-(x), \mu_F^+(x)], \text{ for all } x \in H.$$

Definition 3.1. An interval valued fuzzy set F of H is called an interval valued (α, β) -fuzzy subalgebraic hypersystem of H with $\alpha \neq \in \wedge q$, if it satisfies for all $t_i \in (0, 1]$ and $x_i \in H (i = 1, 2, \dots, n)$,

(HF2) $U(x_i; \widetilde{t}_i)\alpha F$ implies $U(y; \text{rmin}\{\widetilde{t}_1, \widetilde{t}_2, \dots, \widetilde{t}_n\})\beta F$, for all $y \in f(x_1, x_2, \dots, x_n)$.

Let F be an interval valued fuzzy set of H such that $\widetilde{\mu}_F(x) \leq [0.5, 0.5]$ for all $x \in H$. Let $x \in H$ and $t \in (0, 1]$ be such that $U(x; \widetilde{t}) \in \wedge q F$, then $\widetilde{\mu}_F(x) \geq \widetilde{t}$ and $\widetilde{\mu}_F(x) + \widetilde{t} > [1, 1]$. It follows that $[1, 1] < \widetilde{\mu}_F(x) + \widetilde{t} \leq \widetilde{\mu}_F(x) + \widetilde{\mu}_F(x) = 2\widetilde{\mu}_F(x)$, which implies $\widetilde{\mu}_F(x) > [0.5, 0.5]$. This means that $\{U(x; \widetilde{t}) | U(x; \widetilde{t}) \in \wedge q F\} = \emptyset$. Therefore, the case $\alpha = \in \wedge q$ in Definition 3.1 will be omitted.

Proposition 3.2.

- (i) Every interval valued $(\in \vee q, \in \vee q)$ -fuzzy subalgebraic hypersystem of H is an interval valued $(\in, \in \vee q)$ -fuzzy subalgebraic hypersystem of H ;
- (ii) Every interval valued $(\in, \in)$ -fuzzy subalgebraic hypersystem of H is an interval valued $(\in, \in \vee q)$ -fuzzy subalgebraic hypersystem of H .

The converse of Propositions 3.2 is not true. Consider Klein's 4-group $H = \{e, x, y, z\}$. Defined an interval valued fuzzy set F of H by

$$\widetilde{\mu}_F(e) = [0.6, 0.7], \quad \widetilde{\mu}_F(x) = [0.7, 0.8] \quad \text{and} \quad \widetilde{\mu}_F(y) = \widetilde{\mu}_F(z) = [0.4, 0.5].$$

Then, F is an interval valued $(\in, \in \vee q)$ -fuzzy subalgebraic hypersystem of H . We note that F is not an interval valued (α, β) -fuzzy algebraic hypersystem of H , where $(\alpha, \beta) \neq (\in, \in), (q, \in \vee q), (\in \vee q, \in \vee q)$.

Lemma 3.3.

- (i) If A is a subalgebraic hypersystem of H , then the characteristic function χ_A of A is an interval valued $(\in, \in)$ -fuzzy subalgebraic hypersystem of H ;
- (ii) For any subset A of H , χ_A is an interval valued $(\in, \in \vee q)$ -fuzzy subalgebraic hypersystem of H if and only if A is a subalgebraic hypersystem of H .

Now, we give the main result on a general interval valued (α, β) -fuzzy subalgebraic hypersystem of algebraic hypersystems.

Theorem 3.4. Let F be a non-zero interval valued (α, β) -fuzzy subalgebraic hypersystem of H . Then the set $U(F; [0, 0]) = \{x \in H | \widetilde{\mu}_F(x) > [0, 0]\}$ is a subalgebraic hypersystem of H .

Proof. Let $x_i \in U(F; [0, 0])$, for all $i = 1, 2, \dots, n$, then $\widetilde{\mu}_F(x_i) > [0, 0]$, $i = 1, 2, \dots, n$. Assume that $\widetilde{\mu}_F(y) = [0, 0]$, for some $y \in f(x_1, x_2, \dots, x_n)$. If $\alpha \in \{\in, \in \vee q\}$, then $U(x_i; \widetilde{\mu}_F(x_i))\alpha F$, for some $i = 1, 2, \dots, n$, but $U(y; \text{rmin}\{\widetilde{\mu}_F(x_1), \dots, \widetilde{\mu}_F(x_n)\})\beta F$, for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Note that $U(x_i; [1, 1])q F$ ($i = 1, 2, \dots, n$), but, for all $y \in f(x_1, x_2, \dots, x_n)$, $U(y; \text{rmin}\{[1, 1], [1, 1], \dots, [1, 1]\}) = U(y; [1, 1])\beta F$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, this is a contradiction. Hence,

for all $y \in f(x_1, x_2, \dots, x_n), \widetilde{\mu_F}(y) > [0, 0]$, that is, $y \in U(F; [0, 0])$, and so $f(x_1, x_2, \dots, x_n) \subseteq U(F; [0, 0])$. This completes the proof. ■

Entropy and similarity measure of fuzzy sets are two important topics in fuzzy set theory. Entropy of a fuzzy set describes the fuzziness degree of fuzzy set. Many scholars have studied it from different points of view. We denote IVFSs for the sets of all interval valued (α, β) -fuzzy subalgebraic hypersystems of H .

A real function $E : IVFSs \rightarrow [0, 1]$ is called an entropy on IVFSs, if E satisfies the following properties:

- (1) $E(A) = 0$ if and only if A is a crisp subalgebraic hypersystem,
- (2) $E(A) = 1$ if and only if $\mu_A^-(x) + \mu_A^+(x) = 1$,
- (3) $E(A) \leq E(B)$ if A is less fuzzy than B ,
- (4) $E(A) = E(A^c)$.

For $M = \{x_1, \dots, x_n\}$, we can give the following formulas to calculate the entropy of interval valued (α, β) -fuzzy subalgebraic hypersystem A :

$$E_1(A) = 1 - \frac{1}{n} \sum_{i=1}^n |\mu_A^-(x_i) + \mu_A^+(x_i) - 1|,$$

$$E_2(A) = 1 - \sqrt{\frac{1}{n} \sum_{i=1}^n (\mu_A^-(x_i) + \mu_A^+(x_i) - 1)^2}.$$

A real function $\varphi; IVFSs \times IVFSs \rightarrow [0, 1]$ is called similarity measure of interval valued (α, β) -fuzzy subalgebraic hypersystems, if φ satisfies the following properties:

- (1) $\varphi(A, A^c) = 0$ if A is a crisp subalgebraic hypersystem,
- (2) $\varphi(A, B) = 1 \Leftrightarrow A = B$,
- (3) $\varphi(A, B) = \varphi(B, A)$,
- (4) if $A \subseteq B \subseteq C$, then $\varphi(A, C) \leq \varphi(A, B)$ and $\varphi(A, C) \leq \varphi(B, C)$.

Based on the point of view, then we have the following statement:

Theorem 3.5.

- (i) $\varphi(A^-, (A^+)^c)$ is entropy of interval valued (α, β) -fuzzy subalgebraic hypersystem A ;
- (ii) Suppose φ be similarity measure of interval valued (α, β) -fuzzy subalgebraic hypersystems and $A \in IVFSs$, then $\varphi(A, A^c)$ is entropy of interval valued (α, β) -fuzzy subalgebraic hypersystem A .

4. Interval valued $(\in, \in \vee q)$ -fuzzy subalgebraic hypersystems

In this section, we mainly discuss some fundamental aspects of interval valued $(\in, \in \vee q)$ -fuzzy subalgebraic hypersystems of an algebraic hypersystem H . First, we can extend the concept of fuzzy subalgebraic hypersystems to the concept of interval valued fuzzy subalgebraic hypersystems in an algebraic hypersystem as follows:

Definition 4.1. An interval valued fuzzy set F of H is said to be an interval valued fuzzy subalgebraic hypersystem of H , if for all $x_i \in H$ ($i = 1, 2, \dots, n$),

$$(HF3) \quad \text{rinf}_{y \in f(x_1, x_2, \dots, x_n)} \widetilde{\mu}_F(y) \geq \text{rmin}\{\widetilde{\mu}_F(x_1), \widetilde{\mu}_F(x_2), \dots, \widetilde{\mu}_F(x_n)\}.$$

Example 4.2. Consider Example 2.2 (ii). Define an interval valued fuzzy set F by $\widetilde{\mu}_F(1) = [1, 1]$, $\widetilde{\mu}_F(-1) = [0.5, 0.6]$ and $\widetilde{\mu}_F(i) = \widetilde{\mu}_F(-i) = [0, 0]$. Then F is an interval valued fuzzy subalgebraic hypersystem of H .

Let F be an interval valued fuzzy set. For every $t \in [0, 1]$, the set $U(F; \widetilde{t}) = \{x \in H \mid \widetilde{\mu}_F(x) \geq \widetilde{t}\}$ is called the interval valued level subset of F . Now, we characterize interval valued fuzzy subalgebraic hypersystems by their level subalgebraic hypersystems.

Theorem 4.3. An interval valued fuzzy set F of H is an interval valued fuzzy subalgebraic hypersystem of H if and only if for any $[0, 0] < \widetilde{t} \leq [1, 1]$, $U(F; \widetilde{t}) (\neq \emptyset)$ is a subalgebraic hypersystem of H .

Proof. For every $x_1, x_2, \dots, x_n \in U(F; \widetilde{t})$, we have $\text{rmin}\{\widetilde{\mu}_F(x_1), \dots, \widetilde{\mu}_F(x_n)\} \geq \widetilde{t}$, and so $\text{rinf}\{\widetilde{\mu}_F(y) \mid y \in f(x_1, x_2, \dots, x_n)\} \geq \widetilde{t}$. Therefore, for every $y \in f(x_1, x_2, \dots, x_n)$, we have $y \in U(F; \widetilde{t})$, so $f(x_1, x_2, \dots, x_n) \subseteq U(F; \widetilde{t})$.

Conversely, assume that for every $[0, 0] < \widetilde{t} \leq [1, 1]$, $U(F; \widetilde{t}) (\neq \emptyset)$ is a subalgebraic hypersystem of H . For every $x_1, x_2, \dots, x_n \in H$, we have $\widetilde{\mu}_F(x_i) \geq \text{rmin}\{\widetilde{\mu}_F(x_1), \dots, \widetilde{\mu}_F(x_n)\} = \widetilde{t}_0$. Then, $x_i \in U(F; \widetilde{t}_0)$, for $i = 1, 2, \dots, n$. So, $f(x_1, x_2, \dots, x_n) \subseteq U(F; \widetilde{t}_0)$. Therefore, for every $y \in f(x_1, x_2, \dots, x_n)$, we have $\widetilde{\mu}_F(y) \geq \widetilde{t}_0$ implying

$$\text{rinf}\{\widetilde{\mu}_F(y) \mid y \in f(x_1, x_2, \dots, x_n)\} \geq \text{rmin}\{\widetilde{\mu}_F(x_1), \dots, \widetilde{\mu}_F(x_n)\} \geq \widetilde{t} \quad \blacksquare$$

Further, we define the following concept:

Definition 4.4. An interval valued fuzzy set F of H is said to be an interval valued $(\in, \in \vee q)$ -fuzzy subalgebraic hypersystem of H if for all $t_i \in (0, 1]$ and $x_i \in H$ ($i = 1, 2, \dots, n$),

$$(HF4) \quad U(x_i; \widetilde{t}_i) \in F \text{ implies } U(y; \text{rmin}\{\widetilde{t}_1, \widetilde{t}_2, \dots, \widetilde{t}_n\}) \in \vee q F, \text{ for all } y \in f(x_1, x_2, \dots, x_n).$$

Note that if F is an interval valued fuzzy subalgebraic hypersystem of H according to Definition 4.1, then F is an interval valued $(\in, \in \vee q)$ -fuzzy subalgebraic hypersystem of H according to Definition 4.4. But the converse is not true shown by the following example:

Example 4.5.

(i) Let $H = \{e, a, b\}$ be an algebraic hypersystem defined by the following table:

$\circ$	e	a	b
e	$\{e\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{e, b\}$	$\{a, b\}$
b	$\{b\}$	$\{a, b\}$	$\{e, a\}$

Define an interval valued fuzzy set F by $\widetilde{\mu}_F(e) = [0.8, 0.9]$, $\widetilde{\mu}_F(a) = [0.7, 0.8]$ and $\widetilde{\mu}_F(b) = [0.6, 0.7]$. Then it is easy to see that F is an interval valued $(\in, \in \vee q)$ -fuzzy subalgebraic hypersystem of H , but is not an interval valued fuzzy subalgebraic hypersystem of H .

(ii) Consider Example 2.2 (ii), and define

$$[0.4, 0.5] \leq \widetilde{\mu}_F(i) = \widetilde{\mu}_F(-i) \leq \widetilde{\mu}_F(-1) = \widetilde{\mu}_F(1).$$

Then, F not only is an interval valued $(\in, \in \vee q)$ -fuzzy subalgebraic system of H , but also is an interval valued fuzzy subalgebraic system of H .

(iii) Consider Example 2.2 (iii), define an interval valued fuzzy set F by $\widetilde{\mu}_F(x) = [0.7, 0.8]$, $\widetilde{\mu}_F(y) = [0.5, 0.6]$ and $\widetilde{\mu}_F(z) = \widetilde{\mu}_F(t) = [0.3, 0.4]$, then F is an interval valued $(\in, \in \vee q)$ -fuzzy subalgebraic system of H .

Theorem 4.6. *An interval valued fuzzy set F of H is an interval valued $(\in, \in \vee q)$ -fuzzy subalgebraic hypersystem of H if and only if for all $x_i \in H (i = 1, 2, \dots, n)$,*

$$(HF5) \quad \text{rmin}\{\widetilde{\mu}_F(x_1), \widetilde{\mu}_F(x_2), \dots, \widetilde{\mu}_F(x_n), [0.5, 0.5]\} \leq \text{rinf}_{y \in f(x_1, x_2, \dots, x_n)} \widetilde{\mu}_F(y).$$

Proof. Assume that F is an interval valued $(\in, \in \vee q)$ -fuzzy subalgebraic hypersystem of H . Let $x_i \in H (i = 1, 2, \dots, n)$, we consider the following cases:

$$(i) \quad \text{rmin}\{\widetilde{\mu}_F(x_1), \widetilde{\mu}_F(x_2), \dots, \widetilde{\mu}_F(x_n)\} < [0.5, 0.5],$$

$$(ii) \quad \text{rmin}\{\widetilde{\mu}_F(x_1), \widetilde{\mu}_F(x_2), \dots, \widetilde{\mu}_F(x_n)\} \geq [0.5, 0.5].$$

Case (i): Assume that there exists $y \in f(x_1, x_2, \dots, x_n)$ such that

$$\widetilde{\mu}_F(y) < \text{rmin}\{\widetilde{\mu}_F(x_1), \widetilde{\mu}_F(x_2), \dots, \widetilde{\mu}_F(x_n), [0.5, 0.5]\},$$

which implies $\widetilde{\mu}_F(y) < \text{rmin}\{\widetilde{\mu}_F(x_1), \dots, \widetilde{\mu}_F(x_n)\}$. Choose t such that

$$\widetilde{\mu}_F(y) < \widetilde{t} < \text{rmin}\{\widetilde{\mu}_F(x_1), \widetilde{\mu}_F(x_2), \dots, \widetilde{\mu}_F(x_n)\}.$$

Then, $U(x_i; \widetilde{t}) \in F$, but $U(y; \widetilde{t}) \notin \overline{\vee q}F$, which contradicts (HF4).

Case (ii): Assume that $\widetilde{\mu}_F(y) < [0.5, 0.5]$ for some $y \in f(x_1, x_2, \dots, x_n)$. Then, $U(x_i; [0.5, 0.5]) \in F, i = 1, 2, \dots, n$. But $U(y; [0.5, 0.5]) \notin \forall q F$, a contradiction. Hence (HF5) holds.

Conversely, let $U(x_i; \widetilde{t}_i) \in F (i = 1, 2, \dots, n)$, then $\widetilde{\mu}_F(x_i) \geq \widetilde{t}_i (i = 1, 2, \dots, n)$. For any $y \in f(x_1, x_2, \dots, x_n)$, we have

$$\begin{aligned} \widetilde{\mu}_F(y) &\geq \text{rmin}\{\widetilde{\mu}_F(x_1), \widetilde{\mu}_F(x_2), \dots, \widetilde{\mu}_F(x_n), [0.5, 0.5]\} \\ &\geq \text{rmin}\{\widetilde{t}_1, \widetilde{t}_2, \dots, \widetilde{t}_n, [0.5, 0.5]\}. \end{aligned}$$

If $\text{rmin}\{\widetilde{t}_1, \widetilde{t}_2, \dots, \widetilde{t}_n\} > [0.5, 0.5]$, then $\widetilde{\mu}_F(y) \geq [0.5, 0.5]$, which implies $\widetilde{\mu}_F(y) + \text{rmin}\{\widetilde{t}_1, \widetilde{t}_2, \dots, \widetilde{t}_n\} > [1, 1]$.

If $\text{rmin}\{\widetilde{t}_1, \widetilde{t}_2, \dots, \widetilde{t}_n\} \leq [0.5, 0.5]$, then $\widetilde{\mu}_F(y) \geq \text{rmin}\{\widetilde{t}_1, \widetilde{t}_2, \dots, \widetilde{t}_n\}$. Therefore, $U(y; \text{rmin}\{\widetilde{t}_1, \widetilde{t}_2, \dots, \widetilde{t}_n\}) \in \forall q F$ for all $y \in f(x_1, x_2, \dots, x_n)$. This completes the proof. ■

Theorem 4.7. *Let F be an interval valued $(\in, \in \forall q)$ -fuzzy subalgebraic hypersystem of H . Then for all $[0, 0] < \widetilde{t} \leq [0.5, 0.5]$, $U(F; \widetilde{t})$ is an empty set or a subalgebraic hypersystem of H . Conversely, if F is an interval valued fuzzy set of H such that $U(F; \widetilde{t}) (\neq \emptyset)$ is a subalgebraic hypersystem of H for all $[0, 0] < \widetilde{t} \leq [0.5, 0.5]$, then F is an interval valued $(\in, \in \forall q)$ -fuzzy subalgebraic hypersystem of H .*

Proof. Let F be an interval valued $(\in, \in \forall q)$ -fuzzy subalgebraic hypersystem of H and $[0, 0] < \widetilde{t} \leq [0.5, 0.5]$. Let $x_i \in U(F; \widetilde{t}), i = 1, 2, \dots, n$, then $\widetilde{\mu}_F(x_i) \geq \widetilde{t}$. Now,

$$\begin{aligned} \text{rinf}\{\widetilde{\mu}_F(y) | y \in f(x_1, x_2, \dots, x_n)\} &\geq \text{rmin}\{\widetilde{\mu}_F(x_1), \widetilde{\mu}_F(x_2), \dots, \widetilde{\mu}_F(x_n), [0.5, 0.5]\} \\ &\geq \text{rmin}\{\widetilde{t}, [0.5, 0.5]\} = \widetilde{t}. \end{aligned}$$

Therefore, for every $y \in f(x_1, x_2, \dots, x_n)$, we have $\widetilde{\mu}_F(y) \geq \widetilde{t}$, and so $y \in U(F; \widetilde{t})$, which implies, $f(x_1, x_2, \dots, x_n) \subseteq U(F; \widetilde{t})$. Therefore $U(F; \widetilde{t})$ is a subalgebraic hypersystem of H .

Conversely, let F be an interval valued fuzzy set of H such that $U(F; \widetilde{t}) (\neq \emptyset)$ is a subalgebraic hypersystem of H for all $[0, 0] < \widetilde{t} \leq [0.5, 0.5]$. For every $x_i \in H (i = 1, 2, \dots, n)$, we can write

$$\widetilde{\mu}_F(x_i) \geq \text{rmin}\{\widetilde{\mu}_F(x_1), \widetilde{\mu}_F(x_2), \dots, \widetilde{\mu}_F(x_n), [0.5, 0.5]\} = \widetilde{t}_0, \quad i = 1, 2, \dots, n,$$

then $x_i \in U(F; \widetilde{t}_0)$, and so $f(x_1, x_2, \dots, x_n) \subseteq U(F; \widetilde{t}_0)$. Therefore, for every $y \in f(x_1, x_2, \dots, x_n)$, we have $\widetilde{\mu}_F(y) \geq \widetilde{t}_0$, which implies $\widetilde{\mu}_F(y) \geq \widetilde{t}_0$ for all $y \in f(x_1, x_2, \dots, x_n)$. Therefore, F is an interval valued $(\in, \in \forall q)$ -fuzzy subalgebraic hypersystem of H . ■

Naturally, a corresponding result should be considered when $U(F; \widetilde{t})$ is a subalgebraic hypersystem of H for all $[0.5, 0.5] < \widetilde{t} \leq [1, 1]$.

Theorem 4.8. *Let F be an interval valued fuzzy set of H . Then $U(F; \widetilde{t}) (\neq \emptyset)$ is a subalgebraic hypersystem of H for all $[0.5, 0.5] < \widetilde{t} \leq [1, 1]$ if and only if*

$$\begin{aligned}
(\text{HF6}) \quad & \text{rmin}\{\widetilde{\mu}_F(x_1), \widetilde{\mu}_F(x_2), \dots, \widetilde{\mu}_F(x_n)\} \\
& \leq \text{rinf}\{\text{rmax}\{\widetilde{\mu}_F(y), [0.5, 0.5]\} | y \in f(x_1, x_2, \dots, x_n)\}, \\
& \text{for all } x_i \in H, \ i = 1, 2, \dots, n.
\end{aligned}$$

Proof. Assume that $U(F; \tilde{t})$ is a subalgebraic hypersystem of H . If there exist $x_i, y \in H$ ($i = 1, 2, \dots, n$) with $y \in f(x_1, x_2, \dots, x_n)$ such that

$$\text{rmax}\{\widetilde{\mu}_F(y), [0.5, 0.5]\} < \text{rmin}\{\widetilde{\mu}_F(x_1), \widetilde{\mu}_F(x_2), \dots, \widetilde{\mu}_F(x_n)\} = \tilde{t},$$

then $[0.5, 0.5] < \tilde{t} \leq [1, 1]$, $\widetilde{\mu}_F(y) < \tilde{t}$, $x_i \in U(F; \tilde{t})$ ($i = 1, 2, \dots, n$). Since $x_i \in U(F; \tilde{t})$ and $U(F; \tilde{t})$ is a subalgebraic hypersystem, so $f(x_1, x_2, \dots, x_n) \subseteq U(F; \tilde{t})$ and $\widetilde{\mu}_F(y) \geq \tilde{t}$ for all $y \in f(x_1, x_2, \dots, x_n)$, which is a contradiction with $\widetilde{\mu}_F(y) < \tilde{t}$. Therefore,

$$\text{rmin}\{\widetilde{\mu}_F(x_1), \widetilde{\mu}_F(x_2), \dots, \widetilde{\mu}_F(x_n)\} \leq \text{rmax}\{\widetilde{\mu}_F(y), [0.5, 0.5]\},$$

for all $x_i, y \in H$ ($i = 1, 2, \dots, n$) with $y \in f(x_1, x_2, \dots, x_n)$, which implies,

$$\text{rmin}\{\widetilde{\mu}_F(x_1), \widetilde{\mu}_F(x_2), \dots, \widetilde{\mu}_F(x_n)\} \leq \text{rinf}\{\text{rmax}\{\widetilde{\mu}_F(y), [0.5, 0.5]\} | y \in f(x_1, x_2, \dots, x_n)\},$$

for all $x_i \in H$. Hence (HF6) holds.

Conversely, assume that $[0.5, 0.5], [1, 1]$ and $x_i \in U(F; \tilde{t})$ ($i = 1, 2, \dots, n$). Then,

$$\begin{aligned}
[0.5, 0.5] < \tilde{t} & \leq \text{rmin}\{\widetilde{\mu}_F(x_1), \widetilde{\mu}_F(x_2), \dots, \widetilde{\mu}_F(x_n)\} \\
& \leq \text{rinf}\{\text{rmax}\{\widetilde{\mu}_F(y), [0.5, 0.5]\} | y \in f(x_1, x_2, \dots, x_n)\}.
\end{aligned}$$

It follows that, for every $y \in f(x_1, x_2, \dots, x_n)$,

$$[0.5, 0.5] < \tilde{t} \leq \text{rmax}\{\widetilde{\mu}_F(y), [0.5, 0.5]\},$$

and so $\tilde{t} \leq \widetilde{\mu}_F(y)$, which implies $y \in U(F; \tilde{t})$. Hence $f(x_1, x_2, \dots, x_n) \subseteq U(F; \tilde{t})$, that is, $U(F; \tilde{t})$ is a subalgebraic hypersystem of H . ■

Let F be an interval valued fuzzy set of an algebraic hypersystem H and $J = \{\alpha | \alpha \in (0, 1] \text{ and } U(F; [0.5, 0.5]) \text{ is an empty set or a subalgebraic hypersystem of } H\}$. In particular, if $J = (0, 1]$, then F is an ordinary interval valued fuzzy subalgebraic hypersystem of H (Theorem 4.3); if $J = (0, 0.5)$, F is an interval valued $(\in, \in \vee q)$ -fuzzy subalgebraic hypersystem of H (Theorem 4.7).

In [25], Yuan et al. gave the definition of a fuzzy subgroup with thresholds which is a generalization of Rosenfeld's fuzzy subgroup, and Bhkat and Das's fuzzy subgroup. Based on [25], we can extend the concept of a fuzzy subgroup with thresholds to the concept of fuzzy subalgebraic hypersystems with thresholds in the following way:

Definition 4.9. Let $s, t \in [0, 1]$ and $\tilde{s} < \tilde{t}$, then an interval valued fuzzy set F of H is called an interval valued fuzzy subalgebraic hypersystem with thresholds $(\tilde{s}, \tilde{t})$ of H if it satisfies:

$$\begin{aligned}
(\text{HF7}) \quad & \text{rmin}\{\widetilde{\mu}_F(x_1), \widetilde{\mu}_F(x_2), \dots, \widetilde{\mu}_F(x_n), \widetilde{t}\} \\
& \leq \inf\{\text{rmax}\{\widetilde{\mu}_F(y), \widetilde{s}\} | y \in f(x_1, x_2, \dots, x_n)\},
\end{aligned}$$

for all $x_i \in H$, $i = 1, 2, \dots, n$.

Remark. If F is an interval valued fuzzy subalgebraic hypersystem with thresholds of H , then we can conclude that F is an ordinary interval valued fuzzy subalgebraic hypersystem when $\widetilde{s} = [0, 0]$, $\widetilde{t} = [1, 1]$; and F is an interval valued $(\in, \in \vee q)$ -fuzzy subalgebraic hypersystem when $\widetilde{s} = [0, 0]$, $\widetilde{t} = [0.5, 0.5]$.

Now, we characterize interval valued fuzzy subalgebraic hypersystems with thresholds by their level subalgebraic hypersystems.

Theorem 4.10. *An interval valued fuzzy set F of H is an interval valued fuzzy subalgebraic hypersystem with thresholds $(\widetilde{s}, \widetilde{t})$ of H if and only if $U(F; \widetilde{\alpha})$ ($\neq \emptyset$) is a subalgebraic hypersystem of H for all $\widetilde{s} \leq \widetilde{\alpha} \leq \widetilde{t}$.*

Proof. Let F be an interval valued fuzzy subalgebraic hypersystem with thresholds $(\widetilde{s}, \widetilde{t})$ of H and $\widetilde{s} \leq \widetilde{\alpha} \leq \widetilde{t}$. Let $x_i \in U(F; \widetilde{\alpha})$, then $\widetilde{\mu}_F(x_i) \geq \widetilde{\alpha}$, $i = 1, 2, \dots, n$. Now

$$\begin{aligned}
& \text{rmin}\{\widetilde{\mu}_F(x_1), \widetilde{\mu}_F(x_2), \dots, \widetilde{\mu}_F(x_n), \widetilde{t}\} \\
& \leq \text{rinf}\{\text{rmax}\{\widetilde{\mu}_F(y), \widetilde{s}\} | y \in f(x_1, x_2, \dots, x_n)\} \\
& \geq \text{rmin}\{\widetilde{\alpha}, \widetilde{t}\} \geq \widetilde{\alpha} > \widetilde{s}.
\end{aligned}$$

So, for every $\alpha \in f(x_1, x_2, \dots, x_n)$, we have $\text{rmax}\{\widetilde{\mu}_F(y), \widetilde{s}\} > \widetilde{\alpha} > \widetilde{s}$, which implies $\widetilde{\mu}_F(y) > \widetilde{\alpha}$, and so $y \in U(F; \widetilde{\alpha})$. Hence $f(x_1, x_2, \dots, x_n) \subseteq U(F; \widetilde{\alpha})$. Therefore, $U(F; \widetilde{\alpha})$ is a subalgebraic hypersystem of H for all $\widetilde{\alpha} \in (\widetilde{s}, \widetilde{t}]$.

Conversely, let F be an interval valued fuzzy set of H such that $U(F; \widetilde{\alpha})$ ($\neq \emptyset$) is a subalgebraic hypersystem of H for all $\widetilde{s} \leq \widetilde{\alpha} \leq \widetilde{t}$. If there exist x_i ($i = 1, 2, \dots, n$), $y \in H$ with $y \in f(x_1, x_2, \dots, x_n)$ such that $\text{rmax}\{\widetilde{\mu}_F(y), \widetilde{s}\} < \text{rmin}\{\widetilde{\mu}_F(x_1), \dots, \widetilde{\mu}_F(x_n), \widetilde{t}\} = \widetilde{\alpha}$, then $\widetilde{\alpha} \in (\widetilde{s}, \widetilde{t}]$, $\widetilde{\mu}_F(y) < \widetilde{\alpha}$, $x_i \in U(F; \widetilde{\alpha})$, $i = 1, 2, \dots, n$. Since $U(F; \widetilde{\alpha})$ is a subalgebraic hypersystem of H and $x_i \in U(F; \widetilde{\alpha})$ for all $y \in f(x_1, x_2, \dots, x_n)$, so $f(x_1, x_2, \dots, x_n) \subseteq U(F; \widetilde{\alpha})$. Hence $\widetilde{\mu}_F(y) \geq \widetilde{\alpha}$, for all $y \in f(x_1, x_2, \dots, x_n)$. This is a contradiction with $\widetilde{\mu}_F(y) < \widetilde{\alpha}$. Therefore $\text{rmin}\{\widetilde{\mu}_F(x_1), \dots, \widetilde{\mu}_F(x_n), \widetilde{t}\} \leq \text{rmax}\{\widetilde{\mu}_F(y), \widetilde{s}\}$, for all $x_i, y \in H$ with $y \in f(x_1, x_2, \dots, x_n)$. This proves that F is an interval valued fuzzy subalgebraic hypersystem with thresholds $(\widetilde{s}, \widetilde{t})$ of H . ■

5. Implication-Based interval valued fuzzy subalgebraic hypersystems

Fuzzy logic is an extension of set theoretic variables (or terms of the linguistic variable truth). Some operators, like $\wedge, \vee, \neg, \rightarrow$ in fuzzy logic are also defined by using truth tables, the extension principle can be applied to derive definitions of the operators.

In the fuzzy logic, truth value of fuzzy proposition P is denoted by $[P]$. In the following, we display the fuzzy logical and corresponding set-theoretical notions:

$$\begin{aligned}
[x \in A] &= A(x); \\
[x \notin A] &= 1 - A(x); \\
[P \wedge Q] &= \min\{[P], [Q]\}; \\
[P \vee Q] &= \max\{[P], [Q]\}; \\
[P \rightarrow Q] &= \min\{1, 1 - [P] + [Q]\}; \\
[\forall x P(x)] &= \inf[P(x)]; \\
\models P &\text{ if and only if } [P] = 1 \text{ for all valuations.}
\end{aligned}$$

Of course, various implication operators have been defined. We only show a selection of them in the next table, α denotes the degree of truth (or degree of membership) of the premise, β the respective values for the consequence, and I the resulting degree of truth for the implication:

Name	Definition of Implication Operators
Early Zadeh	$I_m(\alpha, \beta) = \max\{1 - \alpha, \min\{\alpha, \beta\}\}$
Lukasiewicz	$I_a(\alpha, \beta) = \min\{1, 1 - \alpha + \beta\}$
Standard Star(Godel)	$I_g(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha \leq \beta \\ \beta & \text{if } \alpha > \beta \end{cases}$
Contraposition of Godel	$I_{cg}(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha \leq \beta \\ 1 - \alpha & \text{if } \alpha > \beta \end{cases}$
Gaines-Rescher	$I_{gr}(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha \leq \beta \\ 0 & \text{if } \alpha > \beta \end{cases}$
Kleene-Dienes	$I_b(\alpha, \beta) = \max\{1 - \alpha, \beta\}$

The “quality” of these implication operators could be evaluated either empirically or axiomatically.

In the following definition, we considered the implication operators in the Lukasiewicz system of continuous-valued logic.

Definition 5.1. An interval valued fuzzy set F of H is called an interval valued fuzzifying subalgebraic hypersystem of H if it satisfies, for all $x_i \in H, i = 1, 2, \dots, n$,

$$\models [\inf\{x_i \in F\} \rightarrow [\forall y \in f(x_1, x_2, \dots, x_n), y \in F]].$$

Clearly, Definition 5.1 is equivalent to Definition 4.1. Therefore, an interval valued fuzzifying subalgebraic hypersystem is an ordinary interval valued fuzzy subalgebraic hypersystem.

Now, we introduce the concept of interval valued t -tautology, i.e.,

$$\models_{\tilde{t}} P \text{ if and only if } [P] \geq \tilde{t} \text{ for all valuations.}$$

Based on [25], we can extend the concept of implication-based fuzzy subalgebraic hypersystems in the following way:

Definition 5.2. Let F be an interval valued fuzzy set of H and $t \in (0, 1]$ is a fixed number. Then F is called a t -implication-based interval valued fuzzy subalgebraic hypersystem of H , if for all $x_i \in H$, $i = 1, 2, \dots, n$,

$$\models_{\tilde{t}} [\text{rinf}\{x_i \in F\} \rightarrow [\forall y \in f(x_1, x_2, \dots, x_n), y \in F]].$$

Now, let I be an implication operator, then we have

Corollary 5.3. *An interval valued fuzzy set F of H is a t -implication-based interval valued fuzzy subalgebraic hypersystem of H if and only if, for all $x_i \in H$, $i = 1, 2, \dots, n$,*

$$I(\text{rinf}\{\widetilde{\mu_F}(x_i)\}, \text{rinf}\{\widetilde{\mu_F}(y) | y \in f(x_1, x_2, \dots, x_n)\}) \geq \tilde{t}.$$

Let F be an interval valued fuzzy set of H , then we have the following results:

Theorem 5.4.

- (i) *Let $I = I_{gr}$, then F is an 0.5-implication-based fuzzy interval valued subalgebraic hypersystem of H if and only if F is an interval valued fuzzy subalgebraic hypersystem with thresholds $(\tilde{r} = [0, 0], \tilde{s} = [1, 1])$ of H ;*
- (ii) *Let $I = I_g$, then F is an 0.5-implication-based interval valued fuzzy subalgebraic hypersystem of H if and only if F is an interval valued fuzzy subalgebraic hypersystem with thresholds $(\tilde{r} = [0, 0], \tilde{s} = [0.5, 0.5])$ of H ;*
- (iii) *Let $I = I_{cg}$, then F is an 0.5-implication-based interval valued fuzzy subalgebraic hypersystem of H if and only if F is an interval valued fuzzy subalgebraic hypersystem with thresholds $(\tilde{r} = [0.5, 0.5], \tilde{s} = [1, 1])$ of H .*

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NEW RESULTS ON REMOTALITY IN BANACH SPACES

M. Sababheh

*Department of Science and Humanities
Princess Sumaya University For Technology
Al Jubaiha, Amman 11941
Jordan
e-mail: sababheh@yahoo.com or sababheh@psut.edu.jo*

R. Khalil

*Department of Mathematics
Jordan University
Al Jubaiha, Amman 11942
Jordan
e-mail: roshdi@ju.edu.jo*

Abstract. A set E in a Banach space X is called remotal if for each $x \in X$, there exists an $e \in E$ such that $\|x - e\| = \sup\{\|x - e\| : e \in E\}$. If e is unique, E is called uniquely remotal. One of the main results of this paper is: a weakly closed bounded set E in a reflexive Banach space is uniquely remotal if and only if the weak closed convex hull of E is uniquely remotal.

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1. Introduction

Let X be any normed space and let E be a closed bounded subset of X . For $x \in X$ we define $D(x, E) = \sup\{\|x - e\| : e \in E\}$. We say that E is remotal in X if, for every $x \in X$ there exists $e \in E$ such that $\|x - e\| = D(x, E)$. It is clear that if X is finite dimensional then any closed bounded (compact) subset E is remotal. In fact, it is proved in [8] that “Every closed bounded subset of the normed space X is remotal if and only if X is finite dimensional”. Many questions regarding remotality have been raised and some partial results are available, we refer the reader to [3], [8] and [7].

For $x \in X$, we shall denote by $F(x, E)$ the set of all $e \in E$ such that $\|x - e\| = D(x, E)$. For a set $E \subset X$ we denote the convex hull of E by $[E]$, the closed convex hull by $\overline{[E]}$ and the weak closed convex hull of E by $\overline{[E]}^w$. For $x, y \in X$ we denote the set $\{tx + (1 - t)y : t \in (0, 1)\}$ by $L(x, y)$ and the set $\{tx + (1 - t)y : t \in [0, 1]\}$

by $L[x, y]$. For $x \in X$ and $r > 0$, $S(x, r) = \{y \in X : \|y - x\| = r\}$ and $B(x, r) = \{y \in X : \|y - x\| \leq r\}$. A point $e \in E$ is called an extreme point of E if whenever $e \in L[x, y]$ for some $x, y \in E$ we have $e = x = y$. The set of all extreme points of E is denoted by $Ext(E)$. Finally, for a closed bounded set $E \subset X$ we denote the set $\{e \in E : \text{there exists } x \in X \text{ such that } e \in F(x, E)\}$ by $F(E)$.

A strongly related concept is the proximality concept; a set $E \subset X$ is said to be proximal if for each $x \in X$ there exists $e \in E$ such that $\|x - e\| = \inf\{\|x - y\| : y \in E\}$.

2. Basic results

The proof of the following Lemma can be found in [8].

Lemma 2.1 *Let X be a Banach space, $x \in X$ and let E be a closed bounded subset of X . Then $D(x, E) = D(x, \overline{[E]})$.*

A similar result for $\overline{[E]}^w$ is valid, namely we have the following result:

Lemma 2.2 *Let E be a closed bounded subset of the Banach space X , then $D(x, E) = D(x, \overline{[E]}^w)$ for all $x \in X$.*

Proof. Since $E \subseteq \overline{[E]}^w$ we clearly have $D(x, E) \leq D(x, \overline{[E]}^w)$. On the other hand, let $z \in \overline{[E]}^w$ and let $(z_n) \subset [E]$ be such that $z_n \xrightarrow{w} z$, then $\|x - z\| \leq \liminf \|x - z_n\| \leq D(x, [E])$. Since this is true for any $z \in \overline{[E]}^w$, we infer that $D(x, \overline{[E]}^w) \leq D(x, [E])$. But, by Lemma 2.1 we get that $D(x, \overline{[E]}^w) \leq D(x, E)$. This completes the proof of the lemma. ■

The proof of the following Lemma can be found in [3].

Lemma 2.3 *Let $x \in X$ and suppose that $D(x, E)$ is attained by a unique element $e \in E$, then e is an extreme point of E .*

The proof of the following Corollary can be found in [8].

Corollary 2.4 *Let E be a closed bounded subset of the normed space X and let $x \in X$ be such that $F(x, E) \neq \emptyset$. If $e \in F(x, E)$ is such that $e \in L(y, z)$ for some $y, z \in E$, then $y, z \in F(x, E)$.*

The following proposition gives us more.

Proposition 2.5 *Let E be a closed bounded convex subset of the normed space X and let $e \in F(x, E)$ for some $x \in X$. If $e \in L(y, z)$ for some $y, z \in E$ then $L[y, z] \subset F(x, E)$.*

Proof. By Corollary 2.4 we know that $y, z \in F(x, E)$. So, let $e' \in L(y, z)$ and suppose that $e' \neq e$. Let $t, t' \in (0, 1)$ be such that

$$(2.1) \quad e' = y + t'(z - y) \text{ and } e = y + t(z - y).$$

We assert that $e' \in F(x, E)$. For, we have three cases: either $t = t'$ or $t < t'$ or $t > t'$.

If $t = t'$ then $e = e'$ and we are done. If $t < t'$, then (2.1) implies that

$$e = y + \frac{t}{t'}(e' - y) \Rightarrow e \in L(e', y).$$

But according to Corollary 2.4 we have $e' \in F(x, E)$. As for the third case, if $t > t'$ then (2.1) implies

$$e = z + \frac{1-t}{1-t'}(e' - z) \Rightarrow e \in L(e', z).$$

Again Corollary 2.4 gives $e' \in F(x, E)$. This completes the discussion of the three cases and finishes the proof. $\blacksquare$

Observe that the convexity condition in proposition 2.5 can be removed to get the following Corollary.

Corollary 2.6 *Let E be a closed bounded subset of the normed space X and let $e \in F(x, E)$ for some $x \in X$. If $e \in L(y, z)$ for some $y, z \in E$, then $L[y, z] \cap E \subset F(x, E)$.*

It is trivial to show that if $e \in F(x, E)$ then e is a boundary point of E . This observation together with Corollary 2.4 gives the following result.

Proposition 2.7 *Let X be a normed space, $x \in X$ and E be a closed bounded convex subset of X . If $e \in F(x, E)$ is such that $e \in L(x_1, x_2)$ for some $x_1, x_2 \in E$ then $L[x_1, x_2] \subset \partial E$; the boundary of E .*

Proposition 2.8 *Let X be a normed space, $x \in X$ and let E be a closed bounded subset of X . If $e \in F(x, E)$ then $e \in F(x + t(x - e), E)$ for all $t \geq 0$.*

Proof. Denote $x + t(x - e)$ by z and Observe that for any $e' \in E$ we have

$$\begin{aligned} \|z - e'\| &\leq \|x - e'\| + t\|x - e\| \leq \|x - e\| + t\|x - e\| \\ &= (1 + t)\|x - e\| = \|x + t(x - e) - e\| = \|z - e\|. \end{aligned}$$

This completes the proof. $\blacksquare$

The following proposition gives us an equivalent condition that a point $e \in E$ is a farthest point from other points of X . We leave the easy proof to the reader.

Proposition 2.9 *Let E be a closed bounded subset of a normed space X and let $e \in E$. Then $e \in F(E)$ if and only if there exists $x_0 \in X$ and $r > 0$ such that $e \in S(x_0, r)$ and $E \subset B(x_0, r)$. In this case we have $e \in F(x_0, E)$ and $D(x_0, E) = r$.*

Observe that Corollary 2.6 and propositions 2.7 and 2.9 give us a clear image of farthest points.

The proof of the following proposition is left to the reader.

Proposition 2.10 *For $x \in X$, a normed space, let*

$$A_x = \bigcap_{e \in E} \{z \in E : \|z - x\| \geq \|x - e\|\}.$$

Then

- i) $F(x, E) = A_x$; *including the case when they are both empty.*
- ii) $F(E) = \bigcup_{x \in X} A_x$.

3. Convexity results

Convex sets play an important role in the subject specially when X is a reflexive space. The following result gives us the relationship between remotality of the set and of its convex hull. We remark that the result of the Proposition is known in the literature, see [1].

Proposition 3.1 *Let E be a closed bounded subset of the normed space X . Then E is remotal if and only if $[E]$ is remotal.*

Proof. Suppose that E is remotal, then $[E]$ is remotal by virtue of lemma 2.1. For the converse suppose that $[E]$ is remotal and let $x \in X$ and $y \in F(x, [E])$. If $y \in E$ then $y \in F(x, E)$. If $y \notin E$ then there exist $\alpha_1, \dots, \alpha_n \in (0, 1)$ and $y_1, \dots, y_n \in E$ such that $y = \sum_{k=1}^n \alpha_k y_k$. But then

$$D(x, E) = \|x - y\| = \left\| \sum_{k=1}^n \alpha_k (x - y_k) \right\| \leq \sum_{k=1}^n \alpha_k \|x - y_k\|.$$

If $\|x - y_k\| < D(x, E)$ for any k then we would have $D(x, E) < D(x, E)$. Thus, $\|x - y_k\| = D(x, E)$ for all $k = 1, \dots, n$. This proves that $y_k \in F(x, E)$. Since this is valid for all $x \in X$ we infer that E is remotal. ■

Since $[E]$ is not necessarily closed even if E is closed, a natural question that comes to mind is: Is it true that E is remotal if and only if $\overline{[E]}$ is remotal? In [8], this question was answered affirmatively when X is a reflexive space. Unfortunately, we do not have an answer for the general case.

Uniquely remotal sets are of special interest because of their connection to the geometry of Banach spaces. One conjecture concerning the problem is the following.

Conjecture 3.2 *Let X be a Banach space and let E be a closed bounded subset of X . If E is uniquely remotal in X then E is a singleton.*

Many attempts were made to solve this conjecture and many partial results were obtained. We refer the reader to [7] and [4] for some results.

The following two results discuss the relation between unique remotality of the set and the convex hull of it. The importance of these two results is a natural consequence of the easiness of studying convex sets rather than studying general sets. For example, closed bounded convex sets are weakly compact in reflexive spaces. In fact this observation was the key in proving that every closed bounded convex set in a reflexive space is proximal.

Proposition 3.3 *Let E be a closed bounded subset of the Banach space X . Then E is uniquely remotal if and only if $[E]$ is uniquely remotal.*

Proof. If $[E]$ is uniquely remotal then trivially E is uniquely remotal by virtue of lemma 2.1 and proposition 3.1 and the fact that $E \subset [E]$. For the converse, suppose that E is uniquely remotal and let $\hat{x} \in F(x, E)$ for given $x \in X$. Then surely $[E]$ is remotal and $\hat{x} \in F(x, [E])$. Let $y \in F(x, [E])$. We assert that $y \in F(x, E)$ and this will finish the proof. If $y \in E$ then we are done. If $y \in [E] \setminus E$ then there exist $\alpha_1, \dots, \alpha_n \in (0, 1)$ with the property that $\sum_{i=1}^n \alpha_i = 1$ and $e_1, \dots, e_n \in E$ such that $y = \sum_{i=1}^n \alpha_i e_i$. Now,

$$D(x, E) = \|x - y\| = \left\| \sum_{i=1}^n \alpha_i (x - e_i) \right\| \leq \sum_{i=1}^n \alpha_i \|x - e_i\|.$$

We know that $\|x - e_i\| \leq D(x, E)$ for each i . If for any i we have $\|x - e_i\| < D(x, E)$ then we would have $D(x, E) = \|x - y\| < D(x, E)$ which is a contradiction. This proves that $e_i \in F(x, E), \forall i$. Since E is uniquely remotal we have $e_i = \hat{x}$ for all i . That is $y = \hat{x}$. This completes the proof. ■

We remark that the result of this result is known in the literature, see [6].

Observe that this result says nothing about the relationship between unique remotality of E and of $\overline{[E]}$, the closed convex hull of E ! In the following result we prove this relation but under the conditions that X is reflexive and that E is weakly closed.

Theorem 3.4 *Let X be a reflexive Banach space and let E be a weakly closed bounded subset of X . Then, E is uniquely remotal if, and only if $\overline{[E]}^w$ is uniquely remotal.*

Proof. Suppose that $\overline{[E]}^w$ is uniquely remotal and let $x \in X$. We assert that $F(x, E)$ is attained by a unique element $\hat{x} \in E$. Indeed, let $\hat{x} \in \overline{[E]}^w$ be the unique element of $\overline{[E]}^w$ such that $D(x, \overline{[E]}^w) = \|x - \hat{x}\|$. But according to lemma 2.3, $\hat{x}$ is an extreme point of $\overline{[E]}^w$. Since X is reflexive and E is weakly closed, we infer that all extreme points of $\overline{[E]}^w$ are elements of E , see [5]. Hence $\hat{x} \in E$. By lemma 2.2 we get $\hat{x} \in F(x, E)$.

It is clear that $F(x, E)$ is a singleton or otherwise $F(x, \overline{[E]}^w)$ would have more than one element which is a contradiction.

For the converse, suppose that E is uniquely remotal and let $x \in X$. Since $D(x, E) = D(x, \overline{E}^w)$, by Lemma 2.2, we have $F(x, \overline{E}^w) \neq \emptyset$. Let $y \in F(x, E) \cap F(x, \overline{E}^w)$ and note that this intersection is a singleton because E is uniquely remotal. Let r denote $D(x, E)$ and let $\hat{y} \in F(x, \overline{E}^w)$. We assert that $\hat{y} = y$. If $\hat{y}$ is an extreme point of $\overline{E}^w$ then $\hat{y} \in E$, hence $\hat{y} = y$. Thus, it suffices to show that $\hat{y}$ is an extreme point of $\overline{E}^w$.

Let H be the hyperplane that supports $S(x, r)$ at $\hat{y}$ and let $f \in X^*$ be such that $H = \{w \in X : f(w) = \lambda\}$ where $\lambda \in \mathbb{R}$. Without loss of generality, assume that $S(x, r) \subset \{z \in X : f(z) \leq \lambda\}$. We assert that $M := F(x, \overline{E}^w) \cap H$ is a weakly closed convex subset of X . Indeed, let $(m_n) \subset M$ be such that $m_n \rightarrow m$ weakly. We claim that $m \in M$: Since $f \in X^*$ and $m_n \xrightarrow{w} m$ we infer that $f(m_n) \rightarrow f(m)$. But $f(m_n) = \lambda$ because $m_n \in H$. Hence $f(m) = \lambda$ and $m \in H$. Further, $\|m_n - x\| = r$, $\forall n$. But since $m_n \xrightarrow{w} m$ we have $\|m - x\| \leq r$. If $\|m - x\| < r$ then we have a contradiction to the facts that H supports $S(x, r)$ and $m \in H$. Hence, $\|m - x\| = r$. This completes the proof that $m \in M$ and M is weakly closed. For convexity, suppose that $w_1, w_2 \in M$ and let $\alpha \in (0, 1)$ and let $w = \alpha w_1 + (1 - \alpha)w_2$. Since $w_1, w_2 \in H$ we have $f(w_1) = f(w_2) = \lambda$ and hence $f(w) = \lambda$. That is $w \in H$. Moreover, since $w \in (w_1, w_2)$ we have necessarily $w \in F(x, \overline{E}^w)$ by virtue of Corollary 2.4. This proves our claim that M is convex.

In fact, M is an extremal subset of $F(x, \overline{E}^w)$. Indeed, if $w \in M$ is such that $w \in (w_1, w_2)$ for some $w_1, w_2 \in F(x, \overline{E}^w)$ then $w_1, w_2 \in S(x, r)$ and $f(w) = \alpha f(w_1) + (1 - \alpha)f(w_2)$ for some $\alpha \in (0, 1)$. But $f(w) = \lambda$ and $f(w_i) \leq \lambda$ for $i = 1, 2$. Hence $f(w_1) = f(w_2) = \lambda$. That is, $w_1, w_2 \in H$ hence they are in M . This proves that M is extremal.

Now being weakly closed and bounded in a reflexive space, M is the closed convex hull of its extreme points. But the extreme points of M (an extremal set) are extreme points of $\overline{E}^w$, hence of E because X is reflexive and E is weakly closed.

So, if M contains more than one element then the set of extreme points of M contains more than one element, say y_1 and y_2 . Then $y_1, y_2 \in E$ and $y_1, y_2 \in S(x, r)$. This contradicts the fact that E is uniquely remotal. This shows that M is a singleton. In fact, $M = \{\hat{y}\}$. Consequently, $\hat{y}$ is an extreme point of M , hence of $\overline{E}^w$. This completes the proof of the Theorem. ■

Question: If E is a closed bounded subset of the normed space X , is it true that E is uniquely remotal if and only if $\overline{E}$ is uniquely remotal?

4. Further results

Definition 4.1 If E is a subset of the Banach space X , we say that E is sectionally uniquely remotal (s.u.r.) if $E \cap Y$ is uniquely remotal in Y for every finite dimensional subspace $Y \subseteq X$. Here we use the convention that $\emptyset$ is uniquely remotal in any space.

Proposition 4.2 *E is s.u.r. if and only if E is a singleton.*

Proof. If E is a singleton then the result is clear. For the converse, suppose that $E \cap Y$ is uniquely remotal in Y for every finite dimensional space $Y \subseteq X$. Suppose in the way of contrary that E contains two elements, say $e_1, e_2 \in E$ and let $Y = \text{span}\{e_1, e_2\}$. By assumption, $E \cap Y$ is uniquely remotal in Y . But $E \cap Y$ is compact in Y , hence $E \cap Y$ is a singleton [4]. But $E \cap Y \supset \{e_1, e_2\}$ which is a contradiction. This proves that E must be a singleton. ■

Proposition 4.3 *If $E \subset X$ is such that $\text{span}(E)$ is finite dimensional and if E is uniquely remotal in X , then E is a singleton.*

Closed bounded subsets E of Banach spaces need not be remotal, however elements $x \in X$ which attains their farthest distance from E always exist. In the sequel, we give some criteria for elements $x \in X$ which attains their farthest distance.

Definition 4.4 Let X be a Banach space and let E be a closed bounded subset of X . If $x \in X$, we say that E is x -compact if $\text{diam}(E \setminus B_n(x)) \rightarrow 0$ as $n \rightarrow \infty$. Here $B_n(x) = \{y \in X : \|y - x\| \leq D(x, E) - \frac{1}{n}\}$.

Proposition 4.5 *Let E be a closed bounded subset of a Banach space X . If E is x -compact, for some $x \in X$, then $D(x, E)$ is attained.*

Proof. Let $(e_n) \subset E$ be such that $\|x - e_n\| \rightarrow r := D(x, E)$. The sequence (e_n) can be selected so that $\|x - e_n\| \uparrow r$. Let $\epsilon > 0$ and let $N \in \mathbb{N}$ be such that $\text{diam}(E \setminus B_N(x)) < \epsilon$. Such N exists because E is x -compact. Now let $M \in \mathbb{N}$ be such that $\|x - e_n\| > r - \frac{1}{N}$. Then for $n, m > M$ we have $\|e_n - e_m\| < \epsilon$ because $e_n, e_m \in E \setminus B_N(x)$. This shows that (e_n) is a Cauchy sequence in a Banach space E . Hence $e_n \rightarrow e \in E$ and $\|x - e\| = \lim \|x - e_n\| = r$. In other words, $D(x, E)$ is attained by e . ■

Corollary 4.6 *Let E be a closed bounded subset of a Banach space X . If E is x -compact for every $x \in X$, then E is remotal.*

We should remark that the converse of this Corollary is not necessarily true. That is, there are remotal sets which are not x -compact for some $x \in X$. The unit sphere provides a good and easy example on this observation. In fact, the unit sphere in any Banach space is remotal but is not 0-compact.

Proposition 4.7 *If the closed bounded set E is x -compact for some x in the Banach space X , then $D(x, E)$ is attained by a unique element $e \in E$.*

Proof. Let $x \in X$ and suppose that E is x -compact. We know that $D(x, E)$ is attained by some $e \in E$, see proposition 4.5. Suppose that $D(x, E)$ is attained by e and e' and we are to show that $e = e'$. We now that $\|x - e\| = \|x - e'\| > D(x, E) - \frac{1}{N}$ for all $N \in \mathbb{N}$. But this implies that $\text{diam}(E \setminus B_N(x)) \geq \|e - e'\|$. But since E is x -compact, $\text{diam}(E \setminus B_N(x)) \rightarrow 0$. This implies that $\|e - e'\| = 0$ as claimed. ■

The converse of this proposition is not necessarily true as can be seen from the following example: Let $X = \ell^2$ and let

$$E = \overline{\left\{ \frac{n}{n+1} \delta_n \right\}} \cup \{\delta_1\}.$$

It can be easily seen that E is not 0-compact. However, $D(0, E)$ is attained by the unique element $\delta_1 \in E$.

Definition 4.8 Let X be a Banach space, E be a closed bounded subset of X and let $x \in X$. We say that E is partially x -compact if there exists $F \subset E$ such that $D(x, F) = D(x, E) := r$ and such that F is x -compact.

Theorem 4.9 *The closed bounded set E is partially x -compact for some x in the Banach space X if, and only if $D(x, E)$ is attained.*

Proof. Suppose that $D(x, E)$ is attained by some element $e \in E$ and take $F = \{e\}$. Then $D(x, E) = D(x, F)$ and, clearly, F is x -compact. For the converse, suppose that E is partially x -compact and let F be as in Definition 4.8. Then since F is x -compact, $D(x, F)$ is attained by some $e \in F \subset E$. This completes the proof. ■

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THE HYPERBOLIC MENELAUS THEOREM IN THE POINCARÉ DISC MODEL OF HYPERBOLIC GEOMETRY

Florentin Smarandache

*Department of Mathematics
University of New Mexico
Gallup, NM 87301
USA
e-mail: smarand@unm.edu*

Cătălin Barbu

*Vasile Alecsandri College
Bacău, str. Vasile Alecsandri, nr. 37, cod 600011
Romania
e-mail: kafka_mate@yahoo.com*

Abstract. In this note, we present the hyperbolic Menelaus theorem in the Poincaré disc of hyperbolic geometry.

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1. Introduction

Hyperbolic Geometry appeared in the first half of the 19th century as an attempt to understand Euclid's axiomatic basis of Geometry. It is also known as a type of non-Euclidean Geometry, being in many respects similar to Euclidean Geometry. Hyperbolic Geometry includes similar concepts as distance and angle. Both these geometries have many results in common but many are different.

There are known many models for Hyperbolic Geometry, such as: Poincaré disc model, Poincaré half-plane, Klein model, Einstein relativistic velocity model, etc. The hyperbolic geometry is a non-euclidian geometry. Menelaus of Alexandria was a Greek mathematician and astronomer, the first to recognize geodesics on a curved surface as natural analogs of straight lines. Here, in this study, we present a proof of Menelaus's theorem in the Poincaré disc model of hyperbolic geometry.

The well-known Menelaus theorem states that if l is a line not through any vertex of a triangle ABC such that l meets BC in D , CA in E , and AB in F , then $\frac{DB}{DC} \cdot \frac{EC}{EA} \cdot \frac{FA}{FB} = 1$ [1]. This result has a simple statement but it is of great interest. We just mention here few different proofs given by A. Johnson [2], N.A. Court [3], C. Coşniţă [4], A. Ungar [5]. F. Smarandache (1983) has generalized the Theorem of Menelaus for any polygon with $n \geq 4$ sides as follows: If a line l intersects the n -gon $A_1A_2\dots A_n$ sides $A_1A_2, A_2A_3, \dots$, and A_nA_1 respectively in the points $M_1, M_2, \dots$, and M_n , then $\frac{M_1A_1}{M_1A_2} \cdot \frac{M_2A_2}{M_2A_3} \cdot \dots \cdot \frac{M_nA_n}{M_nA_1} = 1$ [6].

We begin with the recall of some basic geometric notions and properties in the Poincaré disc. Let D denote the unit disc in the complex z -plane, i.e.

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

The most general Möbius transformation of D is

$$z \rightarrow e^{i\theta} \frac{z_0 + z}{1 + \bar{z}_0 z} = e^{i\theta}(z_0 \oplus z),$$

which induces the Möbius addition $\oplus$ in D , allowing the Möbius transformation of the disc to be viewed as a Möbius left gyro-translation

$$z \rightarrow z_0 \oplus z = \frac{z_0 + z}{1 + \bar{z}_0 z}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $z, z_0 \in D$, and $\bar{z}_0$ is the complex conjugate of z_0 . Let $Aut(D, \oplus)$ be the automorphism group of the grupoid $(D, \oplus)$. If we define

$$gyr : D \times D \rightarrow Aut(D, \oplus), gyr[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\bar{b}}{1 + \bar{a}b},$$

then is true gyro-commutative law

$$a \oplus b = gyr[a, b](b \oplus a).$$

A gyro-vector space $(G, \oplus, \otimes)$ is a gyro-commutative gyro-group $(G, \oplus)$ that obeys the following axioms:

- (1) $gyr[\mathbf{u}, \mathbf{v}]\mathbf{a} \cdot gyr[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a} \cdot \mathbf{b}$ for all points $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in G$.
- (2) G admits a scalar multiplication, $\otimes$, possessing the following properties.
For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $\mathbf{a} \in G$:

$$(G1) \quad 1 \otimes \mathbf{a} = \mathbf{a}$$

$$(G2) \quad (r_1 + r_2) \otimes \mathbf{a} = r_1 \otimes \mathbf{a} \oplus r_2 \otimes \mathbf{a}$$

$$(G3) \quad (r_1 r_2) \otimes \mathbf{a} = r_1 \otimes (r_2 \otimes \mathbf{a})$$

$$(G4) \quad \frac{|r| \otimes \mathbf{a}}{\|r \otimes \mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$$

$$(G5) \quad \text{gyr}[\mathbf{u}, \mathbf{v}](r \otimes \mathbf{a}) = r \otimes \text{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{a}$$

$$(G6) \quad \text{gyr}[r_1 \otimes \mathbf{v}, r_1 \otimes \mathbf{v}] = 1$$

- (3) Real vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of one-dimensional "vectors"

$$\|G\| = \{\pm \|\mathbf{a}\| : \mathbf{a} \in G\} \subset \mathbb{R}$$

with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in G$,

$$(G7) \quad \|r \otimes \mathbf{a}\| = |r| \otimes \|\mathbf{a}\|$$

$$(G8) \quad \|\mathbf{a} \oplus \mathbf{b}\| \leq \|\mathbf{a}\| \oplus \|\mathbf{b}\|.$$

Theorem 1 (The law of gyroines in Möbius gyrovector spaces). *Let ABC be a gyrotriangle in a Möbius gyrovector space $(V_s, \oplus, \otimes)$ with vertices $A, B, C \in V_s$, sides $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{V}_s$, and side gyrolengths $a, b, c \in (-s, s)$, $\mathbf{a} = \ominus B \oplus C$, $\mathbf{b} = \ominus C \oplus A$, $\mathbf{c} = \ominus A \oplus B$, $a = \|\mathbf{a}\|$, $b = \|\mathbf{b}\|$, $c = \|\mathbf{c}\|$, and with gyroangles α, β , and γ at the vertices A, B , and C . Then*

$$\frac{a_\gamma}{\sin \alpha} = \frac{b_\gamma}{\sin \beta} = \frac{c_\gamma}{\sin \gamma},$$

where $v_\gamma = \frac{v}{1 - \frac{v^2}{s^2}}$ [7, p. 267].

Definition 2 The hyperbolic distance function in D is defined by the equation

$$d(a, b) = |a \ominus b| = \left| \frac{a - b}{1 - \overline{a}b} \right|.$$

Here, $a \ominus b = a \oplus (-b)$, for $a, b \in D$.

For further details we refer to the recent book of A. Ungar [5].

2. Main results

In this section, we prove the Menelaus's theorem in the Poincaré disc model of hyperbolic geometry.

Theorem 3 (The Menelaus's Theorem for Hyperbolic Gyrotriangle). *If l is a gyroline not through any vertex of an gyrotriangle ABC such that l meets BC in D , CA in E , and AB in F , then*

$$\frac{(AF)_\gamma}{(BF)_\gamma} \cdot \frac{(BD)_\gamma}{(CD)_\gamma} \cdot \frac{(CE)_\gamma}{(AE)_\gamma} = 1.$$

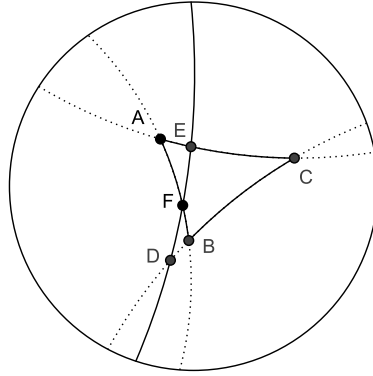


Figure 1

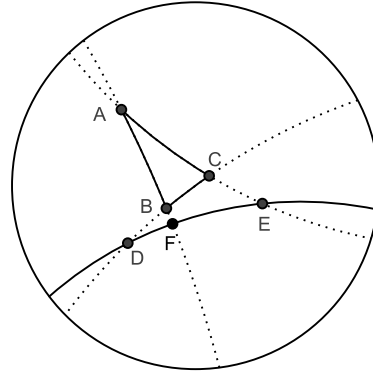


Figure 2

Proof. In function of the position of the gyroline l intersect internally a side of ABC triangle and the other two externally (See Figure 1), or the line l intersect all three sides externally (See Figure 2).

If we consider the first case, the law of gyrosines (See Theorem 1), gives for the gyrotriangles AEF , BFD , and CDE , respectively

$$(1) \quad \frac{(AE)_\gamma}{(AF)_\gamma} = \frac{\sin \widehat{AFE}}{\sin \widehat{AEF}},$$

$$(2) \quad \frac{(BF)_\gamma}{(BD)_\gamma} = \frac{\sin \widehat{FDB}}{\sin \widehat{DFB}},$$

and

$$(3) \quad \frac{(CD)_\gamma}{(CE)_\gamma} = \frac{\sin \widehat{DEC}}{\sin \widehat{EDC}},$$

where $\sin \widehat{AFE} = \sin \widehat{DFB}$, $\sin \widehat{EDC} = \sin \widehat{FDB}$, and $\sin \widehat{AEF} = \sin \widehat{DEC}$, since gyroangles $\widehat{AEF}$ and $\widehat{DEC}$ are supplementary. Hence, by (1), (2) and (3), we have

$$(4) \quad \frac{(AE)_\gamma}{(AF)_\gamma} \cdot \frac{(BF)_\gamma}{(BD)_\gamma} \cdot \frac{(CD)_\gamma}{(CE)_\gamma} = \frac{\sin \widehat{AFE}}{\sin \widehat{AEF}} \cdot \frac{\sin \widehat{FDB}}{\sin \widehat{DFB}} \cdot \frac{\sin \widehat{DEC}}{\sin \widehat{EDC}} = 1,$$

the conclusion follows. The second case is treated similar to the first. ■

Naturally, one may wonder whether the converse of the Menelaus theorem exists.

Theorem 4 (Converse of Menelaus's Theorem for Hyperbolic Gyro-triangle). *If D lies on the gyroline BC , E on CA , and F on AB such that*

$$(5) \quad \frac{(AF)_\gamma}{(BF)_\gamma} \cdot \frac{(BD)_\gamma}{(CD)_\gamma} \cdot \frac{(CE)_\gamma}{(AE)_\gamma} = 1,$$

then D, E , and F are collinear.

Proof. Relabelling if necessary, we may assume that the gyropoint D lies beyond B on BC . If E lies between C and A , then the gyroline ED cuts the gyroside AB , at F' say. Applying Menelaus's theorem to the gyrotriangle ABC and the gyroline $E - F' - D$, we get

$$(6) \quad \frac{(AF')_\gamma}{(BF')_\gamma} \cdot \frac{(BD)_\gamma}{(CD)_\gamma} \cdot \frac{(CE)_\gamma}{(AE)_\gamma} = 1.$$

From (5) and (6), we get $\frac{(AF)_\gamma}{(BF)_\gamma} = \frac{(AF')_\gamma}{(BF')_\gamma}$. This equation holds for $F = F'$. Indeed, if we take $x := |\ominus A \oplus F'|$ and $c := |\ominus A \oplus B|$, then we get $c \ominus x = |\ominus F' \oplus B|$. For $x \in (-1, 1)$ define

$$(7) \quad f(x) = \frac{x}{1 - x^2} : \frac{c \ominus x}{1 - (c \ominus x)^2}.$$

Because $c \ominus x = \frac{c - x}{1 - cx}$, then $f(x) = \frac{x(1 - c^2)}{(c - x)(1 - cx)}$. Since the following equality holds

$$(8) \quad f(x) - f(y) = \frac{c(1 - c^2)(1 - xy)}{(c - x)(1 - cx)(c - y)(1 - cy)}(x - y),$$

we get $f(x)$ is an injective function and this implies $F = F'$, so D, E, F are collinear.

There are still two possible cases. The first is if we suppose that the gyropoint F lies on the gyroside AB , then the gyroline DF cuts the gyrosegment AC in the gyropoint E' . The second possibility is that E is not on the gyroside AC , E lies beyond C . Then DE cuts the gyroline AB in the gyropoint F' . In each case a similar application of Menelaus gives the result. ■

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ON THE FINITE GROUPS WITH AVERAGE LENGTH 3 OF CONJUGACY CLASSES¹

Xianglin Du

*School of Mathematics and Statistics
Chongqing Three Gorges University
Wanzhou, Chongqing, 404100
P.R. China
e-mail: duxianglin2003@yahoo.com.cn*

Abstract. This article studies the problem of average length of conjugacy classes of finite groups, and classifies all finite groups with the average length 3 of conjugacy classes.

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1. Introduction

Let G be a finite group, $k(G)$ be the conjugacy number of G . Let $\mu(G) = |G|/k(G)$. By classes equation: $|G| = c_1 + c_2 + \dots + c_k$, where c_i are length of conjugacy classes of elements of $G, i = 1, 2, \dots, k$. In fact $\mu(G) = |G|/k(G)$ is average length of conjugacy classes of the finite group G . We know that conjugacy classes length can show some character of the group. Further more, the average length of a group has strong restriction to the group. Shi ([1]) proved that if $Z(G) = 1$, then $\mu(G) = 2$ if and only if $G/Z(G) \cong S_3$. Du ([2]) generalized this result that: if $|Z(G)|$ is a odd, then $\mu(G) = 2$ if and only if $G/Z(G) \cong S_3$. Du and Qian ([3]) proved that if $G' \not\cong Z_6$, then $\mu(G) = 3$ if and only if $G/Z(G) \cong A_4, D_{18}, G_{18}$ and for any $x, y \in G, [x, y] \notin Z^*(G)$ (where $G_{18} \cong \langle a, b, c \mid a^3 = b^3 = c^2 = 1, ab = ba, c^{-1}ac = a^{-1}, c^{-1}bc = b^{-1} \rangle$, G_{18} is a group of order 18, it contains six conjugacy classes, there are $\{1\}, \{a, a^2\}, \{b, b^2\}, \{ab, a^2b^2\}, \{ab^2, a^2b\}, \{c, ca, ca^2, cb, cb^2, cab, ca^2b, cab^2, ca^2b^2\}$)

This paper generalizes the results of [3], we will get rid of the condition of $G' \not\cong Z_6$ in paper [3] and have the same results as paper [3].

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For the sake of convenience, let $G_{18} = \langle a, b, c \mid a^3 = b^3 = c^2 = 1, ab = ba, c^{-1}ac = a^{-1}, c^{-1}bc = b^{-1} \rangle$, $G^* = G - \{1\}$, $k(G)$ be the number of conjugacy classes of elements of G , a^G be the conjugacy class containing a , $\text{Irr}(G)$ be the set of all irreducible characters of G , $\text{Irr}^*(G)$ be the set of nonlinear irreducible characters of G . Let χ_P be the character of P by χ restricting on P , θ^G be the induced character of G , where $\chi \in \text{Irr}(G)$, $\theta \in \text{Irr}(G)$. Throughout this paper, all groups are finite.

2. Preliminaries

We need the following lemmas in this paper.

Lemma 2.1. [1]

- (1) Let G be a finite nonAbelian group, then $\mu(G) \geq 8/5$.
- (2) Let $G = A \times B$, where A and B are finite groups. Then $\mu(G) = \mu(A)\mu(B)$.

Lemma 2.2. [3] Let H be a subgroup of a finite group G . Then

- (1) $\mu(H) \leq \mu(G)$, and the equality holds if and only if for any $\chi \in \text{Irr}(G)$, $\chi_H \in \text{Irr}(H)$, and $H' = G'$.
- (2) If $H \triangleleft G$, then $\mu(G/H) \leq \mu(G)$. $\mu(G/H) = \mu(G)$ if and only if $H \leq Z(G)$, and moreover, for any $x, y \in G$, $[x, y] \notin H^*$.

Lemma 2.3. [4] Let G be a finite group. Then $|G| = \sum_{\chi \in \text{Irr}(G)} \chi^2(1)$.

Lemma 2.4. Let G is a finite group. Then

- (1) $\mu(G) = 2$ if and only if $G/Z(G) \simeq S_3$.
- (2) Suppose $G' \not\cong Z_6$. Then $\mu(G) = 3$ if and only if $G/Z(G) \cong A_4, D_{18}, G_{18}$, and for any $x, y \in G$, $[x, y] \notin Z(G) - \{1\}$.

Proof. See [3], Theorem 3.3.

3. Main theorem

Theorem 3.1. Suppose that G is a finite group, if $\mu(G) = 3$, then $G' \not\cong Z_6$.

Proof. If $G' \cong Z_6 = Z_2 \times Z_3$, then $Z_2 \leq Z(G)$. Since $G' \cong Z_6$, G is solvable. Let $G = HL$, where H is a $\{2, 3\}$ -Hall subgroup, L is a $\{2, 3\}'$ -Hall subgroup. It is easy to see that $H \triangleleft G$ and L is an Abelian group. Moreover, we prove that $L \leq Z(G)$.

Let $Q \in Syl_3(G)$ and let $Z_3 = Q \cap G' = \langle u \rangle$, then $\langle u \rangle \text{ char } G' \triangleleft G$. So $\langle u \rangle \triangleleft G$.

Firstly, we prove that $L = C_L(u) \triangleleft G$. Because $\langle u \rangle \triangleleft G$, we have $C_L(u) \triangleleft G$. By n-c theorem, $L/C_L(u) \leq Aut(\langle u \rangle) = Z_2$, it follows that $C_L(u) = L$ since L is a $\{2, 3\}'$ -group.

Secondly, we prove that $L \leq Z(G)$.

For any element $a \in L$, $g \in G$, Since $a^{-1}a^g \in G' = Z_2 \times \langle u \rangle$, we have $a^g = ab$, where $b \in Z_2 \times \langle u \rangle$. Since $Z_2 \leq Z(G)$ and $L = C_L(u)$, we have $ab = ba$. If $b \neq 1$, then a^g is a $\{2, 3\}'$ -element, and the order of ab can be divided by 2 or 3, this is a contradiction. So $b = 1$, $a^g = a$, and $L \leq Z(G)$ follows.

Therefore, $G = H \times L$. By Lemma 2.1, we have $\mu(G) = \mu(H)\mu(L) = \mu(H)$.

In the following proof, we only consider $G = H = PQ$, where $P \in Syl_2(G)$, $Q \in Syl_3(G)$, clearly, $Q \triangleleft G$. Let $|G| = 2^n \cdot 3^m$. We will complete the proof of Theorem 3.1 in four steps.

Step 1. $\langle u \rangle \not\leq Z(G)$.

Proof. Because $G' = Z_2 \times Z_3$, we have $P' = Z_2 = \langle z \rangle \leq Z(G)$. If $\langle u \rangle \leq Z(G)$, for any $a \in P$, $g \in G$, $g^{-1}aga^{-1} \in G' = Z_2 \times \langle u \rangle$. It implies that $g^{-1}ag = axy$, where $x \in Z_2$, $y \in \langle u \rangle$ and $xy \in Z(G)$. Since $\langle u \rangle$ is a subgroup of order 3, we have $g^{-1}a^3g = (ax)^3 \in P$. P is Sylow-2 subgroup of G implies that $g^{-1}ag \in P$. So $P \triangleleft G$, and $G = P \times Q$. In this case P and Q are not Abelian since $G' = Z_2 \times Z_3$. Furthermore, we have $P' = Z_2$, $Q' = Z_3$. Now we compute $k(Q)$, the number of conjugacy classes of Q . Since $|Q/Q'| = 3^{m-1}$ we have $k(Q) > 3^{m-1}$. Let $k(Q) = k$, clearly, for any $a \in Q$, $|a^Q| = 1$, or 3, and the number of conjugacy classes of elements with length 1 of Q is $|Z(Q)|$. Therefore,

$$3^m = |Q| = |Z(Q)| + 3(k - |Z(Q)|) = 3k - 2|Z(Q)|.$$

So we can conclude that $\mu(Q) = |Q|/k = 3 - 2|Z(Q)|/k$. Since $|Z(Q)| \leq 3^{m-2}$ and $k > |Q/Q'| = 3^{m-1}$, we have

$$\mu(Q) = |Q|/k > 3 - 2 \cdot 3^{m-2}/3^{m-1} = 3 - 2/3 > 2.$$

We know from Lemma 2.1 (1) that $\mu(P) \geq 8/5$, therefore

$$3 = \mu(G) = \mu(P \times Q) = \mu(P) \cdot \mu(Q) > \mu(P) \geq (8/5) \cdot 2 = 16/5 > 3,$$

a contradiction.

Step 2. Q is an Abelian subgroup.

Proof. We have know that $\langle u \rangle \triangleleft G$, so for any $g \in G$, $u^g = u^i \in \langle u \rangle$, $i = 1, 2$. It follows that $|G : C_G(u)| = |u^G| = 2$. Let $C_G(u) = P_1Q$, where $|P : P_1| = 2$, which implies that $P_1 \triangleleft P$. Let $P = \langle a, P_1 \rangle$. For any $x \in P_1$, $y \in Q$, we have $x^{-1}x^y \in Q \cap G' = \langle u \rangle$. Thus $x^y = u^i$. If $i \neq 0$, then the left is an elements of order 2 and the right is an element of order 6. Therefore, $i = 0$ and $xy = yx$. That means that $C_G(u) = P_1Q = P_1 \times Q$. Since $P = \langle a, P_1 \rangle$ and $a \notin C_G(u)$,

$|a^G| = 3$, or 6. Therefore $3^{m-1} \mid |C_G(a)|$. Let Q_1 is a Sylow-3 subgroup of $C_G(a)$, then $|Q : Q_1| = 3$. We prove Q_1 is an Abelian subgroup. For any $x, y \in Q_1$, $x^{-1}x^y = u^i \in Q_1 \cap G' \leq \langle u \rangle$. $i \neq 0$ means that $u \in Q_1 \leq C_G(a)$, and $a \in C_G(u)$ follows, it is a contradiction. So $i = 0$, that is $xy = yx$. Therefore Q_1 is an Abelian. Clearly, $u \in Z(Q)$. So $Q = \langle u, Q_1 \rangle$, and Q is an Abelian subgroup.

Step 3. $G = (P\langle u \rangle) \times Q_1$.

Proof. Since $Q_1 \leq C_G(a)$ and $P_1Q = P_1 \times Q$, $Q_1 \leq C_G(P)$. Therefore, $Q_1 \triangleleft G$. Clearly, $Q_1 \cap \langle u \rangle = 1$, so $Q = \langle u \rangle \times Q_1$ and $Q_1 \triangleleft Z(G)$. Therefore $G = (P\langle u \rangle) \times Q_1$.

Step 4. The finial contradiction.

Proof. By Step 3, $G = (P\langle u \rangle) \times Q_1$, by Lemma 2.1, $\mu(G) = \mu(P\langle u \rangle) \cdot \mu(Q_1) = \mu(P\langle u \rangle)$. So in the following proof, we can think $G = P\langle u \rangle$. Because $\langle u \rangle$ is an abelian normal subgroup of G , for any $\chi \in Irr(G)$, we have $\chi(1) \mid |G : \langle u \rangle| = 2^n$. (See [4].)

Remember that

$$Z_2 = \langle z \rangle \leq Z(G).$$

Now, we compute the number of irreducible characters of G .

The irreducible characters of $\bar{G} = G/Z_2$ may be identified as irreducible characters of G . For any $\chi \in Irr(G)$, if $z \in \ker \chi$, then $\chi \in Irr(G/Z_2)$. Therefore, $Irr(G) = Irr(G/Z_2) \cup \{\chi \in Irr(G) \mid z \notin \ker \chi\}$. If $\chi(1) = 1$, since $z \in G'$, $\chi(z) = 1$ and $z \in \ker \chi$, it follows that $\chi \in Irr(G/Z_2)$. So for any $\chi \in Irr(G)$, if $z \notin \ker \chi$, then $\chi(1) > 1$.

Let $\bar{G} = G/Z_2$, $\bar{A} = \bar{P}_1\bar{Q}$, then $\bar{A}$ is an Abelian normal subgroup of $\bar{G}$. Hence $\chi(1) \mid |\bar{G} : \bar{A}| = |P : P_1| = 2$ ([4]). Therefore any $\chi \in Irr(\bar{G})$, $\chi(1) = 1$, or 2. Thus

$$2^{n-1} \cdot 3 = |\bar{G}| = |\bar{G}/\bar{G}'| + |Irr^*(\bar{G})| \cdot 2^2.$$

Since $|\bar{G}/\bar{G}'| = |\bar{P}| = 2^{n-1}$, it is easy to see that

$$|Irr^*(\bar{G})| = 2^{n-2}.$$

Therefore $\bar{G}$ has exactly $2^{n-1} + 2^{n-2} = 3 \cdot 2^{n-2}$ irreducible characters. We know that G has exactly $|G|/3 = 2^n$ irreducible characters, therefore G has $2^n - 3 \cdot 2^{n-2} = 2^{n-2}$ other irreducible characters χ with $z \notin \ker \chi$.

Let $\chi \in Irr(G)$ with $z \notin \ker \chi$, ρ be the irreducible C-representation affording χ .

Since $P' = Z_2 = \langle z \rangle \leq Z(G)$, for $a \notin Z(P)$, there exists $g \in P$ such that $a^g = az$. $z \in Z(G)$ implies that $\rho(z) = \omega I$, where I is an unit matrix with $\chi(1)$ degree, ω is a root of unity. So

$$\chi(a) = \text{tr} \rho(a) = \text{tr} \rho(a^g) = \text{tr} \rho(az) = \text{tr}(\rho(z)\rho(a)) = \text{tr}(\omega \rho(a)) = \omega \chi(a).$$

$z \notin \ker \chi$ implies that $\omega \neq 1$. Therefore $\chi(a) = 0$.

We restrict χ in subgroup P . Then we have

$$(*) \quad (\chi, \chi)_P = \frac{1}{|P|} \sum_{a \in P} \chi(a)\chi(a^{-1}) = \frac{|Z(P)|\chi^2(1)}{|P|}$$

Hence

$$\chi^2(1) = |P|/|Z(P)|(\chi, \chi)_P.$$

Since P is non-Abelian,

$$|P : Z(P)| = 2^{n-s} \geq 4,$$

where $|Z(P)| = 2^s$. If $(\chi, \chi)_P > 1$, then $\chi(1) \geq 2^2$. If $\chi(1) = 2$, then $(\chi, \chi)_P = 1$ and $\chi \in Irr(P)$.

If any $\chi \in Irr(G)$ with $z \notin \ker \chi$, $\chi(1) \geq 2^2$. Then we have

$$3.2^n = |G| = |G/Z_2| + \sum_{z \notin \ker \chi} \chi^2(1) \geq 3.2^{n-1} + 2^{n-2}.2^4$$

Thus $3.2^{n-1} \geq 8.2^{n-1}$, a contradiction.

Therefore there exists at least one $\chi \in Irr(G)$ with $z \notin \ker \chi$, $\chi(1) = 2$.

Now we prove, for any $\chi \in Irr(G)$ with $z \notin \ker \chi$, $\chi(1) = 2$ or $\chi(1) = 4$. Let $\theta \in Irr(G)$ with $\theta(1) = 2$, $z \notin \ker \theta$, we have known that $\theta \in Irr(P)$, and by (*), $|P : Z(P)| = 4$. Also by (*), for any $\theta \in Irr^*(P)$, we have $\theta(1) = 2$.

Since $|P|/|Z(P)| = 4$, by (*) for any $\chi \in Irr(G)$ with $\chi(1) > 2$, $z \notin \ker \chi$, we have $(\chi, \chi)_P > 1$. Restricting χ on subgroup P , let $\chi_P = n_1\theta_1 + \dots + n_t\theta_t$, where $\theta_i \in Irr(P)$. Then there is a $\theta_i(1) = 2$ since $z \notin \ker \chi$. Let it be $\theta = \theta_1$, by Frobenius Reciprocity theorem, the induced character $\theta_1^G = n_1\chi + \dots + m_t\chi_t$. We know that $\theta_1^G(1) = |G : P|\theta_1(1) = 3.2 = 6$, $\chi(1) > 2$ and $\chi(1)$ is power of 2, therefore the only possible is $\chi(1) = 4$.

Suppose that there exists n_1 irreducible characters $\chi \in Irr(G)$ with $\chi(1) = 2$, $z \notin \ker \chi$, n_2 irreducible characters $\chi \in Irr(G)$ with $\chi(1) = 4$, $z \notin \ker \chi$, then $n_1 + n_2 = 2^{n-2}$. Therefore we have

$$\begin{aligned} 3.2^n = |G| &= |G/Z_2| + \sum_{z \notin \ker \chi} \chi^2(1) = 3.2^{n-1} \\ &+ \sum_{z \notin \ker \chi} \chi^2(1) = 3.2^{n-1} + n_1.2^2 + n_2.2^4. \end{aligned}$$

From here we have $2^{n-3} = 3.n_2$. It is a final contradiction.

Theorem 3.2. Suppose that G is a finite group, Then $\mu(G) = 3$ if and only if $G/Z(G) \cong A_4, D_{18}, G_{18}$, and for any $x, y \in G$, $[x, y] \notin Z^*(G)$.

Proof. By Theorem 3.1, $G' \not\cong Z_6$. So by Lemma 2.4(2), the Theorem is true. The proof is completed.

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ON (λ, μ) -FUZZY SUBHYPERLATTICES¹

Yuming Feng

Qingsong Zeng

Huiling Duan

*School of Mathematics and Statistics
Chongqing Three-Gorges University
Wanzhou, Chongqing, 404100
P.R. China*

*e-mails: yumingfeng25928@163.com
zqs2008@sina.com
huilingduan25928@163.com*

Abstract. We first introduce the concepts of (λ, μ) -fuzzy subhyperlattices and (λ, μ) -fuzzy ideals. Secondly, we list some equivalent conditions of them. Lastly, we prove that the Cartesian product of two (λ, μ) -fuzzy subhyperlattices is still a (λ, μ) -fuzzy subhyperlattice. This paper can be seen as a generalization of [1].

Keywords: Cartesian product; (λ, μ) -fuzzy; subhyperlattice; ideal.

1. Introduction and preliminaries

The concept of fuzzy sets was first introduced by Zadeh [18] in 1965. The theory of fuzzy sets has been developed fast and has many applications in many branches of sciences. In mathematics, the study of fuzzy algebraic structures was first initiated by pioneer paper of Rosenfeld[11]. He first studied the fuzzy subgroup of a group and since then, many researchers have been engaged in extending the concepts and results of abstract algebra based on fuzzy sets.

Hyperstructure theory was first introduced in 1934 by Marty at the 8th Congress of Scandinavian Mathematicians (see [10]). Later on, hyperstructures have been developed in both pure and applied sciences. A comprehensive review of the theory of hyperstructures can be found in [2, 3]. Hyperstructures are generalizations of classic structures. For example, hypergroup [2] is a generalization of group, hyperlattice[9] and superlattice[6] are generalizations of lattice and so on.

Fuzzy hyperstructures have been introduced rather recently. Corsini and Toffan studied fuzzy hypergroups in [4], Hasankhani and Zahedi studied fuzzy hyper-rings in [8], Serafimidis, Konstantinidou and Kehagias studied fuzzy hyperlattices in [12] and so on.

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Recently, Yuan [17] introduced the concept of fuzzy subgroup with thresholds. A fuzzy subgroup with thresholds λ and μ is also called a (λ, μ) -fuzzy subgroup. Yao continued to research (λ, μ) -fuzzy normal subgroups, (λ, μ) -fuzzy quotient subgroups and (λ, μ) -fuzzy subrings in [14, 15, 16]. Ali and Ray discussed the product of fuzzy sublattices in [1].

In this paper, we introduced the concept of fuzzy hyperlattice with thresholds. Let us recall some definitions and notions.

By a fuzzy subset of a nonempty set X we mean a mapping from X to the unit interval $[0, 1]$. If A is a fuzzy subset of X , then we denote $A_\alpha = \{x \in X | A(x) \geq \alpha\}$ for all $\alpha \in [0, 1]$.

A partial hypergroupoid $\langle H; * \rangle$ is a nonempty set H with a function from $H \times H$ to the set of subsets of H , i.e.,

$$\begin{aligned} * : H \times H &\rightarrow \mathbf{P}(H) \\ (x, y) &\rightarrow x * y. \end{aligned}$$

A hypergroupoid is a nonempty set H , endowed with a hyperoperation, that is a function from $H \times H$ to the set of nonempty subsets of H .

If $A, B \in \mathbf{P}(H) - \{\emptyset\}$, then we define $A * B = \cup\{a * b | a \in A, b \in B\}$, $x * B = \{x\} * B$ and $A * y = A * \{y\}$.

Definition 1.1 ([5],[7]) Let H be a nonempty set, $\sqcup : H \times H \rightarrow P^*(H)$ be a hyperoperation, where $P(H)$ is the power set of H and $P^*(H) = P(H) - \{\emptyset\}$ and $\wedge : H \times H \rightarrow H$ be an operation. Then $(H, \sqcup, \wedge)$ is called a hyperlattice if for all $a, b, c \in H$:

- (1) $a \in a \sqcup a, a = a \wedge a$;
- (2) $a \sqcup b = b \sqcup a, a \wedge b = b \wedge a$;
- (3) $(a \sqcup b) \sqcup c = a \sqcup (b \sqcup c), (a \wedge b) \wedge c = a \wedge (b \wedge c)$;
- (4) $a \in (a \sqcup b) \wedge a, a \in (a \wedge b) \sqcup a$;
- (5) $b \in a \sqcup b \Leftrightarrow a = a \wedge b \Leftrightarrow a \leq b$.

The readers can consult [2],[13] to learn more about hyperstructures and fuzzy sets.

Throughout this paper, we will always assume that $0 \leq \lambda < \mu \leq 1$.

2. (λ, μ) -fuzzy subhyperlattices

Throughout this section H always denotes a hyperlattice. The meet, hyperjoin and partial order of H , will be denoted as $\wedge, \sqcup$, and $\leq$, respectively.

Definition 2.1 A fuzzy subset A of a hyperlattice H is said to be a (λ, μ) -fuzzy subhyperlattice of H if $\forall a, b \in H$,

$$A(a \wedge b) \vee \lambda \geq (A(a) \wedge A(b)) \wedge \mu$$

and

$$\inf_{t \in a \sqcup b} A(t) \vee \lambda \geq (A(a) \wedge A(b)) \wedge \mu.$$

Remark 2.2 From the previous definition, we know that a fuzzy subhyperlattice is a $(0, 1)$ -fuzzy subhyperlattice.

Theorem 2.3 *Let A be a fuzzy subset of H . Then the following are equivalent:*

- (1) A is a (λ, μ) -fuzzy subhyperlattice of H ;
- (2) A_α is a subhyperlattice of H , for any $\alpha \in (\lambda, \mu]$, where $A_\alpha \neq \emptyset$.

Proof. 1. (1) $\Rightarrow$ (2). Let A be a (λ, μ) -fuzzy subhyperlattice of H . For any $\alpha \in (\lambda, \mu]$, such that $A_\alpha \neq \emptyset$, we need to show that $x \wedge y \in A_\alpha$ and $x \sqcup y \subseteq A_\alpha$, for all $x, y \in A_\alpha$.

From $x \in A_\alpha$ we know that $A(x) \geq \alpha$. And similarly we obtain that $A(y) \geq \alpha$. Thus $A(x \wedge y) \vee \lambda \geq (A(x) \wedge A(y)) \wedge \mu \geq \alpha \wedge \mu = \alpha$. Note that $\lambda < \alpha$ and so $x \wedge y \in A_\alpha$.

From $A(x) \geq \alpha$ and $A(y) \geq \alpha$ we know that $\inf_{t \in x \sqcup y} A(t) \vee \lambda \geq (A(x) \wedge A(y)) \wedge \mu \geq \alpha \wedge \mu = \alpha$ and $\lambda < \alpha$, we conclude that $\inf_{t \in x \sqcup y} A(t) \geq \alpha$. So $A(t) \geq \alpha$ for any $t \in x \sqcup y$. Thus $x \sqcup y \subseteq A_\alpha$.

2. (2) $\Rightarrow$ (1). If there exist $x_0, y_0 \in H$ such that $A(x_0 \wedge y_0) \vee \lambda < \alpha = (A(x_0) \wedge A(y_0)) \wedge \mu$, then $\alpha \in (\lambda, \mu]$, $A(x_0) \wedge A(y_0) \geq \alpha$. So $x_0 \in A_\alpha$ and $y_0 \in A_\alpha$. But $A(x_0 \wedge y_0) < \alpha$, that is $x_0 \wedge y_0 \notin A_\alpha$. This is a contradiction with that A_α is a subhyperlattice of H . Thus $A(x \wedge y) \vee \lambda \geq (A(x) \wedge A(y)) \wedge \mu$ holds for all $x, y \in H$.

Again, if there exist $x_0, y_0 \in H$ such that $\inf_{t \in x_0 \sqcup y_0} A(t) \vee \lambda < \alpha = (A(x_0) \wedge A(y_0)) \wedge \mu$, then $\alpha \in (\lambda, \mu]$, $A(x_0) \wedge A(y_0) \geq \alpha$. So $x_0 \in A_\alpha$ and $y_0 \in A_\alpha$. But $\inf_{t \in x_0 \sqcup y_0} A(t) < \alpha$, that is $A(t) < \alpha$ for some $t \in x_0 \sqcup y_0$. So $x_0 \sqcup y_0 \not\subseteq A_\alpha$. This is a contradiction with that A_α is a subhyperlattice of H . Thus $\inf_{t \in x \sqcup y} A(t) \vee \lambda \geq (A(x) \wedge A(y)) \wedge \mu$ holds for all $x, y \in H$. ■

3. (λ, μ) -fuzzy ideals

Definition 3.1 Let $(H, \sqcup, \wedge)$ be a hyperlattice. A nonempty subset I of H is called an ideal of H if for all $a, b \in H$,

$$a, b \in I \Rightarrow a \sqcup b \subseteq I$$

and

$$a \in H, b \in I \Rightarrow a \wedge b \in I.$$

Proposition 3.2 *Suppose I is a subset of a hyperlattice H , then the following are equivalent for all $a, b \in H$,*

- (1) $a \in H, b \in I \Rightarrow a \wedge b \in I$;
- (2) $a \in I$ and $b \leq a \Rightarrow b \in I$.

Proof. (1) $\Rightarrow$ (2). If $b \leq a$, then $b = a \wedge b$. From (1) we know that $a \wedge b \in I$. And so $b \in I$.

(2) $\Rightarrow$ (1). From $a \wedge b \leq b \in I$ and (2) we know that $a \wedge b \in I$. ■

Definition 3.3 A fuzzy subset A of a hyperlattice H is a (λ, μ) -fuzzy ideal of H if for all $a, b \in H$,

$$A(a \wedge b) \vee \lambda \geq (A(a) \vee A(b)) \wedge \mu$$

and

$$\inf_{t \in a \sqcup b} A(t) \vee \lambda \geq (A(a) \wedge A(b)) \wedge \mu.$$

Proposition 3.4 Suppose A is a fuzzy subset of a hyperlattice H , then the following are equivalent for all $a, b \in H$,

- (1) $A(a \wedge b) \vee \lambda \geq (A(a) \vee A(b)) \wedge \mu$;
- (2) $a \leq b \Rightarrow A(a) \vee \lambda \geq A(b) \wedge \mu$.

Proof. (1) $\Rightarrow$ (2). If $a \leq b$, then $a \wedge b = a$. Thus $A(a) \vee \lambda = A(a \wedge b) \vee \lambda \geq (A(a) \vee A(b)) \wedge \mu \geq A(b) \wedge \mu$.

(2) $\Rightarrow$ (1). From $a \wedge b \leq a$ we know that $A(a \wedge b) \vee \lambda \geq A(a) \wedge \mu$ and from $a \wedge b \leq b$ we conclude that $A(a \wedge b) \vee \lambda \geq A(b) \wedge \mu$. Thus $A(a \wedge b) \vee \lambda \geq (A(a) \wedge \mu) \vee (A(b) \wedge \mu) = (A(a) \vee A(b)) \wedge \mu$.

Hence, we complete the proof. $\blacksquare$

Theorem 3.5 Let A be a (λ, μ) -fuzzy subhyperlattice of H . Then the following are equivalent:

- (1) A is a (λ, μ) -fuzzy ideal of H ;
- (2) A_α is an ideal of H , for any $\alpha \in (\lambda, \mu]$, where $A_\alpha \neq \emptyset$.

Proof. (1) $\Rightarrow$ (2). Let A be a (λ, μ) -fuzzy ideal of H . For any $\alpha \in (\lambda, \mu]$, such that $A_\alpha \neq \emptyset$, we need to show that $x \wedge y \in A_\alpha$, for all $x \in A_\alpha$ and $y \in H$.

From $A(x) \geq \alpha$ we obtain that $A(x \wedge y) \vee \lambda \geq (A(x) \vee A(y)) \wedge \mu \geq \alpha$. Note that $\lambda < \alpha$, we conclude that $A(x \wedge y) \geq \alpha$. So $x \wedge y \in A_\alpha$.

(2) $\Rightarrow$ (1). If there exist $x_0, y_0 \in H$ such that $A(x_0 \wedge y_0) \vee \lambda < \alpha = (A(x_0) \vee A(y_0)) \wedge \mu$, then $\alpha \in (\lambda, \mu]$, $A(x_0) \vee A(y_0) \geq \alpha$. So $x_0 \in A_\alpha$ or $y_0 \in A_\alpha$. But $A(x_0 \wedge y_0) < \alpha$, that is $x_0 \wedge y_0 \notin A_\alpha$. This is a contradiction with that A_α is an ideal of H . Thus $A(x \wedge y) \vee \lambda \geq (A(x) \vee A(y)) \wedge \mu$ holds for all $x, y \in H$.

The proof is ended. $\blacksquare$

Example 3.6 Let $H = \{0, a, b, 1\}$ and define $\sqcup$ and $\wedge$ as following

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\sqcup$	0	a	b	1
0	{0}	{a}	{b}	{1}
a	{a}	{0, a}	{1}	{b, 1}
b	{b}	{1}	{0, b}	{a, 1}
1	{1}	{b, 1}	{a, 1}	H

Then $(H, \sqcup, \wedge, 0, 1)$ is a bounded distributive hyperlattice (see Example 1.8 of [9]).

Consider the following fuzzy subset of H , defined by

H	0	a	b	1
A	1.0	0.2	0.3	0.2

Then A is a $(0.2, 1.0)$ -fuzzy ideal of H . Also A is a $(\lambda, 1.0)$ -fuzzy ideal of H , where $0 \leq \lambda \leq 0.2$.

We give the definition of (λ, μ) -fuzzy prime ideal as following.

Definition 3.7 A proper (λ, μ) -fuzzy ideal of H is called a (λ, μ) -fuzzy prime ideal, if for all $a, b \in H$,

$$(A(a) \vee A(b)) \vee \lambda \geq A(a \wedge b) \wedge \mu.$$

Example 3.8 Consider the bounded distributive hyperlattice H of the previous example, and consider the following fuzzy subsets of H defined by

H	0	a	b	1
A	0.25	0.16	0.25	0.16

Then A is a $(0.16, 0.25)$ -fuzzy prime ideal of H .

Theorem 3.9 Let A be a (λ, μ) -fuzzy subhyperlattice of H . Then the following are equivalent:

- (1) A is a (λ, μ) -fuzzy prime ideal of H ;
- (2) A_α is a prime ideal of H , for any $\alpha \in (\lambda, \mu]$, where A_α is a proper ideal of H .

Proof. (1) $\Rightarrow$ (2). Let A be a (λ, μ) -fuzzy prime ideal of H . For any $\alpha \in (\lambda, \mu]$, such that A_α is a proper ideal of H , we need to show that $x \wedge y \in A_\alpha \Rightarrow x \in A_\alpha$ or $y \in A_\alpha$. From $x \wedge y \in A_\alpha$ we obtain that $A(x \wedge y) \geq \alpha$. So $(A(x) \vee A(y)) \vee \lambda \geq A(x \wedge y) \wedge \mu \geq \alpha \wedge \mu = \alpha$. Note that $\lambda < \alpha$, we conclude that $A(x) \vee A(y) \geq \alpha$. Thus $x \in A_\alpha$ or $y \in A_\alpha$.

(2) $\Rightarrow$ (1). If there exist $x_0, y_0 \in H$ such that $(A(x_0) \vee A(y_0)) \vee \lambda < \alpha = A(x_0 \wedge y_0) \wedge \mu$, then $\alpha \in (\lambda, \mu]$, $A(x_0 \wedge y_0) \geq \alpha$. So $x_0 \wedge y_0 \in A_\alpha$. But $A(x_0) \vee A(y_0) < \alpha$, that is $x_0 \notin A_\alpha$ and $y_0 \notin A_\alpha$. This is a contradiction with that A_α is a prime ideal of H . Thus $(A(x) \vee A(y)) \vee \lambda \geq A(x \wedge y) \wedge \mu$ holds for all $x, y \in H$.

The proof is ended. ■

4. Cartesian product of (λ, μ) -fuzzy subhyperlattices

Let H_1 and H_2 be two hyperlattices. The Cartesian product of H_1 and H_2 is defined by $H_1 \times H_2 \doteq \{(x, y) | x \in H_1, y \in H_2\}$.

For $(a, b), (c, d) \in H_1 \times H_2$, we define

$$(a, b) \leq (c, d) \doteq \begin{cases} a \leq c \\ b \leq d \end{cases},$$

$$(a, b) \wedge (c, d) \doteq (a \wedge c, b \wedge d)$$

and

$$(a, b) \sqcup (c, d) \doteq \cup_{t_1 \in a \sqcup c, t_2 \in b \sqcup d} (t_1, t_2)$$

Proposition 4.1 *Let H_1 and H_2 be two hyperlattices. Then $(H_1 \times H_2, \sqcup, \wedge)$ is a hyperlattice.*

Proof. For all $(a, b), (c, d), (e, f) \in H_1 \times H_2$, we have

- (1) From $(a, b) \sqcup (a, b) = \cup_{t_1 \in a \sqcup a, t_2 \in b \sqcup b} (t_1, t_2)$, $a \in a \sqcup a$ and $b \in b \sqcup b$ we know that $(a, b) \in (a, b) \sqcup (a, b)$.

It is obvious that $(a, b) \wedge (a, b) = (a \wedge a, b \wedge b) = (a, b)$.

- (2) $(a, b) \sqcup (c, d) = \cup_{t_1 \in a \sqcup c, t_2 \in b \sqcup d} (t_1, t_2) = \cup_{t_1 \in c \sqcup a, t_2 \in d \sqcup b} (t_1, t_2) = (c, d) \sqcup (a, b)$.

$(a, b) \wedge (c, d) = (a \wedge c, b \wedge d) = (c \wedge a, d \wedge b) = (c, d) \wedge (a, b)$.

- (3) $((a, b) \sqcup (c, d)) \sqcup (e, f) = \cup_{t_1 \in a \sqcup c, t_2 \in b \sqcup d} (t_1, t_2) \sqcup (e, f) = \cup_{t_1 \in (a \sqcup c) \sqcup e, t_2 \in (b \sqcup d) \sqcup f} (t_1, t_2) = \cup_{t_1 \in a \sqcup (c \sqcup e), t_2 \in b \sqcup (d \sqcup f)} (t_1, t_2) = (a, b) \sqcup ((c, d) \sqcup (e, f))$.

$((a, b) \wedge (c, d)) \wedge (e, f) = (a \wedge c, b \wedge d) \wedge (e, f) = ((a \wedge c) \wedge e, (b \wedge d) \wedge f) = (a \wedge (c \wedge e), b \wedge (d \wedge f)) = (a, b) \wedge ((c, d) \wedge (e, f))$.

- (4) $((a, b) \sqcup (c, d)) \wedge (a, b) = (\cup_{t_1 \in a \sqcup c, t_2 \in b \sqcup d} (t_1, t_2)) \wedge (a, b) = \cup_{t_1 \in a \sqcup c, t_2 \in b \sqcup d} (t_1 \wedge a, t_2 \wedge b) = \cup_{t_1 \in (a \sqcup c) \wedge a, t_2 \in (b \sqcup d) \wedge b} (t_1, t_2) \ni (a, b)$.

$((a, b) \wedge (c, d)) \sqcup (a, b) = (a \wedge c, b \wedge d) \sqcup (a, b) = \cup_{t_1 \in (a \wedge c) \sqcup a, t_2 \in (b \wedge d) \sqcup b} (t_1, t_2) \ni (a, b)$.

- (5) $(a, b) \in (a, b) \sqcup (c, d) \Leftrightarrow (a, b) \in \cup_{t_1 \in a \sqcup c, t_2 \in b \sqcup d} (t_1, t_2) \Leftrightarrow \left\{ \begin{array}{l} a \in a \sqcup c \\ b \in b \sqcup d \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} c = a \wedge c \\ d = b \wedge d \end{array} \right\} \Leftrightarrow (c, d) = (a, b) \wedge (c, d)$.

$(c, d) = (a, b) \wedge (c, d) \Leftrightarrow \left\{ \begin{array}{l} c = a \wedge c \\ d = b \wedge d \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} c \leq a \\ d \leq b \end{array} \right\} \Leftrightarrow (c, d) \leq (a, b)$. ■

Theorem 4.2 *Let A be a (λ, μ) -fuzzy subhyperlattice of the hyperlattice H_1 and B be a (λ, μ) -fuzzy subhyperlattice of the hyperlattice H_2 . Then $A \times B$ is a (λ, μ) -fuzzy subhyperlattice of the hyperlattice $H_1 \times H_2$, where*

$$(A \times B)(x, y) \doteq A(x) \wedge B(y), \quad \forall (x, y) \in H_1 \times H_2.$$

Proof. Let $(a, b), (c, d) \in H_1 \times H_2$. Then

$$\begin{aligned} (A \times B)\{(a, b) \wedge (c, d)\} \vee \lambda &= (A \times B)(a \wedge c, b \wedge d) \vee \lambda \\ &= \{A(a \wedge c) \wedge B(b \wedge d)\} \vee \lambda \\ &= \{A(a \wedge c) \vee \lambda\} \wedge \{B(b \wedge d) \vee \lambda\} \\ &\geq \{(A(a) \wedge A(c)) \wedge \mu\} \wedge \{(B(b) \wedge B(d)) \wedge \mu\} \\ &= \{A(a) \wedge A(c) \wedge B(b) \wedge B(d)\} \wedge \mu \\ &= \{A(a) \wedge B(b) \wedge A(c) \wedge B(d)\} \wedge \mu \\ &= \{(A \times B)(a, b) \wedge (A \times B)(c, d)\} \wedge \mu \end{aligned}$$

and

$$\begin{aligned}
\left(\inf_{t \in (a,b) \sqcup (c,d)} (A \times B)(t) \right) \vee \lambda &= \left(\inf_{t_1 \in a \sqcup c, t_2 \in b \sqcup d} (A \times B)(t_1, t_2) \right) \vee \lambda \\
&= \left(\inf_{t_1 \in a \sqcup c, t_2 \in b \sqcup d} \{A(t_1) \wedge B(t_2)\} \right) \vee \lambda \\
&= \inf_{t_1 \in a \sqcup c, t_2 \in b \sqcup d} \{(A(t_1) \vee \lambda) \wedge (B(t_2) \vee \lambda)\} \\
&\geq \{(A(a) \wedge A(c)) \wedge \mu\} \wedge \{(B(b) \wedge B(d)) \wedge \mu\} \\
&= \{A(a) \wedge A(c) \wedge B(b) \wedge B(d)\} \wedge \mu \\
&= \{A(a) \wedge B(b) \wedge A(c) \wedge B(d)\} \wedge \mu \\
&= \{(A \times B)(a, b) \wedge (A \times B)(c, d)\} \wedge \mu
\end{aligned}$$

Hence $A \times B$ is a (λ, μ) -fuzzy subhyperlattice of the hyperlattice $H_1 \times H_2$. $\blacksquare$

The following example shows that the product of two (λ, μ) -fuzzy ideals is not necessarily a (λ, μ) -fuzzy ideal.

Example 4.3 Let $H = \{a, b, c, d, e\}$. $\sqcup$ and $\wedge$ are given by the following tables

$\wedge$	a	b	c	d	e
a	a	b	c	d	e
b	b	b	d	d	b
c	c	d	c	d	c
d	d	d	d	d	d
e	e	b	c	d	e

$\sqcup$	a	b	c	d	e
a	$\{a, e\}$	$\{a, e\}$	$\{a, e\}$	$\{a, e\}$	$\{a, e\}$
b	$\{a, e\}$	b	$\{a, e\}$	b	$\{a, e\}$
c	$\{a, e\}$	$\{a, e\}$	c	c	$\{a, e\}$
d	$\{a, e\}$	b	c	d	$\{a, e\}$
e	$\{a, e\}$	$\{a, e\}$	$\{a, e\}$	$\{a, e\}$	$\{a, e\}$

It is easy to verify that H is a hyperlattice.

Consider the following fuzzy set A and B of H , respectively.

H	a	b	c	d	e
A	0.2	0.2	0.9	0.4	0.2

H	a	b	c	d	e
B	0.4	0.4	0.8	0.8	0.4

Clearly, A and B are $(0, 1)$ -fuzzy ideals of H . For $(b, c), (c, d) \in H \times H$, we have

$$(A \times B)(b, c) = A(b) \wedge B(c) = 0.2$$

and

$$(A \times B)(c, d) = A(c) \wedge B(d) = 0.8.$$

Therefore

$$\{(A \times B)(b, c) \vee (A \times B)(c, d)\} \wedge 1 = 0.8.$$

On the other hand, we have

$$(A \times B)\{(b, c) \wedge (c, d)\} \vee 0 = (A \times B)(b \wedge c, c \wedge d)(A \times B)(d, d) \vee 0 = 0.4.$$

Thus $(A \times B)\{(b, c) \wedge (c, d)\} \vee 0 \not\geq \{(A \times B)(b, c) \vee (A \times B)(c, d)\} \wedge 1$. Hence $A \times B$ is not a $(0, 1)$ -fuzzy ideal of $H \times H$.

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BLOCKWISE REPEATED LOW-DENSITY BURST ERROR CORRECTING LINEAR CODES

Bal Kishan Dass

*Department of Mathematics
University of Delhi
Delhi - 110 007
India
e-mail: dassbk@rediffmail.com*

Surbhi Madan¹

*Department of Mathematics
Shivaji College (University of Delhi)
Raja Garden
New Delhi - 110 027
India
e-mail: surbhimadan@gmail.com*

Abstract. The paper presents necessary and sufficient condition on the number of parity-check digits required for the existence of a linear code capable of correcting errors in the form of 2-repeated low-density bursts occurring within a sub-block. An illustration of a code of length 24 correcting all 2-repeated low-density bursts of length 3 or less with weight 2 or less occurring within a sub-block of length 12 has also been provided.

Keywords: error locating codes, error correction, burst errors, repeated burst errors, low-density repeated burst errors.

AMS Subject Classification: 94B20, 94B65, 94B25.

1. Introduction

In the theory of error control coding, codes have been developed to detect, correct and/or to locate various kinds of errors. Amongst these, burst errors have played a dominant role and have been studied extensively by many authors. Most of the earlier studies in this direction have been made with respect to the following definition of a burst:

Definition 1. A burst of length b is a vector whose all non-zero components are among some b consecutive components, the first and the last of which is non-zero.

¹Corresponding author.

Depending upon the type of channel used during the process of transmission the nature of burst errors differ. It has been observed that in very busy communication channels, errors repeat themselves. Recently, repeated bursts have been introduced and studied by Berardi, Dass and Verma [1]. An m -repeated burst of length b is defined as follows:

Definition 2. An m -repeated burst of length b is a vector of length n whose only non-zero components are confined to m distinct sets of b consecutive components, the first and the last component of each set being non-zero.

Certain situations like lightening or other disturbances which induce burst errors usually operate in a way that over a given length some digits are received correctly whereas others are corrupted. Such situations led to the development of codes dealing with errors that are bursts of length b or less with weight w or less ($w \leq b$), known as low-density bursts (refer Wyner [14]). A study of low-density burst error detecting and correcting linear codes has been made by Sharma and Dass [12] and Dass [2]. Different situations demanded the development of codes which correct those errors that are repeated low-density burst errors of length b or less with weight w or less. A study of such codes was initiated by Dass and Verma [6]. A 2-repeated low-density burst of length b with weight w ($w \leq b$) is defined as follows:

Definition 3. A 2-repeated low-density burst of length b with weight w is a vector of length n whose only non-zero components are confined to two distinct sets of b consecutive components, the first and the last component of each set being non-zero, with w ($w \leq b$) non-zero components within each set of such b consecutive components.

For example, (01023000132400) is a 2-repeated low-density burst of length 4 with weight 3 over $GF(5)$.

Wolf and Elspas [13] introduced the coding technique called *error-locating codes* (EL Codes). The concept of error location coding lies midway between error detection and error correction. Error location technique provides an attractive alternative to the conventional error detection in decision feedback communications. Wolf and Elspas [13] obtained results in the form of bounds over the number of parity-check digits required for binary codes capable of detecting and locating a single sub-block containing random errors. Further, Dass [3], [4] studied codes locating burst errors and low-density burst errors. In our earlier papers [7], [9] the authors have obtained bounds for codes locating 2-repeated burst errors and 2-repeated low-density burst errors occurring within a single sub-block. This paper extends the study further to the correction of 2-repeated low-density burst errors occurring within a sub-block. The development of codes correcting repeated low-density burst errors within a sub-block economizes in the number of parity-check digits in comparison to the usual low-density burst error locating codes.

In this paper lower and upper bounds on the number of parity check digits required for the existence of such a code are obtained. The paper concludes with an illustration of such a code. Throughout the paper, we consider a block of n

digits, consisting of r check digits and $k = n - r$ information digits, subdivided into s mutually exclusive sub-blocks, each sub-block contains $t = n/s$ digits.

2. Bounds for linear codes correcting 2-repeated low-density bursts

An (n, k) linear EL code over $GF(q)$ capable of detecting and locating a single sub-block containing 2-repeated low-density burst of length b or less with weight w or less must satisfy the following two conditions:

- (i) The syndrome resulting from the occurrence of any 2-repeated low-density burst of length b or less with weight w or less within any one sub-block must be non-zero.
- (ii) The syndrome resulting from the occurrence of any 2-repeated low-density burst of length b or less with weight w or less within a single sub-block must be distinct from the syndrome resulting likewise from any 2-repeated low-density burst of length b or less with weight w or less within *any other* sub-block.

Further, an (n, k) linear code over $GF(q)$ capable of correcting an error requires the syndromes of any two vectors to be distinct irrespective of whether they belong to the same sub-block or to different sub-blocks. So, in order to correct 2-repeated low-density bursts of length b or less with weight w or less lying within a sub-block the following conditions need to be satisfied:

- (iii) The syndrome resulting from the occurrence of any 2-repeated low-density burst of length b or less with weight w or less within a single sub-block must be distinct from the syndrome resulting from any other 2-repeated low-density burst of length b or less with weight w or less within the same sub-block.
- (iv) The syndrome resulting from the occurrence of any 2-repeated low-density burst of length b or less with weight w or less within a single sub-block must be distinct from the syndrome resulting likewise from any 2-repeated low-density burst of length b or less with weight w or less within *any other* sub-block.

Remark 1. We observe that condition (iv) is the same as condition (ii). Also, for computational purposes condition (i) is taken care of by condition (iii). So we need to consider conditions (iii) and (iv) or equivalently conditions (ii) and (iii) for correction of the said type of errors.

We first obtain a lower bound over the number of parity check digits required for such a code.

Theorem 1. *The number of check digits r required for an (n, k) linear code over $GF(q)$, subdivided into s sub-blocks of length t each, that corrects 2-repeated low-density bursts of length b or less with weight w or less lying within a single corrupted sub-block is bounded from below by*

$$\begin{aligned}
 (1) \quad r \geq & \log_q \left\{ 1 + s \left[(q-1)[1+(q-1)]^{(b-1, w-1)} \left(\binom{t-2b+2}{2} (q-1)[1+(q-1)]^{(b-1, w-1)} \right. \right. \right. \\
 & + \binom{t-2b+1}{1} [1+(q-1)]^{(b-1, \min(w, b-1))} \\
 & + \sum_{i=t-2b+2}^{t-b-w+1} [1+(q-1)]^{(t-i-b+1, w)} + \sum_{i=t-b-w+2}^{t-b+1} q^{t-i-b+1} \Big) \\
 & + \left(\binom{t-2b+2}{1} \sum_{k_1=0}^{b-2} \sum_{r_4, r_5, r_6} + \sum_{i=t-2b+3}^{t-b} \sum_{k_1=0}^{t-i-b} \sum_{r_4, r_5, r_6} \right) \binom{k_1}{r_4} \binom{b-k_1-1}{r_5} \binom{k_1}{r_6} \\
 & \cdot (q-1)^{r_4+r_5+r_6+2} \\
 & + \binom{t-b+1}{1} (q-1)[1+(q-1)]^{(b-1; w, \min(2w-1, b-1))} \\
 & \left. + [1+(q-1)]^{(b-1, \min(2w, b-1))} - 1 \right\},
 \end{aligned}$$

where $0 \leq r_4 \leq w-1$, $1 \leq r_5 \leq 2w-2$, $0 \leq r_6 \leq w-2$, $r_4+r_5 \geq w$, $r_4+r_5+r_6 \leq 2w-2$.

Proof. Let V be an (n, k) linear code over $GF(q)$ that corrects 2-repeated low-density bursts of length b or less with weight w or less within a single corrupted sub-block. The maximum number of distinct syndromes available using r check digits is q^r . The proof proceeds by first counting the number of syndromes that are required to be distinct by the two conditions and then setting this number less than or equal to q^r .

Since the code is capable of correcting all errors which are 2-repeated low-density bursts of length b or less with weight w or less within any single sub-block, the syndrome produced by a 2-repeated low-density burst of length b or less with weight w or less in a given sub-block must be distinct from any such syndrome likewise resulting from another 2-repeated low-density burst of length b or less with weight w or less in the same sub-block (refer to condition (iii)). Moreover, syndromes produced by 2-repeated low-density bursts of length b or less with weight w or less in different sub-blocks must also be distinct by condition (iv). Thus, the syndromes of vectors which are 2-repeated low-density bursts of length b or less with weight w or less, whether in the same sub-block or in different sub-blocks, must be distinct. Since there are

$$\begin{aligned}
 & (q-1)[1+(q-1)]^{(b-1, w-1)} \left(\binom{t-2b+2}{2} (q-1)[1+(q-1)]^{(b-1, w-1)} \right. \\
 & \quad \left. + \binom{t-2b+1}{1} [1+(q-1)]^{(b-1, \min(w, b-1))} \right)
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=t-2b+2}^{t-b-w+1} [1 + (q-1)]^{(t-i-b+1, w)} + \sum_{i=t-b-w+2}^{t-b+1} q^{t-i-b+1} \Big) \\
& + \left(\binom{t-2b+2}{1} \sum_{k_1=0}^{b-2} \sum_{r_4, r_5, r_6} + \sum_{i=t-2b+3}^{t-b} \sum_{k_1=0}^{t-i-b} \sum_{r_4, r_5, r_6} \right) \binom{k_1}{r_4} \binom{b-k_1-1}{r_5} \binom{k_1}{r_6} \\
& \cdot (q-1)^{r_4+r_5+r_6+2} \\
& + \binom{t-b+1}{1} (q-1) [1 + (q-1)]^{(b-1, w, \min(2w-1, b-1))} \\
& + [1 + (q-1)]^{(b-1, \min(2w, b-1))} - 1
\end{aligned}$$

2-repeated low-density burst of length b or less with weight w or less within one sub-block of length t excluding the vector of all zeros [5], where $0 \leq r_4 \leq w-1$, $1 \leq r_5 \leq 2w-2$, $0 \leq r_6 \leq w-2$, $r_4 + r_5 \geq w$, $r_4 + r_5 + r_6 \leq 2w-2$, and as there are s sub-blocks in all, we must have at least

$$\begin{aligned}
& 1 + s \left[(q-1) [1 + (q-1)]^{(b-1, w-1)} \left(\binom{t-2b+2}{2} (q-1) [1 + (q-1)]^{(b-1, w-1)} \right. \right. \\
& + \binom{t-2b+1}{1} [1 + (q-1)]^{(b-1, \min(w, b-1))} \\
& + \sum_{i=t-2b+2}^{t-b-w+1} [1 + (q-1)]^{(t-i-b+1, w)} + \sum_{i=t-b-w+2}^{t-b+1} q^{t-i-b+1} \Big) \\
& + \left(\binom{t-2b+2}{1} \sum_{k_1=0}^{b-2} \sum_{r_4, r_5, r_6} + \sum_{i=t-2b+3}^{t-b} \sum_{k_1=0}^{t-i-b} \sum_{r_4, r_5, r_6} \right) \binom{k_1}{r_4} \binom{b-k_1-1}{r_5} \binom{k_1}{r_6} \\
& \cdot (q-1)^{r_4+r_5+r_6+2} \\
& + \binom{t-b+1}{1} (q-1) [1 + (q-1)]^{(b-1, w, \min(2w-1, b-1))} \\
& \left. + [1 + (q-1)]^{(b-1, \min(2w, b-1))} - 1 \right]
\end{aligned}$$

where $0 \leq r_4 \leq w-1$, $1 \leq r_5 \leq 2w-2$, $0 \leq r_6 \leq w-2$, $r_4 + r_5 \geq w$, $r_4 + r_5 + r_6 \leq 2w-2$, distinct syndromes including the all zeros syndrome. Therefore, we must have

$$\begin{aligned}
q^r & \geq 1 + s \left[(q-1) [1 + (q-1)]^{(b-1, w-1)} \left(\binom{t-2b+2}{2} (q-1) [1 + (q-1)]^{(b-1, w-1)} \right. \right. \\
& + \binom{t-2b+1}{1} [1 + (q-1)]^{(b-1, \min(w, b-1))} \\
& + \sum_{i=t-2b+2}^{t-b-w+1} [1 + (q-1)]^{(t-i-b+1, w)} + \sum_{i=t-b-w+2}^{t-b+1} q^{t-i-b+1} \Big)
\end{aligned}$$

$$\begin{aligned}
& + \left(\binom{t-2b+2}{1} \sum_{k_1=0}^{b-2} \sum_{r_4, r_5, r_6} + \sum_{i=t-2b+3}^{t-b} \sum_{k_1=0}^{t-i-b} \sum_{r_4, r_5, r_6} \right) \binom{k_1}{r_4} \binom{b-k_1-1}{r_5} \binom{k_1}{r_6} \\
& \cdot (q-1)^{r_4+r_5+r_6+2} \\
& + \binom{t-b+1}{1} (q-1) [1 + (q-1)]^{(b-1; w, \min(2w-1, b-1))} \\
& + [1 + (q-1)]^{(b-1, \min(2w, b-1))} - 1 \Big]
\end{aligned}$$

Taking logarithm on both the sides we get the result as stated in (1). $\blacksquare$

Remark 2. By taking $s = 1$ the bound obtained in (1) reduces to

$$\begin{aligned}
& \log_q \left((q-1) [1 + (q-1)]^{(b-1, w-1)} \left(\binom{t-2b+2}{2} (q-1) [1 + (q-1)]^{(b-1, w-1)} \right. \right. \\
& + \binom{t-2b+1}{1} [1 + (q-1)]^{(b-1, \min(w, b-1))} \\
& + \sum_{i=t-2b+2}^{t-b-w+1} [1 + (q-1)]^{(t-i-b+1, w)} + \sum_{i=t-b-w+2}^{t-b+1} q^{t-i-b+1} \Big) \\
& + \left(\binom{t-2b+2}{1} \sum_{k_1=0}^{b-2} \sum_{r_4, r_5, r_6} + \sum_{i=t-2b+3}^{t-b} \sum_{k_1=0}^{t-i-b} \sum_{r_4, r_5, r_6} \right) \binom{k_1}{r_4} \binom{b-k_1-1}{r_5} \binom{k_1}{r_6} \\
& \cdot (q-1)^{r_4+r_5+r_6+2} \\
& + \binom{t-b+1}{1} (q-1) [1 + (q-1)]^{(b-1; w, \min(2w-1, b-1))} \\
& + [1 + (q-1)]^{(b-1, \min(2w, b-1))} - 1 \Big)
\end{aligned}$$

which coincides with the result for correction of 2-repeated low-density bursts obtained by Dass and Verma [5].

Remark 3. For $w = b$ the bound obtained in (1) reduces to

$$\log_q \left\{ 1 + s \left[q^{2b-2} \left\{ q + (q-1)^2 \binom{t-2b+2}{2} + (q-1) \binom{t-2b+1}{1} \right\} - 1 \right] \right\}.$$

which coincides with the lower bound on the number of parity check digits required for the blockwise correction of 2-repeated bursts [8].

Several other particular cases by fixing up the parameters may also be deduced which would result into known results obtained earlier by various authors.

In the following result, we derive another bound on the number of check digits required for the existence of such a code. The proof is based on the technique

used to establish Varshamov-Gilbert-Sacks bound by constructing a parity check matrix for such a code (refer Sacks [11], also Theorem 4.7, Peterson and Weldon [10]). This technique not only ensures the existence of such a code but also gives a method for the construction of the code.

Theorem 2. *An (n, k) linear code over $GF(q)$ capable of correcting 2-repeated low-density burst of length b or less with weight w or less occurring within a single sub-block of length t ($4b < t$) can always be constructed using r check digits, where r is the smallest integer satisfying the inequality*

$$\begin{aligned}
 q^r \geq & \left\{ [1 + (q - 1)]^{(b-1, w-1)} \pi_{3, t-b}^{b, w} \right. \\
 & + \left(\sum_{k_1=1}^{b-1} \sum_{r_1, r_2, r_3} \binom{b-k_1-1}{r_1} \binom{k_1}{r_2} \binom{b-k_1-1}{r_3} (q-1)^{r_1+r_2+r_3+1} \right) \pi_{2, t-2b+1}^{b, w} \\
 & + \sum_{k_1=1}^{b-1} \sum_{r_4, r_5, r_6, r_7, r_8} \binom{b-k_1-1}{r_4} \binom{k_1}{r_5} \binom{b-k_1-1}{r_6} \binom{k_1}{r_7} \binom{b-k_1-1}{r_8} \\
 & \times (q-1)^{r_4+r_5+r_6+r_7+r_8+1} \pi_{1, t-3b+2}^{b, w} \\
 & + \sum_{k_1=1}^{b-1} \sum_{r_9, r_{10}, \dots, r_{15}} \binom{b-k_1-1}{r_9} \binom{k_1}{r_{10}} \binom{b-k_1-1}{r_{11}} \binom{k_1}{r_{12}} \binom{b-k_1-1}{r_{13}} \\
 & \times \binom{k_1}{r_{14}} \binom{b-k_1-1}{r_{15}} (q-1)^{r_9+r_{10}+\dots+r_{15}+1} \\
 & + \sum_{k_1=1}^{b-1} \sum_{r_{16}, r_{17}, \dots, r_{20}} \binom{b-k_1-1}{r_{16}} \binom{k_1}{r_{17}} \binom{b-k_1-1}{r_{18}} \binom{k_1}{r_{19}} \binom{b-k_1-1}{r_{20}} \\
 & \times (q-1)^{r_{16}+r_{17}+r_{18}+r_{19}+r_{20}+1} \\
 & + \sum_{k_1=1}^{b-1} \sum_{r_{21}, r_{22}, r_{23}} \binom{b-k_1-1}{r_{21}} \binom{k_1}{r_{22}} \binom{b-k_1-1}{r_{23}} (q-1)^{r_{21}+r_{22}+r_{23}+1} \pi_{1, t-3b+2}^{b, w} \\
 & + \sum_{k_1=1}^{b-1} \sum_{r_{24}, r_{25}, r_{26}} \binom{b-k_1-1}{r_{24}} \binom{k_1}{r_{25}} \binom{b-k_1-1}{r_{26}} (q-1)^{r_{24}+r_{25}+r_{26}+1} \pi_{1, t-2b+1}^{b, w} \\
 & + \sum_{k_1=1}^{b-1} \sum_{r_{27}, r_{28}, \dots, r_{31}} \binom{b-k_1-1}{r_{27}} \binom{k_1}{r_{28}} \binom{b-k_1-1}{r_{29}} \binom{k_1}{r_{30}} \binom{b-k_1-1}{r_{31}} \\
 & \times (q-1)^{r_{27}+r_{28}+r_{29}+r_{30}+r_{31}+1} \\
 & + \sum_{k_1=1}^{b-1} \sum_{r_{32}, r_{33}, r_{34}} \binom{b-k_1-1}{r_{32}} \binom{k_1}{r_{33}} \binom{b-k_1-1}{r_{34}} (q-1)^{r_{32}+r_{33}+r_{34}+1} \\
 & + [1 + (q-1)]^{(b-1; w, 2w-2)} \pi_{2, t-2b+1}^{b, w} + \binom{b-1}{2w-1} (q-1)^{2w-1} \pi_{2, t-b}^{b, w}
 \end{aligned}
 \tag{2}$$

$$\begin{aligned}
& + \sum_{k_1=1}^{b-1} \sum_{r_{35}, r_{36}, r_{37}} \binom{b-k_1-1}{r_{35}} \binom{k_1}{r_{36}} \binom{b-k_1-1}{r_{37}} (q-1)^{r_{35}+r_{36}+r_{37}+1} \pi_{1,t-2b+1}^{b,w} \\
& + \sum_{k_1=1}^{b-1} \sum_{r_{38}, r_{39}, \dots, r_{42}} \binom{b-k_1-1}{r_{38}} \binom{k_1}{r_{39}} \binom{b-k_1-1}{r_{40}} \binom{k_1}{r_{41}} \binom{b-k_1-1}{r_{42}} \\
& \times (q-1)^{r_{38}+r_{39}+r_{40}+r_{41}+r_{42}+1} \\
& + \sum_{k_1=1}^{b-1} \sum_{r_{43}, r_{44}, r_{45}} \binom{b-k_1-1}{r_{43}} \binom{k_1}{r_{44}} \binom{b-k_1-1}{r_{45}} (q-1)^{r_{43}+r_{44}+r_{45}+1} \\
& + [1 + (q-1)]^{(b-1; 2w, 3w-2)} \pi_{1,t-2b+1}^{b,w} + \binom{b-1}{3w-1} (q-1)^{3w-1} \pi_{1,t-b}^{b,w} \\
& + \sum_{k_1=1}^{b-1} \sum_{r_{46}, r_{47}, r_{48}} \binom{b-k_1-1}{r_{46}} \binom{k_1}{r_{47}} \binom{b-k_1-1}{r_{48}} (q-1)^{r_{46}+r_{47}+r_{48}+1} \\
& + [1 + (q-1)]^{(b-1; 3w, \min(4w-1, b-1))} \Big\} \\
& + \left\{ \left([1 + (q-1)]^{(b-1, w-1)} \left\{ q^{w-1} ((q-1)(t-b-w+1) + 1) \right. \right. \right. \\
& \quad + (q-1)^2 \sum_{i=w+1}^b (t-b-i+1) [1 + (q-1)]^{(i-2, w-2)} \Big\} + \sum_{i=w}^{2w-1} \binom{b-1}{i} (q-1)^i \\
& \quad + \sum_{k=1}^{b-1} \sum_{r'_1, r'_2, r'_3} \binom{b-k-1}{r'_1} \binom{k}{r'_2} \binom{b-k-1}{r'_3} (q-1)^{r'_1+r'_2+r'_3+1} \Big) \\
& \times (s-1) \cdot \left((q-1) [1 + (q-1)]^{(b-1, w-1)} \left(\binom{t-2b+2}{2} (q-1) [1 + (q-1)]^{(b-1, w-1)} \right. \right. \\
& \quad + \binom{t-2b+1}{1} [1 + (q-1)]^{(b-1, \min(w, b-1))} + \sum_{i=t-2b+2}^{t-b-w+1} [1 + (q-1)]^{(t-i-b+1, w)} \\
& \quad \left. \left. + \sum_{i=t-b-w+2}^{t-b+1} q^{t-i-b+1} \right) \right. \\
& + \left(\left(\binom{t-2b+2}{1} \sum_{k_1=0}^{b-2} \sum_{r'_4, r'_5, r'_6} + \sum_{i=t-2b+3}^{t-b} \sum_{k_1=0}^{t-i-b} \sum_{r'_4, r'_5, r'_6} \right) \binom{k_1}{r'_4} \binom{b-k_1-1}{r'_5} \binom{k_1}{r'_6} \right. \\
& \quad \cdot (q-1)^{r'_4+r'_5+r'_6+2} \\
& + \binom{t-b+1}{1} (q-1) [1 + (q-1)]^{(b-1; w, \min(2w-1, b-1))} \\
& \left. \left. + [1 + (q-1)]^{(b-1, \min(2w, b-1))} - 1 \right) \right\},
\end{aligned}$$

where $0 \leq r_1 \leq w-2$, $1 \leq r_2 \leq 2w-2$, $0 \leq r_3 \leq w-1$, $r_2 + r_3 \geq w$, $r_1 + r_2 + r_3 \leq 2w-2$; $0 \leq r_4 \leq w-2$, $1 \leq r_5 \leq 2w-2$, $0 \leq r_6 \leq 2w-3$, $0 \leq r_7 \leq 2w-2$, $0 \leq r_8 \leq 2w-2$, $2 \leq r_5 + r_6 \leq 2w-2$, $w \leq r_7 + r_8 \leq 2w-2$, $2w \leq r_5 + r_6 + r_7 + r_8 \leq 3w-2$, $2w \leq r_4 + r_5 + \dots + r_8 \leq 3w-2$; $0 \leq r_9 \leq w-2$, $1 \leq r_{10} \leq 2w-2$, $0 \leq r_{11} \leq 2w-3$, $0 \leq r_{12} \leq 2w-2$, $0 \leq r_{13} \leq 2w-2$, $0 \leq r_{14} \leq 2w-2$, $0 \leq r_{15} \leq 2w-2$, $2 \leq r_{10} + r_{11} \leq 2w-2$, $2 \leq r_{12} + r_{13} \leq 2w-2$, $w \leq r_{14} + r_{15} \leq 2w-2$, $2w \leq r_{12} + r_{13} + r_{14} + r_{15} \leq 3w-2$, $3w \leq r_{10} + r_{11} + \dots + r_{15} \leq 4w-2$, $3w \leq r_9 + r_{10} + \dots + r_{15} \leq 4w-2$; $1 \leq r_{16} \leq 2w-2$, $0 \leq r_{17} \leq 2w-2$, $0 \leq r_{18} \leq 2w-2$, $0 \leq r_{19} \leq 2w-2$, $0 \leq r_{20} \leq 2w-2$, $2 \leq r_{17} + r_{18} \leq 2w-2$, $w \leq r_{19} + r_{20} \leq 2w-2$, $2w \leq r_{17} + r_{18} + r_{19} + r_{20} \leq 3w-2$, $3w-1 \leq r_{16} + r_{17} + \dots + r_{20} \leq 4w-2$; $1 \leq r_{21} \leq 2w-3$, $0 \leq r_{22} \leq 2w-2$, $0 \leq r_{23} \leq 2w-2$, $w \leq r_{22} + r_{23} \leq 2w-2$, $2w-1 \leq r_{21} + r_{22} + r_{23} \leq 3w-3$; $w \leq r_{24} \leq 2w-2$, $0 \leq r_{25} \leq 2w-2$, $0 \leq r_{26} \leq 2w-2$, $w \leq r_{25} + r_{26} \leq 2w-2$, $r_{24} + r_{25} + r_{26} = 3w-2$; $0 \leq r_{27} \leq w-2$, $1 \leq r_{28} \leq 3w-2$, $0 \leq r_{29} \leq 3w-3$, $0 \leq r_{30} \leq 2w-2$, $0 \leq r_{31} \leq 2w-2$, $w+2 \leq r_{28} + r_{29} \leq 3w-2$, $w \leq r_{30} + r_{31} \leq 2w-2$, $3w \leq r_{28} + r_{29} + r_{30} + r_{31} \leq 4w-2$, $3w \leq r_{27} + r_{28} + \dots + r_{31} \leq 4w-2$; $w+1 \leq r_{32} \leq 3w-2$, $0 \leq r_{33} \leq 2w-2$, $0 \leq r_{34} \leq 2w-2$, $w \leq r_{33} + r_{34} \leq 2w-2$, $3w-1 \leq r_{32} + r_{33} + r_{34} \leq 4w-2$; $0 \leq r_{35} \leq w-2$, $1 \leq r_{36} \leq 3w-2$, $0 \leq r_{37} \leq 3w-3$, $r_{36} + r_{37} \geq 2w$, $r_{35} + r_{36} + r_{37} \leq 3w-2$; $0 \leq r_{38} \leq w-2$, $1 \leq r_{39} \leq 2w-2$, $0 \leq r_{40} \leq 2w-3$, $0 \leq r_{41} \leq 3w-2$, $0 \leq r_{42} \leq 3w-2$, $2 \leq r_{39} + r_{40} \leq 2w-2$, $2w \leq r_{41} + r_{42} \leq 3w-2$, $3w \leq r_{39} + r_{40} + r_{41} + r_{42} \leq 4w-2$, $3w \leq r_{38} + r_{39} + \dots + r_{42} \leq 4w-2$; $1 \leq r_{43} \leq 2w-2$, $0 \leq r_{44} \leq 3w-2$, $0 \leq r_{45} \leq 3w-2$, $2w \leq r_{44} + r_{45} \leq 3w-2$, $3w-1 \leq r_{43} + r_{44} + r_{45} \leq 4w-2$; $0 \leq r_{46} \leq w-2$, $1 \leq r_{47} \leq 4w-2$, $0 \leq r_{48} \leq 3w-1$, $r_{47} + r_{48} \geq 3w$, $r_{46} + r_{47} + r_{48} \leq 4w-2$;
 $0 \leq r'_1 \leq w-2$, $1 \leq r'_2 \leq 2w-2$, $0 \leq r'_3 \leq w-1$, $r'_2 + r'_3 \geq w$, $r'_1 + r'_2 + r'_3 \leq 2w-2$;
 $0 \leq r'_4 \leq w-1$, $1 \leq r'_5 \leq 2w-2$, $0 \leq r'_6 \leq w-2$, $r'_4 + r'_5 \geq w$, $r'_4 + r'_5 + r'_6 \leq 2w-2$
and $\pi_{m,n}^{b,w}$ denotes the number of m -repeated low-density bursts of length b or less with weight w or less ($w \leq b$) in a vector of length n , $[1+x]^{(m,r)}$ denotes the incomplete binomial expansion of $(1+x)^m$ upto the term x^r in ascending powers of x and $[1+x]^{(m;r_1,r_2)}$ denotes the incomplete binomial expansion of $(1+x)^m$ from the term x^{r_1} to the term x^{r_2} ($r_1 < r_2$).

Proof. We shall prove the result by constructing an appropriate $(n-k) \times n$ parity check matrix H for the desired code. Suppose that the columns of the first $s-1$ sub-blocks of H and the first $j-1$ columns $h_1, h_2, \dots, h_{j-1}$ of the s^{th} sub-block have been appropriately added. We lay down conditions to add the j^{th} column h_j to the s^{th} sub-block as follows:

Since the code is to correct 2-repeated low-density bursts of length b or less with weight w or less within a single sub-block, therefore, by condition (iii), the syndrome of any 2-repeated low-density burst in any sub-block must be different from the syndrome resulting from any other such burst within the same sub-block. Therefore the j^{th} column h_j can be added provided that h_j is not a linear combination of $w-1$ or fewer columns from the immediately preceding $b-1$ or fewer columns of H together with any w or fewer columns chosen from three sets

of b or fewer consecutive columns each amongst the first $j - 1$ columns. In other words,

$$(3) \quad \begin{aligned} h_j \neq & (\alpha_1 h_i + \alpha_2 h_{i+1} + \cdots + \alpha_{w-1} h_{i+w-2}) \\ & + (\beta_1 h_{i_1} + \beta_2 h_{i_1+1} + \cdots + \beta_w h_{i_1+w-1}) \\ & + (\gamma_1 h_{i_2} + \gamma_2 h_{i_2+1} + \cdots + \gamma_w h_{i_2+w-1}) \\ & + (\delta_1 h_{i_3} + \delta_2 h_{i_3+1} + \cdots + \delta_w h_{i_3+w-1}), \end{aligned}$$

where $\alpha_i, \beta_i, \gamma_i$ and $\delta_i \in GF(q)$ and the h_i are any $w - 1$ or less columns amongst $h_{j-b+1}, \dots, h_{j-1}$ and h_{i_1}, h_{i_2} and h_{i_3} are any w or less columns each from three sets of b or less consecutive columns amongst all the preceding $j - 1$ columns.

The number of linear combinations corresponding to the right hand side of (3) is (refer Dass and Verma [5])

$$(4) \quad \begin{aligned} & [1 + (q - 1)]^{(b-1, w-1)} \pi_{3, j-b}^{b, w} \\ & + \left(\sum_{k_1=1}^{b-1} \sum_{r_1, r_2, r_3} \binom{b-k_1-1}{r_1} \binom{k_1}{r_2} \binom{b-k_1-1}{r_3} (q-1)^{r_1+r_2+r_3+1} \right) \pi_{2, j-2b+1}^{b, w} \\ & + \sum_{k_1=1}^{b-1} \sum_{r_4, r_5, r_6, r_7, r_8} \binom{b-k_1-1}{r_4} \binom{k_1}{r_5} \binom{b-k_1-1}{r_6} \binom{k_1}{r_7} \binom{b-k_1-1}{r_8} \\ & \times (q-1)^{r_4+r_5+r_6+r_7+r_8+1} \pi_{1, j-3b+2}^{b, w} \\ & + \sum_{k_1=1}^{b-1} \sum_{r_9, r_{10}, \dots, r_{15}} \binom{b-k_1-1}{r_9} \binom{k_1}{r_{10}} \binom{b-k_1-1}{r_{11}} \binom{k_1}{r_{12}} \binom{b-k_1-1}{r_{13}} \\ & \times \binom{k_1}{r_{14}} \binom{b-k_1-1}{r_{15}} (q-1)^{r_9+r_{10}+\dots+r_{15}+1} \\ & + \sum_{k_1=1}^{b-1} \sum_{r_{16}, r_{17}, \dots, r_{20}} \binom{b-k_1-1}{r_{16}} \binom{k_1}{r_{17}} \binom{b-k_1-1}{r_{18}} \binom{k_1}{r_{19}} \binom{b-k_1-1}{r_{20}} \\ & \times (q-1)^{r_{16}+r_{17}+r_{18}+r_{19}+r_{20}+1} \\ & + \sum_{k_1=1}^{b-1} \sum_{r_{21}, r_{22}, r_{23}} \binom{b-k_1-1}{r_{21}} \binom{k_1}{r_{22}} \binom{b-k_1-1}{r_{23}} (q-1)^{r_{21}+r_{22}+r_{23}+1} \pi_{1, j-3b+2}^{b, w} \\ & + \sum_{k_1=1}^{b-1} \sum_{r_{24}, r_{25}, r_{26}} \binom{b-k_1-1}{r_{24}} \binom{k_1}{r_{25}} \binom{b-k_1-1}{r_{26}} (q-1)^{r_{24}+r_{25}+r_{26}+1} \pi_{1, j-2b+1}^{b, w} \\ & + \sum_{k_1=1}^{b-1} \sum_{r_{27}, r_{28}, \dots, r_{31}} \binom{b-k_1-1}{r_{27}} \binom{k_1}{r_{28}} \binom{b-k_1-1}{r_{29}} \binom{k_1}{r_{30}} \binom{b-k_1-1}{r_{31}} \\ & \times (q-1)^{r_{27}+r_{28}+r_{29}+r_{30}+r_{31}+1} \\ & + \sum_{k_1=1}^{b-1} \sum_{r_{32}, r_{33}, r_{34}} \binom{b-k_1-1}{r_{32}} \binom{k_1}{r_{33}} \binom{b-k_1-1}{r_{34}} (q-1)^{r_{32}+r_{33}+r_{34}+1} \end{aligned}$$

$$\begin{aligned}
& + [1 + (q - 1)]^{(b-1;w,2w-2)} \pi_{2,j-2b+1}^{b,w} + \binom{b-1}{2w-1} (q-1)^{2w-1} \pi_{2,j-b}^{b,w} \\
& + \sum_{k_1=1}^{b-1} \sum_{r_{35}, r_{36}, r_{37}} \binom{b-k_1-1}{r_{35}} \binom{k_1}{r_{36}} \binom{b-k_1-1}{r_{37}} (q-1)^{r_{35}+r_{36}+r_{37}+1} \pi_{1,j-2b+1}^{b,w} \\
& + \sum_{k_1=1}^{b-1} \sum_{r_{38}, r_{39}, \dots, r_{42}} \binom{b-k_1-1}{r_{38}} \binom{k_1}{r_{39}} \binom{b-k_1-1}{r_{40}} \binom{k_1}{r_{41}} \binom{b-k_1-1}{r_{42}} \\
& \times (q-1)^{r_{38}+r_{39}+r_{40}+r_{41}+r_{42}+1} \\
& + \sum_{k_1=1}^{b-1} \sum_{r_{43}, r_{44}, r_{45}} \binom{b-k_1-1}{r_{43}} \binom{k_1}{r_{44}} \binom{b-k_1-1}{r_{45}} (q-1)^{r_{43}+r_{44}+r_{45}+1} \\
& + [1 + (q - 1)]^{(b-1;2w,3w-2)} \pi_{1,j-2b+1}^{b,w} + \binom{b-1}{3w-1} (q-1)^{3w-1} \pi_{1,j-b}^{b,w} \\
& + \sum_{k_1=1}^{b-1} \sum_{r_{46}, r_{47}, r_{48}} \binom{b-k_1-1}{r_{46}} \binom{k_1}{r_{47}} \binom{b-k_1-1}{r_{48}} (q-1)^{r_{46}+r_{47}+r_{48}+1} \\
& + [1 + (q - 1)]^{(b-1;3w, \min(4w-1, b-1))}
\end{aligned}$$

where conditions on $r_1 \cdots r_{48}$ are as stated in (2).

Further, by condition (iv), h_j can be added to the s^{th} sub-block provided that h_j is not a linear combination of w or fewer columns out of the immediately preceding $b-1$ or fewer columns together with w or fewer columns out of one set of b or fewer consecutive columns from amongst the first $j-1$ columns together with linear combinations of w or fewer columns out of any two sets of b or fewer consecutive columns each within *any other* sub-block, i.e.,

$$\begin{aligned}
h_j \neq & (\alpha'_1 h_i + \alpha'_2 h_{i+1} + \cdots + \alpha'_{w-1} h_{i+w-2}) \\
& + (\beta'_1 h_{i_1} + \beta'_2 h_{i_1+1} + \cdots + \beta'_w h_{i_w}) \\
& + (\gamma'_1 h_{p_1} + \gamma'_2 h_{p_1+1} + \cdots + \gamma'_w h_{p_1+w-1}) \\
& + (\delta'_1 h_{p_2} + \delta'_2 h_{p_2+1} + \cdots + \delta'_w h_{p_2+w-1}),
\end{aligned}$$

where $\alpha'_i, \beta'_i, \gamma'_i, \delta'_i \in GF(q)$, not all γ'_i, δ'_i zero and h_i are any $w-1$ columns amongst $h_{j-b+1}, h_{j-b+2}, \dots, h_{j-1}$ and h'_{i_1} s are any w columns from a set of b consecutive columns from the previously chosen $j-1$ columns of s^{th} sub-block and both h_{p_1} 's and h_{p_2} 's are sets of w columns from any b consecutive columns each from *any other* sub-block.

The number of ways in which the coefficients α'_i and β'_i can be chosen is [2, 6]

$$\begin{aligned}
(5) \quad & \left([1 + (q-1)]^{(b-1, w-1)} \left\{ q^{w-1} ((q-1)(j-b-w+1) + 1) + \right. \right. \\
& (q-1)^2 \sum_{i=w+1}^b (j-b-i+1) [1 + (q-1)]^{(i-2, w-2)} \left. \right\} + \sum_{i=w}^{2w-1} \binom{b-1}{i} (q-1)^i \\
& \left. + \sum_{k=1}^{b-1} \sum_{r'_1, r'_2, r'_3} \binom{b-k-1}{r'_1} \binom{k}{r'_2} \binom{b-k-1}{r'_3} (q-1)^{r'_1+r'_2+r'_3+1} \right),
\end{aligned}$$

where conditions on r'_1, r'_2, r'_3 are as stated in (2).

Also, the number of linear combinations corresponding to the last two terms on the right hand side of (5) is the same as the number of 2-repeated low-density bursts of length b or less with weight w or less within a sub-block of length t , excluding the vector of all zeros and this number in a sub-block of length t , is [5]

$$\begin{aligned}
(6) \quad & (q-1)[1 + (q-1)]^{(b-1, w-1)} \left(\binom{t-2b+2}{2} (q-1)[1 + (q-1)]^{(b-1, w-1)} \right. \\
& + \binom{t-2b+1}{1} [1 + (q-1)]^{(b-1, \min(w, b-1))} \\
& + \sum_{i=t-2b+2}^{t-b-w+1} [1 + (q-1)]^{(t-i-b+1, w)} + \sum_{i=t-b-w+2}^{t-b+1} q^{t-i-b+1} \left. \right) \\
& + \left(\binom{t-2b+2}{1} \sum_{k_1=0}^{b-2} \sum_{r_4, r_5, r_6} + \sum_{i=t-2b+3}^{t-b} \sum_{k_1=0}^{t-i-b} \sum_{r_4, r_5, r_6} \right) \binom{k_1}{r_4} \binom{b-k_1-1}{r_5} \binom{k_1}{r_6} \\
& \cdot (q-1)^{r_4+r_5+r_6+2} \\
& + \binom{t-b+1}{1} (q-1)[1 + (q-1)]^{(b-1; w, \min(2w-1, b-1))} \\
& + [1 + (q-1)]^{(b-1, \min(2w, b-1))} - 1,
\end{aligned}$$

where conditions on r'_4, r'_5, r'_6 are as stated in (2).

Since there are $s-1$ previously chosen sub-blocks, therefore number of such linear combinations becomes

$$(7) \quad (s-1) \cdot \text{expr}(6).$$

Thus, according to condition (iv), the number of linear combinations to which h_j can not be equal to is the product computed in $\text{expr}(5)$ and $\text{expr}(7)$. i.e.

$$(8) \quad \text{expr}(5) \cdot \text{expr}(7).$$

Thus, for blockwise correction of 2-repeated low-density burst errors, the total number of linear combinations that h_j can not be equal to is the sum of linear

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MULTIDIMENSIONAL GENERATING RELATIONS SUGGESTED BY A GENERATING RELATION FOR HYPER-BESSEL FUNCTIONS

M.A. Pathan

*Department of Mathematics
University of Botswana
Private Bag 0022, Gaborone
Botswana
e-mail: mapathan@gmail.com*

Maged G. Bin-Saad

*Department of Mathematics
Aden University
P.O. Box 6014, Khormaksar, Aden
Yemen
e-mail: mgbinsaad@yahoo.com*

Abstract. The authors derive a general theorem on multidimensional generating functions involving arbitrary coefficients. By appropriately specializing these coefficients a number of (known and new) results are shown to follow as applications of the theorem.

Keywords and phrases. multidimensional generating functions; hyper-bessel functions; multivariable hypergeometric functions; multinomial expansion

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1. Introduction and notations

We define the generalized hypergeometric series ${}_A F_B$ [6] with A numerator and B denominator parameters by

$$(1.1) \quad {}_A F_B [(a_A); (b_B); x] = \sum_{m=0}^{\infty} \frac{(a_A)_m}{(b_B)_m} \frac{x^m}{m!}, \quad x \in C,$$

where $(a_A) = a_1, a_2, \dots, a_A$; $b_j \neq 0, -1, -2, \dots$; ($j = 1, 2, \dots, B$) and $(a)_n$ denotes the Pochhammer symbol given by

$$(a)_0 = 1 \quad \text{and} \quad (a)_n = a(a+1)\dots(a+n-1), \quad (n = 1, 2, 3, \dots).$$

A generating function for hyper-Bessel function $J_{m,n,p}(x)$ due to Pathan [3, p. 40-41, Eq. (2.1)] is recalled here in the following form:

$$(1.2) \quad \exp \left[x \left(y + s + t - \frac{1}{yst} \right) \right] = \sum_{m,n,p=-\infty}^{\infty} J_{m,n,p}(x) y^m s^n t^p,$$

where

$$(1.3) \quad J_{m,n,p}(x) = \frac{x^{m+n+p}}{\Gamma(m+1)\Gamma(n+1)\Gamma(p+1)} {}_0F_3 \left[-; m+1, n+1, p+1; -x^4 \right].$$

Motivated by the aforementioned work of Pathan [3], we aim here at presenting a new general theorem on multidimensional generating relations with essentially arbitrary coefficients. Many earlier (known) results given by [3], [4], [7] and [8] are shown to be special cases of our main result.

For convenience, a few conventions and some notations are introduced here:

(1) Boldface letters ($\mathbf{p}_r$) and ($\mathbf{k}_s$) denote vectors of dimension r and s , respectively; for instance, we have

$$(1.4) \quad \mathbf{p}_r = (p_1, \dots, p_r) \text{ and } \mathbf{k}_s = (k_1, \dots, k_s).$$

(2) The symbol $\Omega(p, k)$ denotes a double sequence and the symbol $\Omega(\mathbf{p}_r, \mathbf{k}_s)$ denotes a multiple $(r+s)$ -dimensional sequence as follows:

$$(1.5) \quad \Omega(\mathbf{p}_r, \mathbf{k}_s) = \Omega(p_1, \dots, p_r; k_1, \dots, k_s).$$

(3) Throughout our present investigation, sufficient conditions to ensure absolute convergence of the series involved are understood to hold true for the essentially arbitrary double sequences and the essentially arbitrary multiple sequences of the types described above.

(4) Quite frequently, multiple series are written in simplified notation. Thus

$$(1.6) \quad \begin{aligned} \sum_{\mathbf{p}_r=0}^{\infty} & \text{ means } \sum_{p_1, \dots, p_r=0}^{\infty}, \\ \sum_{\mathbf{p}_r=0}^{\mathbf{n}} & \text{ means } \sum_{p_1=0}^{n_1} \dots \sum_{p_r=0}^{n_r} \text{ and} \\ \sum_{\mathbf{p}_r, \mathbf{k}_s=0}^{\infty} & \text{ means } \sum_{p_1, \dots, p_r=0}^{\infty} \times \sum_{k_1, \dots, k_s=0}^{\infty}. \end{aligned}$$

(5) The symbols M_k^s , N_k^s and R_k^s are defined by

$$(1.7) \quad M_k^s = m_k + \dots + m_s, N_k^s = n_k + \dots + n_s \text{ and } R_k^s = r_k + \dots + r_s$$

respectively. In particular, we have

$$(1.8) \quad M_1^s = m_1 + \dots + m_s, N_1^s = n_1 + \dots + n_s \text{ and } R_1^s = r_1 + \dots + r_s.$$

2. A class of multidimensional generating relations

Our main results on generating functions involving bilateral series are given by the following theorem:

Theorem. *Let $\Omega(p, k)$ be a suitably bounded double sequence of complex numbers then*

$$\begin{aligned}
 (2.1) \quad & \sum_{p,k=0}^{\infty} \Omega(p, k) \frac{u^p}{p!} \frac{\left(x \left(y + s + t + \frac{z}{yst} \right) \right)^k}{k!} \\
 &= \sum_{m,n,r=-\infty}^{\infty} \frac{(xy)^m (xs)^n (xt)^r}{\Gamma(m+1) \Gamma(n+1) \Gamma(r+1)} \\
 & \quad \sum_{p,k=0}^{\infty} \Omega(p, 4k+m+n+r) \frac{u^p}{p!} \frac{x^{4k} z^k}{(m+1)_k (n+1)_k (r+1)_k k!},
 \end{aligned}$$

provided that each member of (2.1) exists.

More generally, let $\Omega(\mathbf{p}_r, \mathbf{k}_s)$ be a suitably bounded multiple $(r+s)$ dimensional sequence of complex numbers, then

$$\begin{aligned}
 (2.2) \quad & \sum_{\mathbf{p}_r, \mathbf{k}_s=0}^{\infty} \Omega(\mathbf{p}_r, \mathbf{k}_s) \prod_{j=1}^r \left\{ \frac{u_j^{p_j}}{p_j!} \right\} \prod_{j=1}^s \left\{ \frac{x_j^{k_j} \left(y_j + s_j + t_j + \frac{z_j}{y_j s_j t_j} \right)^{k_j}}{k_j!} \right\} \\
 &= \sum_{\mathbf{m}_s, \mathbf{n}_s, \mathbf{r}_s=-\infty}^{\infty} \prod_{j=1}^s \left\{ \frac{(x_j y_j)^{m_j} (x_j s_j)^{n_j} (x_j t_j)^{r_j}}{\Gamma(m_j+1) \Gamma(n_j+1) \Gamma(r_j+1)} \right\} \\
 & \quad \cdot \sum_{\mathbf{p}_r, \mathbf{k}_s=0}^{\infty} \Omega(\mathbf{p}_r, 4\mathbf{k}_s + \mathbf{m}_s + \mathbf{n}_s + \mathbf{r}_s) \\
 & \quad \cdot \prod_{j=1}^r \left\{ \frac{u_j^{p_j}}{p_j!} \right\} \cdot \prod_{j=1}^s \left\{ \frac{x_j^{4k_j} z_j^{k_j}}{(m_j+1)_{k_j} (n_j+1)_{k_j} (r_j+1)_{k_j} k_j!} \right\},
 \end{aligned}$$

provided that each member of (2.2) exists.

Proof. Denote, for convenience, the first member of (2.1) by $f(u, x, y, s, t, z)$. Then, as a consequence of the binomial theorem, it is easily seen that:

$$(2.3) \quad f(u, x, y, s, t, z) = \sum_{p,k,m,n,r=0}^{\infty} \Omega(p, k+m+n+r) \frac{u^p x^{k+m+n+r} y^{m-k} s^{n-k} t^{r-k} z^k}{p! k! m! n! r!}.$$

Upon replacing the summation indices m, n and r in (2.3) by $m+k, n+k$ and $r+k$, respectively, if we rearrange the resulting series (which can be justified by

absolute convergence of the series involved), we are led finally to assertion (2.1). The derivation of assertion (2.2) runs parallel to that of (2.1). We skip the details involved.

3. Applications of the theorem

First, in its special case when

$$u \mapsto 0, \quad z = -1 \quad \text{and} \quad \Omega(p, k) = 1,$$

assertion (2.1) would obviously correspond to the generating function (1.2).

Secondly, upon setting

$$u \mapsto 0, \quad z = -1 \quad \text{and} \quad \Omega(p, k) = \Omega_1(p) \Omega_2(k),$$

where

$$\Omega_1(p) = 1 \quad \text{and} \quad \Omega_2(k) = \frac{1}{(b_B)_k},$$

we shall obtain an extension of the result of Pathan (1.2) in the following form:

$$(3.1) \quad {}_0F_B \left[-; b_1, b_2, \dots, b_B; x \left(y + s + t - \frac{1}{yst} \right) \right] = \sum_{m,n,r=-\infty}^{\infty} J_{m,n,r}(b_B; x) y^m s^n t^r,$$

where

$$(3.2) \quad J_{m,n,r}(b_B; x) = {}_0F_{4B+3} \left[-; \frac{D_B}{4}, \frac{D_B+1}{4}, \frac{D_B+2}{4}, \frac{D_B+3}{4}, m+1, n+1, r+1; -x^4 \right],$$

$$(D_B = (b_B) + m + n + r).$$

Now, we consider some applications of assertion (2.2). By setting

$$z_j \mapsto -1, \quad (j = 1, 2, \dots, s), \quad u_i \mapsto 0, \quad (i = 1, 2, \dots, r) \quad \text{and} \quad \Omega(\mathbf{p}_r, \mathbf{k}_s) = 1,$$

(2.2) immediately yields the following simple consequence of Pathan's generating function (1.2):

$$(3.3) \quad \exp \left[\sum_{j=1}^s x_j \left(y_j + s_j + t_j - \frac{1}{y_j s_j t_j} \right) \right]$$

$$= \sum_{\mathbf{m}_s, \mathbf{n}_s, \mathbf{r}_s = -\infty}^{\infty} \prod_{j=1}^s \{ J_{m_j, n_j, r_j}(x_j) y_j^{m_j} s_j^{n_j} t_j^{r_j} \},$$

where

$$J_{m_j, n_j, r_j}(x_j) = \frac{x_j^{m_j + n_j + r_j}}{\Gamma(m_j + 1) \Gamma(n_j + 1) \Gamma(r_j + 1)}$$

$${}_0F_3 \left[-; m_j + 1, n_j + 1, r_j + 1; -x_j^4 \right], \quad (j = 1, 2, \dots, s).$$

Next, if we let

$$z_j \mapsto -z_j s_j^2 t_j, \quad x_j \mapsto 1 \quad \text{and} \quad t_j \mapsto 0, \quad (j = 1, 2, \dots, s),$$

in (2.2) and making use of the series rearrangement technique illustrated fairly fully in [10, Ch. 2], then we shall obtain a generating relation in the following form

$$\begin{aligned} & \sum_{\mathbf{p}_r, \mathbf{k}_s=0}^{\infty} \Omega(\mathbf{p}_r, \mathbf{k}_s) \cdot \prod_{j=1}^r \left\{ \frac{u_j^{p_j}}{p_j!} \right\} \prod_{j=1}^s \left\{ \frac{(y_j + s_j - \frac{z_j s_j}{y_j})^{k_j}}{k_j!} \right\} \\ (3.4) \quad &= \sum_{\mathbf{m}_s=-\infty}^{\infty} \sum_{\mathbf{n}_s=0}^{\infty} \prod_{j=1}^s \left\{ \frac{y_j^{m_j} s_j^{n_j}}{\Gamma(m_j + 1) \Gamma(n_j + 1)} \right\} \sum_{\mathbf{p}_r=0}^{\infty} \sum_{\mathbf{k}_s=0}^{\mathbf{n}} \Omega(\mathbf{p}_r, \mathbf{k}_s + \mathbf{m}_s + \mathbf{n}_s) \\ & \cdot \prod_{j=1}^r \left\{ \frac{u_j^{p_j}}{p_j!} \right\} \cdot \prod_{j=1}^s \left\{ \binom{n_j}{k_j} \frac{(-z_j)^{k_j}}{(m_j + 1)_{k_j}} \right\}. \end{aligned}$$

Indeed, except for some obvious notional variations, it is the main generating relation of Srivastava et al. [7, p. 10, Equation (2.2)], which is a generalization of another known result of Srivastava et al. [8, p.477, Eq. (2,2)]. Further, according to the multinomial expansion [9, p. 329, Eq. 9.4(220)]:

$$(1 - x_1 - \dots - x_r)^{-\nu} = \sum_{k_1, \dots, k_r=0}^{\infty} (\nu)_{k_1+\dots+k_r} \frac{x_1^{k_1}}{k_1!} \dots \frac{x_r^{k_r}}{k_r!}, \quad (|x_1 + \dots + x_r| < 1);$$

(2.2) with $z_j \mapsto z_j s_j t_j$, ($j = 1, 2, \dots, s$), and

$$\Omega(\mathbf{p}_r, \mathbf{k}_s) = \frac{(a)_{k_1+\dots+k_s+p_1+\dots+p_r} (b_1)_{p_1} \dots (b_r)_{p_r}}{(c_1)_{p_1} \dots (c_r)_{p_r}},$$

would yield a generating relation involving Lauricella function $F_A^{(n)}$ of n -variables (see [10, p. 35, Eq. 1.4(1)]) and Exton function of n -variables ${}^{(p)}H_4^{(n)}$ (see [1, p.97, Eq. 3.4(3.5.2)])

$$\begin{aligned} & (1 - X)^{-a} F_A^{(r)} \left[a, b_1, \dots, b_r; c_1, \dots, c_r; \frac{u_1}{(1 - X)}, \dots, \frac{u_r}{(1 - X)} \right] \\ (3.5) \quad &= \sum_{\mathbf{m}_s=-\infty}^{\infty} \sum_{\mathbf{n}_s, \mathbf{r}_s=0}^{\infty} \prod_{j=1}^s \left\{ \frac{y_j^{m_j} s_j^{n_j} t_j^{r_j} x_j^{m_j+n_j+r_j}}{\Gamma(m_j + 1) \Gamma(n_j + 1) \Gamma(r_j + 1)} \right\} (a)_{M_1^s + N_1^s + R_1^s} \\ & {}^{(r)}H_4^{(r+s)} \left[a_{M_1^s + N_1^s + R_1^s}, b_1, \dots, b_r; m_1 + 1, \dots, m_s + 1, c_1, \dots, c_r; \right. \\ & \quad \left. z_1 x_1^2, \dots, z_s x_s^2, u_1, \dots, u_r \right], \end{aligned}$$

where

$$(3.6) \quad X = \sum_{j=1}^s x_j \left(y_j + s_j + t_j + \frac{z_j}{y_j} \right)$$

and M_1^s, N_1^s and R_1^s are given by (1.8). Furthermore, if in assertion (2.2) we set

$$s = r, \quad z_j \mapsto z_j y_j^2 s_j t_j, \quad (s_j, t_j) \mapsto (0, 0), \quad u_j \mapsto \frac{u_j}{y_j}, \quad (j = 1, 2, \dots, r),$$

and

$$(3.7) \quad \Omega(\mathbf{p}_r, \mathbf{k}_r) = (a)_{k_1+\dots+k_r} (b_1)_{p_1} \dots (b_r)_{p_r},$$

we obtain a generating relation for Lauricella function $F_A^{(n)}$ in the form

$$\left(1 - \sum_{j=1}^r x_j y_j (1 + z_j)\right)^{-a} \prod_{j=1}^r \left\{ \left(1 - \frac{u_j}{y_j}\right)^{-b_j} \right\} = \sum_{\mathbf{m}_r=-\infty}^{\infty} \prod_{j=1}^r \left\{ \frac{[x_j y_j (1 + z_j)]^{m_j}}{\Gamma(m_j + 1)} \right\} (a)_{M_1^r}$$

$$(3.8) \quad F_A^{(r)} [a_{M_1^r}, b_1, \dots, b_r; m_1 + 1, \dots, m_r + 1; u_1 x_1 (1 + z_1), \dots, u_r x_r (1 + z_r)].$$

Equation (3.8) provides a generalization of a known result [9, p. 325, Eq. 6.5(91)].

In order to derive another consequence of assertion (2.2), we now set

$$s = r, \quad z_j \mapsto z_j s_j t_j, \quad (s_j, t_j) \mapsto (0, 0) \quad \text{and} \quad u_j \mapsto u_j y_j, \quad (j = 1, 2, 3, \dots, r).$$

We thus obtain the following elegant result

$$(3.9) \quad \begin{aligned} & \sum_{\mathbf{p}_r, \mathbf{k}_r=0}^{\infty} \Omega(\mathbf{p}_r, \mathbf{k}_r) \prod_{j=1}^r \left\{ \frac{(u_j y_j)^{p_j} x_j^{k_j} (y_j + z_j)^{k_j}}{p_j! k_j!} \right\} \\ &= \sum_{\mathbf{m}_r=0}^{\infty} \prod_{j=1}^r \frac{(y_j x_j)^{m_j}}{m_j!} \sum_{\mathbf{k}_r=0}^{\infty} \sum_{\mathbf{p}_r=0}^{\mathbf{m}} \Omega(\mathbf{p}_r, \mathbf{k}_r + \mathbf{m}_r - \mathbf{p}_r) \\ & \quad \prod_{j=1}^r \left\{ \binom{m_j}{p_j} \left(\frac{u_j}{x_j} \right)^{p_j} \frac{(z_j x_j)^{k_j}}{k_j!} \right\}. \end{aligned}$$

Now, if in (3.9), we set $\Omega(\mathbf{p}_r, \mathbf{k}_r) = \frac{(a)_{k_1+\dots+k_r} (b_1)_{p_1+k_1} \dots (b_r)_{p_r+k_r} (d_1)_{p_1} \dots (d_r)_{p_r}}{(c_1)_{p_1+k_1} \dots (c_r)_{p_r+k_r}}$,

we find for the Erdélyi series $H_{n,p}$ (see [9, p. 36, Eq. 1.4(19)]; see also [6]) that

$$(3.10) \quad \begin{aligned} & \sum_{\mathbf{k}_r=0}^{\infty} \prod_{j=1}^r \left\{ \frac{(b_j)_{k_j} [x_j (y_j + z_j)]^{k_j}}{(c_j)_{k_j} k_j!} {}_2F_1[b_j + k_j, d_j; c_j + k_j] \right\} (a)_{K_1^r} \\ &= \sum_{\mathbf{m}_r=0}^{\infty} \prod_{j=1}^r \left\{ \frac{(x_j y_j)^{m_j} (b_j)_{m_j}}{(c_j)_{m_j} m_j!} \right\} (a)_{M_1^r} \end{aligned}$$

$$H_{2r,r} [a_{M_1^r}, b_1 + m_1, \dots, b_r + m_r, -m_1, \dots, -m_r, d_1, \dots, d_r;$$

$$c_1 + m_1, \dots, c_r + m_r; z_1 x_1, \dots, z_r x_r, -u_1/x_1, \dots, -u_r/x_r].$$

For Exton's functions ${}^{(k)}E_D^{(n)}(1)$ and ${}^{(k)}E_D^{(n)}(2)$ (see [1, p. 104, Eq. (3.6.1) and p. 89, Eq. 3.4(3.4.2)]), formula (3.9) similarly yields the following generating relations:

$$\begin{aligned}
& {}_0F_1 \left[\begin{matrix} ---; \\ a; \end{matrix} \sum_{j=1}^r x_j(y_j + z_j) \right] {}_0F_1 \left[\begin{matrix} ---; \\ c; \end{matrix} \sum_{j=1}^q u_j y_j \right] {}_0F_1 \left[\begin{matrix} ---; \\ d; \end{matrix} \sum_{j=q+1}^r u_j y_j \right] \\
&= \sum_{\mathbf{m}_r, \mathbf{k}_r=0}^{\infty} \prod_{j=1}^r \left\{ \frac{(x_j y_j)^{m_j} (x_j z_j)^{k_j}}{m_j! k_j!} \right\} \frac{1}{(a) - M_1^r + K_1^r} \\
(3.11) \quad & {}_{(1)}^{(q)}E_D^{(r)} \left[1 - a - M_1^r - K_1^r, -m_1, \dots, -m_r; c, d; \frac{u_1}{x_1}, \dots, \frac{u_r}{x_r} \right],
\end{aligned}$$

$$(q < r, q = 1, 2, 3, \dots),$$

$$\begin{aligned}
& \left[1 - \sum_{j=1}^q u_j y_j \right]^{-a} \left[1 - \sum_{j=q+1}^r u_j y_j \right]^{-b} \left[1 - \sum_{j=1}^r x_j(y_j + z_j) \right]^{-c} \\
&= \sum_{\mathbf{m}_r, \mathbf{k}_r=0}^{\infty} \prod_{j=1}^r \left\{ \frac{(x_j y_j)^{m_j} (x_j z_j)^{k_j}}{m_j! k_j!} \right\} (c) - M_1^r + k_1^r
\end{aligned}$$

$$\begin{aligned}
(3.12) \quad & {}_{(2)}^{(q)}E_D^{(r)} \left[a, b, -, -m_1, \dots, m_r; 1 - c - M_1^r + K_1^r; \frac{u_1}{x_1}, \dots, \frac{u_r}{x_r} \right], \\
& (q < r, q = 1, 2, 3, \dots),
\end{aligned}$$

Moreover, if in the formula (3.9), we let

$$(3.13) \quad \Omega(\mathbf{p}_r, \mathbf{k}_r) = \Omega_1(\mathbf{p}_r) \Omega_2(\mathbf{k}_r)$$

the left-hand side of (3.10) would reduce at once to a product of two multiple series with essentially arbitrary coefficients. Thus, by assigning suitable special values to the coefficients $\Omega_1(\mathbf{p}_r)$ and $\Omega_2(\mathbf{k}_r)$, we can derive a number of generating functions involving the product of multiple functions and polynomials.

For instance, in view of the following definition of the multiple Bessel polynomials introduced by Pathan and Bin-Saad [4, p. 91, Equation (2.1)]:

$$(3.14) \quad y_{m_1, \dots, m_n}^{(\alpha_1, \dots, \alpha_n; \beta)}(x_1, \dots, x_n) = \sum_{k_1=0}^{m_1} \dots \sum_{k_n=0}^{m_n} (1 + \beta + \nu)_{k_1 + \dots + k_n} \prod_{j=1}^n \left\{ \binom{m_j}{k_j} x_j^{k_j} \right\},$$

where $\nu = \alpha_1 m_1 + \dots + \alpha_n m_n$ by appropriately choosing the multiple sequences $\Omega(\mathbf{p}_r, \mathbf{k}_r)$ in accordance with the definition (3.14), we shall find

$$\begin{aligned}
& y_{s_1, \dots, s_r}^{(\alpha_1, \dots, \alpha_r; \beta)}(x_1(y_1 + z_1), \dots, x_r(y_r + z_r)) \times y_{l_1, \dots, l_r}^{(\gamma_1, \dots, \gamma_r; \delta)}(u_1 y_1, \dots, u_r y_r) \\
&= \sum_{\mathbf{m}_r, \mathbf{k}_r=0}^{\infty} \prod_{j=1}^r \left\{ \frac{(x_j y_j)^{m_j} (x_j z_j)^{k_j} (-s_j)_{k_j + m_j}}{m_j! k_j!} \right\} (1 + \beta + \nu_1)_{M_1^r + K_1^r}
\end{aligned}$$

$$(3.15) \quad F_{1:1;\dots;1}^{1:2;\dots;2} \left[\begin{matrix} 1 + \delta + \nu_2 : -l_1, -m_1; \dots; -l_r, -m_r; \\ -\beta - \nu_1 - M_1^r - K_1^r : 1 + s_1 - m_1 - k_1, \dots, 1 + s_1 - m_r - k_r; \end{matrix} \quad \frac{u_1}{x_1}, \dots, \frac{u_r}{x_r} \right],$$

$\nu_1 = \alpha_1 s_1 + \dots + \alpha_r s_r$, $\nu_2 = \gamma_1 l_1 + \dots + \gamma_r l_r$, $(s_j, l_j) = 1, 2, 3, \dots$, $(j = 1, 2, 3, \dots, r)$, where $F_{p:q_1;\dots;q_{1n}}^{l:m_1;\dots;m_n}[x_1, \dots, x_n]$ is the generalized Kampé de Fériet series of n -variables [9, p. 38, (24) and (25)].

We find it worthwhile to conclude by mentioning that the general assertion (2.2) can be applied further fairly easily to derive a considerable wide variety of generating functions for double, triple and several multivariable hypergeometric polynomials including, for example, Hermite polynomials, Whittaker's function, Humbert's function, the generalized Kampé de Fériet function of two variables, the general triple hypergeometric series (cf. [5] and [9]) and Exton's triple series (see [2]).

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RARELY b -CONTINUOUS FUNCTIONS

Saeid Jafari

*College of Vestsjaelland
South Herrestraede 11, 4200 Slagelse
Denmark
e-mail: jafari@stofanet.dk*

Uğur Şengül

*Department of Mathematics
Faculty of Science and Letters
Marmara University
34722 Göztepe-İstanbul
Turkey
usengul@marmara.edu.tr*

Abstract. In this paper we introduce a new class of functions called rarely b -continuous. Some characterizations and several properties concerning rare b -continuity are obtained.

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1. Introduction and preliminaries

In 1979, Popa [15] introduced the notion of rarely continuous functions as a generalization of weak continuity. The function has been further investigated by Long and Herrington in [12] and by various authors [5], [6], [7], [8], [9], [16]. The first author of this article introduced and investigated weak b -continuity [17] as a generalization of weak continuity. The purpose of the present paper is to introduce concept of rare b -continuity in topological spaces as a generalization of rare continuity and weak b -continuity. We investigate several properties of rarely b -continuous functions. Rare b -continuity implied by rare precontinuity and rare quasi continuity and implies rare β -continuity. The notion of $I.b$ -continuity is also introduced which is weaker than b -continuity and stronger than b -continuity. It is shown that if Y is a regular space, then the function $f : X \rightarrow Y$ is $I.b$ -continuous on X if and only if f is rarely b -continuous on X .

Throughout this paper, X and Y are topological spaces. Recall that a rare set R is a set R such that $\text{int}(R) = \emptyset$. A subset S of a space (X, τ) is called regular open [18] (resp. regular closed [18]) if $S = \text{int}(cl(S))$ (resp. $S = cl(\text{int}(S))$).

A subset S of a space (X, τ) is called semi-open [11] (resp. preopen [13], α -open [14], semi-preopen [2] or β -open [1], b -open [3] or γ -open [4]) if $S \subset cl(\text{int}(S))$ (resp. $S \subset \text{int}(cl(S))$, $S \subset \text{int}(cl(\text{int}(S)))$, $S \subset cl(\text{int}(cl(S)))$, $S \subset cl(\text{int}(S)) \cup \text{int}(cl(S))$). The complement of a semi-open (resp. preopen, α -open, β -open, b -open) set is said to be semi-closed (resp., preclosed, α -closed, β -closed, b -closed).

The family of all open (resp., regular open, semi-open, preopen, α -open, β -open, b -open) sets of X denoted by $O(X)$ (resp., $RO(X)$, $SO(X)$, $PO(X)$, $\alpha O(X)$, $\beta O(X)$, $BO(X)$).

The family of all b -closed sets of X is denoted by $BC(X)$ and the family of all b -open (resp. open, regular open) sets of X containing a point $x \in X$ is denoted by $BO(X, x)$ (resp., $O(X, x)$, $RO(X, x)$).

If S is a subset of a space X , then the b -closure of S , denoted by $bcl(S)$, is the smallest b -closed set containing S . The b -interior of S , denoted by $bint(S)$ is the largest b -open set contained in S . Our next definition contains some types of functions used throughout this paper.

Definition 1 A function $f : X \rightarrow Y$ is called:

- (a) Weakly continuous [10] (resp. weakly b -continuous [17]) if for each $x \in X$ and each open set G containing $f(x)$, there exists $U \in O(X, x)$ (resp., $U \in BO(X, x)$) such that $f(U) \subset cl(G)$.
- (b) b -continuous [4] if $f^{-1}(V)$ is b -open in X for every open set V of Y ;
- (c) Rarely continuous [15] (resp., rarely precontinuous [8], rarely quasicontinuous [16], rarely β -continuous [7]) at $x \in X$ if for each $G \in O(Y, f(x))$, there exists a rare set R_G with $G \cap cl(R_G) = \emptyset$ and $U \in O(X, x)$ (resp., $U \in PO(X, x)$, $U \in SO(X, x)$, $U \in \beta O(X, x)$) such that $f(U) \subset G \cup R_G$.

2. Rarely b -continuous functions

Definition 2 A function $f : X \rightarrow Y$ is called rarely b -continuous at $x \in X$ if for each open set $G \subset Y$ containing $f(x)$, there exists a rare set R_G with $G \cap cl(R_G) = \emptyset$ and $U \in BO(X, x)$ such that $f(U) \subset G \cup R_G$.

Theorem 3 The following statements are equivalent for a function $f : X \rightarrow Y$:

- (a) The function is rarely b -continuous at $x \in X$.
- (b) For each $G \in O(Y, f(x))$, there exists a rare set R_G with $G \cap cl(R_G) = \emptyset$ such that

$$x \in bint(f^{-1}(G \cup R_G)).$$

(c) For each $G \in O(Y, f(x))$, there exists a rare set R_G with $cl(G) \cap R_G = \emptyset$ such that

$$x \in bint(f^{-1}(cl(G) \cup R_G)).$$

(d) For each $G \in RO(Y, f(x))$, there exists a rare set R_G with $G \cap cl(R_G) = \emptyset$ such that

$$x \in bint(f^{-1}(G \cup R_G)).$$

(e) For each $G \in O(Y, f(x))$, there exists $U \in BO(X, x)$ such that

$$int(f(U) \cap (Y - G)) = \emptyset.$$

(f) For each $G \in O(Y, f(x))$, there exists $U \in BO(X, x)$ such that

$$int(f(U)) \subset cl(G).$$

Proof. (a) $\Rightarrow$ (b): Let $x \in X$ and $G \in O(Y, f(x))$. Then, there exists a rare set R_G with $G \cap cl(R_G) = \emptyset$ and $U \in BO(X, x)$ such that $f(U) \subset G \cup R_G$. It follows that $x \in U \subset f^{-1}(G \cup R_G)$, then we have $x \in bint(f^{-1}(G \cup R_G))$.

(b) $\Rightarrow$ (c): Suppose that $G \in O(Y, f(x))$. Then there exists a rare set R_G with $G \cap cl(R_G) = \emptyset$ such that $x \in bint(f^{-1}(G \cup R_G))$. Since $G \cap cl(R_G) = \emptyset$, $R_G \subset Y - G$ where $Y - G = (Y - cl(G)) \cup (cl(G) - G)$. Now, we have $R_G \subset (R_G \cap (Y - cl(G))) \cup (cl(G) - G)$. Set $R^* = R_G \cap (Y - cl(G))$. It follows that R^* is a rare set with $cl(G) \cap R^* = \emptyset$. Therefore, $x \in bint(f^{-1}(G \cup R_G)) \subset bint(f^{-1}(cl(G) \cup R^*))$.

(c) $\Rightarrow$ (d): Assume that $x \in X$ and $G \in RO(Y, f(x))$. Then, there exists a rare set R_G with $cl(G) \cap R_G = \emptyset$ such that $x \in bint(f^{-1}(cl(G) \cup R_G))$. Set $R^* = R_G \cup (cl(G) - G)$. It follows that R^* is a rare set and $G \cap cl(R^*) = \emptyset$. Hence $x \in bint(f^{-1}(cl(G) \cup R_G)) = bint[f^{-1}(G \cup (cl(G) - G) \cup R_G)] = bint[f^{-1}(G \cup R^*)]$.

(d) $\Rightarrow$ (e): Let $G \in O(Y, f(x))$. Then, using $f(x) \in G \subset int(cl(G))$ and the fact that $int(cl(G)) \in RO(Y, f(x))$, there exists a rare set R_G with $int(cl(G)) \cap cl(R_G) = \emptyset$ such that $x \in bint[f^{-1}(int(cl(G)) \cup R_G)]$. Suppose $U = bint[f^{-1}(int(cl(G)) \cup R_G)]$. Then, $U \in BO(X, x)$ and, therefore, $f(U) \subset int(cl(G)) \cup R_G$. We have $int[f(U) \cap (Y - G)] = int(f(U)) \cap (int(Y - G)) \subset int[cl(G) \cup R_G] \cap (Y - cl(G)) \subset ((cl(G) \cup int(R_G)) \cap (Y - cl(G))) = \emptyset$.

(e) $\Rightarrow$ (f): Since $int(f(U) \cap (Y - G)) = int(f(U)) \cap int(Y - G) = int[f(U)] \cap (Y - cl(G)) = \emptyset$, we have $int[f(U)] \subset cl(G)$.

(f) $\Rightarrow$ (a): $G \in O(Y, f(x))$. Then, by (f), there exists $U \in BO(X, x)$ such that $int[f(U)] \subset cl(G)$. Then, $f(U) = [f(U) - int(f(U))] \cup int(f(U)) \subset [f(U) - int(f(U))] \cup cl(G) = [f(U) - int(f(U))] \cup G \cup (cl(G) - G)$. Set $R^* = [f(U) - int(f(U))] \cap (Y - G)$ and $R^{**} = (cl(G) - G)$. Then, R^* and R^{**} are rare sets. Moreover, $R_G = R^* \cup R^{**}$ is a rare set and $cl(R_G) \cap G = \emptyset$ and $f(U) \subset G \cup R_G$. ■

Theorem 4 A function $f : X \rightarrow Y$ is rarely b -continuous if and only if $f^{-1}(G) \subset bint(f^{-1}(G \cup R_G))$ for every open set G in Y , where R_G is a rare set with $G \cap cl(R_G) = \emptyset$.

Proof. Clear from the Theorem 3. ■

Remark 5 Rare b -continuity is implied by rare quasi-continuity and rare precontinuity, and implies rare β -continuity, but the converse implications are not true in general as the following examples shows.

Example 6 Let τ be the usual topology for $\mathbb{R}$ and for $A = [0, 1] \cup ((1, 2) \cap \mathbb{Q})$ define $\sigma = \{\emptyset, \mathbb{R}, A, \mathbb{R} - A\}$. Then, the identity function $f : (\mathbb{R}, \tau) \rightarrow (\mathbb{R}, \sigma)$ is rarely b -continuous but it is neither rarely quasi-continuous nor rarely precontinuous.

Example 7 Let τ be the usual topology for $\mathbb{R}$ and $\sigma = \{\emptyset, \mathbb{R}, [1, 2) \cap \mathbb{Q}\}$. Then, the identity function $f : (\mathbb{R}, \tau) \rightarrow (\mathbb{R}, \sigma)$ is rarely β -continuous but it is not rarely b -continuous.

Definition 8 A function $f : X \rightarrow Y$ is called $I.b$ -continuous at $x \in X$ if for each open set $G \subset Y$ containing $f(x)$, there exists a b -open set U containing x such that $\text{int}[f(U)] \subset G$.

If f has this property at each point $x \in X$, then we say that f is $I.b$ -continuous on X .

Remark 9 It is clear that $I.b$ -continuity is weaker than b -continuity and stronger than rare b -continuity.

Theorem 10 Let Y be a regular space. Then the function $f : X \rightarrow Y$ is $I.b$ -continuous on X if and only if f is rarely b -continuous on X .

Proof. *Necessity* is clear.

Sufficiency. Let f be rarely b -continuous on X . Suppose that $f(x) \in G$, where G is an open set in Y and $x \in X$. By the regularity of Y , there exists an open set G_1 in Y such that $f(x) \in G_1$ and $\text{cl}(G_1) \subset G$. Since f is rarely b -continuous, then there exists $U \in \text{BO}(X, x)$ such that $\text{int}[f(U)] \subset \text{cl}(G_1)$. This implies $\text{int}[f(U)] \subset G$ which means that f is $I.b$ -continuous on X . ■

Definition 11 A function $f : X \rightarrow Y$ is called strongly b -open if for every $U \in \text{BO}(X)$, $f(U)$ is open.

Theorem 12 If a function $f : X \rightarrow Y$ is strongly b -open and rarely b -continuous then f is weakly b -continuous.

Proof. Suppose that $x \in X$ and G is any open set of Y containing $f(x)$. Since f is rarely b -continuous, there exists a rare set R_G with $G \cap \text{cl}(R_G) = \emptyset$ and $U \in \text{BO}(X, x)$ such that $f(U) \subset G \cup R_G$. Then $f(U) \cap (Y - \text{cl}(G)) \subset R_G$. Since f is strongly b -open $f(U) \cap (Y - \text{cl}(G))$ is open. But the rare set R_G has no interior point. Then $f(U) \cap (Y - \text{cl}(G)) = \emptyset$. This implies that $f(U) \subset \text{cl}(G)$. Hence f is weakly b -continuous. ■

Lemma 13 (Andrijevic [3]) *The intersection of an open set and a b -open set is a b -open set.*

Theorem 14 *If a function $f : X \rightarrow Y$ is rarely b -continuous at x and for each open set G containing $f(x)$, $f^{-1}(cl(R_G))$ is closed in X , then f is b -continuous at x where R_G is a rare set with $G \cap cl(R_G) = \emptyset$.*

Proof. Let $G \in O(Y, f(x))$. Since f is rarely b -continuous at x , there exist a rare set R_G with $G \cap cl(R_G) = \emptyset$ and $U \in BO(X, x)$ such that $f(U) \subset G \cup R_G$. Since $G \cap cl(R_G) = \emptyset$, we have

$$f(x) \notin cl(R_G) \text{ and } x \in X - f^{-1}(cl(R_G)).$$

Set $V = U \cap (X - f^{-1}(cl(R_G)))$ then, by Lemma 13,

$$V \in BO(X, x) \text{ and } f(V) \subset f(U) \cap (Y - cl(R_G)) \subset G.$$

Therefore, f is b -continuous at x . ■

Theorem 15 *If a function $f : X \rightarrow Y$ is rarely b -continuous then the graph function $g : X \rightarrow X \times Y$, defined by $g(x) = (x, f(x))$ for every $x \in X$ is rarely b -continuous.*

Proof. Suppose that $x \in X$ and W is any open set containing $g(x)$. Then there exist open sets U and V in X and Y respectively such that $(x, f(x)) \in U \times V \subset W$. Since f is rarely b -continuous, there exists $G \in BO(X, x)$ such that $int[f(G)] \subset cl(V)$. Let $O = U \cap G$. By Lemma 13, $O \in BO(X, x)$ and we have

$$int[g(O)] \subset int[U \times f(G)] \subset U \times cl(V) \subset cl(W).$$

Therefore, g is rarely b -continuous. ■

Definition 16 *A topological space (X, τ) is said to be b -compact [4] if every b -open cover of X has a finite subcover.*

Definition 17 Let $\mathcal{A} = \{G_i\}$ be a class of subsets of X . By rarely union sets [5] of $\mathcal{A}$ we mean $\{G_i \cup R_{G_i}\}$, where each R_{G_i} is a rare set such that each of $\{G_i \cap cl(R_{G_i})\}$ is empty.

Definition 18 A topological space (X, τ) is called rarely almost compact [5] if each open cover of X has a finite subfamily whose rarely union sets cover the space.

Definition 19 A subset K of a space X is said to be:

- (a) b -compact relative to X [4] if for every cover $\{V_\alpha : \alpha \in I\}$ of K by b -open sets of X , there exists a finite subset I_0 of I such that $K \subset \cup\{V_\alpha : \alpha \in I_0\}$,

- (b) rarely almost compact relative to X [5] if for every cover of K by open sets of X , there exists a finite subfamily whose rarely union sets cover K .

Theorem 20 *Let $f : X \rightarrow Y$ be rarely b -continuous and K be a b -compact set in X . Then $f(K)$ is a rarely almost compact subset of Y .*

Proof. Suppose that $\mathcal{G}$ is an open cover of $f(K)$. Set $\mathcal{G}^* = \{V \in \mathcal{G} : V \cap f(K) \neq \emptyset\}$. Then $\mathcal{G}^*$ is an open cover of $f(K)$. Hence for each $x \in K$, there is some $V_x \in \mathcal{G}^*$ such that $f(x) \in V_x$. Since f is rarely b -continuous there exist a rare set R_{V_x} with $V_x \cap cl(R_{V_x}) = \emptyset$ and a b -open set U_x containing x such that $f(U_x) \subset V_x \cup R_{V_x}$. Hence there is a subfamily $\{U_{x_i}\}_{x_i \in K_0}$ which covers K , where K_0 is a finite subset of K . The subfamily $\{V_{x_i} \cup R_{V_{x_i}}\}_{x_i \in K_0}$ also covers $f(K)$. ■

Lemma 21 *If $g : Y \rightarrow Z$ is continuous and one-to-one, then g preserves rare sets [12].*

Theorem 22 *If $f : X \rightarrow Y$ is a rarely b -continuous surjection and $g : Y \rightarrow Z$ is continuous and one-to-one, then $g \circ f : X \rightarrow Z$ is rarely b -continuous.*

Proof. Suppose that $x \in X$ and $g(f(x)) \in V$, where V is open set Z . By hypothesis, g is continuous, therefore there exists an open set $G \subset Y$ containing $f(x)$ such that $g(G) \subset V$. Since f is rarely b -continuous, there exists rare set R_G with $G \cap cl(R_G) = \emptyset$ and a b -open set U containing x such that $f(U) \subset G \cup R_G$. It follows from Lemma 21 that $g(R_G)$ is a rare set in Z . Since R_G is a subset of $Y - G$ and g is injective, we have $cl(g(R_G)) \cap V = \emptyset$. This implies that $g(f(U)) \subset V \cup g(R_G)$. Hence the result follows. ■

Definition 23 A function $f : X \rightarrow Y$ is called pre- b -open if for every $U \in BO(X)$, $f(U) \in BO(Y)$.

Theorem 24 *If $f : X \rightarrow Y$ is a pre b -open surjection and $g : Y \rightarrow Z$ a function such that $g \circ f : X \rightarrow Z$ is rarely b -continuous. Then g is rarely b -continuous.*

Proof. Let $y \in Y$ and $x \in X$ such that $f(x) = y$. Let G be an open set containing $g(f(x))$. Then there exists a rare set R_G with $G \cap cl(R_G) = \emptyset$ and a b -open set U containing x such that $g(f(U)) \subset G \cup R_G$. But $f(U)$ is a b -open set containing $f(x) = y$ such that $g(f(U)) = (g \circ f)(U) \subset G \cup R_G$. This shows that g is rarely b -continuous at y . ■

Definition 25 A function $f : X \rightarrow Y$ satisfies interiority rare b condition if $bint(f^{-1}(G \cup R_G)) \subset f^{-1}(G)$ for each open set G in Y , where R_G is a rare set with $G \cap cl(R_G) = \emptyset$.

Theorem 26 *If $f : X \rightarrow Y$ is rarely b -continuous and satisfies interiority rare b condition then f is b -continuous.*

Proof. Since f is rarely b -continuous by Theorem 4, we have

$$f^{-1}(G) \subset \text{bint}(f^{-1}(G \cup R_G)),$$

where G is an open set in Y and R_G is a rare set with $G \cap \text{cl}(R_G) = \emptyset$. On the other hand by the interiority rare b condition we have $\text{bint}(f^{-1}(G \cup R_G)) \subset f^{-1}(G)$. Therefore $f^{-1}(G)$ is b -open in X and consequently f is b -continuous. ■

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A STUDY ON AUGMENTED GRADED RINGS¹

Mashhoor Refai

Professor of Mathematics
Vice President for Academic Affairs
Princess Sumaya University for Technology
Amman, Jordan
e-mail: m.refai@psut.edu.jo

Abstract. In this paper, we study some properties of augmented graded rings and give the relationships between augmented graded rings and other types of well known strongly graded rings.

Keywords: augmented graded rings, augmented graded modules, strongly graded rings.

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1. Introduction

Let G be a group with identity e . A ring R is said to be G -graded ring if there exist additive subgroups R_g of R such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The G -graded ring is denoted by (R, G) . We denote by $\text{supp}(R, G)$ the support of G which is defined to be $\{g \in G : R_g \neq 0\}$. The elements of R_g are called homogeneous of degree g and R_e (the identity component of R) is a subring of R and $1 \in R_e$ (see [4]). For $x \in R$, x can be written uniquely as $\sum_{g \in G} x_g$ where x_g is the component of x in R_g . Also we write $h(R) = \bigcup_{g \in G} R_g$.

One of the most important problems in graded ring theory is to study the link between a certain property for R or (R, G) and R_e (the identity component of R). One can think about grading R_e by a group G . These types of rings appear naturally when we deal with the group ring of G over a G -graded ring R .

In [5], we studied the G -graded rings in which the identity component is itself a G -graded ring satisfying some related conditions with the graduation of G . We called these rings augmented G -graded rings. Also, we gave the relationship between these new rings and the stronger properties of G -graded rings given in [6].

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This simple effort is useful for studying those rings in detail and that is a good tool to solve many problems in Graded Ring Theory. In [6], we defined three successively stronger properties that a grading may have and investigated the relationship between these new strongly gradings and the stronger nondegenerate and faithful properties which are motivated by the work of Cohen and Rowen.

In this paper, we follow the work done in [5], [6] to give further properties of augmented graded rings. Also, we give some relationships between these rings and other types of graded rings like nondegenerate, faithful, strong, first strong, etc.

2. Preliminaries

In this section, we give some preliminaries concerning graded rings. For more details, one can look in [4], [5], [6], [8].

Definition 2.1. Let R be a G -graded ring. Then (R, G) is said to be left (resp., right) nondegenerate if for every $r \in R - \{0\}$, $(rR)_e = \{(rx)_e : x \in R\} \neq 0$ (resp., if for every $r \in R - \{0\}$, $(Rr)_e = \{(xr)_e : x \in R\} \neq 0$). Otherwise, (R, G) is left degenerate (resp., right degenerate). Also, (R, G) is nondegenerate if it is both left and right nondegenerate.

Lemma 2.2. Let R be a G -graded ring and $a_g \in R_g$ with $g \in G$. Then

1. $(a_g R)_e = a_g R_{g^{-1}}$.
2. $(R a_g)_e = R_{g^{-1}} a_g$.

Proposition 2.3. Let R be a G -graded ring. Then (R, G) is nondegenerate if and only if $R_{g^{-1}} a_g \neq 0$ and $a_g R_{g^{-1}} \neq 0$ for all $a_g \in R_g - \{0\}$.

Corollary 2.4. Suppose (R, G) is nondegenerate. If $g \in \text{supp}(R, G)$ then $g^{-1} \in \text{supp}(R, G)$.

Definition 2.5. Let R be a G -graded ring. Then (R, G) is said to be left (resp., right) faithful if for each $a_g \in R_g - \{0\}$, $a_g R_h \neq 0$ for all $g, h \in G$ (resp., $R_h a_g \neq 0$ for all $g, h \in G$). We say that (R, G) is faithful if it is both left and right faithful.

Definition 2.6. A G -graded ring (R, G) is said to be strongly graded if $R_g R_h = R_{gh}$ for all $g, h \in G$.

Proposition 2.7. Let R be a G -graded ring. Then, (R, G) is strong if and only if $R_g R_{g^{-1}} = R_e$ for all $g \in G$. This is equivalent to $1 \in R_g R_{g^{-1}}$ for all $g \in G$.

Definition 2.8. Let R be a G -graded ring. Then, (R, G) is first strong if $1 \in R_g R_{g^{-1}}$, for all $g \in \text{supp}(R, G)$, or, equivalently, if $R_g R_{g^{-1}} = R_e$, for all $g \in \text{supp}(R, G)$.

Remark 2.9. Every strongly graded is first strong and faithful but the converse is not true (see [6]). Indeed, (R, G) is strong if and only if it is first strong and $\text{supp}(R, G) = G$.

Proposition 2.10. *If (R, G) is first strong then $\text{supp}(R, G)$ is a subgroup of G .*

Proposition 2.11. *Let R be a first strongly graded ring. Suppose R_g is simple R_e -submodule of R for all $g \in G$. Then R is gr-simple.*

Proof. Suppose R_g is simple R_e -module for all $g \in G$. Let $0 \neq Y \subseteq R$ be a graded R -submodule of R . Then there exists $g \in G$ such that $Y_g = Y \cap R_g \neq 0$. But $Y_g \subseteq R_g$ is a nonzero R_e -submodule. Hence $Y_g = Y \cap R_g = R_g$ or $R_g \subseteq Y$ and then $R_e = R_{g^{-1}}R_g \subseteq R_{g^{-1}}Y \subseteq Y$. Thus $1 \in Y$. Since 1 is invertible we get $Y = R$. Therefore, R is gr-simple. ■

Definition 2.12. Let R be a G -graded ring. Then (R, G) is said to be second strong if $\text{supp}(R, G)$ is a monoid in G and $R_g R_h = R_{gh}$ for all $g, h \in \text{supp}(R, G)$.

Remark 2.13. Every first strongly graded ring is second strong but the converse is not true in general (see [6]).

Definition 2.14. A G -graded ring R is said to be left semifaitful if

1. $\text{supp}(R, G)$ is a subgroup of G .
2. For all $g, h \in \text{supp}(R, G)$ and $a_g \in R_g - \{0\}$ we have $a_g R_h \neq 0$.

Similarly we define right semifaitful graded rings. So, (R, G) is called semifaitful if it is both left and right semifaitful.

Proposition 2.15. *Every semifaitful graded ring is nondegenerate.*

Proof. Suppose (R, G) is semifaitful. Let $g \in \text{supp}(R, G)$. Then $g^{-1} \in \text{supp}(R, G)$ and hence $a_g R_{g^{-1}} \neq 0$ for all $a_g \in R_g - \{0\}$. Thus (R, G) is left nondegenerate. Similarly we prove the right nondegeneracy. ■

Corollary 2.16. *Every first strongly graded ring is semifaitful.*

Proof. Assume (R, G) is first strong. By Proposition 2.10, $\text{supp}(R, G)$ is a subgroup of G . Let $g, h \in \text{supp}(R, G)$ and $a_g \in R_g - \{0\}$. Suppose for the contrary $a_g R_h = 0$. Thus $0 = a_g R_h R_{h^{-1}} = a_g R_e$. Hence $a_g = 0$ which contradicts the fact $a_g \neq 0$ and hence R is left semifaitful. Similarly, we show R is right semifaitful. ■

Proposition 2.17. *Let R be a G -graded ring. Then (R, G) is faithful if and only if (R, G) is semifaitful and $\text{supp}(R, G) = G$.*

Definition 2.18. A ring R is said to be augmented G -graded ring if it satisfies the following conditions:

1. $R = \bigoplus_{g \in G} R_g$ where R_g is an additive subgroup of R and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$.

2. If R_e is the identity component of the graduation, then $R_e = \bigoplus_{g \in G} R_{e-g}$ where R_{e-g} is an additive subgroup of R_e and $R_{e-g}R_{e-h} \subseteq R_{e-gh}$ for all $g, h \in G$.
3. For each $g \in G$, there exists $r_g \in R_g$ such that $R_g = \bigoplus_{h \in G} R_{e-h}r_g$. We assume $r_e = 1$.
4. If $g, h \in G$ and r_g, r_h are both nonzero, then $r_g r_h = r_{gh}$ and for all $x, y \in R_e$ we have $(xr_g)(yr_h) = xy r_{gh}$.

Remark 2.19. By the last definition we have:

1. Condition 3 of the definition implies $R_h = R_e r_h$ for all $h \in G$.
2. R_g is a G -graded R_e -module with the usual multiplication on R and with the graduation $R_{g-h} = R_{e-h}r_g$ for all $h \in G$.
3. $R_{g-h}R_{g'-h'} \subseteq R_{gg'-hh'}$ for all $g, g', h, h' \in G$.
Also, $R_{g-h}R_{g'-h'} = R_{e-h}r_g R_{e-h'}r_{g'}$.
4. If $r_g, r_{g^{-1}}$ are both nonzero, then $r_g R_e = R_e r_g = R_g$.

Corollary 2.20. *Let R be an augmented G -graded ring where $\text{supp}(R, G) = G$. Then (R, G) is strong and hence faithful.*

3. Properties of augmented graded rings

In this section, we give some properties of the augmented graded rings as well as its relationships to other kinds of graded rings.

Proposition 3.1. *Let R be a G -graded ring such that $\text{supp}(R, G)$ is a subgroup of G . Then (R, G) is first strong.*

Proof. Let $g \in \text{supp}(R, G)$. Then $g^{-1} \in \text{supp}(R, G)$, i.e., $R_g \neq 0$ and $R_{g^{-1}} \neq 0$. Since $R_g = R_e r_g$ and $R_{g^{-1}} = R_e r_{g^{-1}}$, we get $r_g \neq 0$ and $r_{g^{-1}} \neq 0$. Hence $r_g r_{g^{-1}} = r_{gg^{-1}} = r_e = 1$ and then $1 \in R_g R_{g^{-1}}$. Therefore, R is first strong. ■

Proposition 3.2. *Suppose R is augmented G -graded ring such that $\text{supp}(R, G)$ is a monoid in G . Then (R, G) is second strong.*

Proof. Let $g, h \in \text{supp}(R, G)$. Then $r_g \neq 0$ and $r_h \neq 0$. So, $r_g r_h = r_{gh}$ implies $r_{gh} \in R_g R_h$ and hence $R_{gh} = R_e r_{gh} \subseteq R_e R_g R_h = R_g R_h \subseteq R_{gh}$. Thus $R_g R_h = R_{gh}$ for all $g, h \in \text{supp}(R, G)$. Since $\text{supp}(R, G)$ is a monoid in G we have (R, G) is second strong. ■

Corollary 3.3. *If (R, G) is augmented graded ring then $R_g R_h = R_{gh}$ for all $g, h \in \text{supp}(R, G)$.*

Corollary 3.4. *Suppose (R, G) is augmented graded ring such that R has no zero divisors in $h(R)$. Then (R, G) is second strong.*

Proof. If $g, h \in \text{supp}(R, G)$ then $r_g \neq 0$ and $r_h \neq 0$ and hence $r_{gh} = r_g r_h \neq 0$ where $r_g, r_h \in h(R)$. Thus $gh \in \text{supp}(R, G)$ and hence $\text{supp}(R, G)$ is a monoid in G . By Proposition 3.2, (R, G) is second strong. ■

Proposition 3.5. *Let R be an augmented G -graded ring. Then, the following are equivalent:*

1. (R, G) is nondegenerate.
2. $\text{supp}(R, G)$ is a subgroup of G .
3. (R, G) is first strong.
4. (R, G) is semifaitful.

Proof. (1) implies (2): Assume (R, G) is nondegenerate graded ring. Let $g, h \in \text{supp}(R, G)$. Then $g^{-1} \in \text{supp}(R, G)$ and hence $r_g r_{g^{-1}} = r_e = 1$ where $R_g = R_e r_g$ for $g \in G$. Suppose for the contrary $gh \notin \text{supp}(R, G)$. Then $r_{gh} = 0$. But $r_{gh} = r_g r_h$ implies $r_g r_h = 0$ and hence $r_{g^{-1}} r_g r_h = 0$ or $r_h = 0$ and then $h \notin \text{supp}(R, G)$ which is a contradiction. Therefore, $\text{supp}(R, G)$ is a monoid in G . By Corollary 2.4, $\text{supp}(R, G)$ is a subgroup of G .

(2) implies (3): Follows from Proposition 3.1.

(3) implies (4): Follows from Proposition 2.15.

(4) implies (1): Follows from Proposition 2.14. ■

Corollary 3.6. *Let (R, G) be an augmented G -graded ring. Then the following are equivalent:*

1. $\text{supp}(R, G) = G$.
2. (R, G) is strongly graded.
3. (R, G) is faithful.

Proof. Follows directly from Remark 2.9 and Proposition 3.5. ■

Proposition 3.7. *Let R be an augmented G -graded ring such that $\text{supp}(R, G) = G$. Then the following are equivalent:*

1. (R, G) is faithful.
2. (R, G) is semifaitful.
3. (R, G) is nondegenerate.
4. (R, G) is first strong.
5. (R, G) is second strong.
6. (R, G) is strong.

Proof. (1) implies (2): Follows from Proposition 2.17.

(2) implies (3): Follows from Proposition 2.14.

(3) implies (4): Follows from Proposition 3.5.

(4) implies (5): Follows from Remark 2.13.

(5) implies (6): Follows from Corollary 2.20.

(6) implies (1): Follows from Remark 2.9. $\blacksquare$

Corollary 3.8. *If (R, G) is augmented and $\text{supp}(R, G) = G$, then by Corollary 2.20, (R, G) is faithful, semif faithful, nondegenerate, strong, first strong and second strong at the same time.*

After giving the relationships between the augmented graded rings and other types of strongly graduations, in addition to, nondegeneracy, faithfulness and semif faithfulness, we present some properties of the augmented graded rings.

Proposition 3.9. *Let R be an augmented G -graded ring such that r_g is an R_e -torsion free element (i.e., $mr_g \neq 0$ for all $m \in R_e$) for all $g \in \text{supp}(R, G)$. Then R_g is cyclic projective R_e -submodule of R and hence R is projective R_e -module. In particular, R and R_g are free R_e -modules.*

Proof. Let $g \in G$. If $g \notin \text{supp}(R, G)$, then done. Suppose $g \in \text{supp}(R, G)$. Then $r_g \neq 0$ and $R_g = R_e r_g$. Since r_g is R_e -torsion free element, r_g is linearly independent over R_e and then R_g is cyclic projective R_e -module being free R_e -module. Since $R = \bigoplus_{g \in G} R_g$, the set $F = \{r_g : g \in \text{supp}(R, G)\}$ forms a basis of R as an R_e -module. Thus R is free and hence projective. $\blacksquare$

Corollary 3.10. *By Proposition 3.9, if $\text{supp}(R, G)$ is finite, then R is finitely generated free R_e -module. Clearly, $R \cong R_e^{|\text{supp}(R, G)|} = R_e^{|F|}$ where $F = \{r_g : g \in \text{supp}(R, G)\}$.*

Proposition 3.11. *Let R be an augmented G -graded ring such that $\text{supp}(R, G)$ is a monoid in G . Then R is commutative if and only if R_e is commutative and $\text{supp}(R, G)$ is abelian.*

Proof. Suppose R is commutative. Then clearly R_e is commutative. Let $g, h \in \text{supp}(R, G)$. Then $r_g \neq 0$ and $r_h \neq 0$. But $r_{gh} = r_g r_h = r_h r_g = r_{hg}$. Since $gh, hg \in \text{supp}(R, G)$, $r_{gh} = r_{hg} \neq 0$. Thus $gh = hg$ and $\text{supp}(R, G)$ is abelian monoid in G . For the converse, suppose R_e is commutative and $gh = hg$ for all $g, h \in \text{supp}(R, G)$. We prove R is commutative step by step.

Step 1: We show $x_g y_h = y_h x_g$ for all $g, h \in G$. Let $g, h \in G$, $x_g \in R_g = R_e r_g$, $y_h \in R_h = R_e r_h$. If either $g \notin \text{supp}(R, G)$ or $h \notin \text{supp}(R, G)$, then done. Suppose $g, h \in \text{supp}(R, G)$. Then $gh = hg$ and hence $r_{gh} = r_{hg}$. Suppose $x_g = x r_g$ and $y_h = y r_h$ for some $x, y \in R_e$. Then $x_g y_h = (x r_g)(y r_h) = x y r_{gh} = y x r_{hg} = (y r_h)(x r_g) = y_h x_g$.

Step 2: We claim that $x y_g = y_g x$ for all $x \in R$, $y_g \in R_g$ and $g \in G$. Fix $g \in G$. Suppose $x \in R$ and $y_g \in R_g$. If $g \notin \text{supp}(R, G)$, then $y_g = 0$ and done. Assume $g \in \text{supp}(R, G)$ and let $x = \sum_{h \in G} x_h$ where $x_h \in R_h$ for all $h \in G$. Then

$$x y_g = \left(\sum_{h \in G} x_h \right) y_g = \sum_{h \in G} (x_h y_g) = \sum_{h \in G} (y_g x_h) = y_g \left(\sum_{h \in G} x_h \right) = y_g x.$$

Step 3: We now show that $xy = yx$ for all $x, y \in R$. Let $x, y \in R$ and suppose $y = \sum_{g \in G} y_g$ where $y_g \in R_g$ for all $g \in G$. Then we have

$$xy = x \left(\sum_{g \in G} y_g \right) = \sum_{g \in G} xy_g = \sum_{g \in G} y_g x = \left(\sum_{g \in G} y_g \right) x = yx.$$

Therefore, R is commutative. ■

Proposition 3.12. *Let R be an augmented G -graded ring. Then R_g is simple left R_e -module for all $g \in G$ if and only if R_e is simple left R_e -module.*

Proof. Suppose R_e is simple left R_e -module and $g \in G$. If $g \notin \text{supp}(R, G)$, then $R_g = 0$ is simple R_e -module. Suppose $g \in \text{supp}(R, G)$ and let $r_g \in R_g$ such that $R_g = R_e r_g \neq 0$. Let $M \subseteq R_g$ be a nonzero R_e -submodule. We show $M = X r_g$ where X is an R_e -submodule of R_e , i.e., X is a left ideal in R_e . Let $X = \{x \in R_e : x r_g \in M\}$. Then clearly, $0 \in X$, and hence $X \neq \emptyset$. Since $M \subseteq R_g$ and $M \neq 0$, there exists $a_g \in M - \{0\}$ such that $a_g = x r_g$ for some $x \in R_e$. So, $x \in X - \{0\}$, i.e., $X \neq 0$. Let $x, y \in X$. Then $x r_g, y r_g \in M$. Hence $(x - y) r_g = x r_g - y r_g \in M$. Since $x - y \in R_e$, $x - y \in X$, i.e., X is a subgroup of R_e . Let $x \in X$ and $y_e \in R_e$. Then $x r_g \in M$ and hence $(y_e x) r_g = y_e (x r_g) \in R_e M = M$. Since $y_e x \in R_e$, $y_e x \in X$. Thus X is an R_e -submodule of R_e . Now, if $m \in M \subseteq R_g$, then $m = y r_g$ for some $y \in R_e$. So, $y \in X$ and $m \in X r_g$. If $m \in X r_g$, then $m = x r_g$ for some $x \in X$. By the definition of X we have $m \in M$. Therefore, $M = X r_g$. Since R_e is simple R_e -module, $X = R_e$ and hence $M = R_e r_g = R_g$, i.e., R_g is simple R_e -module for all $g \in G$. The converse is obvious. ■

Corollary 3.13. *Let R be an augmented G -graded ring. If R is gr -simple, then R_g is simple left R_e -module for all $g \in G$.*

Proof. Follows directly from Proposition 3.12. ■

Corollary 3.14. *Suppose R is an augmented G -graded ring and $N \subseteq R_g$ is R_e -submodule. Then $N = X r_g$ where X is a left ideal in R_e for all $g \in G$.*

Corollary 3.15. *Suppose R is an augmented G -graded ring such that $\text{supp}(R, G)$ is a subgroup of G . Then the following are equivalent:*

1. (R, G) is gr -simple.
2. R_e is simple ring (R_e -module).
3. R_g is simple R_e -module for all $g \in G$.

Proof. (1) implies (2): Trivial.

(2) implies (3): Follows from Proposition 3.12.

(3) implies (1): Follows from Proposition 2.11. ■

Remark 3.16. Suppose R is an augmented G -graded ring such that $\text{supp}(R, G)$ is a subgroup of G . Then we have the following:

R is simple ring $\Rightarrow R$ is gr-simple ring $\Leftrightarrow R_g$ is simple R_e -module for all $g \in G \Leftrightarrow R_e$ is simple ring.

Proposition 3.17. Let R be an augmented G -graded ring such that $\text{supp}(R, G) = H$ is a subgroup of G and M be a G -graded left R -module. Let x be a homogeneous element in M . Then x is R -torsion free if and only if x is R_e -torsion free.

Proof. Suppose $x \in h(M)$ and x is R -torsion free. Then x is R_e -torsion free. Conversely, suppose $x \in h(M)$ and x is R_e -torsion free. Let $\omega \in R$ such that $\omega x = 0$. Suppose $\omega = \sum_{g \in G} \omega_g = \sum_{g \in H} \omega_g$. Hence $\sum_{g \in H} \omega_g x = 0$ and so $\omega_g x = 0$ for all $g \in H$. Let $g \in H$ and assume $\omega_g = r_g \lambda^{(g)} = \lambda^{(g)} r_g$ where $\lambda^{(g)} \in R_e$ for all $g \in H$ (Notice that H is a subgroup of G). So, $(r_g \lambda^{(g)})x = 0$ or $r_g(\lambda^{(g)}x) = 0$ and hence $R_e r_g(\lambda^{(g)}x) = 0$, i.e., $R_g(\lambda^{(g)}x) = 0$. Thus $R_{g^{-1}}R_g(\lambda^{(g)}x) = 0$. So, $R_e(\lambda^{(g)}x) = 0$, i.e., $\lambda^{(g)}x = 0$. Hence $\lambda^{(g)} = 0$ for all $g \in H$. Therefore, $\omega_g = 0$ for all $g \in H$ and hence $\omega = 0$, i.e., x is R -torsion free. ■

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WEAK LATTICES¹**Ivan Chajda**

*Palacký University Olomouc
Department of Algebra and Geometry
Trida 17.listopadu 12
77146 Olomouc
Czech Republic
ivan.chajda@upol.cz*

Helmut Länger

*Vienna University of Technology
Institute of Discrete Mathematics and Geometry
Wiedner Hauptstraße 8-10
1040 Vienna
Austria
h.laenger@tuwien.ac.at*

Abstract. The ordered set induced by a BCK-algebra $\mathcal{A}$ can be equipped with a binary term operation on $\mathcal{A}$ such that the resulting structure is a so-called weak semilattice. If this structure is endowed with an antitone involution we can introduce a second binary operation and the structure arising this way is called a weak lattice. Properties of weak lattices and weak semilattices are investigated and connections to directoids and semilattices are established. Moreover, a derived structure similar to basic algebras is introduced and called a skew basic algebra. An axiomatization of these algebras is presented. It is shown that every bounded poset can be organized into a weak lattice and the number of non-isomorphic weak lattices of cardinality less than five is determined.

Keywords: BCK-algebra, weak semilattice, weak lattice, directoid, semilattice, antitone involution, skew basic algebra, de Morgan laws.

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1. Introduction

Let $(L, \leq)$ be a poset. For $x, y \in L$ let $U(x, y)$ respectively $L(x, y)$ denote the set of all upper respectively lower bounds of $\{x, y\}$. The poset $(L, \leq)$ is called *up-directed* respectively *down-directed* if $U(x, y) \neq \emptyset$ respectively $L(x, y) \neq \emptyset$ for all $x, y \in L$. A lattice may be regarded as an up- and down-directed poset where for each $x, y \in L$ the set $U(x, y)$ has a smallest and the set $L(x, y)$ a greatest element.

Several generalizations of lattices can be found in literature, see e. g. [5] and [6]. Let $(L, \leq)$ be a poset being both up-directed and down-directed. For each $x, y \in L$ we can pick up an arbitrary element $x \sqcup y$ of $U(x, y)$ and $x \sqcap y$ of $L(x, y)$ with the only constraint that $x \sqcup y = y$ and $x \sqcap y = x$ whenever $x \leq y$. The resulting algebra is the so-called λ -lattice introduced by V. Snášel in [6]. λ -lattices can be axiomatized as follows:

Definition 1.1 (cf. [6]) A λ -lattice is an algebra $(L, \sqcup, \sqcap)$ of type $(2, 2)$ satisfying the following axioms:

$$\begin{array}{ll} \text{(L1)} & x \sqcup x = x \\ \text{(L2)} & x \sqcup y = y \sqcup x \\ \text{(L3)} & x \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z \\ \text{(L4)} & x \sqcup (x \sqcap y) = x \\ \text{(L1')} & x \sqcap x = x \\ \text{(L2')} & x \sqcap y = y \sqcap x \\ \text{(L3')} & x \sqcap ((x \sqcap y) \sqcap z) = (x \sqcap y) \sqcap z \\ \text{(L4')} & x \sqcap (x \sqcup y) = x \end{array}$$

λ -lattices are applied e. g. in the theory of so-called non-associative MV-algebras (cf. e. g. [3]) where it is shown that every non-associative MV-algebra is a λ -lattice with respect to the induced order.

Another interesting structure on an up-directed poset (or, alternatively, on a down-directed poset) was introduced by J. Ježek and R. Quackenbush (cf. [5]), namely a so-called *directoid* (more precisely, a *join-directoid*) which is defined as follows:

Definition 1.2 (cf. [5]) A *directoid* (more precisely, a *join-directoid*) is a groupoid $(L, \sqcup)$ satisfying the following axioms:

$$\begin{array}{ll} \text{(D1)} & x \sqcup x = x \\ \text{(D2)} & (x \sqcup y) \sqcup x = x \sqcup y \\ \text{(D3)} & y \sqcup (x \sqcup y) = x \sqcup y \\ \text{(D4)} & x \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z \end{array}$$

Dually, one can define so-called *meet-directoids*. Directoids can be characterized by the existence of a certain partial order relation.

Lemma 1.1 A groupoid $(L, \sqcup)$ is a (join-)directoid if and only if there exists a partial ordering $\leq$ on L satisfying

$$\begin{array}{ll} \text{(JD1)} & x, y \leq x \sqcup y \\ \text{(JD2)} & \text{If } x \leq y \text{ then } x \sqcup y = y \sqcup x = y. \end{array}$$

Proof. First assume $(L, \sqcup)$ to be a join-directoid. Define $x \leq y$ by $x \sqcup y = y$. (D1) implies reflexivity of $\leq$. If $x \leq y \leq x$ then $x \sqcup y = y$ and $y \sqcup x = x$ and hence $x = y \sqcup x = (y \sqcup x) \sqcup y = x \sqcup y = y$ according to (D2). If, finally, $x \leq y \leq z$ then $x \sqcup y = y$ and $y \sqcup z = z$ and hence $x \sqcup z = x \sqcup (y \sqcup z) = x \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z = y \sqcup z = z$ according to (D4) showing $x \leq z$. Since $x \sqcup (x \sqcup y) = x \sqcup ((x \sqcup x) \sqcup y) = (x \sqcup x) \sqcup y = x \sqcup y$ according to (D1) and (D4) we have $x \leq x \sqcup y$. Moreover, $y \sqcup (x \sqcup y) = x \sqcup y$ according to (D3) and hence $y \leq x \sqcup y$. Finally, $x \leq y$ implies $x \sqcup y = y$ according to the definition of $\leq$ and $y \sqcup x = (x \sqcup y) \sqcup x = x \sqcup y = y$ according to (D2) and the definition of $\leq$. The rest of the proof is clear. ■

Remark 1.1 Obviously, the partial order relation $\leq$ on a (join-)directoid mentioned in Lemma 1.1 is uniquely determined by $(L, \sqcup)$ via $x \leq y$ if $x \sqcup y = y$. $(L, \leq)$ is called the *poset corresponding to* $(L, \sqcup)$.

It is immediately clear that in a directoid, $x \sqcup y$ is an arbitrarily chosen element of $U(x, y)$ having the property that $x \sqcup y = y \sqcup x = y$ whenever $x \leq y$. But, contrary to the case of λ -lattices, $x \sqcup y$ need not coincide with $y \sqcup x$ in general. In other words, for any $x, y \in L$ there are picked up some elements $x \sqcup y$ and $y \sqcup x$ of $U(x, y)$ such that $x \sqcup y = y \sqcup x = y$ in case $x \leq y$.

2. Weak semilattices

It turns out that also the concept of a directoid is not general enough for some investigations of algebras of non-classical logic (cf. [2]). Let us recall the necessary concepts. In logics, BCK-algebras play an important role. They reflect certain properties of the implication operation and are defined as follows:

Definition 2.1 (cf. [4]) A BCK-algebra is an algebra $\mathcal{A} = (A, \rightarrow, 1)$ of type $(2, 0)$ satisfying the following axioms:

$$\begin{aligned} (x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) &= 1 \\ x \rightarrow ((x \rightarrow y) \rightarrow y) &= 1 \\ x \rightarrow x &= 1 \\ x \rightarrow y = 1 \text{ and } y \rightarrow x = 1 &\text{ together imply } x = y. \end{aligned}$$

In [2] the following was proved:

Proposition 2.1 (cf. [2]) *If $(A, \rightarrow, 1)$ is a BCK-algebra and $x \sqcup y := (x \rightarrow y) \rightarrow y$ then*

- (W1) $x \sqcup x = x$.
- (W2) $x \sqcup y = y$ and $y \sqcup x = x$ together imply $x = y$.
- (W3) $x \sqcup (x \sqcup y) = y \sqcup (x \sqcup y) = (x \sqcup y) \sqcup y = x \sqcup y$.
- (W4) $(x \sqcup z) \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z$.

This motivates the following

Definition 2.2 A *weak join-semilattice* is a groupoid $(L, \sqcup)$ satisfying (W1)-(W4).

Now we characterize weak join-semilattices by the existence of a certain partial order relation.

Theorem 2.1 A groupoid $(L, \sqcup)$ is a weak join-semilattice if and only if (P1) holds:

$$(P1) \quad (x \sqcup y) \sqcup y = x \sqcup y$$

and if there exists a partial ordering $\leq$ on L satisfying (P2)-(P4):

$$(P2) \quad x, y \leq x \sqcup y$$

$$(P3) \quad x \leq y \text{ implies } x \sqcup y = y.$$

$$(P4) \quad x \leq y \text{ implies } x \sqcup z \leq y \sqcup z.$$

Proof. Let $a, b, c \in L$.

First, assume $(L, \sqcup)$ to be a weak join-semilattice.

(P1): follows from (W3).

Now, define $x \leq y$ by $x \sqcup y = y$. Then $\leq$ is reflexive and because of (W2) it is antisymmetric. If $a \leq b \leq c$ then $a \sqcup b = b$ and $b \sqcup c = c$ and hence

$$\begin{aligned} a \sqcup c &= (a \sqcup c) \sqcup c = (a \sqcup c) \sqcup (b \sqcup c) = (a \sqcup c) \sqcup ((a \sqcup b) \sqcup c) \\ &= (a \sqcup b) \sqcup c = b \sqcup c = c \end{aligned}$$

according to (W3) and (W4), i. e. $a \leq c$ showing transitivity of $\leq$. Hence $(L, \leq)$ is a poset.

(P2): follows from (W3).

(P3): follows from the definition of $\leq$.

(P4): If $a \leq b$ then $a \sqcup b = b$ and hence

$$(a \sqcup c) \sqcup (b \sqcup c) = (a \sqcup c) \sqcup ((a \sqcup b) \sqcup c) = (a \sqcup b) \sqcup c = b \sqcup c$$

according to (W4), i. e. $a \sqcup c \leq b \sqcup c$.

Conversely, assume $(L, \sqcup)$ to satisfy (P1) and assume there exists a partial ordering $\leq$ on L satisfying (P2)-(P4).

(W1): follows from reflexivity of $\leq$ and (P3).

(W2): If $a \sqcup b = b$ and $b \sqcup a = a$ then, according to (P2), $a \leq a \sqcup b = b$ and $b \leq b \sqcup a = a$ and hence by antisymmetry of $\leq$, $a = b$.

(W3): Because of (P2), $a \leq a \sqcup b$ which according to (P3) implies $a \sqcup (a \sqcup b) = a \sqcup b$ showing the first part of (W3). Analogously, the second part of (W3) follows. The last part of (W3) follows from (P1).

(W4): $a \leq a \sqcup b$ according to (P2) whence by (P4) $a \sqcup c \leq (a \sqcup b) \sqcup c$ showing $(a \sqcup c) \sqcup ((a \sqcup b) \sqcup c) = (a \sqcup b) \sqcup c$ according to (P3). ■

Remark 2.1 Obviously, the partial order relation $\leq$ on a weak join-semilattice mentioned in Theorem 2.1 is uniquely determined by $(L, \sqcup)$ via $x \leq y$ if $x \sqcup y = y$. $(L, \leq)$ is called the *poset corresponding to* $(L, \sqcup)$.

Next, we define a bounded weak join-semilattice with sectionally antitone involutions.

Definition 2.3 A *bounded weak join-semilattice with sectionally antitone involutions* is an algebra of the form $(L, \sqcup, 0, 1, (^a; a \in L))$ where $(L, \sqcup)$ is a weak join-semilattice with corresponding partial ordering $\leq$, 0 is the smallest and 1 the greatest element of $(L, \leq)$ and for every $a \in L$, $x \mapsto x^a$ is an antitone involution of $([a, 1], \leq)$, i. e. $x \leq y$ implies $y^a \leq x^a$ and $(x^a)^a = x$ for all $x, y \in [a, 1]$.

3. Skew basic algebras

Now we introduce total algebras which will turn out to correspond to bounded weak join-semilattices with sectionally antitone involutions in a natural bijective way.

Definition 3.1 A *skew basic algebra* is an algebra $(A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying (A1) – (A8):

$$(A1) \quad \neg\neg x = x$$

$$(A2) \quad \neg x \oplus x = 1$$

$$(A3) \quad x \oplus 0 = 0 \oplus x = x$$

$$(A4) \quad \neg(\neg(x \oplus y) \oplus y) \oplus y = x \oplus y$$

$$(A5) \quad \neg(\neg x \oplus y) \oplus y = x \text{ and } \neg(\neg y \oplus x) \oplus x = y \text{ together imply } x = y.$$

$$(A6) \quad \neg(x \oplus (\neg(x \oplus y) \oplus y)) \oplus (\neg(x \oplus y) \oplus y) = \neg(x \oplus y) \oplus y$$

$$(A7) \quad \neg(\neg x \oplus y) \oplus y = x \text{ and } \neg(z \oplus x) \oplus x = \neg z \text{ together imply} \\ \neg(\neg(\neg x \oplus y) \oplus (z \oplus y)) \oplus (z \oplus y) = \neg x \oplus y$$

$$(A8) \quad \neg(\neg(\neg(x \oplus (\neg(y \oplus z) \oplus z)) \oplus (\neg(y \oplus z) \oplus z)) \oplus (\neg(x \oplus z) \oplus z)) \oplus (\neg(x \oplus z) \oplus z) = \\ \neg(x \oplus (\neg(y \oplus z) \oplus z)) \oplus (\neg(y \oplus z) \oplus z)$$

Here and in the following, 1 is an abbreviation for $\neg 0$.

Now, one can formulate and prove the mentioned natural bijective correspondence.

Theorem 3.1 *If $\mathcal{L} = (L, \sqcup, 0, 1, (^a; a \in L))$ is a bounded weak join-semilattice with sectionally antitone involutions and one defines*

$$\begin{aligned} x \oplus y &:= (y \sqcup x^0)^y \\ \neg x &:= x^0 \end{aligned}$$

then $\mathbf{S}(\mathcal{L}) := (L, \oplus, \neg, 0)$ is a skew basic algebra. If, conversely, $\mathcal{A} = (A, \oplus, \neg, 0)$ is a skew basic algebra and one defines

$$\begin{aligned} x \sqcup y &:= \neg(\neg y \oplus x) \oplus x \\ 1 &:= \neg 0 \\ x^y &:= \neg x \oplus y \end{aligned}$$

then $\mathbf{L}(\mathcal{A}) := (A, \sqcup, 0, 1, (^a; a \in A))$ is a bounded weak join-semilattice with sectionally antitone involutions. Moreover, $\mathbf{L}(\mathbf{S}(\mathcal{L})) = \mathcal{L}$ for every bounded weak join-semilattice $\mathcal{L}$ with sectionally antitone involutions and $\mathbf{S}(\mathbf{L}(\mathcal{A})) = \mathcal{A}$ for every skew basic algebra $\mathcal{A}$.

Proof. Let $a, b, c \in L$.

First, assume $\mathcal{L} = (L, \sqcup, 0, 1, (^a; a \in L))$ to be a bounded weak join-semilattice with sectionally antitone involutions and define

$$\begin{aligned} x \oplus y &:= (y \sqcup x^0)^y \\ \neg x &:= x^0 \end{aligned}$$

Let $(L, \leq)$ denote the poset corresponding to $(L, \sqcup)$. Then

$$(1) \quad a \oplus b \geq b.$$

Moreover, $\neg(\neg b \oplus a) \oplus a = (a \sqcup (a \sqcup b)^a)^a = (a \sqcup b)^{aa} = a \sqcup b$ and if $a \geq b$ then $\neg a \oplus b = (b \sqcup a)^b = a^b$. We check axioms (A1)-(A8):

$$(A1) \quad \neg \neg a = a^{00} = a$$

$$(A2) \quad \neg a \oplus a = a^a = 1$$

$$(A3) \quad a \oplus 0 = (0 \sqcup a^0)^0 = a^{00} = a \text{ and } 0 \oplus a = (a \sqcup 0^0)^a = (a \sqcup 1)^a = 1^a = a$$

$$(A4) \quad \neg(\neg(a \oplus b) \oplus b) \oplus b = b \sqcup (a \oplus b) = a \oplus b \text{ according to (1)}$$

$$(A5) \quad \neg(\neg a \oplus b) \oplus b = a \text{ and } \neg(\neg b \oplus a) \oplus a = b \text{ together imply } b \sqcup a = a \text{ and } a \sqcup b = b \text{ whence } a = b \text{ according to (W2)}$$

$$(A6) \quad \neg(a \oplus (\neg(a \oplus b) \oplus b)) \oplus (\neg(a \oplus b) \oplus b) = \neg(a \oplus (b \sqcup a^0)) \oplus (b \sqcup a^0) = (b \sqcup a^0) \sqcup a^0 = b \sqcup a^0 = \neg(a \oplus b) \oplus b \text{ according to (W3)}$$

$$(A7) \quad \neg(\neg a \oplus b) \oplus b = a \text{ and } \neg(c \oplus a) \oplus a = \neg c \text{ together imply } b \sqcup a = a \text{ and } a \sqcup c^0 = c^0 \text{ whence } b \leq a \leq c^0 \text{ which implies } (c^0)^b \leq a^b. \text{ Hence } \neg(\neg(\neg a \oplus b) \oplus (c \oplus b)) \oplus (c \oplus b) = (c^0)^b \sqcup a^b = a^b = \neg a \oplus b.$$

$$\begin{aligned}
\text{(A8)} \quad & \neg(\neg(\neg(a \oplus (\neg(b \oplus c) \oplus c)) \oplus (\neg(b \oplus c) \oplus c)) \oplus (\neg(a \oplus c) \oplus c)) \oplus (\neg(a \oplus c) \oplus c) = \\
& \neg(\neg(\neg(a \oplus (c \sqcup b^0)) \oplus (c \sqcup b^0)) \oplus (c \sqcup a^0)) \oplus (c \sqcup a^0) = \neg(\neg((c \sqcup b^0) \sqcup \\
& a^0) \oplus (c \sqcup a^0)) \oplus (c \sqcup a^0) = (c \sqcup a^0) \sqcup ((c \sqcup b^0) \sqcup a^0) = (c \sqcup b^0) \sqcup a^0 = \\
& \neg(a \oplus (\neg(b \oplus c) \oplus c)) \oplus (\neg(b \oplus c) \oplus c) \text{ according to (W4)}
\end{aligned}$$

Hence, $\mathbf{S}(\mathcal{L})$ is a skew basic algebra.

Conversely, assume $\mathcal{A} = (A, \oplus, \neg, 0)$ to be a skew basic algebra and define

$$\begin{aligned}
x \sqcup y &:= \neg(\neg y \oplus x) \oplus x \\
1 &:= \neg 0 \\
x^y &:= \neg x \oplus y
\end{aligned}$$

Then $a^0 = \neg a \oplus 0 = \neg a$ according to (A3) and $(b \sqcup a^0)^b = \neg(\neg(a \oplus b) \oplus b) \oplus b = a \oplus b$ according to (A4). We check axioms (W1)-(W4):

- (W1) $a \sqcup a = \neg(\neg a \oplus a) \oplus a = \neg 1 \oplus a = 0 \oplus a = a$ according to (A2), (A1) and (A3)
- (W2) $a \sqcup b = b$ and $b \sqcup a = a$ together imply $\neg(\neg b \oplus a) \oplus a = b$ and $\neg(\neg a \oplus b) \oplus b = a$ whence according to (A5) $b = a$, i. e. $a = b$.
- (W3) $a \sqcup (a \sqcup b) = \neg(\neg(\neg(\neg b \oplus a) \oplus a) \oplus a) \oplus a = \neg(\neg b \oplus a) \oplus a = a \sqcup b$ according to (A4). Now put $z = 0$ in (A8). Then one obtains $\neg(\neg(\neg(x \oplus \neg y) \oplus \neg y) \oplus \neg x) \oplus \neg x = \neg(x \oplus \neg y) \oplus \neg y$. Hence $b \sqcup (a \sqcup b) = \neg(\neg(\neg(\neg b \oplus a) \oplus a) \oplus b) \oplus b = \neg(\neg b \oplus a) \oplus a = a \sqcup b$. Finally, $(a \sqcup b) \sqcup b = \neg(\neg b \oplus (\neg(\neg b \oplus a) \oplus a)) \oplus (\neg(\neg b \oplus a) \oplus a) = \neg(\neg b \oplus a) \oplus a = a \sqcup b$ according to (A6).
- (W4) $(a \sqcup c) \sqcup ((a \sqcup b) \sqcup c) = \neg(\neg(\neg(\neg c \oplus (\neg(\neg b \oplus a) \oplus a)) \oplus (\neg(\neg b \oplus a) \oplus a)) \oplus (\neg(\neg c \oplus a) \oplus a)) \oplus (\neg(\neg c \oplus a) \oplus a) = \neg(\neg c \oplus (\neg(\neg b \oplus a) \oplus a)) \oplus (\neg(\neg b \oplus a) \oplus a) = (a \sqcup b) \sqcup c$ according to (A8)

Hence $(L, \sqcup)$ is a weak join-semilattice. Let $(L, \leq)$ denote its corresponding poset. Then

$a \geq b$ implies $b \sqcup a^b = \neg(\neg(\neg a \oplus b) \oplus b) \oplus b = \neg a \oplus b = a^b$ according to (A4) which implies $b \leq a^b$.

$c \leq a \leq b$ implies $c \sqcup a = a$ and $a \sqcup b = b$ whence $\neg(\neg a \oplus c) \oplus c = a$ and $\neg(\neg b \oplus a) \oplus a = b = \neg \neg b$ according to (A1) and hence $b^c \sqcup a^c = \neg(\neg(\neg a \oplus c) \oplus (\neg b \oplus c)) \oplus (\neg b \oplus c) = \neg a \oplus c = a^c$ according to (A7) whence $b^c \leq a^c$.

$0 \sqcup a = \neg(\neg a \oplus 0) \oplus 0 = \neg \neg a \oplus 0 = \neg \neg a = a$ according to (A3) and (A1) whence $0 \leq a$.

$a \sqcup 1 = \neg(\neg 1 \oplus a) \oplus a = \neg(0 \oplus a) \oplus a = \neg a \oplus a = 1$ according to (A1), (A3) and (A2) whence $a \leq 1$.

Finally, $b \leq a$ implies $a^{bb} = \neg(\neg a \oplus b) \oplus b = b \sqcup a = a$.

Hence, $\mathbf{L}(\mathcal{A})$ is a bounded weak semilattice with sectionally antitone involutions.

Conversely, let $\mathcal{L} = (L, \sqcup, 0, 1, ({}^a; a \in L))$ be a bounded weak semilattice with sectionally antitone involutions and put $\mathbf{L}(\mathbf{S}(\mathcal{L})) = (L, \cup, 0, \bar{1}, ({}_a; a \in L))$. Then

$$a \cup b = \neg(\neg b \oplus a) \oplus a = (a \sqcup (a \sqcup b)^a)^a = (a \sqcup b)^{aa} = a \sqcup b.$$

Hence the posets corresponding to $(L, \sqcup)$ respectively $(L, \cup)$ coincide.

Let $(L, \leq)$ denote this poset.

Moreover, $\bar{1} = \neg 0 = 0^0 = 1$.

If $a \leq b$ then $a \sqcup b = b$ whence $b_a = \neg b \oplus a = (a \sqcup b)^a = b^a$.

Hence $\mathbf{L}(\mathbf{S}(\mathcal{L})) = \mathcal{L}$.

Now let, finally, $\mathcal{A} = (A, \oplus, \neg, 0)$ be a skew basic algebra and put $\mathbf{S}(\mathbf{L}(\mathcal{A})) = (A, +, ', 0)$. Then

$a + b = (b \sqcup a^0)^b = \neg(\neg(\neg(\neg a \oplus 0) \oplus b) \oplus b) \oplus b = \neg(\neg(\neg \neg a \oplus b) \oplus b) \oplus b = \neg(\neg(a \oplus b) \oplus b) \oplus b = a \oplus b$ according to (A3), (A1) and (A4).

Finally, $a' = a^0 = \neg a \oplus 0 = \neg a$ according to (A3).

Hence $\mathbf{S}(\mathbf{L}(\mathcal{A})) = \mathcal{A}$. ■

Dually to the notion of a weak join-semilattice we define the notion of a weak meet-semilattice:

Definition 3.2 A *weak meet-semilattice* is a groupoid $(L, \sqcap)$ satisfying (W1')-(W4'):

$$(W1') \quad x \sqcap x = x$$

$$(W2') \quad x \sqcap y = x \text{ and } y \sqcap x = y \text{ together imply } x = y.$$

$$(W3') \quad (x \sqcap y) \sqcap x = (x \sqcap y) \sqcap y = x \sqcap (x \sqcap y) = x \sqcap y$$

$$(W4') \quad (z \sqcap (x \sqcap y)) \sqcap (z \sqcap y) = z \sqcap (x \sqcap y)$$

Corollary 3.1 *Dually to the case of weak join-semilattices it can be proved that a groupoid $(L, \sqcap)$ is a weak meet-semilattice if and only if (P1') holds:*

$$(P1') \quad x \sqcap (x \sqcap y) = x \sqcap y$$

and if there exists a partial ordering $\leq$ on L satisfying (P2') – (P4'):

$$(P2') \quad x \sqcap y \leq x, y.$$

$$(P3') \quad x \leq y \text{ implies } x \sqcap y = x.$$

$$(P4') \quad x \leq y \text{ implies } z \sqcap x \leq z \sqcap y.$$

$\leq$ is uniquely determined by $(L, \sqcap)$ via $x \leq y$ if $x \sqcap y = x$. $(L, \leq)$ is called the poset corresponding to $(L, \sqcap)$.

4. Weak lattices

Now, we define the notion of a weak lattice.

Definition 4.1 A *weak lattice* is an algebra $(L, \sqcup, \sqcap)$ of type $(2, 2)$ satisfying (W1)-(W5) and (W1')-(W5') where

$$(W5) \quad x \sqcap (x \sqcup y) = x.$$

$$(W5') \quad (x \sqcap y) \sqcup y = y.$$

Weak lattices can also be characterized by the existence of a certain partial order relation.

Lemma 4.1 *An algebra $(L, \sqcup, \sqcap)$ of type $(2, 2)$ is a weak lattice if and only if (P1) and (P1') hold and if there exists a partial ordering $\leq$ on L satisfying (P2)-(P4) and (P2')-(P4').*

Proof. Let $a, b \in L$.

First assume $(L, \sqcup, \sqcap)$ to be a weak lattice and define $x \leq y$ by $x \sqcup y = y$ and $x \sqsubseteq y$ by $x \sqcap y = x$. Then $a \leq b$ implies $a \sqcup b = b$ and hence $a \sqcap b = a \sqcap (a \sqcup b) = a$ according to (W5), i.e., $a \sqsubseteq b$, and, conversely, $a \sqsubseteq b$ implies $a \sqcap b = a$ and hence $a \sqcup b = (a \sqcap b) \sqcup b = b$ according to (W5'), i.e., $a \leq b$. This shows that both partial orderings coincide. According to Theorem 2.1 and Corollary 3.1, (P1) and (P1') hold and there exists a partial ordering $\leq$ on L satisfying (P2)-(P4) and (P2')-(P4'). The rest of the proof follows from Theorem 2.1 and Corollary 3.1. ■

Remark 4.1 Obviously, the partial order relation $\leq$ mentioned in Lemma 4.1 is uniquely determined by $(L, \sqcup, \sqcap)$ via $x \leq y$ if and only if $x \sqcup y = y$ if and only if $x \sqcap y = x$. $(L, \leq)$ is called the *poset corresponding to $(L, \sqcup, \sqcap)$* .

Example 4.1 Consider the four-element chain $0 < a < b < 1$ and define the operations $\sqcup$ and $\sqcap$ on $L := \{0, a, b, 1\}$ as follows:

$\sqcup$	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	1	b	1
1	1	1	1	1

$\sqcap$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	0	b	b
1	0	a	b	1

It is easy to check that $(L, \sqcup, \sqcap)$ is a weak lattice which is not a directoid (neither w. r. t. $\sqcup$ nor w. r. t. $\sqcap$) since e. g. $a \leq b$ and $a \sqcup b = b$, but $b \sqcup a = 1 \neq b$. Of course, the order of the original chain coincides with the order induced by $(L, \sqcup, \sqcap)$.

Example 4.2 If $(L, \leq, 0, 1)$ is a bounded poset and one defines

$$x \sqcup y := \begin{cases} y & \text{if } x \leq y \\ 1 & \text{otherwise} \end{cases}$$

$$x \sqcap y := \begin{cases} x & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

then $(L, \sqcup, \sqcap)$ is a weak lattice. This can be seen as follows:

Let $a, b, c \in L$. We check (P1)-(P4):

(P1): If $a \leq b$ then $(a \sqcup b) \sqcup b = b \sqcup b = b = a \sqcup b$ and if $a \not\leq b$ then $(a \sqcup b) \sqcup b = 1 \sqcup b = 1 = a \sqcup b$.

(P2): If $a \leq b$ then $a \leq b = a \sqcup b$ and $b = a \sqcup b$ and if $a \not\leq b$ then $a \leq 1 = a \sqcup b$ and $b \leq 1 = a \sqcup b$.

(P3): If $a \leq b$ then $a \sqcup b = b$.

(P4): If $a \leq b \leq c$ then $a \leq c$ and hence $a \sqcup c = c = b \sqcup c$ and if $a \leq b \not\leq c$ then $a \sqcup c \leq 1 = b \sqcup c$.

The dual conditions follow analogously.

According to Lemma 4.1, $(L, \sqcup, \sqcap)$ is a weak lattice.

Now, we introduce a unary operation on weak lattices.

Definition 4.2 A *weak lattice with an antitone involution* is an algebra $(L, \sqcup, \sqcap, ')$ of type $(2, 2, 1)$ such that $(L, \sqcup, \sqcap)$ is a weak lattice and $'$ is an antitone involution. If, in addition, $(L, \sqcup, \sqcap, ')$ satisfies the de Morgan laws $(x \sqcup y)' = y' \sqcap x'$ and $(x \sqcap y)' = y' \sqcup x'$ then it is called a *weak lattice with an antitone involution satisfying the de Morgan laws*.

Example 4.3 Consider the weak lattice $(L, \sqcup, \sqcap)$ of Example 4.1 and define a unary operation $'$ on L by $0' := 1$, $a' := b$, $b' := a$ and $1' := 0$. One can easily verify that $(L, \sqcup, \sqcap, ')$ is a weak lattice with an antitone involution.

Remark 4.2 That $'$ is antitone follows from the de Morgan laws: If $x \leq y$ then $x \sqcup y = y$ which implies $y' \sqcap x' = (x \sqcup y)' = y'$ whence $y' \leq x'$.

That there exist weak lattices with an antitone involution not satisfying the de Morgan laws is shown by the following example:

Example 4.4 Let L and $\sqcup$ be defined as in Example 4.1, $(L, \sqcap)$ a meet-semilattice, i.e.,

$\sqcap$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

and $'$ be defined as in Example 4.3. Then $(L, \sqcup, \sqcap, ')$ is a weak lattice with an antitone involution not satisfying the de Morgan laws since $(b \sqcup a)' = 1' = 0 \neq a = b \sqcap a = a' \sqcap b'$.

Now, it can be proved that every weak join-semilattice together with an antitone involution induces in a natural way a weak lattice with an antitone involution satisfying the de Morgan laws:

Theorem 4.1 Let $(L, \sqcup)$ be a weak join-semilattice and $'$ an antitone involution of the corresponding poset $(L, \leq)$. Define $x \sqcap y := (y' \sqcup x')'$. Then $(L, \sqcup, \sqcap, ')$ is a weak lattice with an antitone involution satisfying the de Morgan laws.

Proof. There hold (W1)-(W4). (W1')-(W4') can be easily proved, e.g.:

$$(W1'): x \sqcap x = (x' \sqcup x')' = x'' = x \text{ according to (W1).}$$

Now

$$(W5): x \sqcap (x \sqcup y) = ((x \sqcup y)' \sqcup x')' = x'' = x \text{ since } x \leq x \sqcup y \text{ and hence } (x \sqcup y)' \leq x'.$$

$$(W5'): (x \sqcap y) \sqcup y = (y' \sqcup x')' \sqcup y'' = y'' = y \text{ since } y' \leq y' \sqcup x' \text{ and hence } (y' \sqcup x')' \leq y''.$$

$$(x \sqcup y)' = (x'' \sqcup y'')' = y' \sqcap x'$$

$$(x \sqcap y)' = (y' \sqcup x')'' = y' \sqcup x'. \quad \blacksquare$$

Remark 4.3 It was proved in [2] that in a BCK-algebra $(A, \rightarrow, 1)$ one can define a partial ordering $\leq$ by $x \leq y$ if $x \rightarrow y = 1$ and it holds $x \leq y$ if and only if $x \sqcup y = y$ where $x \sqcup y = (x \rightarrow y) \rightarrow y$. Moreover, 1 is the greatest element of $(A, \leq)$. If $(A, \leq)$ has a smallest element 0 then the term operation $'$ defined by $x' := x \rightarrow 0$ is antitone.

Now, we introduce BCK-algebras with 0 satisfying the double negation law.

Definition 4.3 (cf. [1]) A BCK-algebra $(A, \rightarrow, 1)$ with 0 is said to satisfy the *double negation law* if $(x \rightarrow 0) \rightarrow 0 = x$.

In fact, in logics represented by BCK-algebras the unary operation $\neg x = x \rightarrow 0$ is considered as the connective negation. Thus the double negation law expresses the fact that $\neg\neg x = x$ (contrary to the intuitionistic case where only $\neg\neg\neg x = \neg x$ and where it can happen that $\neg\neg x \neq x$). BCK-algebras satisfying the double negation law were treated in [1]. Here we can state

Theorem 4.2 *If $(A, \rightarrow, 1)$ is a BCK-algebra with 0 satisfying the double negation law and*

$$x \sqcup y := (x \rightarrow y) \rightarrow y$$

$$x \sqcap y := (((y \rightarrow 0) \rightarrow (x \rightarrow 0)) \rightarrow (x \rightarrow 0)) \rightarrow 0$$

$$x' := x \rightarrow 0$$

then $(A, \sqcup, \sqcap, ')$ is a weak lattice with an antitone involution satisfying the de Morgan laws.

Proof. This follows from Proposition 2.1. $\blacksquare$

Theorem 4.3 *Up to isomorphism there exist exactly one one-element weak lattice, one two-element weak lattice, four three-element weak lattices and 74 four-element weak lattices.*

Proof. The case concerning weak lattices of cardinality less than three is trivial.

Let us first assume $L = \{0, a, 1\}$ with $0 < a < 1$.

(P2), (P3), (P2') and (P3'): These conditions hold if and only if operation tables for $\sqcup$, respectively $\sqcap$, look as follows:

$\sqcup$	0	a	1
0	0	a	1
a	b	a	1
1	1	1	1

$\sqcap$	0	a	1
0	0	0	0
a	0	a	a
1	0	c	1

where $b \in \{a, 1\}$ and $c \in \{0, a\}$.

(P1): This condition holds if $x \in \{0, 1\}$ or $y = 1$ or $x \leq y$.

If $(x, y) = (a, 0)$ then $(x \sqcup y) \sqcup y = (a \sqcup 0) \sqcup 0 = b \sqcup 0 = b = a \sqcup 0 = x \sqcup y$.

Hence (P1) holds.

(P1'): This condition holds if $x = 0$ or $y \in \{0, 1\}$ or $x \leq y$.

If $(x, y) = (1, a)$ then $x \sqcap (x \sqcap y) = 1 \sqcap (1 \sqcap a) = 1 \sqcap c = c = 1 \sqcap a = x \sqcap y$.

Hence (P1') holds.

(P4) and (P4'): These conditions can be easily checked.

Since all four possibilities for b, c yield pairwise non-isomorphic weak lattices, up to isomorphism there exist exactly four three-element weak lattices.

Now we turn to the four-element case. Let $L = \{0, a, b, 1\}$.

First we consider the case $0 < a, b < 1$ with a and b being incomparable.

(P2), (P3), (P2') and (P3'): These conditions hold if and only if operation tables for $\sqcup$, respectively $\sqcap$, look as follows:

$\sqcup$	0	a	b	1
0	0	a	b	1
a	c	a	1	1
b	d	1	b	1
1	1	1	1	1

$\sqcap$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	e	f	1

where $c \in \{a, 1\}$, $d \in \{b, 1\}$, $e \in \{0, a\}$ and $f \in \{0, b\}$.

(P1): This condition holds if $x \in \{0, 1\}$ or $y = 1$ or $x \leq y$.

If $(x, y) = (a, 0)$ then $(x \sqcup y) \sqcup y = (a \sqcup 0) \sqcup 0 = c \sqcup 0 = c = a \sqcup 0 = x \sqcup y$.

If $(x, y) = (a, b)$ then $(x \sqcup y) \sqcup y = (a \sqcup b) \sqcup b = 1 \sqcup b = 1 = a \sqcup b = x \sqcup y$.

If $(x, y) = (b, 0)$ then $(x \sqcup y) \sqcup y = (b \sqcup 0) \sqcup 0 = d \sqcup 0 = d = b \sqcup 0 = x \sqcup y$.

If $(x, y) = (b, a)$ then $(x \sqcup y) \sqcup y = (b \sqcup a) \sqcup a = 1 \sqcup a = 1 = b \sqcup a = x \sqcup y$.

Hence (P1) holds.

(P1'): This condition holds if $x = 0$ or $y \in \{0, 1\}$ or $x \leq y$.

If $(x, y) = (a, b)$ then $x \sqcap (x \sqcap y) = a \sqcap (a \sqcap b) = a \sqcap 0 = 0 = a \sqcap b = x \sqcap y$.

If $(x, y) = (b, a)$ then $x \sqcap (x \sqcap y) = b \sqcap (b \sqcap a) = b \sqcap 0 = 0 = b \sqcap a = x \sqcap y$.

If $(x, y) = (1, a)$ then $x \sqcap (x \sqcap y) = 1 \sqcap (1 \sqcap a) = 1 \sqcap e = e = 1 \sqcap a = x \sqcap y$.

If $(x, y) = (1, b)$ then $x \sqcap (x \sqcap y) = 1 \sqcap (1 \sqcap b) = 1 \sqcap f = f = 1 \sqcap b = x \sqcap y$.

Hence (P1') holds.

Since every homomorphism of a weak lattice is order-preserving, the only non-trivial isomorphism between our algebras could be the interchanging of a and b . We have the following cases: $(c, d, e, f) \in \{a, 1\} \times \{b, 1\} \times \{0, a\} \times \{0, b\}$. Now the weak lattices corresponding to the following quadruples are isomorphic:

$$\begin{aligned} &(a, b, 0, b), (a, b, a, 0) \\ &(a, 1, 0, 0), (1, b, 0, 0) \\ &(a, 1, 0, b), (1, b, a, 0) \\ &(a, 1, a, 0), (1, b, 0, b) \\ &(a, 1, a, b), (1, b, a, b) \\ &(1, 1, 0, b), (1, 1, a, 0) \end{aligned}$$

So we have 10 pairwise non-isomorphic four-element weak lattices whose corresponding order is not a chain.

Finally, consider the case $0 < a < b < 1$.

(P2), (P3), (P2') and (P3'): These conditions hold if and only if operation tables for $\sqcup$, respectively $\sqcap$, look as follows:

$$\begin{array}{c|cccc} \sqcup & 0 & a & b & 1 \\ \hline 0 & 0 & a & b & 1 \\ a & c & a & b & 1 \\ b & d & e & b & 1 \\ 1 & 1 & 1 & 1 & 1 \end{array} \quad \begin{array}{c|cccc} \sqcap & 0 & a & b & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ a & 0 & a & a & a \\ b & 0 & f & b & b \\ 1 & 0 & g & h & 1 \end{array}$$

where $c \in \{a, b, 1\}$, $d, e \in \{b, 1\}$, $f, g \in \{0, a\}$ and $h \in \{0, a, b\}$.

(P1): This condition holds if $x \in \{0, 1\}$ or $y = 1$ or $x \leq y$.

If $(x, y) = (a, 0)$ then $(x \sqcup y) \sqcup y = (a \sqcup 0) \sqcup 0 = c \sqcup 0$ and $x \sqcup y = a \sqcup 0 = c$.

If $(x, y) = (b, 0)$ then $(x \sqcup y) \sqcup y = (b \sqcup 0) \sqcup 0 = d \sqcup 0 = d = b \sqcup 0 = x \sqcup y$.

If $(x, y) = (b, a)$ then $(x \sqcup y) \sqcup y = (b \sqcup a) \sqcup a = e \sqcup a = e = b \sqcup a = x \sqcup y$.

Hence (P1) holds if and only if $(c, d) \neq (b, 1)$.

(P1'): This condition holds if $x = 0$ or $y \in \{0, 1\}$ or $x \leq y$.

If $(x, y) = (b, a)$ then $x \sqcap (x \sqcap y) = b \sqcap (b \sqcap a) = b \sqcap f = f = b \sqcap a = x \sqcap y$.

If $(x, y) = (1, a)$ then $x \sqcap (x \sqcap y) = 1 \sqcap (1 \sqcap a) = 1 \sqcap g = g = 1 \sqcap a = x \sqcap y$.

If $(x, y) = (1, b)$ then $x \sqcap (x \sqcap y) = 1 \sqcap (1 \sqcap b) = 1 \sqcap h$ and $x \sqcap y = 1 \sqcap b = h$.

Hence (P1') holds if and only if $(g, h) \neq (0, a)$.

(P4): This condition holds if and only if $(c, d) \neq (1, b)$.

(P4'): This condition holds if and only if $(g, h) \neq (a, 0)$.

Summing up, we have that the algebra with the above operation tables is a weak lattice if and only if

$$\begin{aligned} ((c, d), e, f, (g, h)) \in & \{(a, b), (a, 1), (b, b), (1, 1)\} \times \{b, 1\} \\ & \times \{0, a\} \times \{(0, 0), (a, a), (0, b), (a, b)\}. \end{aligned}$$

Hence we have $4 \cdot 2 \cdot 2 \cdot 4 = 64$ possibilities. The corresponding algebras are pairwise non-isomorphic. Together this yields $10 + 64 = 74$ pairwise non-isomorphic four-element weak lattices. ■

Finally, we introduce some axioms.

Definition 4.4 For an algebra $(L, \sqcup, \sqcap)$ we define (W6), (W6'), (W7) and (W7') as follows:

$$(W6) \quad (x \sqcup y) \sqcup x = x \sqcup y$$

$$(W6') \quad y \sqcap (x \sqcap y) = x \sqcap y$$

$$(W7) \quad y \sqcap (x \sqcup y) = y$$

$$(W7') \quad (x \sqcap y) \sqcup x = x$$

Next, we show some implications between some axioms:

Lemma 4.2 *There hold (i)-(v):*

- (i) (W3) and (W5) together imply (W7).
- (ii) (W3') and (W5') together imply (W7').
- (iii) (W1) and (W5) together imply (W1').
- (iv) (W2) and (W5') together imply (W2').
- (v) (W6) implies (W2).

Proof. (i): $y \sqcap (x \sqcup y) = y \sqcap (y \sqcup (x \sqcup y)) = y$.

(ii): $(x \sqcap y) \sqcup x = ((x \sqcap y) \sqcap x) \sqcup x = x$. This follows analogously.

(iii): $x \sqcap x = x \sqcap (x \sqcup x) = x$.

(iv): $x \sqcap y = x$ and $y \sqcap x = y$ together imply $x \sqcup y = (x \sqcap y) \sqcup y = y$ and $y \sqcup x = (y \sqcap x) \sqcup x = x$ whence $x = y$.

(v): $x \sqcup y = y$ and $y \sqcup x = x$ together imply $x = y \sqcup x = (x \sqcup y) \sqcup x = x \sqcup y = y$. ■

Lemma 4.3 *There hold (i) and (ii):*

- (i) Any join-directoid satisfying (W4) is a join-semilattice.

(ii) Any weak join-semilattice satisfying (W6) is a join-semilattice.

Proof. (i) If $(L, \sqcup)$ is a join-directoid, $(L, \leq)$ denotes the corresponding poset, $a, b, c \in L$ and $a, b \leq c$ then

$$\begin{aligned} (a \sqcup b) \sqcup c &= (a \sqcup b) \sqcup (c \sqcup b) \\ &= (a \sqcup b) \sqcup ((a \sqcup c) \sqcup b) \\ &= (a \sqcup c) \sqcup b = c \sqcup b = c, \end{aligned}$$

i. e. $a \sqcup b \leq c$.

(ii) If $(L, \sqcup)$ is a weak join-semilattice, $(L, \leq)$ denotes the corresponding poset, $a, b \in L$ and $a \leq b$ then $b \sqcup a = (a \sqcup b) \sqcup a = a \sqcup b = b$ and according to Lemma 1.1 and Theorem 2.1, $(L, \sqcup)$ is a join-directoid and according to Lemma 4.3 (i) a join-semilattice. ■

Corollary 4.1 *There hold (i) and (ii):*

(i) *If $(L, \sqcup)$ is a weak join-semilattice satisfying (W5) and (W6), $(L, \leq)$ denotes the corresponding poset, ' is an antitone involution of $(L, \leq)$ and one defines $x \sqcap y := (x' \sqcup y')'$ for all $x, y \in L$ then $(L, \sqcup, \sqcap)$ is a lattice.*

(ii) *Any weak lattice satisfying (W6) and (W6') is a lattice.*

Proof. (i): According to Lemma 4.3 (ii), $(L, \sqcup)$ is a join-semilattice and hence $(L, \sqcap)$ a meet-semilattice. Because of (W5) one absorption law holds. The other one follows by duality.

(ii): Let $(L, \sqcup, \sqcap)$ be a weak lattice. According to Lemma 4.3 (ii) and its dual $(L, \sqcup)$ and $(L, \sqcap)$ are semilattices. Because of (W5) and (W5') the absorption laws hold. ■

Remark 4.4 Since (W3) and $x \sqcup y = y \sqcup x$ together imply (W6), according to Lemma 4.3 (ii) it follows that any commutative weak join-semilattice is a join-semilattice and according to Corollary 4.1 (ii) that every commutative weak lattice is a lattice. If in Definition 2.2 (W2) is replaced by the stronger axiom $(x \sqcup y) \sqcup x = x \sqcup y$ or if the axiom $(x \sqcup y) \sqcup x = x \sqcup (x \sqcup y)$ is included, then the weak join-semilattice becomes a join-semilattice.

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CHARACTERIZATION OF HYPER BCI -ALGEBRA OF ORDER 3**R. Ameri**

*Department of Mathematics
University of Tehran
Tehran
Iran
e-mail: rez_ameri@yahoo.com*

A. Radfar

*Department of Mathematics
Payamenoor University
Tehran
Iran
e-mail: ateferadfar@yahoo.com*

R.A. Borzooei

*Department of Mathematics
Shahid Beheshti University
Tehran
Iran
e-mail: borzooei@sbu.ac.ir*

Abstract. In this paper, first we introduce the concepts of weak hyper BCI -algebras and strong hyper BCI -algebras. Then by using that concepts, we characterize all of the hyper BCI -algebras of order 3 up to isomorphism.

Keywords: hyper BCI -algebra, weak hyper BCI -algebra, strong hyper BCI -algebra.

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1. Introduction

The study of BCK -algebras was initiated by Y. Imai and K. Iséki [4] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculi. The hyperstructure theory (called also multialgebras) was introduced in 1934 by F. Marty [7] at the 8th Congress of Scandinavian Mathematicians. Since then many researchers have worked on algebraic hyperstructures and developed it. A recent book [3] contains a wealth of applications. Via this book, Corsini and Leoreanu presented some of the numerous applications of algebraic hyperstructures, especially those from the last fifteen years, to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence and probabilities. Hyperstructures have many applications to several

sectors of both pure and applied sciences. In [1], Y.B. Jun et al. applied the hyper structures to *BCK*-algebras, and introduced the notion of a hyper *BCK*-algebra which is a generalization of *BCK*-algebra and investigated some related properties. In [6], X.X. Long applied the hyper structure to *BCI*-algebras and introduce the concepts of hyper *BCI*-algebras which is a generalization of *BCI*-algebras. Now, in this note we define the notions of weak hyper *BCI*-algebras and strong hyper *BCI*-algebras and we obtain some related results which have been mentioned in this paper.

2. Preliminary

Definition 2.1. [6] An algebra $(X, *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions:

- (BCI-1): $((x * y) * (x * z)) * (z * y) = 0$,
- (BCI-2): $x * 0 = x$,
- (BCI-3): $x * y = 0$ and $y * x = 0$ imply $x = y$,

for any $x, y, z \in X$. A *BCI*-algebra X is called *p-semisimple BCI-algebra* if $0 * (0 * x) = x$, for all $x \in X$.

Definition 2.2. [6] Let H be a nonempty set and " $\circ$ " be a hyper operation on H . Then H is called a *hyper BCI-algebra*, if it contains a constant 0 and satisfies the following conditions:

- (B₁) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
- (B₂) $(x \circ y) \circ z = (x \circ z) \circ y$,
- (B₃) $x \ll x$,
- (B₄) $x \ll y$ and $y \ll x \Rightarrow x = y$,
- (B₅) $0 \circ (0 \circ x) \ll x, \quad x \neq 0$,

for all $x, y, z \in H$.

By a *hyper BCK-algebra* we mean a nonempty set H endowed with a hyper operation " $\circ$ " and a constant 0 which satisfy axioms (B₁), (B₂), (B₄) and $x \circ H \ll \{x\}$, for all $x \in H$. It is easy to see that every hyper *BCK*-algebra is a hyper *BCI*-algebra.

Let $(H, \circ, 0)$ be a hyper *BCI*-algebra. By H^+ we mean

$$H^+ = \{x \in H \mid 0 \in 0 \circ x\}.$$

We note that $0 \in H^+$, thus $H^+ \neq \emptyset$.

Definition 2.3. [5] Let I be a nonempty subset of hyper *BCI*-algebra H and $0 \in I$. Then I is called a

- (i) weak hyper *BCI*-ideal of H if $xoy \subseteq I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$,
- (ii) hyper *BCI*-ideal of H if $xoy \ll I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$,
- (iii) strong hyper *BCI*-ideal of H if $xoy \approx I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$, where $xoy \approx I$ means $xoy \cap I \neq \emptyset$.

By $W(H)$ we means the class of all weak hyper *BCI*-ideals, by $I(H)$ we mean the class of all hyper *BCI*-ideals and by $S(H)$ we means the class of all strong hyper *BCI*-ideals of H .

Theorem 2.4. [2] *There are 19 hyper BCK-algebras of order 3 up to isomorphism.*

Proposition 2.5. [6] *In any hyper BCI-algebra, the following hold:*

- (i) $x \ll x \circ o$,
- (ii) $A \ll A$,
- (iii) $y \ll z$ implies $x \circ z \ll x \circ y$,

for all $x, y, z \in H$ and for all nonempty subsets A and B of H .

Definition 2.6. [2] Let $(H_1, \circ_1, 0_1)$ and $(H_2, \circ_2, 0_2)$ be two hyper *BCI*-algebras and $f : H_1 \rightarrow H_2$ be a function. Then f is said to be a homomorphism if and only if

$$f(x \circ_1 y) = f(x) \circ_2 f(y), \text{ for all } x, y \in H_1.$$

If f is one to one (onto) we say that f is a monomorphism (epimorphism) and if f is both one to one and onto, we say that f is an isomorphism and $(H_1, \circ_1, 0_1)$ and $(H_2, \circ_2, 0_2)$ are isomorphic.

3. Some properties of hyper *BCI*-algebras

Remark 3.1. We note that if $(H, *, 0)$ is a *BCI*-algebra and we define $x \circ y = \{x * y\}$, then $(H, \circ, 0)$ is a hyper *BCI*-algebra. Hence hyper *BCI*-algebras are a generalization of *BCI*-algebras

Theorem 3.2. *In any hyper BCI-algebra, the following hold:*

- (i) if $x \ll 0$, then $x = 0$,
- (ii) if $A \ll \{0\}$, then $A = \{0\}$,
- (iii) if $A \circ A = \{0\}$, then A is singleton,
- (iv) $x \in x \circ 0$,
- (v) $x \in 0 \circ 0$ imply that $x \in H^+$,
- (vi) if $H^+ = \{0\}$, then $0 \circ 0 = \{0\}$,
- (vii) if $0 \circ 0 = \{0\}$, then $0 \circ x = \{0\}$, for all $x \in H^+$,

for all $x \in H$ and for all nonempty subset A of H .

Proof. (i) Let $x \ll 0$. Thus $0 \in x \circ 0$. By (B_1) , $0 \in 0 \circ 0(s)(0 \circ 0) \circ (x \circ 0) \ll 0 \circ x$. Hence there is $t \in 0 \circ x$ such that $0 \ll t$ and so $0 \in 0 \circ t$. By (B_5) , $0 \in 0 \circ t(s)0 \circ (0 \circ x) \ll \{x\}$. Hence $0 \ll x$. By (B_4) , we imply that $x = 0$.

(ii) Let $A \ll \{0\}$. Then for all $a \in A$, $a \ll 0$. By (i), we imply that $a = 0$. Hence $A = \{0\}$.

(iii) Let $A \circ A = \{0\}$ and $x, y \in A$. Then $x \circ y = \{0\}$ and $y \circ x = \{0\}$. By (B_4) , $x = y$. Therefore, A is singleton.

(iv) By (B_2) and (B_3) , $0 \in 0 \circ 0(s)(x \circ x) \circ 0 = (x \circ 0) \circ x$. Thus there is $t \in x \circ 0$ such that $0 \in t \circ x$ and so $t \ll x$. On the other hands $0 \in (x \circ 0) \circ t = (x \circ t) \circ 0$. Hence there is an element $a \in x \circ t$ such that $0 \in a \circ 0$. Thus $a \ll 0$. By (i), $a = 0$ and so $0 \in x \circ t$. Hence $x \ll t$. By (B_4) , $x = t$. Therefore, $x \in x \circ 0$.

(v) Let $x \in 0 \circ 0$. By (B_3) and (B_2) , $0 \in (0 \circ 0) \circ x = (0 \circ x) \circ 0$. Thus there exist $a \in 0 \circ x$ such that $0 \in a \circ 0$ and so $a \ll 0$. By (i), $a = 0$. Hence $0 \in 0 \circ x$. Therefore, $x \in H^+$.

(vi) By (v), the proof is easy.

(vii) Let $0 \circ 0 = \{0\}$ and $x \in H^+$. By (B_1) , $(0 \circ x) \circ (0 \circ x) \ll 0 \circ 0 = \{0\}$. By (ii), $(0 \circ x) \circ (0 \circ x) = \{0\}$. By (iii), $0 \circ x$ is singleton. Since $x \in H^+$, $0 \in 0 \circ x$. Thus $0 \circ x = \{0\}$. ■

Theorem 3.3. *Let H be a hyper BCI-algebra and $x \circ x = \{0\}$, for all $x \in H$. Then H is a BCI-algebra.*

Proof. By Remark 3.1, it is sufficient to prove that $x \circ y$ is singleton, for all $x, y \in H$. By (B_1) ,

$$(x \circ y) \circ (x \circ y) \ll x \circ x = \{0\}.$$

By Theorem 3.2(iii), $x \circ y$ is singleton. Therefore, H is a BCI-algebra. ■

Lemma 3.4. *Let H be a hyper BCI-algebra and $H^+ = \{0\}$. Then the following hold:*

- (i) if $x \ll y$, then $x = y$,
- (ii) if $A \ll B$, then $A(s)B$,
- (iii) $0 \circ (0 \circ x) = \{x\}$,
- (iv) $x \circ x = \{0\}$,

for all $x, y \in H$ and for all nonempty subsets A, B of H .

Proof. (i) Let H be a hyper BCI-algebra, $H^+ = \{0\}$ and $x \ll y$. By Theorem 3.2(vi), $0 \circ 0 = \{0\}$. By $0 \in x \circ y$, (B_1) and (B_3) , $\{0\} = 0 \circ 0(s)(y \circ y) \circ (x \circ y) \ll y \circ x$. Hence there exist $a \in y \circ x$ such that $0 \ll a$ and so $0 \in 0 \circ a$. Thus $a \in H^+ = \{0\}$. Hence $a = 0$ and so $y \ll x$. By (B_4) , $x = y$.

(ii) Let $A \ll B$. Then, for all $a \in A$ there exist $b \in B$ such that $a \ll b$. By (i), $a = b$. Therefore, $A(s)B$.

(iii) By (B_5) and (ii), the proof is easy.

(iv) By (iii), $0 \circ (0 \circ x) = \{x\}$, for all $x \in H$. Thus $0 \circ y = \{x\}$, for all $y \in 0 \circ x$. By (iii), $0 \circ x = 0 \circ (0 \circ y) = \{y\}$. Thus $0 \circ x = \{y\}$ and $0 \circ y = \{x\}$. By $0 \circ 0 = \{0\}$ and by (B₁), $x \circ x = (0 \circ y) \circ (0 \circ y) \ll 0 \circ 0 = \{0\}$.

Now, by Theorem 3.2(ii), $x \circ x = \{0\}$. ■

Theorem 3.5. *Let H be a hyper *BCI*-algebra and $H^+ = \{0\}$. Then H is a *p*-semisimple *BCI*-algebra.*

Proof. Let H be a hyper *BCI*-algebra and $H^+ = \{0\}$. By Lemma 3.4(iv), $x \circ x = \{0\}$ and so by Theorem 3.3, H is a *BCI*-algebra. By Lemma 3.4(iii), $0 \circ (0 \circ x) = \{x\}$. Therefore, H is a *p*-semisimple *BCI*-algebra. ■

Theorem 3.6. *If $f : (H_1, \circ_1, 0_1) \longrightarrow (H_2, \circ_2, 0_2)$ is an isomorphism of hyper *BCI*-algebras, then*

- (i) *if $0_1 \circ_1 0_1 = \{0_1\}$, then $f(0_1) = 0_2$,*
- (ii) *if $x \in 0_1 \circ_1 x$, for all $x \in 0_1 \circ_1 0_1$, then $f(0_1) = 0_2$.*

Proof. (i) Let $(H_1, \circ_1, 0_1)$ and $(H_2, \circ_2, 0_2)$ be two hyper *BCI*-algebras. Then $f(0_1) = f(0_1 \circ_1 0_1) = f(0_1) \circ_2 f(0_1)$ and $0_2 \in f(0_1) \circ_2 f(0_1)$. Thus $f(0_1) = 0_2$.

(ii) Let $f(y) = 0_2$ and $f(0_1) = x$. By $0_2 \in f(0_1) \circ_2 f(0_1) = f(0_1 \circ_1 0_1)$, we imply that $y \in 0_1 \circ_1 0_1$. By hypothesis, $y \in 0_1 \circ y$ and by Theorem 3.2(iv), $y \in y \circ_1 0_1$. Thus $f(y) \in f(0_1) \circ_2 f(y)$ and $f(y) \in f(y) \circ_2 f(0_1)$. Hence $0_2 \in x \circ_2 0_2$ and $0_2 \in 0_2 \circ_1 x$. By (B₄), $x = 0_2$. Therefore, $f(0_1) = 0_2$. ■

In what follows, first we introduce the concepts of weak hyper *BCI*-algebras and strong hyper *BCI*-algebras. Then, we find some results on (strong) weak hyper *BCI*-algebras of order 3. Finally, we characterize the hyper *BCI*-algebras of order 3.

Definition 3.7. Let H be a hyper *BCI*-algebra. Then the set $S_k = \{x \in H : x \circ H \ll \{x\}\}$ is defined as hyper *BCK*-part of H . If $H \neq S_k$, then H is known as a proper hyper *BCI*-algebra.

A hyper *BCI*-algebra H is called a

- (i) weak proper hyper *BCI*-algebra if H is proper and $H^+ = H$. In other word if 0 is the smallest element of H ,
- (ii) strong proper hyper *BCI*-algebra if $H^+ \neq H$. We note that if $x \notin H^+$, then $0 \notin 0 \circ x$. Thus $0 \circ x \not\ll \{0\}$. Therefore, $0 \circ H \not\ll \{0\}$ and $(H, \circ, 0)$ is proper.

4. Characterization of weak proper hyper *BCI*-algebra of order 3

Lemma 4.1. *Let $H = \{0, a, b\}$ be a weak proper hyper *BCI*-algebra. Then, the following hold:*

- (i) $0 \circ 0 \neq \{0, a, b\}$,
- (ii) if $0 \circ 0 = \{0, a\}$, then $a \ll b$,
- (iii) if $0 \circ a = \{0\}$, then $0 \circ 0 = \{0\}$.

Proof. (i) Let $0 \circ 0 = \{0, a, b\}$. Since H is a weak proper hyper BCI -algebra, then $0 \in 0 \circ b$. By (B_5) ,

$$\{0, a, b\} = 0 \circ 0(s)0 \circ (0 \circ b) \ll \{b\}.$$

Thus $a \ll b$. By the similar way, $b \ll a$. By (B_4) , $a = b$, which is a contradiction.

(ii) Let $0 \circ 0 = \{0, a\}$. By $b \in H^+$ and (B_5) , $\{0, a\} = 0 \circ 0(s)0 \circ (0 \circ b) \ll \{b\}$. Therefore, $a \ll b$.

(iii) Let $0 \circ a = \{0\}$. By (B_1) , $0 \circ 0(s)(0 \circ a) \circ (0 \circ a) \ll a \circ a = \{0\}$. Now, by Theorem 3.2(ii), $0 \circ 0 = \{0\}$. ■

Lemma 4.2. *Let $H = \{0, a, b\}$ be a weak proper hyper BCI -algebra. Then the following hold:*

- (i) if $0 \circ 0 \not\ll \{0\}$, then $(H, \circ, 0)$ is a chain,
- (ii) if $0 \circ a \not\ll \{0\}$, then $0 \circ 0 \not\ll \{0\}$,
- (iii) if $a \circ 0 \not\ll \{a\}$, then $a \ll b$,
- (iv) if $a \circ a \not\ll \{a\}$, then $a \circ 0 \not\ll \{a\}$,
- (v) if $a \circ b \not\ll \{a\}$, then $a \circ 0 \not\ll \{a\}$.

Proof. (i) Let $H = \{0, a, b\}$ be a weak proper hyper BCI -algebra. Then $0 \ll a$ and $0 \ll b$. It is remind to prove that a and b are comparable. If $0 \circ 0 \not\ll 0$, then $0 \circ 0 = \{0, a\}$ or $\{0, b\}$ or $\{0, a, b\}$. By Lemma 4.1(i), $0 \circ 0 \neq \{0, a, b\}$. Thus $0 \circ 0 = \{0, a\}$ or $\{0, b\}$. If $0 \circ 0 = \{0, a\}$, then by Lemma 4.1(ii), $a \ll b$ and if $0 \circ 0 = \{0, b\}$, $b \ll a$. Therefore, H is a chain.

(ii) If $0 \circ a \not\ll \{0\}$, then $0 \circ a \neq \{0\}$. By Theorem 3.2(vii), $0 \circ 0 \neq \{0\}$ and so $0 \circ 0 \not\ll \{0\}$.

(iii) If $a \circ 0 \not\ll \{a\}$, then $b \in a \circ 0$ and $b \not\ll a$. By Theorems 3.2(i),(iv), $a \in a \circ 0$ and $0 \notin a \circ 0$. Thus $a \circ 0 = \{a, b\}$. By (B_2) , $0 \in b \circ b(s)(a \circ 0) \circ b = (a \circ b) \circ 0$. Hence $0 \in (a \circ b) \circ 0$. It means that there is an element $x \in a \circ b$ such that $0 \in x \circ 0$ and so $x \ll 0$. By Theorem 3.2(i), $x = 0$ and so $0 \in a \circ b$. Therefore, $a \ll b$.

(iv) If $a \circ a \not\ll \{a\}$, then $b \in a \circ a$ and $b \not\ll a$. By Theorem 3.2(iv) and (B_1) ,

$$b \in b \circ 0(s)(a \circ a) \circ (0 \circ a) \ll a \circ 0.$$

Thus there is an element $t \in a \circ 0$ such that $b \ll t$. From $b \not\ll a$ and $b \not\ll 0$ we imply that $t = b$. Hence $b \in a \circ 0$ and $b \not\ll a$. Therefore, $a \circ 0 \not\ll \{a\}$.

(v) If $a \circ b \not\ll a$, then $b \in a \circ b$ and $b \not\ll a$. We have $b \in b \circ 0(s)(a \circ b) \circ (0 \circ b) \ll a \circ 0$. Thus there is an element $t \in a \circ 0$ such that $b \ll t$. From $b \not\ll a$ and $b \not\ll 0$ we imply that $t = b$. Hence $b \in a \circ 0$ and $b \not\ll a$. Therefore, $a \circ 0 \not\ll a$. ■

Theorem 4.3. *Every weak proper hyper BCI -algebra of order three is a chain.*

Proof. Let $H = \{0, a, b\}$ be a weak proper hyper BCI-algebra. It is clear that $0 \ll a$ and $0 \ll b$. It is remind to prove that a and b are comparable. Since H is proper, then

$$0 \circ H \not\ll \{0\} \text{ or } a \circ H \not\ll \{a\} \text{ or } b \circ H \not\ll \{b\}.$$

So we will investigate three cases.

Case 1. If $0 \circ H \not\ll \{0\}$, then $0 \circ 0 \not\ll \{0\}$ or $0 \circ a \not\ll \{0\}$ or $0 \circ b \not\ll \{0\}$. By Lemma 4.2(i) and (ii), we imply that H is a chain.

Case 2. If $a \circ H \not\ll \{a\}$, then $a \circ 0 \not\ll \{a\}$ or $a \circ a \not\ll \{a\}$ or $a \circ b \not\ll \{a\}$. By Lemma 4.2(iii),(iv) and (v), we imply that H is a chain.

Case 3. $b \circ H \not\ll \{b\}$ is similar to the case two.

Therefore, in every case H is a chain. ■

Proposition 4.4. *Let $H = \{0, a, b\}$ be a weak proper hyper BCI-algebra. Then, the following hold:*

- (i) if $0 \circ 0 = \{0, a\}$, then $0 \circ a = 0 \circ b = \{0, a\}$,
- (ii) $0 \circ 0 = 0 \circ a = 0 \circ b = \{0\}$ or $\{0, a\}$ or $\{0, b\}$.

Proof. (i) Let $0 \circ 0 = \{0, a\}$. By Lemma 4.1(ii), $a \ll b$. If $0 \circ a = \{0\}$, then by Lemma 4.1, $0 \circ 0 = \{0\}$ which is a contradiction. By the similar way $0 \circ b \neq \{0\}$. Thus $a \in 0 \circ a$ or $b \in 0 \circ a$ and $a \in 0 \circ b$ or $b \in 0 \circ b$. If $b \in 0 \circ a$, then $0 \circ b(s)0 \circ (0 \circ a) \ll \{a\}$. From $a \ll b$ we imply that $a \in 0 \circ b$ and $b \notin 0 \circ b$. Hence $0 \circ b = \{0, a\}$. Thus $b \in 0 \circ a(s)(0 \circ b) \circ (0 \circ b) \ll 0 \circ 0 = \{0, a\}$. It means that $b \ll 0$ or $b \ll a$ which is a contradiction. Thus $b \notin 0 \circ a$. By $0 \circ a \neq \{0\}$ we conclude that $0 \circ a = \{0, a\}$. If $b \in 0 \circ b$, then $b \in 0 \circ b(s)(0 \circ b) \circ (0 \circ b) \ll 0 \circ 0 = \{0, a\}$. Hence $b \ll 0$ or $b \ll a$ which both of them are contradiction. Therefore, $0 \circ b = \{0, a\}$.

(ii) By (i) and Lemma 4.1(i),(iii), the proof is easy. ■

Theorem 4.5. *Let $H = \{0, a, b\}$ be a weak proper hyper BCI-algebra. Then the following hold:*

- (i) if $0 \circ 0 = \{0\}$, then $a \circ 0 = \{a, b\}$ and $b \circ 0 = \{b\}$ or $a \circ 0 = \{a\}$ and $b \circ 0 = \{a, b\}$,
- (ii) if $0 \circ 0 = \{0, a\}$, then $a \circ 0 = \{a\}$ and $b \circ 0 = \{b\}$.

Proof. (i) Let $0 \circ 0 = \{0\}$. Then, by Proposition 4.4, $0 \circ a = 0 \circ b = 0 \circ 0 = \{0\}$. Thus $0 \circ H \ll \{0\}$. Since H is proper, then $a \circ H \not\ll \{a\}$ or $b \circ H \not\ll \{b\}$. Without loss of generality, let $a \circ H \not\ll \{a\}$. By Lemma 4.2(iii),(iv),(v), we imply that $a \circ 0 \not\ll \{a\}$ and $a \ll b$ and so $a \circ 0 = \{a, b\}$. By Theorem 3.2, $b \in b \circ 0$ and from $0 \ll b$ we conclude that $0 \notin b \circ 0$. If $a \in b \circ 0$, then $0 \in a \circ a(s)(b \circ 0) \circ a = (b \circ a) \circ 0$. Hence $0 \in (b \circ a) \circ 0$. Since $0 \notin a \circ 0$ and $0 \notin b \circ 0$, $0 \in b \circ a$ and so $b \ll a$ which is a contradiction. Thus $a \notin b \circ 0$ and so $b \circ 0 = \{b\}$. By the similar way if $b \circ H \not\ll \{b\}$, then $a \circ 0 = \{a\}$ and $b \circ 0 = \{a, b\}$.

(ii) Let $0 \circ 0 = \{0, a\}$. By Lemma 4.4, $0 \circ b = 0 \circ a = \{0, a\}$ and by Lemma 4.1, $a \ll b$. By Theorem 3.2, $a \in a \circ 0$ and $b \in b \circ 0$. If $b \in a \circ 0$, then

$$b \in a \circ 0(s)(0 \circ a) \circ (0 \circ a) \ll 0 \circ 0 = \{0, a\}.$$

Hence $b \ll 0$ or $b \ll a$ which is a contradiction. Thus $b \notin a \circ 0$ and so $a \circ 0 = \{a\}$.

If $a \in b \circ 0$, then

$$0 \in a \circ a(s)(b \circ 0) \circ a = (b \circ a) \circ 0.$$

Thus there is an element $x \in b \circ a$ such that $0 \in x \circ 0$ and so $x \ll 0$. By Theorem 3.2, $x = 0$ and so $0 \in b \circ a$. Hence $b \ll a$ which is a contradiction. Therefore, $0 \circ b = \{b\}$. ■

Theorem 4.6. *Let $H = \{0, a, b\}$ be a weak proper hyper BCI-algebra, $0 \circ 0 = \{0\}$ and $a \circ 0 = \{a, b\}$. Then $b \in a \circ a$, $b \in b \circ a$, $a \circ b \neq \{0, a\}$ and $b \circ b \neq \{0, a\}$.*

Proof. By Propositions 4.5 and 4.4,

$$0 \circ a = 0 \circ b = 0 \circ 0 = \{0\}, \quad a \circ 0 = \{a, b\}, \quad b \circ 0 = \{b\} \text{ and } a \ll b.$$

Thus $b \in a \circ 0(s)(b \circ a) \circ (a \circ a) \ll b \circ a$. From $a \ll b$ and $0 \ll b$ we imply that $b \in b \circ a$. By (B_1) , $b \in b \circ a(s)(a \circ 0) \circ (a \circ 0) \ll a \circ a$. Therefore, $b \in a \circ a$. To prove $a \circ b \neq \{0, a\}$ and $b \circ b \neq \{0, a\}$ we will prove that if $a \in a \circ b$, then $b \in a \circ b$ and if $a \in b \circ b$, then $b \in b \circ b$. Let $a \in a \circ b$, then

$$(1) \quad b \in a \circ 0(s)(a \circ b) \circ (b \circ b) \ll a \circ b.$$

Since $a \ll b$ and $0 \ll b$, then by (1), $b \in a \circ b$.

Let $a \in b \circ b$, then $b \in a \circ 0(s)(b \circ b) \circ (b \circ b) \ll b \circ b$. Hence $b \in b \circ b$. ■

Theorem 4.7. *Let $H = \{0, a, b\}$ be a weak proper hyper BCI-algebra and $0 \circ 0 = \{0, a\}$. Then*

$$(i) \quad a \circ a = a \circ b = \{0, a\},$$

$$(ii) \quad a \in b \circ b,$$

$$(iii) \quad \text{if } b \in b \circ b, \text{ then } b \in b \circ a.$$

Proof. (i) By Propositions 4.5 and 4.4 we conclude that

$$0 \circ a = 0 \circ b = 0 \circ 0 = \{0, a\}, \quad a \circ 0 = \{a\}, \quad b \circ 0 = \{b\} \text{ and } a \ll b.$$

By (B_1) , $a \in 0 \circ 0(s)(a \circ b) \circ (a \circ b) \ll a \circ a$. Hence $a \ll a \circ a$ and so $a \in a \circ a$ or $b \in a \circ a$. If $b \in a \circ a$, then

$$b \in a \circ a(s)(0 \circ 0) \circ (0 \circ 0) \ll 0 \circ 0 = \{0, a\}.$$

Thus $b \ll 0$ or $b \ll a$ which both of them are contradiction. Hence $b \notin a \circ a$ and so $a \circ a = \{0, a\}$. Also by (B_1) , $\{0, a\} = 0 \circ (b \circ a)(s)(a \circ a) \circ (b \circ a) \ll a \circ b$.

Hence $a \ll a \circ b$. Therefore, $a \in a \circ b$ or $b \in a \circ b$. If $b \in a \circ b$, then $b \in b \circ 0(s)(a \circ b) \circ (0 \circ b) \ll a \circ 0 = \{a\}$. It means that $b \ll a$ which is a contradiction. Thus $b \notin a \circ b$. Since $a \in a \circ b$ and $a \ll b$, $a \circ b = \{0, a\}$.

(ii) By (B_2) , $a \in 0 \circ 0(s)(b \circ b) \circ 0 = (b \circ 0) \circ b = b \circ b$. Thus $a \in b \circ b$.

(iii) If $b \in b \circ b$, then $b \in b \circ b(s)(b \circ 0) \circ (a \circ 0) \ll b \circ a$. Therefore, $b \in b \circ a$. ■

Theorem 4.8. Let $H = \{0, a, b\}$ and $f : (H, \circ_1, 0) \longrightarrow (H, \circ_2, 0)$ be a non identity isomorphism of weak proper hyper *BCI*-algebras, then

(i) $f(0) = 0$,

(ii) if $a \ll_1 b$, then $b \ll_2 a$.

Proof. (i) By Lemma 4.4 and Theorem 3.6, the proof is clear.

(ii) Let $a \ll_1 b$. Since $f(0) = 0$ and f is not identity, then $f(a) = b$ and $f(b) = a$. By $0 \in a \circ_1 b$, $f(0) \in f(a) \circ_2 f(b)$ and so $0 \in b \circ_2 a$. Therefore, $b \ll_2 a$. ■

By Theorems 4.8 and 4.3 any weak proper hyper *BCI*-algebra is a chain and any chain $0 \ll a \ll b$ is isomorph by a chain $0 \ll b \ll a$. So if we let $H = \{0, a, b\}$ and $0 \ll a \ll b$, then we can find all weak proper hyper *BCI*-algebra of order 3 up to isomorphism.

According Theorems 4.4, 4.5, 4.6 and 4.7 we have two following structures:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a, b\}$	$\{0, b\}$ or $\{0, a, b\}$	$\{0\}, \{0, b\}$ or $\{0, a, b\}$
b	$\{b\}$	$\{b\}$ or $\{a, b\}$	$\{0\}, \{0, b\}$ or $\{0, a, b\}$

$\circ$	0	a	b
0	$\{0, a\}$	$\{0, a\}$	$\{0, a\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{a\}, \{b\}$ or $\{a, b\}$	$\{0, a\}$ or $\{0, a, b\}$

Case 1. Let $0 \circ 0 = 0 \circ a = 0 \circ b = \{0\}$, $a \circ 0 = \{a, b\}$, $b \circ 0 = \{b\}$.

Let $a \circ a = \{0, b\}$. If $a \in b \circ a$, then

$$a \in b \circ a(s)(a \circ 0) \circ a = (a \circ a) \circ 0 = 0 \circ 0 \cup b \circ 0 = \{0, b\}.$$

which is a contradiction. Hence $b \circ a = \{b\}$. If $a \circ b = \{0\}$, then

$$b \circ b(s)(a \circ a) \circ (b \circ a) \ll a \circ b = \{0\}.$$

By Theorem 3.2, $b \circ b = \{0\}$ and we get the following Cayley table which is a weak proper hyper *BCI*-algebra.

$\circ_1$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a, b\}$	$\{0, b\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0\}$

$$W(H) = I(H) = S(H) = \{\{0\}, \{0, a\}, H\}.$$

If $a \circ b = \{0, b\}$, then by $\{0\} = 0 \circ b(s)b \circ b$ and (B_3) ,

$$b \circ b = 0 \circ b \cup b \circ b = \{0, b\} \circ b = (a \circ a) \circ b = (a \circ b) \circ a = \{0, b\} \circ a = \{0, b\}.$$

Thus $b \circ b = \{0, b\}$ and we get the following weak proper hyper BCI -algebra.

$\circ_2$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a, b\}$	$\{0, b\}$	$\{0, b\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$

$$W(H) = I(H) = S(H) = \{\{0\}, \{0, a\}, H\}.$$

If $a \circ b = \{0, a, b\}$, then

$$b \circ b = 0 \circ b \cup b \circ b = \{0, b\} \circ b = (a \circ a) \circ b = (a \circ b) \circ a = \{0, a, b\} \circ a = \{0, b\}.$$

Thus $b \circ b = \{0, b\}$ and we get the following weak proper hyper BCI -algebra.

$\circ_3$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a, b\}$	$\{0, b\}$	$\{0, a, b\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$

$$I(H) = S(H) = \{\{0\}, \{0, a\}, H\} \text{ and } W(H) = \{\{0\}, \{0, b\}, \{0, a\}, H\}.$$

Let $a \circ a = \{0, a, b\}$ and $b \circ a = \{b\}$. If $a \circ b = \{0\}$, then $b \circ b(s)(a \circ 0) \circ (b \circ 0) \ll a \circ b = \{0\}$. By Theorem 3.2, $b \circ b = \{0\}$ and we get the next weak proper hyper BCI -algebra.

$\circ_4$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a, b\}$	$\{0, a, b\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0\}$

$$W(H) = I(H) = S(H) = \{\{0\}, \{0, a\}, H\}.$$

If $a \circ b = \{0, b\}$, then $a \circ b \cup b \circ b = \{a, b\} \circ b = (a \circ 0) \circ b = (a \circ b) \circ 0 = \{0, b\}$. Hence $\{0, b\} \cup b \circ b = \{0, b\}$. which means that $b \circ b = \{0\}$ or $b \circ b = \{0, b\}$ which both of them are weak proper hyper BCI -algebras.

$\circ_5$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a, b\}$	$\{0, a, b\}$	$\{0, b\}$
b	$\{b\}$	$\{b\}$	$\{0\}$

$\circ_6$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a, b\}$	$\{0, a, b\}$	$\{0, b\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$

$$W(H, \circ_5) = I(H, \circ_5) = S(H, \circ_5) = W(H, \circ_6) = I(H, \circ_6) = S(H, \circ_6) = \{\{0\}, \{0, a\}, H\}.$$

If $a \circ b = \{0, a, b\}$, then $b \circ b = \{0\}, \{0, b\}$ or $\{0, a, b\}$ which all of them are weak proper hyper *BCI*-algebras.

$\circ_7$	0	a	b	$\circ_8$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$	0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a, b\}$	$\{0, a, b\}$	$\{0, a, b\}$	a	$\{a, b\}$	$\{0, a, b\}$	$\{0, a, b\}$
b	$\{b\}$	$\{b\}$	$\{0\}$	b	$\{b\}$	$\{b\}$	$\{0, b\}$

$$I(H, \circ_7) = S(H, \circ_7) = I(H, \circ_8) = S(H, \circ_8) = \{\{0\}, \{0, a\}, H\},$$

$$W(H, \circ_7) = W(H, \circ_8) = \{\{0\}, \{0, b\}, \{0, a\}, H\}.$$

$\circ_9$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a, b\}$	$\{0, a, b\}$	$\{0, a, b\}$
b	$\{b\}$	$\{b\}$	$\{0, a, b\}$

$$I(H) = S(H) = \{\{0\}, \{0, a\}, H\} \text{ and } W(H) = \{\{0\}, \{0, b\}, \{0, a\}, H\}.$$

Let $a \circ a = \{0, a, b\}$ and $b \circ a = \{a, b\}$. By (B₁), $(a \circ a) \circ (b \circ a) \ll a \circ b$. Hence $\{0, a, b\} \ll a \circ b$. We imply that $b \in a \circ b$. Thus $a \circ b = \{0, b\}$ or $\{0, a, b\}$. If $a \circ b = \{0, b\}$, then by $(a \circ a) \circ b = (a \circ b) \circ a$ we conclude that $\{0, b\} \cup b \circ b = \{0, a, b\}$. Hence $a \in b \circ b$. By Theorem 4.7(iii), $b \circ b = \{0, a, b\}$. But in this case $(a \circ 0) \circ b \neq (a \circ b) \circ 0$. Hence $a \circ b \neq \{0, b\}$.

If $a \circ b = \{0, a, b\}$, then $b \circ b = \{0\}, \{0, b\}$ or $\{0, a, b\}$. If $b \circ b = \{0\}$, then $(b \circ a) \circ b \neq (b \circ b) \circ a$. If $b \circ b = \{0, b\}$ or $\{0, a, b\}$, then we get two following weak proper hyper *BCI*-algebras.

$\circ_{10}$	0	a	b	$\circ_{11}$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$	0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a, b\}$	$\{0, a, b\}$	$\{0, a, b\}$	a	$\{a, b\}$	$\{0, a, b\}$	$\{0, a, b\}$
b	$\{b\}$	$\{a, b\}$	$\{0, b\}$	b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

$$I(H, \circ_{10}) = S(H, \circ_{10}) = I(H, \circ_{11}) = S(H, \circ_{11}) = \{\{0\}, \{0, a\}, H\},$$

$$W(H, \circ_{10}) = W(H, \circ_{11}) = \{\{0\}, \{0, b\}, \{0, a\}, H\}.$$

Case 2. Let $0 \circ 0 = 0 \circ a = 0 \circ b = \{0, a\}$, $a \circ 0 = \{a\}$, $b \circ 0 = \{b\}$ and $a \circ a = a \circ b = \{0, a\}$.

If $b \circ b = \{0, a\}$, then $b \circ a = \{b\}, \{a\}$ or $\{a, b\}$. If $b \circ a = \{a, b\}$, then $(b \circ a) \circ (b \circ a) \not\leq b \circ b$. The other cases are weak proper hyper *BCI*-algebras with the following Cayley tables.

$\circ_{12}$	0	a	b	$\circ_{13}$	0	a	b
0	$\{0, a\}$	$\{0, a\}$	$\{0, a\}$	0	$\{0, a\}$	$\{0, a\}$	$\{0, a\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$	a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{b\}$	$\{0, a\}$	b	$\{b\}$	$\{a\}$	$\{0, a\}$

$$I(H, \circ_{12}) = S(H, \circ_{12}) = \{\{0\}, \{0, a\}, H\}, \quad I(H, \circ_{13}) = S(H, \circ_{13}) = \{\{0\}, H\}, \\ W(H, \circ_{12}) = \{\{0\}, \{0, b\}, \{0, a\}, H\} \text{ and } W(H, \circ_{13}) = \{\{0\}, \{0, b\}, H\}.$$

If $b \circ b = \{0, a, b\}$, then by Theorem 4.7, $b \in b \circ a$. Thus $b \circ a = \{b\}$ or $\{a, b\}$, which both of them are weak proper hyper BCI -algebra with the following Cayley tables:

$\circ_{14}$	0	a	b	$\circ_{15}$	0	a	b
0	$\{0, a\}$	$\{0, a\}$	$\{0, a\}$	0	$\{0, a\}$	$\{0, a\}$	$\{0, a\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$	a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{b\}$	$\{0, a, b\}$	b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

$$I(H, \circ_{14}) = S(H, \circ_{14}) = I(H, \circ_{15}) = \{\{0\}, \{0, a\}, H\}, \quad S(H, \circ_{15}) = \{\{0\}, H\}, \\ W(H, \circ_{14}) = W(H, \circ_{15}) = \{\{0\}, \{0, b\}, \{0, a\}, H\}.$$

5. Characterization of strong proper hyper BCI -algebra of order 3

Let $H = \{0, a, b\}$ and $(H, \circ, 0)$ be a strong proper hyper BCI -algebra. By definition of strong proper hyper BCI -algebra, $H^+ \neq H$. Thus $H^+ = \{0\}$ or $H^+ = \{0, a\}$.

Theorem 5.1. *Let $H = \{0, a, b\}$ be a strong proper hyper BCI -algebra and $H^+ = \{0, a\}$. Then, the following hold:*

- (i) $0 \circ b = \{b\}$,
- (ii) $0 \circ 0 = \{0\}$ or $\{0, a\}$,
- (iii) if $0 \circ 0 = \{0\}$, then $0 \circ a = \{0\}$,
- (iv) if $0 \circ 0 = \{0, a\}$, then $0 \circ a = \{0, a\}$,
- (v) $0 \circ a = 0 \circ 0 = \{0\}$ or $\{0, a\}$,
- (vi) if $0 \circ 0 = \{0, a\}$, then $a \in b \circ b$ and $a \in a \circ a$.

Proof. (i) Since $b \notin H^+$, $0 \notin 0 \circ b$. If $a \in 0 \circ b$, then by $a \in H^+$ and (B_5) , $0 \in 0 \circ a(s)0 \circ (0 \circ b) \ll \{b\}$, which is a contradiction. Hence $a \notin 0 \circ b$. Since $b \notin H^+$, $0 \notin 0 \circ b$. Therefore, $0 \circ b = \{b\}$.

(ii) If $b \in 0 \circ 0$, then by Theorem 3.2(i) and (B_2) ,

$$0 \notin b \circ 0 = (0 \circ b) \circ 0 = (0 \circ 0) \circ b \supseteq b \circ b \ni 0,$$

which is a contradiction. Thus $b \notin 0 \circ 0$ and so $0 \circ 0 = \{0\}$ or $0 \circ 0 = \{0, a\}$.

(iii) By Theorem 3.2(vii), the proof is easy.

(iv) Let $0 \circ 0 = \{0, a\}$. If $0 \circ a = \{0\}$, then by (iv), $0 \circ 0 = \{0\}$ which is a contradiction. Since $a \in H^+$, $0 \in 0 \circ a$. It is sufficient to prove that $b \notin 0 \circ a$. On the contrary let $b \in 0 \circ a$. By (i) and (B_2) ,

$$b \circ a = (0 \circ b) \circ a = (0 \circ a) \circ b \supseteq 0 \circ b \cup b \circ b \ni \{0, b\}.$$

Thus $b \in b \circ a$ and $0 \in b \circ a$. Hence $b \ll a$ and so $a \not\ll b$.

By (B_1) and (B_3) ,

$$a \in 0 \circ 0(s)(a \circ a) \circ (b \circ a) \ll a \circ b.$$

Thus $a \in a \circ b$. On the other hand,

$$\{0, a\} = 0 \circ 0(s)(b \circ b) \circ (b \circ b) \ll b \circ b.$$

Hence $a \in b \circ b$. By (B_1) ,

$$a \in a \circ b(s)(b \circ b) \circ (0 \circ b) \ll b \circ 0.$$

Since $b \ll a$, then $a \in b \circ 0$. By Theorem 3.2, $b \circ 0 = \{a, b\}$. By (B_1) and (i),

$$0 \in a \circ a(s)(0 \circ 0) \circ (b \circ 0) \ll 0 \circ b = \{b\}.$$

Thus $0 \ll b$, which is a contradiction. Hence $b \notin 0 \circ a$. Therefore, $0 \circ a = \{0, a\}$.

(v) By (iii) and (iv), the proof is easy.

(vi) Let $0 \circ 0 = \{0, a\}$. By (v), $0 \circ a = \{0, a\}$. By (B_1) ,

$$\{0, a\} = 0 \circ 0(s)(a \circ a) \circ (a \circ a) \ll a \circ a.$$

Hence there is $t \in a \circ a$ such that $a \ll t$.

Let $t = b$ and so $b \in a \circ a$ and $a \ll b$. Thus $b \not\ll a$. By (B_2) ,

$$0 \notin b \circ a = (0 \circ b) \circ a = (0 \circ a) \circ b \supseteq a \circ b \ni 0,$$

which is a contradiction. Thus $t = a$ and so $a \in a \circ a$. By similar way from

$$\{0, a\} = 0 \circ 0(s)(b \circ b) \circ (b \circ b) \ll b \circ b,$$

we imply that $a \in b \circ b$. ■

Theorem 5.2. *Let $H = \{0, a, b\}$ be a strong proper hyper *BCI*-algebra and $H^+ = \{0, a\}$. If $0 \circ 0 = \{0, a\}$, then the following hold:*

- (i) $b \circ a = \{b\}$,
- (ii) $a \circ b = \{b\}$,
- (iii) $b \circ 0 = \{b\}$,
- (iv) $a \circ 0 = \{a\}$,
- (v) $a \circ a = \{0, a\}$,
- (vi) $b \circ b = \{0, a\}$.

Proof. If $0 \circ 0 = \{0, a\}$, then by Theorem 5.1, $0 \circ a = \{0, a\}$, $0 \circ b = \{b\}$, $a \in b \circ b$ and $a \in a \circ a$.

(i) By (B_2) ,

$$(1) \quad b \circ a = (0 \circ b) \circ a = (0 \circ a) \circ b = \{0, a\} \circ b = \{b\} \cup a \circ b.$$

Thus $b \in b \circ a$. If $0 \in a \circ b$, then by (1), $0 \in b \circ a$. It means that $a \ll b$ and $b \ll a$. Therefore, $a = b$ which is a contradiction. Hence $0 \notin a \circ b$ and $0 \notin b \circ a$. If $b \circ a = \{a, b\}$, then $a \circ a(s)(0 \circ a) \circ (b \circ a) \ll 0 \circ b = \{b\}$. Hence $a \circ a = \{b\}$ which is a contradiction. Therefore, $b \circ a = \{b\}$.

(ii) By $b \circ a = \{b\}$ and (1), $a \circ b = \{b\}$.

(iii) By Theorem 3.2(i),(iii), $b \in b \circ 0$ and $0 \notin b \circ 0$. By (B_2) ,

$$b \circ 0 = (b \circ a) \circ 0 = (b \circ 0) \circ a.$$

If $a \in b \circ 0$, then we conclude that $0 \in b \circ 0$ which is a contradiction. Hence $b \circ 0 = \{b\}$.

(iv) By Theorem 3.2, $a \in a \circ 0$ and $0 \notin a \circ 0$. If $b \in a \circ 0$, then

$$b = 0 \circ b(s)(0 \circ 0) \circ (a \circ 0) \ll 0 \circ a = \{0, a\}.$$

Thus $b \ll a$. But in (1) we proved that $a \not\ll b$ and $b \not\ll a$. Thus $b \notin a \circ 0$ and $a \circ 0 = \{a\}$.

(v) By Theorem 5.1(vii), $a \in a \circ a$. If $b \in a \circ a$, then

$$b \in a \circ a(s)(0 \circ a) \circ (0 \circ a) \ll 0 \circ 0 = \{0, a\},$$

which is a contradiction. Thus $a \circ a = \{0, a\}$.

(vi) By Theorem 5.1(vii), $a \in b \circ b$. If $b \in b \circ b$, then

$$b \in b \circ b(s)(0 \circ b) \circ (0 \circ b) \ll 0 \circ 0 = \{0, a\},$$

which is a contradiction. Thus $b \circ b = \{0, a\}$. ■

By Theorem 5.2, if $0 \circ 0 = \{0, a\}$ we just have one weak proper hyper *BCI*-algebra with the following Cayley table:

$\circ_{16}$	0	a	b
0	$\{0, a\}$	$\{0, a\}$	$\{b\}$
a	$\{a\}$	$\{0, a\}$	$\{b\}$
b	$\{b\}$	$\{b\}$	$\{0, a\}$

$$W(H) = I(H) = S(H) = \{\{0\}, \{0, a\}, H\}.$$

Theorem 5.3. Let $H = \{0, a, b\}$ and $(H, \circ, 0)$ be a strong proper hyper *BCI*-algebra and $H^+ = \{0, a\}$. If $0 \circ 0 = \{0\}$, then the following hold:

- (i) $b \circ a = \{b\}$,
- (ii) $b \circ b = \{0\}$,
- (iii) $b \circ 0 = \{b\}$,
- (iv) $a \circ 0 = \{a\}$,
- (v) $a \circ b = \{b\}$,
- (vi) $a \circ a = \{0, a\}$ or $\{0\}$.

Proof. If $0 \circ 0 = \{0\}$, then by Theorem 5.1, $0 \circ a = \{0\}$ and $0 \circ b = \{b\}$.

(i) By (B₂), $b \circ a = (0 \circ b) \circ a = (0 \circ a) \circ b = 0 \circ b = \{b\}$. Thus $b \circ a = \{b\}$.

(ii) By (B₁), $b \circ b(s)(0 \circ b) \circ (0 \circ b) \ll 0 \circ 0 = \{0\}$. By Theorem 3.2(ii), $b \circ b = \{0\}$.

(iii) By Theorem 3.2, $b \in b \circ 0$ and we have $(b \circ 0) \circ (b \circ 0) \ll b \circ b = \{0\}$. By Theorem 3.2(ii), $(b \circ 0) \circ (b \circ 0) = \{0\}$ and so by Theorem 3.2(iii), $b \circ 0$ is singleton. By Theorem 3.2(iv), we imply that $b \circ 0 = \{b\}$.

(iv) By Theorem 3.2, $a \in a \circ 0$. If $b \in a \circ 0$, then $b = 0 \circ b(s)(0 \circ 0) \circ (a \circ 0) \ll 0 \circ a = \{0\}$. It means that $b \ll 0$ which is a contradiction. Hence $a \circ 0 = \{a\}$.

(v) By (B₁), (i) and (ii), $0 \circ (a \circ b) = (b \circ b) \circ (a \circ b) \ll b \circ a = \{b\}$. Thus $0 \circ (a \circ b) = \{b\}$ or $\{a\}$. From $0 \circ 0 = 0 \circ a = \{0\}$ and $0 \circ b = \{b\}$ we conclude that $a \circ b = \{b\}$.

(vi) By (B₁), (ii) and (v), $0 = b \circ b = (a \circ b) \circ (a \circ b) \ll a \circ a$. Since $b \notin H^+$, $a \circ a = \{0\}$ or $\{0, a\}$. ■

By Theorem 5.3, if $0 \circ 0 = \{0\}$ we can have two strong proper hyper *BCI*-algebras and we get two following strong proper hyper *BCI*-algebras.

$\circ_{17}$	0	a	b	$\circ_{18}$	0	a	b
0	{0}	{0}	{b}	0	{0}	{0}	{b}
a	{a}	{0}	{b}	a	{a}	{0, a}	{b}
b	{b}	{b}	{0}	b	{b}	{b}	{0}

$$W(H, \circ_{17}) = I(H, \circ_{17}) = S(H, \circ_{17}) = W(H, \circ_{18}) = I(H, \circ_{18}) = S(H, \circ_{18}) = \{\{0\}, \{0, a\}, H\}.$$

Theorem 5.4. Let $H = \{0, a, b\}$ and $f : (H, \circ_1, 0) \longrightarrow (H, \circ_2, 0)$ be a non identity isomorphism of strong proper hyper *BCI*-algebras, then

- (i) $f(0) = 0$,
- (ii) if $(H, \circ_1)^+ = \{0, a\}$, then $(H, \circ_2)^+ = \{0, b\}$.

Proof. (i) By Theorems 5.2, 5.3, 3.5 and 3.6, the proof is routine and we are omitted.

(ii) Let $(H, \circ_1)^+ = \{0, a\}$. Since $f(0) = 0$ and f is not identity, $f(a) = b$ and $f(b) = a$. By $0 \in 0 \circ_1 a$, $f(0) \in f(0) \circ_2 f(a)$ and so $0 \in 0 \circ_2 b$. Therefore, $(H, \circ_2)^+ = \{0, b\}$. ■

Let $H = \{0, a, b\}$ and H be a strong proper hyper BCI -algebra. Then $H^+ = \{0\}$ or $\{0, a\}$ or $\{0, b\}$. If $(H, \circ_1)^+ = \{0, a\}$, then by Theorem 5.4, $(H, \circ_1)$ is isomorph by a strong proper hyper BCI -algebra $(H, \circ_2)$ $(H, \circ_2)^+ = \{0, b\}$.

Let $(H, \circ, 0)$ be a strong proper hyper BCI -algebra and $H^+ = \{0\}$. Then by Theorem 3.5, H is p-semisimple BCI -algebra and we get the only p-semisimple BCI -algebra of order 3 with the following Cayley table:

$\circ_{19}$	0	a	b
0	$\{0\}$	$\{b\}$	$\{a\}$
a	$\{a\}$	$\{0\}$	$\{b\}$
b	$\{b\}$	$\{a\}$	$\{0\}$

$$W(H) = I(H) = S(H) = \{\{0\}, H\}.$$

6. Characterization of hyper BCI -algebra of order 3

Theorem 6.1. *The number of proper hyper BCI -algebra of order 3 is 19.*

Theorem 6.2. *The number of hyper BCI -algebra of order 3 is 38.*

Proof. By Theorem 2.4, there are 19 hyper BCK -algebra of order 3 up to isomorphism. In this note we proved that there are 19 proper hyper BCI -algebra of order 3 up to isomorphism. Also every hyper BCK -algebra is hyper BCI -algebra. Thus there are 38 hyper BCI -algebra of order 3. ■

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NUMERICAL SOLUTION OF SERIES $L-C-R$ EQUATION BASED ON HAAR WAVELET

Naresh Berwal

*Department of Mathematics
IES, IPS Academy
Indore 452010
India
e-mail: nareshberwal.019@gmail.com*

Dinesh Panchal

*Department of Mathematics DAVV
Indore, 452001
India
e-mail: dkuma_panc@yahoo.co.in*

C.L. Parihar

*Indian Academy of Mathematics
kaushaliya puri
Indore, 452001
India
e-mail: profparihar@hotmail.com*

Abstract. Haar wavelet is the simplest and computer oriented tool for solving ordinary differential equations and partial differential equations. Numerical solution of Series $L-C-R$ is very useful in many engineering branches. In this paper we shall discuss the numerical solution of series $L-C-R$ circuit with Haar wavelet method. We shall find charge in series $L-C-R$ circuit at different times. Two different cases show the accuracy of Haar method.

Keywords: Haar wavelet, $L-C-R$ circuit, linear ordinary differential equation, Matlab.
2010 Mathematical Subject Classification: 65T60, 34G10, 65D25.

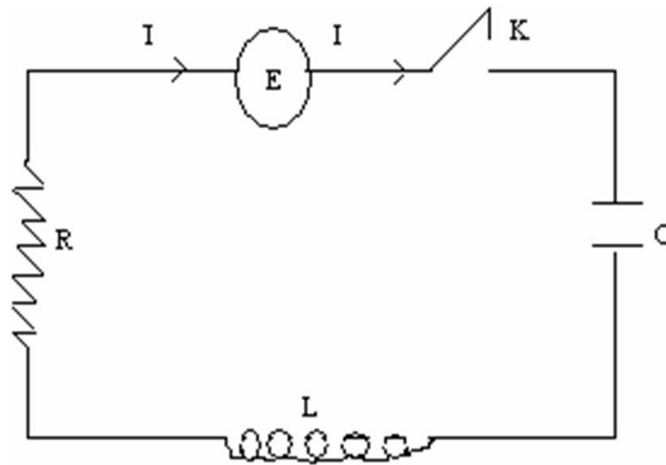
1. Introduction

Wavelet transform and wavelet analysis are recently developed as mathematical tools. The roots of wavelet find in various disciplines such as image compression, data compression, denoising data, solution of initial & boundary value problems etc. First Chen and Hsiao [1] gave a method for solving linear systems of ordinary differential equations and partial differential equations based on haar wavelet. Haar wavelet is the simplest orthonormal wavelet with compact support. Lepik [6], [7],

[8] applied Haar wavelet in solving nonlinear integro-differential equation and partial differential equations. In recent years wavelet transform is very useful in the field of numerical approximation. Due to simplicity, Haar wavelet had become an effective tool for solving ordinary differential equations and partial differential equations.

Haar wavelet can be applied to compute the charge on the capacitors and currents as function of time. A simple electrical circuit consists of the following elements connected in series with a key K .

- (i) A battery which supplied an electromotive force (E.M.F).
- (ii) An inductor that has inductance L .
- (iii) a resistance R and
- (iv) a capacitor that has capacitance C .



Series L-C-R Circuit

The differential equation of above figure is

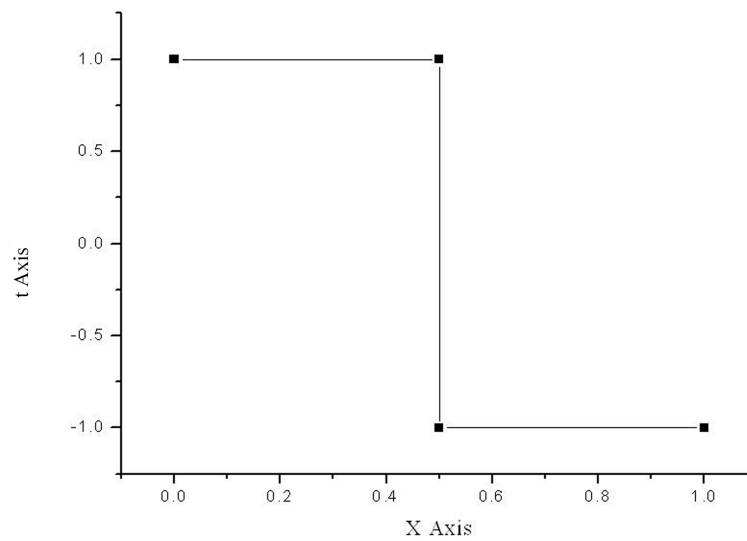
$$(1.1) \quad L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = E \quad Q(0) = 0, Q'(0) = 0. \quad I = \frac{dQ}{dt}$$

where Q is charge, R is resistance, L is inductance, C is capacitance, w is angular velocity and E is electromotive force.

2. Haar wavelet

The Haar wavelet was first introduced by Alfred Haar [5] in 1910. Haar wavelet is a certain sequence of rescaled “square-shaped” function which together forms a wavelet family or basis. Haar wavelet is defined for $t \in [0, 1)$

$$\psi(x) = \begin{cases} 1 & 0 \leq t < \frac{1}{2} \\ -1 & \frac{1}{2} \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

*Haar wavelet*

Haar wavelet family is defined for $t \in [0, 1)$ as follows:

$$(2.1) \quad h_i(t) = \begin{cases} 1 & \text{for } t \in [\eta_1, \eta_2) \\ -1 & \text{for } t \in [\eta_2, \eta_3) \\ 0 & \text{otherwise} \end{cases}$$

where $\eta_1 = \frac{K}{m}$, $\eta_2 = \frac{K+0.5}{m}$, $\eta_3 = \frac{K+1}{m}$. The integer $m = 2^j$ ($j = 0, 1, \dots, J$) indicates the level of the wavelet; $k = 0, 1, \dots, m-1$ is the translation parameter. The maximal level of relation is J . The index i in (1) is calculated according to the formula $i = m + k + 1$; In the case of minimal values $m = 1, k = 0$, we have $i = 2$. The maximum value of i is $i = 2^{j+1} = M$. It is assumed that the value $i = 1$ corresponding to the scaling function for which $h_1 = 1$ for $t \in [0, 1)$.

3. Approximation of function

Any square integrable function $Q(t)$ can be expanded by haar series of infinite terms in the interval $[0, 1)$

$$Q(t) = \sum_{i=1}^{\infty} a_i h_i(t)$$

where a_i , $i = 1, 2, \dots$, are haar coefficients. If $Q(t)$ be is approximate as piecewise constant during each subinterval, then $Q(t)$ will be terminated at finite M terms, that is

$$Q(t) = \sum_{i=1}^M a_i h_i(t).$$

4. Haar wavelet method

We follow work done by Phang Chang and Phang Piau [2], for solving n^{th} order linear ordinary differential equation

$$a_1 Q^{(n)}(t) + a_2 Q^{(n-1)}(t) + \cdots + a_n Q(t) = f(t),$$

where $t \in [A, B)$.

Let $M = 2^{j+1}$. The interval $[A, B)$ will be divided into M subintervals, hence $\Delta t = \frac{A-B}{M}$, the values at each point can be estimated by following 5 stapes.

1. Let $Q^{(n)}(t) = \sum_{i=1}^M a_i h_i(t)$ where h is haar wavelet and a_i , $i = 1, 2, \dots$, are the wavelet coefficients.
2. Obtain appropriate v order of $Q(t)$ by integration of $Q^{(n)}(t)$ with respect to t from 0 to t .
3. Substitute value of $Q^{(n)}(t)$ and all the values of $Q^{(v)}(t)$ into the given differential equation.
4. Calculate the wavelet coefficients a_i .
5. Obtain the numerical solution for $Q(t)$.

5. Computing $p_{i,v}(t)$

By Hsiao-Chen [1] method we know that

$$(5.1) \quad p_{1,i}(t) = \int_0^t h_i(t) dt,$$

$$(5.2) \quad p_{v,i}(t) = \int_0^t p_{v-1,i}(t) dt, \quad v = 2, 3, \dots$$

Carrying out these integrations with the aid equation (2.1), we have

$$p_{1,i}(t) = \begin{cases} t - \eta_1 & \text{for } t \in [\eta_1, \eta_2), \\ \eta_3 - t & \text{for } t \in [\eta_2, \eta_3), \\ 0 & \text{elsewhere,} \end{cases}$$

$$p_{2,i}(t) = \begin{cases} \frac{1}{2}(t - \eta_1)^2 & \text{for } t \in [\eta_1, \eta_2), \\ \frac{1}{4m^2} - \frac{1}{2}(\eta_3 - t)^2 & \text{for } t \in [\eta_2, \eta_3), \\ \frac{1}{4m^2} & \text{for } t \in [\eta_3, 1), \\ 0 & \text{elsewhere.} \end{cases}$$

Similarly, we can find other values of p .

6. Solution of Series $L-C-R$ equation using Haar wavelet

The calculation is done by using 3 levels of Haar wavelet.

If an alternating E.M.F. $E \sin(\omega t)$ is applied to an inductance L and a capacitance C in series, then the differential equation will be (where $R = 0$)

$$(6.1) \quad L \frac{d^2 Q}{dt^2} + \frac{Q}{C} = E \sin \omega t \quad Q(0) = 0, \quad Q'(0) = 0.$$

Now, the complete solution of differential equation (6.1) is

$$Q(t) = C_1 \cos\left(\frac{t}{\sqrt{LC}}\right) + C_2 \sin\left(\frac{t}{\sqrt{LC}}\right) + \frac{EC}{(1 - LC\omega^2)} \sin(\omega t)$$

Also at $Q(0) = 0$ and $Q'(0) = 0$ we have,

$$\begin{aligned} C_1 &= 0 \\ C_2 &= -\frac{EC\omega}{(1 - LC\omega^2)} \end{aligned}$$

Therefore, the exact solution of equation (6.1) is

$$(6.2) \quad Q(t) = -\frac{EC\omega}{(1 - LC\omega^2)} \sqrt{LC} \sin\left(\frac{t}{\sqrt{LC}}\right) + \frac{EC}{(1 - LC\omega^2)} \sin(\omega t)$$

Haar solution

Let

$$(6.3) \quad Q''(t) = \sum_{i=1}^M a_i h_i(t)$$

Now integrating above equation with respect to t from 0 to t then we have

$$Q'(t) = \sum_{i=1}^M a_i p_{1,i}(t) + Q'(0)$$

Again integrating above equation with respect to t from 0 to t then we get

$$Q(t) = \left[\sum_{i=1}^M a_i p_{2,i}(t) \right] + tQ'(0) + Q(0)$$

After using boundary condition we get following result

$$(6.4) \quad Q(t) = \left[\sum_{i=1}^M a_i p_{2,i}(t) \right]$$

Now, from equation (6.1)

$$L \sum_{i=1}^M a_i h_i(t) + \frac{1}{C} \left[\sum_{i=1}^M a_i p_{2,i}(t) \right] = E \sin(wt)$$

$$(6.5) \quad \sum_{i=1}^M a_i \left[h_i(t) + \frac{p_{2,i}(t)}{LC} \right] = \frac{E \sin(wt)}{L}$$

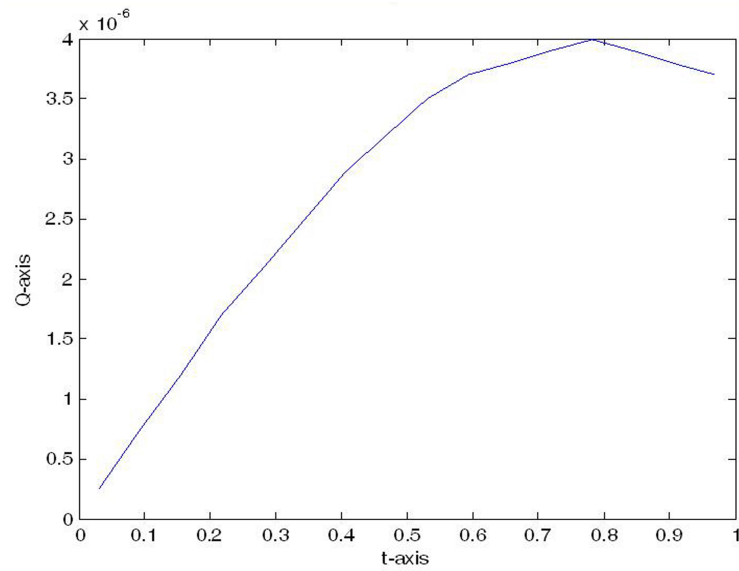
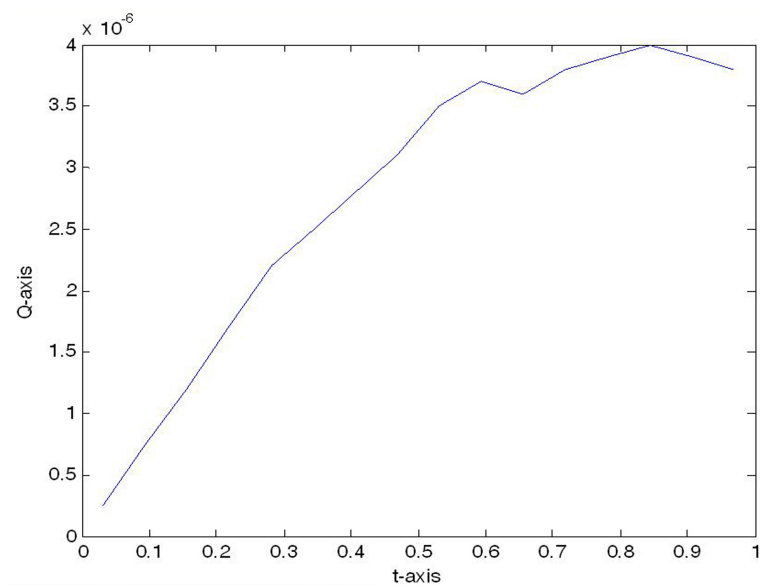
After solving system of linear differential equation we get wavelet coefficients a_i .

Case I. If $E = 2v$, $w = 2$ radian, $C = 2\mu F$, and $L = 1 H$. Then equation (6.5) becomes

$$\sum_{i=1}^M a_i \left[h_i(t) + 10^6 \frac{p_{2,i}(t)}{2} \right] = 2 \sin(2t).$$

Table I
Numerical solution of Case I

t /32	Haar solution	Exact solution
1	0.00000025	0.00000025
3	0.00000074	0.00000074
5	0.0000012	0.0000012
7	0.0000017	0.0000017
9	0.0000022	0.0000021
11	0.0000025	0.0000025
13	0.0000028	0.0000029
15	0.0000031	0.0000032
17	0.0000035	0.0000035
19	0.0000037	0.0000037
21	0.0000036	0.0000038
23	0.0000038	0.0000039
25	0.0000039	0.0000040
27	0.0000040	0.0000039
29	0.0000039	0.0000038
31	0.0000038	0.0000037

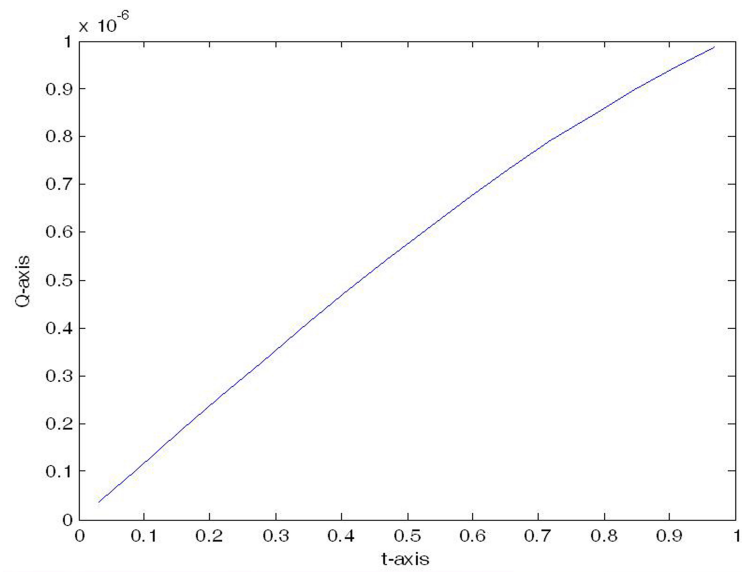
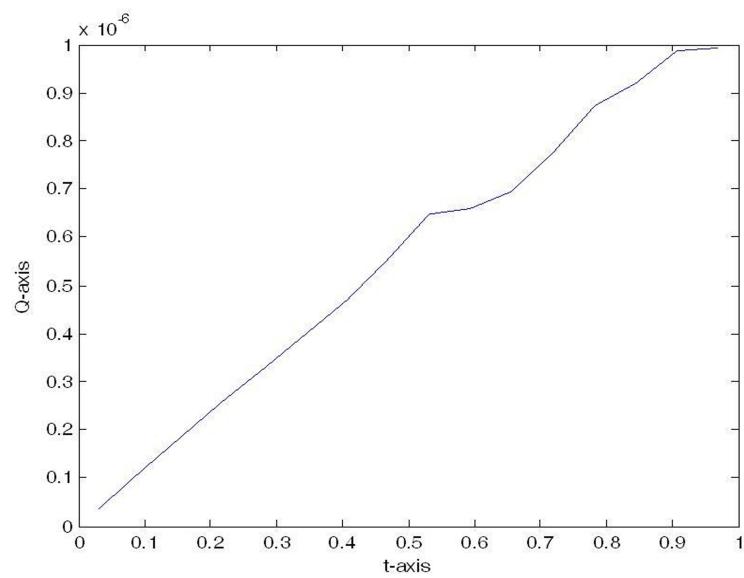
*Exact Figure**Haar Figure*

Case II. If $E = 4v$, $w = 1$ radian, $C = 0.3\mu F$, and $L = 2H$. Then equation (6.5) becomes

$$\sum_{i=1}^M a_i \left[h_i(t) + 10^6 \frac{p_{2,i}(t)}{0.6} \right] = 2 \sin(t).$$

Table II
Numerical solution of Case II

t /32	Haar solution	Exact solution
1	0.000000036	0.000000037
3	0.000000011	0.000000011
5	0.000000018	0.000000018
7	0.000000026	0.000000026
9	0.000000033	0.000000033
11	0.000000039	0.000000040
13	0.000000047	0.000000047
15	0.000000055	0.000000054
17	0.000000064	0.000000060
19	0.000000065	0.000000067
21	0.000000069	0.000000073
23	0.000000077	0.000000079
25	0.000000087	0.000000084
27	0.000000091	0.000000089
29	0.000000098	0.000000094
31	0.000000099	0.000000098

*Exact Figure**Haar Figure*

7. Conclusion

The main goal of this paper is to demonstrate the Haar wavelet method is a powerful tool for solving linear ordinary differential equation by taking series $L-C-R$ equation. The algorithm and procedure have been applied by using Haar wavelet method for solving ODE's. The result is compared with the exact solution. It is worth mentioning that Haar solution provides excellent result even for small values of M ($M = 16$). For large values of M ($M = 32, M = 64$), we can also obtain the results closer to exact values.

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SIMPLIFIED MARGINAL LINEARIZATION METHOD IN AUTONOMOUS LIENARD SYSTEMS

Weijing Zhao¹

*Faculty of Electronic Information and Electrical Engineering
Dalian University of Technology
Dalian, Liaoning, 116024
P.R. China
and
College of Air Traffic Management
Civil Aviation University of China
Tianjin, 300300
P.R. China*

Hongxing Li

*Faculty of Electronic Information and Electrical Engineering
Dalian University of Technology
Dalian, Liaoning, 116024
P.R. China*

Yuming Feng

*School of Mathematics and Statistics
Chongqing Three Gorges University
Chongqing, 404100
P.R. China*

Abstract. In this paper, a simplified marginal linearization method in autonomous Lienard systems is proposed. The new method simplified coefficients of each equation, leads to little calculation, and the time and space complexity are reduced. At last, the simulation results show that the simplified marginal linearization method in autonomous Lienard systems is of high approximation precision.

Keywords: autonomous Lienard systems; marginal linearization; fuzzy systems; rectangle wave.

AMS Subject Classification: 03B52; 65L05.

1. Introduction

In the fields of science and technology, many theoretical issues in physics have been summarized into a large number of ordinary differential equations, most of them

¹Corresponding author. Tel: +86 411 84706694. Email: 693509394@qq.com (Weijing Zhao).

are nonlinear differential equations. In the history of radio and vacuum tube technology, Lienard systems (equations) were intensely studied as they can be used to model oscillating circuits. H. Cartan and E. Cartan [1] first studied the existence of periodic solutions for differential equation $L\ddot{i}(t) + (r - \psi(t))\dot{i}(t) + \frac{i}{C} = 0$ in telecommunications technology issues, where L, r, C are positive constant, representing inductance, resistance and capacitance respectively. Van der Pol [2] first proposed the famous Van der Pol equation $\ddot{y}(t) + \mu(y^2 - 1)\dot{y}(t) + y = 0$ ($\mu > 0$) when he studied the equal-amplitude oscillation of triode. In 1928, a French engineer Alfred-Marie Lienard [3] generalized an extensive one: $\ddot{y}(t) + f(y)\dot{y}(t) + g(y) = 0$, i.e., so-called Lienard systems. Lienard systems are widely applied to atmospheric dynamics, physics, biology and other fields. Unfortunately, it is difficult to use it for the simple reason that analytical solutions can't be presented for majority of them. Meanwhile many experts have made great efforts to solve this nonlinear systems problem from different aspects, such as stability of the solution [4], [5], boundedness of solution [6], [7], limit cycle [8], [9], etc.

Since Zadeh [10] first presented the concept of fuzzy sets in 1965, a variety of applications of fuzzy logic have been implemented in various fields ranging from industrial control to financial management. For example, there is a considerable amount of work on hyperoperations defined through fuzzy sets. This study was initiated by Corsini in [14] and then continued by him together with Leoreanu in [15], [16]. Most notably, fuzzy systems have been successfully applied to control vague, incomplete, and ill-defined systems. Li [11] revealed interpolation mechanism of fuzzy control, i.e., the fuzzy control algorithms used commonly at present are all regarded as some interpolation functions. Li [12] first proposed modelling method based on fuzzy inference (MMFI) for fuzzy control systems, i.e., fuzzy inference is used on a controlled object, and fuzzy inference rule base is transferred into HX equations. It has shown that the mathematical model of a system formed by MMFI can approximate the real mathematical model of the system that is formed by mechanism modelling method. In order to solve the problem that each HX equation is a nonlinear equation, marginal linearization method in modeling on fuzzy control systems is proposed in [13]. This method turned HX equations into some kind of linear differential equations of linear differential equations with constant coefficients. So it provides a way to get approximately analytical solution of nonlinear equations initial value problem.

In this paper, we introduce a simplified marginal linearization method in autonomous Lienard systems. Simplified coefficients of the equations are given, and it needs to solve $p - 1$ equations in each segment instead of solving $(p - 1)(q - 1)$ equations in each piece. So the problem of autonomous Lienard systems is partially resolved from marginal linearization aspect.

The rest of the paper is organized as follows. Some useful concepts and notations are briefly reviewed in Section 2. In Section 3, the proposed simplified marginal linearization method in autonomous Lienard systems is discussed in detail. The simulation experiments of the new method is described in Section 4. Finally, conclusions are drawn in Section 5.

2. Preliminaries

In this section, some useful concepts and notations are introduced.

Definition 2.1. ([6], [9]) Let f and g be two continuous functions on $\mathbb{R}$, with g satisfies Lipschitz condition in any finite interval then the second order ordinary differential equation of the form $\ddot{y}(t) + f(y)\dot{y}(t) + g(y) = 0$ is called the autonomous Lienard systems(equations).

Note 2.1. In the previous definition, the hypothesis of f and g are two continuous functions on $\mathbb{R}$, with g satisfies Lipschitz condition in any finite interval guarantees the existence and uniqueness of autonomous Lienard equations.

Lemma 2.1. ([11], [12], [13]) Let $Y=[a_1, b_1]$, $\dot{Y}=[a_2, b_2]$ and $\ddot{Y}=[a_3, b_3]$ respectively be the universe of $y(t)$, $\dot{y}(t)$ and $\ddot{y}(t)$, and $\mathcal{A}=\{A_i\}_{(1 \leq i \leq p)}$, $\mathcal{B}=\{B_j\}_{(1 \leq j \leq q)}$, $\mathcal{C}=\{C_{ij}\}_{(1 \leq i \leq p, 1 \leq j \leq q)}$ respectively be the fuzzy partition(a group of base elements) of corresponding universe, where $A_i \in \mathcal{F}(Y)$, $B_j \in \mathcal{F}(\dot{Y})$ and $C_{ij} \in \mathcal{F}(\ddot{Y})$, which are called base element and y_i , $\dot{y}_j$, $\ddot{y}_{ij}$ are respectively the peakpoints of A_i , B_j , C_{ij} , and with the condition: $a_1 \leq y_1 < y_2 < \dots < y_p \leq b_1$, $a_2 \leq \dot{y}_1 < \dot{y}_2 < \dots < \dot{y}_q \leq b_2$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are regarded as linguistic variables so that a group of fuzzy inference rules is formed as follows:

$$(1) \quad \text{If } y(t) \text{ is } A_i \text{ and } \dot{y}(t) \text{ is } B_j, \text{ then } \ddot{y}(t) \text{ is } C_{ij},$$

where $i = 1, 2, \dots, p$, $j = 1, 2, \dots, q$.

Theorem 2.1. ([13]) The fuzzy logic system based on (1) can be represented as a binary piecewise interpolation function $F(\cdot, \cdot)$

$$(2) \quad \ddot{y}(t) = F(y(t), \dot{y}(t)) \triangleq \sum_{i=1}^p \sum_{j=1}^q A_i(y(t)) B_j(\dot{y}(t)) y_{ij}.$$

Here, A_i are taken as “rectangle wave” membership functions, B_j are taken as “triangle wave” membership functions (see Fig 1 and Fig 2).

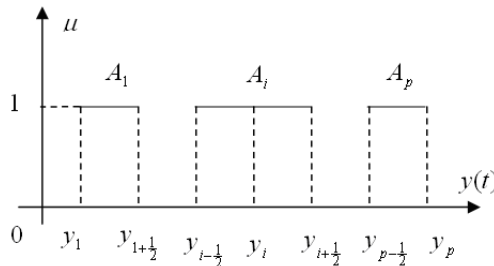


Fig 1 Rectangle wave membership functions of A_i

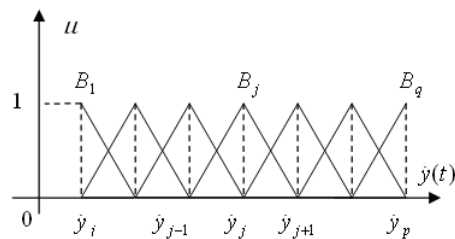


Fig 2 Triangle wave membership functions of B_j

$$(3) \quad A_i(y(t)) = \begin{cases} 1, & y_{i-\frac{1}{2}} \leq y(t) < y_{i+\frac{1}{2}}, \\ 0, & \text{otherwise.} \end{cases}$$

$$(4) \quad B_j(y(t)) = \begin{cases} \frac{\dot{y}(t) - \dot{y}_{j-1}}{\dot{y}_j - \dot{y}_{j-1}}, & \dot{y}_{j-1} \leq \dot{y}(t) \leq \dot{y}_j, \\ \frac{\dot{y}(t) - \dot{y}_{j+1}}{\dot{y}_j - \dot{y}_{j+1}}, & \dot{y}_j \leq \dot{y}(t) \leq \dot{y}_{j+1}, \\ 0, & \text{otherwise.} \end{cases}$$

where $i = 1, 2, \dots, p$. We stipulate that $y_{1-\frac{1}{2}} = y_1$, $y_{p+\frac{1}{2}} = y_p$; and also stipulate $\dot{y}_0 = \dot{y}_1$, $\dot{y}_{q+1} = \dot{y}_q$.

Theorem 2.2. ([13]) **(marginal linearization method)** Under the previous assumptions and conditions of Theorem 2.1, the input-output model of the second order system based on eq. (1) can be represented as a second order differential equation with variable coefficients:

$$(5) \quad \ddot{y}(t) + P_1(y(t), \dot{y}(t))\dot{y}(t) = Q_1(y(t), \dot{y}(t)),$$

where

$$(6) \quad P_1(y(t), \dot{y}(t)) = \sum_{i=1}^p \sum_{j=1}^{q-1} P_1^{(i,j)},$$

$$(7) \quad Q_1(y(t), \dot{y}(t)) = \sum_{i=1}^p \sum_{j=1}^{q-1} Q_1^{(i,j)},$$

and $P_1^{(i,j)}$, $Q_1^{(i,j)}$ are defined as local coefficients on the (i, j) -th piece as follows:

$$(8) \quad P_1^{(i,j)} = \begin{cases} \frac{\ddot{y}_{ij+1} - \ddot{y}_{ij}}{\dot{y}_j - \dot{y}_{j+1}}, & (y(t), \dot{y}(t)) \in [y_{i-\frac{1}{2}}, y_{i+\frac{1}{2}}] \times [\dot{y}_j, \dot{y}_{j+1}], \\ 0, & \text{otherwise.} \end{cases}$$

$$(9) \quad Q_1^{(i,j)} = \begin{cases} \frac{\dot{y}_j \ddot{y}_{ij+1} - \dot{y}_{j+1} \ddot{y}_{ij}}{\dot{y}_j - \dot{y}_{j+1}}, & (y(t), \dot{y}(t)) \in [y_{i-\frac{1}{2}}, y_{i+\frac{1}{2}}] \times [\dot{y}_j, \dot{y}_{j+1}], \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2.3. ([13]) Under condition of Theorem 2.2, when $(y(t), \dot{y}(t)) \in [y_{i-\frac{1}{2}}, y_{i+\frac{1}{2}}] \times [\dot{y}_j, \dot{y}_{j+1}]$, i.e., when on the (i, j) -th piece, equation (5) degenerates into a local equation

$$(10) \quad \ddot{y}(t) + P_1^{(i,j)}\dot{y}(t) = Q_1^{(i,j)}.$$

3. Simplified marginal linearization method in autonomous Lienard systems

In order to deal with the nonlinear system with variable coefficients, marginal linearization method in modeling on fuzzy control systems is proposed in [13]. This method can turn a nonlinear system with variable coefficients into a linear model with variable coefficients in the way that the membership functions of the fuzzy sets in fuzzy partitions of the universes are changed from triangle waves into rectangle waves.

Observe the coefficients in Theorem 2.2, the expression of coefficients in the equations are too complex. And it need to solve $(p-1)(q-1)$ local equations. Therefore, it brings a lot of inconvenience. In this section, we have made an attempt to simplify the calculation process of the autonomous Lienard systems.

Theorem 3.1. (Simplified marginal linearization method in autonomous Lienard system) *Under the condition of eq. (10), autonomous Lienard system $\ddot{y}(t) + f(y)\dot{y}(t) + g(y) = 0$ can be simplified represented as a second order differential equation with variable coefficients:*

$$(11) \quad \ddot{y}(t) + P_1(y(t))\dot{y}(t) = Q_1(y(t)),$$

where

$$(12) \quad P_1(y(t)) = \sum_{i=1}^p P_1^{(i)},$$

$$(13) \quad Q_1(y(t)) = \sum_{i=1}^p Q_1^{(i)},$$

and P_1^i, Q_1^i are defined as local coefficients in the i -th segment as follows:

$$(14) \quad P_1^{(i)} = \begin{cases} f(y_i), & t \in [y_i, y_{i+1}], \\ 0, & \text{otherwise.} \end{cases}$$

$$(15) \quad Q_1^{(i)} = \begin{cases} -g(y_i), & t \in [y_i, y_{i+1}], \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Here we only proof the expression of Q_1^i, P_1^i can be proved similarly. According to definition 2.1, we have that $\ddot{y}(t) + f(y)\dot{y}(t) + g(y) = 0$, therefore,

$$\ddot{y}(t) = -f(y)\dot{y}(t) - g(y).$$

Then, by Lemma 2.1

$$(16) \quad \ddot{y}_{ij} = -f(y_i)\dot{y}_j - g(y_i).$$

According to Definition 2.2, when $(y(t), \dot{y}(t)) \in [y_{i-\frac{1}{2}}, y_{i+\frac{1}{2}}] \times [\dot{y}_j, \dot{y}_{j+1}]$,

$$(17) \quad Q_1^{(i,j)} = \frac{\dot{y}_j \ddot{y}_{ij+1} - \dot{y}_{j+1} \ddot{y}_{ij}}{\dot{y}_j - \dot{y}_{j+1}}.$$

From (16) and (17), we obtain that

$$\begin{aligned} Q_1^{(i,j)} &= \frac{\dot{y}_j \ddot{y}_{ij+1} - \dot{y}_{j+1} \ddot{y}_{ij}}{\dot{y}_j - \dot{y}_{j+1}} = \frac{\dot{y}_j [-f(y_i) \dot{y}_{j+1} - g(y_i)] + \dot{y}_{j+1} [f(y_i) \dot{y}_j + g(y_i)]}{\dot{y}_j - \dot{y}_{j+1}} \\ &= -\frac{g(y_i)(\dot{y}_j - \dot{y}_{j+1})}{\dot{y}_j - \dot{y}_{j+1}} = -g(y_i). \end{aligned}$$

Since $\forall (y(t), \dot{y}(t)) \in [y_{i-\frac{1}{2}}, y_{i+\frac{1}{2}}] \times [\dot{y}_1, \dot{y}_2], [y_{i-\frac{1}{2}}, y_{i+\frac{1}{2}}] \times [\dot{y}_2, \dot{y}_3], \dots, [y_{i-\frac{1}{2}}, y_{i+\frac{1}{2}}] \times [\dot{y}_{q-1}, \dot{y}_q]$, $Q_1^{(i,j)} = -g(y_i)$ always holds, i.e., when $(y(t), \dot{y}(t)) \in [y_{i-\frac{1}{2}}, y_{i+\frac{1}{2}}] \times ([\dot{y}_1, \dot{y}_2] \cup [\dot{y}_2, \dot{y}_3] \cup \dots \cup [\dot{y}_{q-1}, \dot{y}_q]) = [y_{i-\frac{1}{2}}, y_{i+\frac{1}{2}}] \times [a_2, b_2]$,

$$Q_1^{(i,j)} = -g(y_i);$$

when $(y(t), \dot{y}(t)) \notin [y_{i-\frac{1}{2}}, y_{i+\frac{1}{2}}] \times [a_2, b_2]$, $Q_1^{(i,j)} = 0$.

Since $\dot{y}(t) \in [a_2, b_2]$ holds inevitably, we only consider the value of $y(t)$. Therefore,

$$Q_1^{(i)} = \begin{cases} -g(y_i), & t \in [y_i, y_{i+1}], \\ 0, & \text{otherwise.} \end{cases}$$

Corollary 3.1. *As to autonomous Lienard systems $\ddot{y}(t) + f(y)\dot{y}(t) + g(y) = 0$, when $y(t) \in [y_i, y_{i+1}]$, a local equation can be simplified as follows:*

$$(18) \quad \ddot{y}(t) + P_1^{(i)} \dot{y}(t) = Q_1^{(i)}.$$

Its coefficients have nothing to do with $\dot{y}_j$.

Proof. The assertions are trivial consequences of Theorems 2.2 and 3.1.

Corollary 3.2. *As to autonomous Lienard systems $\ddot{y}(t) + f(y)\dot{y}(t) + g(y) = 0$, Let $Y = [a_1, b_1]$, $\dot{Y} = [a_2, b_2]$ and $\ddot{Y} = [a_3, b_3]$ respectively be the universe of $y(t)$, $\dot{y}(t)$ and $\ddot{y}(t)$, and $\mathcal{A} = \{A_i\}_{(1 \leq i \leq p)}$, $\mathcal{C} = \{C_i\}_{(1 \leq i \leq p)}$ respectively be the fuzzy partition (a group of base elements) of corresponding universe, where $A_i \in \mathcal{F}(Y)$, $C_i \in \mathcal{F}(\dot{Y})$, which are called base element and $y_i, \dot{y}_i$ are respectively the peakpoints of A_i, C_i , and with the condition: $a_1 \leq y_1 < y_2 < \dots < y_p \leq b_1$, $\mathcal{A}$ and $\mathcal{C}$ are regarded as linguistic variables so that a group of fuzzy inference rules is formed as follows:*

$$(19) \quad \text{If } y(t) \text{ is } A_i, \text{ then } \ddot{y}(t) \text{ is } C_i,$$

where $i = 1, 2, \dots, p$.

Proof. According to Theorem 3.1, in each segments, local coefficients have nothing to do with $\dot{y}_j$, and $\dot{y}(t) \in [a_2, b_2]$ holds inevitably. As a result, it don't need to fuzzy inference on $\dot{y}(t)$.

Corollary 3.3. *As to autonomous Lienard systems $\ddot{y}(t) + f(y)\dot{y}(t) + g(y) = 0$, when applying the simplified marginal linearization method, it doesn't need fuzzy inference on $\dot{y}(t)$, so the partition of rectangular region $[a_1, b_1] \times [a_2, b_2]$ in $(y(t), \dot{y}(t))$ -plane can be simplified as the partition of closed interval $[a_1, b_1]$ in a real line. Then, it needs to solve $p - 1$ linear time-invariant equations in each segments instead of solving $(p - 1)(q - 1)p$ equations in each piece.*

Proof. It is obvious from Theorem 3.1 and Corollary 3.2.

Note 3.1. Solving autonomous Lienard systems $\ddot{y}(t) + f(y)\dot{y}(t) + g(y) = 0$ by using the method in [13], we have $(p - 1)(q - 1)$ linear time-invariant equations to solve. With p and q increase, the number of the equations grows rapidly. So it is difficult to solve these equations (e.g., when $p = 7, q = 8$, it needs to solve 42 equations). According to Corollary 3.3, as to autonomous Lienard systems $\ddot{y}(t) + f(y)\dot{y}(t) + g(y) = 0$, it only needs to solve p equations (e.g., when $p = 7, q = 8$ still holds, it only needs to solve 6 equations). The number of equations to be solved are reduced significantly and makes the program simple, so the time complexity is reduced.

Note 3.2. As to autonomous Lienard systems $\ddot{y}(t) + f(y)\dot{y}(t) + g(y) = 0$, it doesn't need to calculate the value of $\ddot{y}_{ij}$ when applying the simplified marginal linearization method. So the space complexity is reduced for it doesn't need to open memory space for $\ddot{y}_{ij}$.

4. Simulation experiments

The novel method proposed in Section 3 has showed that the simplified marginal linearization method in autonomous Lienard systems lead to little calculation, and the time and space complexity are reduced. This section presents whether the novel method is right and effective.

Given a system, for example, we still regard Van der pol equation in [12] as the real model of the system.

$$(20) \quad \ddot{y}(t) + \mu(y^2(t) - 1)\dot{y}(t) + y(t) = 0,$$

where $\mu = 1$. It's a special autonomous Lienard system.

The program design of the simulation follows the following steps:

Step 1. Determine the universes Y and $\dot{Y}$. By using solution (20), find respectively the maximum and the minimum of $y(t)$ and $\dot{y}(t)$: $y_{\max} = \max \{y(t)\}$, $y_{\min} = \min \{y(t)\}$, $\dot{y}_{\max} = \max \{\dot{y}(t)\}$, $\dot{y}_{\min} = \min \{\dot{y}(t)\}$. In order to allow an acceptable range of error, these maximum and minimum values should be extended to such an extent that we can get the universes: $Y = [a_1, b_1]$, $\dot{Y} = [a_2, b_2]$, where

$$\begin{aligned} a_1 &= y_{\min} - 0.1|y_{\min}|, \quad b_1 = y_{\max} + 0.1|y_{\max}|, \\ a_2 &= \dot{y}_{\min} - 0.1|\dot{y}_{\min}|, \quad b_2 = \dot{y}_{\max} + 0.1|\dot{y}_{\max}|. \end{aligned}$$

Step 2. Calculate the peakpoints. Given a natural number $p > 1$, take $h_1 = (b_1 - a_1)/(p - 1)$. And the isometry partition nodal points y_i are computed by the following equation: $y_i = a_1 + (i - 1)h_1$, $i = 1, 2, \dots, p$.

Step 3. According to (11)-(15), calculate the coefficients in each segment, $P_1^{(i)}$, $Q_1^{(i)}$, $i = 1, 2, \dots, p$.

Step 4. Given initial values $y(0) = y_0$, $\dot{y}(0) = \dot{y}_0$, solve the local equation (18) segment by segment, and p local equations should be solved. For this purpose, let $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$, so the local equation (18) becomes a system of local first order differential equations:

$$(21) \quad \begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -P_1^{(i)}x_2(t) + Q_1^{(i)}. \end{cases}$$

By using Matlab 6.5, we can easily find the solution to the whole and draw the plots for the curves of $x_1(t)$, $x_2(t)$ and of the phase plane $(x_1(t), x_2(t))$. At the same time, draw the plots for the solution to the real model (20) and the curves of its phase plane, and compare these plots.

Example 1. Take the initial values $x_1(0) = 2$, $x_2(0) = 0$, and let $T = 20$, $p = 7$; Here the state curves $x_1(t)$ and $x_2(t)$, phase plan curves $(x_1(t), x_2(t))$ and the comparison with corresponding curves on real model are shown in Fig 3–Fig 5.

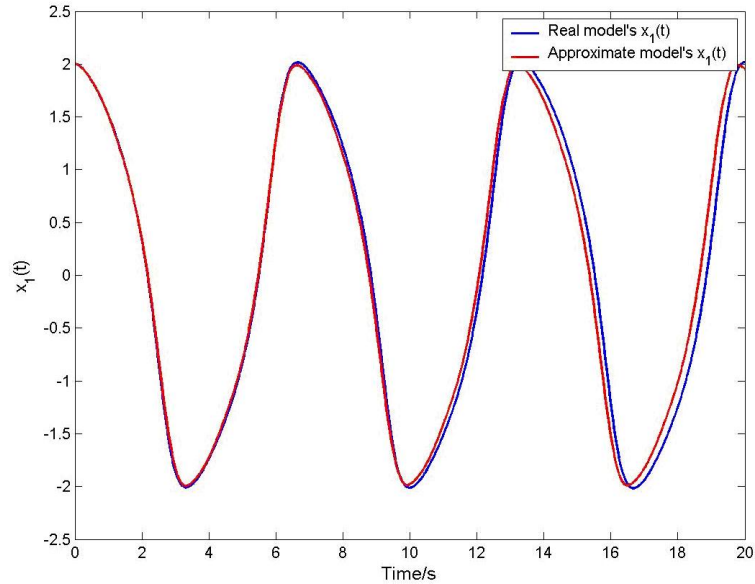


Fig 3 Simulation curves of state $x_1(t)$ under $p = 7$

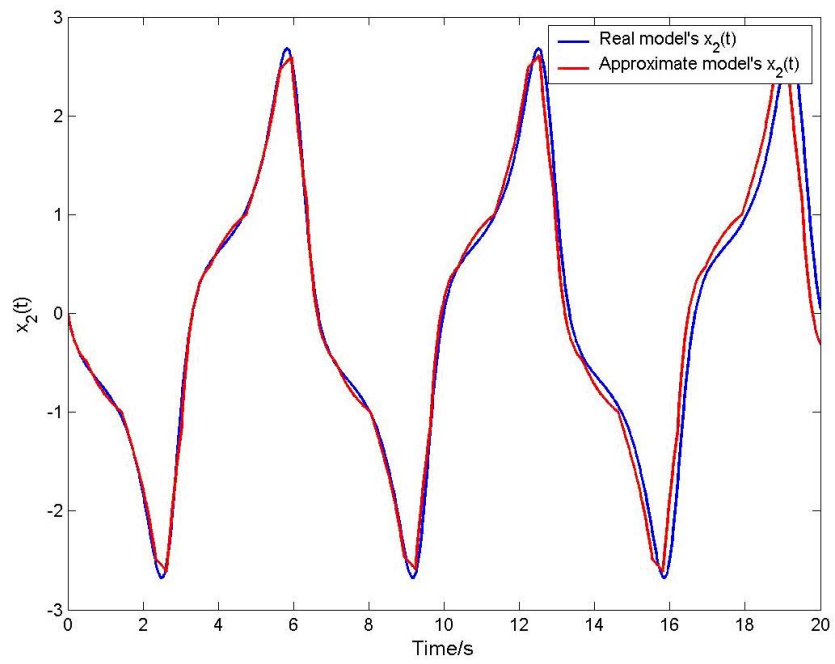


Fig 4 Simulation curves of state $x_2(t)$ under $p = 7$

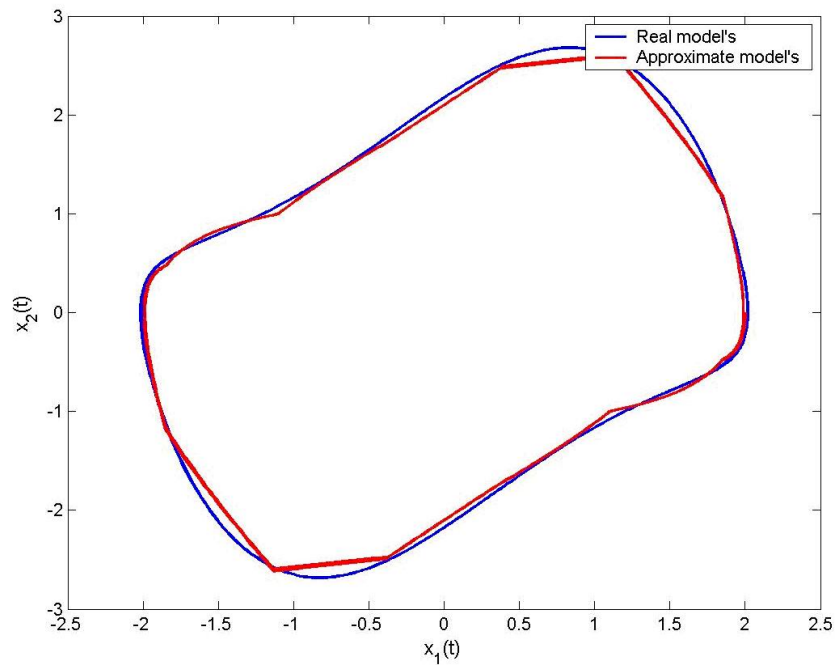


Fig 5 Simulation curves of phase plane $(x_1(t), x_2(t))$ under $p = 7$

Example 2. Let $p = 14$ and other parameters be the same as ones in the former example. The simulation results are shown in Fig 6-Fig 8. It has shown that more fuzzy inference rules are used in simulation, the error will be smaller. It's obvious that the curves of approximate model almost coincide with the ones of real model, which means that the simplified marginal linearization method in autonomous Lienard systems is of high approximation precision. In other words, it is reliable algorithm.

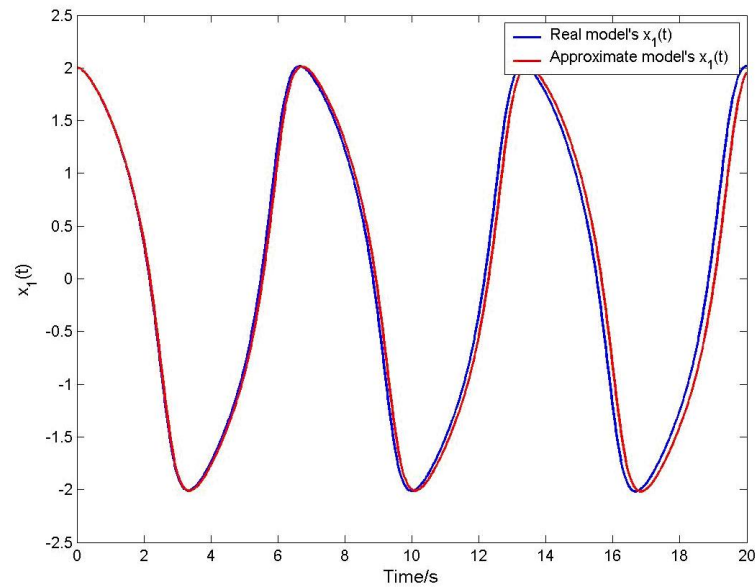


Fig 6 Simulation curves of state $x_1(t)$ under $p = 14$

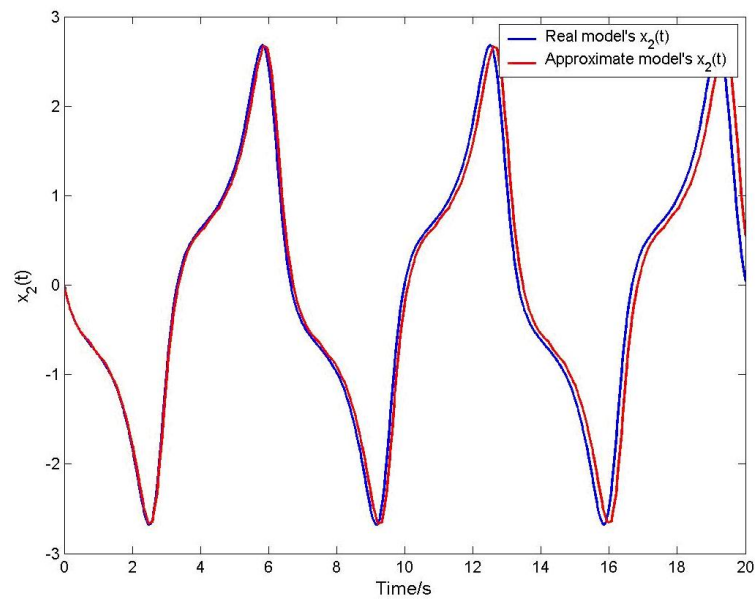


Fig 7 Simulation curves of state $x_2(t)$ under $p = 14$

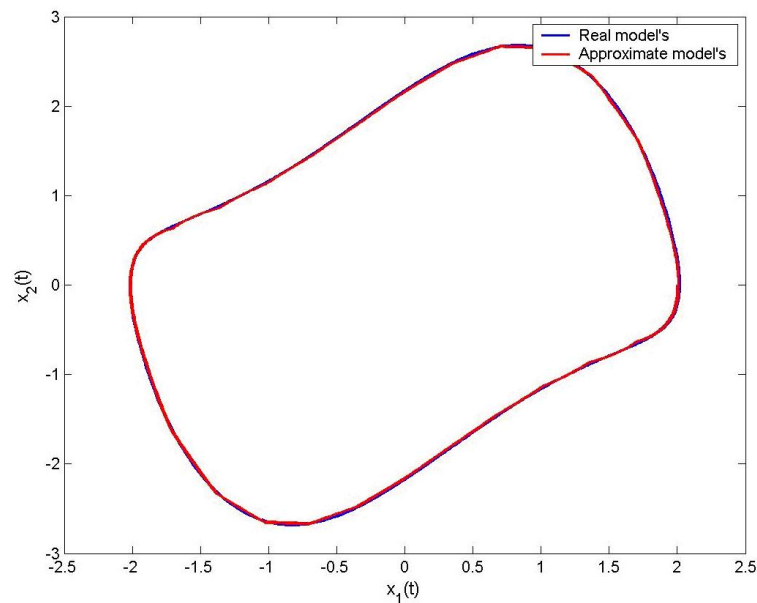


Fig 8 Simulation curves of phase plane $(x_1(t), x_2(t))$ under $p = 14$

5. Conclusions

In this paper, the simplified marginal linearization method in autonomous Lienard systems is proposed. The novel method simplified coefficients of the each equations, leads to little calculation, and the time and space complexity is reduced. From the perspective of fuzzy inference, two-dimensional fuzzy inference of Y and $\dot{Y}$ is simplified as fuzzy inference of Y only. The simulation results show that simplified marginal linearization method in autonomous Lienard system is high approximation precision. Consequently, the theory of marginal linearization is riched.

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DISTRIBUTIONAL AND TEMPERED DISTRIBUTIONAL DIFFRACTION FRESNEL TRANSFORMS AND THEIR EXTENSION TO BOEHMIAN SPACES

S.K.Q. Al-Omari

*Department of Applied Sciences
Faculty of Engineering Technology
Al-Balqa Applied University
Amman 11134
Jordan
e-mail: s.k.q.alomari@fet.edu.jo*

Abstract. In [22], authors investigate the diffraction Fresnel transform on certain space of tempered distributions. Further, they extend their results to a context of Boehmian spaces. In this paper, we discuss various spaces of Boehmians. Spaces, so obtained, can handle the Fresnel transform in some approach. The extended transform and its inverse are therefore considered satisfactory and, are well recognized. Further theorems are also established in some detail.

Keywords: diffraction Fresnel transform; tempered distribution; Boehmian space; generalized function.

1991 Mathematics Subject Classification: Primary 54C40, 14E20; Secondary 46E25, 20C20.

1. Window on diffraction Fresnel transform

The diffraction Fresnel transform (optical Fresnel transform) is introduced as a four parameter class of linear integral transforms [22]. The diffraction Fresnel transform is adopted to express mathematically the Fraunhofer diffraction, when the kernel is $\exp i\zeta\tau$ and, it describes a Fresnel diffraction, when the kernel is $\exp(\zeta - \tau)^2$ [10]. Moreover, the transform under consideration is of great importance in electromagnetic, acoustic, and other wave propagation problems which represent the solution of the wave equation under a variety of circumstances.

At optical frequencies, the diffraction Fresnel transform can model a broad class of optical systems including thin lenses, sections of free space, in the Fresnel approximation, and arbitrary concatenations, which, sometimes, referred to as first order optical systems; see, for example, [12], [16], [20], [21].

It is necessary to know that the transform under consideration has been referred to by various names such as: quadratic-phase integrals; see, for example, [16], fractional Fourier transform; see, for example [5], [6], [11], [12], [13] and [17], generalized Fresnel transforms; see [10], and some others.

The classical diffraction Fresnel transform of a function $f(\varsigma)$ is defined by

$$(1) \quad \mathcal{F}_d f(\tau) = \int_{\mathcal{R}} \mathcal{E}(\alpha_1, \gamma_1, \gamma_2, \alpha_2; \varsigma, \tau) f(\varsigma) d\varsigma.$$

where

$$(2) \quad \mathcal{E}(\alpha_1, \gamma_1, \gamma_2, \alpha_2; \varsigma, \tau) := \mathcal{E} := \frac{1}{\sqrt{2\pi i \gamma_1}} \exp\left(\frac{i}{2\gamma_1} (\alpha_1 \varsigma^2 - 2\varsigma\tau + \alpha_2 \tau^2)\right)$$

is the transform kernel with real parameters, $\alpha_1, \gamma_1, \alpha_2$ and γ_2 . The parameters $(\alpha_1, \gamma_1, \alpha_2$ and $\gamma_2)$ are elements of more ray transfer matrix M describing optical systems, $\alpha_1 \alpha_2 - \gamma_1 \gamma_2 = 1$. For more details, if the parameters $(\alpha_1, \gamma_1, \alpha_2$ and $\gamma_2)$ are written in matrix form

$$\begin{pmatrix} \alpha_1 & \gamma_1 \\ \gamma_2 & \alpha_2 \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

then the diffraction Fresnel transform becomes a fractional Fourier transform [24].

In an earlier paper [22], authors have established that the diffraction Fresnel transform maps the space of rapidly decreasing functions into itself and further applied it to a space of Boehmians. The present work represents an effort of us to obtain more general spaces of Boehmians that can adequately handle the four parameters with a different approach. We build upon analysis of [22]. We describe four spaces of Boehmians for the diffraction Fresnel transform with careful attention. An avenue towards this end is to first define the diffraction Fresnel transform for Boehmians and derive its properties. Definition of the inverse transform, when it exists, is also established.

Let $\mathcal{S}$ be the space of rapidly decreasing functions over $\mathcal{R}$ (the space of rapid descent) [7]. Then the diffraction Fresnel transform of $\psi \in \mathcal{S}(\mathcal{R})$ is also in $\mathcal{S}(\mathcal{R})$ [22, Theorem 2.1]. The Parseval relation for the diffraction Fresnel transform, compared to Fourier transforms, is interpreted to mean

$$(3) \quad \langle \mathcal{F}_d f, \psi \rangle = \langle f, \mathcal{F}_d \psi \rangle,$$

where $\psi \in \mathcal{S}$, $f \in \mathcal{S}'(\mathcal{R})$, $\mathcal{S}'(\mathcal{R})$ being the dual space of $\mathcal{S}(\mathcal{R})$ of distributions of slow growth over $\mathcal{R}$. Hence, from equation (3), it may be noted that $\mathcal{F}_d f \in \mathcal{S}'(\mathcal{R})$ for every $f \in \mathcal{S}'(\mathcal{R})$.

In brief details, we spread the paper over six sections. The diffraction Fresnel transform is reviewed in Section 1. Section 2 presents a general construction of Boehmians. The first Bohmian space is presented in Section 3. The tempered distributional diffraction space of Boehmians is given in Section 4. The extended diffraction Fresnel transform of a Bohmian and its properties are established in Section 5. In Section 6, we give another approach of the cited transform to

Boehmian spaces. With regret to have some repetition in the analysis due to their necessity.

2. Boehmian Spaces and General Construction

To make this work self contained as much as possible we recall the following: Let $\mathcal{G}$ be a linear space and $\mathcal{H}$ be a subspace of $\mathcal{G}$. Assume, to each pair (f, ϕ) of elements, $f \in \mathcal{G}$ and $\phi \in \mathcal{H}$, is assigned the product $f * \phi$ such that the following conditions are satisfied:

- (1) $\phi, \psi \in \mathcal{H} \Rightarrow \phi * \psi \in \mathcal{H}$ and $\phi * \psi = \psi * \phi$.
- (2) $f \in \mathcal{G}, \phi, \psi \in \mathcal{H} \Rightarrow (f * \phi) * \psi = f * (\phi * \psi)$.
- (3) If $f, g \in \mathcal{G}, \phi \in \mathcal{H}$, then $(f + g) * \phi = f * \phi + g * \phi$
- (4) If $k \in \mathcal{R}$, then $k(f * \phi) = (kf) * \phi = f * (k\phi)$.

Let Δ be a family of sequences from $\mathcal{H}$ such that:

- (1) If $f, g \in \mathcal{G}, (\epsilon_n) \in \Delta$ and $f * \epsilon_n = g * \epsilon_n$ ($n = 1, 2, \dots$), then $f = g$.
- (2) $(\epsilon_n), (\tau_n) \in \Delta \Rightarrow (\epsilon_n * \tau_n) \in \Delta$.

Each element of Δ will be called *delta sequence*.

Consider the class $\mathcal{A}$ of pairs of sequences defined by

$$\mathcal{A} = \{((f_n), (\epsilon_n)) : (f_n) \subseteq \mathcal{G}^{\mathbf{N}}, (\epsilon_n) \in \Delta\},$$

for each $n \in \mathbf{N}$, the set of natural numbers. The pair $((f_n), (\epsilon_n)) \in \mathcal{A}$ is said to be a *quotient of sequences*, denoted by $\frac{f_n}{\epsilon_n}$, if

$$(4) \quad f_n * \epsilon_m = f_m * \epsilon_n, \forall n, m \in \mathbf{N}.$$

Two quotients of sequences $\frac{f_n}{\epsilon_n}$ and $\frac{g_n}{\tau_n}$ are said to be *equivalent*, $\frac{f_n}{\epsilon_n} \sim \frac{g_n}{\tau_n}$, if

$$(5) \quad f_n * \epsilon_m = g_m * \tau_n, \forall n, m \in \mathbf{N}.$$

The relation $\sim$ is an equivalent relation on $\mathcal{A}$ and hence, splits $\mathcal{A}$ into equivalence classes. The equivalence class containing $\frac{f_n}{\epsilon_n}$ is denoted by $\left[\frac{f_n}{\epsilon_n} \right]$. These equivalence classes are called *Boehmians* and the *space of all Boehmians* is denoted by $\mathcal{H}_\beta$.

The sum and multiplication by a scalar of two Boehmians can be defined in a natural way

$$\left[\frac{f_n}{\epsilon_n} \right] + \left[\frac{g_n}{\tau_n} \right] = \left[\frac{f_n * \tau_n + g_n * \epsilon_n}{\epsilon_n * \tau_n} \right]$$

and

$$a \left[\frac{f_n}{\epsilon_n} \right] = \left[\frac{af_n}{\epsilon_n} \right], a \in C, \text{ the field of complex numbers.}$$

The operation $*$ and differentiation are defined by $\left[\frac{f_n}{\epsilon_n} \right] * \left[\frac{g_n}{\tau_n} \right] = \left[\frac{f_n * g_n}{\epsilon_n * \tau_n} \right]$ and $\mathcal{D}^\alpha \left[\frac{f_n}{\epsilon_n} \right] = \left[\frac{\mathcal{D}^\alpha f_n}{\epsilon_n} \right]$. Many a time, $\mathcal{G}$ is equipped with a notion of convergence. The intrinsic relationship between the notion of convergence and the product $*$ are given by:

- (1) If $f_n \rightarrow f$ as $n \rightarrow \infty$ in $\mathcal{G}$ and, $\phi \in \mathcal{H}$ is any fixed element, then $f_n * \phi \rightarrow f * \phi$ in $\mathcal{G}$ (as $n \rightarrow \infty$).
- (2) If $f_n \rightarrow f$ as $n \rightarrow \infty$ in $\mathcal{G}$ and $(\epsilon_n) \in \Delta$, then $f_n * \epsilon_n \rightarrow f$ in $\mathcal{G}$ (as $n \rightarrow \infty$).

The operation $*$ can be extended to $\mathcal{H}_\beta \times \mathcal{H}$ by the following definition.

Definition 2.1. If $\left[\frac{f_n}{\epsilon_n} \right] \in \mathcal{H}_\beta$ and $\phi \in \mathcal{H}$, then $\left[\frac{f_n}{\epsilon_n} \right] * \phi = \left[\frac{f_n * \phi}{\epsilon_n} \right]$.

In $\mathcal{H}_\beta$, two types of convergence, δ -convergence and Δ -convergence, are defined as follows:

Definition 2.2. A sequence of Boehmians (β_n) in $\mathcal{H}_\beta$ is said to be δ -convergent to a Boehmian β in $\mathcal{H}_\beta$, denoted by $\beta_n \xrightarrow{\delta} \beta$, if there exists a delta sequence (ϵ_n) such that $(\beta_n * \epsilon_n), (\beta * \epsilon_n) \in \mathcal{G}$, $\forall k, n \in \mathbf{N}$, and

$$(\beta_n * \epsilon_k) \rightarrow (\beta * \epsilon_k) \text{ as } n \rightarrow \infty, \text{ in } \mathcal{G}, \text{ for every } k \in \mathbf{N}.$$

The following lemma is equivalent for the statement of δ -convergence.

Lemma 2.3. $\beta_n \xrightarrow{\delta} \beta$ ($n \rightarrow \infty$) in $\mathcal{H}_\beta$ if and only if there is $f_{n,k}, f_k \in \mathcal{G}$ and $\epsilon_k \in \Delta$ such that $\beta_n = \left[\frac{f_{n,k}}{\epsilon_k} \right], \beta = \left[\frac{f_k}{\epsilon_k} \right]$ and for each $k \in \mathbf{N}$,

$$f_{n,k} \rightarrow f_k \text{ as } n \rightarrow \infty \text{ in } \mathcal{G}.$$

Definition 2.4. A sequence of Boehmians (β_n) in $\mathcal{H}_\beta$ is said to be Δ -convergent to a Boehmian β in $\mathcal{H}_\beta$, denoted by $\beta_n \xrightarrow{\Delta} \beta$, if there exists a $(\epsilon_n) \in \Delta$ such that $(\beta_n - \beta) * \epsilon_n \in \mathcal{G}, \forall n \in \mathbf{N}$, and $(\beta_n - \beta) * \epsilon_n \rightarrow 0$ as $n \rightarrow \infty$ in $\mathcal{G}$.

See, for examples, [1]–[4], [8], [14]–[15] and [22]–[23].

3. The tempered space $\mathcal{H}_{\beta_1}(\mathcal{S}', \mathcal{D}, \Delta, \bullet)$ of Boehmians

To start our mission of extending diffraction Fresnel transforms to Boehmian spaces we first discuss the space $\mathcal{H}_{\beta_1}(\mathcal{S}', \mathcal{D}, \bullet, \Delta)$ of Boehmians with the notation $\bullet$ described by [23, Definition 2.2]

$$(6) \quad \langle f \bullet \sigma, \psi \rangle = \langle f, \psi \star \check{\sigma} \rangle$$

for every $f \in \mathcal{S}'$, $\sigma \in \mathcal{D}$, the Schwartz space of test functions of bounded supports, $\check{\sigma}(t) = \sigma(-t)$, $t \in \mathcal{R}$, $\psi \in \mathcal{S}$ is arbitrary and $\star$ acts for the usual convolution product. The righthand side of Equ.(6) is clearly well-defined, since $\psi \star \check{\sigma} \in \mathcal{S}$. Hence, $f \bullet \sigma \in \mathcal{S}'$ is justified.

Following, are results proved for defining $\mathcal{H}_{\beta_1}$.

Theorem 3.1. *Let $f \in \mathcal{S}'$ and $\sigma \in \mathcal{D}$ then $f \star \check{\sigma} \in \mathcal{S}'$.*

Proof. By aid of [18, Theorem 1.14.1 (iii), p. 26], $f \star \check{\sigma} \in \mathcal{O}_M \subset \mathcal{S}'$, $\mathcal{O}_M$ is the space of multipliers of $\mathcal{S}'$. Hence the theorem.

By Δ we denote the subset of $\mathcal{D}$ of all sequences with the following properties:

- (1) $\int_{\mathcal{R}} \epsilon_n(\varsigma) d\varsigma = 1, n \in \mathbf{N}$.
- (2) $\int_{\mathcal{R}} |\epsilon_n(\varsigma)| d\varsigma \leq M, 0 < M \in \mathcal{R}$.
- (3) $\text{supp} \epsilon_n \rightarrow 0$ as $n \rightarrow \infty$, $\text{supp} \epsilon_n = \{\varsigma : \epsilon_n(\varsigma) \neq 0, \forall n \in \mathbf{N}\}$.

Each member (ϵ_n) in Δ is said to be delta sequence.

Following are needful:

- (1) If $(\epsilon_n), (\alpha_n) \in \Delta$, then

$$\int_{\mathcal{R}} (\epsilon_n \star \check{\alpha}_n)(\varsigma) d\varsigma = \int_{\mathcal{R}} \epsilon_n(\varsigma) d\varsigma \int_{\mathcal{R}} \alpha_n(y) dy = 1;$$

- (2) If $(\epsilon_n), (\alpha_n) \in \Delta$ then

$$\int_{\mathcal{R}} |(\epsilon_n \star \check{\alpha}_n)(\varsigma)| d\varsigma \leq \int_{\mathcal{R}} |\epsilon_n(\varsigma)| d\varsigma \int_{\mathcal{R}} |\check{\alpha}_n(\tau)| d\tau \leq M_1 M_2,$$

for some constants M_1 and M_2 where $\int_{\mathcal{R}} |\epsilon_n| \leq M_1, \int_{\mathcal{R}} |\alpha_n| \leq M_2$.

- (3) $\text{supp}(\epsilon_n \star \check{\alpha}_n) \subseteq \text{supp} \epsilon_n + \text{supp} \check{\alpha}_n \rightarrow 0$ as $n \rightarrow \infty$.

Moreover, it is easy to see that for each $(\epsilon_n) \in \Delta$, $(\check{\epsilon}_n) \in \Delta$ as well.

Lemma 3.2. *If $f \in \mathcal{S}'$ and $\sigma_1, \sigma_2 \in \mathcal{D}$ then the following hold:*

- (1) $\sigma_1 \bullet \sigma_2 = \sigma_1 \star \sigma_2 = \sigma_2 \bullet \sigma_1$; (2) $f \bullet (\sigma_1 \bullet \sigma_2) = (f \bullet \sigma_1) \bullet \sigma_2$.

For proof see [23, Lemma 2.5].

Lemma 3.3. *Let $f, g \in \mathcal{S}'$, $(\epsilon_n) \in \Delta$ and $f \bullet \epsilon_n = g \bullet \epsilon_n$ then $f = g$ in $\mathcal{S}'$, for every $n \in \mathbf{N}$.*

Proof. At first, certainly $\psi \star \check{\epsilon}_n \rightarrow \psi$ as $n \rightarrow \infty$ for each $(\epsilon_n) \in \Delta$ and $\psi \in \mathcal{S}$. Hence, from the hypothesis of this lemma, $\langle f \bullet \epsilon_n, \psi \rangle = \langle g \bullet \epsilon_n, \psi \rangle, \psi \in \mathcal{S}$. Therefore, by equation (6), we get

$$\langle f, \psi \star \check{\epsilon}_n \rangle = \langle g, \psi \star \check{\epsilon}_n \rangle.$$

Thus

$$\langle f, \psi \rangle = \langle g, \psi \rangle.$$

for all $\psi \in \mathcal{S}$. This completes the proof of the lemma.

The desired space $\mathcal{H}_{\beta_1}(\mathcal{S}', \mathcal{D}, \Delta, \bullet)$, briefly $\mathcal{H}_{\beta_1}$, of Boehmians is described. Addition, multiplication by a scalar and the operation $\bullet$, in $\mathcal{H}_{\beta_1}(\mathcal{S}', \mathcal{D}, \Delta, \bullet)$, are respectively defined in the usual way:

$$\left[\frac{f_n}{\epsilon_n} \right] + \left[\frac{g_n}{\alpha_n} \right] = \left[\frac{f_n \bullet \alpha_n + g_n \bullet \epsilon_n}{\epsilon_n \bullet \alpha_n} \right],$$

$$k \left[\frac{f_n}{\epsilon_n} \right] = \left[\frac{k f_n}{\epsilon_n} \right], \text{ for all } k \in \mathcal{R},$$

and

$$\left[\frac{f_n}{\epsilon_n} \right] \bullet \left[\frac{g_n}{\alpha_n} \right] = \left[\frac{f_n \bullet g_n}{\epsilon_n \bullet \alpha_n} \right].$$

Lemma 3.4. Let $f_n \rightarrow f \in \mathcal{S}', \sigma \in \mathcal{D}$ then $f_n \bullet \sigma \rightarrow f \bullet \sigma$ as $n \rightarrow \infty$.

Proof. Let $\psi \in \mathcal{S}$ be arbitrary then the fact that $\psi \star \check{\sigma} \in \mathcal{S}$ suggests to write

$$\begin{aligned} \langle f_n \bullet \sigma - f \bullet \sigma, \psi \rangle &= \langle (f_n - f) \bullet \sigma, \psi \rangle \\ &= \langle f_n - f, \psi \star \check{\sigma} \rangle. \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

This completes the proof.

Lemma 3.5. Let $f_n \rightarrow f$ in $\mathcal{S}', (\epsilon_n) \in \Delta$ then $f_n \bullet \epsilon_n \rightarrow f$ as $n \rightarrow \infty$.

Proof. Using properties of delta sequences and the integral operator $\int$, we get

$$\begin{aligned} \langle f_n \bullet \epsilon_n - f, \psi \rangle &= \langle f_n \bullet \epsilon_n - f \bullet \epsilon_n, \psi \rangle \\ &= \langle (f_n - f) \bullet \epsilon_n, \psi \rangle \\ &= \langle f_n - f, \psi \star \check{\epsilon}_n \rangle \\ &\rightarrow \langle f_n - f, \psi \rangle \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

This completes the proof of the lemma.

Theorem 3.6. The mapping $\mathcal{S}' \rightarrow \mathcal{H}_{\beta_1}(\mathcal{S}', \mathcal{D}, \Delta, \bullet)$ defined by $f \rightarrow \left[\frac{f \bullet \epsilon_n}{\epsilon_n} \right]$ is a continuous imbedding with respect to δ convergence.

Proof. Let $\left[\frac{f \bullet \epsilon_n}{\epsilon_n} \right] = \left[\frac{g \bullet \alpha_n}{\alpha_n} \right]$ then $(f \bullet \epsilon_n) \bullet \alpha_m = (g \bullet \alpha_m) \bullet \epsilon_m$. In particular, $(f \bullet \epsilon_n) \bullet \alpha_n = (g \bullet \alpha_n) \bullet \epsilon_n$. Allowing $n \rightarrow \infty$ and using Lemma 3.5 implies that the mapping is one-to-one. Next, let $f_n \rightarrow 0$ as $n \rightarrow \infty$ in $\mathcal{S}'$, then, $\langle f_n, \psi \rangle \rightarrow 0$ for all $\psi \in \mathcal{S}$. Hence, $\langle f_n \bullet \epsilon_n, \psi \rangle = \langle f_n, \psi \star \check{\epsilon}_n \rangle \rightarrow \langle f_n, \psi \rangle \rightarrow 0$ as $n \rightarrow \infty$ in $\mathcal{S}'$. Therefore, $\left[\frac{f \bullet \epsilon_n}{\epsilon_n} \right] \rightarrow 0$ as $n \rightarrow \infty$ in $\mathcal{H}_{\beta_1}(\mathcal{S}', \mathcal{D}, \Delta, \bullet)$. This establishes that the mapping is continuous with respect to δ convergence. The proof is completed.

4. The Diffraction Space $\mathcal{H}_{\beta_2}(\mathcal{S}', \mathcal{D}, \Delta, \wedge)$ of Boehmians

In this section we present a space of Boehmians by a different operation.

Between $\mathcal{S}'$ and $\mathcal{D}$ we define a mapping $\wedge$ by

$$(f \wedge \sigma)(\tau) = \int_{\mathcal{R}} \mathcal{F}_d f(\tau - y) \sigma(y) dy$$

where $f \in \mathcal{S}'$ and $\sigma \in \mathcal{D}$ are arbitrary.

Lemma 4.1. Let $f \in \mathcal{S}'$ and $\sigma \in \mathcal{D}$ then $\mathcal{F}_d(f \bullet \sigma)(\tau) = (f \wedge \sigma)(\tau)$.

Proof. Let $f \in \mathcal{S}'$. Employing equation (3) and equation (6) yield

$$\begin{aligned} \langle \mathcal{F}_d(f \bullet \sigma)(\tau), \psi(\tau) \rangle &= \langle f \bullet \sigma(\tau), \mathcal{F}_d \psi(\tau) \rangle \\ &= \langle f(\tau), (\mathcal{F}_d \psi \star \check{\sigma})(\tau) \rangle, \\ &= \left\langle f(\tau), \int_{\mathcal{R}} \mathcal{F}_d \psi(\varsigma) \sigma(\varsigma - \tau) d\varsigma \right\rangle \end{aligned}$$

The substitution $\varsigma - \tau = y$ and, the translation property of distributions, through y , jointly with equation (3), yield

$$\begin{aligned} \langle \mathcal{F}_d(f \bullet \sigma)(\tau), \psi(\tau) \rangle &= \left\langle f(\tau), \int_{\mathcal{R}} \mathcal{F}_d \psi(\tau + y) \sigma(y) dy \right\rangle \\ &= \int_{\mathcal{R}} \langle f(\tau - y), \mathcal{F}_d \psi(\tau) \rangle \sigma(y) dy \\ &= \int_{\mathcal{R}} \langle \mathcal{F}_d f(\tau - y), \psi(\tau) \rangle \sigma(y) dy \\ &= \left\langle \int_{\mathcal{R}} \mathcal{F}_d f(\tau - y) \sigma(y) dy, \psi(\tau) \right\rangle \\ &= \langle (f \wedge \sigma)(\tau), \psi(\tau) \rangle, \end{aligned}$$

where $\psi \in \mathcal{S}$ is arbitrary. The proof of this theorem is completed.

Lemma 4.2. $f \wedge \sigma \in \mathcal{S}'$, for every $f \in \mathcal{S}, \sigma \in \mathcal{D}$.

Proof. Let $f \in \mathcal{S}'$, $\sigma \in \mathcal{D}$ then $f \bullet \sigma \in \mathcal{S}'$, by equation (6). Hence, $f \wedge \sigma = \mathcal{F}_d(f \bullet \sigma) \in \mathcal{S}'$. This completes the proof of the lemma.

Lemma 4.3. *If $f \in \mathcal{S}'$, $\sigma_1, \sigma_2 \in \mathcal{D}$ then the following are true:*

- (1) $\sigma_1 \wedge \sigma_2 = \sigma_2 \wedge \sigma_1$; (2) $f \wedge (\sigma_2 \wedge \sigma_1) = (f \wedge \sigma_1) \wedge \sigma_2$.

Proof. We attempt to proof the first part of this lemma since the proof of second part is similar. For, let $\sigma_1, \sigma_2 \in \mathcal{D}$. Using Lemma 3.2, $\sigma_1 \wedge \sigma_2 = \mathcal{F}_d(\sigma_1 \bullet \sigma_2) = \mathcal{F}_d(\sigma_2 \bullet \sigma_1) = \sigma_2 \wedge \sigma_1$. Hence the lemma.

Lemma 4.4. *If $f_1, f_2 \in \mathcal{S}'$, $(\epsilon_n) \in \Delta$ and $f_1 \wedge \epsilon_n = f_2 \wedge \epsilon_n$ then $f_1 = f_2$.*

Proof of this lemma follows from Theorem 4.1 and the fact that

$$\mathcal{F}_d \epsilon_n \rightarrow \frac{i}{2\gamma_1 \sqrt{2\pi i \gamma_1}} \alpha_2 \tau^2 \text{ as } n \rightarrow \infty.$$

The relationship between convergence and the operation $\wedge$ is given by:

Lemma 4.5. *If $f_n \rightarrow f \in \mathcal{S}'$, $\sigma \in \mathcal{D}$ then $f_n \wedge \sigma \rightarrow f \wedge \sigma$ as $n \rightarrow \infty$.*

Lemma 4.6. *If $f_n \rightarrow f \in \mathcal{S}'$, $(\epsilon_n) \in \Delta$ then $f_n \wedge \epsilon_n \rightarrow f$ as $n \rightarrow \infty$.*

Proof of Lemma 4.4 and Lemma 4.5 is straightforward from Lemma 4.1, detailed proof thus avoided.

Theorem 4.7. *The mapping $\mathcal{S}' \rightarrow \mathcal{H}_{\beta_2}(\mathcal{S}', \mathcal{D}, \Delta, \bullet)$ defined by $f \rightarrow \left[\frac{f \bullet \epsilon_n}{\epsilon_n} \right]$ is a continuous imbedding with respect to δ convergence.*

Proof. (See Theorem 3.6.)

By the above conclusions, the desired space $\mathcal{H}_{\beta_2}(\mathcal{S}', \mathcal{D}, \Delta, \wedge)$ is considered. We preserve the operations of addition, multiplication by a scalar and the operation $\wedge$ between Boehmians.

The operation $\wedge$ can be extended to $\mathcal{H}_{\beta_2} \times \mathcal{D}$ by the following definition:

Definition 4.7. If $\left[\frac{f_n}{\epsilon_n} \right] \in \mathcal{H}_{\beta_2}$ and $\phi \in \mathcal{D}$, then $\left[\frac{f_n}{\epsilon_n} \right] \wedge \phi = \left[\frac{f_n \wedge \phi}{\epsilon_n} \right]$.

In $\mathcal{H}_{\beta_2}$, convergence is defined as:

Definition 4.8. A sequence of Boehmians (β_n) in $\mathcal{H}_{\beta_2}$ is δ convergent to a Bohmian β in $\mathcal{H}_{\beta_2}$, if there exists (ϵ_n) such that $(\beta_n \wedge \epsilon_n), (\beta \wedge \epsilon_n) \in \mathcal{H}_{\beta_2}$, $\forall k$, $n \in \mathbf{N}$, and $(\beta_n \wedge \epsilon_k) \rightarrow (\beta \wedge \epsilon_k)$ as $n \rightarrow \infty$, in $\mathcal{H}_{\beta_2}$, for every $k \in \mathbf{N}$.

The following lemma is equivalent to δ convergence.

Lemma 4.9. $\beta_n \xrightarrow{\delta} \beta$ in $\mathcal{H}_{\beta_2}$ if and only if there is $f_{n,k}, f_k \in \mathcal{S}'$ and $(\epsilon_k) \in \Delta$ such that $\beta_n = \left[\frac{f_{n,k}}{\epsilon_k} \right]$, $\beta = \left[\frac{f_k}{\epsilon_k} \right]$ and $\forall k \in \mathbf{N}$, $f_{n,k} \rightarrow f_k$ as $n \rightarrow \infty$ in $\mathcal{S}'$.

Definition 4.10. (β_n) in $\mathcal{H}_{\beta_2}$ is Δ convergent to β in $\mathcal{H}_{\beta_2}$ if there is a $(\epsilon_n) \in \Delta$ such that $(\beta_n - \beta) \wedge \epsilon_n \in \mathcal{H}_{\beta_2}, \forall n \in \mathbf{N}$, and $(\beta_n - \beta) \wedge \epsilon_n \rightarrow 0$ as $n \rightarrow \infty$ in $\mathcal{H}_{\beta_2}$.

5. The Diffraction transform of $\mathcal{H}_{\beta_1}(\mathcal{S}', \mathcal{D}, \Delta, \bullet)$

Let $\left[\frac{f_n}{\epsilon_n}\right] \in \mathcal{H}_{\beta_1}(\mathcal{S}', \mathcal{D}, \Delta, \bullet)$. Since operations we introduced on $\mathcal{H}_{\beta_1}$ and $\mathcal{H}_{\beta_2}$ ($\bullet$ and $\wedge$, respectively), are mathematically not equal, the typical Boehmian $\left[\frac{f_n}{\epsilon_n}\right] \in \mathcal{H}_{\beta_1}$, is denoted by $\left[\frac{f_n^*}{\epsilon_n}\right]$ in the space $\mathcal{H}_{\beta_2}$. We may agree this will make no confusion in notations. Restrictions on parameters here are that $\alpha_1 = \alpha_2$.

Definition 5.1. The diffraction Fresnel transform $\hat{\mathcal{F}}_d^* : \mathcal{H}_{\beta_1} \rightarrow \mathcal{H}_{\beta_2}$ of $\left[\frac{f_n}{\epsilon_n}\right] \in \mathcal{H}_{\beta_1}(\mathcal{S}', \mathcal{D}, \Delta, \bullet)$ is defined by

$$(7) \quad \hat{\mathcal{F}}_d^* \left(\left[\frac{f_n}{\epsilon_n} \right] \right) = \left[\frac{f_n^*}{\epsilon_n} \right],$$

in $\mathcal{H}_{\beta_2}$.

Theorem 5.2. $\hat{\mathcal{F}}_d^* : \mathcal{H}_{\beta_1} \rightarrow \mathcal{H}_{\beta_2}$ is well-defined, linear and independent of the representative.

Proof. Let $\left[\frac{f_n}{\epsilon_n}\right] \in \mathcal{H}_{\beta_1}$ then $f_n \bullet \epsilon_m = f_m \bullet \epsilon_n, \forall n, m \in \mathbf{N}$. Applying the diffraction Fresnel transform and using Theorem 4.1 yield $f_n^* \wedge \epsilon_m = f_m^* \wedge \epsilon_n$, for every $m, n \in \mathbf{N}$. Therefore $\left[\frac{f_n^*}{\epsilon_n}\right] \in \mathcal{H}_{\beta_2}$. To show $\hat{\mathcal{F}}_d^*$ is well defined, let $\left[\frac{g_n}{\alpha_n}\right] = \left[\frac{f_n}{\epsilon_n}\right]$ in $\mathcal{H}_{\beta_1}$ then, $f_n \bullet \alpha_m = g_m \bullet \epsilon_n$. Once again, applying the diffraction Fresnel transform and Theorem 4.1 yields $f_n^* \wedge \alpha_m = g_m^* \wedge \epsilon_n, \forall n, m \in \mathbf{N}$. Hence,

$$\left[\frac{f_n^*}{\epsilon_n} \right] = \left[\frac{g_m^*}{\alpha_n} \right].$$

Let $k_1, k_2 \in \mathcal{R}$. Theorem 4.1. implies

$$\hat{\mathcal{F}}_d^* \left(k_1 \left[\frac{f_n}{\epsilon_n} \right] + k_2 \left[\frac{g_n}{\alpha_n} \right] \right) = \left[\frac{k_1 f_n^* \wedge \epsilon_n + k_2 g_m^* \wedge \alpha_n}{\epsilon_n \wedge \alpha_n} \right].$$

That is

$$\hat{\mathcal{F}}_d^* \left(k_1 \left[\frac{f_n}{\epsilon_n} \right] + k_2 \left[\frac{g_n}{\alpha_n} \right] \right) = k_1 \left[\frac{f_n^*}{\epsilon_n} \right] + k_2 \left[\frac{g_m^*}{\alpha_n} \right].$$

This completes the proof of the theorem.

Theorem 5.3. $\hat{\mathcal{F}}_d^* : \mathcal{H}_{\beta_1} \rightarrow \mathcal{H}_{\beta_2}$ is a bijection map.

Proof. Let $\left[\frac{f_n^*}{\epsilon_n}\right] = \left[\frac{g_n^*}{\alpha_n}\right] \in \mathcal{H}_{\beta_2}$, then $f_n^* \wedge \alpha_m = g_n^* \wedge \epsilon_n, \forall n$. From Theorem 4.1, $\mathcal{F}_d(f_n \bullet \alpha_m) = \mathcal{F}_d(g_n \bullet \epsilon_n)$. Since $\mathcal{F}_d$ is one-one, on $\mathcal{S}'$, we get $f_n \bullet \alpha_m = g_n \bullet \epsilon_n$. Hence, $\left[\frac{f_n}{\epsilon_n}\right] = \left[\frac{g_n}{\alpha_n}\right]$. Finally, let $\left[\frac{f_n^*}{\epsilon_n}\right] \in \mathcal{H}_{\beta_2}$ then the Boehmian $\left[\frac{f_n}{\epsilon_n}\right] \in \mathcal{H}_{\beta_1}$ satisfies $\hat{\mathcal{F}}_d^* \left(\left[\frac{f_n}{\epsilon_n}\right] \right) = \left[\frac{f_n^*}{\epsilon_n}\right]$.

Proof is, therefore, completed.

The inversion formula $\left(\hat{\mathcal{F}}_d^*\right)^{-1}$ for $\hat{\mathcal{F}}_d^*$ is defined by the following definition:

Definition 5.4. Let $\left[\frac{f_n^*}{\epsilon_n}\right] \in \mathcal{H}_{\beta_2}$ then the inverse transform of $\hat{\mathcal{F}}_d^*$ is defined by

$$(8) \quad \left(\hat{\mathcal{F}}_d^*\right)^{-1} \left(\left[\frac{f_n^*}{\epsilon_n}\right] \right) = \left[\frac{f_n}{\epsilon_n}\right],$$

in $\mathcal{H}_{\beta_1}$.

Theorem 5.5. $\left(\hat{\mathcal{F}}_d^*\right)^{-1} : \mathcal{H}_{\beta_2} \rightarrow \mathcal{H}_{\beta_1}$ is well-defined, linear and bijection.

This theorem can be proved by analysis which is alike to that employed for Theorem 5.2.

Theorem 5.6. $\hat{\mathcal{F}}_d^* : \mathcal{H}_{\beta_1} \rightarrow \mathcal{H}_{\beta_2}, \left(\hat{\mathcal{F}}_d^*\right)^{-1} : \mathcal{H}_{\beta_2} \rightarrow \mathcal{H}_{\beta_1}$ are continuous with respect to δ convergence.

Proof. Let $(\beta_n) \in \mathcal{H}_{\beta_1}, \beta \in \mathcal{H}_{\beta_1}$ be such that $\beta_n \xrightarrow{\delta} \beta$ then, using Lemma 2.3, we can find $f_{n,k}, f_k \in \mathcal{S}', (\epsilon_n) \in \Delta$, such that $\beta_n = \left[\frac{f_{n,k}}{\epsilon_n}\right], \beta = \left[\frac{f_k}{\epsilon_n}\right]$ and $f_{n,k} \rightarrow f_k$ for every $k \in \mathbb{N}$ as $n \rightarrow \infty$ in $\mathcal{S}'$. Since $\mathcal{F}_d$ is continuous from $\mathcal{S}'$ into $\mathcal{S}'$, it follows $\mathcal{F}_d(f_{n,k}) \rightarrow \mathcal{F}_d(f_k)$ as $n \rightarrow \infty$ in $\mathcal{S}'$. Hence, $\left[\frac{f_{n,k}^*}{\epsilon_n}\right] \rightarrow \left[\frac{f_k^*}{\epsilon_n}\right]$ as $n \rightarrow \infty$.

To prove the second part of the theorem, let $\tilde{\beta}, (\tilde{\beta}_n) \in \mathcal{H}_{\beta_2}$ then, as above, there are $f_{n,k}^*, f_k^* \in \mathcal{S}'$ such that $\tilde{\beta}_n = \left[\frac{f_{n,k}^*}{\epsilon_n}\right], \tilde{\beta} = \left[\frac{f_k^*}{\epsilon_n}\right]$ and, $f_{n,k}^* \rightarrow f_k^*$ for every k , as $n \rightarrow \infty$. Hence $f_{n,k} \rightarrow f_k$ as $n \rightarrow \infty$ for every $k \in \mathbb{N}$ in $\mathcal{S}'$. Thus,

$$\left[\frac{f_{n,k}}{\epsilon_n}\right] = \left(\hat{\mathcal{F}}_d^*\right)^{-1} \left(\left[\frac{f_{n,k}^*}{\epsilon_n}\right] \right) \rightarrow \left[\frac{f_k}{\epsilon_n}\right] = \left(\hat{\mathcal{F}}_d^*\right)^{-1} \left(\left[\frac{f_k^*}{\epsilon_n}\right] \right)$$

as $n \rightarrow \infty$ for every k .

Hence, the theorem.

Theorem 5.7. $\hat{\mathcal{F}}_d^* : \mathcal{H}_{\beta_1} \rightarrow \mathcal{H}_{\beta_2}, \left(\hat{\mathcal{F}}_d^*\right)^{-1} : \mathcal{H}_{\beta_2} \rightarrow \mathcal{H}_{\beta_1}$ are continuous with respect to Δ convergence.

Proof. We prove the first part of the theorem since the second part of the theorem can be proved similarly. Let $\beta_n \xrightarrow{\Delta} \beta \in \mathcal{H}_{\beta_1}$ as $n \rightarrow \infty$ then there is $(f_n) \in \mathcal{S}'$ such that

$$(\beta_n - \beta) \bullet \epsilon_n = \left[\frac{f_n \bullet \epsilon_k}{\epsilon_k} \right] \rightarrow 0 \text{ and } f_n \rightarrow 0 \text{ as } n \rightarrow \infty, (\epsilon_n) \in \Delta. \text{ Thus,}$$

$$\begin{aligned} \left(\hat{\mathcal{F}}_d^* \beta_n - \hat{\mathcal{F}}_d^* \beta \right) \wedge \epsilon_n &= \hat{\mathcal{F}}_d^* (\beta_n - \beta) \wedge \epsilon_n \\ &= \left[\frac{f_n^* \wedge \epsilon_k}{\epsilon_k} \right] \\ &= f_n^* \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

This completes the proof of the theorem.

6. The diffraction Fresnel transform of $\mathcal{H}_{\beta_3}(\mathcal{E}', \mathcal{D}, \Delta, \bullet)$

By $\mathcal{E}$ we denote the space of smooth functions of arbitrary support on $\mathcal{R}$ and, $\mathcal{E}'$ its strong dual of distributions of compact support. By aid of the kernel method, the diffraction Fresnel transform of $f \in \mathcal{E}'$ has been extended to [22]

$$(9) \quad \mathcal{F}_d f(\tau) = \frac{1}{\sqrt{2\pi i \gamma_1}} \left\langle f(t), \exp \frac{i(\alpha_1 \zeta^2 - 2\tau \zeta + \alpha_2 \tau^2)}{2\gamma_1} \right\rangle,$$

where $f \in \mathcal{E}'$ is arbitrary.

Let $\mathcal{H}_{\beta_3}(\mathcal{E}', \mathcal{D}, \Delta, \bullet)$ be the Boehmian space with $\mathcal{E}'$, as linear space, $\mathcal{D}$, as a subspace of $\mathcal{E}'$, and $\bullet$, as an operation between $\mathcal{E}'$ and $\mathcal{D}$. Denote by $\mathcal{D}_F$ the set of analytic functions that are diffraction Fresnel transforms of compactly supported distributions in $\mathcal{E}'$.

Convergence in $\mathcal{D}_F$ is defined as follows:

$\mathcal{T}_n \rightarrow \mathcal{T}$ in $\mathcal{D}_F$ if there are distributions $v_n, v \in \mathcal{E}'$ such that $\mathcal{T}_n = \mathcal{F}_d v_n$, $\mathcal{T} = \mathcal{F}_d v$ and $v_n \rightarrow v$ in $\mathcal{E}'$.

Between $\mathcal{D}_F$ and $\mathcal{D}$ we introduce a mapping defined by

$$(10) \quad (\mathcal{T} \Upsilon \sigma)(\tau) = \int_{\mathcal{R}} \mathcal{T}(\tau - y) \sigma(y) dy,$$

where $\mathcal{T} \in \mathcal{D}_F$ and $\sigma \in \mathcal{D}$ are arbitrary.

Theorem 6.1. Let $\mathcal{T} \in \mathcal{D}_F$, $\mathcal{T} = \mathcal{F}_d v$, $v \in \mathcal{E}'$, and $\sigma \in \mathcal{D}$ then

$$\mathcal{F}_d(f \bullet \sigma)(\tau) = (\mathcal{T} \Upsilon \sigma)(\tau).$$

Proof. Let $\mathcal{T} \in \mathcal{D}_F, \mathcal{T} = \mathcal{F}_d v, v \in \mathcal{E}'$ then, employing Equ.(10) yields

$$\begin{aligned}
 \mathcal{F}_d(v \bullet \sigma)(\tau) &= \left\langle (v \bullet \sigma)(\varsigma), \frac{1}{\sqrt{2\pi i \gamma_1}} \exp\left(\frac{i}{2\gamma_1}(\varsigma^2 - 2\varsigma\tau + \tau^2)\right) \right\rangle \\
 &= \frac{1}{\sqrt{2\pi i \gamma_1}} \left\langle v(\varsigma), \left(\exp\left(\frac{i}{2\gamma_1}(\varsigma^2 - 2\varsigma\tau + \tau^2)\right) \star \check{\sigma}\right)(\varsigma) \right\rangle \\
 &= \frac{1}{\sqrt{2\pi i \gamma_1}} \left\langle v(\varsigma), \int_{\mathcal{R}} \exp\left(\frac{i}{2\gamma_1}(t^2 - 2t\tau + \tau^2)\right) \sigma(t - \varsigma) dt \right\rangle, \\
 &= \frac{1}{\sqrt{2\pi i \gamma_1}} \left\langle v(\varsigma), \int_{\mathcal{R}} \exp\left(\frac{i}{2\gamma_1}((y + \varsigma) - 2(y + \varsigma)\tau + \tau^2)\right) \sigma(y) dy \right\rangle \\
 &= \frac{1}{\sqrt{2\pi i \gamma_1}} \int_{\mathcal{R}} \left\langle v(\varsigma), \exp\left(\frac{i}{2\gamma_1}(\varsigma^2 - 2(\tau - y)\varsigma + (\tau - y)^2)\right) \right\rangle \sigma(y) dy \\
 &= \int_{\mathcal{R}} \mathcal{T}(\tau - y) \sigma(y) dy \\
 &= (\mathcal{T} \Upsilon \sigma)(\tau),
 \end{aligned}$$

where $\mathcal{T} = \mathcal{F}_d v, \sigma \in \mathcal{D}$ and $\star$ is the convolution product.

The proof of this theorem is therefore completed.

Lemma 6.2. $\mathcal{T} \Upsilon \sigma \in \mathcal{D}_F$, for every $\mathcal{T} \in \mathcal{D}_F$ and $\sigma \in \mathcal{D}$.

Proof. Let $v \in \mathcal{E}', \sigma \in \mathcal{D}$, then $v \bullet \sigma \in \mathcal{E}'$. From Theorem 6.1 we get $\mathcal{T} \Upsilon \sigma = \mathcal{F}_d(v \bullet \sigma) \in \mathcal{D}_F, \mathcal{T} = \mathcal{F}_d v$. Hence, we proved the lemma.

Lemma 6.3. If $\mathcal{T} \in \mathcal{D}_F, \sigma_1, \sigma_2 \in \mathcal{D}$ then the following are true:

$$(1) \sigma_1 \Upsilon \sigma_2 = \sigma_2 \Upsilon \sigma_1; (2) \mathcal{T} \Upsilon (\sigma_2 \Upsilon \sigma_1) = (\mathcal{T} \Upsilon \sigma_1) \Upsilon \sigma_2.$$

Proof of this theorem is a straightforward consequence of Theorem 6.1 and properties of the integral operator $\int$.

Lemma 6.4. If $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{D}_F, (\epsilon_n) \in \Delta$ and $\mathcal{T}_1 \Upsilon \epsilon_n = \mathcal{T}_2 \Upsilon \epsilon_n$ then $\mathcal{T}_1 = \mathcal{T}_2$.

Lemma 6.5. If $\mathcal{T}_n \rightarrow \mathcal{T} \in \mathcal{D}_F, \sigma \in \mathcal{D}$ then $\mathcal{T}_n \Upsilon \sigma \rightarrow \mathcal{T} \Upsilon \sigma$ as $n \rightarrow \infty$.

Lemma 6.6. If $\mathcal{T}_n \rightarrow \mathcal{T} \in \mathcal{D}_F, (\epsilon_n) \in \Delta$ then $\mathcal{T}_n \Upsilon \epsilon_n \rightarrow \mathcal{T}$ as $n \rightarrow \infty$.

Proofs of Lemma 6.4-Lemma 6.6 are straightforward.

From the above conclusions, the space $\mathcal{H}_{\beta_4}(\mathcal{D}_F, \mathcal{D}, \Delta, \Upsilon), \mathcal{H}_{\beta_4}$, can be considered as a Boehmian space. In $\mathcal{H}_{\beta_4}$, as earlier, addition, multiplication by a scalar, the operation Υ and differentiation are normally defined.

The operation $\Upsilon : \mathcal{H}_{\beta_4} \times \mathcal{D} \rightarrow \mathcal{H}_{\beta_4}$ is given by the following definition:

Definition 6.7. $\left[\frac{\mathcal{T}_n}{\epsilon_n}\right] \Upsilon \phi = \left[\frac{\mathcal{T}_n \Upsilon \phi}{\epsilon_n}\right], \left[\frac{\mathcal{T}_n}{\epsilon_n}\right] \in \mathcal{H}_{\beta_4}, \phi \in \mathcal{D}.$

Definition 6.8. A sequence $(\beta_n) \in \mathcal{H}_{\beta_4}$ is δ -convergent to $\beta \in \mathcal{H}_{\beta_4}$, if there exists (ϵ_n) such that $(\beta_n \vee \epsilon_n), (\beta \vee \epsilon_n) \in \mathcal{H}_{\beta_4}$, $\forall k, n \in \mathbf{N}$, and $(\beta_n \vee \epsilon_k) \rightarrow (\beta \vee \epsilon_k) \in \mathcal{H}_{\beta_4}$ as $n \rightarrow \infty$, for every $k \in \mathbf{N}$.

In other words,

Lemma 6.9. $\beta_n \xrightarrow{\delta} \beta$ in $\mathcal{H}_{\beta_4}$ if and only if there is $\mathcal{T}_{n,k}, \mathcal{T}_k \in \mathcal{D}_F$ and $(\epsilon_k) \in \Delta$ such that $\beta_n = \left[\frac{\mathcal{T}_{n,k}}{\epsilon_k} \right], \beta = \left[\frac{\mathcal{T}_k}{\epsilon_k} \right]$ and $\forall k \in \mathbf{N}, \mathcal{T}_{n,k} \rightarrow \mathcal{T}_k \in \mathcal{D}_F$ as $n \rightarrow \infty$.

Definition 6.10. $(\beta_n) \in \mathcal{H}_{\beta_4}$ is Δ -convergent to $\beta \in \mathcal{H}_{\beta_4}$ if there is a $(\epsilon_n) \in \Delta$ such that $(\beta_n - \beta) \vee \epsilon_n \in \mathcal{H}_{\beta_4}, \forall n \in \mathbf{N}$, and $(\beta_n - \beta) \vee \epsilon_n \rightarrow 0$ as $n \rightarrow \infty$ in $\mathcal{H}_{\beta_4}$.

Definition 6.11. The extended diffraction Fresnel transform of $\left[\frac{f_n}{\epsilon_n} \right]$ is defined by

$$(11) \quad \hat{\mathcal{F}}_d \left(\left[\frac{v_n}{\epsilon_n} \right] \right) = \left[\frac{\mathcal{T}_n}{\epsilon_n} \right],$$

where $\mathcal{T}_n = \mathcal{F}_d v_n$.

Theorem 6.12. $\hat{\mathcal{F}}_d : \mathcal{H}_{\beta_3} \rightarrow \mathcal{H}_{\beta_4}$ is well-defined, linear and independent of the representative.

Proof. Let $\left[\frac{v_n}{\epsilon_n} \right] \in \mathcal{H}_{\beta_3}$ then $v_n \bullet \epsilon_m = v_m \bullet \epsilon_n, \forall n, m \in \mathbf{N}$. Applying the diffraction Fresnel transform and using Theorem 6.1 yield

$$\mathcal{T}_n \vee \epsilon_m = \mathcal{T}_m \vee \epsilon_n,$$

where $\mathcal{T}_n = \mathcal{F}_d v_n, \mathcal{T}_m = \mathcal{F}_d v_m$, for every $m, n \in \mathbf{N}$. Therefore $\left[\frac{\mathcal{T}_n}{\epsilon_n} \right] \in \mathcal{H}_{\beta_4}$. To

show $\hat{\mathcal{F}}_d$ is well defined, let $\left[\frac{g_n}{\alpha_n} \right] = \left[\frac{v_n}{\epsilon_n} \right]$ in $\mathcal{H}_{\beta_3}$ then $v_n \bullet \alpha_m = g_m \bullet \epsilon_n$.

Once again, applying the diffraction Fresnel transform and using Theorem 6.1 yield $\mathcal{T}_n \vee \alpha_m = H_m \vee \epsilon_n$ where, $\mathcal{T}_n = \mathcal{F}_d v_n$ and $H_n = \mathcal{F}_d g_n, \forall n \in \mathbf{N}$. Hence, $\left[\frac{\mathcal{T}_n}{\epsilon_n} \right] = \left[\frac{H_n}{\alpha_n} \right]$.

Next, let $k_1, k_2 \in \mathcal{R}$ then, using Theorem 6.1, we have

$$\hat{\mathcal{F}}_d \left(k_1 \left[\frac{v_n}{\epsilon_n} \right] + k_2 \left[\frac{g_n}{\alpha_n} \right] \right) = \left[\frac{k_1 \mathcal{T}_n \vee \epsilon_n + k_2 H_n \vee \alpha_n}{\epsilon_n \vee \alpha_n} \right].$$

That is $\hat{\mathcal{F}}_d \left(k_1 \left[\frac{v_n}{\epsilon_n} \right] + k_2 \left[\frac{g_n}{\alpha_n} \right] \right) = k_1 \left[\frac{\mathcal{T}_n}{\epsilon_n} \right] + k_2 \left[\frac{H_n}{\alpha_n} \right], \mathcal{T}_n = \mathcal{F}_d v_n, H_n = \mathcal{F}_d g_n$.

This completes the proof of the theorem.

Theorem 6.13. The mapping $\hat{\mathcal{F}}_d : \mathcal{H}_{\beta_3} \rightarrow \mathcal{H}_{\beta_4}$ is bijection.

Proof. Assume that $\left[\frac{\mathcal{T}_n}{\epsilon_n}\right] = \left[\frac{H_n}{\alpha_n}\right]$, where $\mathcal{T}_n = \mathcal{F}_d v_n, H_n = \mathcal{F}_d g_n, \forall n$, then $\mathcal{T}_n \vee \alpha_m = H_m \vee \epsilon_n$. By Theorem 6.1, $\mathcal{F}_d(v_n \bullet \alpha_m) = \mathcal{F}_d(g_n \bullet \epsilon_n)$ and hence $v_n \bullet \alpha_m = g_n \bullet \epsilon_n$. Hence, $\left[\frac{v_n}{\epsilon_n}\right] = \left[\frac{g_n}{\alpha_n}\right]$. Finally, let $\left[\frac{\mathcal{T}_n}{\epsilon_n}\right] \in \mathcal{H}_{\beta_4}$ then $\mathcal{T}_n = \mathcal{F}_d v_n$, for some v_n and all n .

Therefore, $\left[\frac{v_n}{\epsilon_n}\right] \in \mathcal{H}_{\beta_3}$ satisfies $\hat{\mathcal{F}}_d\left(\left[\frac{v_n}{\epsilon_n}\right]\right) = \left[\frac{\mathcal{T}_n}{\epsilon_n}\right] \in \mathcal{H}_{\beta_4}$.

The theorem is completely proved.

Definition 6.14. Let $\left[\frac{\mathcal{T}_n}{\epsilon_n}\right] \in \mathcal{H}_{\beta_4}$. The *inverse* of $\hat{\mathcal{F}}_d$ is defined by

$$(12) \quad \left(\hat{\mathcal{F}}_d\right)^{-1}\left(\left[\frac{\mathcal{T}_n}{\epsilon_n}\right]\right) = \left[\frac{v_n}{\epsilon_n}\right],$$

in $\mathcal{H}_{\beta_1}, \mathcal{T}_n = \hat{\mathcal{F}}_d v_n, v_n \in \mathcal{E}'$.

Theorem 6.15. $\left(\hat{\mathcal{F}}_d\right)^{-1} : \mathcal{H}_{\beta_4} \rightarrow \mathcal{H}_{\beta_3}$ is well-defined, linear and bijection mapping.

This theorem can be proved easily by using the same analysis employed for Theorem 6.12. Detailed proof is, thus, avoided.

Theorem 6.16. $\hat{\mathcal{F}}_d : \mathcal{H}_{\beta_3} \rightarrow \mathcal{H}_{\beta_4}, \left(\hat{\mathcal{F}}_d\right)^{-1} : \mathcal{H}_{\beta_4} \rightarrow \mathcal{H}_{\beta_3}$ are continuous with respect to δ convergence.

Proof. Let $(\beta_n) \in \mathcal{H}_{\beta_1}, \beta \in \mathcal{H}_{\beta_1}$ be such that $\beta_n \xrightarrow{\delta} \beta$. By using Lemma 2.3, there can be found $v_{n,k}, v_k \in \mathcal{E}', (\epsilon_n) \in \Delta$, such that $\beta_n = \left[\frac{v_{n,k}}{\epsilon_n}\right], \beta = \left[\frac{v_k}{\epsilon_n}\right]$ and $v_{n,k} \rightarrow v_k$ for every $k \in \mathbf{N}$ as $n \rightarrow \infty$ in $\mathcal{E}'$. Continuity of $\mathcal{F}_d$ implies $\mathcal{F}_d(v_{n,k}) \rightarrow \mathcal{F}_d(v_k)$ as $n \rightarrow \infty$ in $\mathcal{D}_F$. Hence, $\left[\frac{\mathcal{T}_{n,k}}{\epsilon_n}\right] \rightarrow \left[\frac{\mathcal{T}_k}{\epsilon_n}\right]$, where $\mathcal{T}_{n,k} = \mathcal{F}_d(v_{n,k}), \mathcal{T}_k = \mathcal{F}_d(v_k)$. Next, let $\tilde{\beta}, (\tilde{\beta}_n) \in \mathcal{H}_{\beta_4}$ then, there are $\mathcal{T}_{n,k}, \mathcal{T}_k \in \mathcal{D}_F$ such that $\tilde{\beta}_n = \left[\frac{\mathcal{T}_{n,k}}{\epsilon_n}\right], \tilde{\beta} = \left[\frac{\mathcal{T}_k}{\epsilon_n}\right], \mathcal{T}_{n,k} = \mathcal{F}_d(v_{n,k}), \mathcal{T}_k = \mathcal{F}_d(v_k)$, and $\mathcal{T}_{n,k} \rightarrow \mathcal{T}_k$ for every k , as $n \rightarrow \infty$. Hence $v_{n,k} \rightarrow v_k$ as $n \rightarrow \infty$ for every $k \in \mathbf{N}$ in $\mathcal{E}'$. Thus,

$$\left[\frac{v_{n,k}}{\epsilon_n}\right] = \left(\hat{\mathcal{F}}_d\right)^{-1}\left(\left[\frac{\mathcal{T}_{n,k}}{\epsilon_n}\right]\right) \rightarrow \left[\frac{v_k}{\epsilon_n}\right] = \left(\hat{\mathcal{F}}_d\right)^{-1}\left(\left[\frac{\mathcal{T}_k}{\epsilon_n}\right]\right)$$

as $n \rightarrow \infty$ for every k . Hence, the theorem.

Theorem 6.17. $\hat{\mathcal{F}}_d : \mathcal{H}_{\beta_3} \rightarrow \mathcal{H}_{\beta_4}, \left(\hat{\mathcal{F}}_d\right)^{-1} : \mathcal{H}_{\beta_4} \rightarrow \mathcal{H}_{\beta_3}$ are continuous with respect to Δ convergence.

Proof of this theorem follows from repeated analysis employed for Theorem 5.6.

Theorem 6.18. *The mappings $\mathcal{D}_F \rightarrow \mathcal{H}_{\beta_4}, \mathcal{T} \rightarrow \left[\frac{T \vee \epsilon_n}{\epsilon_n} \right]$ and, $\mathcal{E}' \rightarrow \mathcal{H}_{\beta_3}, f \rightarrow \left[\frac{f \bullet \epsilon_n}{\epsilon_n} \right]$ are continuous imbedding mappings with respect to δ convergence.*

See Theorem 3.6 for similar proof.

Conclusion.

The diffraction Fresnel transform which is a generalization of the Fresnel transform is extended to spaces of generalized functions, namely, Boehmian spaces. The extended transform maintains most of its general properties which have in the classical sense.

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ON ρ -HOMEOMORPHISMS IN TOPOLOGICAL SPACES**C. Devamanoharan****S. Pious Missier**

*Post Graduate and Research Department of Mathematics
V.O. Chidambaram College
Thoothukudi 628 008
Tamil Nadu
India
e-mail: kanchidev@gmail.com
spmissier@yahoo.com*

S. Jafari

*College of Vestsjaelland South
Herrestraede 11
4200 Slagelse
Denmark
e-mail: jafari@stofanet.dk*

Abstract. In this paper, we first introduce a new class of closed map called ρ -closed map. Moreover, we introduce a new class of homeomorphism called a ρ -homeomorphism. We also introduce another new class of closed map called ρ^* -closed map and introduce a new class of homeomorphism called a ρ^* -homeomorphism and prove that the set of all ρ^* -homeomorphisms forms a group under the operation of composition of maps.

Keywords and phrases: ρ -closed map, ρ -open map, ρ -homeomorphism, ρ^* -closed map, ρ^* -homeomorphism.

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1. Introduction

The notion homeomorphism plays a very important role in topology. By definition, a homeomorphism between two topological spaces X and Y is a bijection map $f : X \rightarrow Y$ when both f and f^{-1} are continuous. In the course of generalizations of the notion of homeomorphism, Maki et al. [35] introduced g -homeomorphisms

and gc -homeomorphisms in topological spaces. Devi et al. [13], [14] studied semi-generalized homeomorphisms and generalized semi-homeomorphisms and also they have introduced α -homeomorphisms in topological spaces. Jafari et al. [28] introduced $\tilde{g}$ -homeomorphisms in topological spaces.

In this chapter, we first introduce ρ -closed maps in topological spaces, ρ -open maps and then we introduce and study ρ -homeomorphisms. We prove that the concepts of ρ -homeomorphism and of homeomorphism (resp. g -homeomorphism, semihomomorphism, prehomomorphism) are independent. We also introduce ρ^* -closed map and ρ^* -homeomorphism. It turns out that the set of all ρ^* -homeomorphisms forms a group under the operation of composition of maps.

2. Preliminaries

Throughout this paper (X, τ) , (Y, σ) and (Z, η) will always denote topological spaces on which no separation axioms are assumed, unless otherwise mentioned. When A is a subset of (X, τ) , $\text{cl}(A)$ and $\text{int}(A)$ denote the closure and the interior of the set A , respectively.

We recall the following definitions and some results, which are used in the sequel.

Definition 2.1. Let (X, τ) be a topological space. A subset A of a space (X, τ) is called:

1. preopen [20] if $A \subseteq \text{int}(\text{cl}(A))$ and preclosed if $\text{cl}(\text{int}(A)) \subseteq A$.
2. semiopen [18] if $A \subseteq \text{cl}(\text{int}(A))$ and semiclosed if $\text{int}(\text{cl}(A)) \subseteq A$.
3. semipreopen [1] if $A \subseteq \text{cl}(\text{int}(\text{cl}(A)))$ and semipreclosed if $\text{int}(\text{cl}(\text{int}(A))) \subseteq A$.
4. regular open if $A = \text{int}(\text{cl}(A))$ and regular closed if $A = \text{cl}(\text{int}(A))$.

Definition 2.2. Let (X, τ) be a topological space. A subset A of a space (X, τ) is called:

1. generalized closed (briefly g -closed) [19] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .
2. generalized preclosed (briefly gp -closed) [25] if $\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .
3. generalized preregular closed (briefly gpr -closed) [13] if $\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in (X, τ) .
4. πgp -closed [27] if $\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and U is π -open in X .
5. ω -closed [32] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open in X .
6. $\wedge g$ -closed [33] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open in X .

7. $*g$ -closed [36] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is $\hat{g}$ -open in X .
8. $\#g$ -semi closed (briefly $\#gs$ -closed) [35] if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and U is $*g$ -open in X .
9. $\tilde{g}$ -closed set [16] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is $\#gs$ -open in X .
10. ρ -closed set [6] if $\text{pcl}(A) \subseteq \text{Int}(U)$ whenever $A \subseteq U$ and U is $\tilde{g}$ -open in (X, τ) .
11. ρ_s -closed set [6] if $\text{pcl}(A) \subseteq \text{Int}(\text{cl}(U))$ whenever $A \subseteq U$ and U is $\tilde{g}$ -open in (X, τ) .
12. π -open [37] if it is a finite union of regular open sets. The complement of a π -open set is said to be π -closed.

The complements of the above mentioned sets are called their respective open set.

Definition 2.3. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called :

1. semi-continuous [18] if $f^{-1}(V)$ is semiopen in (X, τ) for every open set V in (Y, σ) ;
2. pre-continuous [20] (resp. g -continuous [4], ω -continuous [32], gp -continuous [2], gpr -continuous [12], πgp -continuous [28], $\#g$ -semicontinuous [35], $\tilde{g}$ -continuous [30]), if $f^{-1}(F)$ is Preclosed (resp. g -closed, ω -closed, gp -closed, gpr -closed, πgp -closed, $\#gs$ -closed, $\tilde{g}$ -closed) in (X, τ) for every closed set F in (Y, σ) ;
3. contra-continuous [9] if $f^{-1}(V)$ is closed in (X, τ) for every open set V in (Y, σ) ;
4. M-precontinuous [38] if $f^{-1}(V)$ is preclosed in (X, τ) for every preclosed set V in (Y, σ) ;
5. RC-continuous [12] if $f^{-1}(V)$ is regular closed in (X, τ) for every open set V in (Y, σ) ;
6. ρ -continuous [7] (resp. ρ_s -continuous [7]) if $f^{-1}(V)$ is ρ -closed (resp. ρ_s -closed) in (X, τ) for every closed set V in (Y, σ) ;
7. ρ -irresolute [7] (resp. $\tilde{g}$ -irresolute [30]), if $f^{-1}(A)$ is ρ -closed (resp. $\tilde{g}$ -closed) in (X, τ) for every ρ -closed (resp. $\tilde{g}$ -closed) set A in (Y, σ) ;
8. contra-open [4] if $f(V)$ is closed in (Y, σ) for every open set V in (X, τ) ;
9. M-preclosed [22] if $f(V)$ is preclosed in (Y, σ) for every preclosed set V in (X, τ) ;

10. preclosed [46] (resp. ω -closed [59], g -closed [37], gp -closed [46], gpr -closed [47], πgp -closed [50], gs -closed, $\tilde{g}$ -closed [27]), if $f(F_1)$ is preclosed (resp. ω -closed, g -closed, gp -closed, gpr -closed, πgp -closed, gs -closed, $\tilde{g}$ -closed), in (Y, σ) for every closed set F_1 in (X, τ) .

Definition 2.4. A space (X, τ) is called: a $T_{1/2}$ space [32] (resp. T_ω space [59], $gsT^{#1/2}$ space [71], $T\tilde{g}$ -space [56], ρ - T_s space [10]), if every g -closed (resp. ω -closed, $\#g$ -semi-closed, $\sim g$ -closed, ρ_s -closed) set is closed in (X, τ) .

Definition 2.5. A bijective function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called a

1. homeomorphism if f is both open and continuous.
2. generalized homeomorphism (briefly g -homeomorphism) [35] if f is both g -open and g -continuous.
3. semi-homeomorphism [13] if f is both continuous and semi-open.
4. pre-homeomorphism [40] if f is both M -precontinuous and M -preopen.
5. gp -homeomorphism if f is both gp -continuous and gp -open.
6. gpr -homeomorphism if f is both gpr -continuous and gpr -open.
7. πgp -homeomorphism if f is both πgp -continuous and πgp -open.

Definition 2.6.

- (i) Let (X, τ) be a topological space and $A \subseteq X$. We define the ρ -closure of A [10] (briefly ρ -cl(A)) to be the intersection of all ρ -closed sets containing A .
- (ii) Let (X, τ) be a topological space and $A \subseteq X$. We define the ρ -interior of A [10] (briefly ρ -int(A)) to be the union of all ρ -open sets contained in A .
- (iii) A topological space (X, τ) is ρ -compact [11] if every ρ -open cover of X has a finite subcover.
- (iv) Let (X, τ) be a topological space. Let x be a point of (X, τ) and V be a subset of X . Then V is called a ρ -open neighbourhood (simply ρ -neighbourhood) [11] of x in (X, τ) if there exists a ρ -open set U of (X, τ) such that $x \in U \subseteq V$.

Proposition 2.7. [10] *Let (X, τ) be a topological space and $A \subseteq X$. The following properties are hold:*

- (i) *If A is ρ -closed then ρ -cl(A) = A . The converse is not true.*
- (ii) *If $A \subset B$ then ρ -cl(A) $\subset$ ρ -cl(B).*

Theorem 2.8.

- (i) [10] *Every open and preclosed subset of (X, τ) is ρ -closed.*

- (ii) [10] Every ρ -closed set is gp -closed (resp. gpr -closed, πgp -closed, ρ_s -closed) set.
- (iii) [11] If $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\tilde{g}$ -irresolute and M -preclosed function, then $f(A)$ is ρ -closed in (Y, σ) for every ρ -closed set A of (X, τ) .
- (iv) [11] Every ρ -continuous function is gp -continuous (resp. gpr -continuous, πgp -continuous, ρ_s -continuous) function.
- (v) [11] Every ρ -closed subset of a ρ -compact space X is ρ -compact relative to X .
- (vi) [11] If $f : (X, \tau) \rightarrow (Y, \sigma)$ is ρ -irresolute and a subset A of X is ρ -compact relative to X then its image $f(A)$ is ρ -compact relative to Y .
- (vii) [53] If $A \subseteq Y \subseteq X$ where A is $\tilde{g}$ -open in Y and Y is $\tilde{g}$ -open in X then A is $\tilde{g}$ -open in X .

3. ρ -closed maps

Definition 3.1. A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be ρ -closed if the image of every closed set in (X, τ) is ρ -closed in (Y, σ) .

Example 3.2.

- (i) Let $X = Y = \{a, b, c, d, e\}$, $\tau = \{\phi, \{a, b\}, \{c, d\}, \{a, b, c, d\}, X\}$, $\sigma = \{\phi, \{b\}, \{d, e\}, \{b, d, e\}, \{a, c, d, e\}, Y\}$. Define a map $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = d$, $f(b) = e$, $f(c) = b$, $f(d) = c$, $f(e) = a$. Then f is a ρ -closed map.
- (ii) Let $X=Y=\{a, b, c, d, e\}$, $\tau=\{\phi, \{a, b\}, \{c, d\}, \{a, b, c, d\}, X\}$, $\sigma = \{\phi, \{b\}, \{d, e\}, \{b, d, e\}, \{a, c, d, e\}, Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then f is not a ρ -closed map. Since for the closed set $V = \{e\}$ in (X, τ) , $f(V) = \{e\}$, which is not a ρ -closed set in (Y, σ) .

Theorem 3.3. Every Contra-closed map and Preclosed map $f : (X, \tau) \rightarrow (Y, \sigma)$ is ρ -closed map.

Proof. Let V be a closed set in (X, τ) . Then $f(V)$ is open and preclosed in (Y, σ) . Hence by Theorem 2.4(i), $f(V)$ is ρ -closed in (Y, σ) . Therefore f is a ρ -closed map. ■

The converse of this theorem need not be true as seen from the following example.

Example 3.4. As in Example 3.2(i), f is a ρ -closed map but neither contra-closed map nor preclosed map. Since for the closed set $V = \{a, b, e\}$ in (X, τ) , $f(V) = \{a, d, e\}$ is neither preclosed nor open in (Y, σ) .

Theorem 3.5. Every ρ -closed map $f : (X, \tau) \rightarrow (Y, \sigma)$ is a gp -closed (resp. gpr -closed, πgp -closed) map.

Proof. Let V be a closed set in (X, τ) . Then $f(V)$ is a ρ -closed set in (Y, σ) . By Theorem 2.4(ii), $f(V)$ is gp -closed in (Y, σ) (resp. By Theorem 2.4(ii), $f(V)$ is gpr -closed in (Y, σ) , by Theorem 2.4(ii), $f(V)$ is πgp -closed in (Y, σ)). Hence f is a gp -closed (resp. gpr -closed, πgp -closed) map. ■

The converse of this theorem need not be true as seen from the following examples.

Example 3.6.

- (i) Let $X = Y = \{a, b, c, d, e\}$, $\tau = \{\phi, \{a, b\}, \{a, b, d\}, \{a, b, c, d\}, \{a, b, d, e\}, X\}$, $\sigma = \{\phi, \{b, c, d\}, \{a, b, c, d\}, \{b, c, d, e\}, Y\}$. Define $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = c$; $f(b) = e$; $f(c) = a$; $f(d) = b$; $f(e) = d$. Then the function f is a gp -closed map but not ρ -closed map. Since for the closed set $V = \{e\}$ in (X, τ) , $f(V) = \{d\}$, is a gp -closed set but not a ρ -closed set in (Y, σ) .
- (ii) Let $X = Y = \{a, b, c, d, e\}$, $\tau = \{\phi, \{a, b\}, \{a, b, d\}, \{a, b, c, d\}, \{a, b, d, e\}, X\}$, $\sigma = \{\phi, \{a, b\}, \{c, d\}, \{a, b, c, d\}, Y\}$. Define f as in Example 3.6(i), the function f is gpr -closed map but not ρ -closed map. Since for all the closed sets in (X, τ) , its images are all gpr -closed sets in (X, σ) but no one is ρ -closed set in (Y, σ) .
- (iii) As in Example 3.6(i), define $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = c$; $f(b) = b$; $f(c) = a$; $f(d) = e$; $f(e) = d$. Then the function f is a πgp -closed map but not ρ -closed map. Since for the closed set $V = \{a, d\}$ is πgp -closed set but not a ρ -closed set in (Y, σ) .

The following examples show that the concepts of closed map and of ρ -closed map are independent.

Example 3.7.

- (i) As in Example 3.2(i), f is a ρ -closed map but not a closed map. Since for the closed set $v = \{e\}$ in (X, τ) , $f(V) = \{a\}$ is ρ -closed but not closed in (Y, σ) .
- (ii) Let τ be the collection of subsets of N consisting of ϕ and all subsets of N of the form $\{n, n+1, n+2, \dots\}$, $n \in N$. Then τ is a topology for N . Let (Z, κ) be the digital topology. Let $f : (N, \tau) \rightarrow (Z, \kappa)$ be the constant map defined by $f(x) = \{4\}$ for each $x \in N$. The image of each closed set in (N, τ) is $\{4\}$ which is closed in (X, κ) , but not ρ -closed in (X, κ) , because there is a $\tilde{g}$ -open set $U = \{1, 2, 3, 4\}$ containing $\{4\}$, is not open in (X, κ) such that $\text{pcl}(\{4\}) = \{4\} \not\subseteq \text{int}(U) = \{1, 2, 3\}$. Therefore f is a closed map but not a ρ -closed map.

The following examples show that the concepts of g -closed map and of ρ -closed map are independent.

Example 3.8.

- (i) Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a, c\}, X\}$, $\sigma = \{\phi, \{a\}, \{a, b\}, Y\}$. Define a map $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = c$; $f(b) = b$; $f(c) = a$. Then f is a ρ -closed map but not g -closed map. since for the closed set $V = \{b\}$ in (X, τ) , $f(V) = \{b\}$ is ρ -closed but not g -closed in (Y, σ) .
- (ii) As in Example 3.7(ii), f is a g -closed map but not a ρ -closed map.

The following example shows that the composition of two ρ -closed maps need not be ρ -closed.

Example 3.9. Let $X=Y=Z=\{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b, c\}, X\}$, $\sigma = \{\phi, \{a, c\}, Y\}$, $\eta = \{\phi, \{a\}, \{a, b\}, Z\}$. Define $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = f(b) = b$; $f(c) = a$ and define $g : (Y, \sigma) \rightarrow (Z, \eta)$ by $g(a) = c$; $g(b) = b$; $g(c) = a$. Then both f and g are ρ -closed maps but their composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is not a ρ -closed map, since for the closed set $V = \{b, c\}$ in (X, τ) , $g \circ f(V) = \{a, b\}$, which is not a ρ -closed set in (Z, η) .

Theorem 3.10. *If $f : (X, \tau) \rightarrow (Y, \sigma)$ is ρ -closed, $g : (Y, \sigma) \rightarrow (Z, \eta)$ is ρ -closed and (Y, σ) is a ρ - T_s space then their composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is ρ -closed.*

Proof. Let V be a closed set in (X, τ) . Then $f(V)$ is a ρ -closed set in (Y, σ) . Since (Y, σ) is ρ - T_s and by Theorem 2.4(ii), $f(V)$ is a closed set in (Y, σ) . Hence $g(f(V)) = (g \circ f)(V)$ is a ρ -closed in (Z, η) . Therefore $g \circ f$ is a ρ -closed map. ■

Theorem 3.11. *If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a $\tilde{g}$ -closed (resp. g -closed, ω -closed, gs -closed) map, $g : (Y, \sigma) \rightarrow (Z, \eta)$ is a ρ -closed map and Y is $T\tilde{g}$ -space (resp. $T_{1/2}$ space, T_w space, $gsT_{1/2}^\#$ space) then their composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is a ρ -closed map.*

Proof. Let V be a closed set in (X, τ) . Then $f(V)$ is a $\tilde{g}$ -closed (resp. g -closed, ω -closed, gs -closed) set in (Y, σ) . Since (Y, σ) is a $T\tilde{g}$ -space (resp. $T_{1/2}$ space, T_w space, $gsT_{1/2}^\#$ space), therefore $f(V)$ is a closed set in (Y, σ) . Since g is ρ -closed, $g(f(V)) = (g \circ f)(V)$ is ρ -closed in (Z, η) . Therefore $g \circ f$ is a ρ -closed map. ■

Theorem 3.12. *If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a $\tilde{g}$ -closed and contra-closed map, $g : (Y, \sigma) \rightarrow (Z, \eta)$ is an M -preclosed and open map then their composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is ρ -closed map.*

Proof. Let V be a closed set in (X, τ) . Then $f(V)$ is $\tilde{g}$ -closed and open in (Y, σ) . Since every $\tilde{g}$ -closed is preclosed and g is M -preclosed and open, hence $g(f(V)) = (g \circ f)(V)$ is preclosed and open in (Z, η) . By Theorem 2.4(i), $(g \circ f)(V)$ is ρ -closed in (Z, η) . Therefore $g \circ f$ is a ρ -closed map. ■

Theorem 3.13. *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a closed map and $g : (Y, \sigma) \rightarrow (Z, \eta)$ be a ρ -closed map then their composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is ρ -closed.*

Proof. Let V be a closed set in (X, τ) . Then $f(V)$ is a closed set in (Y, σ) . Hence $g(f(V)) = (g \circ f)(V)$ is ρ -closed set in (Z, η) . Therefore $g \circ f$ is a ρ -closed map. ■

If f is ρ -closed map and g is closed, then their composition need not be a ρ -closed map as seen from the following example.

Example 3.14. Let $X=Y=Z=\{a, b, c\}$, $\tau=\{\phi, \{a\}, \{b, c\}, X\}$, $\sigma=\{\phi, \{a, c\}, Y\}$, $\eta=\{\phi, \{c\}, \{a, c\}, Z\}$. Define $f : (X, \tau) \rightarrow (Y, \sigma)$ be $f(a) = f(b) = c$; $f(c) = b$ and define $g : (Y, \sigma) \rightarrow (Z, \eta)$ be the identity map. Then f is a ρ -closed map and g is a closed map. But their composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is not a ρ -closed map. Since for the closed set $V = \{a\}$ in (X, τ) , $(g \circ f)(V) = g(f(V)) = g(c) = \{c\}$, which is not is ρ -closed set in (Z, η) .

Theorem 3.15. *If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a ρ -closed, $g : (Y, \sigma) \rightarrow (Z, \eta)$ is M -preclosed and $\tilde{g}$ -irresolute map then $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is ρ -closed.*

Proof. Let V be a closed set in (X, τ) . Then $f(V)$ is a ρ -closed set in (Y, σ) . Hence by Theorem 2.4(iii), $g(f(V)) = (g \circ f)(V)$ ρ -closed in (Z, η) . Therefore $g \circ f$ is a ρ -closed map. ■

Theorem 3.16. *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \eta)$ be two mappings such that their composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ be a ρ -closed mapping. Then the following statements are true if:*

1. f is continuous and surjective then g is ρ -closed.
2. g is ρ -irresolute, injective then f is ρ -closed.
3. f is $\tilde{g}$ -continuous, surjective and (X, τ) is a $T\tilde{g}$ -space, then g is ρ -closed.
4. f is g -continuous, surjective and (X, τ) is a $T_{1/2}$ space then g is ρ -closed.
5. f is ρ -continuous, surjective and (X, τ) is a ρ - T_s space then g is ρ -closed.

Proof. 1. Let A be a closed set in (Y, σ) . Since f is continuous, $f^{-1}(A)$ is closed in (X, τ) and since $g \circ f$ is ρ -closed, $(g \circ f)(f^{-1}(A)) = g(A)$ is a ρ -closed in (Z, η) , since f is surjective. Therefore, g is a ρ -closed map.

2. Let A be a closed set in (X, τ) . Since $g \circ f$ is ρ -closed, then $(g \circ f)(A)$ is ρ -closed in (Z, η) . Since g is ρ -irresolute, then $g^{-1}(g \circ f)(A)$ is ρ -closed in (Y, σ) , since g is injective. Thus, f is a ρ -closed map.

3. Let A be a closed set of (Y, σ) . Since f is $\tilde{g}$ -continuous, $f^{-1}(A)$ is $\tilde{g}$ -closed in (X, τ) . Since (X, τ) is a $T\tilde{g}$ -space, $f^{-1}(A)$ is closed in (X, τ) , since $g \circ f$ is ρ -closed, $(g \circ f)(f^{-1}(A)) = g(A)$ is ρ -closed in (Z, η) , since f is surjective. Thus g is a ρ -closed map.

4. Let A be a closed set of (Y, σ) . Since f is g -continuous, $f^{-1}(A)$ is g -closed in (X, τ) . Since (X, τ) is a $T_{1/2}$ -space, $f^{-1}(A)$ is closed in (X, τ) , since $g \circ f$ is ρ -closed, $(g \circ f)(f^{-1}(A)) = g(A)$ is ρ -closed in (Z, η) , since f is surjective. Thus g is a ρ -closed map.

5. Let A be a closed set (Y, σ) . Since f is ρ -continuous, $f^{-1}(A)$ is ρ -closed in (X, τ) . Since (X, τ) is a ρ - T_s space and by Theorem 2.4(ii), $f^{-1}(A)$ is closed in (X, τ) . Since $g \circ f$ is ρ -closed, $(g \circ f)(f^{-1}(A)) = g(A)$ is ρ -closed in (Z, η) . Since f is surjective. Thus, g is a ρ -closed map. ■

As for the restriction f_A of a map $f : (X, \tau) \rightarrow (Y, \sigma)$ to a subset A of (X, τ) , we have the following.

Theorem 3.17. *Let (X, τ) and (Y, σ) be any topological spaces, Then if:*

1. $f : (X, \tau) \rightarrow (Y, \sigma)$ is ρ -closed and A is a closed subset of (X, τ) then $f_A : (A, \tau_A) \rightarrow (Y, \sigma)$ is ρ -closed.
2. $f : (X, \tau) \rightarrow (Y, \sigma)$ is ρ -closed and $A = f^{-1}(B)$, for some closed set B of (Y, σ) , then $f_A : (A, \tau_A) \rightarrow (Y, \sigma)$ is ρ -closed.

Proof. 1. Let B be a closed set of (A, τ_A) . Then $B = A \cap F$ for some closed set F of (X, τ) and so B is closed in (X, τ) . Since f is ρ -closed, then $f(B)$ is ρ -closed in (Y, σ) . But $f(B) = f_A(B)$ and therefore f_A is a ρ -closed map.

2. Let F be a closed set of (A, τ_A) . Then $F = A \cap H$ for some closed set H of (X, τ) . Now $f_A(F) = f(F) = f(A \cap H) = f(f^{-1}(B) \cap H) = B \cap f(H)$. Since f is ρ -closed, $f(H)$ is ρ -closed in (Y, σ) and so $B \cap f(H)$ is ρ -closed in (Y, σ) . Therefore f_A is a ρ -closed map. ■

Theorem 3.18. *A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is ρ -closed if and only if for each subset S of (Y, σ) and for each open set U containing $f^{-1}(S)$ there is a ρ -open set V of (Y, σ) such that $S \subset V$ and $f^{-1}(V) \subset U$.*

Proof. Suppose that f is a ρ -closed map. Let $S \subset Y$ and U be an open subset of (X, τ) such that $f^{-1}(S) \subset U$. Then $V = (f(U^c))^c$ is a ρ -open set containing S such that $f^{-1}(V) \subset U$. For the converse. Let S be a closed set of (X, τ) . Then $f^{-1}((f(S))^c) \subset S^c$ and S^c is open. By assumption, there exists a ρ -open set V of (Y, σ) such that $(f(S))^c \subset V$ and $f^{-1}(V) \subset S^c$ and so $S \subset (f^{-1}(V))^c$. Hence $V^c \subset f(S) \subset f(f^{-1}(V)^c) \subset V^c$ which implies $f(S) = V^c$. Since V^c is ρ -closed in (Y, σ) , $f(S)$ is ρ -closed in (Y, σ) and therefore f is ρ -closed. ■

Theorem 3.19. *If a mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is ρ -closed then $\rho\text{-cl}(f(A)) \subseteq f(\text{cl}(A))$ for every subset A of (X, τ) .*

Proof. Suppose that f is ρ -closed and $A \subseteq X$, then $f(\text{cl}(A))$ is ρ -closed in (Y, σ) . Hence by Proposition 2.7(i), $\rho\text{-cl}(f(\text{cl}(A))) = f(\text{cl}(A))$. Also $(A) \subseteq f(\text{cl}(A))$, and by Proposition 2.7(ii), we have, $\rho\text{-cl}(f(A)) \subseteq \rho\text{-cl}(f(\text{cl}(A))) = f(\text{cl}(A))$. ■

The converse of this theorem need not be true as seen from the following example.

Example 3.20. Let $X = \{a, b, c\} = Y$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$, $\sigma = \{\emptyset, \{a\}, \{a, b\}, Y\}$. Define $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = c$; $f(b) = a$; $f(c) = b$. For every subset A of X , we have $\rho\text{-cl}(f(A)) \subseteq f(\text{cl}(A))$. But f is not a ρ -closed map. Since for the closed set $V = \{b, c\}$ in (X, τ) , $f(V) = \{a, b\}$ is not a ρ -closed set in (Y, σ) .

4. ρ -Open maps

Definition 4.1. A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be ρ -open map if the image $f(A)$ is ρ -open in (Y, σ) for every open set A in (X, τ) .

Theorem 4.2. For any bijection $f : (X, \tau) \rightarrow (Y, \sigma)$, the following statements are equivalent.

1. $f^{-1} : (Y, \sigma) \rightarrow (X, \tau)$ is ρ -continuous
2. f is a ρ -open map and
3. f is a ρ -closed map.

Proof. (1)→(2). Let U be an open set of (X, τ) . By assumption, $(f^{-1})^{-1}(U) = f(U)$ is ρ -open in (Y, σ) and so f is a ρ -open map.

(2)→(3). Let V be a closed set of (X, τ) . Then V^c is open in (X, τ) . By assumption $f(V^c) = (f(V))^c$ is ρ -open in (Y, σ) and therefore $f(V)$ is ρ -closed in (Y, σ) . Hence f is a ρ -closed map.

(3)→(1) Let V be a closed set of (X, τ) . By assumption $f(V)$ is ρ -closed in (Y, σ) . But $f(V) = (f^{-1})^{-1}(V)$ and therefore f^{-1} is ρ -continuous on (Y, σ) . ■

Theorem 4.3. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be mapping. If f is a ρ -open mapping, then for each $x \in X$ and for each neighbourhood U of x in (X, τ) , there exists a ρ -neighbourhood W of $f(x)$ in (Y, σ) such that $W \subset f(U)$.

Proof. Let $x \in X$ and U be an arbitrary neighbourhood of x . Then there exists an open set V in (X, τ) such that $x \in V \subseteq U$. By assumption, $f(V)$ is a ρ -open set in (Y, σ) . Further, $f(x) \in f(V) \subseteq f(U)$, clearly $f(U)$ is a ρ -neighbourhood of $f(x)$ in (Y, σ) and so the theorem holds, by taking $W = f(V)$. ■

The converse of this theorem need not be true as seen from the following example.

Example 4.4. As in example 4.4, Let $U = \{a, b, c, d\}$ be an open set in (X, τ) and $f(a) = a$. Then $a \in U$ and for each $a = f(a) \in f(U) = \{a, c, d, e\}$, by assumption, there exists a ρ -neighbourhood $W_a = \{a, c, d, e\}$ of a in (Y, σ) such that $W_a \subseteq f(U)$. But $f(U)$ is not a ρ -open set in (Y, σ) .

Theorem 4.5. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is ρ -open if and only if for any subset B of (Y, σ) and for any closed set S containing $f^{-1}(B)$, there exists a ρ -closed set A of (Y, σ) containing B such that $f^{-1}(A) \subseteq S$.

Proof. Similar to Theorem 3.18. ■

5. ρ -Homeomorphisms

Definition 5.1. A bijection $f : (X, \tau) \rightarrow (Y, \sigma)$ is called ρ -homeomorphism if f is both ρ -continuous and ρ -open.

Example 5.2. Let $X=Y=\{a, b, c\}$, $\tau=\{\phi, \{a\}, \{a, b\}, X\}$, $\sigma=\{\phi, \{b\}, \{b, c\}, Y\}$. Define a map $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = c$, $f(b) = b$, $f(c) = a$. Then f is a ρ -homeomorphism.

Theorem 5.3. Every ρ -homeomorphism is a gp -homeomorphism (resp. gpr -homeomorphism, πgp -homeomorphism).

Proof. By Theorem 2.4(iv), every ρ -continuous map is gp -continuous (resp. gpr -continuous, πgp -continuous) and also by Theorem 2.4(ii), every ρ -open map is gp -open (resp. gpr -open, πgp -open), the proof follows. ■

The converse of the above theorem need not be true as seen from the following example.

Example 5.4.

- (i) Let $X = Y = \{a, b, c, d, e\}$, $\tau = \{\phi, \{b, c, d\}, \{a, b, c, d\}, \{b, c, d, e\}, X\}$ and $\sigma = \{\phi, \{a, b, e\}, X\}$. Define $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = e$, $f(b) = c$, $f(c) = d$, $f(d) = a$, $f(e) = b$. Then f is gp -homeomorphism but not ρ -homeomorphism. Observe that for the closed set $V = \{c, d\}$ in (Y, σ) , $f^{-1}(V) = \{b, c\}$ is gp -closed but not ρ -closed in (X, τ) , that is f is not ρ -continuous.
- (ii) Let $X=Y=\{a, b, c\}$, $\tau=\{\phi, \{a\}, \{a, b\}, \{c, a\}, X\}$, $\sigma = \{\phi, \{b\}, \{c, a\}, Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be an identity map. Then f is πgp -homeomorphism but not ρ -homeomorphism. Since for the closed set $V = \{c, a\}$ in (Y, σ) , $f^{-1}(V) = \{c, a\}$ is not ρ -closed in (X, τ) , that is f is not ρ -continuous.
- (iii) Let $X = \{a, b, c, d, e\}$, $Y = \{a, b, c, d\}$, $\tau = \{\phi, \{c\}, \{e\}, \{a, b\}, \{c, e\}, \{a, b, c\}, \{a, b, e\}, \{a, b, c, e\}, X\}$, $\sigma = \{\phi, \{c, d\}, Y\}$. Define $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = f(c) = a$, $f(b) = f(e) = b$, $f(d) = c$. Then f is gpr -homeomorphism but not ρ -homeomorphism. Since for the closed set $V = \{a, b\}$ in (Y, σ) , $f^{-1}(V) = \{a, b, c, e\}$ is not ρ -closed in (X, τ) , that is f is not ρ -continuous.

Theorem 5.5. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be both a contra-open and contra-continuous function; further let f be a gp -homeomorphism. Then f is a ρ -homeomorphism.

Proof. Let U be open in (X, τ) . Then $f(U)$ is gp -open in (Y, σ) . Hence $Y - f(U)$ is gp -closed in (Y, σ) . Since f is contra-open, then $f(U)$ is closed in (Y, σ) and so $Y - f(U)$ is open in (Y, σ) . By Theorem 2.2 [29], $Y - f(U)$ is preclosed in (Y, σ) and by Theorem 2.4(i), $Y - f(U)$ is ρ -closed in (Y, σ) , that is $f(U)$ is ρ -open in (Y, σ) . Hence f is ρ -open. Let V be closed in (Y, σ) . Then $f^{-1}(V)$ is gp -closed in (X, τ) . Since f is contra-continuous, then $f^{-1}(V)$ is open in (X, τ) . By Theorem 2.2 [29] and by Theorem 2.4(i), $f^{-1}(V)$ is ρ -closed in (X, τ) . Hence f is ρ -continuous. Since f is ρ -continuous and ρ -open, therefore f is ρ -homeomorphism. ■

Definition 5.6. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called

1. contra- π -open (resp. regular-contra-open), if $f(U)$ is π -closed (resp. regular closed) in (Y, σ) for every open set U in (X, τ) .
2. contra- π -continuous, if $f^{-1}(V)$ is π -open in (X, τ) for every closed set V in (Y, σ) .

Theorem 5.7. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be both a contra- π -open and contra- π -continuous function; further, let f be a πgp -homeomorphism. Then f is a ρ -homeomorphism.

Proof. Let U be open in (X, τ) . Then $f(U)$ is πgp -open in (Y, σ) . Hence $Y - f(U)$ is πgp -closed in (Y, σ) . Since f is contra- π -open, then $f(U)$ is π -closed in (Y, σ) and so $Y - f(U)$ is π -open in (Y, σ) . By Theorem 2.4 [53], $Y - f(U)$ is preclosed in (Y, σ) and since every π -open is open and by Theorem 2.4(i), $Y - f(U)$ is ρ -closed in (Y, σ) , that is $f(U)$ is ρ -open in (Y, σ) . Hence f is ρ -open. Let V be closed in (Y, σ) . Then $f^{-1}(V)$ is πgp -closed in (X, τ) . Since f is contra- π -continuous, then $f^{-1}(V)$ is π -open in (X, τ) . By Theorem 2.4 [53] and since every π -open is open and by Theorem 2.4(i), $f^{-1}(V)$ is ρ -closed in (X, τ) . Hence f is ρ -continuous. Since f is ρ -continuous and ρ -open, therefore f is ρ -homeomorphism. ■

Theorem 5.8. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be both a contra-regular open and RC -continuous function; further, let f be a gpr -homeomorphism. Then f is a ρ -homeomorphism.

Proof. Let U be open in (X, τ) . Then $f(U)$ is gpr -open in (Y, σ) . Hence $Y - f(U)$ is gpr -closed in (Y, σ) . Since f is contra-regular open, then $f(U)$ is regular closed in (Y, σ) and so $Y - f(U)$ is regular open in (Y, σ) . By Theorem 3.10 [22], $Y - f(U)$ is preclosed in (Y, σ) and since every regular open is open and by Theorem 2.4(i), $Y - f(U)$ is ρ -closed in (Y, σ) , that is $f(U)$ is ρ -open in (Y, σ) . Hence f is ρ -open. Let V be closed in (Y, σ) . Then $f^{-1}(V)$ is gpr -closed in (X, τ) . Since f is completely contra-continuous, then $f^{-1}(V)$ is regular open in (X, τ) . By Theorem 3.10 [22] and since every regular open is open and by Theorem 2.4(i), $f^{-1}(V)$ is ρ -closed in (X, τ) . Hence f is ρ -continuous. Since f is ρ -continuous and ρ -open, therefore f is ρ -homeomorphism. ■

Theorem 5.9. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be both a contra-open and contra-continuous function; let f be a pre-homeomorphism. Then f is a ρ -homeomorphism.

Proof. Let U be open in (X, τ) . Then $f(U)$ is preopen in (Y, σ) . Hence $Y - f(U)$ is preclosed in (Y, σ) . Since f is contra-open, then $f(U)$ is closed in (Y, σ) and so $Y - f(U)$ is open in (Y, σ) . By Theorem 2.4(i), $Y - f(U)$ is ρ -closed in (Y, σ) , that is $f(U)$ is ρ -open in (Y, σ) . Hence f is ρ -open. Let V be closed in (Y, σ) . Then $f^{-1}(V)$ is preclosed in (X, τ) . Since f is contra-continuous, then $f^{-1}(V)$ is open in (X, τ) . By Theorem 2.4(i), $f^{-1}(V)$ is ρ -closed in (X, τ) . Hence f is ρ -continuous. Since f is ρ -continuous and ρ -open, therefore f is ρ -homeomorphism. ■

The concepts of ρ -homeomorphism and of homeomorphism are independent as can be seen from the following examples.

Example 5.10.

- (i) Let $X = \{a, b, c\} = Y$, $\tau = \{\phi, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{a, b\}, Y\}$. The identity map f on X is ρ -homeomorphism but not a homeomorphism, because it is not continuous.
- (ii) Let (R, τ) be a usual topology and $((0, 1), \tau^*)$, where τ^* denotes that relativized usual topology on $(0, 1)$. Let $f : (R, \tau) \rightarrow ((0, 1), \tau^*)$ be a map defined by $f(x) = 2x-1/x(x-1)$. Then f is a homeomorphism but not a ρ -homeomorphism, because it is not ρ -continuous.

The concepts of ρ -homeomorphism and of g -homeomorphism are independent as can be seen from the following examples.

Example 5.11.

- (i) Let $X = \{a, b, c\} = Y$, $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$, $\sigma = \{\phi, \{b\}, \{a, b\}, Y\}$. Define $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = b$, $f(b) = a$, $f(c) = c$. Then f is a ρ -homeomorphism but not g -homeomorphism. Since for the open set $V = \{a, c\}$ in (X, τ) , $f(V) = \{b, c\}$ is not g -open in (Y, σ) .
- (ii) As in Example 5.10(ii), f is a g -homeomorphism but not ρ -homeomorphism, because it is not ρ -continuous.

The concepts of ρ -homeomorphism and of semi-homeomorphism are independent as can be seen from the following examples.

Example 5.12.

- (i) Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a, b\}, X\}$, $\sigma = \{\phi, \{a\}, \{a, b\}, Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be an identity map. Then f is a ρ -homeomorphism. But f is not a semi-homeomorphism. Since for the closed set $V = \{b, c\}$ in (Y, σ) , $f^{-1}(V) = \{b, c\}$, which is not closed in (X, τ) . Therefore f is not a continuous map.
- (ii) Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b, c\}, X\}$, $\sigma = \{\phi, \{c\}, \{a, b\}, Y\}$. Define $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = c$, $f(b) = a$, $f(c) = b$. Then f is a semihomeomorphism. But f is not a ρ -homeomorphism. Since for the closed set $V = \{c\}$ in (Y, σ) , $f^{-1}(V) = \{a\}$, which is not ρ -closed in (X, τ) . Therefore f is not a ρ -continuous map.

The concepts of ρ -homeomorphism and of pre-homeomorphism are independent as can be seen from the following examples.

Example 5.13.

- (i) Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{a, b\}, X\}$, $\sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, Y\}$. Define $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = b$, $f(b) = a$, $f(c) = c$. Then f is a ρ -homeomorphism. But f is not a pre-homeomorphism. Since for the closed set $V = \{b, c\}$ in (Y, σ) , $f^{-1}(V) = \{c, a\}$, which is not preclosed in (X, τ) . Therefore f is not a pre-continuous map.

- (ii) Let $X = \{a, b, c, d, e\}$ with the topology $\tau = \{\phi, \{b, c, d\}, \{a, b, c, d\}, \{b, c, d, e\}, X\}$ and $Y = \{a, b\}$ with the discrete topology σ . Define $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(c) = f(d) = a$, $f(a) = f(b) = f(e) = b$. Then f is pre-homeomorphism but not ρ -homeomorphism. Since for the closed set $V = \{a\}$ in (Y, σ) , $f^{-1}(V) = \{c, d\}$ is not ρ -closed in (X, τ) , that is f is not ρ -continuous.

Theorem 5.14. *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a bijection ρ -continuous map. Then the following statements are equivalent.*

1. f is a ρ -open map.
2. f is a ρ -homeomorphism.
3. f is a ρ -closed map.

Proof. (1)→(2) By hypothesis and by assumption, proof is obvious.

(2)→(3) Let V be a closed set in (X, τ) . Then V^c is open in (X, τ) . By hypothesis, $f(V^c) = (f(V))^c$ is ρ -open in (Y, σ) . That is, $f(V)$ is ρ -closed in (Y, σ) . Therefore f is a ρ -closed map.

(3)→(1) Let V be an open set in (X, τ) . Then V^c is closed in (X, τ) . By hypothesis, $f(V^c) = (f(V))^c$ is ρ -closed in (Y, σ) . That is, $f(V)$ is ρ -open in (Y, σ) . Therefore f is a ρ -open map. ■

The composition of two ρ -homeomorphism maps need not be a ρ -homeomorphism as can be seen from the following example.

Example 5.15. Let $X = Y = Z = \{a, b, c\}$, $\tau = \{\phi, \{a, b\}, X\}$, $\sigma = \{\phi, \{a\}, \{a, b\}, Y\}$, $\eta = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, Z\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be an identity map and define $g : (Y, \sigma) \rightarrow (Z, \eta)$ by $g(a) = b$, $g(b) = a$, $g(c) = c$. Then both f and g are ρ -homeomorphisms, but their composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is not a ρ -homeomorphism. Since for the closed set $V = \{a\}$ in (Z, η) , $(g \circ f)^{-1}(V) = \{b\}$, which is not a ρ -closed set in (X, τ) . Therefore $g \circ f$ is not a ρ -continuous map and so $g \circ f$ is not a ρ -homeomorphism.

Theorem 5.16. *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a ρ -homeomorphism. Let A be an open ρ -closed subset of X and let B be a closed subset of Y such that $f(A) = B$. Assume that $\rho C(X, \tau)$ (the class of all ρ -closed sets of (X, τ)) be closed under finite intersections. Then the restriction $f_A : (A, \tau_A) \rightarrow (B, \sigma_B)$ is a ρ -homeomorphism.*

Proof. We have to show that f_A is a bijection, f_A is a ρ -open map and f_A is a ρ -continuous map.

(i) Since f is one-one, f_A is also one-one. Also since $f(A) = B$ we have $f_A(A) = B$ so that f_A is onto and hence f_A is a bijection.

(ii) Let U be an open set of (A, τ_A) . Then $U = A \cap H$, for some open set H in (X, τ) . Since f is one-one, then $f(U) = f(A \cap H) = f(A) \cap f(H) = B \cap f(H)$. Since f is ρ -open and H is an open set in (X, τ) , then $f(H)$ is a ρ -open set in (Y, σ) . Therefore $f(U)$ is a ρ -open set in (B, σ_B) . Hence f_A is a ρ -open map.

(iii) Let V be a closed set in (B, σ_B) . Then $V = B \cap K$, for some closed set K in (Y, σ) . Since B is a closed set in (Y, σ) , then V is a closed set in (Y, σ) . By hypothesis and assumption, $f^{-1}(V) \cap A = H_1$ (say) is a ρ -closed set in (X, τ) . Since $f_A^{-1}(V) = H_1$, it is sufficient to show that H_1 is a ρ -closed set in (A, τ_A) . Let G_1 be $\tilde{g}$ -open in (A, τ_A) such that $H_1 \subseteq G_1$. Then by hypothesis and by Lemma 2.4(vii), G_1 is $\tilde{g}$ -open in X . Since H_1 is a ρ -closed set in (X, τ) , we have $\text{Pcl}_X(H_1) \subseteq \text{Int}(G_1)$. Since A is open and by Lemma 2.10 [23], $\text{Pcl}_A(H_1) = \text{Pcl}_X(H_1) \cap A \subseteq \text{Int}(G_1) \cap A = \text{Int}(G_1) \cap \text{Int}(A) = \text{Int}(G_1 \cap A) \subseteq \text{Int}(G_1)$ and so $H_1 = f_A^{-1}(V)$ is ρ -closed set in (A, τ_A) . Therefore f_A is a ρ -continuous map. Hence f_A is a ρ -homeomorphism. ■

Definition 5.17. A topological space (X, τ) is called a ρ -Hausdorff if for each pair x, y of distinct points of X , there exists ρ -open neighbourhoods U_1 and U_2 of x and y , respectively, that are disjoint.

Theorem 5.18. Let (X, τ) be a topological space and let (Y, σ) be a ρ -Hausdorff space. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a one-one ρ -irresolute map. Then (X, τ) is also a ρ -Hausdorff space.

Proof. Let x_1, x_2 be any two distinct points of X . Since f is one-one, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$. Let $y_1 = f(x_1)$, $y_2 = f(x_2)$ so that $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$. Then $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Since (Y, σ) is ρ -Hausdorff, then there exists ρ -open sets U_1 and U_2 of (Y, σ) such that $y_1 \in U_1$, $y_2 \in U_2$ and $U_1 \cap U_2 = \phi$. Since f is ρ -irresolute, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are ρ -open sets of (X, τ) . Now $f^{-1}(U_1) \cap f^{-1}(U_2) = f^{-1}(U_1 \cap U_2) = f^{-1}(\phi) = \phi$, and $y_1 \in U_1$ implies $f^{-1}(y_1) \in f^{-1}(U_1)$ implies $x_1 \in f^{-1}(U_1)$, $y_2 \in U_2$ implies $f^{-1}(y_2) \in f^{-1}(U_2)$ implies $x_2 \in f^{-1}(U_2)$. Thus it is shown that for every pair of distinct points x_1, x_2 of X , there exists disjoint ρ -open sets $f^{-1}(U_1)$ and $f^{-1}(U_2)$ such that $x_1 \in f^{-1}(U_1)$ and $x_2 \in f^{-1}(U_2)$. Accordingly, the space (X, τ) is a ρ -Hausdorff space. ■

Theorem 5.19. Every ρ -compact subset A of a ρ -Hausdorff space X is ρ -closed. Assume that $\rho O(X, \tau)$ (the class of all ρ -open sets of (X, τ)) be closed under finite intersections.

Proof. We shall show that $X - A$ is ρ -open. Let $x \in X - A$. Since X is ρ -Hausdorff, for every $y \in A$, there exists disjoint ρ -open neighbourhoods U_y and V_y of x and y such that $U_y \cap V_y = \phi$. Now the collection $\{V_y / y \in A\}$ is a ρ -open cover of A , since A is compact, there exists a finite subcover $\{y_i, i = 1, \dots, n\}$ such that $A \subset \cup \{V_{y_i}, i = 1, \dots, n\}$.

Let $U = \cap \{U_{y_i}, i = 1, \dots, n\}$ and $V = \cup \{V_{y_i}, i = 1, \dots, n\}$. Then, by assumption, U is an ρ -open neighbourhood of x . Clearly, $U \cap V = \phi$, hence $U \cap A = \phi$, thus $U \subset X - A$, which means $X - A$ is ρ -open, therefore A is ρ -closed. ■

Theorem 5.20. Let (X, τ) a topological space and let (Y, σ) be a ρ -Hausdorff space. Assume that $\rho O(X, \tau)$ (the class of all ρ -open sets of (X, τ)) be closed under finite intersections. If f, g are ρ -irresolute maps of X into Y , then the set $A = \{x \in X : f(x) = g(x)\}$ is a ρ -closed subset of (X, τ) .

Proof. We shall show that $X - A$ is a ρ -open subset of (X, τ) . Now $X - A = \{x \in X : f(x) \neq g(x)\}$. Let $p \in X - A$. Set $y_1 = f(p)$, $y_2 = g(p)$. By the definition of $X - A$, we have $y_1 \neq y_2$. Thus y_1, y_2 are two distinct points of Y . Since (Y, σ) is a ρ -Hausdorff space, there exists ρ -open sets U_1, U_2 of (Y, σ) such that $y_1 = f(p) \in U_1$, $y_2 = g(p) \in U_2$ and $U_1 \cap U_2 = \phi$. Therefore $p \in f^{-1}(U_1)$, $p \in g^{-1}(U_2)$, so that $p \in f^{-1}(U_1) \cap g^{-1}(U_2) = W$ (say). Since f and g are ρ -irresolute maps, $f^{-1}(U_1)$ and $g^{-1}(U_2)$ are ρ -open sets of (X, τ) and by assumption W is a ρ -open set containing p . We will now show that $W \subset X - A$. Let $y \in W$, since $U_1 \cap U_2 = \phi$, then $f(y) \neq g(y)$ and hence from the definition of $X - A$, $y \in X - A$. Therefore $W \subset X - A$, which means $X - A$ is a ρ -open set. It follows that A is a ρ -closed subset of (X, τ) . ■

We define another new class of maps called ρ^* -closed maps.

Definition 5.21. A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be a ρ^* -closed map if the image $f(A)$ is ρ -closed in (Y, σ) for every ρ -closed set A in (X, τ) .

Example 5.22. As in Example 3.2(i), f is a ρ^* -closed map.

Theorem 5.23. If $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\tilde{g}$ -irresolute and M -preclosed functions then f is a ρ^* -closed map.

Proof. By Theorem 2.4(iii), the theorem follows. ■

Theorem 5.24. Every ρ -closed map is a ρ^* -closed map if (X, τ) is ρ - T_S space.

Proof. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a ρ -closed map and V be a ρ -closed set in (X, τ) . Since (X, τ) is a ρ - T_S space and by Theorem 2.4(ii), V is a closed set in (X, τ) and since f is ρ -closed, then $f(V)$ is a ρ -closed set in (Y, σ) . Hence f is a ρ^* -closed map. ■

We next introduce a new class of maps called ρ^* -homeomorphisms. This class of maps is closed under composition of maps.

Definition 5.25. A bijection $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be ρ^* -homeomorphism if both f and f^{-1} are ρ -irresolute.

Example 5.26. Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\}$, $\sigma = \{\phi, \{b\}, Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = f(b) = a$, $f(c) = c$. Then f is a ρ^* -homeomorphism.

Theorem 5.27. A bijective ρ -irresolute map of a ρ -compact space X onto a ρ -Hausdorff space Y is a ρ^* -homeomorphism.

Proof. Let (X, τ) be a ρ -compact space and (Y, σ) be a ρ -Hausdorff space. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a bijective ρ -irresolute map. We have to show that f is a ρ^* -homeomorphism. We need only to show that f^{-1} is a ρ -irresolute map. Let F be a ρ -closed set in (X, τ) . Since (X, τ) is a ρ -compact space, then by Theorem 2.4(v), F is a ρ -compact subset of (X, τ) . Since f is ρ -irresolute and by Theorem 2.4(vi), $f(F)$ is a ρ -compact subset of (Y, σ) . Since (Y, σ) is a ρ -Hausdorff space and assume that $\rho O(X, \tau)$ be closed under finite intersections, then by Theorem 5.19, $f(F)$ is a ρ -closed set in (Y, σ) . Hence f is a ρ^* -homeomorphism. ■

Theorem 5.28. *If $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \eta)$ are ρ^* -homeomorphisms then their composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is also ρ^* -homeomorphism.*

Proof. Let V be a ρ -closed set in (Z, η) . Now $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$. Since g is a ρ^* -homeomorphism, then $g^{-1}(V)$ is a ρ -closed set in (Y, σ) and since f is a ρ^* -homeomorphism, then $f^{-1}(g^{-1}(V))$ is a ρ -closed set in (X, τ) . Therefore $g \circ f$ is ρ -irresolute. Also for a ρ -closed set F in (X, τ) , we have $(g \circ f)(F) = g(f(F))$. Since f is a ρ^* -homeomorphism, then $f(F)$ is a ρ -closed set in (Y, σ) and since g is a ρ^* -homeomorphism, then $g(f(F))$ is a ρ -closed set in (Z, η) . Therefore $(g \circ f)^{-1}$ is ρ -irresolute. Hence $g \circ f$ is a ρ^* -homeomorphism. ■

Let Γ be a collection of all topological spaces. We introduce a relation, say “ $\equiv_{\rho^*}$ ”, into the family Γ as follows: for two elements (X, τ) and (Y, σ) of Γ , (X, τ) is ρ^* -homeomorphic to (Y, σ) , say $(X, \tau) \equiv_{\rho^*} (Y, \sigma)$, if there exists a ρ^* -homeomorphism $f : (X, \tau) \rightarrow (Y, \sigma)$. Then, we have the following theorem on the relation “ $\equiv_{\rho^*}$ ”.

Theorem 5.29. *The relation $\equiv_{\rho^*}$ above is an equivalence relation in the collection of all topological spaces Γ .*

Proof. (i) For any element $(X, \tau) \in \Gamma$, $(X, \tau) \equiv_{\rho^*} (X, \tau)$ holds. Indeed, the identity function $I_x : (X, \tau) \rightarrow (X, \tau)$ is a ρ^* -homeomorphism.

(ii) Suppose $(X, \tau) \equiv_{\rho^*} (Y, \sigma)$, where (X, τ) and $(Y, \sigma) \in \Gamma$. Then, there exists a ρ^* -homeomorphism $f : (X, \tau) \rightarrow (Y, \sigma)$. By definition it is seen that $f^{-1} : (Y, \sigma) \rightarrow (X, \tau)$ is a ρ^* -homeomorphism and $(Y, \sigma) \equiv_{\rho^*} (X, \tau)$.

(iii) Suppose that $(X, \tau) \equiv_{\rho^*} (Y, \sigma)$ and $(Y, \sigma) \equiv_{\rho^*} (Z, \eta)$, where $(X, \tau), (Y, \sigma)$ and $(Z, \eta) \in \Gamma$. By Theorem 5.28, it is shown that $(X, \tau) \equiv_{\rho^*} (Z, \eta)$. ■

We denote the family of all ρ^* -homeomorphism of a topological space (X, τ) onto itself by $\rho^*-h(X, \tau)$.

Theorem 5.30. *The set $\rho^*-h(X, \tau)$ is a group under the composition of maps.*

Proof. Define a binary operation $\Upsilon : \rho^*-h(X, \tau) \times \rho^*-h(X, \tau) \rightarrow \rho^*-h(X, \tau)$ by $\Upsilon(f, g) = g \circ f$ (the composition of f and g) for all $f, g \in \rho^*-h(X, \tau)$. Then by Theorem 5.28, $g \circ f \in \rho^*-h(X, \tau)$. We know that the composition of maps is associative and the identity map $I : (X, \tau) \rightarrow (X, \tau)$ belonging to $\rho^*-h(X, \tau)$ serves as the identity element. If $f \in \rho^*-h(X, \tau)$ then $f^{-1} \in \rho^*-h(X, \tau)$ such that $f \circ f^{-1} = f^{-1} \circ f = I$ and so inverse exists for each element of $\rho^*-h(X, \tau)$. Therefore $(\rho^*-h(X, \tau), \circ)$ is a group under the operation of composition of maps. ■

Theorem 5.31. *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a ρ^* -homeomorphism. Then f induces an isomorphism from the group $\rho^*-h(X, \tau)$ onto the group $\rho^*-h(Y, \sigma)$.*

Proof. We define a map $\kappa_f : \rho^*-h(X, \tau) \rightarrow \rho^*-h(Y, \sigma)$ by $\kappa_f(\theta) = f \circ \theta \circ f^{-1}$, for every $\theta \in \rho^*-h(X, \tau)$, where f is a given map. By Theorem 5.28, κ_f is well defined in general, because $f \circ \theta \circ f^{-1}$ is a ρ^* -homeomorphism for every ρ^* -homeomorphism $\theta : (X, \tau) \rightarrow (Y, \sigma)$. We have to show that κ_f is a bijective

homomorphism. Bijection of κ_f is clear. Further, for all $\theta_1, \theta_2 \in \rho^*-h(X, \tau)$, $\kappa_f(\theta_1 \circ \theta_2) = f \circ (\theta_1 \circ \theta_2) \circ f^{-1} = (f \circ \theta_1 \circ f^{-1}) \circ (f \circ \theta_2 \circ f^{-1}) = \kappa_f(\theta_1) \circ \kappa_f(\theta_2)$. Therefore, κ_f is a homomorphism and so it is an isomorphism induced by f . ■

Converse of this theorem need not be true as seen from the following example. That is, there exists a map $f : (X, \tau) \rightarrow (Y, \sigma)$ which induces an isomorphism $\kappa_f : \rho^*-h(X, \tau) \rightarrow \rho^*-h(Y, \sigma)$, but not ρ^* -homeomorphism.

Example 5.32. Let $X=Y=\{a, b, c\}$ with $\tau=\{\phi, \{a, b\}, X\}$ and $\sigma = \{\phi, \{b, c\}, Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity function. Then f is not a ρ^* -homeomorphism. But the induced homeomorphism $\kappa_f : \rho^*-h(X, \tau) \rightarrow \rho^*-h(Y, \sigma)$ is an isomorphism, because $\rho^*-h(X, \tau) = \{I_x, \theta_c\} \cong \rho^*-h(Y, \sigma) = \{I_y, \theta_a\}$.

Corollary 5.33.

- (i) If $\rho^*-h(X, \tau) \not\cong \rho^*-h(Y, \sigma)$, then $(X, \tau) \not\cong (Y, \sigma)$.
- (ii) Let $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \eta)$ be two ρ^* -homeomorphisms. Then, $\kappa_{g \circ f} = \kappa_g \circ \kappa_f$ holds and $\kappa_{I_x} = 1$ is the identity isomorphism.

Proof. (i) (resp.(ii)) It is obvious from Theorem 5.31 (resp. the definition of κ_f etc). ■

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GENERALIZED QUASI-COINCIDENCE IN FUZZY SUB-HYPERMODULES

R. Ameri

*School of Mathematics,
Statistics and Computer Sciences
University of Tehran
P.O. Box 14155-6455, Tehran
Iran
e-mail: rameri@ut.ac.ir*

H. Hedayati

*Department of Mathematics
Faculty of Basic Science
Babol University of Technology
Babol
Iran
e-mail: hedayati143@yahoo.com*

M. Norouzi

*Department of Mathematics
Faculty of Mathematical Sciences
University of Mazandaran
Babolsar
Iran
e-mail: m.norouzi@umz.ac.ir*

Abstract. We consider a general form of the notion of quasi-coincidence of a fuzzy point with a fuzzy set. We introduce the notions of $(\in, q_k)$ -fuzzy sub-hypermodule and $(\in, \in \vee q_k)$ -fuzzy sub-hypermodule of a given hypermodule, and investigate several properties of these notions.

Keywords: $(\in, q_k)$ -fuzzy sub-hypermodule, $(\in, \in \vee q_k)$ -fuzzy sub-hypermodule, $(\in \vee q_k)$ -level sub-hypermodule.

1. Introduction

The concept of hyperstructure was first introduced by Marty at the eighth Congress of Scandinavian Mathematicians in 1934 when he defined hypergroups and started to analyse their properties ([14]). Because there are extensive applications in many branches of mathematics and applied sciences, the theory of algebraic hyperstructures (or hypersystems) has nowadays become a well-established branch

in algebraic theory. Later on, people have developed the semi-hypergroups, which are the simplest algebraic hyperstructures having closure and associativity properties. A comprehensive review of the theory of hyperstructures can be found in [6], [7] and [20].

The theory of fuzzy sets, proposed by Zadeh [22] in 1965, has provided a useful mathematical tool for describing the behavior of systems that are too complex or illdefined to admit precise mathematical analysis by classical methods and tools. Murali [17] proposed a definition of a fuzzy point belonging to fuzzy subset under a natural equivalence on fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [18], played a vital role to generate some different types of fuzzy subsets. It is worth pointing out that Bhakat and Das [3,4] initiated the concept of (α, β) -fuzzy subgroup by using the “belong to” relation $(\in)$ and “quasi-coincident with” relation (q) between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in \vee q)$ -fuzzy subgroup. In fact, these notions were originally introduced by Pu and Liu in [18]. The $(\in, \in \vee q)$ -fuzzy subgroup is an important generalization of the fuzzy subgroups defined by Rosenfeld [19] and their structure was described by Bhakat and Das in [4].

In recent years, there has been considerable interest in the relationships between the fuzzy sets and the algebraic hyperstructures. Fuzzy hyperstructures are a direct generalization of the concept of fuzzy algebras (fuzzy groups, fuzzy rings, fuzzy modules etc). This approach can be extended to fuzzy hypergroups. For example, given a crisp hypergroup $(H, \circ)$ and a fuzzy subset μ in H , then we say that μ is a fuzzy subhypergroup of H if every level set of μ say μ_t , is a (crisp) subhypergroup of H . This was initiated by Zahedi et al. [24] and continued by Ameri and Hedayati [2], Davvaz and Corsini [8], Davvaz et al. [9], Zhan et al. [25], [26] and so on.

In this paper, we consider more general form of the notion of quasi-coincidence of a fuzzy point with a fuzzy set. Thus, this is a natural generalization of the fuzzy sub-hypermodules. As a generalization of $(\in, \in \vee q)$ -fuzzy sub-hypermodules, we introduce the notions of $(\in, q_k)$ -fuzzy sub-hypermodules and $(\in, \in \vee q_k)$ -fuzzy sub-hypermodules and investigate some of the interesting properties of $(\in, \in \vee q_k)$ -fuzzy sub-hypermodules. Finally, we consider $(\in \vee q_k)$ -level sub-hypermodule of a fuzzy set, and prove some related results.

2. Preliminaries

A *hyperstructure* is a non-empty set H together with a mapping “ $\circ$ ” : $H \times H \longrightarrow P^*(H)$, where $P^*(H)$ is the set of all the non-empty subsets of H . If $x \in H$ and $A, B \in P^*(H)$, then by $A \circ B$, $A \circ x$ and $x \circ B$, we mean $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$, $A \circ x = A \circ \{x\}$ and $x \circ B = \{x\} \circ B$, respectively (For more details see [6], [7]).

Now, we call hyperstructure $(H, +)$ a *canonical hypergroup* ([16]) if the following axioms are satisfied:

- (1) For every $x, y, z \in H$, $x + (y + z) = (x + y) + z$;
- (2) For every $x, y \in H$, $x + y = y + x$;
- (3) There exists an element $0 \in H$, such that $0 + x = x$, for all $x \in H$;
- (4) For every $x \in H$, there exists a unique element $x' \in H$, such that $0 \in x + x'$ (we call the element x' the *opposite* of x and it is denoted by $-x$).

Definition 2.1. [26] A *hyperring* is an algebraic hyperstructure $(R, +, \cdot)$ which satisfies the following axioms:

- (i) $(R, +)$ is a canonical hypergroup;
- (ii) $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element;
- (iii) The multiplication is distributive with respect to the hyperoperation “+”.

Let $(R, +, \cdot)$ be a hyperring and A a non-empty subset of R . Then A is called a *sub-hyperring* of R if $(A, +, \cdot)$ itself is a hyperring.

Definition 2.2. [1] A non-empty set M is called a *left hypermodule* over a hyperring R (R -hypermodule) if $(M, +)$ is a canonical hypergroup and there exists the map “ $\cdot$ ” : $R \times M \longrightarrow P^*(M)$ by $(r, m) \mapsto r \cdot m$ such that, for all $r_1, r_2 \in R$ and $m_1, m_2 \in M$, we have

- (i) $r_1(m_1 + m_2) = r_1m_1 + r_1m_2$;
- (ii) $(r_1 + r_2)m_1 = r_1m_1 + r_2m_1$;
- (iii) $(r_1r_2)m_1 = r_1(r_2m_1)$.

Example 2.3. Let M be an R -module over a unitary ring R and G a subgroup of the multiplicative semigroup of R satisfying the condition $aGbG = abG$, for every $a, b \in R$. Note that this condition is equivalent to the normality of G only if $R \setminus \{0\}$ is a group which appears in the case of division rings. Now, we introduce in M an equivalence relation “ $\sim$ ” defined as follows:

$$x \sim y \iff x = ty, \quad t \in G.$$

Let $\overline{M}$ be the set of the equivalence classes of M with respect to $\sim$. Then, a hyperoperation “ $\oplus$ ” can be endowed in $\overline{M}$ by

$$\overline{x} \oplus \overline{y} = \{\overline{w} \in \overline{M} \mid \overline{w} \subseteq \overline{x} + \overline{y}\},$$

i.e., $\overline{x} \oplus \overline{y}$ consists of all the classes $\overline{w} \in \overline{M}$ which are contained in the set-wise sum of $\overline{x}$ and $\overline{y}$. Thus, $(\overline{M}, \oplus)$ becomes a canonical hypergroup. Now, we suppose that $\overline{R}$ is the quotient hyperring of R by G . Consider an external composition from $\overline{R} \times \overline{M}$ to $\overline{M}$ defined by

$$\overline{a} \cdot \overline{x} = \overline{ax} \quad \text{for all } \overline{a} \in \overline{R}, \overline{x} \in \overline{M}.$$

Then the above composition satisfies the axioms of the hypermodule and so $\overline{M}$ becomes a hypermodule over hyperring $\overline{R}$. It was proved by Massouros ([15]) that this hypermodule is strongly related with the analytic projective geometries as well as with the Euclidean spherical geometries (for details see [15]).

In what follows, all hypermodules are left hypermodules. Recall that a non-empty subset A of a hypermodule M is a *sub-hypermodule* if $(A, +, \cdot)$ is a hypermodule.

Definition 2.4. [26] A fuzzy subset μ of a hypermodule M over a hyperring R is a *fuzzy sub-hypermodule* of M if

- (1) $\inf_{z \in x+y} \mu(z) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in M$;
- (2) $\mu(-x) \geq \mu(x)$, for all $x \in M$;
- (3) $\mu(ax) \geq \mu(x)$, for all $a \in R$ and $x \in M$.

If μ is a fuzzy sub-hypermodule of M , clearly we have

$$\mu(-x) = \mu(x), \quad \inf_{z \in x-y} \mu(z) \geq \min\{\mu(x), \mu(y)\}, \quad \text{for all } x, y \in M.$$

Let M be an R -hypermodule. Then, for a fuzzy subset μ of M , the level subset μ_t is defined by

$$\mu_t = \{x \in M \mid \mu(x) \geq t\}; \quad t \in [0, 1].$$

Theorem 2.5. [26] Let μ be a fuzzy subset of an R -hypermodule M . Then the following are equivalent:

- (1) μ is a fuzzy sub-hypermodule of M ;
- (2) Each non-empty level subset of μ is a sub-hypermodule of M .

A fuzzy set μ of a hypermodule M of the form

$$\mu(y) = \begin{cases} t (\neq 0), & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases}$$

is said to be a *fuzzy point* with *support* x and *value* t and is denoted by $(x)_t$. A fuzzy point $(x)_t$ is said to "belong to" (resp. be *quasi-coincident with*) a fuzzy set μ , written as $(x)_t \in \mu$ (resp. $(x)_t q\mu$) if $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$). If $(x)_t \in \mu$ or $(x)_t q\mu$, then we write $(x)_t \in \vee q\mu$. If $(x)_t \in \mu$ and $(x)_t q\mu$, then we write $(x)_t \in \wedge q\mu$. The symbol $\overline{\in \vee q}$ means that $\in \vee q$ does not hold.

3. Generalization of $(\in, \in \vee q)$ -fuzzy sub-hypermodules

Let k denote an arbitrary element of $(0, 1]$ unless otherwise specified. To say that $(x)_t q_k \mu$, we mean $\mu(x) + t + k > 1$. To say that $(x)_t \in \vee q_k \mu$, we mean $(x)_t \in \mu$ or $(x)_t q_k \mu$. The symbol $\overline{\in \vee q_k}$ means that $\in \vee q_k$ does not hold.

Definition 3.1. A fuzzy set μ of M is called an $(\in, \in \vee q_k)$ -fuzzy sub-hypermodule of M if, for all $t_1, t_2 \in [0, 1)$, $a \in R$ and $x, y \in M$

- (1) $(x)_{t_1} \in \mu, (y)_{t_2} \in \mu \implies (z)_{\min(t_1, t_2)} \in \vee q_k \mu$, for all $z \in x + y$;
- (2) $(x)_{t_1} \in \mu \implies (-x)_{t_1} \in \vee q_k \mu$;
- (3) $(x)_{t_1} \in \mu \implies (ax)_{t_1} \in \vee q_k \mu$.

Similarly, $(\in, \in \vee q)$ -fuzzy sub-hypermodule, $(\in, q)$ -fuzzy sub-hypermodule and $(\in, \in)$ -fuzzy sub-hypermodule are defined (see [26]).

Theorem 3.2. A fuzzy set μ of hypermodule M over hyperring R is an $(\in, \in \vee q_k)$ -fuzzy sub-hypermodule of M if and only if, for all $x, y \in M$ and $a \in R$ the following conditions hold:

- (1) $\mu(z) \geq \min \left\{ \mu(x), \mu(y), \frac{1-k}{2} \right\}$, for all $z \in x + y$;
- (2) $\mu(-x) \geq \min \left\{ \mu(x), \frac{1-k}{2} \right\}$;
- (3) $\mu(ax) \geq \min \left\{ \mu(x), \frac{1-k}{2} \right\}$.

Proof. Let μ be an $(\in, \in \vee q_k)$ -fuzzy sub-hypermodule of M . Assume that (1) is not valid. Then there exist $a, b \in M$ and $z \in a + b$ such that

$$\mu(z) < \min \left\{ \mu(a), \mu(b), \frac{1-k}{2} \right\}.$$

If $\min \{ \mu(a), \mu(b) \} < \frac{1-k}{2}$, then

$$\mu(z) < \min \{ \mu(a), \mu(b) \}.$$

Hence $\mu(z) < t \leq \min \{ \mu(a), \mu(b) \}$ for some $t \in (0, 1)$. It follows that $(a)_t \in \mu$ and $(b)_t \in \mu$, but $(z)_t \notin \mu$. Moreover, $\mu(z) + t < 2t < \frac{1-k}{2} + \frac{1-k}{2} = 1-k$, and so $(z)_t \notin \overline{q_k} \mu$. Consequently, $(z)_t \notin \vee \overline{q_k} \mu$, this is a contradiction.

If $\min \{ \mu(a), \mu(b) \} \geq \frac{1-k}{2}$, then

$$\mu(a) \geq \frac{1-k}{2}, \mu(b) \geq \frac{1-k}{2} \text{ and } \mu(z) < \frac{1-k}{2}.$$

Thus $(a)_{\frac{1-k}{2}} \in \mu$ and $(b)_{\frac{1-k}{2}} \in \mu$, but $(z)_{\frac{1-k}{2}} \notin \mu$. Also,

$$\mu(z) + \frac{1-k}{2} < \frac{1-k}{2} + \frac{1-k}{2} = 1-k,$$

i.e., $(z)_{\frac{1-k}{2}} \overline{q_k}\mu$. Hence $(z)_{\frac{1-k}{2}} \in \overline{\vee q_k}\mu$, again, a contradiction. Therefore (1) is valid.

Assume that (2) is not valid. Then there exists $a \in M$, such that

$$\mu(-a) < \min \left\{ \mu(a), \frac{1-k}{2} \right\}.$$

If $\mu(a) < \frac{1-k}{2}$, then $\mu(-a) < \mu(a)$. Hence $\mu(-a) < t \leq \mu(a)$ for some $t \in (0, 1)$. It follows that $(a)_t \in \mu$, but $(-a)_t \notin \mu$. Moreover, $\mu(-a) + t < 2t < 1 - k$ and so $(-a)_t \overline{q_k}\mu$. Consequently $(-a)_t \in \overline{\vee q_k}\mu$, this is a contradiction. If $\mu(a) \geq \frac{1-k}{2}$, then $\mu(-a) < \frac{1-k}{2}$. Thus $(a)_{\frac{1-k}{2}} \in \mu$, but $(-a)_{\frac{1-k}{2}} \notin \mu$. Also,

$$\mu(-a) + \frac{1-k}{2} < \frac{1-k}{2} + \frac{1-k}{2} = 1 - k,$$

i.e., $(-a)_{\frac{1-k}{2}} \overline{q_k}\mu$. Hence $(-a)_{\frac{1-k}{2}} \in \overline{\vee q_k}\mu$, again, a contradiction. Therefore, (2) is valid. Similarly, (3) is valid.

Conversely, suppose that (1), (2) and (3) are valid. Let $x, y \in M$ and $t_1, t_2 \in (0, 1]$ be such that $(x)_{t_1} \in \mu$ and $(y)_{t_2} \in \mu$. Then, for all $z \in x + y$,

$$\mu(z) \geq \min \left\{ \mu(x), \mu(y), \frac{1-k}{2} \right\} \geq \min \left\{ t_1, t_2, \frac{1-k}{2} \right\}.$$

Assume that $t_1 \leq \frac{1-k}{2}$ or $t_2 \leq \frac{1-k}{2}$. Then $\mu(z) \geq \min(t_1, t_2)$, which implies that $(z)_{\min(t_1, t_2)} \in \mu$. Now, suppose that $t_1 > \frac{1-k}{2}$ and $t_2 > \frac{1-k}{2}$. Then $\mu(z) \geq \frac{1-k}{2}$ and thus

$$\mu(z) + \min(t_1, t_2) > \frac{1-k}{2} + \frac{1-k}{2} = 1 - k,$$

i.e., $(z)_{\min(t_1, t_2)} q_k\mu$. Hence $(z)_{\min(t_1, t_2)} \in \vee q_k\mu$.

Let $x \in M$ and $t \in (0, 1]$ be such that $(x)_t \in \mu$. Then

$$\mu(-x) \geq \min \left\{ t, \frac{1-k}{2} \right\}.$$

Assume that $t \leq \frac{1-k}{2}$. Then $\mu(-x) \geq t$, which implies that $(-x)_t \in \mu$. Now, suppose that $t > \frac{1-k}{2}$. Then $\mu(-x) \geq \frac{1-k}{2}$, and so

$$\mu(-x) + t > \frac{1-k}{2} + \frac{1-k}{2} = 1 - k,$$

i.e., $(-x)_t q_k\mu$. Hence $(-x)_t \in \vee q_k\mu$. Similarly, $(x)_t \in \mu$ implies $(ax)_t \in \vee q_k\mu$ for all $a \in R$. Therefore μ is an $(\in, \in \vee q_k)$ -fuzzy sub-hypermodule of M . ■

Corollary 3.3. *A fuzzy set μ of M is an $(\in, \in \vee q)$ -fuzzy sub-hypermodule of M if only if for all $x, y \in M$ and $a \in R$ the following conditions hold:*

- (1) $\mu(z) \geq \min\{\mu(x), \mu(y), 0.5\}$, for all $z \in x + y$;
- (2) $\mu(-x) \geq \min\{\mu(x), 0.5\}$;
- (3) $\mu(ax) \geq \min\{\mu(x), 0.5\}$.

Proof. It follows taking $k = 0$ in Theorem 3.2. ■

Theorem 3.4. *Let μ be a fuzzy set of M . Then μ is an $(\in, \in \vee q_k)$ -fuzzy sub-hypermodule of M if only if the level subset $\mu_t = \{x \in M \mid \mu(x) \geq t\}$ is a sub-hypermodule of M for all $t \in \left(0, \frac{1-k}{2}\right]$.*

Proof. Assume that μ is an $(\in, \in \vee q_k)$ -fuzzy sub-hypermodule of M . Let $t \in \left(0, \frac{1-k}{2}\right]$ and $x, y \in \mu_t$. Then $(x)_t \in \mu$ and $(y)_t \in \mu$. Let $z \notin \mu_t$ for some $z \in x + y$. It follows that $(z)_t \notin \mu$. Moreover, $\mu(z) + t < 2t \leq \frac{1-k}{2} + \frac{1-k}{2} = 1-k$, and so $(z)_t \notin \overline{q_k}\mu$. Consequently, $(z)_t \notin \overline{\vee q_k}\mu$, this is a contradiction. Hence, $z \in \mu_t$ for all $z \in x + y$. Therefore $x + y \subseteq \mu_t$.

Let $x \in \mu_t$. Then $\mu(x) \geq t$. It follows from Theorem 3.2 that

$$\mu(-x) \geq \min\left\{\mu(x), \frac{1-k}{2}\right\} \geq \min\left\{t, \frac{1-k}{2}\right\} = t,$$

so that $-x \in \mu_t$. Consequently $(\mu_t, +)$ is a sub-hypergroup of $(M, +)$. Similarly, for all $x \in \mu_t$ and $a \in R$, we have $ax \in \mu_t$, by Theorem 3.2. Therefore μ_t is a sub-hypermodule of M .

Conversely, suppose that μ_t is a sub-hypermodule of M for all $t \in \left(0, \frac{1-k}{2}\right]$. Let (1) of Theorem 3.2 be not valid, then there exist $a, b \in M$ and $z \in a + b$ such that

$$\mu(z) < \min\left\{\mu(a), \mu(b), \frac{1-k}{2}\right\}.$$

Hence we can take $t \in (0, 1)$ such that

$$\mu(z) < t \leq \min\left\{\mu(a), \mu(b), \frac{1-k}{2}\right\}.$$

Then $t \in \left(0, \frac{1-k}{2}\right]$ and $a, b \in \mu_t$. Since μ_t is a sub-hypermodule of M , it follows that $z \in \mu_t$, so that $\mu(z) \geq t$. This is a contradiction. Therefore (1) of Theorem 3.2 is valid. Similarly, (2) and (3) of Theorem 3.2 are valid. Consequently, μ is an $(\in, \in \vee q_k)$ -fuzzy sub-hypermodule of M by Theorem 3.2. ■

Corollary 3.5. *Let μ be a fuzzy set of M . Then μ is an $(\in, \in \vee q)$ -fuzzy sub-hypermodule of M if and only if the level subset $\mu_t = \{x \in M \mid \mu(x) \geq t\}$ is a sub-hypermodule of M for all $t \in (0, 0.5]$.*

Proof. In Theorem 3.4, taking $k = 0$. ■

Example 3.6. Consider the set $M = \{0, a, b, c\}$ with the following table:

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Hence $(M, +)$ is a commutative group. Since every commutative group is a $\mathbb{Z}$ -module therefore M is a $\mathbb{Z}$ -module ($\mathbb{Z}$ is the set of all integers). By Example 2.3, consider $\overline{M} = \{\overline{0}, \overline{a}, \overline{b}, \overline{c}\}$ with the following table:

$\oplus$	$\overline{0}$	$\overline{a}$	$\overline{b}$	$\overline{c}$
$\overline{0}$	$\overline{0}$	$\overline{a}$	$\overline{b}$	$\overline{c}$
$\overline{a}$	$\overline{a}$	$\overline{0}$	$\overline{c}$	$\overline{b}$
$\overline{b}$	$\overline{b}$	$\overline{c}$	$\overline{0}$	$\overline{a}$
$\overline{c}$	$\overline{c}$	$\overline{b}$	$\overline{a}$	$\overline{0}$

Thus $(\overline{M}, \oplus)$ is a canonical hypergroup. Therefore, $(\overline{M}, \oplus, \cdot)$ is a hypermodule over hyperring $\overline{\mathbb{Z}}$. Let μ be a fuzzy set of hypermodule $\overline{M} = \{\overline{0}, \overline{a}, \overline{b}, \overline{c}\}$ defined by $\mu(\overline{0}) = 0.5$, $\mu(\overline{a}) = 0.8$, and $\mu(\overline{b}) = \mu(\overline{c}) = 0.3$.

- (1) If $k = 0.4$, then $\mu_t = \overline{M}$ for all $t \in (0, 0.3]$. Hence μ is an $(\in, \in \vee q_{0.4})$ -fuzzy sub-hypermodule of $\overline{M}$ by Theorem 3.4.
- (2) If $k = 0.2$, then

$$\mu_t = \begin{cases} \overline{M}, & \text{if } t \in (0, 0.3] \\ \{\overline{0}, \overline{a}\}, & \text{if } t \in (0.3, 0.4]. \end{cases}$$

Since $\overline{M}$ and $\{\overline{0}, \overline{a}\}$ are sub-hypermodule of $\overline{M}$, μ is an $(\in, \in \vee q_{0.2})$ -fuzzy sub-hypermodule of $\overline{M}$ by Theorem 3.4.

Example 3.7. Let $\overline{M}$ be the hypermodule given in Example 3.6. Let μ be a fuzzy set of hypermodule $\overline{M}$ defined by $\mu(\overline{0}) = 0.45$, $\mu(\overline{a}) = \mu(\overline{c}) = 0.4$, and $\mu(\overline{b}) = 0.48$. If $k = 0.04$, then

$$\mu_t = \begin{cases} \overline{M}, & \text{if } t \in (0, 0.4] \\ \{\overline{0}, \overline{b}\}, & \text{if } t \in (0.4, 0.45] \\ \{\overline{b}\}, & \text{if } t \in (0.45, 0.48]. \end{cases}$$

Note that μ_t is not a sub-hypermodule of $\overline{M}$ for $t \in (0.45, 0.48]$. Hence μ is not an $(\in, \in \vee_{q_{0.04}})$ -fuzzy sub-hypermodule of $\overline{M}$.

Theorem 3.8. *Every $(\in, \in)$ -fuzzy sub-hypermodule of M is an $(\in, \in \vee_{q_k})$ -fuzzy sub-hypermodule of M .*

Proof. Let μ be an $(\in, \in)$ -fuzzy sub-hypermodule of M . Let (1) of Theorem 3.2 is not valid, then there exist $a, b \in M$ and $z \in a + b$ such that

$$\mu(z) < \min \left\{ \mu(a), \mu(b), \frac{1-k}{2} \right\}.$$

Hence we can take $t \in (0, 1)$ such that

$$\mu(z) < t \leq \min \left\{ \mu(a), \mu(b), \frac{1-k}{2} \right\}.$$

Thus $(a)_t \in \mu$ and $(b)_t \in \mu$, then $\mu(z) \geq t$. This is a contradiction. Therefore (1) of Theorem 3.2 is valid. Similarly, (2) and (3) of Theorem 3.2 are valid. Consequently μ is an $(\in, \in \vee_{q_k})$ -fuzzy sub-hypermodule of M by Theorem 3.2. ■

The next corollary immediately follow from theorem 3.8, by taking $k = 0$.

Corollary 3.9. *Every $(\in, \in)$ -fuzzy sub-hypermodule of M is an $(\in, \in \vee_q)$ -fuzzy sub-hypermodule of M .*

The converse of Theorem 3.8 is not true as seen in the following example.

Example 3.10. Consider the $(\in, \in \vee_{q_{0.4}})$ -fuzzy sub-hypermodule of $\overline{M}$ given in Example 3.6. Then μ is not an $(\in, \in)$ -fuzzy sub-hypermodule of $\overline{M}$ since, $(\overline{a})_{0.62} \in \mu$ and $(\overline{a})_{0.66} \in \mu$, but $(\overline{0})_{\min(0.62, 0.66)} = (\overline{a} \oplus \overline{a})_{\min(0.62, 0.66)} \notin \mu$, because $\mu(\overline{0}) = 0.5 < 0.62$.

4. Properties of $(\in, \in \vee_{q_k})$ -fuzzy sub-hypermodules

Definition 4.1. A fuzzy set μ of M is called an $(\in, q_k)$ -fuzzy sub-hypermodule of M if, for all $t_1, t_2 \in (0, 1]$, $a \in R$ and $x, y \in M$

- (1) $(x)_{t_1} \in \mu, (y)_{t_2} \in \mu \implies (z)_{\min(t_1, t_2)} \in q_k \mu$, for all $z \in x + y$;
- (2) $(x)_t \in \mu \implies (-x)_t \in q_k \mu$;
- (3) $(x)_t \in \mu \implies (ax)_t \in q_k \mu$.

Theorem 4.2. *Every $(\in, q_k)$ -fuzzy sub-hypermodule of M is an $(\in, \in \vee_{q_k})$ -fuzzy sub-hypermodule of M .*

Proof. Let μ be an $(\in, q_k)$ -fuzzy sub-hypermodule of M . If (1) of Theorem 3.2 is not valid, then there exists $a, b \in M$ and $z \in a + b$ such that

$$\mu(z) < \min \left\{ \mu(a), \mu(b), \frac{1-k}{2} \right\}.$$

Hence we can take $t \in (0, 1)$ such that

$$\mu(z) < t \leq \min \left\{ \mu(a), \mu(b), \frac{1-k}{2} \right\}.$$

Thus $(a)_t \in \mu$ and $(b)_t \in \mu$, then $(z)_t \in \mu$, but $\mu(z) + t < 2t < \frac{1-k}{2} + \frac{1-k}{2} = 1-k$, this is a contradiction. Therefore (1) of Theorem 3.2 is valid. Similarly, (2) and (3) of Theorem 3.2 are valid. Consequently μ is an $(\in, \in \vee q_k)$ -fuzzy sub-hypermodule of M by Theorem 3.2. ■

Taking $k = 0$ in Theorem 4.2, we have the following corollary.

Corollary 4.3. *Every $(\in, q)$ -fuzzy sub-hypermodule of M is an $(\in, \in \vee q)$ -fuzzy sub-hypermodule of M .*

The next example shows that the converse of Theorem 4.2 does not hold.

Example 4.4. Consider the $(\in, \in \vee q_{0.2})$ -fuzzy sub-hypermodule of $\overline{M}$ given in Example 3.6. Note that $(\bar{a})_{0.4} \in \mu$ and $(\bar{b})_{0.25} \in \mu$, but $(\bar{a} \oplus \bar{b})_{\min(0.4, 0.25)} = (\bar{c})_{0.25} \notin \mu$, because $\mu(\bar{c}) + 0.25 + 0.2 < 1$. Therefore, μ is not an $(\in, q_{0.2})$ -fuzzy sub-hypermodule of $\overline{M}$.

Theorem 4.5. *Let M be a hypermodule. If $0 \leq k < r < 1$, then every $(\in, \in \vee q_k)$ -fuzzy sub-hypermodule of M is an $(\in, \in \vee q_r)$ -fuzzy sub-hypermodule of M .*

Proof. Straightforward. ■

The following example shows that if $0 \leq k < r < 1$, then an $(\in, \in \vee q_r)$ -fuzzy sub-hypermodule of M may not be an $(\in, \in \vee q_k)$ -fuzzy sub-hypermodule of M .

Example 4.6. Let $\overline{M}$ and μ be as in Example 3.7. If $r = 0.16$ and $k = 0.04$, then

$$\mu_t = \begin{cases} \overline{M}, & \text{if } t \in (0, 0.4] \\ \{\bar{0}, \bar{b}\}, & \text{if } t \in (0.4, 0.42]. \end{cases}$$

Since $\overline{M}$ and $\{\bar{0}, \bar{b}\}$ are sub-hypermodules of $\overline{M}$, then μ is an $(\in, \in \vee q_{0.16})$ -fuzzy sub-hypermodule of $\overline{M}$ by Theorem 3.4. But μ is not an $(\in, \in \vee q_{0.04})$ -fuzzy sub-hypermodule of $\overline{M}$ (see Example 3.7).

Let S be a subset of hypermodule M . Consider a fuzzy set μ_s in M where for all $x \in M$ defined by

$$\mu_s(x) = \begin{cases} 1, & \text{if } x \in S \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 4.7. *A non-empty subset S of hypermodule M is a sub-hypermodule of M if and only if the fuzzy set μ_s in M is an $(\in, \in \vee q_k)$ -fuzzy sub-hypermodule of M .*

Proof. Let S be a sub-hypermodule of M . Then $(\mu_s)_t$ is clearly a sub-hypermodule of M for all $t \in \left(0, \frac{1-k}{2}\right]$. Hence μ_s is an $(\in, \in \vee q_k)$ -fuzzy sub-hypermodule of M by Theorem 3.4.

Conversely, assume that μ_s is an $(\in, \in \vee q_k)$ -fuzzy sub-hypermodule of M . Let $x, y \in S$. Then for all $z \in x + y$

$$\mu_s(z) \geq \min \left\{ \mu_s(x), \mu_s(y), \frac{1-k}{2} \right\} = \min \left(1, \frac{1-k}{2} \right) = \frac{1-k}{2}.$$

Since $k \in [0, 1)$, $\mu_s(z) = 1$ for all $z \in x + y$ and so $z \in S$. Hence $x + y \subseteq S$.

Let $x \in S$. Then

$$\mu_s(-x) \geq \min \left\{ \mu_s(x), \frac{1-k}{2} \right\} = \min \left(1, \frac{1-k}{2} \right) = \frac{1-k}{2}.$$

Since $k \in [0, 1)$, $\mu_s(-x) = 1$ and so $-x \in S$. Similarly, $ax \in S$ for all $a \in R$. Therefore S is a sub-hypermodule of M . ■

Theorem 4.8. *Let S be a sub-hypermodule of M . Then for every $t \in \left(0, \frac{1-k}{2}\right]$, there exists an $(\in, \in \vee q_k)$ -fuzzy sub-hypermodule μ of M such that $\mu_t = S$.*

Proof. Let μ be a fuzzy set of hypermodule M defined by

$$\mu(x) = \begin{cases} t, & \text{if } x \in S \\ 0, & \text{otherwise} \end{cases}$$

for all $x \in M$, where $t \in \left(0, \frac{1-k}{2}\right]$. Obviously, $\mu_t = S$.

If (1) of Theorem 3.2 is not valid, then there exist $a, b \in M$ and $z \in a + b$ such that

$$\mu(z) < \min \left\{ \mu(a), \mu(b), \frac{1-k}{2} \right\}.$$

Hence we can take $t \in (0, 1)$ such that

$$\mu(z) < t \leq \min \left\{ \mu(a), \mu(b), \frac{1-k}{2} \right\}.$$

Since $\#Im(\mu) = 2$, it follows that $\mu(z) = 0$ and $\min \left\{ \mu(a), \mu(b), \frac{1-k}{2} \right\} = t$.

Hence $\mu(a) = t = \mu(b)$, and so $a, b \in S$. Since S is sub-hypermodule of M , $z \in S$. Thus $\mu(z) = t$, which is a contradiction. Therefore (1) of Theorem 3.2

is valid. Similarly, (2) and (3) of Theorem 3.2 are valid. Consequently μ is an $(\in, \in \vee q_k)$ -fuzzy sub-hypermodule of M by Theorem 3.2. ■

Taking $k = 0$ in Theorem 4.8, we have the following corollary.

Corollary 4.9. *Let S be a sub-hypermodule of M . Then for every $t \in (0, 0.5]$, there exists an $(\in, \in \vee q)$ -fuzzy sub-hypermodule μ of M such that $\mu_t = S$.*

Theorem 4.10. *Let μ be an $(\in, \in \vee q_k)$ -fuzzy sub-hypermodule of M such that $\mu(x) < \frac{1-k}{2}$ for all $x \in M$. Then μ is an $(\in, \in)$ -fuzzy sub-hypermodule of M .*

Proof. Let $(x)_{t_1} \in \mu$, $(y)_{t_2} \in \mu$ and $z \in x + y$. By assumption, we have

$$\mu(z) \geq \min \left\{ \mu(x), \mu(y), \frac{1-k}{2} \right\} = \min \{ \mu(x), \mu(y) \} \geq \min(t_1, t_2).$$

Thus $(z)_{\min(t_1, t_2)} \in \mu$.

Let $(x)_t \in \mu$. By assumption, we have

$$\mu(-x) \geq \min \left\{ \mu(x), \frac{1-k}{2} \right\} = \mu(x) \geq t.$$

Thus $(-x)_t \in \mu$. Similarly, for all $(x)_t \in \mu$ and $r \in R$ we have $(rx)_t \in \mu$. Therefore μ is an $(\in, \in)$ -fuzzy sub-hypermodule of M . ■

Taking $k = 0$ in Theorem 4.10, we have the following corollary.

Corollary 4.11. *Let μ be an $(\in, \in \vee q)$ -fuzzy sub-hypermodule of M such that $\mu(x) < 0.5$ for all $x \in M$. Then μ is an $(\in, \in)$ -fuzzy sub-hypermodule of M .*

Theorem 4.12. *Let $\{ \mu_i \mid i \in A \}$ be a family of $(\in, \in \vee q_k)$ -fuzzy sub-hypermodules of M . Then $\mu = \bigcap_{i \in A} \mu_i$ is an $(\in, \in \vee q_k)$ -fuzzy sub-hypermodule of M .*

Proof. Let $x, y \in M$ and $t_1, t_2 \in (0, 1]$ be such that $(x)_{t_1} \in \mu$ and $(y)_{t_2} \in \mu$. Assume that $(z)_{\min(t_1, t_2)} \notin \overline{\vee q_k \mu}$ for some $z \in x + y$. Then $\mu(z) < \min(t_1, t_2)$ and $\mu(z) + \min(t_1, t_2) \leq 1 - k$, which imply that

$$\mu(z) < \frac{1-k}{2}. \quad (*)$$

Let

$$\Psi_1 = \{ i \in A \mid (z)_{\min(t_1, t_2)} \in \mu_i \} \text{ and }$$

$$\Psi_2 = \{ i \in A \mid (z)_{\min(t_1, t_2)} \notin q_k \mu_i \} \cap \{ j \in A \mid (z)_{\min(t_1, t_2)} \notin \overline{\mu_j} \}.$$

Then, $A = \Psi_1 \cup \Psi_2$ and $\Psi_1 \cap \Psi_2 = \emptyset$. If $\Psi_2 = \emptyset$, then $(z)_{\min(t_1, t_2)} \in \mu_i$ for all $i \in A$, that is, $\mu_i(z) \geq \min(t_1, t_2)$ for all $i \in A$, which yields $\mu(z) \geq \min(t_1, t_2)$. This is a contradiction. Hence, $\Psi_2 \neq \emptyset$, and so for every $i \in \Psi_2$ we have $\mu_i(z) < \min(t_1, t_2)$

and $\mu_i(z) + \min(t_1, t_2) > 1 - k$. It follows that $\min(t_1, t_2) > \frac{1-k}{2}$. Now, $(x)_{t_1} \in \mu$ implies $\mu(x) \geq t_1$ and thus $\mu_i(x) \geq \mu(x) \geq t_1 \geq \min(t_1, t_2) > \frac{1-k}{2}$ for all $i \in A$. Similarly, $\mu_i(y) \geq \frac{1-k}{2}$ for all $i \in A$.

Next, suppose that $t = \mu_i(z) < \frac{1-k}{2}$. Taking $t < r < \frac{1-k}{2}$, we get $(x)_r \in \mu_i$ and $(y)_r \in \mu_i$, but $(z)_r \notin \bigvee q_k \mu_i$. This contradicts that μ_i is an $(\in, \in \bigvee q_k)$ -fuzzy sub-hypermodule of M . Hence, $\mu_i(z) \geq \frac{1-k}{2}$ for all $i \in A$, and so $\mu(z) \geq \frac{1-k}{2}$ which contradicts $(*)$. Therefore, $(z)_{\min(t_1, t_2)} \in \bigvee q_k \mu$ for all $z \in x + y$. Similarly, $(-x)_t \in \bigvee q_k \mu$ and $(ax)_t \in \bigvee q_k \mu$ for all $x \in M$ and $a \in R$. Consequently, $\mu = \bigcap_{i \in A} \mu_i$ is an $(\in, \in \bigvee q_k)$ -fuzzy sub-hypermodule of M . ■

Taking $k = 0$ in Theorem 4.12, we have the following corollary.

Corollary 4.13. *Let $\{\mu_i \mid i \in A\}$ be a family of $(\in, \in \bigvee q)$ -fuzzy sub-hypermodules of M . Then $\mu = \bigcap_{i \in A} \mu_i$ is an $(\in, \in \bigvee q)$ -fuzzy sub-hypermodule of M .*

The following example shows that there exists $k \in [0, 1)$ such that the union of two $(\in, \in \bigvee q_k)$ -fuzzy sub-hypermodules of M may not be an $(\in, \in \bigvee q_k)$ -fuzzy sub-hypermodule of M .

Example 4.14. Let $\overline{M} = \{\overline{0}, \overline{a}, \overline{b}, \overline{c}\}$ be a hypermodule given in Example 3.6 and μ an $(\in, \in \bigvee q_{0.2})$ -fuzzy sub-hypermodule of $\overline{M}$ described in Example 3.6. Let ν be a fuzzy set in $\overline{M}$ defined by $\nu(\overline{0}) = 0.4$, $\nu(\overline{a}) = \nu(\overline{c}) = 0.3$, and $\nu(\overline{b}) = 0.5$. Then

$$\nu_t = \begin{cases} \overline{M}, & \text{if } t \in (0, 0.3] \\ \{\overline{0}, \overline{b}\}, & \text{if } t \in (0.3, 0.4] \end{cases}$$

Since $\overline{M}$ and $\{\overline{0}, \overline{b}\}$ are sub-hypermodules of $\overline{M}$, so ν is an $(\in, \in \bigvee q_{0.2})$ -fuzzy sub-hypermodule of $\overline{M}$ by Theorem 3.4. The union $\mu \cup \nu$ of μ and ν is given by $\mu \cup \nu(\overline{0}) = 0.5$, $\mu \cup \nu(\overline{a}) = 0.8$, $\mu \cup \nu(\overline{b}) = 0.5$, and $\mu \cup \nu(\overline{c}) = 0.3$. Hence

$$(\mu \cup \nu)_t = \begin{cases} \overline{M}, & \text{if } t \in (0, 0.3] \\ \{\overline{0}, \overline{a}, \overline{b}\}, & \text{if } t \in (0.3, 0.4] \end{cases}$$

Since $\{\overline{0}, \overline{a}, \overline{b}\}$ is not a sub-hypermodule of M , it follows that $\mu \cup \nu$ is not an $(\in, \in \bigvee q_{0.2})$ -fuzzy sub-hypermodule of $\overline{M}$ by Theorem 3.4.

For any fuzzy set μ in M and $t \in (0, 1]$, we denote

$$\langle \mu \rangle_t = \{x \in M \mid (x)_t \in q_k \mu\} \quad \text{and} \quad [\mu]_t = \{x \in M \mid (x)_t \in \bigvee q_k \mu\}.$$

Obviously, $[\mu]_t = \mu_t \cup \langle \mu \rangle_t$.

Theorem 4.15. *Let μ be a fuzzy set in hypermodule of M . Then μ is an $(\in, \in \vee q_k)$ -fuzzy sub-hypermodule of M if and only if $[\mu]_t$ is a sub-hypermodule of M for all $t \in (0, 1]$.*

We call $[\mu]_t$ an $(\in \vee q_k)$ -level sub-hypermodule of μ .

Proof. Assume that μ is an $(\in, \in \vee q_k)$ -fuzzy sub-hypermodule of M and let $x, y \in [\mu]_t$ for $t \in (0, 1]$. Then $(x)_t \in \vee q_k \mu$ and $(y)_t \in \vee q_k \mu$, that is, $\mu(x) \geq t$ or $\mu(x) + t > 1 - k$, and $\mu(y) \geq t$ or $\mu(y) + t > 1 - k$. Using Theorem 3.2, for all $x, y \in M$, and $a \in R$ the following axioms are satisfied:

$$(1) \quad \mu(z) \geq \min \left\{ \mu(x), \mu(y), \frac{1-k}{2} \right\}; \quad \forall z \in x + y;$$

$$(2) \quad \mu(-x) \geq \min \left\{ \mu(x), \frac{1-k}{2} \right\};$$

$$(3) \quad \mu(ax) \geq \min \left\{ \mu(x), \frac{1-k}{2} \right\}.$$

Case 1. $\mu(x) \geq t$ and $\mu(y) \geq t$. If $t > \frac{1-k}{2}$, then for all $z \in x + y$

$$\mu(z) \geq \min \left\{ \mu(x), \mu(y), \frac{1-k}{2} \right\} = \frac{1-k}{2}.$$

Hence $\mu(z) + t > \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$, and so $(z)_t \in q_k \mu$. Moreover,

$$\mu(-x) \geq \min \left\{ \mu(x), \frac{1-k}{2} \right\} = \frac{1-k}{2}.$$

Thus $\mu(-x) + t > \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$, and so $(-x)_t \in q_k \mu$. Similarly, $(ax)_t \in q_k \mu$ for all $a \in R$. If $t \leq \frac{1-k}{2}$, then for all $z \in x + y$

$$\mu(z) \geq \min \left\{ \mu(x), \mu(y), \frac{1-k}{2} \right\} \geq t,$$

and thus $(z)_t \in \mu$. Similarly, $(-x)_t \in \mu$ and $(ax)_t \in \mu$ for all $a \in R$. Therefore $(z)_t \in \vee q_k \mu$, $(-x)_t \in \vee q_k \mu$, and $(ax)_t \in \vee q_k \mu$ for all $a \in R$. Hence $x + y \subseteq [\mu]_t$, $-x \in [\mu]_t$, and $ax \in [\mu]_t$ for all $a \in R$.

Case 2. $\mu(x) \geq t$ and $\mu(y) + t > 1 - k$. If $t > \frac{1-k}{2}$, then for all $z \in x + y$

$$\begin{aligned} \mu(z) &\geq \min \left\{ \mu(x), \mu(y), \frac{1-k}{2} \right\} \geq \min \left\{ \mu(y), \frac{1-k}{2} \right\} \\ &> \min \left(1 - k - t, \frac{1-k}{2} \right) = 1 - k - t \end{aligned}$$

and so $(z)_t q_k \mu$. Moreover,

$$\mu(-x) \geq \min \left\{ \mu(x), \frac{1-k}{2} \right\} \geq \min \left(t, \frac{1-k}{2} \right) = \frac{1-k}{2},$$

and thus $(-x)_t q_k \mu$. Similarly, $(ax)_t q_k \mu$ for all $a \in R$. If $t \leq \frac{1-k}{2}$, then for all $z \in x + y$

$$\mu(z) \geq \min \left\{ \mu(x), \mu(y), \frac{1-k}{2} \right\} \geq \min \left\{ t, 1-k-t, \frac{1-k}{2} \right\} = t.$$

Hence $(z)_t \in \mu$. Furthermore, $(-x)_t \in \mu$ and $(ax)_t \in \mu$ for all $a \in R$. Thus $(z)_t \in \vee q_k \mu$, $(-x)_t \in \vee q_k \mu$, and $(ax)_t \in \vee q_k \mu$ for all $a \in R$. Then $x + y \subseteq [\mu]_t$, $-x \in [\mu]_t$, and $ax \in [\mu]_t$ for all $a \in R$.

Case 3. $\mu(x) + t > 1 - k$ and $\mu(y) \geq t$. Similar to the case 2.

Case 4. $\mu(x) + t > 1 - k$ and $\mu(y) + t > 1 - k$. If $t > \frac{1-k}{2}$, then for all $z \in x + y$

$$\mu(z) \geq \min \left\{ \mu(x), \mu(y), \frac{1-k}{2} \right\} > \min \left(1-k-t, \frac{1-k}{2} \right) = 1-k-t.$$

Thus $(z)_t q_k \mu$. Moreover, $(-x)_t q_k \mu$ and $(ax)_t q_k \mu$ for all $a \in R$. If $t \leq \frac{1-k}{2}$, then for all $z \in x + y$

$$\mu(z) \geq \min \left\{ \mu(x), \mu(y), \frac{1-k}{2} \right\} \geq \min \left(1-k-t, \frac{1-k}{2} \right) = \frac{1-k}{2} \geq t,$$

and so $(z)_t \in \mu$. Similarly, $(-x)_t \in \mu$ and $(ax)_t \in \mu$ for all $a \in R$. Hence $(z)_t \in \vee q_k \mu$, $(-x)_t \in \vee q_k \mu$, and $(ax)_t \in \vee q_k \mu$ for all $a \in R$. Then $x + y \subseteq [\mu]_t$, $-x \in [\mu]_t$ and $ax \in [\mu]_t$ for all $a \in R$. Consequently, $[\mu]_t$ is a sub-hypermodule of M .

Conversely, let μ be a fuzzy set in hypermodule of M and $t \in (0, 1]$ be such that $[\mu]_t$ is a sub-hypermodule of M . Let there exists $a, b \in M$ and $z \in a + b$ such that

$$\mu(z) < t \leq \min \left\{ \mu(a), \mu(b), \frac{1-k}{2} \right\}$$

for some $t \in (0, 1)$. Then $a, b \in \mu_t \subseteq [\mu]_t$, which implies that $z \in [\mu]_t$ (since $a + b \subseteq [\mu]_t$). Hence $\mu(z) \geq t$ or $\mu(z) + t + k > 1$, a contradiction. Thus $\mu(z) \geq \min \left\{ \mu(a), \mu(b), \frac{1-k}{2} \right\}$ for all $a, b \in M$ and $z \in a + b$. Therefore (1) of Theorem 3.2 is valid. Similarly, (2) and (3) of Theorem 3.2 are valid. Consequently μ is an $(\in, \in \vee q_k)$ -fuzzy sub-hypermodule of M by Theorem 3.2. ■

A fuzzy set μ of hypermodule M is said to be *proper* if $Im(\mu)$ has at least two elements. Two fuzzy sets are said to be *equivalent* if they have same family of level subsets. Otherwise, they are said to be *non-equivalent*.

Theorem 4.16. *Let μ be an $(\in, \in \vee q_k)$ -fuzzy sub-hypermodule of M such that $\#\left\{\mu(x) \mid \mu(x) < \frac{1-k}{2}\right\} \geq 2$. Then there exist two proper non-equivalent $(\in, \in \vee q_k)$ -fuzzy sub-hypermodule of M such that μ can be expressed as the union of them.*

Proof. Let $\left\{\mu(x) \mid \mu(x) < \frac{1-k}{2}\right\} = \{t_1, t_2, \dots, t_r\}$, where $t_1 > t_2 > \dots > t_r$ and $r \geq 2$. Then the chain of $(\in \vee q_k)$ -level sub-hypermodules of μ is

$$[\mu]_{\frac{1-k}{2}} \subseteq [\mu]_{t_1} \subseteq [\mu]_{t_2} \subseteq \dots \subseteq [\mu]_{t_r} = M.$$

Define two fuzzy sets ν and γ of M by

$$\nu(x) = \begin{cases} t_1, & \text{if } x \in [\mu]_{t_1} \\ t_2, & \text{if } x \in [\mu]_{t_2} \setminus [\mu]_{t_1} \\ \dots & \\ t_r, & \text{if } x \in [\mu]_{t_r} \setminus [\mu]_{t_{r-1}} \end{cases} \quad \gamma(x) = \begin{cases} \mu(x), & \text{if } x \in [\mu]_{\frac{1-k}{2}} \\ k, & \text{if } x \in [\mu]_{t_2} \setminus [\mu]_{\frac{1-k}{2}} \\ t_3, & \text{if } x \in [\mu]_{t_3} \setminus [\mu]_{t_2} \\ \dots & \\ t_r, & \text{if } x \in [\mu]_{t_r} \setminus [\mu]_{t_{r-1}} \end{cases}$$

respectively, where $t_3 < k < t_2$. Then ν and γ are $(\in, \in \vee q_k)$ -fuzzy sub-hypermodules of M , and $\nu, \gamma \leq \mu$. The chain of $(\in \vee q_k)$ -level sub-hypermodules of ν and γ are, respectively, given by

$$[\mu]_{t_1} \subseteq [\mu]_{t_2} \subseteq \dots \subseteq [\mu]_{t_r} \quad \text{and} \quad [\mu]_{\frac{1-k}{2}} \subseteq [\mu]_{t_2} \subseteq \dots \subseteq [\mu]_{t_r}.$$

Therefore ν and γ are non-equivalent and clearly $\mu = \nu \cup \gamma$. This completes the proof. $\blacksquare$

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SOME MODULAR EQUATIONS IN THE FORM OF SCHLÄFLI¹**M.S. Mahadeva Naika**

*Department of Mathematics
Bangalore University
Central College Campus
Bengaluru - 560 001
India
e-mail: msmnaika@rediffmail.com*

K. Sushan Bairy

*PG Department of Mathematics
Vijaya College
R.V. Road, Basavanagudi
Bengaluru - 560 004
India
e-mail: ksbaury@rediffmail.com*

Abstract. On page 90 of his first notebook, S. Ramanujan records Schläfli-type modular equations for degrees 3, 5, 7, 11, 13, 17 and 19. In this paper, we establish Schläfli-type modular equations for degrees 11, 13, 17 and 19 which are recorded by Ramanujan in his first notebook. We also establish several new Schläfli-type modular equations of degrees 2, 4, 9, 15, 23, 25, 29, 31, 47 and 71. As an application, we deduce some explicit evaluations of Ramanujan-Weber class invariants.

Keywords: Schläfli-type modular equation, Class invariant.

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1. Introduction

In [13], Schläfli established modular equations for degrees 3, 5, 7, 11, 13, 17 and 19 which were also recorded by S. Ramanujan on page 90 of his first notebook [12]. In [11], Ramanathan gave a proof of the equation of degree 11. In [14], G.N. Watson gave a proof of Schläfli-type modular equation of degree thirteen and also examined Schläfli-type modular equations in [15]. Using the theory of modular forms, Berndt [2, pp. 231, 282, 315], [3, pp. 378–379] has verified these modular equations. Recently, William Hart [6] has proved several Schläfli-type modular equations of degrees 2, 3, 5, 7, 11, 13 and 17 by using modular forms in different levels.

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We define

$$(1.1) \quad K(k) := \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}} = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2} k^{2n} = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right),$$

where $0 < k < 1$ and ${}_2F_1$ is the ordinary or Gaussian hypergeometric function defined by

$${}_2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \quad 0 \leq |z| < 1,$$

where

$$(a)_0 = 1, \quad (a)_n = a(a+1) \cdots (a+n-1) \quad \text{for } n \text{ a positive integer}$$

and a, b, c are complex numbers such that $c \neq 0, -1, -2, \dots$. The number k is called the modulus of K , and $k' := \sqrt{1-k^2}$ is called the complementary modulus. Let K, K', L and L' denote the complete elliptic integrals of the first kind associated with the moduli k, k', l and l' , respectively. Suppose that the equality

$$(1.2) \quad n \frac{K'}{K} = \frac{L'}{L}$$

holds for some positive integer n . Then a modular equation of degree n is a relation between the moduli k and l which is induced by (1.2). Following Ramanujan, set $\alpha = k^2$ and $\beta = l^2$. Then we say β is of degree n over α . The multiplier m is defined by

$$(1.3) \quad m = \frac{K}{L}.$$

Ramanujan's class invariant G_n is defined by

$$(1.4) \quad G_n := 2^{-1/4} q^{-1/24} \chi(q) = \{4\alpha(1-\alpha)\}^{-1/24},$$

where

$$\chi(q) = (-q; q^2)_{\infty}, \quad q = \exp(-\pi\sqrt{n})$$

and

$$(a; q)_{\infty} := \prod_{n=0}^{\infty} (1 - aq^n), \quad |q| < 1.$$

M.S. Mahadeva Naika [8] and Mahadeva Naika and K. Sushan Bairy [9] have obtained several new explicit evaluations of the Ramanujan-Weber class invariants using modular equations.

In this paper, we establish several modular equations in the form of Schläfli for degrees 2, 4, 9, 11, 13, 15, 17, 19, 23, 25, 29, 31, 47 and 71. As an application, we obtain several explicit evaluations of Ramanujan-Weber class invariants.

2. Preliminary results

In this section, we collect several results which are useful in proving our main Schläfli-type modular equations.

Lemma 2.1. [2, Eq. (24.21), p. 215] *If β is of degree 2 over α , then*

$$(2.1) \quad \beta = \left(\frac{1 - \sqrt{1 - \alpha}}{1 + \sqrt{1 - \alpha}} \right)^2.$$

Lemma 2.2. [2, Eq. (24.22), p. 215] *If β is of degree 4 over α , then*

$$(2.2) \quad \beta = \left(\frac{1 - \sqrt[4]{1 - \alpha}}{1 + \sqrt[4]{1 - \alpha}} \right)^4.$$

Lemma 2.3. [3, Entry 62, 63, 64, 65, Ch.36, pp. 387–388] *Let*

$$(2.3) \quad u = 1 - \sqrt{\alpha\beta} - \sqrt{(1 - \alpha)(1 - \beta)},$$

$$(2.4) \quad v = 64 \left[\sqrt{\alpha\beta} + \sqrt{(1 - \alpha)(1 - \beta)} - \sqrt{\alpha\beta(1 - \alpha)(1 - \beta)} \right],$$

$$(2.5) \quad w = 32\sqrt{\alpha\beta(1 - \alpha)(1 - \beta)}.$$

1. *If β is of degree 9 over α , then*

$$(2.6) \quad u^6 - w(14u^3 + uv) - 3w^2 = 0.$$

2. *If β is of degree 13 over α , then*

$$(2.7) \quad \sqrt{u}(u^3 + 8w) - \sqrt{w}(11u^2 + v) = 0.$$

3. *If β is of degree 17 over α , then*

$$(2.8) \quad u^3 - w^{1/3}(10u^2 + v) + 13w^{2/3}u + 12w = 0.$$

4. *If β is of degree 29 over α , then*

$$(2.9) \quad \sqrt{u}(u^2 + 17uw^{1/3} - 9w^{2/3}) - w^{1/6}(9u^2 + v - 13uw^{1/3} + 15w^{2/3}) = 0.$$

Lemma 2.4. [3, Entry 53, 54, 55, 56, Ch.36, p. 385] *Let*

$$(2.10) \quad U = 1 \pm (\alpha\beta)^{1/8} \pm [(1 - \alpha)(1 - \beta)]^{1/8},$$

$$(2.11) \quad V = 4 \left[(\alpha\beta)^{1/8} + \{(1 - \alpha)(1 - \beta)\}^{1/8} \pm \{\alpha\beta(1 - \alpha)(1 - \beta)\}^{1/8} \right],$$

$$(2.12) \quad W = 4\{\alpha\beta(1 - \alpha)(1 - \beta)\}^{1/8}.$$

1. Let U , V and W be given by (2.10)–(2.12), with the plus signs taken. If β is of degree 15 over α , then

$$(2.13) \quad U(U^2 - V) + W = 0.$$

2. Let U , V and W be given by (2.10)–(2.12), with the plus signs taken. If β is of degree 31 over α , then

$$(2.14) \quad U^2 - V = \sqrt{UW}.$$

3. Let U , V and W be given by (2.10)–(2.12), with the plus signs taken. If β is of degree 47 over α , then

$$(2.15) \quad U^2 - V - UW^{1/3} - 2W^{2/3} = 0.$$

4. Let U , V and W be given by (2.10)–(2.12), with the minus signs taken. If β is of degree 71 over α , then

$$(2.16) \quad U^3 - W^{1/3}(4U^2 + V) + 2UW^{2/3} - W = 0.$$

Lemma 2.5. [2, Entry 7, Ch. 20, p. 363] If β is of degree 11 over α , then

$$(2.17) \quad (\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} + 2[16\alpha\beta(1-\alpha)(1-\beta)]^{1/12} = 1.$$

Lemma 2.6. [3, Entry 58, Ch. 36, p. 386] Let

$$(2.18) \quad A = 1 - (\alpha\beta)^{1/4} - [(1-\alpha)(1-\beta)]^{1/4},$$

$$(2.19) \quad B = 16 \left[(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} - \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4} \right],$$

$$(2.20) \quad C = 16\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4}.$$

If β is of degree 19 over α , then

$$(2.21) \quad A^5 - 7A^2C - BC = 0.$$

Lemma 2.7. [2, Entry 15, Ch. 20, p. 411] If β is of degree 23 over α , then

$$(2.22) \quad (\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} + 2^{2/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24} = 1.$$

Lemma 2.8. [2, Entry 15 (i), (ii), Ch. 19, p. 291] If β is of degree 25 over α , then

$$(2.23) \quad \left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8} - 2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/12} = (mm')^{1/2},$$

$$(2.24) \quad \left(\frac{\alpha}{\beta}\right)^{1/8} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/8} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8} - 2\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/12} = \frac{5}{(mm')^{1/2}}.$$

Lemma 2.9 (Identity Theorem). *Suppose $f(z)$ is analytic in a domain D , and that $\{z_n\}$ is a sequence of distinct points converging to a point z_0 in D . If $f(z_n) = 0$ for each n , then $f(z) \equiv 0$ throughout D .*

3. Main results

In this section, we prove Schläfli-type modular equations of degrees 11, 13, 17 and 19 recorded by Ramanujan in his notebooks. We also establish several new Schläfli-type modular equations of degrees 2, 4, 9, 15, 23, 25, 29, 31, 47 and 71.

First, we set

$$(3.1) \quad P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/24}$$

and

$$(3.2) \quad Q := \left\{ \frac{\beta(1-\beta)}{\alpha(1-\alpha)} \right\}^{1/24},$$

where β is of degree n over α .

Using (3.1) and (3.2), we obtain the following lemma.

Lemma 3.1. *We have*

$$(3.3) \quad \alpha = \frac{1+r}{2}$$

and

$$(3.4) \quad \beta = \frac{1+s}{2},$$

where, $r = \pm \sqrt{1 - \frac{P^{12}}{Q^{12}}}$ and $s = \pm \sqrt{1 - P^{12}Q^{12}}$.

Theorem 3.1. *If β is of degree 2 over α , then*

$$(3.5) \quad \frac{2^6}{P^8} \left[Q^{12} + \frac{1}{Q^{12}} \right] + \left[P^{12} - \frac{2^8}{P^{12}} \right] - 16 \left[2P^4 - \frac{15}{P^4} \right] = 0.$$

where P and Q are defined as in (3.1) and (3.2) respectively.

Proof. Equation (2.1) can be written as

$$(3.6) \quad 1 - a - b = ab,$$

where $a = \sqrt{1-\alpha}$ and $b = \sqrt{\beta}$. Squaring both sides of equation (3.6), we find that

$$(3.7) \quad 1 - 2a - 2b + a^2 + 2ab + b^2 - a^2b^2 = 0.$$

Isolating the terms containing a on one side of the equation (3.7), squaring both sides and then using (3.3) and (3.4), we deduce that

$$(3.8) \quad r - 2rs + rs^2 + 33 + s^2 - 48b + 30s - 16sb = 0.$$

Again isolating the terms containing b on one side of the equation (3.8), squaring both sides and then using (3.3) and (3.4), we find that

$$(3.9) \quad \begin{aligned} & 56rsP^{12}Q^{24} + 8P^{12} - 16Q^{12} + 16Q^{12}s - 16rQ^{12} - 8P^{12}s + 16rsQ^{12} \\ & + 80P^{12}Q^{24} - 2P^{24}Q^{36} + 4sP^{24}Q^{12} - 48rP^{12}Q^{24} - 72sP^{12}Q^{24} \\ & - 2rP^{24}Q^{36} - 8P^{24}Q^{12} + P^{36}Q^{24} = 0. \end{aligned}$$

Eliminating r and s in the same manner, we deduce that

$$(3.10) \quad \begin{aligned} & (-256Q^{12} + P^{24}Q^{12} + 64P^4 + 240Q^{12}P^8 - 32Q^{12}P^{16} \\ & + 64Q^{24}P^4)(-15360P^{12}Q^{12} + P^{48}Q^{24} + 32Q^{24}P^{40} + 784Q^{24}P^{32} \\ & - 64Q^{36}P^{28} - 64P^{28}Q^{12} + 7168P^{24}Q^{24} - 4096Q^{36}P^{20} - 4096Q^{12}P^{20} \\ & + 49408Q^{24}P^{16} - 15360Q^{36}P^{12} + 4096Q^{48}P^8 + 69632Q^{24}P^8 \\ & + 4096P^8 + 16384Q^{36}P^4 + 16384Q^{12}P^4 + 65536Q^{24}) = 0. \end{aligned}$$

By examining the factors near $q = 0$, it can be seen that there is a neighbourhood about the origin, where the first factor vanishes but the second factor does not. By the Identity Theorem 2.9, the first factor vanishes identically. Hence, we obtain (3.5). $\blacksquare$

Theorem 3.2. *If β is of degree 4 over α , then*

$$(3.11) \quad \begin{aligned} & P^{24} + 256 \left(Q^8 + \frac{1}{Q^8} \right) \left(2575 - \frac{2^{12}}{P^{24}} \right) \\ & + 128 \left(Q^4 + \frac{1}{Q^4} \right) \left(17P^{12} + \frac{36864}{P^{12}} \right) \\ & + \frac{16384}{P^{12}} \left[Q^{28} + \frac{1}{Q^{28}} + 32 \left(Q^{20} + \frac{1}{Q^{20}} \right) + 241 \left(Q^{12} + \frac{1}{Q^{12}} \right) \right] \\ & - 16384 \left(Q^{16} + \frac{1}{Q^{16}} \right) - 1855488 = 0. \end{aligned}$$

where P and Q are defined as in (3.1) and (3.2) respectively.

Proof of the identity (3.11) is similar to the proof of the identity (3.5) given above except that in place of result (2.1), result (2.2) is used.

Theorem 3.3. *If β is of degree 9 over α , then*

$$(3.12) \quad \begin{aligned} & 64 \left(P^{12} + \frac{1}{P^{12}} \right) \left[1 + Q^3 + \frac{1}{Q^3} \right] \\ & = 9324 + \left(Q^{18} + \frac{1}{Q^{18}} \right) + 28 \left(Q^{15} + \frac{1}{Q^{15}} \right) \\ & + 298 \left(Q^{12} + \frac{1}{Q^{12}} \right) + 1548 \left(Q^9 + \frac{1}{Q^9} \right) \\ & + 4383 \left(Q^6 + \frac{1}{Q^6} \right) + 7704 \left(Q^3 + \frac{1}{Q^3} \right), \end{aligned}$$

where P and Q are defined as in (3.1) and (3.2) respectively.

Proof. Using (3.1) in (2.3) – (2.5), we find that

$$(3.13) \quad u = 1 - a_1 - \frac{P^{12}}{4a_1},$$

$$(3.14) \quad v = 64a_1 + \frac{16P^{12}}{a_1} - 16P^{12},$$

$$(3.15) \quad w = 8P^{12},$$

where $a_1 = \{\alpha\beta\}^{1/2}$. Using (3.13)–(3.15) in (2.6), we deduce that

$$(3.16) \quad \begin{aligned} & 428032a_2^4a_1P^{12} - 782336a_2^3a_1P^{12} - 195584a_2^2a_1P^{24} \\ & + 6688a_2a_1P^{48} - 428032a_2^3P^{12} - 195584a_2^3a_1P^{24} - 6144a_2^2a_1P^{12} \\ & - 48896a_2^2a_1P^{36} - 81920a_2^4a_1 - 1280a_2a_1P^{36} + 48896a_2^2P^{36} \\ & + 3840a_2^2P^{24} + 240a_2P^{48} + 121344a_2^3P^{24} + P^{72} - 24a_1P^{60} \\ & + 782336a_2^4P^{12} - 24576a_2^5a_1 + 3840a_2^4P^{24} \\ & + 6144a_2^5P^{12} + 1280a_2^3P^{36} + 240a_2^2P^{48} \\ & + 24a_2P^{60} - 24576a_2^3a_1 + 4096a_2^3 + 61440a_2^4 \\ & + 61440a_2^5 + 4096a_2^6 = 0, \end{aligned}$$

where $a_2 = \alpha\beta$. Isolating the terms involving a_1 on one side of the equation, squaring both sides and eliminating r and s , we find that

$$(3.17) \quad \begin{aligned} & (-64P^{24}Q^{18} + 64P^{24}Q^{21} - 64Q^{18} + 64Q^{15} - 28P^{12}Q^{33} + P^{12}Q^{36} \\ & + 298P^{12}Q^6 - 28P^{12}Q^3 + 64P^{24}Q^{15} + 64Q^{21} - 7704P^{12}Q^{21} + P^{12} \\ & + 4383P^{12}Q^{12} - 1548P^{12}Q^9 + 9324P^{12}Q^{18} - 7704P^{12}Q^{15} \\ & - 1548P^{12}Q^{27} + 298P^{12}Q^{30} + 4383P^{12}Q^{24})(-3456P^{36}Q^{18} - 4096Q^{36} \\ & + 196812P^{24}Q^{18} + 10882P^{24}Q^{12} + 4096P^{48}Q^{30} + 188P^{24}Q^{66} \\ & + 493551P^{24}Q^{24} - 1048968P^{24}Q^{30} - 623232P^{36}Q^{30} + 778752P^{12}Q^{36} \\ & + 188P^{24}Q^6 + P^{24} + 4096P^{48}Q^{42} + P^{24}Q^{72} + 778752P^{36}Q^{36} \\ & + 493551P^{24}Q^{48} + 10882P^{24}Q^{60} - 4096P^{48}Q^{36} + 163584P^{36}Q^{24} \\ & + 163584P^{36}Q^{48} + 2030364P^{24}Q^{36} + 163584P^{12}Q^{48} - 3456P^{12}Q^{18} \\ & + 4096Q^{30} + 4096Q^{42} - 623232P^{36}Q^{42} - 1048968P^{24}Q^{42} \\ & - 623232P^{12}Q^{30} + 163584P^{12}Q^{24} + 196812P^{24}Q^{54} - 623232P^{12}Q^{42} \\ & - 3456P^{36}Q^{54} - 3456P^{12}Q^{54})(-64P^{24}Q^{18} - 64P^{24}Q^{21} - 64Q^{18} \\ & - 64Q^{15} + 28P^{12}Q^{33} + P^{12}Q^{36} + 298P^{12}Q^6 + 28P^{12}Q^3 \\ & - 64P^{24}Q^{15} - 64Q^{21} + 7704P^{12}Q^{21} + P^{12} + 4383P^{12}Q^{12} \\ & + 1548P^{12}Q^9 + 9324P^{12}Q^{18} + 7704P^{12}Q^{15} + 1548P^{12}Q^{27} \\ & + 298P^{12}Q^{30} + 4383P^{12}Q^{24}) = 0. \end{aligned}$$

Putting $n = 1/9$ in (3.1) and (3.2), we find that

$$(3.18) \quad P = \frac{1}{G_9^2} \quad \text{and} \quad Q = 1.$$

Using (3.18) in (3.17), we find that the last factor vanishes for the specific value of $q = e^{-\pi\sqrt{1/9}}$, then the last factor vanishes in a neighbourhood of $q = e^{-\pi\sqrt{1/9}}$. This proves the theorem. $\blacksquare$

Theorem 3.4. [12, p. 90], [3, Entry 41, p. 378] *If β is of degree 11 over α , then*

$$(3.19) \quad Q^6 + \frac{1}{Q^6} = 2\sqrt{2} \left(\frac{2}{P^5} - \frac{11}{P^3} + \frac{22}{P} - 22P + 11P^3 - 2P^5 \right),$$

where P and Q are defined as in (3.1) and (3.2) respectively.

Proof. Using (3.1) in (2.17), we find that

$$(3.20) \quad \{\alpha\beta\}^{1/4} + \frac{P^6}{\{16\alpha\beta\}^{1/4}} = 1 - 2P^2.$$

Squaring both sides of (3.20), we find that

$$(3.21) \quad \{\alpha\beta\}^{1/2} + \frac{P^{12}}{\{16\alpha\beta\}^{1/2}} = (1 - 2P^2)^2 - P^6.$$

Again squaring both sides of (3.21), we find that

$$(3.22) \quad \{\alpha\beta\} + \frac{P^{24}}{\{16\alpha\beta\}} = \left((1 - 2P^2)^2 - P^6 \right)^2 - \frac{P^{12}}{2}.$$

Using (3.3) and (3.4) in (3.22), we find that

$$(3.23) \quad \begin{aligned} & 32P^{20}Q^{12} - 4576P^{14}Q^{12} - P^{10}Q^{24} - 352P^2Q^{12} - 352P^{18}Q^{12} \\ & + 8096P^{12}Q^{12} - P^{10} + 1672P^{16}Q^{12} + 1672P^4Q^{12} - 9746P^{10}Q^{12} \\ & + 8096P^8Q^{12} + 32Q^{12} - 4576P^6Q^{12} = 0. \end{aligned}$$

Simplifying the above equation, we obtain the required result (3.19).

Theorem 3.5. [12, p. 90], [3, Entry 41, p. 378] *If β is of degree 13 over α , then*

$$(3.24) \quad \begin{aligned} & \left(Q^7 + \frac{1}{Q^7} \right) + 13 \left(Q^5 + \frac{1}{Q^5} \right) + 52 \left(Q^3 + \frac{1}{Q^3} \right) \\ & + 78 \left(Q + \frac{1}{Q} \right) = 8 \left(P^6 - \frac{1}{P^6} \right), \end{aligned}$$

where P and Q are defined as in (3.1) and (3.2) respectively.

Proof. Using (3.13)–(3.15) in (2.7), we deduce that

$$\begin{aligned}
 & 11712512a_2^3a_1P^{24} + 15610880a_2^2a_1P^{36} - 2030144a_2^2a_1P^{48} \\
 & + 15610880a_2^3a_1P^{36} + P^{84} + 249774080a_2^4a_1P^{12} \\
 & + 13590528a_2^5a_1P^{12} + 16384a_2^7 + 53088a_2a_1P^{60} - 21504a_2^2a_1P^{24} \\
 & + 13590528a_2^3a_1P^{12} - 32482304a_2^4a_1P^{24} - 28a_1P^{72} - 16384a_2^3a_1 \\
 (3.25) \quad & + 28672a_2^3P^{12} + 129929216a_2^4P^{12} + 32482304a_2^3P^{24} + 8960a_2^2P^{36} \\
 & + 129929216a_2^5P^{12} - 11712512a_2^4P^{24} - 2928128a_2^3P^{36} - 344064a_2^4a_1 \\
 & - 573440a_2^5a_1 - 114688a_2^6a_1 + 2030144a_2^2P^{48} + 28672a_2^6P^{12} \\
 & + 21504a_2^5P^{24} + 8960a_2^4P^{36} + 2240a_2^3P^{48} + 336a_2^2P^{60} + 114688a_2^4 \\
 & + 573440a_2^5 + 344064a_2^6 - 2240P^{48}a_2a_1 + 28a_2P^{72} + 336a_2P^{60} = 0.
 \end{aligned}$$

Isolating the terms involving a_1 on one side of the above equation, squaring on both sides and then eliminating r and s , we find that

$$\begin{aligned}
 & (8Q^7P^{12} + P^6 - 8Q^7 + 52Q^4P^6 + P^6Q^{14} + 52P^6Q^{10} + 13Q^2P^6 \\
 & + 78Q^6P^6 + 13P^6Q^{12} + 78P^6Q^8)(-8Q^7P^{12} + P^6 + 8Q^7 + 52Q^4P^6 \\
 & + 78P^6Q^8 + P^6Q^{14} + 52P^6Q^{10} + 13Q^2P^6 + 78Q^6P^6 \\
 & + 13P^6Q^{12})(15574P^{12}Q^{12} + 4888Q^8P^{12} + 64Q^{14}P^{24} - 26Q^2P^{12} \\
 & + 64Q^{14} - 26Q^{26}P^{12} + P^{12} + 273Q^{24}P^{12} + Q^{28}P^{12} + 15574Q^{16}P^{12} \\
 & - 1508Q^{22}P^{12} - 10244Q^{18}P^{12} - 1508Q^6P^{12} - 18044P^{12}Q^{14} \\
 & + 4888Q^{20}P^{12} - 10244Q^{10}P^{12} + 273Q^4P^{12})(1378P^{12}Q^{12} - 624P^6Q^{13} \\
 & + 1612Q^8P^{12} + 64Q^{14}P^{24} + 104Q^9P^{18} - 13Q^2P^{12} + 64Q^{14} \\
 & + 624Q^{13}P^{18} + 8Q^7P^{18} - 2080Q^{10}P^{12} + 117Q^4P^{12} - 13Q^{26}P^{12} \\
 & + 8Q^{21}P^{18} - 8Q^{21}P^6 - 624Q^{15}P^6 + 624Q^{15}P^{18} + 117Q^{24}P^{12} \\
 & + Q^{28}P^{12} - 520Q^{22}P^{12} - 2080Q^{18}P^{12} - 104P^6Q^9 + P^{12} - 520Q^6P^{12} \\
 & - 974P^{12}Q^{14} + 832P^6Q^{11} + 104Q^{19}P^{18} + 1612Q^{20}P^{12} - 104Q^{19}P^6 \\
 (3.26) \quad & + 1378Q^{16}P^{12} - 832Q^{11}P^{18} - 832Q^{17}P^{18} - 8Q^7P^6 \\
 & + 832Q^{17}P^6)(1378P^{12}Q^{12} + 624P^6Q^{13} + 1612Q^8P^{12} + 64Q^{14}P^{24} \\
 & - 104Q^9P^{18} - 13Q^2P^{12} + 64Q^{14} - 624Q^{13}P^{18} - 8Q^7P^{18} \\
 & - 2080Q^{10}P^{12} + 117Q^4P^{12} - 13Q^{26}P^{12} - 8Q^{21}P^{18} + 8Q^{21}P^6 \\
 & + 624Q^{15}P^6 - 624Q^{15}P^{18} + 117Q^{24}P^{12} + Q^{28}P^{12} - 520Q^{22}P^{12} \\
 & - 2080Q^{18}P^{12} + 104P^6Q^9 + P^{12} - 520Q^6P^{12} - 974P^{12}Q^{14} \\
 & - 832P^6Q^{11} - 104Q^{19}P^{18} + 1612Q^{20}P^{12} + 104Q^{19}P^6 + 1378Q^{16}P^{12} \\
 & + 832Q^{11}P^{18} + 832Q^{17}P^{18} + 8Q^7P^6 - 832Q^{17}P^6)(4096Q^{28}P^{48} + P^{24} \\
 & + 2200848Q^{14}P^{24} - 17472P^{36}Q^{38} + 26Q^2P^{24} + 96512Q^{20}P^{36} \\
 & + 655616Q^{32}P^{12} + 5115604Q^{16}P^{24} + 625664Q^{22}P^{36} - 3328Q^{16}P^{36} \\
 & + 96512P^{36}Q^{36} + 5115604Q^{40}P^{24} + 4096Q^{28} - 2309632P^{36}Q^{28} \\
 & + 8411936Q^{18}P^{24} - 17472Q^{18}P^{36} + 11103846P^{24}Q^{24} + 655616Q^{24}P^{36} \\
 & + 167752Q^{10}P^{24} + 403Q^4P^{24} + 30433Q^{48}P^{24} + 23542962Q^{28}P^{24}
 \end{aligned}$$

$$\begin{aligned}
& - 64Q^{42}P^{12} - 996736Q^{26}P^{12} - 64Q^{14}P^{36} + 18564676Q^{30}P^{24} \\
& + 167752Q^{46}P^{24} - 3328P^{36}Q^{40} + 655616Q^{24}P^{12} + 2200848Q^{42}P^{24} \\
& + 11103846Q^{32}P^{24} - 2309632Q^{28}P^{12} + 18564676Q^{26}P^{24} \\
& + 704444Q^{12}P^{24} + 625664Q^{22}P^{12} - 64P^{36}Q^{42} - 17472Q^{18}P^{12} \\
& + 625664Q^{34}P^{36} + 26Q^{54}P^{24} + 655616Q^{32}P^{36} + 9171448Q^{34}P^{24} \\
& + 9835644Q^{36}P^{24} - 64P^{12}Q^{14} - 996736Q^{30}P^{12} + 4082Q^{50}P^{24} \\
& - 996736Q^{30}P^{36} - 996736P^{36}Q^{26} + 9171448Q^{22}P^{24} + 4082Q^6P^{24} \\
& - 17472Q^{38}P^{12} + 96512Q^{20}P^{12} + 8411936Q^{38}P^{24} + 625664Q^{34}P^{12} \\
& + 403Q^{52}P^{24} + 96512Q^{36}P^{12} - 3328Q^{16}P^{12} - 3328Q^{40}P^{12} \\
& + 704444Q^{44}P^{24} + 9835644Q^{20}P^{24} + 30433Q^8P^{24} + Q^{56}P^{24}) = 0.
\end{aligned}$$

By examining the factors near $q = 0$, it can be seen that there is a neighbourhood about the origin, where the first factor vanish but the other factors does not. By the Identity Theorem, the first factor vanishes identically. This proves the theorem. $\blacksquare$

Theorem 3.6. *If β is of degree 15 over α , then*

$$\begin{aligned}
& Q^{36} + \frac{1}{Q^{36}} + 2 \left(Q^{24} + \frac{1}{Q^{24}} \right) \left[360\sqrt{2} \left(P^9 + \frac{1}{P^9} \right) \right. \\
& + 11580 \left(P^6 + \frac{1}{P^6} \right) + 40328\sqrt{2} \left(P^3 + \frac{1}{P^3} \right) + 93915 \Big] \\
& - \left(Q^{12} + \frac{1}{Q^{12}} \right) \left[32 \left\{ 32\sqrt{2} \left(P^{21} + \frac{1}{P^{21}} \right) - 720 \left(P^{18} + \frac{1}{P^{18}} \right) \right. \right. \\
& + 2520\sqrt{2} \left(P^{15} + \frac{1}{P^{15}} \right) - 5430 \left(P^{12} + \frac{1}{P^{12}} \right) \\
& + 5195\sqrt{2} \left(P^9 + \frac{1}{P^9} \right) - 13815 \left(P^6 + \frac{1}{P^6} \right) \\
& + 22590\sqrt{2} \left(P^3 + \frac{1}{P^3} \right) \Big\} + 34993 \Big] \\
& = 16 \left\{ 49050\sqrt{2} \left(P^3 + \frac{1}{P^3} \right) - 112365 \left(P^6 + \frac{1}{P^6} \right) \right. \\
& + 107730\sqrt{2} \left(P^9 + \frac{1}{P^9} \right) - 34200 \left(P^{12} + \frac{1}{P^{12}} \right) \\
& + 22560\sqrt{2} \left(P^{15} + \frac{1}{P^{15}} \right) - 20192 \left(P^{18} + \frac{1}{P^{18}} \right) \\
& + 2880\sqrt{2} \left(P^{21} + \frac{1}{P^{21}} \right) - 256 \left(P^{24} + \frac{1}{P^{24}} \right) \Big\} - 5028660,
\end{aligned} \tag{3.27}$$

where P and Q are defined as in (3.1) and (3.2) respectively.

Proof of the identity (3.27) is similar to the proof of the identity (3.24) given above except that in place of result (2.7), result (2.13) is used.

Theorem 3.7. [12, p. 90], [3, Entry 41, p. 378] *If β is of degree 17 over α , then*

$$(3.28) \quad \begin{aligned} & \left(Q^9 + \frac{1}{Q^9}\right) + 119 \left(Q^3 + \frac{1}{Q^3}\right) - 34 \left(Q^6 + \frac{1}{Q^6}\right) - 16 \left(P^8 + \frac{1}{P^8}\right) \\ & + 68 \left(Q^3 + \frac{1}{Q^3}\right) \left(P^4 + \frac{1}{P^4}\right) + 136 \left(P^4 + \frac{1}{P^4}\right) + 340 = 0, \end{aligned}$$

where P and Q are defined as in (3.1) and (3.2) respectively.

The proof of the identity (3.28) is similar to the proof of the identity (3.24), except that in place of result (2.7), result (2.8) is used.

Theorem 3.8. [12, p. 90], [3, Entry 41, pp. 378, 379] *If β is of degree 19 over α , then*

$$(3.29) \quad \begin{aligned} & Q^{10} + \frac{1}{Q^{10}} + 114 \left(Q^6 + \frac{1}{Q^6}\right) + 190\sqrt{2} \left(Q^4 + \frac{1}{Q^4}\right) \left(P^3 - \frac{1}{P^3}\right) \\ & + 19 \left(Q^2 + \frac{1}{Q^2}\right) \left(\frac{8}{P^6} - 5 + 8P^6\right) - 4\sqrt{2} \left(\frac{4}{P^9} + \frac{19}{P^3} - 19P^3 - 4P^9\right) = 0, \end{aligned}$$

where P and Q are defined as in (3.1) and (3.2) respectively.

Proof. Using (3.1) in (2.18)–(2.20), we find that

$$(3.30) \quad A = 1 - c_1 - \frac{1}{2} \frac{P^6}{c_1},$$

$$(3.31) \quad B = 16c_1 + 8 \frac{P^6}{c_1} - 8P^6,$$

$$(3.32) \quad C = 8P^6.$$

where $c_1 = \{\alpha\beta\}^{1/4}$. Using (3.30)–(3.32) in (2.21), we find that

$$(3.33) \quad \begin{aligned} & -130240a_1P^{36}a_2 - 640a_2P^{24}a_1 + 5120a_1a_2^4P^6 + 8064a_1P^{30}a_2^2 \\ & + 15360a_1a_2^3P^{18} - 1161088a_1P^{24}a_2^2 + 1024a_2^5 + 2078720a_2^3P^6a_1 \\ & + 129920a_2P^{30}a_1 + 119808a_2^2P^6a_1 - 2083840a_1a_2^3P^{12} \\ & + 2966528a_2^2P^{18}a_1 - 1932288a_2^2P^{12}a_1 + 960a_1P^{42}a_2 - 1024a_1a_2^2 \\ & + 10240a_2^4 + 5120a_2^3 + 1280a_2^2P^{12} + 351232a_2^2P^{18} + 160a_2P^{36} \\ & - 834176a_2^2P^{24} + 1404928a_2^3P^6 - 3336704a_2^3P^{12} - 10240a_1a_2^3 \\ & + 9152a_2P^{42} + 461056a_2^2P^{30} + 1844224a_2^3P^{18} + 20a_1P^{54} \\ & - 5120a_1a_2^4 + 180P^{48}a_2 + 3360P^{36}a_2^2 + 13440a_2^3P^{24} \\ & + 585728a_2^4P^6 + 11520a_2^4P^{12} - 20a_1P^{48} + P^{60} = 0. \end{aligned}$$

Isolating the terms involving a_1 to one side of the above equation and squaring both sides, substituting $a_2 = (1+r)(1+s)/4$ and eliminating r and s , we find that

$$\begin{aligned}
& (-37240P^{24}Q^{28} + P^{18} + 41344Q^{20}P^{30} + 41344Q^{20}P^6 - 208848Q^{20}P^{24} \\
& + 449388Q^{20}P^{18} - 208848Q^{20}P^{12} + 304Q^8P^{24} + 12806Q^8P^{18} + 304Q^8P^{12} \\
& + 10944Q^{24}P^6 + 149321Q^{24}P^{18} - 39824Q^{24}P^{12} + 10944P^{30}Q^{24} \\
& - 37240P^{12}Q^{12} - 37240Q^{12}P^{24} + 228P^{18}Q^{36} - 512Q^{20} + P^{18}Q^{40} \\
& - 39824P^{24}Q^{24} + 122550Q^{12}P^{18} + 304P^{24}Q^{32} - 39824Q^{16}P^{12} + 10944Q^{16}P^6 \\
& - 39824Q^{16}P^{24} + 10944Q^{16}P^{30} + 149321Q^{16}P^{18} + 228Q^4P^{18} + 12806P^{18}Q^{32} \\
& - 512Q^{20}P^{36} + 304P^{12}Q^{32} - 37240Q^{28}P^{12} + 122550Q^{28}P^{18})(-8137026944P^{24}Q^{48} \\
& + P^{36} + 5807166992P^{30}Q^{28} - 13769395456P^{48}Q^{36} + 39178P^{36}Q^{72} \\
& - 13769395456P^{24}Q^{36} + 16270080P^{48}Q^{60} + 100704256P^{60}Q^{48} \\
& - 46751414915P^{36}Q^{48} + 100704256P^{12}Q^{48} + 16270080P^{24}Q^{60} \\
& - 1895116544P^{24}Q^{28} - 472858624P^{60}Q^{36} - 8137026944P^{48}Q^{48} \\
& + 15801984144P^{36}Q^{24} - 1479092392P^{36}Q^{60} + 16316592P^{42}Q^{64} \\
& + 136087747P^{36}Q^{64} + 81472P^{48}Q^{64} + 74749420608P^{48}Q^{40} - 472858624P^{60}Q^{44} \\
& - 16857729024P^{54}Q^{40} + 727401472P^{54}Q^{48} + 5603328P^{66}Q^{36} - 3210240P^{54}Q^{24} \\
& + 421689344Q^{20}P^{30} + 16270080Q^{20}P^{24} + 512Q^{20}P^{18} - 13769395456P^{48}Q^{44} \\
& - 8618720720P^{42}Q^{24} + 31548679200P^{42}Q^{48} + 5603328P^{66}Q^{44} - 3210240Q^{56}P^{18} \\
& + 3193459200P^{18}Q^{44} + 512P^{54}Q^{60} + 31761710544P^{30}Q^{36} + 727401472P^{18}Q^{48} \\
& + 421689344P^{30}Q^{60} + 5603328P^6Q^{44} + 31761710544P^{30}Q^{44} \\
& + 421689344P^{42}Q^{60} - 304(Q^8P^{30} + Q^8P^{42} + P^{30}Q^{72} + P^{42}Q^{72}) \\
& + 39178Q^8P^{36} + 31761710544P^{42}Q^{44} + 31548679200P^{30}Q^{48} \\
& - 13769395456P^{24}Q^{44} - 43661221076P^{36}Q^{44} + 81472Q^{16}P^{24} \\
& + 136087747Q^{16}P^{36} + 81472Q^{16}P^{48} + 16316592Q^{16}P^{42} \\
& + 31761710544P^{42}Q^{36} + 419578368Q^{52}P^{18} - 311296(Q^{52}P^{60} \\
& + Q^{52}P^{12}) - 3210240Q^{24}P^{18} + 512P^{18}Q^{60} - 10043781520Q^{52}P^{36} \\
& + 5807166992Q^{52}P^{30} - 8618720720P^{30}Q^{24} - 228Q^{76}P^{36} \\
& + 3193459200(P^{54}Q^{44} + P^{54}Q^{36}) - 42336256P^{66}Q^{40} \\
& + 3193459200P^{18}Q^{36} + 1803415552P^{60}Q^{40} + 249431687698P^{36}Q^{40} \\
& - 183111246912P^{30}Q^{40} + 512Q^{20}P^{54} + 74749420608P^{24}Q^{40} \\
& - 16857729024P^{18}Q^{40} + 1249370688P^{24}Q^{24} + 262144Q^{40} \\
& - 8618720720Q^{56}P^{30} + 15801984144Q^{56}P^{36} + 1803415552P^{12}Q^{40} \\
& + 81472Q^{64}P^{24} - 1895116544Q^{52}P^{48} + 5807166992Q^{52}P^{42} \\
& + 419578368Q^{52}P^{54} - 42336256P^6Q^{40} - 143792Q^{12}P^{30} \\
& - 8137026944P^{24}Q^{32} - 8137026944Q^{32}P^{48} - 46751414915Q^{32}P^{36} \\
& - 143792Q^{12}P^{42} + 419578368P^{54}Q^{28} - 10043781520P^{36}Q^{28} \\
& + 5807166992P^{42}Q^{28} - 1895116544P^{48}Q^{28} - 311296P^{60}Q^{28} \\
& - 43661221076P^{36}Q^{36} + 1249370688P^{48}Q^{24} + 16316592Q^{16}P^{30} \\
& - 228Q^4P^{36} - 472858624P^{12}Q^{36} + 727401472P^{18}Q^{32} + Q^{80}P^{36} \\
& - 1479092392Q^{20}P^{36} + 100704256P^{12}Q^{32} - 311296Q^{28}P^{12} \\
& + 419578368Q^{28}P^{18} - 2674668(Q^{68}P^{36} + P^{36}Q^{12}) - 143792Q^{68}P^{30} \\
& + 5603328Q^{36}P^6 - 143792Q^{68}P^{42} + 16270080Q^{20}P^{48} \\
& + 31548679200Q^{32}P^{42} + 100704256Q^{32}P^{60} + 727401472Q^{32}P^{54} \\
& + 16316592Q^{64}P^{30} + 1249370688Q^{56}P^{24} - 3210240Q^{56}P^{54} \\
& - 8618720720Q^{56}P^{42} + 1249370688Q^{56}P^{48} - 1895116544Q^{52}P^{24} \\
& + 262144Q^{40}P^{72} + 31548679200P^{30}Q^{32} + 421689344Q^{20}P^{42} \\
& - 183111246912P^{42}Q^{40} - 472858624P^{12}Q^{44}) = 0.
\end{aligned}$$

By examining the factors near $q = 0$, it can be seen that there is a neighbourhood about the origin, where the first factor vanishes but the other factors does not. By the Identity Theorem, the first factor vanishes identically. This proves the theorem. $\blacksquare$

Theorem 3.9. *If β is of degree 23 over α , then*

$$(3.35) \quad \begin{aligned} Q^{12} + \frac{1}{Q^{12}} + 4223122 = & 8 \left[4\sqrt{2} \left(P^{11} + \frac{1}{P^{11}} \right) - 92 \left(P^{10} + \frac{1}{P^{10}} \right) \right. \\ & + 506\sqrt{2} \left(P^9 + \frac{1}{P^9} \right) - 3588 \left(P^8 + \frac{1}{P^8} \right) + 9292\sqrt{2} \left(P^7 + \frac{1}{P^7} \right) \\ & - 37605 \left(P^6 + \frac{1}{P^6} \right) + 61916\sqrt{2} \left(P^5 + \frac{1}{P^5} \right) - 170292 \left(P^4 + \frac{1}{P^4} \right) \\ & \left. + 199042\sqrt{2} \left(P^3 + \frac{1}{P^3} \right) - 400108 \left(P^2 + \frac{1}{P^2} \right) + 348404\sqrt{2} \left(P + \frac{1}{P} \right) \right], \end{aligned}$$

where P and Q are defined as in (3.1) and (3.2) respectively.

The proof of the identity (3.35) is similar to the proof of the identity (3.19), except that in place of result (2.17), result (2.22) is used.

Theorem 3.10. *If β is of degree 25 over α , then*

$$(3.36) \quad \begin{aligned} 64 \left(P^{12} + \frac{1}{P^{12}} \right) \left[1 + \left(Q + \frac{1}{Q} \right) + \left(Q^2 + \frac{1}{Q^2} \right) \right] = & \left(Q^{15} + \frac{1}{Q^{15}} \right) \\ & + 26 \left(Q^{14} + \frac{1}{Q^{14}} \right) + 301 \left(Q^{13} + \frac{1}{Q^{13}} \right) + 2076 \left(Q^{12} + \frac{1}{Q^{12}} \right) \\ & + 9726 \left(Q^{11} + \frac{1}{Q^{11}} \right) + 33880 \left(Q^{10} + \frac{1}{Q^{10}} \right) + 94480 \left(Q^9 + \frac{1}{Q^9} \right) \\ & + 222580 \left(Q^8 + \frac{1}{Q^8} \right) + 456305 \left(Q^7 + \frac{1}{Q^7} \right) + 827530 \left(Q^6 + \frac{1}{Q^6} \right) \\ & + 1346255 \left(Q^5 + \frac{1}{Q^5} \right) + 1985080 \left(Q^4 + \frac{1}{Q^4} \right) + 2668655 \left(Q^3 + \frac{1}{Q^3} \right) \\ & + 3285730 \left(Q^2 + \frac{1}{Q^2} \right) + 3718805 \left(Q + \frac{1}{Q} \right) + 3875380, \end{aligned}$$

where P and Q are defined as in (3.1) and (3.2) respectively.

Proof. Using (3.2) in (2.23) and (2.24), we find that

$$(3.37) \quad \begin{aligned} 2d_1^2 Q^3 + d_1^4 - d_1^3 - 2d_1^3 Q + Q^6 - Q^3 d_1 - 2Q^4 d_1 - Q^6 d_1 - Q^3 d_1^3 \\ + 2Q^4 d_1^2 - 2Q^5 d_1 - 2Q^2 d_1^3 + 2Q^2 d_1^2 = 0, \end{aligned}$$

where $d_1 = \left(\frac{\beta}{\alpha} \right)^{1/8}$. Substituting $d_1^2 = d_2$ and isolating the terms involving d_1 on one side of equation, squaring both sides and again substituting $d_2^2 = d_3$ and $d_3^2 = d_4$, we find that

$$\begin{aligned}
& -4d_3Q^2d_2 - 4d_3d_2Q - 4d_3d_2Q^5 - 4d_3Q^4d_2 - d_3d_2Q^6 - 2d_3Q^9 - 6d_2Q^3d_3 \\
(3.38) \quad & -4Q^8d_3 - 4Q^4d_3 - 2d_3Q^3 - d_2d_3 + d_4 - 6d_2Q^9 - 6d_3Q^6 - 4Q^{10}d_2 - 4Q^8d_2 \\
& - d_2Q^6 - 4d_2Q^7 - d_2Q^{12} - 4d_2Q^{11} + Q^{12} - 8d_3Q^7 - 8d_3Q^5 = 0.
\end{aligned}$$

Now isolating the terms involving d_2 on one side of the above equation, squaring both sides, we find that

$$\begin{aligned}
& -d_4d_3 - Q^{12}d_3 - d_3Q^{24} - 48d_3Q^{21} - 104d_3Q^{17} - 114d_3Q^{18} + 2d_4Q^{18} \\
& - 80Q^{20}d_3 + Q^{24} - 80Q^{16}d_3 + 2d_4Q^6 - 48d_3Q^{15} + 2d_4Q^{12} - 104Q^{19}d_3 \\
(3.39) \quad & - 24Q^{14}d_3 - 8Q^{13}d_3 - 24d_3Q^{22} - 8d_3Q^{23} - 8d_4d_3Q^{11} - 104d_3Q^5d_4 \\
& - 80Q^4d_3d_4 - 48d_3Q^9d_4 - 80Q^8d_3d_4 - 104d_4d_3Q^7 - 48d_3Q^3d_4 \\
& - 114d_3Q^6d_4 - 24Q^2d_4d_3 - 8d_4d_3Q - d_4d_3Q^{12} - 24d_4d_3Q^{10} + d_4^2 = 0,
\end{aligned}$$

where $d_4 = \frac{\beta}{\alpha}$. Isolating the terms involving d_3 on one side of the above equation and squaring on both sides and eliminating r and s , we find that

$$\begin{aligned}
& (-64Q^{15} - 64Q^{17} - 64Q^{16} - 64Q^{14} - 64Q^{13} + 2668655P^{12}Q^{12} + P^{12} \\
& + 94480P^{12}Q^{24} + 2076Q^3P^{12} + 26QP^{12} + 222580Q^7P^{12} \\
& + 456305Q^8P^{12} + 9726Q^4P^{12} + 301Q^2P^{12} + 1346255Q^{10}P^{12} \\
& + 827530Q^9P^{12} + 94480Q^6P^{12} + 1985080Q^{11}P^{12} + 33880Q^5P^{12} \\
(3.40) \quad & + Q^{30}P^{12} + 2668655Q^{18}P^{12} + 26Q^{29}P^{12} + 33880Q^{25}P^{12} + 2076Q^{27}P^{12} \\
& + 9726Q^{26}P^{12} + 222580Q^{23}P^{12} + 827530Q^{21}P^{12} + 456305Q^{22}P^{12} \\
& + 3718805Q^{14}P^{12} + 3718805Q^{16}P^{12} + 1346255Q^{20}P^{12} + 1985080Q^{19}P^{12} \\
& + 3285730Q^{13}P^{12} + 3875380Q^{15}P^{12} + 301Q^{28}P^{12} + 3285730Q^{17}P^{12} \\
& - 64P^{24}Q^{14} - 64P^{24}Q^{16} - 64P^{24}Q^{13} - 64P^{24}Q^{15} - 64Q^{17}P^{24}) \\
& (Q^2 + Q + 1)^4(Q^2 + 1)^4(Q^8 - Q^4 + 1)^2(Q^4 + 1)^2(Q + 1)^2 = 0.
\end{aligned}$$

By examining the factors near $q = 0$, it can be seen that there is a neighbourhood about the origin, where the first factor vanish but the other factors does not. By the Identity Theorem, the first factor vanishes identically. This proves the theorem. $\blacksquare$

Theorem 3.11. *If β is of degree 29 over α , then*

$$\begin{aligned}
& Q^{15} + \frac{1}{Q^{15}} + 128 \left(P^{14} - \frac{1}{P^{14}} \right) - 58 \left(P^2 - \frac{1}{P^2} \right) \left(Q^{12} + \frac{1}{Q^{12}} \right) \\
& + 29 \left[36 \left(P^4 + \frac{1}{P^4} \right) + 185 \right] \left(Q^9 + \frac{1}{Q^9} \right) - 6264 \left(P^6 - \frac{1}{P^6} \right) \left(Q^6 + \frac{1}{Q^6} \right) \\
(3.41) \quad & + 58 \left[184 \left(P^8 + \frac{1}{P^8} \right) - 394 \left(P^4 + \frac{1}{P^4} \right) + 649 \right] \left(Q^3 + \frac{1}{Q^3} \right) \\
& = 24940 \left(P^2 - \frac{1}{P^2} \right) - 10672 \left(P^6 - \frac{1}{P^6} \right) + 6496 \left(P^{10} - \frac{1}{P^{10}} \right),
\end{aligned}$$

where P and Q are defined as in (3.1) and (3.2) respectively.

Theorem 3.12. *If β is of degree 31 over α , then*

$$\begin{aligned}
 (3.42) \quad & Q^{16} + \frac{1}{Q^{16}} + 775 \left(Q^{12} + \frac{1}{Q^{12}} \right) + 38657 \left(Q^8 + \frac{1}{Q^8} \right) \\
 & - 252526 \left(Q^4 + \frac{1}{Q^4} \right) + 537726 = 128\sqrt{2} \left(P^{15} + \frac{1}{P^{15}} \right) \\
 & - 5952 \left(P^{12} + \frac{1}{P^{12}} \right) - \left(P^9 + \frac{1}{P^9} \right) \left[6448\sqrt{2} \left(Q^4 + \frac{1}{Q^4} \right) - 31744\sqrt{2} \right] \\
 & - \left(P^6 + \frac{1}{P^6} \right) \left[1984 \left(Q^8 + \frac{1}{Q^8} \right) - 50840 \left(Q^4 + \frac{1}{Q^4} \right) + 157976 \right] \\
 & - \left(P^3 + \frac{1}{P^3} \right) \left[62\sqrt{2} \left(Q^{12} + \frac{1}{Q^{12}} \right) - 15872\sqrt{2} \left(Q^8 + \frac{1}{Q^8} \right) \right. \\
 & \left. + 119536\sqrt{2} \left(Q^4 + \frac{1}{Q^4} \right) - 274660\sqrt{2} \right],
 \end{aligned}$$

where P and Q are defined as in (3.1) and (3.2) respectively.

Theorem 3.13. *If β is of degree 47 over α , then*

$$\begin{aligned}
 (3.43) \quad & Q^{24} + \frac{1}{Q^{24}} + 188 \left(Q^{12} - \frac{1}{Q^{12}} \right) \left[8\sqrt{2} \left(P^{11} + \frac{1}{P^{11}} \right) \right. \\
 & + 744 \left(P^{10} + \frac{1}{P^{10}} \right) + 11048\sqrt{2} \left(P^9 + \frac{1}{P^9} \right) + 168664 \left(P^8 + \frac{1}{P^8} \right) \\
 & + 805016\sqrt{2} \left(P^7 + \frac{1}{P^7} \right) + 5327756 \left(P^6 + \frac{1}{P^6} \right) \\
 & + 13001656\sqrt{2} \left(P^5 + \frac{1}{P^5} \right) + 48688888 \left(P^4 + \frac{1}{P^4} \right) \\
 & + 71813704\sqrt{2} \left(P^3 + \frac{1}{P^3} \right) + 169828168 \left(P^2 + \frac{1}{P^2} \right) \\
 & \left. + 162800104\sqrt{2} \left(P + \frac{1}{P} \right) + 254642055 \right] \\
 & = 2048\sqrt{2} \left(P^{23} + \frac{1}{P^{23}} \right) - 96256\sqrt{2} \left(P^{21} + \frac{1}{P^{21}} \right) \\
 & - 240640 \left[P^{20} + \frac{1}{P^{20}} \right] + 1756672\sqrt{2} \left[P^{19} + \frac{1}{P^{19}} \right] \\
 & + 9072128 \left[P^{18} + \frac{1}{P^{18}} \right] - 9709824\sqrt{2} \left[P^{17} + \frac{1}{P^{17}} \right] \\
 & - 125866752 \left[P^{16} + \frac{1}{P^{16}} \right] - 101562112\sqrt{2} \left[P^{15} + \frac{1}{P^{15}} \right] \\
 & + 625579776 \left[P^{14} + \frac{1}{P^{14}} \right] + 1536739072\sqrt{2} \left[P^{13} + \frac{1}{P^{13}} \right]
 \end{aligned}$$

$$\begin{aligned}
& + 1149034944 \left[P^{12} + \frac{1}{P^{12}} \right] - 5082091200\sqrt{2} \left[P^{11} + \frac{1}{P^{11}} \right] \\
& - 17928597440 \left[P^{10} + \frac{1}{P^{10}} \right] - 8133057472\sqrt{2} \left[P^9 + \frac{1}{P^9} \right] \\
& + 23387154880 \left[P^8 + \frac{1}{P^8} \right] + 43138741696\sqrt{2} \left[P^7 + \frac{1}{P^7} \right] \\
& + 61416509280 \left[P^6 + \frac{1}{P^6} \right] + 2130217778\sqrt{2} \left[P^5 + \frac{1}{P^5} \right] \\
& + 11813501888 \left[P^4 + \frac{1}{P^4} \right] + 4974868032\sqrt{2} \left[P^3 + \frac{1}{P^3} \right] \\
& - 40687005120 \left[P^2 + \frac{1}{P^2} \right] - 96512465088\sqrt{2} \left[P + \frac{1}{P} \right] \\
& - 189300816838,
\end{aligned}$$

where P and Q are defined as in (3.1) and (3.2) respectively.

Theorem 3.14. *If β is of degree 71 over α , then*

$$\begin{aligned}
& Q^{36} + \frac{1}{Q^{36}} + 142 \left(Q^{24} + \frac{1}{Q^{24}} \right) \left(401715285197 - 250672328128\sqrt{2} \left[P + \frac{1}{P} \right] \right. \\
& + 243012755392 \left[P^2 + \frac{1}{P^2} \right] - 90783237160\sqrt{2} \left[P^3 + \frac{1}{P^3} \right] \\
& + 51543826752 \left[P^4 + \frac{1}{P^4} \right] - 10877163840\sqrt{2} \left[P^5 + \frac{1}{P^5} \right] \\
& + 3301519972 \left[P^6 + \frac{1}{P^6} \right] - 342806080\sqrt{2} \left[P^7 + \frac{1}{P^7} \right] \\
& + 45053504 \left[P^8 + \frac{1}{P^8} \right] - 1643912\sqrt{2} \left[P^9 + \frac{1}{P^9} \right] \\
(3.44) \quad & \left. + 51914 \left[P^{10} + \frac{1}{P^{10}} \right] - 192\sqrt{2} \left[P^{11} + \frac{1}{P^{11}} \right] \right) \\
& + 71 \left(Q^{12} + \frac{1}{Q^{12}} \right) \left(2048\sqrt{2} \left[P^{23} + \frac{1}{P^{23}} \right] + 942080 \left[P^{22} + \frac{1}{P^{22}} \right] \right. \\
& + 5676032\sqrt{2} \left[P^{21} + \frac{1}{P^{21}} \right] - 187541504 \left[P^{20} + \frac{1}{P^{20}} \right] \\
& + 1097148416\sqrt{2} \left[P^{19} + \frac{1}{P^{19}} \right] - 9176226304 \left[P^{18} + \frac{1}{P^{18}} \right] \\
& + 32585199616\sqrt{2} \left[P^{17} + \frac{1}{P^{17}} \right] - 204517744640 \left[P^{16} + \frac{1}{P^{16}} \right] \\
& \left. + 566617288448\sqrt{2} \left[P^{15} + \frac{1}{P^{15}} \right] - 2751003146240 \left[P^{14} + \frac{1}{P^{14}} \right] \right)
\end{aligned}$$

$$\begin{aligned}
& + 5807932620800\sqrt{2} \left[P^{13} + \frac{1}{P^{13}} \right] - 21186828070080 \left[P^{12} + \frac{1}{P^{12}} \right] \\
& + 33186228353536\sqrt{2} \left[P^{11} + \frac{1}{P^{11}} \right] - 88563043373568 \left[P^{10} + \frac{1}{P^{10}} \right] \\
& + 98954852869312\sqrt{2} \left[P^9 + \frac{1}{P^9} \right] - 176963128626688 \left[P^8 \right. \\
& \left. + \frac{1}{P^8} \right] + 108271940623872\sqrt{2} \left[P^7 + \frac{1}{P^7} \right] - 7129338452448 \left[P^6 + \frac{1}{P^6} \right] \\
& - 221694467125760\sqrt{2} \left[P^5 + \frac{1}{P^5} \right] + 819155626103296 \left[P^4 + \frac{1}{P^4} \right] \\
& - 1026292194878784\sqrt{2} \left[P^3 + \frac{1}{P^3} \right] + 2083094118350336 \left[P^2 + \frac{1}{P^2} \right] \\
& - 1805589616637440\sqrt{2} \left[P + \frac{1}{P} \right] + 2727816983432377 \Big) \\
& = 131072\sqrt{2} \left[P^{35} + \frac{1}{P^{35}} \right] - 9306112 \left[P^{34} + \frac{1}{P^{34}} \right] + 153550848\sqrt{2} \left[P^{33} + \frac{1}{P^{33}} \right] \\
& - 3150118912 \left[P^{32} + \frac{1}{P^{32}} \right] + 22730178560\sqrt{2} \left[P^{31} + \frac{1}{P^{31}} \right] - 248314986496 \left[P^{30} \right. \\
& \left. + \frac{1}{P^{30}} \right] + 1082027458560\sqrt{2} \left[P^{29} + \frac{1}{P^{29}} \right] - 7839822381056 \left[P^{28} + \frac{1}{P^{28}} \right] \\
& + 24407413309440\sqrt{2} \left[P^{27} + \frac{1}{P^{27}} \right] - 133963714543616 \left[P^{26} + \frac{1}{P^{26}} \right] \\
& + 330092418940928\sqrt{2} \left[P^{25} + \frac{1}{P^{25}} \right] - 1479732762742784 \left[P^{24} + \frac{1}{P^{24}} \right] \\
& + 3046058228477952\sqrt{2} \left[P^{23} + \frac{1}{P^{23}} \right] - 11604272147963904 \left[P^{22} + \frac{1}{P^{22}} \right] \\
& + 20576178256737280\sqrt{2} \left[P^{21} + \frac{1}{P^{21}} \right] - 68251974000062464 \left[P^{20} + \frac{1}{P^{20}} \right] \\
& + 106291599722672128\sqrt{2} \left[P^{19} + \frac{1}{P^{19}} \right] - 311894723269365248 \left[P^{18} + \frac{1}{P^{18}} \right] \\
& + 432349292593594368\sqrt{2} \left[P^{17} + \frac{1}{P^{17}} \right] - 1135412123693654016 \left[P^{16} + \frac{1}{P^{16}} \right] \\
& + 1415543996178500096\sqrt{2} \left[P^{15} + \frac{1}{P^{15}} \right] - 3358715145903804416 \left[P^{14} + \frac{1}{P^{14}} \right] \\
& + 3800077608784535552\sqrt{2} \left[P^{13} + \frac{1}{P^{13}} \right] - 8218354561861466496 \left[P^{12} + \frac{1}{P^{12}} \right] \\
& + 8512046427768020736\sqrt{2} \left[P^{11} + \frac{1}{P^{11}} \right] - 16925549916428891904 \left[P^{10} + \frac{1}{P^{10}} \right] \\
& + 16187505728909967008\sqrt{2} \left[P^9 + \frac{1}{P^9} \right] - 29846446514225130752 \left[P^8 + \frac{1}{P^8} \right]
\end{aligned}$$

$$\begin{aligned}
& + 26572617280356920576\sqrt{2} \left[P^7 + \frac{1}{P^7} \right] - 45769170606955635920 \left[P^6 + \frac{1}{P^6} \right] \\
& + 38178953157178087680\sqrt{2} \left[P^5 + \frac{1}{P^5} \right] - 61755339772352116992 \left[P^4 + \frac{1}{P^4} \right] \\
& + 48456136115051360544\sqrt{2} \left[P^3 + \frac{1}{P^3} \right] - 73802624665517984512 \left[P^2 + \frac{1}{P^2} \right] \\
& + 54558586002509042432\sqrt{2} \left[P + \frac{1}{P} \right] - 78308975348446520116.
\end{aligned}$$

where P and Q are defined as in (3.1) and (3.2) respectively.

Proofs of the identities (3.41)–(3.44) are similar to the proof of (3.24), except that in place of result (2.7), result (2.9) is used to prove (3.41); result (2.14) is used to prove (3.42); result (2.15) is used to prove (3.43) and the result (2.16) is used to prove (3.44).

Remark 3.1. The identities (3.19), (3.24), (3.28) and (3.29) have been verified using modular forms in [3, Entry 41, pp. 378, 379] and other identities (3.5), (3.11), (3.12), (3.27), (3.35), (3.36), (3.41)–(3.44) are appears to be new.

4. Explicit evaluations of class invariants

In this section, we establish several explicit evaluations of Ramanujan-Weber class invariants G_n using modular equations obtained in Section as an application.

Theorem 4.1. *We have*

$$(4.1) \quad G_2^8 = \frac{\sqrt{2} + 1}{2},$$

$$(4.2) \quad G_4^8 = \sqrt{2} \left(\frac{1 + \sqrt{2}}{2} \right)^2,$$

$$(4.3) \quad G_{16}^8 = \left(\frac{\sqrt{2} + 1}{4} \right) \left(8 + 4\sqrt{2} + \sqrt[4]{2}(6\sqrt{2} + 3) \right),$$

$$(4.4) \quad G_9^3 = \frac{\sqrt{3} + 1}{\sqrt{2}},$$

$$(4.5) \quad G_{11}^2 = \frac{1}{2^{1/6}3^{1/2}} \left[(3\sqrt{3} - \sqrt{11})^{1/3} + (3\sqrt{3} + \sqrt{11})^{1/3} \right],$$

$$(4.6) \quad G_{13}^4 = \frac{3 + \sqrt{13}}{2},$$

$$(4.7) \quad G_{15}^6 = \sqrt{2} \left(\frac{1 + \sqrt{5}}{\sqrt{2}} \right)^2,$$

$$(4.8) \quad G_{17} = \left(\frac{1}{4} + \frac{\sqrt{17}}{4} + \frac{1}{2} \sqrt{\frac{1}{2} + \frac{\sqrt{17}}{2}} \right)^{1/2},$$

$$(4.9) \quad G_{19} = \frac{1}{\sqrt{3}} \left(2\sqrt{2} + \left(\frac{23}{\sqrt{2}} - 3\sqrt{\frac{57}{2}} \right)^{1/3} + \left(\frac{23}{\sqrt{2}} + 3\sqrt{\frac{57}{2}} \right)^{1/3} \right)^{1/2},$$

$$(4.10) \quad G_{23} = \left(\frac{2\sqrt{2} + (25\sqrt{2} - 3\sqrt{138})^{1/3} + (25\sqrt{2} + 3\sqrt{138})^{1/3}}{3} \right)^{1/2},$$

$$(4.11) \quad G_{25} = \frac{\sqrt{5} + 1}{2},$$

$$(4.12) \quad G_{29}^4 = \frac{1}{6} \left\{ 9 + 3^{1/3} \left[(207 - 16\sqrt{87})^{1/3} + (207 + 16\sqrt{87})^{1/3} \right] + \sqrt{36 + \left(-9 - 3^{1/3} \left[(207 - 16\sqrt{87})^{1/3} - (207 + 16\sqrt{87})^{1/3} \right] \right)^2} \right\},$$

$$(4.13) \quad G_{31}^2 = \frac{1}{3} \left\{ \sqrt{2} + (47\sqrt{2} - 3\sqrt{186})^{1/3} + (47\sqrt{2} + 3\sqrt{186})^{1/3} \right\},$$

$$(4.14) \quad G_{81}^3 = (47 - 2\sqrt{3})^{2/3} \left(\frac{43 + 27\sqrt{3}}{169} \right) + (47 - 2\sqrt{3})^{1/3} \left(\frac{15 + 7\sqrt{3}}{13} \right) + 3 + 2\sqrt{3},$$

$$(4.15) \quad G_{121} = \frac{1}{12} \left[4\sqrt{2} + 2^{5/6} 11^{1/3} \left((7 - 3\sqrt{3})^{1/3} + (7 + 3\sqrt{3})^{1/3} \right) + \sqrt{144 + \left[-4\sqrt{2} - 2^{5/6} 11^{1/3} \left((7 - 3\sqrt{3})^{1/3} - (7 + 3\sqrt{3})^{1/3} \right) \right]^2} \right].$$

Proof of (4.1). Putting $n = \frac{1}{2}$, using the fact that $G_n = G_{1/n}$ and by the definition of G_n in (3.5), we find that

$$(4.16) \quad (4G^{16} - 4G^8 - 1)(2G^8 + 1)^2(2G^4 - 1)^2(2G^4 + 1)^2 = 0,$$

where $G := G_2$. But

$$(4.17) \quad 4G^{16} - 4G^8 - 1 = 0.$$

Solving the above equation and $G_2 > 1$, we obtain the required result (4.1). ■

Proofs of the identities (4.2)–(4.15) are similar to the proof of (4.1), except that in place of result (3.5), result (3.11) is used to evaluate (4.2) and (4.3); result (3.12) is used to prove (4.4) and (4.14); result (3.19) is used to prove (4.5) and (4.15); result (3.24) is used to prove (4.6); result (3.27) is used to prove (4.7); result (3.28) is used to prove (4.8); result (3.29) is used to prove (4.9); result (3.35) is used to prove (4.10); result (3.36) is used to prove (4.11); result (3.41) is used to prove (4.12) and result (3.42) is used to prove (4.13).

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COMMON FIXED POINTS FOR WEAKLY COMPATIBLE MAPPINGS AND APPLICATIONS IN DYNAMIC PROGRAMMING

Hemant Kumar Pathak

*School of Studies in Mathematics
Pt. Ravishankar Shukla University
Raipur (C.G.), 492010
India
e-mail: hkpathak05@gmail.com*

Rakesh Tiwari¹

*Department Of Mathematics
Govt. V.Y.T.PG.Autonomous College
Durg (C.G.), 491001
India
e-mail: rakeshtiwari66@gmail.com*

Abstract. In this note, we establish a common fixed point theorem for a quadruple of self mappings on a complete metric space satisfying weak compatibility and a generalized Φ -contraction. Our main result improves and extends some known results. As an application, we use our main result to obtain common solutions of certain functional equations arising in dynamic programming. We also discuss an illustrative example to validate all the conditions of the main result in dynamic programming.

Keywords and phrases: common fixed points, weakly compatible mappings, dynamic programming.

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1. Introduction

In 1986, the notion of compatible mappings which generalizes commuting mappings, was introduced by Jungck [6]. Further, in 1998, the more general class of mappings called weakly compatible mappings was introduced by Jungck and Rhoades [7]. Recall that self mappings S and T of a metric space (X, d) are called weakly compatible if $Sx = Tx$ for some $x \in X$ implies that $STx = TStx$.

Bellman and Lee [1] initiated the basic form of the functional equation arising in dynamic programming as follows:

$$f(x) = \sup_{y \in D} \{A(x, y, f(a(x, y)))\}, \quad x \in S.$$

¹Corresponding author.

In 1984, Bhakta and Mitra [2] obtained some existence theorem for the following functional equation which arises in multistage decision process related to dynamic programming

$$f(x) = \sup_{y \in D} \{r(x, y) + f(c(x, y))\}, \quad x \in S.$$

In 2003, Liu and Ume [8] provided sufficient conditions which insure the existing and uniqueness for solution for the functional equation

$$f(x) = \text{opt}_{y \in D} \{u[p(x, y) + f(a(x, y))] + v \text{opt}[q(x, y), f(b(x, y))]\}, \quad x \in S.$$

Several existence and uniqueness results of solution and common solution for some functional equations and systems of functional equations in dynamic programming are discussed by Liu et al. [9].

In 2010, Singh and Mishra [13] established coincidence and fixed point theorems for a new class of contractive, nonexpansive and hybrid contractions mappings. Applications regarding the existence of solutions of certain functional equations are also discussed.

Recently, Jiang et al. [5] studied the properties of solutions of the following functional equation arising in dynamic programming of multistage decision process:

$$f(x) = \text{opt}_{y \in D} \{p(x, y), q(x, y)f(a(x, y)), r(x, y), f(b(x, y)), s(x, y)f(c(x, y))\}, \quad \forall x \in S.$$

Bondar et al. [3] proved some common fixed point theorems for two pairs of mappings and some applications are given in dynamic programming.

Most recently, Pathak et al. [10] introduced the following two functional equations arising in dynamic programming of multistage decision process:

$$f(x) = \text{opt}_{y \in D} \text{opt} \{p(x, y) + A(x, y, f(a(x, y))), q(x, y)\}, \quad \forall x \in S,$$

and

$$f(x) = \text{opt}_{y \in D} \text{opt} \{p_1(x, y) + q(x, y)f(a(x, y)), p_2(x, y) + r(x, y)f(b(x, y))\}, \quad \forall x \in S.$$

In this paper, we prove some common fixed point theorem for a quadruple of self mappings of a complete metric space satisfying weak compatibility condition and a generalized Φ -contraction. Subsequently, we use our main theorem to obtain common solutions of certain functional equations arising in dynamic programming.

2. Preliminaries

In what follows, we denote by Φ the collection of all the functions $\varphi : [0, \infty) \rightarrow [0, \infty)$ which are upper semicontinuous from the right, non-decreasing and satisfy $\limsup_{s \rightarrow t+} \varphi(s) < t$, $\varphi(t) < t$, for all $t > 0$.

Let X denote a metric space endowed with metric d and let $\mathbb{N}$ denote the set of natural numbers.

Now, let A, B, S and T be self-mappings of X such that

$$(2.1) \quad A(X) \subset T(X) \quad \text{and} \quad B(X) \subset S(X)$$

$$(2.2) \quad \begin{aligned} & [d^p(Ax, By) + a d^p(Sx, Ty)]d^p(Ax, By) \\ & \leq a \max\{d^p(Ax, Sx)d^p(By, Ty), d^q(Ax, Ty)d^{q'}(By, Sx)\} \\ & + \max\left\{\varphi_1(d^{2p}(Sx, Ty)), \varphi_2(d^r(Ax, Sx)d^{r'}(By, Ty)), \right. \\ & \varphi_3(d^s(Ax, Ty)d^{s'}(By, Sx)), \\ & \left. \varphi_4\left(\frac{1}{2}[d^l(Ax, Ty)]d^{l'}(Ax, Sx) + d^l(By, Sx)\right)d^{l'}(By, Ty)\right\}, \end{aligned}$$

for all $x, y \in X$, $\varphi_i \in \Phi$ ($i = 1, 2, 3, 4$), $a, p, q, q', r, r', s, s', l, l' \geq 0$ and $2p = q + q' = r + r' = s + s' = l + l' \leq 1$. Condition (2.2) is commonly called a generalized Φ -contraction.

Now, we pick $x_0 \in X$. Since $A(X) \subset T(X)$, we can choose a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Again, since $B(X) \subset S(X)$ for $x_1 \in X$, we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$. Continuing in this way, we can construct a sequence $\{y_n\}$ in X such that

$$(2.3) \quad y_{2n} = Tx_{2n+1} = Ax_{2n} \quad \text{and} \quad y_{2n+1} = Sx_{2n+2} = Bx_{2n+1} \quad (n \in \mathbb{N} \cup \{0\}).$$

First, we prove the following lemmas:

Lemma 2.1. *Let us suppose $d_n = d(y_n, y_{n-1})$, $n \in \mathbb{N}$. Then, $\lim_{n \rightarrow \infty} d_n = 0$.*

Proof. In (2.2), putting $x = x_{2n}$ and $y = x_{2n+1}$ and using (2.3), we get

$$\begin{aligned} & [d^p(Ax_{2n}, Bx_{2n+1}) + a d^p(Sx_{2n}, Tx_{2n+1})]d^p(Ax_{2n}, Bx_{2n+1}) \\ & \leq a \max\left\{d^p(Ax_{2n}, Sx_{2n})d^p(Bx_{2n+1}, Tx_{2n+1}), d^q(Ax_{2n}, Tx_{2n+1}) \right. \\ & \left. d^{q'}(Bx_{2n+1}, Sx_{2n})\right\} + \max\left\{\varphi_1(d^{2p}(Sx_{2n}, Tx_{2n+1})), \varphi_2(d^r(Ax_{2n}, Sx_{2n}) \right. \\ & \left. d^{r'}(Bx_{2n+1}, Tx_{2n+1})), \varphi_3(d^s(Ax_{2n}, Tx_{2n+1})d^{s'}(Bx_{2n+1}, Sx_{2n})), \right. \\ & \left. \varphi_4\left(\frac{1}{2}[d^l(Ax_{2n}, Tx_{2n+1})d^{l'}(Ax_{2n}, Sx_{2n}) + d^l(Bx_{2n+1}, Sx_{2n})d^{l'}(Bx_{2n+1}, Tx_{2n+1})]\right)\right\}, \end{aligned}$$

or

$$\begin{aligned} [d_{2n+1}^p + ad_{2n}^p]d_{2n+1}^p & \leq a \max\{d_{2n+1}^p d_{2n}^p, 0\} + \max\left\{\varphi_1(d_{2n}^{2p}), \varphi_2(d_{2n}^r d_{2n+1}^{r'}), \right. \\ & \left. \varphi_3(0), \varphi_4\left(\frac{1}{2}[d_{2n+1}^l + d_{2n}^l d_{2n+1}^{l'}]\right)\right\} \\ & \leq ad_{2n+1}^p d_{2n}^p + \max\left\{\varphi_1(d_{2n}^{2p}), \varphi_2(d_{2n}^r d_{2n+1}^{r'}), \right. \\ & \left. \varphi_3(0), \varphi_4\left(\frac{1}{2}[d_{2n+1}^l d_{2n+1}^{l'} + d_{2n}^l d_{2n+1}^{l'}]\right)\right\}, \end{aligned}$$

which implies

$$(2.4) \quad d_{2n+1}^{2p} \leq \max \left\{ \varphi_1(d_{2n}^{2p}), \varphi_2(d_{2n}^r d_{2n+1}^{r'}), \varphi_3(0), \varphi_4 \left(\frac{1}{2} [d_{2n+1}^{l+l'} + d_{2n}^l d_{2n+1}^{l'}] \right) \right\}.$$

If $d_{2n+1} > d_{2n}$ then, we have

$$d_{2n+1}^{2p} \leq \max \left\{ \varphi_1(d_{2n+1}^{2p}), \varphi_2(d_{2n+1}^{r+r'}), \varphi_3(0), \varphi_4 \left(\frac{1}{2} [d_{2n+1}^{l+l'} + d_{2n+1}^{l+l'}] \right) \right\} \leq \varphi_i(d_{2n+1}^{2p})$$

($i = 1, 2, 4$).

This, together with a well known result of Chang [4], which states that, if $\varphi_i \in \Phi$, where $i \in I$ (some indexing set), then there exists a $\varphi \in \Phi$ such that $\max\{\varphi_i, i \in I\} \leq \varphi(t)$, for all $t > 0$, imply $d_{2n+1}^{2p} < d_{2n+1}^{2p}$, a contradiction. Consequently, we have $d_{2n+1} \leq d_{2n}$, for all $n \in \mathbb{N}$, and

$$(2.5) \quad d_{2n+1} \leq \varphi(d_{2n}) \quad \text{for all } n \in \mathbb{N} \text{ and some } \varphi \in \Phi.$$

Similarly, for $x = x_{2n+2}$ and $y = x_{2n+1}$, we have

$$(2.6) \quad d_{2n+2}^{2p} \leq \max \left\{ \varphi_1(d_{2n+1}^{2p}), \varphi_2(0), \varphi_3(0), \varphi_4 \left(\frac{1}{2} [d_{2n+2}^{l+l'} + d_{2n+1}^l d_{2n+2}^{l'}] \right) \right\}.$$

A similar argument applied to (2.6) will give

$$(2.7) \quad d_{2n+2} \leq \varphi(d_{2n+1}) \quad \text{for all } n \in \mathbb{N},$$

where $\varphi \in \Phi$ is assumed to be same as in the previous case. Therefore, for all $n \in \mathbb{N}$, we have $d_{n+1} \leq \varphi(d_n)$, and by Lemma 2 of [4], we have $\lim_{n \rightarrow \infty} d_n = 0$.

Lemma 2.2. *The sequence $\{y_n\}$ defined in (2.3) is a Cauchy sequence.*

Proof. We prove that the subsequence $\{y_{2n}\}$ of the sequence $\{y_n\}$ is a Cauchy sequence. On the contrary, let us suppose that $\{y_{2n}\}$ is not Cauchy. Then, there exists an $\epsilon > 0$ such that for each even integer $2k$ there exist even integers $2m(k)$, $2n(k)$ ($n \in \mathbb{N}$) with $2m(k) > 2n(k) \geq 2k$, such that

$$(2.8) \quad d(y_{2n(k)}, y_{2m(k)}) > \epsilon \quad \text{and} \quad d(y_{2n(k)}, y_{2m(k)-2}) \leq \epsilon,$$

that is, $2m(k)$ is the least positive even integer such that $2m(k) > 2n(k)$ and

$$d(y_{2n(k)}, y_{2m(k)-2}) \leq \epsilon.$$

Hence, for each even integer $2k$, we have

$$\begin{aligned} \epsilon &< d(y_{2n(k)}, y_{2m(k)}) \\ &\leq d(y_{2n(k)}, y_{2m(k)-2}) + d(y_{2m(k)-2}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2m(k)}) \\ &< \epsilon + d_{2m(k)-1} + d_{2m(k)}. \end{aligned}$$

Hence, by Lemma 2.1 and (2.8) it follows that

$$(2.9) \quad \lim_{n \rightarrow \infty} d(y_{2n(k)}, y_{2m(k)}) = \epsilon.$$

By making use of the triangle inequalities, for $\rho \in [0, 1]$, we have

$$\begin{aligned} d^\rho(y_{2m(k)+2}, y_{2n(k)+1}) &\leq d^\rho(y_{2m(k)+2}, y_{2m(k)+1}) + d^\rho(y_{2m(k)+1}, y_{2m(k)}) \\ &\quad + d^\rho(y_{2m(k)}, y_{2n(k)}) + d^\rho(y_{2n(k)}, y_{2n(k)+1}), \end{aligned}$$

or

$$d^\rho(y_{2m(k)+2}, y_{2n(k)+1}) - d^\rho(y_{2m(k)}, y_{2n(k)}) \leq d_{2m(k)+2}^\rho + d_{2m(k)+1}^\rho + d_{2n(k)+1}^\rho.$$

And

$$\begin{aligned} d^\rho(y_{2m(k)}, y_{2n(k)}) &\leq d^\rho(y_{2m(k)}, y_{2m(k)+1}) + d^\rho(y_{2m(k)+1}, y_{2m(k)+2}) \\ &\quad + d^\rho(y_{2m(k)+2}, y_{2n(k)+1}) + d^\rho(y_{2n(k)+1}, y_{2n(k)}), \end{aligned}$$

or

$$d^\rho(y_{2m(k)}, y_{2n(k)}) - d^\rho(y_{2m(k)+2}, y_{2n(k)+1}) \leq d_{2m(k)+1}^\rho + d_{2m(k)+2}^\rho + d_{2n(k)+1}^\rho.$$

Thus, we obtain

$$(2.10) \quad |d^\rho(y_{2m(k)}, y_{2n(k)}) - d^\rho(y_{2m(k)+2}, y_{2n(k)+1})| \leq d_{2m(k)+1}^\rho + d_{2m(k)+2}^\rho + d_{2n(k)+1}^\rho.$$

Similarly we have

$$(2.11) \quad |d^\rho(y_{2m(k)+1}, y_{2n(k)}) - d^\rho(y_{2n(k)}, y_{2m(k)})| \leq d_{2m(k)+1}^\rho,$$

$$(2.12) \quad |d^\rho(y_{2m(k)+1}, y_{2n(k)+1}) - d^\rho(y_{2n(k)}, y_{2m(k)})| \leq d_{2n(k)+1}^\rho + d_{2m(k)+1}^\rho,$$

and

$$(2.13) \quad |d^\rho(y_{2m(k)+2}, y_{2n(k)}) - d^\rho(y_{2n(k)}, y_{2m(k)})| \leq d_{2m(k)+1}^\rho + d_{2m(k)+2}^\rho.$$

By Lemma 2.2 and inequalities (2.10)–(2.13), we have

$$\begin{aligned} \lim_{k \rightarrow \infty} d^\rho(y_{2m(k)+2}, y_{2n(k)+1}) &= \lim_{k \rightarrow \infty} d^\rho(y_{2m(k)+1}, y_{2n(k)}) \\ &= \lim_{k \rightarrow \infty} d^\rho(y_{2m(k)+1}, y_{2n(k)+1}) \\ &= \lim_{k \rightarrow \infty} d^\rho(y_{2m(k)+2}, y_{2n(k)}) \\ &= \epsilon. \end{aligned} \quad (2.14)$$

Now, using (2.2) with $x = x_{2m(k)+2}$ and $y = x_{2n(k)+1}$ along with (2.3) and a rearrangement, we obtain

$$\begin{aligned}
& [d^p(Ax_{2m(k)+2}, Bx_{2n(k)+1}) + a d^p(Sx_{2m(k)+2}, Tx_{2n(k)+1})]d^p(Ax_{2m(k)+2}, Bx_{2n(k)+1}) \\
& \leq a \max\{d^p(Ax_{2m(k)+2}, Sx_{2m(k)+2})d^p(Bx_{2n(k)+1}, Tx_{2n(k)+1}), \\
& \quad d^q(Ax_{2m(k)+2}, Tx_{2n(k)+1})d^{q'}(Bx_{2n(k)+1}, Sx_{2m(k)+2})\} \\
& \quad + \max\left\{\varphi_1(d^{2p}(Sx_{2m(k)+2}, Tx_{2n(k)+1})), \right. \\
& \quad \varphi_2(d^r(Ax_{2m(k)+2}, Sx_{2m(k)+2})d^{r'}(Bx_{2n(k)+1}, Tx_{2n(k)+1})), \\
& \quad \varphi_3(d^s(Ax_{2m(k)+2}, Tx_{2n(k)+1})d^{s'}(Bx_{2n(k)+1}, Sx_{2m(k)+2})), \\
& \quad \varphi_4\left(\frac{1}{2}[d^l(Ax_{2m(k)+2}, Tx_{2n(k)+1})d^{l'}(Ax_{2m(k)+2}, Sx_{2m(k)+2}) \right. \\
& \quad \left. \left. + d^l(Bx_{2n(k)+1}, Sx_{2m(k)+2})d^{l'}(Bx_{2n(k)+1}, Tx_{2n(k)+1})\right]\right\},
\end{aligned}$$

or

$$\begin{aligned}
& [d^p(y_{2m(k)+2}, y_{2n(k)+1}) + a d^p(y_{2m(k)+1}, y_{2n(k)})]d^p(y_{2m(k)+2}, y_{2n(k)+1}) \\
& \leq a \max\{d^p(y_{2m(k)+2}, y_{2m(k)+1})d^p(y_{2n(k)+1}, y_{2n(k)}), \\
& \quad d^q(y_{2m(k)+2}, y_{2n(k)+1})d^{q'}(y_{2n(k)+1}, y_{2m(k)+1})\} \\
& \quad + \max\left\{\varphi_1(d^{2p}(y_{2m(k)+1}, y_{2n(k)})), \right. \\
& \quad \varphi_2(d^r(y_{2m(k)+2}, y_{2m(k)+1})d^{r'}(y_{2n(k)+1}, y_{2n(k)})), \\
& \quad \varphi_3(d^s(y_{2m(k)+2}, y_{2n(k)})d^{s'}(y_{2n(k)+1}, y_{2m(k)+1})), \\
& \quad \varphi_4\left(\frac{1}{2}[d^l(y_{2m(k)+2}, y_{2n(k)})d^{l'}(y_{2m(k)+2}, y_{2m(k)+1}) \right. \\
& \quad \left. \left. + d^l(y_{2n(k)+1}, y_{2m(k)+1})d^{l'}(y_{2n(k)+1}, y_{2n(k)})\right]\right\}.
\end{aligned}$$

Letting $k \rightarrow \infty$ and using Lemma 2.1, (2.9) and (2.14) and the fact that $\varphi_i \in \Phi$ ($i = 1, 2, 3, 4$), we have $\epsilon^{2p} + a\epsilon^{2p} \leq a\epsilon^{q+q'} + \max\{\varphi_1(\epsilon^{2p}), \varphi_2(0), \varphi_3(\epsilon^{s+s'}), \varphi_4(0)\}$, or $\epsilon^{2p} \leq \max\{\varphi_1(\epsilon^{2p}), \varphi_2(0), \varphi_3(\epsilon^{s+s'}), \varphi_4(0)\}$, or $\epsilon^{2p} \leq \varphi(\epsilon^{2p}) < \epsilon^{2p}$, a contradiction. Hence, $\{y_{2n}\}$ is a Cauchy sequence in X . This proves that $\{y_n\}$ is Cauchy in X .

3. Main results

The following theorems are our main results of this section.

Theorem 3.1. *Let A, B, S and T be self mappings of a complete metric space X satisfying (2.1) and (2.2). If the pairs (A, S) and (B, T) are weakly compatible and $T(X)$ or $S(X)$ is closed, then A, B, S and T have a unique common fixed point in X .*

Proof. Since X is complete, it follows from Lemma 2.2 that the sequence $\{y_n\}$ converges to a point z in X . Consequently, the subsequences $\{Ax_{2n}\}$, $\{Bx_{2n-1}\}$, $\{Sx_{2n}\}$, $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converge to the same limit z .

Now, suppose that $T(X)$ is closed. Then, since $\{Tx_{2n+1}\} \subset T(X)$, there exists a point $u \in X$ such that $z = Tu$. Then, by using (2.2) with $x = x_{2n}$ and $y = u$, we have

$$\begin{aligned} & [d^p(Ax_{2n}, Bu) + ad^p(Sx_{2n}, Tu)]d^p(Ax_{2n}, Bu) \\ & \leq a \max \left\{ d^p(Ax_{2n}, Sx_{2n})d^p(Bu, Tu), d^q(Ax_{2n}, Tu)d^{q'}(Bu, Sx_{2n}) \right\} \\ & + \max \left\{ \varphi_1(d^{2p}(Sx_{2n}, Tu)), \varphi_2(d^r(Ax_{2n}, Sx_{2n})d^{r'}(Bu, Tu)), \right. \\ & \quad \varphi_3(d^s(Ax_{2n}, Tu)d^{s'}(Bu, Sx_{2n})), \\ & \quad \left. \varphi_4\left(\frac{1}{2}[d^l(Ax_{2n}, Tu)d^{l'}(Ax_{2n}, Sx_{2n}) + d^l(Bu, Sx_{2n})d^{l'}(Bu, Tu)]\right) \right\}, \end{aligned}$$

letting $k \rightarrow \infty$, we obtain

$$\begin{aligned} [d^p(z, Bu) + a d^p(z, z)]d^p(z, Bu) & \leq a \max \{ d^p(z, z)d^p(Bu, z), d^q(z, z)d^{q'}(Bu, z) \} \\ & + \max \left\{ \varphi_1(d^{2p}(z, z)), \varphi_2(d^r(z, z)d^{r'}(Bu, z)), \right. \\ & \quad \varphi_3(d^s(z, z)d^{s'}(Bu, z)), \varphi_4\left(\frac{1}{2}[d^l(z, z)d^{l'}(z, z) \right. \\ & \quad \left. \left. + d^l(Bu, z)d^{l'}(Bu, z)]\right) \right\}, \end{aligned}$$

or

$$d^{2p}(z, Bu) \leq \max \left\{ \varphi_1(0), \varphi_2(0), \varphi_3(0), \varphi_4\left(\frac{1}{2}d^{l+l'}(Bu, z)\right) \right\},$$

or

$$\begin{aligned} d^{2p}(z, Bu) & \leq \max \left\{ \varphi_1(d^{2p}(z, Bu)), \varphi_2(d^{r+r'}(z, Bu)), \right. \\ & \quad \left. \varphi_3(d^{s+s'}(z, Bu)), \varphi_4\left(\frac{1}{2}d^{l+l'}(Bu, z)\right) \right\} \\ & \leq \varphi(d^{2p}(z, Bu)) \\ & < d^{2p}(z, Bu), \end{aligned}$$

a contradiction. This implies that $z = Bu$. Therefore, $Tu = z = Bu$. Hence, it follows by the weak compatibility of the pair (B, T) that $BTu = TBu$, that is $Bz = Tz$.

Now, we shall show that z is a common fixed point of B and T . For this put $x = x_{2n}$ and $y = z$ in (2.2), we have

$$\begin{aligned} & [d^p(Ax_{2n}, Bz) + a d^p(Sx_{2n}, Tz)]d^p(Ax_{2n}, Bz) \\ & \leq a \max \left\{ d^p(Ax_{2n}, Sx_{2n})d^p(Bz, Tz), d^q(Ax_{2n}, Tz)d^{q'}(Bz, Sx_{2n}) \right\} \\ & + \max \{ \varphi_1(d^{2p}(Sx_{2n}, Tz)), \varphi_2(d^r(Ax_{2n}, Sx_{2n})d^{r'}(Bz, Tz)), \\ & \quad \varphi_3(d^s(Ax_{2n}, Tz)d^{s'}(Bz, Sx_{2n})), \\ & \quad \varphi_4\left(\frac{1}{2}[d^l(Ax_{2n}, Tz)d^{l'}(Ax_{2n}, Sx_{2n}) + d^l(Bz, Sx_{2n})d^{l'}(Bz, Tz)]\right) \}. \end{aligned}$$

Letting $n \rightarrow \infty$, we get

$$[d^p(z, Bz) + a d^p(z, Tz)] d^p(z, Bz) \leq a \max\{d^p(z, z) d^p(Bz, Tz), d^q(z, Tz) d^{q'}(Bz, z)\} \\ + \max \text{Big}\{\varphi_1(d^{2p}(z, Tz)), \varphi_2(d^r(z, z) d^{r'}(Bz, Tz)), \varphi_3(d^s(z, Tz) d^{s'}(Bz, z)), \\ \varphi_4\left(\frac{1}{2}[d^l(z, Tz) d^{l'}(z, z) + d^l(Bz, z) d^{l'}(Bz, Tz)]\right)\},$$

or

$$d^{2p}(z, Bz) + a d^{2p}(z, Bz) \\ \leq a d^{q+q'}(Bz, z) + \max\{\varphi_1(d^{2p}(z, Bz)), \varphi_2(0), \varphi_3(d^{s+s'}(z, Bz)), \varphi_4(0)\},$$

or

$$(1 + a) d^{2p}(z, Bz) \leq a d^{q+q'}(Bz, z) \\ + \max\{\varphi_1(d^{2p}(z, Bz)), \varphi_2(0), \varphi_3(d^{s+s'}(z, Bz)), \varphi_4(0)\},$$

or

$$d^{2p}(z, Bz) \leq \frac{a}{1+a} d^{q+q'}(Bz, z) \\ + \frac{1}{1+a} \max\{\varphi_1(d^{2p}(z, Bz)), \varphi_2(0), \varphi_3(d^{s+s'}(z, Bz)), \varphi_4(0)\} \\ < d^{2p}(z, Bz),$$

a contradiction. So $z = Bz = Tz$. Thus z is a common fixed point of B and T .

Similarly, we can prove that z is a common fixed point of A and S . Thus, z is the common fixed point of A , B , S and T . The uniqueness of z as a common fixed point of A , B , S and T can easily be verified.

Remark 3.2 If we assume $S(X)$ to be closed then the above theorem also remains valid. We find the same result if $A(X)$ or $B(X)$ is assumed to be closed by (2.1).

Remark 3.3 Our Theorem 3.1 extends Theorem 2.1 of Pathak et al.[7].

In Theorem 3.1, if we put $a = 0$ and $\varphi_i(t) = ht$ ($i=1, 2, 3, 4$), where $0 < h < 1$, we get the following corollary:

Corollary 3.4. *Let A , B , S and T be self mappings of a complete metric space X satisfying (2.1) and (2.2').*

$$(2.2') \quad d^{2p}(Ax, By) \leq h \max \left\{ (d^{2p}(Sx, Ty), d^r(Ax, Sx) d^{r'}(By, Ty), d^s(Ax, Ty) \right. \\ \left. d^{s'}(By, Sx)), \frac{1}{2}[d^l(Ax, Ty) d^{l'}(Ax, Sx) + d^l(By, Sx) d^{l'}(By, Ty)] \right\}$$

for all $x, y \in X$, $\varphi_i \in \Phi$ ($i = 1, 2, 3, 4$), $a, p, q, q', r, r', s, s', l, l' \geq 0$ and $2p = q + q' = r + r' = s + s' = l + l' \leq 1$. If the pairs (A, S) and (B, T) are weakly compatible and $T(X)$ or $S(X)$ is closed, then A , B , S and T have a unique common fixed point in X .

Especially, when $\max\{d^{2p}(Sx, Ty), d^r(Ax, Sx)d^{r'}(By, Ty), d^s(Ax, Ty)d^{s'}(By, Sx), \frac{1}{2}[d^l(Ax, Ty)d^{l'}(Ax, Sx) + d^l(By, Sx)d^{l'}(By, Ty)]\} = d^{2p}(Sx, Ty)$, we get Corollary 3.9 of Pathak et al.[9].

In Theorem 3.1, if we take $S = T = I_X$ (the identity mapping on X), then we have the following corollary:

Corollary 3.5. *Let A and B be self mappings of a complete metric space X satisfying the following condition:*

$$\begin{aligned} & [d^p(Ax, By) + a d^p(x, y)]d^p(Ax, By) \\ & \leq a \max\{d^p(Ax, x)d^p(By, y), d^q(Ax, y)d^{q'}(By, x)\} \\ & + \max\left\{\varphi_1(d^{2p}(x, y)), \varphi_2(d^r(Ax, x)d^{r'}(By, y)), \varphi_3(d^s(Ax, y)d^{s'}(By, x)), \right. \\ & \left. \varphi_4\left(\frac{1}{2}[d^l(Ax, y)d^{l'}(Ax, x) + d^l(By, x)d^{l'}(By, y)]\right)\right\} \end{aligned}$$

for all $x, y \in X, \varphi_i \in \Phi$ ($i = 1, 2, 3, 4$), $a, p, q, q', r, r', s, s', l, l' \geq 0$ and $2p = q + q' = r + r' = s + s' = l + l' \leq 1$, then A and B have a unique common fixed point in X .

As an immediate consequences of Theorem 3.1 with $S = T$, we have the following:

Corollary 3.6. *Let A, B , and S be self-mappings of X such that*

$$(2.1)' \quad A(X) \cup B(X) \subset S(X)$$

$$\begin{aligned} & [d^p(Ax, By) + a d^p(Sx, Sy)]d^p(Ax, By) \\ & \leq a \max\{d^p(Ax, Sx)d^p(By, Sy), d^q(Ax, Sy)d^{q'}(By, Sx)\} \\ (2.2'') \quad & + \max\left\{\varphi_1(d^{2p}(Sx, Sy)), \varphi_2(d^r(Ax, Sx)d^{r'}(By, Sy)), \right. \\ & \varphi_3(d^s(Ax, Sy)d^{s'}(By, Sx)), \varphi_4\left(\frac{1}{2}[d^l(Ax, Sy)d^{l'}(Ax, Sx) \right. \\ & \left. \left. + d^l(By, Sx)d^{l'}(By, Sy)]\right)\right\} \end{aligned}$$

for all $x, y \in X, \varphi_i \in \Phi$ ($i = 1, 2, 3, 4$), $a, p, q, q', r, r', s, s', l, l' \geq 0$ and $0 \leq 2p = q + q' = r + r' = s + s' = l + l' \leq 1$. If the pairs (A, S) and (B, S) are weakly compatible and $S(X)$ is closed, then A, B and S have a unique common fixed point in X .

The following theorem is an immediate consequence of Theorem 3.1.

Theorem 3.7. *Let S, T and A_n ($n \in \mathbb{N}$) be self mappings of a complete metric space X . Suppose further that the pairs (A_{2n-1}, S) and (A_{2n}, T) are weakly compatible for any $n \in \mathbb{N}$ and*

$$A_{2n-1}(X) \subset T(X) \quad \text{and} \quad A_{2n}(X) \subset S(X).$$

If $S(X)$ or $T(X)$ is closed and that for any $i \in N$, the following condition is satisfied for all $x, y \in X$

$$\begin{aligned} [d^p(A_i x, A_{i+1} y) + a d^p(Sx, Ty)] d^p(A_i x, A_{i+1} y) \\ \leq a \max\{d^p(A_i x, Sx) d^p(A_{i+1} y, Ty), \\ d^q(A_i x, Ty) d^{q'}(A_{i+1} y, Sx)\} + \max\left\{\varphi_1(d^{2p}(Sx, Ty)), \right. \\ \varphi_2(d^r(A_i x, Sx) d^{r'}(A_{i+1} y, Ty)), \varphi_3(d^s(A_i x, Ty) d^{s'}(A_{i+1} y, Sx)), \\ \left. \varphi_4\left(\frac{1}{2}[d^l(A_i x, Ty) d^{l'}(A_i x, Sx) + d^l(A_{i+1} y, Sx) d^{l'}(A_{i+1} y, Ty)]\right)\right\} \end{aligned}$$

where $\varphi_i \in \Phi$ ($i = 1, 2, 3, 4$), $a, p, q, q', r, r', s, s', l, l' \geq 0$ and $0 \leq 2p = q + q' = r + r' = s + s' = l + l' \leq 1$, then S, T and A_n ($n \in \mathbb{N}$) have a common fixed point in X .

4. Applications to existence theorems for functional equations arising in dynamic programming

Throughout this section, we assume that X and Y are Banach spaces, $S \subset X$ is the state space and $D \subset Y$ is the decision space. Let $\mathbb{R} = (-\infty, \infty)$ and $B(S)$ denote the set of all bounded real valued functions on S .

The basic form of the functional equation of dynamic programming is given by Bellman and Lee [1] as follows:

$$f(x) = \text{opt}_y H(x, y, f(T(x, y))),$$

where x and y represent the state and decision vectors respectively, T represents the transformation of the process and $f(x)$ represents the optimal return with initial state x (where opt denotes max or min).

In this section, we study the existence and uniqueness of a common solution of the following functional equations arising in dynamic programming.

$$(4.1) \quad f_i(x) = \sup_{y \in D} H_i(x, y, f_i(T(x, y))), x \in S,$$

$$(4.2) \quad g_i(x) = \sup_{y \in D} F_i(x, y, g_i(T(x, y))), x \in S,$$

where $T : S \times D \rightarrow S$ and $H_i, F_i : S \times D \times \mathbb{R} \rightarrow \mathbb{R}$, $i = 1, 2$.

Suppose the mappings A_i and T_i ($i = 1, 2$) are defined by

$$\begin{aligned} (4.3) \quad A_i h(x) &= \sup_{y \in D} H_i(x, y, h(T(x, y))), \\ T_i k(x) &= \sup_{y \in D} F_i(x, y, k(T(x, y))), \end{aligned}$$

for all $x \in S$; $h, k \in B(S)$, $i = 1, 2$.

Now, we present our main theorems of this section.

Theorem 4.1. Suppose that the following conditions are satisfied:

- (i) H_i and F_i are bounded for $i = 1, 2$.
- (ii) $|H_1(x, y, h(t)) - H_2(x, y, k(t))| \leq M^{-1}(a \max\{|T_1h(t) - A_1h(t)| \cdot |T_2k(t) - A_2k(t)|, |T_1h(t) - A_2k(t)| \cdot |T_2k(t) - A_1h(t)|\} + \max\{\varphi_1(|T_1h(t) - T_2k(t)|), \varphi_2(|T_1h(t) - A_1h(t)|), \varphi_3(|T_2k(t) - A_2k(t)|), \varphi_4(\frac{1}{2}(|T_1h(t) - A_2k(t)| + |T_2k(t) - A_1h(t)|))\})$, for all $(x, y) \in S \times D$, $k \in B(S)$, $t \in S$, $a \geq 0$, where

$$M = [1 + a \sup_{t \in S} |T_1k(t) - T_2h(t)|], \quad \varphi_i \in \Phi \quad (i = 1, 2, 3, 4)$$

and the mappings A_i and T_i ($i = 1, 2$) are defined as in (4.3).

- (iii) For sequences $\{h_n\}, \{k_n\} \subset B(S)$ and $h, k \in B(S)$ with

$$\lim_{n \rightarrow \infty} \sup_{x \in S} |h_n(x) - h(x)| = 0, \quad \lim_{n \rightarrow \infty} \sup_{x \in S} |k_n(x) - k(x)| = 0,$$

there exist $h_i, k_i \in B(S)$ such that $k = T_2h_i$ and $h = T_1k_i$ for $i = 1$ or 2 .

- (iv) For any $h \in B(S)$, there exist $k_1, k_2 \in B(S)$ such that $A_1h(x) = T_2k_2(x)$, $A_2h(x) = T_1k_1(x)$, $x \in S$.
- (v) For any $h, k \in B(S)$, with $A_1h = T_1h$, we have $T_1A_1h = A_1T_1h$ and with $A_2k = T_2k$, we have $T_2A_2k = A_2T_2k$.

Then, the system of functional equations (4.1) and (4.2) have a unique common solution in $B(S)$.

Proof. Obviously, $B(S)$ endowed with the metric

$$d(h, k) = \sup_{x \in D} |h(x) - k(x)| \quad \text{for any } h, k \in B(S)$$

is a complete metric space. Moreover, by condition (i), A_i and T_i are self mappings of $B(S)$ and by condition (iv) it is clear that

$$A_1(B(S)) \subset T_2(B(S)) \quad \text{and} \quad A_2(B(S)) \subset T_1(B(S)).$$

Also, by condition (v), the pairs (A_i, T_i) are weakly compatible for $i = 1, 2$. Moreover, by (4.3) and (i) we have for any $\eta > 0$ there exist $y_i \in D$ ($i = 1, 2$) such that

$$(4.4) \quad A_i h_i(x) < H_i(x_i, y_i, h_i(x)) + \eta,$$

where $x_i = T(x, y_i)$, $i = 1, 2$. Also,

$$(4.5) \quad A_1 h_1(x) \geq H_1(x, y_2, h_1(x_2)),$$

$$(4.6) \quad A_2 h_2(x) \geq H_2(x, y_2, h_2(x_1)),$$

Then, from (4.4), (4.5), (4.6) and (ii), we have

$$\begin{aligned}
 (4.7) \quad & A_1 h_1(x) - A_2 h_2(x) \leq H_1(x, y_1, h_1(x_1)) - H_2(x, y_1, h_2(x_1)) + \eta \\
 & \leq |H_1(x, y_1, h_1(x_1)) - H_2(x, y_1, h_2(x_1))| + \eta \\
 & \leq M^{-1} \left(a \max\{|T_1 h_1(x_1) - A_1 h_1(x_1)|, |T_2 h_2(x_1) - A_2 h_2(x_1)|, \right. \\
 & \quad |T_1 h_1(x_1) - A_2 h_2(x_1)|, |T_2 h_2(x_1) - A_1 h_1(x_1)|\} \\
 & \quad + \max\{\varphi_1(|T_1 h_1(x_1) - T_2 h_2(x_1)|), \varphi_2(|T_1 h_1(x_1) - A_1 h_1(x_1)|), \\
 & \quad \varphi_3(|T_2 h_2(x_1) - A_2 h_2(x_1)|), \varphi_4\left(\frac{1}{2}[|T_1 h_1(x_1) - A_2 h_2(x_1)| \right. \\
 & \quad \left. \left. + |T_2 h_2(x_1) - A_1 h_1(x_1)|]\right)\} \right) \\
 & \leq M^{-1} \left(a \max\{d(T_1 h_1, A_1 h_1)d(T_2 h_2, A_2 h_2), \right. \\
 & \quad d(T_1 h_1, A_2 h_2)d(T_2 h_2, A_1 h_1)\} \\
 & \quad + \max\{\varphi_1(d(T_1 h_1, T_2 h_2)), \varphi_2(d(T_1 h_1, A_1 h_1)), \\
 & \quad \left. \varphi_3(d(T_2 h_2, A_2 h_2)), \varphi_4\left(\frac{1}{2}[d(T_1 h_1, A_2 h_2) + d(T_2 h_2, A_1 h_1)]\right)\} \right) + \eta.
 \end{aligned}$$

From (4.4), (4.5) and (ii), we have

$$\begin{aligned}
 (4.8) \quad & A_1 h_1(x) - A_2 h_2(x) \\
 & \geq -M^{-1} \left(a \max\{d(T_1 h_1, A_1 h_1)d(T_2 h_2, A_2 h_2), d(T_1 h_1, A_2 h_2)d(T_2 h_2, A_1 h_1)\} \right. \\
 & \quad + \max\{\varphi_1(d(T_1 h_1, T_2 h_2)), \varphi_2(d(T_1 h_1, A_1 h_1)), \\
 & \quad \left. \varphi_3(d(T_2 h_2, A_2 h_2)), \varphi_4\left(\frac{1}{2}[d(T_1 h_1, A_2 h_2) + d(T_2 h_2, A_1 h_1)]\right)\} \right) - \eta.
 \end{aligned}$$

Using (4.7) and (4.8), we obtain

$$\begin{aligned}
 (4.9) \quad & |A_1 h_1(x) - A_2 h_2(x)| \\
 & \leq M^{-1} \left(a \max\{d(T_1 h_1, A_1 h_1)d(T_2 h_2, A_2 h_2), d(T_1 h_1, A_2 h_2)d(T_2 h_2, A_1 h_1)\} \right. \\
 & \quad + \max\{\varphi_1(d(T_1 h_1, T_2 h_2)), \varphi_2(d(T_1 h_1, A_1 h_1)), \varphi_3(d(T_2 h_2, A_2 h_2)), \\
 & \quad \left. \varphi_4\left(\frac{1}{2}[d(T_1 h_1, A_2 h_2) + d(T_2 h_2, A_1 h_1)]\right)\} \right) + \eta.
 \end{aligned}$$

Since (4.9) is true for any $x \in S$ and $\eta > 0$ is arbitrary, by taking sup over all $x \in S$ we have,

$$\begin{aligned}
 [1 + a d(T_1 h_1, T_2 h_2)]d(A_1 h_1, A_2 h_2) & \leq \left(a \max\{d(T_1 h_1, A_1 h_1)d(T_2 h_2, A_2 h_2), \right. \\
 & \quad d(T_1 h_1, A_2 h_2)d(T_2 h_2, A_1 h_1)\} + \max\{\varphi_1(d(T_1 h_1, T_2 h_2)), \\
 & \quad \varphi_2(d(T_1 h_1, A_1 h_1)), \varphi_3(d(T_2 h_2, A_2 h_2)), \varphi_4\left(\frac{1}{2}[d(T_1 h_1, A_2 h_2) + d(T_2 h_2, A_1 h_1)]\right)\} \right).
 \end{aligned}$$

Therefore, condition (2.2) is satisfied by mappings A_1, A_2, T_1 and T_2 and hence by Theorem 3.1, they have a common fixed point $h^* \in B(S)$, i.e. $h^*(x)$ is a unique common solution of the functional equations (4.1) and (4.2).

As an immediate consequence of Theorem 4.1 and Corollary 3.5, we have the following:

Theorem 4.2. *Suppose the following conditions are satisfied:*

- (i) H_i is bounded for $i = 1, 2$.
- (ii) $|H_1(x, y, h(t)) - H_2(x, y, k(t))| \leq N^{-1}(a \max\{|h(t) - A_1h(t)|, |k(t) - A_2k(t)|, |h(t) - A_2k(t)|, |k(t) - A_1h(t)|\} + \max\{\varphi_1(|h(t) - k(t)|), \varphi_2(|h(t) - A_1h(t)|), \varphi_3(|k(t) - A_2k(t)|), \varphi_4(\frac{1}{2}(|h(t) - A_2k(t)| + |k(t) - A_1h(t)|)\}))$, for all $(x, y) \in S \times D$, $h, k \in B(S)$, $t \in S$, $a \geq 0$, where $N = [1 + a \sup_{t \in S} |h(t) - k(t)|]$, $\varphi_i \in \Phi$ ($i = 1, 2, 3, 4$) and the mappings A_i are defined as in (4.3).

Then, the functional equations (4.1) and (4.2) have a unique common solution in $B(S)$.

Now, we furnish an example to validate Theorem 4.1.

Example 4.3. Let $X = Y = \mathbb{R}$ be two Banach spaces endowed with the standard norm $\|\cdot\|$ defined by $\|x\| = |x|$ for all $x \in \mathbb{R}$. Let $S = [0, 1] \subset X$ be the state space, $D = [1, \infty) \subset Y$ the decision space and T represents the transformation of the process. Define $T : S \times D \rightarrow S$ by

$$T(x, y) = \frac{x}{y^2 + 1} \text{ for all } x \in S, y \in D.$$

For any $h, k \in B(S)$, define $f_i, g_i : S \rightarrow \mathbb{R}$ ($i = 1, 2$) by

$$f_i(x) = g_i(x) = \frac{1}{4} \left(\frac{x}{x+1} + 1 \right).$$

Define $H_i, F_i : S \times D \times \mathbb{R} \rightarrow \mathbb{R}$ ($i = 1, 2$) by

$$H_i(x, y, t) = \frac{1}{4} \left[\frac{x}{x+1} \sin \left(t \cdot \frac{y}{y+1} \right) + 1 \right]$$

and

$$F_i(x, y, t) = \frac{1}{4} \left[\frac{x}{x+1} \cos \left(t \cdot \frac{y}{y+1} \right) + 1 \right].$$

Clearly, $\|H_i\| \leq \frac{1}{2}$ and $\|F_i\| \leq \frac{1}{2}$. By varying y over D and taking supremum, we see that H_i yield f_i and H_i yield g_i , respectively ($i = 1, 2$), as defined above. Define mappings A_i and T_i ($i = 1, 2$) by

$$\begin{aligned} A_1h(x) &= \sup_{y \in D} H_1(x, y, h(T(x, y))), & A_2k(x) &= \sup_{y \in D} H_2(x, y, k(T(x, y))), \\ T_1h(x) &= \sup_{y \in D} F_1(x, y, h(T(x, y))), & T_2k(x) &= \sup_{y \in D} F_2(x, y, k(T(x, y))), \end{aligned}$$

for all $x \in S; h, k \in B(S)$.

Then, we see that

$$\begin{aligned}
A_1 h(x) &= \sup_{y \in D} H_1(x, y, h(T(x, y))) = \sup_{y \in D} H_1\left(x, y, h\left(\frac{x}{y^2 + 1}\right)\right) \\
&= \sup_{y \in D} \frac{1}{4} \left[\frac{x}{x+1} \sin\left(h\left(\frac{x}{y^2 + 1}\right) \frac{y}{y+1}\right) + 1 \right] = \frac{1}{4} \left(\frac{x}{x+1} + 1 \right) = f_1(x), \\
A_2 k(x) &= \sup_{y \in D} H_2(x, y, k(T(x, y))) = \sup_{y \in D} H_2\left(x, y, k\left(\frac{x}{y^2 + 1}\right)\right) \\
&= \sup_{y \in D} \frac{1}{4} \left[\frac{x}{x+1} \sin\left(k\left(\frac{x}{y^2 + 1}\right) \frac{y}{y+1}\right) + 1 \right] = \frac{1}{4} \left(\frac{x}{x+1} + 1 \right) = f_2(x), \\
T_1 h(x) &= \sup_{y \in D} F_1(x, y, h(T(x, y))) = \sup_{y \in D} F_1\left(x, y, h\left(\frac{x}{y^2 + 1}\right)\right) \\
&= \sup_{y \in D} \frac{1}{4} \left[\frac{x}{x+1} \cos\left(h\left(\frac{x}{y^2 + 1}\right) \frac{y}{y+1}\right) + 1 \right] = \frac{1}{4} \left(\frac{x}{x+1} + 1 \right) = g_1(x), \\
T_2 k(x) &= \sup_{y \in D} F_2(x, y, k(T(x, y))) = \sup_{y \in D} F_2\left(x, y, k\left(\frac{x}{y^2 + 1}\right)\right) \\
&= \sup_{y \in D} \frac{1}{4} \left[\frac{x}{x+1} \cos\left(k\left(\frac{x}{y^2 + 1}\right) \frac{y}{y+1}\right) + 1 \right] = \frac{1}{4} \left(\frac{x}{x+1} + 1 \right) = g_2(x),
\end{aligned}$$

for all $x \in S; h, k \in B(S)$. Also, we see that

$$M = \left[1 + a \sup_{t \in S} |T_1 k(t) - T_2 h(t)| \right] = \left[1 + a \sup_{t \in S} |g_1(x) - g_2(x)| \right] = 1.$$

Further, for any $k \in B(S)$, define

$$h_n(x) = \left(1 - \frac{1}{n}\right) h(x) \quad \text{and} \quad k_n(x) = \left(1 - \frac{1}{n+1}\right) k(x)$$

so that

$$\lim_{n \rightarrow \infty} \sup_{x \in S} |h_n(x) - h(x)| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \sup_{x \in S} |k_n(x) - k(x)| = 0.$$

Now, we observe that, for $\varphi_1(t) = \frac{t}{2}$,

(i) H_i and F_i are bounded for $i = 1, 2$.

$$\begin{aligned}
\text{(ii)} \quad &|H_1(x, y, h(t)) - H_2(x, y, k(t))| \\
&= \left| \frac{1}{4} \left[\frac{x}{x+1} \sin\left(h(t) \frac{y}{y+1}\right) + 1 \right] - \frac{1}{4} \left[\frac{x}{x+1} \sin\left(k(t) \frac{y}{y+1}\right) + 1 \right] \right| \\
&= \frac{1}{4} \left| \frac{x}{x+1} \right| \left| \sin\left(h(t) \frac{y}{y+1}\right) - \sin\left(k(t) \frac{y}{y+1}\right) \right| \\
&= \frac{1}{4} \left| \frac{x}{x+1} \right| \cdot 2 \left| \sin\left(\frac{h(t) \frac{y}{y+1} - k(t) \frac{y}{y+1}}{2}\right) \right| \left| \cos\left(\frac{h(t) \frac{y}{y+1} + k(t) \frac{y}{y+1}}{2}\right) \right|
\end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{4} \left| \frac{x}{x+1} \right| \left| \frac{y}{y+1} \right| |h(t) - k(t)| \\ &\leq \frac{1}{4} \left| \frac{x}{x+1} \right| |h(t) - k(t)| \leq \frac{1}{4} \varphi_1(h(t) - k(t)). \end{aligned}$$

Finally, for any $h \in B(S)$, there exist $k_1, k_2 \in B(S)$ such that

$$A_1 h(x) = T_2 k_2(x), \quad A_2 h(x) = T_1 k_1(x), \quad x \in S.$$

Also, for any $h, k \in B(S)$, with $A_1 h = T_1 h$, we have $T_1 A_1 h = A_1 T_1 h$ and, with $A_2 k = T_2 k$, we have $T_2 A_2 k = A_2 T_2 k$. Thus, all the assumption of Theorem 4.1 are satisfied. So, the system of equations (4.1) and (4.2) have a unique common solution in $B(S)$.

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ON BOUNDEDNESS AND CONTINUITY OF JORDAN, ORDINARY AND QUADRATIC PRODUCT IN ALTERNATIVE SEMI-PRIME ALGEBRAS

A. Tajmouati

*Sidi Mohamed Ben Abdellah University
Faculty of Sciences Dhar Mahraz
Fez
Morocco
e-mail: abdelaziztajmouati@yahoo.fr
atajmouati@gmail.com*

Abstract. In this work we prove that, if A is an alternative semi-prime algebra, which is considered as a complete convex bornological vector space (respectively, completely bornological locally convex space) and its bornology has a net, then there is equivalent between separating boundedness (resp. separating continuity) of Jordan, ordinary product and quadratic product. If A is again topological, then the boundedness is global and if A is Fréchet space, there is an equivalence between the continuity of these three products.

Keywords: bornological algebras, boundedness, Jordan product, quadratic product, derivation, alternative semi-prime algebra, Mackey-convergence, separating space, net in bornological space.

1. Introduction

Let A be an alternative semi-prime $\mathbb{K}$ -algebra with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

In [5], A. Rodriguez Palacios proved that if A is Banach space then the continuity of Jordan product implies those of ordinary product. After, he extended this results in case of Fréchet space by similar techniques.

The goal of this work is to extend this result to class of alternative algebras which are also convex bornological vector space (cbs), [3], [4].

The paper is organized as follows. In the next section we recall some preliminaries. In Section 3, we extend some results defined in normed spaces onto bornological vector spaces case. In particular, we extend the notion of separating space of a linear map between Banach spaces [7] to the case of linear map T

between bornological vector spaces (Definition 3.1). We give the necessary conditions for an operator that is bounded (Corollary 3.1). In Section 4, we prove some properties of derivations which are necessary in the following. Next, in Section 5, inspired by Rodríguez Palacios's technical, we prove the main theorem (Theorem 5.1) and, consequently, we generalize Rodríguez Palacios's theorem (Corollary 5.3). Also, in Section 6, we prove the equivalence between the boundedness of Jordan product and the boundedness of quadratic product (Theorem 6.1).

Furthermore, we study the continuity problem of these products (Theorem 6.2). In the Fréchet case, we conclude the equivalence between the global continuity of ordinary product and quadratic product (Corollary 6.2).

2. Preliminaries

Recall that a bornology on a set X is a family $\mathcal{B}$ of subsets of X such that $\mathcal{B}$ is a covering of X , hereditary under inclusion and stable under finite union.

The pair $(X, \mathcal{B})$ is called bornological set.

A subfamily $\mathcal{B}'$ of $\mathcal{B}$ is said to be base of bornology $\mathcal{B}$, if every element of $\mathcal{B}$ is contained in an element of $\mathcal{B}'$.

Let X and Y be two bornological sets, a map $X \rightarrow Y$ is called bounded if the image of every bounded subset of X is bounded in Y .

A bornology $\mathcal{B}$ on $\mathbb{K}$ -vector space E is said to be vector bornology on E if the maps $(x, y) \mapsto x + y$ and $(\lambda, x) \mapsto \lambda \cdot x$ are bounded.

We call a bornological vector space (b.v.s) any pair $(E, \mathcal{B})$ consisting of a vector space E and a vector bornology $\mathcal{B}$ on E .

A vector bornology on a vector space is called a convex vector bornology if it is stable under the formation of convex hull.

A bornological vector space is said a convex bornological vector space (cbvs) if its bornology is convex.

A sequence $(x_n)_{n \geq 0}$ in bornological vector space (b.v.s) E is said Mackey-convergent to 0 (or converge bornologically to 0) if there exists a bounded set $B \subset E$ such that

$$(\forall \varepsilon > 0) (\exists n_0 \in \mathbb{N}) / (n \geq n_0) \text{ implies } (x_n \in \varepsilon B).$$

If E is (cbvs) then $(x_n)_{n \geq 0}$ is Mackey-convergent to 0 if there exists a bounded disk $B \subset E$ such that $(x_n)_{n \geq 0} \subset E_B$ and $(x_n)_{n \geq 0}$ converges to 0 in E_B .

A set B in a (bvs) E is said M -closed (or b -closed) if every sequence $(x_n)_{n \geq 0} \subseteq B$ Mackey convergent in E its limit belongs to B .

Let E be a (cbvs) and A is a disk in E . A is called a completant disk if the space (E_A, p_A) is a Banach space.

A (cbvs) E is called a complete convex bornological vector space if its bornology has a base consisting of completant disks.

Let E be a (bvs) and $A \subset E$, the bornological closure (briefly b -closure or M -closure) of A denoted $\overline{A}$ is the intersection of all bornologically closed subsets of E containing A .

If $(E, \mathcal{B})$ is a (cbs) then, there exists on E a locally convex topology denoted t_E .

Let (E, T) be a topological vector space (tvs) we denote by b_E its von Neumann bornology given by the topology T . Then (E, b_E) is a (cbvs).

Generally we have $T \neq t_{b_E}$, if there is equality, we say that the topology T is bornological.

Let $(E, \mathcal{B})$ be a (cbvs). The bornology $\mathcal{B}$ is said topological if $\mathcal{B} = b_{t_E}$.

Let E and F be two (lcs) and $u : E \rightarrow F$ a bounded linear map. If the topology of E is bornological then u is continuous.

Let (E, T) be a (lcs), E is called completely bornological if there exists a complete convex bornological space (E_1, B) such that $T = t_{b_{E_1}}$ algebraically and topologically. Evidently, in this case E is bornological.

Since every Fréchet space is completely bornological, then it is bornological [2].

Recall that if E is a (cbvs), a net in E is a family $\mathcal{R}$ of disks of E , $V_{n_1, \dots, n_k}$ with $k, n_1, n_2, \dots, n_k \in \mathbb{N}$, satisfying the condition

$$E = \bigcup_{n_1 \geq 1} V_{n_1} \text{ and } V_{n_1, \dots, n_{k-1}} = \bigcup_{n_k \geq 1} V_{n_1, \dots, n_k} \text{ for all } k > 1.$$

If $\mathcal{B}$ is a separated convex bornology on E , we say that $\mathcal{R}$ and $\mathcal{B}$ are compatible if the following two properties are verified:

- (i) For every sequence $(n_k)_{k \geq 1}$ of integers, there exists a sequence $(\alpha_k)_{k \geq 1}$ of positive reals such that, for each $f_k \in V_{n_1, \dots, n_k}$ and for each $\mu_k \in [0, \alpha_k]$ the series $\sum_{k=1}^{\infty} \mu_k f_k$ converges bornologically in $(E, \mathcal{B})$ and its sum satisfies $\sum_{k=1}^{\infty} \mu_k f_k \in V_{n_1, \dots, n_{k_0}}$ for every $k_0 \in \mathbb{N}$.
- (ii) For every pair $(n_k, \lambda_k)_{k \geq 1}$ consisting of a sequence $(n_k)_{k \geq 1}$ of positive integers and a sequence $(\lambda_k)_{k \geq 1}$ of positive reals, the set $\bigcap_{k \geq 1} \lambda_k V_{n_1, \dots, n_k}$ is bounded in $(E, \mathcal{B})$.

We say that a convex bornological space $(E, \mathcal{B})$ has a net, or that its bornology has a net, if there exists in E a net $\mathcal{R}$ compatible with $\mathcal{B}$. In this case we also say that $\mathcal{R}$ is a net in $(E, \mathcal{B})$ and that $(E, \mathcal{B})$ is a space with a net, see [1], [2].

Recall that every bornology of a (cbvs) having a countable base has a net. Consequently the von Neumann bornology b_E of a Fréchet space has a net.

The bornologically closed graph theorem. [4] *Let $(E, \mathcal{B})$ be a complete (cbvs) and $(E', \mathcal{B}')$ be a (cbvs) such that $\mathcal{B}'$ has a net. Then, every linear map $u : E \rightarrow E'$ with a bornologically closed graph in $E \times E'$ is bounded.*

Let $\mathbb{K}$ be a commutative field of characteristic 0. An algebra over $\mathbb{K}$ is a $\mathbb{K}$ -vector space A with a bilinear map $(x, y) \mapsto x.y$ of $A \times A$ into A . If this product

is associative (resp commutative), we say that the algebra is associative (resp. commutative).

Let $x, y \in A$, define the following maps:

$$\begin{aligned} R_x(y) &= yx \\ L_x(y) &= xy \\ U_x(y) &= 2x(xy) - x^2y \\ U_{x,y} &= \frac{1}{2}(U_{x+y} - U_x - U_y) \end{aligned}$$

It is well-known that $x \mapsto U_x$ is quadratic. $U_x(y)$ is called a quadratic product of x and y .

Let A be an algebra, we denote by A^+ the algebra A equipped with its vector space structure and the product $\circ$ defined by

$$x \circ y = \frac{1}{2}(xy + yx) \text{ for all } x, y \in A.$$

The product $\circ$ is called the Jordan product.

In A^+ , we have the remarkable identities

$$x \circ y = y \circ x \quad \text{and} \quad (x^2 \circ y) \circ x = x^2 \circ (y \circ x).$$

Let A be an algebra, A is called Alternative algebra if

$$\forall x, y \in A, \quad x(xy) = x^2y \text{ and } (yx)x = yx^2.$$

It follows that $U_x(y) = xyx$.

Denotes $[x, y] = xy - yx$, it is easy to verify that:

$$\begin{aligned} [x, [x, y]] &= 4(y \circ (x \circ x) - (y \circ x) \circ x) \quad \text{and} \\ [x, y \circ z] &= y \circ [x, z] + [x, y] \circ z. \end{aligned}$$

An alternative algebra A is semi-prime if $\{0\}$ is the only two sided ideal J of A with $J^2 = 0$. This is equivalent to $aAa = 0$ implies that $a = 0$.

On the other hand, an alternative algebra A is semi-prime if, and only if, A^+ is semi-prime.

A linear mapping D on an algebra A is called a derivation if:

$$D(xy) = D(x)y + xD(y) \text{ for all } x, y \in A$$

Let D be a derivation on an algebra A , we have the Leibnitz rule:

$$\forall x, y \in A, \quad D^n(xy) = \sum_{i=0}^n C_n^i D^i(x) D^{(n-i)}(y).$$

For details, we can see [3].

3. Separating space and properties

In [7], Sinclair studied the necessary conditions for continuity of homomorphisms, derivations and pair of operators acting on a Banach space.

The aim of the present paper is to extend some of this results in case of bornological vector space (bvs) and consequently obtains some techniques to answer the boundedness problem for linear operators.

We extend naturally the notion of separating space of some linear operator S between (bvs) X and (bvs) Y . The notion of separating space characterizes the continuity of linear operators (Corollary 3.1).

Definition 3.1. Let X and Y be two bornological vector spaces, let T be a linear map of X into Y . We called the separating space of T , the subset $\sigma(T)$ of Y defined by:

$$\sigma(T) = \{y \in Y / \exists (x_n)_n \subset X : x_n \xrightarrow{M} 0 \text{ and } Tx_n \xrightarrow{M} y\}$$

Proposition 3.1. Let X and Y be two bornological vector spaces and $T : X \rightarrow Y$ a linear map. Then $\sigma(T) = \{0\}$ if, and only if, the graph of T is b-closed.

Proof. Let $G(T)$ be the graph of T and suppose that $\sigma(T) = \{0\}$.

Let $(x_n, T(x_n))_{n \geq 0} \subset G(T)$ such that $(x_n, T(x_n)) \xrightarrow{M} (x, y) \in X \times Y$.

Then, $x_n \xrightarrow{M} x$ and $T(x_n) \xrightarrow{M} y$, but $T(x_n - x) = T(x_n) - T(x)$.

Hence $T(x_n - x) \xrightarrow{M} y - T(x)$.

Consequently, $(y - T(x)) \in \sigma(T)$ and $y = T(x)$.

Conversely, assume that $G(T)$ is b-closed.

Let $y \in \sigma(T)$. There exists a sequence $(x_n)_{n \geq 0} \subset X$ such that $x_n \xrightarrow{M} 0$ and $T(x_n) \xrightarrow{M} y$. Since $(x_n, T(x_n))_n \subset G(T)$ and $(x_n, T(x_n))_{n \geq 0} \xrightarrow{M} (0, y)$, then $(0, y) \in G(T)$. Therefore $y = 0$. ■

Corollary 3.1. Let X be a complete convex bornological vector space (ccbvs) and Y be a bornological vector space (bvs) such that its bornology has a net, let $T : X \rightarrow Y$ a linear map. Then, T is bounded if, and only if, $\sigma(T) = \{0\}$.

Proof. Using the bornologically closed graph theorem and Proposition 3.1. ■

4. Properties of derivations

Proposition 4.2. Let A be an alternative algebra, assume that A is considered as a bornological vector space (bvs). Let D be a derivation on A . If the ordinary product of A is separating bounded, then the separating space $\sigma(D)$ of D is two-sided ideal in A .

Proof. Let $a \in A$ and $b \in \sigma(D)$. Then there exists a sequence $(b_n)_{n \geq 0} \subset A$ such that $b_n \xrightarrow{M} 0$ and $D(b_n) \xrightarrow{M} b$. By definition of derivation we have

$$D(ab_n) = aD(b_n) + D(a)b_n$$

$$D(b_na) = b_nD(a) + D(b_n)a$$

Since the ordinary product of A is separating bounded, by taking the limit we have: $D(ab_n) \xrightarrow{M} ab$, $D(b_na) \xrightarrow{M} ba$, $ab_n \xrightarrow{M} 0$ and $b_na \xrightarrow{M} 0$.

Therefore, $ab \in \sigma(D)$ and $ba \in \sigma(D)$. ■

Proposition 4.3. *Let A be an alternative algebra, assume that A is considered as a bornological vector space (bvs). Let D be a derivation on A such that its square is bounded. If the ordinary product of A is separating bounded, then $[\sigma(D)]^2 = \{0\}$.*

Proof. Let $a \in A$ and $b \in \sigma(D)$. Then there exists $(b_n)_{n \geq 0} \subset A$ such that

$$b_n \xrightarrow{M} 0 \text{ and } D(b_n) \xrightarrow{M} b.$$

By Leibnitz rule, it follows that

$$D^2(ab_n) = aD^2(b_n) + 2D(a).D(b_n) + D^2(a)b_n.$$

Since D^2 is bounded and the ordinary product of A is separating bounded, taking the limit in sense of Mackey we obtain

$$(\forall a \in A), (\forall b \in \sigma(D)), D(a)b = 0.$$

This shows that $[\sigma(D)]^2 = \{0\}$. ■

Corollary 4.2. *Let A be an alternative semi-prime algebra, assume that A is considered as a complete convex bornological vector space (cbvs) and its bornology has a net. If the ordinary product of A is separating bounded, then every derivation having a bounded square is bounded.*

Proof. By Corollary 3.1, it suffices to prove that $\sigma(D) = \{0\}$.

By Proposition 4.1, $\sigma(D)$ is two sided ideal, and from Proposition 4.2 we have

$$[\sigma(D)]^2 = \{0\}.$$

Since A is semi-prime, then $\sigma(D) = \{0\}$. ■

5. Boundedness and continuity of Jordan and ordinary products

Theorem 5.1. *Let A be an alternative semi-prime algebra, assume that A is considered as a complete convex bornological vector space and its bornology has a net. Then, the Jordan product is separating bounded if, and only if, the ordinary product is separating bounded.*

Proof. Clearly, if the ordinary product is separating bounded then Jordan product is separating bounded.

Conversely, assume that the Jordan product is separating bounded.

Since A is semi-prime, A^+ is again semi-prime [5].

Consider an arbitrary element $a \in A$, and define the map D_a on A :

$$D_a(x) = [a, x] = ax - xa \text{ for all } x \in A$$

D_a is derivation in A^+ , indeed:

$$D_a(xoy) = [a, x \circ y] = x \circ [a, y] + [a, x] \circ y = x \circ D_a(y) + D_a(x) \circ y.$$

D_a^2 is bounded, indeed:

$$D_a^2(x) = [a, [a, x]] = 4(x \circ (a \circ a) - (x \circ a) \circ a).$$

Since A^+ satisfies the conditions of corollary 4.1, it follows that D_a^+ is bounded.

On the other hand, consider the bilinear mapping φ defined on $A \times A$ by

$$\varphi(x, y) = [x, y].$$

Therefore, φ is separating bounded.

Since $ab = a \circ b + \frac{1}{2}\varphi(a, b)$, then ordinary product is separating bounded. ■

Remark 5.1. For example of complete convex bornological vector space (cbvs) such its bornology has a net, we take Fréchet space and complete convex bornological vector space with a countable base.

Lemma 5.1. *Let E be a Mackey-complete convex bornological vector space, F be a topological convex bornological space and G be a bornological vector space. Let $\psi : E \times F \rightarrow G$ be a bilinear map separating bounded. Then, ψ is globally bounded.*

Proof. It is immediate by Theorems 5 and 8 of [1]. ■

Corollary 5.3. *Let A be an alternative semi-prime algebra, assume that A is considered as a complete topological convex bornological vector space and its bornology has a net. Then, the Jordan product is bounded if, and only if, the ordinary product is bounded.*

Proof. It is immediate application of Lemma 5.1 with $E = F = G = A$. ■

Corollary 5.4. *Let A be an alternative semi-prime algebra, assume that A is considered as a completely bornological locally convex space and its bornology has a net. Then, the Jordan product is separating continuous if, and only if, the ordinary product is separating continuous.*

Proof. Assume that Jordan product is separating continuous. Then it is separating bounded. Since A is completely bornological, there is on A a complete convex bornology $\mathcal{B}$. By Theorem 5.1, it follows that ordinary product is separating bounded. Consequently, it is separating continuous.

A similar argument shows that if the ordinary product is separating continuous then also the Jordan product is separating continuous. ■

Corollary 5.5. ([Rodriguez-Palacios's theorem]) *Let A be an alternative semi-prime algebra, assume that A is considered as a Fréchet space. Then, the Jordan product is separating continuous if, and only if, the ordinary product is separating continuous.*

Proof. Since A is a Fréchet space, then it is completely bornological and its bornology has a net [2].

We conclude the result by Corollary 5.2 and Theorem 2.17 of [6]. ■

6. Boundedness and continuity of quadratic product

Theorem 6.2. *Let A be unital alternative algebra, assume that A is considered as a bornological vector space. Then, the Jordan product is bounded if, and only if, the quadratic product is bounded.*

Proof. Since A is unital alternative algebra, then

$$(x, y \in A), U_x(y) = U_x^+(y) = 2x \circ (x \circ y) - (x \circ x) \circ y$$

Therefore, Jordan product is bounded implies that the quadratic product

$$(x, y) \mapsto U_x(y)$$

is bounded.

Conversely, assume that the quadratic product is bounded. Consider the bilinear map defined by:

$$U_{a,b} = \frac{1}{2}(U_{a+b} - U_a - U_b).$$

Denotes by e the unit of A and let $x \in A$. We have:

$$R_x^+ = L_x^+ = U_{x,e} = \frac{1}{2}(U_{x+e} - U_x - U_e).$$

Hence

$$(\forall x, y \in A) \ x \circ y = \frac{1}{2}(U_{x+e}(y) - U_x(y) - U_e(y)).$$

Since A is bornological vector space the map $x \mapsto x + e$ is bounded, consequently the Jordan-product is bounded. $\blacksquare$

Corollary 6.6. *Let A be an unital alternative algebra, assume that A is considered as a complete bornological vector space (respectively, topological) and its bornology has a net. Then, the ordinary product is separating bounded (respectively, bounded) if, and only if, the quadratic product is separating bounded (respectively, bounded).*

Theorem 6.3. *Let A be an unital alternative algebra, assume that A is considered as a bornological locally convex space. Then, the Jordan product is separating continuous if, and only if, the quadratic product is separating continuous.*

Proof. Denotes by b_A the von Neumann bornology of a locally convex topology of A . It follows that (A, b_A) is a convex bornological space. We have

$$(\forall x, y \in A), U_x(y) = U_x^+(y) = 2x \circ (x \circ y) - (x \circ x) \circ y.$$

From this, if the Jordan product is separating continuous then the quadratic product is separating continuous.

Conversely, assume that quadratic product is separating continuous. For a fix $y \in A$ consider the map $\varphi_y : x \mapsto U_x(y)$. Thus, φ_y is separating continuous.

We claim that φ_y is bounded.

Let B be a bounded subset of A and W a neighbourhood of 0 in A . Then there is a neighbourhood V of 0 and a positive real number α such that

$$\varphi_y(V) \subset W \text{ and } B \subset \alpha V.$$

But φ_y is quadratic, hence $\varphi_y(\alpha x) = \alpha^2 \varphi_y(x)$.

Therefore, $\varphi_y(B) \subset \alpha^2 W$, and consequently φ_y is bounded.

On the other hand, the map $y \mapsto U_x(y)$ is linear and continuous, then it is bounded. We conclude that, $(x, y) \mapsto U_x(y)$ is separating bounded.

By Theorem 5.1, we conclude that the Jordan product is separating bounded.

By hypothesis A is bornological, then the Jordan product is separating continuous. So the proof is complete. ■

An immediate consequence, we have the following theorem.

Corollary 6.7. *Let A be unital alternative semi-prime algebra, assume that A is considered as a Fréchet space. Then, the ordinary product is continuous if, and only if, the quadratic product is continuous.*

Proof. It is clear that if the ordinary product is continuous then the quadratic product $(x, y) \mapsto U_x(y) = xyx$ is continuous.

Conversely, suppose that the quadratic product is continuous. Since A is a Fréchet space then it is bornological. Thus, by Theorem 6.2, the Jordan product is separating continuous, and, consequently, it is continuous [6, Theorem 2.17]. By Corollary 6.1, we conclude that the ordinary product is continuous. ■

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ON HYPERRINGS ASSOCIATED WITH BINARY RELATIONS ON SEMIHYPERGROUP

Sanja Jančić Rašović

*Faculty of Natural Sciences and Mathematics
University of Montenegro
Džordža Vašingtona bb, 81000 Podgorica
Montenegro
e-mail: sabu@t-com.me
http://www.pmf.ac.me/*

Abstract. In this paper we construct a class of hyperrings associated with binary relations on a semihypergroup. We establish a connection between the constructed hyperring $(H, +_{\rho_1}, \circ_{\rho_2})$ and the hyperring of multiendomorphisms of hypergroup $(H, +_{\rho_1})$. Also, we analyze subclasses of the constructed class, which are associated with partial orderings on a set of multimappings.

1. Introduction

The hyperstructure theory was introduced by F. Marty at the 8th Congress of Scandinavian Mathematicians held in 1934. A semihypergroup $(H, \circ)$ is a nonempty set H equipped with a hyperoperation $\circ$, that is a map $\circ : H \times H \rightarrow P^*(H)$, where $P^*(H)$ denotes the family of all nonempty subsets of H , and for all $(x, y, z) \in H^3$: $x \circ (y \circ z) = (x \circ y) \circ z$. A semihypergroup is called a hypergroup in the same sense of Marty [1] if for every $a \in H$: $a \circ H = H \circ a = H$. In the above definitions, if $A, B \in P^*(H)$, then $A \circ B$ is given by:

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b$$

$x \circ A$ is used for $\{x\} \circ A$ and $A \circ x$ for $A \circ \{x\}$.

A comprehensive review of the theory of hyperstructures appears in Corsini [2], Corsini and Leoreanu [3] and Vougiouklis [4].

Similar to hypergroups, that are algebraic structures more general than groups, the hyperrings extend the classical notion of rings, substituting both or only one of the binary operations of addition and multiplication by hyperoperations. The principal notion of hyperring theory can be found in Krasner [5], Davvaz [6], [7], [8], Dasic [9], Rota [10], Spartalis [11] and Vougiouklis [12].

The association between hyperstructures and binary relations had been studied mainly in [13], [14], [15], [16], [17], [18]. Chvalina [13], [14] and Hort [19] use ordered structures for the construction of semigroups and hypergroups. Rosenberg [17] associated with any binary relation ρ of the full domain, a hypergrupoid H_ρ and found conditions on ρ , such that H_ρ is a hypergroup. Corsini and Leoreanu [20] study hypergroups and binary relations.

In Section 3 of this paper we obtain a class of strongly distributive hyperring associated with binary relations on semihypergroup. We investigate their morphisms and we also establish connection between the constructed hyperring $(H, +_{\rho_1}, \circ_{\rho_2})$ and the hyperring of multiendomorphisms of a hypergroup $(H, +_{\rho_1})$. Schweizer and Sklar [21], has given a set of postulates designed to describe the algebraic behaviour of ordinary functions under any one of the three operations: addition, multiplication or composition. These postulates define a system which is a partially ordered semigroup with identity, where semigroup operation and the partial order relation are related with certain conditions. We show that these systems generate a subclass of a class that we constructed in Section 3.

2. Preliminaries

In this Section, first we recall some notions, notations and results.

Definition 2.1. A multivalued system $(H, +, \cdot)$ is a hyperring if:

- 1) $(H, +)$ is a hypergroup
- 2) $(H, \cdot)$ is a semihypergroup
- 3) $(\cdot)$ is a distributive with respect to $(+)$, i.e. for all $(x, y, z) \in H^3$ we have:

$$\text{a) } x \cdot (y + z) \subseteq x \cdot y + x \cdot z$$

and

$$\text{b) } (y + z) \cdot x \subseteq y \cdot x + z \cdot x.$$

If in conditions 3a) and 3b) the equality is valid, then the hyperring is called strongly distributive.

Definition 2.2. Let $(A, +, \cdot)$ and $(B, +', \cdot')$ be two hyperrings. A map $f : A \rightarrow B$ is called an inclusion homomorphism if the following conditions are satisfied:

$$f(x + y) \subseteq f(x) +' f(y)$$

and

$$f(x \cdot y) \subseteq f(x) \cdot' f(y)$$

for all $x, y \in A$. A map f is called a strong homomorphism if in the previous conditions the equality is valid.

Definition 2.3 (and Theorem). Let $(H, +)$ be a commutative hypergroup, and $F(H)$ the set of multiendomorphisms of H , i.e.

$$F(H) = \left\{ h : H \rightarrow P^*(H) \mid (\forall x, y \in H) \bigcup_{u \in x+y} h(u) \subseteq h(x) + h(y) \right\}.$$

Let's define for all $(f, g) \in F(H) \times F(H)$:

$$f \oplus g = \{ h \in F(H) \mid (\forall x \in H) h(x) \subseteq f(x) \oplus g(x) \}$$

$$f \odot g = \left\{ h \in F(H) \mid (\forall x \in H) h(x) \subseteq f(g(x)) = \bigcup_{u \in g(x)} f(u) \right\}.$$

Then the structure $(F(H), \oplus, \odot)$ is a hyperring. This hyperring is called a hyperring of multiendomorphisms of hypergroup $(H, +)$. The each binary relation ρ on a set H , a partial hypergroupoid $H_\rho = (H; \circ)$ is associated [17], as follows:

$$\forall (x, z) \in H^2, x \circ x = \{ y \in H \mid (x, y) \in \rho \}, x \circ z = x \circ x \cup z \circ z.$$

By a partial hypergroupoid we mean a non-empty set H endowed with a function from $H \times H$ to the set of subsets of H .

Let

$$D(\rho) = \{ x \in H \mid \exists y \in H : (x, y) \in \rho \}$$

$$R(\rho) = \{ x \in H \mid \exists z \in H : (z, x) \in \rho \}.$$

An element $z \in H$ is called an outer element of ρ if there exists $y \in H$ such that $(y, z) \notin \rho^2$.

Theorem 2.1. [[17]] H_ρ is a hypergroup if and only if:

- 1) $H = D(\rho)$;
- 2) $H = R(\rho)$;
- 3) $\rho \subseteq \rho^2$;
- 4) if z is an outer element of ρ , then $\forall x \in H, (x, z) \in \rho^2 \implies (x, z) \in \rho$.

3. Hyperrings associated with binary relations on semihypergroup

In this Section we construct a class of hyperrings associated with binary relations on semihypergroup. We establish connection between the constructed hyperring $(H, +_{\rho_1}, \circ_{\rho_2})$ and the hyperring of multiendomorphisms of hypergroup $(H, +_{\rho_1})$. Also, we analyze morphisms of obtained class and subclasses which are associated with partial orderings on a set of multimappings.

Theorem 3.1. *Let $(H, \circ)$ be a semihypergroup equipped with binary relations ρ_1 and ρ_2 such that $\rho_1 \subseteq \rho_2$. Let $\rho_i (i = 1, 2)$ be a reflexive and transitive relation such that, for all $x, y, z \in H$,*

$$(1) \quad (x, y) \in \rho_i \implies (\forall b \in y \circ z) (\exists a \in x \circ z) (a, b) \in \rho_i \quad \text{and} \\ (\forall b \in z \circ y) (\exists a \in z \circ x) (a, b) \in \rho_i.$$

We define hyperoperations $+_{\rho_1}$ and $\circ_{\rho_2}$ on H , as follows:

$$x +_{\rho_1} y = \{z \mid (x, z) \in \rho_1 \quad \text{or} \quad (y, z) \in \rho_1\}$$

and

$$x +_{\rho_2} y = \{z \mid \exists a \in x \circ y, (a, z) \in \rho_2\}$$

for all $(x, y) \in H \times H$.

The structure $(H, +_{\rho_1}, \circ_{\rho_2})$ is a strongly distributive hperring.

Proof. By Theorem 2.1, $(H, +_{\rho_1})$ is a hypergroup. Let us prove that $(H, \circ_{\rho_2})$ is a semihypergroup. Since ρ_2 is reflexive, then for any $a \in x \circ y$ it holds $a \in x \circ_{\rho_2} y$ i.e. $x \circ_{\rho_2} y \neq \emptyset$.

Let $x, y, z \in H$. Set:

$$L = (x \circ_{\rho_2} y) \circ_{\rho_2} z = \bigcup_{u \in x \circ_{\rho_2} y} u \circ_{\rho_2} z$$

and

$$D = x \circ_{\rho_2} (y \circ_{\rho_2} z) = \bigcup_{v \in y \circ_{\rho_2} z} x \circ_{\rho_2} v.$$

Suppose $w \in L$. Then there exists $u \in x \circ_{\rho_2} y$ such that $w \in u \circ_{\rho_2} z$. Thus, there exists $a \in x \circ y$ such that $(a, u) \in \rho_2$ and there exists $c \in u \circ z$ such that $(c, w) \in \rho_2$. By condition (1) from $(a, u) \in \rho_2$ and $c \in u \circ z$ it follows that there exists $c' \in a \circ z$ such that $(c', c) \in \rho_2$. Since ρ_2 is transitive, then $(c', w) \in \rho_2$. On the other hand, $c' \in a \circ z \subseteq (x \circ y) \circ z = x \circ (y \circ z)$ and so $c' \in x \circ v$ for some $v \in y \circ z \subseteq y \circ_{\rho_2} z$. So, $w \in x \circ_{\rho_2} v$ while $v \in y \circ_{\rho_2} z$ i.e. $w \in D$.

Thus, $L \subseteq D$. Similarly, we obtain $D \subseteq L$.

Now, we prove the right distributivity of $\circ_{\rho_2}$ with respect to $+_{\rho_1}$.

Let $x, y, z \in H$. Set:

$$L = (x +_{\rho_1} y) \circ_{\rho_2} z = \cup u \circ_{\rho_2} z, \quad \text{while} \quad u \in x +_{\rho_1} y$$

and

$$D = (x \circ_{\rho_2} z) +_{\rho_1} (y \circ_{\rho_2} z) = \cup a +_{\rho_1} b, \quad \text{while} \quad a \in x \circ_{\rho_2} z \quad \text{and} \quad b \in y \circ_{\rho_2} z.$$

If $w \in L$ then there exists $v \in u \circ z$ such that $(v, w) \in \rho_2$ for some $u \in x +_{\rho_1} y$. We have two possibilities:

- (1) If $(x, u) \in \rho_1$ then by condition (1), there exists $a \in x \circ z$ such that $(a, v) \in \rho_1 \subseteq \rho_2$. As ρ_2 is transitive we obtain $(a, w) \in \rho_2$. Since $a \in x \circ z$ then $w \in x \circ_{\rho_2} z$. For any $b \in y \circ_{\rho_2} z$ it holds $w \in w +_{\rho_1} b$, and so $w \in D$.
- (2) If $(y, u) \in \rho_1$ we can similarly prove that $w \in D$. Thus, $L \subseteq D$.

Suppose now $w \in D$. Then, there exist $a, b \in H$ such that $w \in a +_{\rho_1} b$ while $a \in x \circ_{\rho_2} z$ and $b \in y \circ_{\rho_2} z$. That means, $(a, w) \in \rho_1$ or $(b, w) \in \rho_1$ while $(a', a) \in \rho_2$ and $(b', b) \in \rho_2$ for some $a' \in x \circ z$ and $b' \in y \circ z$.

If $(a, w) \in \rho_1 \subseteq \rho_2$, since $(a', a) \in \rho_2$ we obtain $(a', w) \in \rho_2$ while $a' \in x \circ z$ i.e. $w \in x \circ_{\rho_2} z$. As $x \in x +_{\rho_1} y$ it follows $w \in L$.

If $(b, w) \in \rho_1$ similarly we obtain $w \in L$. Thus, $D \subseteq L$.

The left distributivity of $\circ_{\rho_2}$ with respect to $+_{\rho_1}$ can be proved in a similar way. Thus, $(H, +_{\rho_1}, \circ_{\rho_2})$ is a strongly distributive hyperring. ■

Corollary 3.1. *Let $(H, \cdot)$ be a semigroup equipped with binary relations ρ_1 and ρ_2 such that $\rho_1 \subseteq \rho_2$. Let ρ_i ($i = 1, 2$) be a reflexive and transitive relation such that for all $x, y, z \in H$, $(x, y) \in \rho_i$ implies $(x \cdot z, y \cdot z) \in \rho_i$ and $(z \cdot x, z \cdot y) \in \rho_i$.*

If we define hyperoperations $+_{\rho_1}$ and $\circ_{\rho_2}$ on H as follows:

$$x +_{\rho_1} y = \{z \mid (x, z) \in \rho_1 \text{ or } (y, z) \in \rho_1\}$$

and

$$x \circ_{\rho_2} y = \{z \mid (x \cdot y, z) \in \rho_2\}$$

for all $(x, y) \in H \times H$, then the structure $(H, +_{\rho_1}, \circ_{\rho_2})$ is a strongly distributive hyperring.

Throughout the following text the quadruple $(H, \circ, \rho_1, \rho_2)$ will denote a semi-hypergroup $(H, \circ)$ equipped with binary relations ρ_1 and ρ_2 such that ρ_1 and ρ_2 satisfy the conditions of Theorem 3.1.

Definition 3.1. Let the triples (H_1, ρ_1, ρ_2) and $(H_2, \sigma_1, \sigma_2)$ denote the nonempty set H_1 equipped with binary relations ρ_1, ρ_2 and nonempty set H_2 with binary relations σ_1, σ_2 .

(a) The map $\alpha : H_1 \rightarrow H_2$ is said to be isotone if

$$x \rho_i y \implies \alpha(x) \delta_i \alpha(y)$$

for all $x, y \in H_1$ and $i \in \{1, 2\}$.

(b) The map $\alpha : H_1 \rightarrow H_2$ is said to be strongly isotone if

$$\alpha(x) \sigma_i y \iff (\exists x' \in H_1) x \rho_i x' \wedge \alpha(x') = y$$

for all $(x, y) \in H_1 \times H_2$ and $i \in \{1, 2\}$.

The next theorem is generalization of Theorem 4.5 [22].

Theorem 3.2. *Let $(H_1, +_{\rho_1}, \circ_{\rho_2})$ be a hyperring associated with $(H_1, \circ, \rho_1, \rho_2)$ and $(H_2, +_{\sigma_1}, \circ_{\sigma_2})$ be a hyperring associated with $(H_2, \circ, \sigma_1, \sigma_2)$. If $f : (H_1, \circ) \rightarrow (H_2, \circ)$ is an isotone (strongly isotone) homomorphism of semihypergroups $(H_1, \circ)$ and $(H_2, \circ)$, then f is an inclusion (strong) homomorphism of a hyperring $(H_1, +_{\rho_1}, \circ_{\rho_2})$ into hyperring $(H_2, +_{\sigma_1}, \circ_{\sigma_2})$.*

Proof. Let $f : (H_1, \circ) \rightarrow (H_2, \circ)$ be an isotone homomorphism and $x, y \in H_1$. If $w \in f(x +_{\rho_1} y)$ then there exists $z \in H_1$ such that $w = f(z)$ while $(x, z) \in \rho_1$ or $(y, z) \in \rho_1$. Since f is isotone, then $(f(x), f(z) = w) \in \sigma_1$ or $(f(y), f(z) = w) \in \sigma_1$ and so $w \in f(x) +_{\sigma_1} f(y)$.

Now, let $w \in f(x \circ_{\rho_2} y)$. Then there exist $a, z \in H_1$ such that $a \in x \circ y$, $(a, z) \in \rho_2$ and $w = f(z)$. As f is an isotone homomorphism then $f(a) \in f(x) \circ f(y)$ while $(f(a), f(z) = w) \in \sigma_2$ and so $w \in f(x) \circ_{\sigma_2} f(y)$. Thus $f : (H_1, +_{\rho_1}, \circ_{\rho_2}) \rightarrow (H_2, +_{\sigma_1}, \circ_{\sigma_2})$ is an inclusion homomorphism.

If $f : (H_1, \circ) \rightarrow (H_2, \circ)$ is a strongly isotone homomorphism, then obviously f is isotone and as we proved for all $x, y \in H_1$ it holds: $f(x +_{\rho_1} y) \subseteq f(x) +_{\sigma_1} f(y)$ and $f(x \circ_{\rho_2} y) \subseteq f(x) \circ_{\sigma_2} f(y)$. We will prove the converse inclusion.

Suppose $w \in f(x) +_{\sigma_1} f(y)$. Then $(f(x), w) \in \sigma_1$ or $(f(y), w) \in \sigma_1$. It implies that there exists $z \in H_1$ such that $(x, z) \in \rho_1$ or $(y, z) \in \rho_1$ while $f(z) = w$, as f is strongly isotone. Thus, $w = f(z) \in f(x +_{\rho_1} y)$. So, $f(x) +_{\sigma_1} f(y) \subseteq f(x +_{\rho_1} y)$.

Now, let $w \in f(x) \circ_{\sigma_2} f(y)$. Then there exists $u \in f(x) \circ f(y) = f(x \circ y)$ such that $(u, w) \in \sigma_2$. So, there exists $z \in x \circ y$ such that $u = f(z)$ while $(u, w) \in \sigma_2$. As f is strongly isotone, there exists $a \in H_1$ such that $(z, a) \in \rho_2$ and $w = f(a)$. Thus, $a \in x \circ_{\rho_2} y$ and $w = f(a) \in f(x \circ_{\rho_2} y)$. Therefore $f(x) \circ_{\sigma_2} f(y) \subseteq f(x \circ_{\rho_2} y)$.

This completes the proof. $\blacksquare$

Let $(H, \circ)$ be a semihypergroup and ρ be a binary relation on H . If $\{A, B\} \subset P^*(H)$ we write $A \tilde{\rho} B$ to denote that: $(\forall a \in A)(\exists b \in B)$ such that $(b, a) \in \rho$ and $(\forall b \in B)(\exists a \in A)$ such that $(a, b) \in \rho$.

Theorem 3.3. Let $(H, +_{\rho_1}, \circ_{\rho_2})$ be a hyperring associated with $(H, \circ, \rho_1, \rho_2)$ and $(F(H), \oplus, \odot)$ be a hyperring of multiendomorphisms of hypergroup $(H, +_{\rho_1})$. If we define a mapping $\Psi : (H, +_{\rho_1}, \circ_{\rho_2}) \rightarrow (F(H), \oplus, \odot)$ by

$$\Psi(a) = f_a, \forall a \in H,$$

where $f_a : H \rightarrow P^*(H)$ is defined by:

$$f_a(x) = a \circ_{\rho_2} x, \forall x \in H,$$

then:

- (1) Ψ is an inclusion homomorphism.
- (2) If there exists at least one element $x \in H$ such that

$$a \circ x \tilde{\rho}_2 b \circ x \implies a = b$$

for all $a, b \in H$, then Ψ is injective.

Proof. As $(H, +_{\rho_1})$ is a commutative hypergroup there exists the hyperring of its multiendomorphisms $(F(H), \oplus, \odot)$.

Let $a \in H$. We verify that $f_a \in F(H)$. Let $x, y \in H$.

Then:

$$\bigcup_{u \in x +_{\rho_1} y} f_a(u) = \bigcup_{u \in x +_{\rho_1} y} a \circ_{\rho_2} u = a \circ_{\rho_2} (x +_{\rho_1} y) = (a \circ_{\rho_2} x) +_{\rho_1} (a \circ_{\rho_2} y) = f_a(x) +_{\rho_1} f_a(y).$$

(1) Let $a, b \in H$. Set:

$$L = \Psi(a +_{\rho_1} b) = \{f_c | (a, c) \in \rho_1 \text{ or } (b, c) \in \rho_1\}$$

and

$$D = \Psi(a) \oplus \Psi(b) = f_a \oplus f_b = \{h | (\forall x) h(x) \subseteq f_a(x) +_{\rho_1} f_b(x)\}.$$

Let $c \in H$ such that $(a, c) \in \rho_1$ and $x \in H$. Then:

$$f_c(x) = c \circ_{\rho_2} x = \{y | \exists z \in c \circ x, (z, y) \in \rho_2\}.$$

If $y \in f_c(x)$, then $(z, y) \in \rho_2$ for some $z \in c \circ x$ and as $(a, c) \in \rho_1$ there exists $w \in a \circ x$ such that $(w, z) \in \rho_1 \subseteq \rho_2$. By transitivity of ρ_2 we obtain $(w, y) \in \rho_2$ i.e. $y \in a \circ_{\rho_2} x \subseteq f_a(x) +_{\rho_1} f_b(x)$.

If $c \in H$ such that $(b, c) \in \rho_1$, similarly we obtain $f_c \in D$. Thus, $L \subseteq D$.

Now, assume:

$$L = \Psi(a \circ_{\rho_2} b) = \{f_c | c \in a \circ_{\rho_2} b\} = \{f_c | \exists d \in a \circ b, (d, c) \in \rho_2\}$$

and

$$D = \{h | (\forall x) h(x) \subseteq f_a(f_b(x))\}.$$

Let $c \in H$ such that $(d, c) \in \rho_2$ for some $d \in a \circ b$ and $x \in H$. If $y \in f_c(x)$ then $(z, y) \in \rho_2$ for some $z \in c \circ x$, as $(d, c) \in \rho_2$ and $z \in c \circ x$, there exists $w \in d \circ x$ such that $(w, z) \in \rho_2$ and by transitivity of ρ_2 we obtain $(w, y) \in \rho_2$. Thus, $y \in d \circ_{\rho_2} x \subseteq (a \circ b) \circ_{\rho_2} x \subseteq (a \circ_{\rho_2} b) \circ_{\rho_2} x = a \circ_{\rho_2} (b \circ_{\rho_2} x) = f_a(f_b(x))$. Therefore, $f_c(x) \subseteq f_a(f_b(x))$ i.e. $f_c \in D$.

So, $L \subseteq D$.

(2) Let $a \neq b$. Then there exists $x \in H$ such that one of the following is valid:

$$(i) (\exists c \in a \circ x)(\forall d \in b \circ x)(d, c) \notin \rho_2$$

or

$$(ii) (\exists d \in b \circ x)(\forall c \in a \circ x)(c, d) \notin \rho_2.$$

If (i) is valid, then there exists $c \in a \circ x$ such that $c \notin b \circ_{\rho_2} x = f_b(x)$. Since $c \in a \circ x \subseteq a \circ_{\rho_2} x = f_a(x)$, it follows $f_a(x) \not\subseteq f_b(x)$.

If (ii) is valid, then similarly $f_b(x) \not\subseteq f_a(x)$. Hence, $f_a \neq f_b$ i.e. $\Psi(a) \neq \Psi(b)$.

Remark 3.1. If $(H, +_{\rho_1}, \circ_{\rho_2})$ is a hyperring associated with $(H, \cdot, \rho_1, \rho_2)$ where $(H, \cdot)$ is a semigroup with at least one reductive element and ρ_2 is the partial order then Ψ is an inclusion monomorphism of a hyperring $(H, +_{\rho_1}, \circ_{\rho_2})$ into $(F(H), \oplus, \odot)$.

Now, we deal with some classes of hyperrings associated with partial orderings on a semihypergroup (semigroup). Schweizer and Sklar [21] postulated a system $(S, \cdot, \leq)$ where $(S, \cdot)$ is semigroup with identity where partial order $\leq$ satisfies certain conditions, such that these systems are natural generalization of certain algebras of functions. In that case, the set of mappings from the reals to the reals is partially ordered by restriction, i.e. ordinary set inclusion, where the function being regarded as sets of ordered pairs of numbers. We show that such systems generate a suitable subclasses of a class constructed in Theorem 3.1.

Theorem 3.4. *Let $(H, \circ)$ be a semigroup with identity e and let $\leq$ be the partial order on H , connected with semigroup operation with next conditions:*

- (1) *If $a \leq b$ then there exists an element $e_1 \leq e$ such that $a = b \circ e_1$*
- (2) *If $e_2 \leq e$ then $a \circ e_2 \leq a$ and $e_2 \circ a \leq a$.*

Then, there exists a hyperring $(H, +_{\leq}, \circ_{\leq})$ associated with $(H, \cdot, \leq, \leq)$.

Proof. From Theorem 1 [21] it follows that the quadruple $(H, \circ, \leq, \leq)$ satisfies the conditions of Corollary 3.1. ■

Example 1.

- (a) Denote by $P^*(\mathbb{R})$ the set of all nonempty subsets of reals $\mathbb{R}$. Let $M(\mathbb{R})$ be the set of all multimappings from reals to reals, i.e.

$$M(\mathbb{R}) = \{f : A \rightarrow P^*(\mathbb{R}) \mid A \subseteq \mathbb{R}, A \neq \emptyset\}.$$

We can consider function f as the set $\{(x, f(x)) \mid x \in \text{Dom} f\}$. Denote by φ the empty multimapping, i.e. the multimapping that contains no ordered pairs. If $f, g \in M(\mathbb{R})$ and $\text{Dom} f \cap \text{Dom} g \neq \emptyset$, define:

$$f + g = \{(x, f(x) + g(x)) \mid x \in \text{Dom} f \cap \text{Dom} g\}.$$

If $f, g \in M(\mathbb{R}) \cup \varphi$ and $\text{Dom} f \cap \text{Dom} g = \emptyset$, we define:

$$f + g = \varphi.$$

Obviously, $(M(\mathbb{R}) \cup \varphi, +)$ is a semigroup with identity element $o : \mathbb{R} \rightarrow \mathbb{R}$ defined by $o(x) = 0$ for all $x \in \mathbb{R}$.

If we define:

$$f \leq g \iff \text{Dom} f \subseteq \text{Dom} g \quad \text{and} \quad (\forall x \in \text{Dom} f) f(x) = g(x),$$

then $(M(\mathbb{R}) \cup \varphi, +, \leq)$ satisfies conditions of Theorem 3.4. The restriction of o which has the same domain as f serves as element $e_1 \leq o$ such that $f \leq g$ implies $f = g + e_1$. Condition (2) of Theorem 3.4 is, obviously, satisfied. So, there exists a hyperring associated with $(M(\mathbb{R}) \cup \varphi, +, \leq, \leq)$.

It can be noted that condition (1) in Theorem 3.4 is symmetric in this case. Generally, it is not valid (see [21]).

(b) Now, we will define a hyperoperation $\oplus$ on a set $M(\mathbb{R}) \cup \varphi$, as follows:

$$f \oplus g = \{u \mid Dom u = Dom f \cap Dom g, \text{ and } \forall x \in Dom u, u(x) \subseteq f(x) + g(x)\}$$

for all $f, g \in M(\mathbb{R})$ such that $Dom f \cap Dom g \neq \emptyset$, and $f \oplus g = \varphi$ for all $f, g \in M(\mathbb{R}) \cup \varphi$ such that $Dom f \cap Dom g = \emptyset$.

Then, $(M(\mathbb{R}) \cup \varphi, \oplus)$ is a semihypergroup. If we define a binary relation $\leq$ as in a Example 1(a), then, obviously, $\leq$ is a partial order. Let us verify that $(M(\mathbb{R}) \cup \varphi, \oplus, \leq, \leq)$ satisfies condition (1) of Theorem 3.1. Let $f \leq g$ and $b \in g \oplus h$.

If $Dom g \cap Dom h = \emptyset$ then $Dom f \cap Dom h = \emptyset$ and $b = \varphi$. If we put $a = \varphi$ then $a \in f \oplus h$.

If $Dom g \cap Dom h \neq \emptyset$ and $b \in g \oplus h$ then $Dom b = Dom g \cap Dom h$ and for all $x \in Dom b$ it holds $b(x) \subseteq g(x) + h(x)$. Now, we have two possibilities:

- (1) If $Dom f \cap Dom h = \emptyset$ then we put $a = \varphi$ and obviously $a \in f \oplus h$ and $a \leq b$.
- (2) If $Dom f \cap Dom h = A \neq \emptyset$, then we put $A = Dom a$ and $a(x) = b(x)$ for all $x \in A$.

As $Dom f \subseteq Dom g$, then $A \subseteq Dom b$ and we obtain $a \leq b$.

As $a(x) = b(x) \subseteq g(x) + h(x) = f(x) + h(x)$ for all $x \in A$, it follows $a \in f \oplus h$. As $\oplus$ is commutative hyperoperation it follows that condition (1) of Theorem 3.1 is satisfied. Thus, there exists a hyperring associated with $(M(\mathbb{R}) \cup \varphi, \oplus, \leq, \leq)$.

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CENTRALIZERS ON SEMIPRIME GAMMA RINGS

M.F. Hoque

*Department of Mathematics
Pabna Science and Technology University
Pabna-6600
Bangladesh
e-mail: fazlul_math@yahoo.co.in*

A.C. Paul

*Department of Mathematics
Rajshahi University
Rajshahi-6205
Bangladesh
e-mail: acpaulru_math@yahoo.com*

Abstract. Let M be a 2-torsion free semiprime Γ -ring satisfying a certain assumption and let $T : M \rightarrow M$ be an additive mapping such that

$$T(x\alpha y\beta x) = x\alpha T(y)\beta x$$

holds for all $x, y \in M$, and $\alpha, \beta \in \Gamma$. Then we prove that T is a centralizer. We also show that T is a centralizer if M contains a multiplicative identity 1.

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1. Introduction

Let M and Γ be additive abelian groups. If there exists a mapping $(x, \alpha, y) \rightarrow x\alpha y$ of $M \times \Gamma \times M \rightarrow M$, which satisfies the conditions

- (i) $x\alpha y \in M$
- (ii) $(x + y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha + \beta)z = x\alpha z + x\beta z$, $x\alpha(y + z) = x\alpha y + x\alpha z$
- (iii) $(x\alpha y)\beta z = x\alpha(y\beta z)$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$,

then M is called a Γ -ring.

Every ring M is a Γ -ring with $M = \Gamma$. However a Γ -ring need not be a ring. Gamma rings, more general than rings, were introduced by Nobusawa [11]. Bernes [1] weakened slightly the conditions in the definition of Γ -ring in the sense of Nobusawa.

Let M be a Γ -ring. Then an additive subgroup U of M is called a left (right) ideal of M if $M\Gamma U \subseteq U$ ($U\Gamma M \subseteq U$). If U is both a left and a right ideal, then we say U is an ideal of M . Suppose again that M is a Γ -ring. Then M is said to be a 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in M$. An ideal P_1 of a Γ -ring M is said to be prime if for any ideals A and B of M , $A\Gamma B \subseteq P_1$ implies $A \subseteq P_1$ or $B \subseteq P_1$. An ideal P_2 of a Γ -ring M is said to be semiprime if for any ideal U of M , $U\Gamma U \subseteq P_2$ implies $U \subseteq P_2$. A Γ -ring M is said to be prime if $a\Gamma M\Gamma b = (0)$ with $a, b \in M$, implies $a = 0$ or $b = 0$ and semiprime if $a\Gamma M\Gamma a = (0)$ with $a \in M$ implies $a = 0$. Furthermore, M is said to be commutative Γ -ring if $x\alpha y = y\alpha x$ for all $x, y \in M$ and $\alpha \in \Gamma$. Moreover, the set $Z(M) = \{x \in M : x\alpha y = y\alpha x \text{ for all } \alpha \in \Gamma, y \in M\}$ is called the centre of the Γ -ring M .

If M is a Γ -ring, then $[x, y]_\alpha = x\alpha y - y\alpha x$ is known as the commutator of x and y with respect to α , where $x, y \in M$ and $\alpha \in \Gamma$. We make the basic commutator identities:

$$\begin{aligned} [x\alpha y, z]_\beta &= [x, z]_\beta \alpha y + x[\alpha, \beta]_z y + x\alpha[y, z]_\beta \text{ and} \\ [x, y\alpha z]_\beta &= [x, y]_\beta \alpha z + y[\alpha, \beta]_x z + y\alpha[x, z]_\beta, \end{aligned}$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

We consider the following assumption:

$$(A) \quad x\alpha y\beta z = x\beta y\alpha z, \text{ for all } x, y, z \in M, \text{ and } \alpha, \beta \in \Gamma.$$

According to the assumption (A), the above two identities reduce to

$$\begin{aligned} [x\alpha y, z]_\beta &= [x, z]_\beta \alpha y + x\alpha[y, z]_\beta \text{ and} \\ [x, y\alpha z]_\beta &= [x, y]_\beta \alpha z + y\alpha[x, z]_\beta, \end{aligned}$$

which we extensively used.

An additive mapping $T : M \rightarrow M$ is a left (right) centralizer if

$$T(x\alpha y) = T(x)\alpha y \quad (T(x\alpha y) = x\alpha T(y))$$

holds for all $x, y \in M$ and $\alpha \in \Gamma$. A centralizer is an additive mapping which is both a left and a right centralizer. For any fixed $a \in M$ and $\alpha \in \Gamma$, the mapping $T(x) = a\alpha x$ is a left centralizer and $T(x) = x\alpha a$ is a right centralizer. We shall restrict our attention on left centralizer, since all results of right centralizers are the same as left centralizers.

An additive mapping $D : M \rightarrow M$ is called a derivation if

$$D(x\alpha y) = D(x)\alpha y + x\alpha D(y)$$

holds for all $x, y \in M$, and $\alpha \in \Gamma$ and is called a Jordan derivation if

$$D(x\alpha x) = D(x)\alpha x + x\alpha D(x)$$

for all $x \in M$ and $\alpha \in \Gamma$.

An additive mapping $T : M \rightarrow M$ is Jordan left(right) centralizer if

$$T(x\alpha x) = T(x)\alpha x (T(x\alpha x) = x\alpha T(x))$$

for all $x \in M$, and $\alpha \in \Gamma$.

Every left centralizer is a Jordan left centralizer but the converse is not in general true.

An additive mappings $T : M \rightarrow M$ is called a Jordan centralizer if

$$T(x\alpha y + y\alpha x) = T(x)\alpha y + y\alpha T(x),$$

for all $x, y \in M$ and $\alpha \in \Gamma$. Every centralizer is a Jordan centralizer but Jordan centralizer is not in general a centralizer.

Bernes [1], Luh [9] and Kyuno [8] studied the structure of Γ -rings and obtained various generalizations of corresponding parts in ring theory.

Borut Zalar [15] worked on centralizers of semiprime rings and prove that Jordan centralizers and centralizers of this rings coincide. Joso Vukman [12], [13], [14] developed some remarkable results using centralizers on prime and semiprime rings.

Y. Ceven [5] worked on Jordan left derivations on completely prime Γ -rings. He investigated the existence of a nonzero Jordan left derivation on a completely prime Γ -ring that makes the Γ -ring commutative with an assumption. With the same assumption, he showed that every Jordan left derivation on a completely prime Γ -ring is a left derivation on it.

In [6], Fazlul Hoque and A.C. Paul proved that every Jordan centralizer of a 2-torsion free semiprime Γ -ring is a centralizer. Here they also gave an example of a Jordan centralizer which is not a centralizer.

In this paper, we develop some results of J. Vukman [14] in Γ -rings. If M is a 2-torsion free semiprime Γ -ring satisfying the assumption (A) and if $T : M \rightarrow M$ is an additive mapping such that

$$(1) \quad T(x\alpha y\beta x) = x\alpha T(y)\beta x$$

for all $x, y \in M$ and $\alpha, \beta \in \Gamma$, then T is a centralizer. Also, we prove that T is a centralizer if M contains a multiplicative identity 1.

2. Centralizers of Semiprime Gamma Rings

For proving our main results, we need the following Lemmas:

Lemma 2.1 *Suppose M is a semiprime Γ -ring satisfying the assumption (A). Suppose that the relation $a\alpha x\beta b + b\alpha x\beta c = 0$ holds for all $x \in M$, some $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. Then $(a + c)\alpha x\beta b = 0$ is satisfied for all $x \in M$ and $\alpha, \beta \in \Gamma$.*

Proof. Putting $x = x\beta b\alpha y$ in the relation

$$(2) \quad a\alpha x\beta b + b\alpha x\beta c = 0$$

We have

$$(3) \quad a\alpha x\beta b\alpha y\beta b + b\alpha x\beta b\alpha y\beta c = 0$$

On the other hand, a right multiplication by $\alpha y\beta b$ of (2) gives

$$(4) \quad a\alpha x\beta b\alpha y\beta b + b\alpha x\beta c\alpha y\beta b = 0.$$

Subtracting (4) from (3), we have

$$(5) \quad b\alpha x\beta(b\alpha y\beta c - c\alpha y\beta b) = 0.$$

Putting $x = y\beta c\alpha x$ in (5) gives

$$(6) \quad b\alpha y\beta c\alpha x\beta(b\alpha y\beta c - c\alpha y\beta b) = 0$$

Left multiplication by $c\alpha y\beta$ of (5) gives

$$(7) \quad c\alpha y\beta b\alpha x\beta(b\alpha y\beta c - c\alpha y\beta b) = 0$$

Subtracting (7) from (6), we obtain

$$(b\alpha y\beta c - c\alpha y\beta b)\alpha x\beta(b\alpha y\beta c - c\alpha y\beta b) = 0$$

which gives

$$(8) \quad b\alpha y\beta c = c\alpha y\beta b,$$

$y \in M$ and $\alpha, \beta \in \Gamma$. Therefore, $b\alpha x\beta c$ can be replaced by $c\alpha x\beta b$ in (2), which gives

$$a\alpha x\beta b + c\alpha x\beta b = 0$$

i.e.

$$(a + c)\alpha x\beta b = 0$$

Hence, the proof is complete. ■

Lemma 2.2 *Let M be a 2-torsion free semiprime Γ -ring satisfying the assumption (A) and let $T : M \rightarrow M$ be an additive mapping. Suppose that*

$$T(x\alpha y\beta x) = x\alpha T(y)\beta x$$

holds for all $x, y \in M$ and $\alpha, \beta \in \Gamma$. Then

$$(i) \quad [[T(x), x]_\alpha, x]_\beta = 0$$

$$(ii) \quad x\beta[T(x), x]_\alpha \gamma x = 0$$

$$(iii) \quad x\beta[T(x), x]_\alpha = 0$$

$$(iv) \quad [T(x), x]_\alpha \beta x = 0$$

$$(v) \quad [T(x), x]_\alpha = 0.$$

Proof. We prove that (i)

$$(9) \quad [[T(x), x]_\alpha, x]_\beta = 0.$$

For linearization, we put $x + z$ for x in relation (1), we obtain

$$(10) \quad T(x\alpha y \beta z + z\alpha y \beta x) = x\alpha T(y)\beta z + z\alpha T(y)\beta x.$$

Replacing y for x and z for y in (10), we have

$$(11) \quad T(x\alpha x \beta y + y\alpha x \beta x) = x\alpha T(x)\beta y + y\alpha T(x)\beta x.$$

For $z = (x\alpha)^2 x$ relation (10) reduces to

$$(12) \quad \begin{aligned} & T(x\alpha y \beta (x\alpha)^2 x + (x\alpha)^2 x\alpha y \beta x) \\ &= x\alpha T(y)\beta (x\alpha)^2 x + (x\alpha)^2 x\alpha T(y)\beta x. \end{aligned}$$

Putting $y = x\alpha y \beta x$ in (11), we obtain

$$(13) \quad \begin{aligned} & T((x\alpha)^2 x \beta y \beta x + x \beta y \beta (x\alpha)^2 x) \\ &= x\alpha T(x)\beta x\alpha y \beta x + x\alpha y \beta x\alpha T(x)\beta x. \end{aligned}$$

The substitution $x\alpha x \beta y + y\alpha x \beta x$ for y in the relation (1) gives

$$T((x\alpha)^2 x \beta y \beta x + (x\alpha)^2 x \beta y \beta x) = x\alpha T(x\alpha x \beta y + y\alpha x \beta x)\beta x$$

which gives because of (11),

$$(14) \quad \begin{aligned} & T((x\alpha)^2 x \beta y \beta x + x \beta y \beta (x\alpha)^2 x) \\ &= (x\alpha)^2 T(x)\beta y \beta x + x\alpha y \beta T(x)\alpha x \beta x. \end{aligned}$$

Combining (13) with (14), we arrive at

$$(15) \quad x\alpha[T(x), x]_\alpha \beta y \beta x - x\alpha y \beta [T(x), x]_\alpha \beta x = 0.$$

Using (8) in the above relation, we have

$$(16) \quad \begin{aligned} & x\alpha y \beta T(x)\alpha x \beta x - x\alpha y \beta x\alpha T(x)\beta x - x\alpha[T(x), x]_\alpha \beta y \beta x = 0 \\ & T(x)\alpha x\alpha y \beta x \beta x - x\alpha T(x)\beta x\alpha y \beta x - x\alpha[T(x), x]_\alpha \beta y \beta x = 0 \\ & T(x)\alpha x \beta x\alpha y \beta x - x\alpha T(x)\beta x\alpha y \beta x - x\alpha[T(x), x]_\alpha \beta y \beta x = 0 \\ & (T(x)\alpha x - x\alpha T(x))\beta x\alpha y \beta x - x\alpha[T(x), x]_\alpha \beta y \beta x = 0 \\ & [T(x), x]_\alpha \beta x\alpha y \beta x - x\beta[T(x), x]_\alpha \alpha y \beta x = 0 \\ & ([T(x), x]_\alpha \beta x - x\beta[T(x), x]_\alpha)\alpha y \beta x = 0 \\ & [[T(x), x]_\alpha, x]_\beta \alpha y \beta x = 0. \end{aligned}$$

Let $y = y\alpha[T(x), x]_\alpha$ in (16), we have

$$(17) \quad [[T(x), x]_\alpha, x]_\beta \alpha y \alpha [T(x), x]_\alpha \beta x = 0.$$

Right multiplication of (16) by $\alpha[T(x), x]_\alpha$ gives

$$(18) \quad [[T(x), x]_\alpha, x]_\beta \alpha y \beta x \alpha [T(x), x]_\alpha = 0.$$

Subtracting (18) from (17) one obtains

$$[[T(x), x]_\alpha, x]_\beta \alpha y \alpha [[T(x), x]_\alpha, x]_\beta = 0.$$

Since M is semiprime, so (9) follows i.e.

$$[[T(x), x]_\alpha, x]_\beta = 0.$$

Now, we prove the relation (ii):

$$(19) \quad x\beta[T(x), x]_\alpha \gamma x = 0.$$

The linearization of (9) gives

$$\begin{aligned} & [[T(x), x]_\alpha, y]_\beta + [[T(y), x]_\alpha, x]_\beta + [[T(y), x]_\alpha, y]_\beta + [[T(x), y]_\alpha, x]_\beta \\ & + [[T(x), y]_\alpha, y]_\beta + [[T(y), y]_\alpha, x]_\beta = 0. \end{aligned}$$

Putting $x = -x$ in the above relation, we have

$$\begin{aligned} & [[T(x), x]_\alpha, y]_\beta + [[T(y), x]_\alpha, x]_\beta - [[T(y), x]_\alpha, y]_\beta + [[T(x), y]_\alpha, x]_\beta \\ & - [[T(x), y]_\alpha, y]_\beta - [[T(y), y]_\alpha, x]_\beta = 0. \end{aligned}$$

Adding the above two relations, we have

$$2[[T(x), x]_\alpha, y]_\beta + 2[[T(x), y]_\alpha, x]_\beta + 2[[T(y), x]_\alpha, x]_\beta = 0.$$

Since M is 2-torsion free semiprime Γ -ring, so, we have

$$(20) \quad [[T(x), x]_\alpha, y]_\beta + [[T(x), y]_\alpha, x]_\beta + [[T(y), x]_\alpha, x]_\beta = 0.$$

Putting $x\beta y\gamma x$ for y in (20) and using (1), (9), (20) and assumption (A), we have

$$\begin{aligned} 0 &= [[T(x), x]_\alpha, x\beta y\gamma x]_\beta + [[T(x), x\beta y\gamma x]_\alpha, x]_\beta + [x\beta T(y)\gamma x, x]_\alpha, x]_\beta \\ &= x\beta [[T(x), x]_\alpha, y]_\beta \gamma x \\ &\quad + [[T(x), x]_\alpha \beta y\gamma x + x\beta [T(x), y]_\alpha \gamma x + x\beta y\gamma [T(x), x]_\alpha, x]_\beta \\ &\quad + [x\beta [T(y), x]_\alpha \gamma x, x]_\beta \\ &= x\beta [[T(x), x]_\alpha, y]_\beta \gamma x + [T(x), x]_\alpha \beta [y, x]_\beta \gamma x + x\beta [[T(x), y]_\alpha, x]_\beta \gamma x \\ &\quad + x\gamma [y, x]_\beta \beta [T(x), x]_\alpha + x\beta [[T(y), x]_\alpha, x]_\beta \gamma x \\ &= [T(x), x]_\alpha \beta [y, x]_\beta \gamma x + x\gamma [y, x]_\beta \beta [T(x), x]_\alpha \\ &= [T(x), x]_\alpha \beta y\beta x\gamma x - x\gamma x\beta y\beta [T(x), x]_\alpha \\ &\quad + x\gamma y\beta x\beta [T(x), x]_\alpha - [T(x), x]_\alpha \beta x\beta y\gamma x. \end{aligned}$$

Therefore, we have

$$[T(x), x]_{\alpha} \beta y \beta x \gamma x - x \gamma x \beta y \beta [T(x), x]_{\alpha} + x \gamma y \beta x \beta [T(x), x]_{\alpha} - [T(x), x]_{\alpha} \beta x \beta y \gamma x = 0,$$

for all $x, y \in M$, $\alpha, \beta \in \Gamma$, which reduces because of (10) and (15) to

$$[T(x), x]_{\alpha} \beta y \beta x \gamma x - x \gamma x \beta y \beta [T(x), x]_{\alpha} = 0.$$

Left multiplication of the above relation by $x\beta$ gives

$$x\beta [T(x), x]_{\alpha} \beta y \beta x \gamma x - x\beta x \gamma x \beta y \beta [T(x), x]_{\alpha} = 0.$$

One can replace in the above relation according to (15), $x\beta [T(x), x]_{\alpha} \beta y \beta x$ by $x\beta y \beta [T(x), x]_{\alpha} \beta x$ which gives

$$(21) \quad x\beta y \beta [T(x), x]_{\alpha} \beta x \gamma x - x\beta x \beta x \gamma y \beta [T(x), x]_{\alpha} = 0.$$

Left multiplication of the above relation by $T(x)\alpha$ gives

$$(22) \quad T(x)\alpha x \beta y \beta [T(x), x]_{\alpha} \beta x \gamma x - T(x)\alpha x \beta x \beta x \gamma y \beta [T(x), x]_{\alpha} = 0.$$

The substitution $T(x)\alpha y$ for y in (21), we have

$$(23) \quad x\beta T(x)\alpha y \beta [T(x), x]_{\alpha} \beta x \gamma x - x\beta x \beta x \gamma T(x)\alpha y \beta [T(x), x]_{\alpha} = 0.$$

Subtracting (23) from (22), we obtain

$$[T(x), x]_{\alpha} \beta y \beta [T(x), x]_{\alpha} \beta x \gamma x - [T(x), x\beta x \gamma x]_{\alpha} \beta y \beta [T(x), x]_{\alpha} = 0.$$

From the above relation and Lemma 2.1, it follows that

$$([T(x), x\beta x \gamma x]_{\alpha} - [T(x), x]_{\alpha} \beta x \gamma x) \beta y \beta [T(x), x]_{\alpha} = 0,$$

which reduces to

$$(x\beta [T(x), x]_{\alpha} \gamma x + x\beta x \gamma [T(x), x]_{\alpha}) \beta y \beta [T(x), x]_{\alpha} = 0.$$

Relation (9) makes it possible to write $[T(x), x]_{\alpha} \gamma x$ instead of $x\gamma [T(x), x]_{\alpha}$, which means that $x\beta x \gamma [T(x), x]_{\alpha}$ can be replaced by $x\beta [T(x), x]_{\alpha} \gamma x$ in the above relation. Thus we have

$$x\beta [T(x), x]_{\alpha} \gamma x \beta y \beta [T(x), x]_{\alpha} = 0.$$

Right multiplication of the above relation by γx and substitution $y\beta x$ for y gives finally,

$$x\beta [T(x), x]_{\alpha} \gamma x \beta y \beta x \beta [T(x), x]_{\alpha} \gamma x = 0.$$

Hence, by semiprimeness of M , we have

$$x\beta[T(x), x]_\alpha \gamma x = 0.$$

Next, we prove the relation (iii):

$$(24) \quad x\beta[T(x), x]_\alpha = 0, \quad x \in M, \quad \alpha \in \Gamma.$$

First, putting $y\alpha x$ for y in (15), gives because of (19)

$$(25) \quad x\alpha[T(x), x]_\alpha \beta y \alpha x \beta x = 0.$$

The substitution $y\alpha T(x)$ for y in (25), we have

$$(26) \quad x\alpha[T(x), x]_\alpha \beta y \alpha T(x) \alpha x \beta x = 0.$$

Right multiplication of (25) by $\alpha T(x)$,

$$(27) \quad x\alpha[T(x), x]_\alpha \beta y \alpha x \beta x \alpha T(x) = 0.$$

Subtracting (27) from (26) we have

$$\begin{aligned} x\alpha[T(x), x]_\alpha \beta y \alpha (T(x) \alpha x \beta x - x \beta x \alpha T(x)) &= 0 \\ \Rightarrow x\alpha[T(x), x]_\alpha \beta y \alpha [T(x), x \beta x]_\alpha &= 0 \\ \Rightarrow x\alpha[T(x), x]_\alpha \beta y \alpha ([T(x), x]_\alpha \beta x + x \beta [T(x), x]_\alpha) &= 0 \\ \Rightarrow x\beta [T(x), x]_\alpha \alpha y \alpha ([T(x), x]_\alpha \beta x + x \beta [T(x), x]_\alpha) &= 0. \end{aligned}$$

According to (9), one can replace $[T(x), x]_\alpha \beta x$ by $x\beta [T(x), x]_\alpha$, which gives

$$x\beta [T(x), x]_\alpha \alpha y \alpha x \beta [T(x), x]_\alpha = 0, \quad x, y \in M, \quad \alpha, \beta \in \Gamma.$$

Hence, by semiprimeness of M ,

$$x\beta [T(x), x]_\alpha = 0, \quad x, y \in M, \quad \alpha, \beta \in \Gamma.$$

Finally, we prove the relation (v):

$$(28) \quad [T(x), x]_\alpha = 0.$$

From (9) and (24), it follows that

$$[T(x), x]_\alpha \beta x = 0, \quad x \in M, \quad \alpha, \beta \in \Gamma.$$

The linearization of the above relation gives (see how relation (20) was obtained from (9)),

$$[T(x), x]_\alpha \beta y + [T(x), y]_\alpha \beta x + [T(y), x]_\alpha \beta x = 0.$$

Right multiplication of the above relation by $\beta [T(x), x]_\alpha$ gives because of (24),

$$[T(x), x]_\alpha \beta y \beta [T(x), x]_\alpha = 0,$$

which implies

$$[T(x), x]_\alpha = 0. \quad \blacksquare$$

Lemma 2.3 *Let M be Γ -ring satisfying the assumption (A) and let $T : M \rightarrow M$ be an additive mapping such that $T(x\alpha y\beta x) = x\alpha T(y)\beta x$ holds for all $x, y \in M$ and $\alpha, \beta \in \Gamma$. Then*

$$(29) \quad x\alpha(T(x\alpha y + y\alpha x) - T(y)\alpha x - x\alpha T(y))\beta x = 0.$$

Proof. The substitution $x\alpha y + y\alpha x$ for y in (1) gives

$$(30) \quad T(x\alpha x\alpha y\beta x + x\alpha y\alpha x\beta x) = x\alpha T(x\alpha y + y\alpha x)\beta x.$$

On the other hand, we obtain by putting $z = x\alpha x$ in (10), we have

$$T(x\alpha x\alpha y\beta x + x\alpha y\beta x\alpha x) = x\alpha T(y)\alpha x\beta x + x\alpha x\alpha T(y)\beta x,$$

i.e.,

$$(31) \quad T(x\alpha x\alpha y\beta x + x\alpha y\alpha x\beta x) = x\alpha T(y)\alpha x\beta x + x\alpha x\alpha T(y)\beta x.$$

By comparing (30) and (31), we have

$$x\alpha(T(x\alpha y + y\alpha x) - T(y)\alpha x - x\alpha T(y))\beta x = 0.$$

Let $G_\alpha(x, y) = T(x\alpha y + y\alpha x) - T(y)\alpha x - x\alpha T(y)$. Then, it is clear that

$$x\alpha G_\alpha(x, y)\beta x = 0 \text{ and } G_\alpha(x, y) = G_\alpha(y, x).$$

Replacing x for y and using (29), we have

$$y\alpha G_\alpha(x, y)\beta y = 0.$$

We can also prove easily the following results:

- (i) $G_\alpha(x + z, y) = G_\alpha(x, y) + G_\alpha(z, y)$
- (ii) $G_\alpha(x, y + z) = G_\alpha(x, y) + G_\alpha(x, z)$
- (iii) $G_{\alpha+\beta}(x, y) = G_\alpha(x, y) + G_\beta(x, y)$
- (iv) $G_\alpha(-x, y) = -G_\alpha(x, y)$
- (v) $G_\alpha(x, -y) = -G_\alpha(x, y)$. ■

Lemma 2.4 *Let M be a 2-torsion free semiprime Γ -ring satisfying the assumption (A) and let $T : M \rightarrow M$ be an additive mapping. Suppose that*

$$T(x\alpha y\beta x) = x\alpha T(y)\beta x$$

holds for all $x, y \in M$ and $\alpha, \beta \in \Gamma$. Then

$$(a) \quad [G_\alpha(x, y), x]_\alpha = 0$$

$$(b) \quad G_\alpha(x, y) = 0.$$

Proof. First we prove the relation (a):

$$(32) \quad [G_\alpha(x, y), x]_\alpha = 0.$$

The linearization of (28) gives

$$(33) \quad [T(x), y]_\alpha + [T(y), x]_\alpha = 0, \quad x, y \in M, \quad \alpha \in \Gamma.$$

Putting $x\alpha y + y\alpha x$ for y in the above relation and using (28), we obtain

$$\begin{aligned} [T(x), x\alpha y + y\alpha x]_\alpha + [T(x\alpha y + y\alpha x), x]_\alpha &= 0 \\ \Rightarrow x\alpha [T(x), y]_\alpha + [T(x), y]_\alpha \alpha x + [T(x\alpha y + y\alpha x), x]_\alpha &= 0 \\ \Rightarrow [T(x\alpha y + y\alpha x), x]_\alpha + x\alpha [T(x), y]_\alpha + [T(x), y]_\alpha \alpha x &= 0. \end{aligned}$$

According to (33), one can replace $[T(x), y]_\alpha$ by $-[T(y), x]_\alpha$ in the above relation. We have, therefore,

$$[T(x\alpha y + y\alpha x), x]_\alpha - x\alpha [T(y), x]_\alpha - [T(y), x]_\alpha \alpha x = 0,$$

which can be written in the form

$$[T(x\alpha y + y\alpha x) - T(y)\alpha x - x\alpha T(y), x]_\alpha = 0,$$

i.e.,

$$[G_\alpha(x, y), x]_\alpha = 0.$$

The proof is, therefore, complete. ■

Finally, we prove the relation (b):

$$(34) \quad G_\alpha(x, y) = 0.$$

From (29) one obtains (see how (20) was obtained from (9))

$$x\alpha G_\alpha(x, y)\beta z + x\alpha G_\alpha(z, y)\beta x + z\alpha G_\alpha(x, y) = 0.$$

Right multiplication of the above relation by $G_\alpha(x, y)\alpha x$ gives because of (29),

$$(35) \quad x\alpha G_\alpha(x, y)\beta z\beta G_\alpha(x, y)\alpha x = 0.$$

Relation (32) makes it possible to replace in (35), $x\alpha G_\alpha(x, y)$ by $G_\alpha(x, y)\alpha x$. Thus, we have

$$(36) \quad G_\alpha(x, y)\alpha x\beta z\beta G_\alpha(x, y)\alpha x = 0.$$

Therefore, by semiprimeness of M ,

$$(37) \quad G_\alpha(x, y)\alpha x = 0.$$

Of course, we also have

$$(38) \quad x\alpha G_\alpha(x, y) = 0.$$

The linearization of (37) with respect to x gives

$$G_\alpha(x, y)\alpha z + G_\alpha(z, y)\alpha x = 0.$$

Right multiplication of the above relation by $\alpha G_\alpha(x, y)$ gives because of (38),

$$G_\alpha(x, y)\alpha z\alpha G_\alpha(x, y) = 0,$$

which gives

$$G_\alpha(x, y) = 0,$$

i.e.,

$$(39) \quad T(x\alpha y + y\alpha x) = T(y)\alpha x + x\alpha T(y).$$

Hence, the proof is complete. ■

Theorem 2.1 *Let M be a 2-torsion free semiprime Γ -ring satisfying the assumption (A) and let $T : M \rightarrow M$ be an additive mapping. Suppose that*

$$T(x\alpha x\beta x) = x\alpha T(x)\beta x$$

holds for all $x \in M$ and $\alpha, \beta \in \Gamma$. Then T is a centralizer.

Proof. In particular, for $y = x$, the relation (39) reduces to

$$2T(x\alpha x) = T(x)\alpha x + x\alpha T(x).$$

Combining the above relation with (28), we arrive at

$$2T(x\alpha x) = 2T(x)\alpha x, x \in M, \alpha \in \Gamma$$

and

$$2T(x\alpha x) = 2x\alpha T(x), x \in M, \alpha \in \Gamma.$$

Since M is 2-torsion free, so we have

$$T(x\alpha x) = T(x)\alpha x, x \in M, \alpha \in \Gamma$$

and

$$T(x\alpha x) = x\alpha T(x), x \in M, \alpha \in \Gamma.$$

By Theorem 2.1 in [6], it follows that T is a left and also right centralizer which completes the proof of the theorem. ■

Putting $y = x$ in relation (1), we obtain

$$(40) \quad T(x\alpha x\beta x) = x\alpha T(x)\beta x, \quad x \in M, \quad \alpha, \beta \in \Gamma.$$

The question arises whether in a 2-torsion free semiprime Γ -ring the above relation implies that T is a centralizer. Unfortunately, we were unable to answer it affirmative if M has an identity element.

Theorem 2.2 *Let M be a 2-torsion free semiprime Γ -ring with identity element 1 satisfying the assumption (A) and let $T : M \rightarrow M$ be an additive mapping. Suppose that*

$$T(x\alpha x\beta x) = x\alpha T(x)\beta x$$

holds for all $x \in M$ and $\alpha, \beta \in \Gamma$. Then T is a centralizer.

Proof. Putting $x + 1$ for x in relation (40), one obtains after some calculations

$$3T(x\alpha x) + 2T(x) = T(x)\beta x + x\alpha T(x) + x\alpha a\beta x + a\alpha x + x\beta a,$$

where a stands for $T(1)$.

Putting $-x$ for x in the relation above and comparing the relation so obtained with the above relation we have

$$(41) \quad 6T(x\alpha x) = 2T(x)\beta x + 2x\alpha T(x) + 2x\alpha a\beta x$$

and

$$(42) \quad 2T(x) = a\alpha x + x\beta a.$$

We shall prove that $a \in Z(M)$. According to (42) one can replace $2T(x)$ on the right side of (41) by $a\alpha x + x\beta a$ and $6T(x\alpha x)$ on the left side by $3a\alpha x\beta x + 3x\beta x\alpha a$, which gives, after some calculation,

$$a\alpha x\beta x + x\beta x\alpha a - 2x\alpha a\beta x = 0.$$

The above relation can be written in the form

$$(43) \quad [[a, x]_\alpha, x]_\beta = 0; \quad x \in M, \quad \alpha, \beta \in \Gamma.$$

The linearization of the above relation gives

$$(44) \quad [[a, x]_\alpha, y]_\beta + [[a, y]_\alpha, x]_\beta = 0.$$

Putting $y = x\alpha y$ in (44), we obtain because of (43) and (44),

$$\begin{aligned} 0 &= [[a, x]_\alpha, x\alpha y]_\beta + [[a, x\alpha y]_\alpha, x]_\beta \\ &= [[a, x]_\alpha, x]_\beta \beta y + x\alpha [[a, x]_\alpha, y]_\beta + [[a, x]_\alpha \alpha y + x\alpha [a, y]_\alpha, x]_\beta \\ &= x\alpha [[a, x]_\alpha, y]_\beta + [[a, x]_\alpha \alpha y, x]_\beta + [x\alpha [a, y]_\alpha, x]_\beta \\ &= x\alpha [[a, x]_\alpha, y]_\beta + [[a, x]_\alpha, x]_\beta \beta y + [a, x]_\alpha \beta [y, x]_\alpha + x\alpha [[a, y]_\alpha, x]_\beta \\ &= [a, x]_\alpha \beta [y, x]_\alpha. \end{aligned}$$

The substitution $y\beta a$ for y in the above relation gives

$$[a, x]_{\alpha}\beta y\beta[a, x]_{\alpha} = 0,$$

whence it follows $a \in Z(M)$, which reduces (42) to the form $T(x) = a\alpha x$, $x \in M$, $\alpha \in \Gamma$. The proof of the theorem is complete. ■

We conclude with the following conjecture:

Let M be a semiprime Γ -ring with suitable torsion restrictions. Suppose there exists an additive mapping $T : M \rightarrow M$ such that

$$T((x\alpha)^m(x\beta)^n x) = (x\alpha)^m T(x)(\beta x)^n$$

holds for all $x \in M$, $\alpha, \beta \in \Gamma$, where $m \geq 1$, $n \geq 1$ are some integers. Then T is a centralizer.

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INTEGRAL FILTERS AND INTEGRAL BL -ALGEBRAS

Rajab Ali Borzooei

A. Paad

*Department of Mathematics
Shahid Beheshti University
Tehran
Iran
e-mail: borzooei@sbu.ac.ir
a_pad@sbu.ac.ir*

Abstract. In this paper, we introduce the concepts of integral filters and integral BL -algebras. With respect to concepts, we give some related results. In particular, we prove that an integral BL -algebra is a perfect, local, directly indecomposable BL -algebra and SBL -algebra. Also, we give some relations among integral filters and some types of filters in BL -algebras, such as prime, primary, perfect, fantastic, positive implicative and obstinate filters.

Keywords: BL -algebra, integral BL -algebra, BL -algebra with Gödel negation, Gödel algebra, MV -algebra, fantastic, primary, integral, perfect and positive implicative filter.

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1. Introduction

BL -algebras are the algebraic structure for Hájek basic logic [8] in order to investigate many valued logic by algebraic means. His motivations for introducing BL -algebras were of two kinds. The first one was providing an algebraic counterpart of a propositional logic, called Basic Logic, which embodies a fragment common to some of the most important many-valued logics, namely Łukasiewicz Logic, Gödel Logic and Product Logic. This Basic Logic (BL for short) is proposed as "the most general" many-valued logic with truth values in $[0, 1]$ and BL -algebras are the corresponding Lindenbaum-Tarski algebras. The second one was to provide an algebraic mean for the study of continuous t-norms (or triangular norms) on $[0; 1]$. Most familiar example of a BL -algebra is the unit interval $[0, 1]$ endowed with the structure induced by a continuous t-norm. In 1958, Chang [3] introduced the concept of an MV -algebra which is one of the most classes of BL -algebras. Turunen [14] introduced the notion of an implicative filter and a *Boolean* filter

and proved that these notions are equivalent in BL -algebras. *Boolean* filters are an important class of filters, because the quotient BL -algebra induced by these filters are *Boolean* algebras. In this paper we study integral BL -algebras and BL -algebras with Godel negation such that they are local BL -algebras, but the converse is not correct in general. So it is important for us studying this class of BL -algebras. At first, we prove some theorems about local and integral BL -algebras. Also, we show integral MV -algebras are trivial. Moreover, we define integral filters and we prove that a BL -algebra is integral BL -algebra if and only if $\{1\}$ is integral filter. Also, we show in finite BL -algebras, integral filters and perfect filters are equal. In the follow, we define BL -algebras with Godel negation and we prove that these class of BL -algebras are equal to the class integral BL -algebras and in a linearly ordered BL -algebra, a BL -algebra is a SBL -algebra if and only if is a integral BL -algebra. Finally, we study the relationship between integral filters and obstinate filters and we prove that an integral and fantastic filter is a obstinate filter.

2. Preliminaries

In this section, we recollect some definitions and theorems which will be used in the following, and we shall not cite them every time they are used.

Definition 2.1. ([8]) A BL -algebra is an algebra $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2,2,2,2,0,0)$ such that

- (BL1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,
- (BL2) $(L, \odot, 1)$ is a commutative monoid,
- (BL3) $z \leq x \rightarrow y$ if and only if $x \odot z \leq y$, for all $x, y, z \in L$,
- (BL4) $x \wedge y = x \odot (x \rightarrow y)$,
- (BL5) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

We denote $x^n = \overbrace{x \odot \dots \odot x}^{n\text{-times}}$, if $n > 0$ and $x^0 = 1$.

A BL -algebra L is called a *Godel algebra* if $x^2 = x$, for all $x \in L$ and L is called an *MV-algebra* if, $(x^-)^- = x$, for all $x \in L$, where $x^- = x \rightarrow 0$.

Lemma 2.2. ([4],[5],[8]) *In any BL -algebra the following hold:*

- (BL6) $x \leq y$ if and only if $x \rightarrow y = 1$.
- (BL7) $0 \rightarrow x = 1$ and $x \rightarrow x = 1$.
- (BL8) $x \odot y \leq x \wedge y$.
- (BL9) $x^- = 1$ if and only if $x = 0$.
- (BL10) $x^- = x^{---}$.
- (BL11) $x \odot x^- = 0$, $1 \odot x = x$ and $0 \odot x = 0$.
- (BL12) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z$.
- (BL13) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.
- (BL14) $x \leq y$ implies $y \rightarrow z \leq x \rightarrow z$ and $z \rightarrow x \leq z \rightarrow y$.
- (BL15) $(x \wedge y)^- = x^- \vee y^-$.
- (BL16) $x \odot y = 0$ if and only if $x \leq y^-$.

for all $x, y, z \in L$.

We briefly review some types of filters and related theorems that, we refer the reader to [4],[5],[8],[9], [13],[14],[15],[16], for more details.

Definition 2.3. Let L be a BL -algebra and F be a nonempty subset of L . Then

- (i) F is called a *filter* of L , if $x \odot y \in F$, for all $x, y \in F$ and if $x \in F$ and $x \leq y$ then $y \in F$, for all $x, y \in L$.
- (ii) proper filter F is called a *prime filter* of L , if $x \vee y \in F$ implies $x \in F$ or $y \in F$, for all $x, y \in L$.
- (iii) proper filter F is called a *maximal filter* of L , if it is not properly contained in any other proper filter of L .
- (iv) proper filter F is called a *primary filter*, if for all $x, y \in L$, $(x \odot y)^- \in F$ implies $(x^n)^- \in F$ or $(y^n)^- \in F$, for some $n \in \mathbb{N} \cup \{0\}$.
- (v) filter F is called a *Boolean filter*, if $x^- \vee x \in F$, for all $x \in L$.
- (vi) F is called an *implicative filter*, if $1 \in F$ and $x \rightarrow (y \rightarrow z) \in F$ and $x \rightarrow y \in F$, then $x \rightarrow z \in F$, for all $x, y, z \in L$.
- (vii) F is called a *positive implicative filter*, if $1 \in F$ and $x \rightarrow ((y \rightarrow z) \rightarrow y) \in F$ and $x \in F$, then $y \in F$, for all $x, y, z \in L$.
- (viii) F is called a *fantastic filter*, if $1 \in F$ and $z \rightarrow (y \rightarrow x) \in F$ and $z \in F$, then $((x \rightarrow y) \rightarrow y) \rightarrow x \in F$, for all $x, y, z \in L$.

Definition 2.4. Let L be a BL -algebra. Then L is called a *local BL -algebra*, if it has a unique maximal filter.

Note. A filter F of BL -algebra L is maximal and implicative if and only if $x, y \notin F$ imply $x \rightarrow y \in F$ and $y \rightarrow x \in F$, for all $x, y \in L$. This filter is called an *obstinate filter*.

Theorem 2.5. ([11],[12]) Let F of BL -algebra L . Then

- (i) F is a *fantastic filter* of L if and only if $((x \rightarrow 0) \rightarrow 0) \rightarrow x \in F$, for all $x \in L$.
- (ii) if F is an *obstinate filter* of L , then F is a *fantastic filter* of L .

Theorem 2.6. [8] Let F be a filter of BL -algebra L . Then the binary relation $\equiv_F$ which is defined by

$$x \equiv_F y \text{ if and only if } x \rightarrow y \in F \text{ and } y \rightarrow x \in F$$

is a congruence relation on L . Define $\cdot, \rightarrow, \sqcup, \sqcap$ on L/F , the set of all congruence classes of L , as follows:

$$[x] \cdot [y] = [x \odot y], [x] \rightarrow [y] = [x \rightarrow y], [x] \sqcup [y] = [x \vee y], [x] \sqcap [y] = [x \wedge y].$$

Then $(L/F, \cdot, \rightarrow, \sqcup, \sqcap, [0], [1])$ is a BL -algebra which is called *quotient BL -algebra* with respect to F .

Theorem 2.7. ([4],[13]) *Let P be a proper filter of BL -algebra L . Then P is a prime filter if and only if L/P is a BL -chain.*

Definition 2.8. ([5]) Let L be a BL -algebra and $x \in L$. If there exists a smallest positive integer number n such that $x^n = 0$, then we say the order of x is n and we denote by $ord(x) = n$ and we say $ord(x) = \infty$, if no such n exists.

Definition 2.9. ([13]) Let L_1 and L_2 be two BL -algebras. Then the map $f : L_1 \rightarrow L_2$ is called a BL -algebra *homomorphism* if and only if it satisfies the following conditions, for every $x, y \in L_1$:

- (i) $f(0) = 0$,
- (ii) $f(x \odot y) = f(x) \odot f(y)$,
- (iii) $f(x \rightarrow y) = f(x) \rightarrow f(y)$.

If f is a bijective, then the homomorphism f is called BL -algebra *isomorphism*. In this case we write $L_1 \cong L_2$.

The following theorems and definitions are from [6],[16] and the reader can refer to it, for more details.

Theorem 2.10. *Let L be a BL -algebra. The following are equivalent:*

- (i) L is a local BL -algebra,
- (ii) $M(L) = \{x \in L \mid x^n \neq 0, \forall n \in \mathbb{N} \cup \{0\}\} = \{x \in L \mid ord(x) = \infty\}$ is the unique maximal filter of L ,
- (iii) for all $x \in L$, $ord(x) < \infty$ or $ord(x^-) < \infty$.

Theorem 2.11. *Let P be a filter of BL -algebra L . The following are equivalent:*

- (i) L/P is a local BL -algebra,
- (ii) P is a primary filter.

Theorem 2.12. *Let L be a BL -algebra. The following are equivalent:*

- (i) L is a Godel algebra,
- (ii) Any filter of L is an implicative filter of L .

Definition 2.13. A BL -algebra L is called *directly indecomposable* if and only if L is nontrivial and whenever $L \cong L_1 \times L_2$ then either L_1 or L_2 is trivial.

Note. Let $B(L)$ be the *Boolean* algebra of all complemented elements in the distributive lattice $L(L)$.

Theorem 2.14. *Let L be a BL -algebra. Then*

- (i) L is directly indecomposable if and only if $B(L) = \{0, 1\}$.
- (ii) L is directly indecomposable if L is local.

Definition 2.15. Let L be a BL -algebra. Then

- (i) L is a *perfect* BL -algebra, if it is local and for any $x \in L$, $\text{ord}(x) < \infty$ implies $\text{ord}(x^-) = \infty$.
- (ii) proper filter P of L is called *perfect filter*, if for all $x \in L$, $(x^n)^- \in P$, for some $n \in \mathbb{N} \cup \{0\}$ if and only if $((x^-)^m)^- \notin P$, for all $m \in \mathbb{N} \cup \{0\}$.

Theorem 2.16. *Let P be a filter of BL -algebra L . The following are equivalent:*

- (i) L/P is a *perfect* BL -algebra,
- (ii) P is a *perfect filter*.

Theorem 2.17. *Any perfect filter of BL -algebra L is a primary filter.*

Theorem 2.18. [1, 6] *Let L be a BL -algebra. Then*

- (i) a local BL -algebra L is a *perfect* BL -algebra if and only if $MV(L)$ is a *perfect* BL -algebra, where $MV(L) = \{x \in L \mid x^{--} = x\}$.
- (ii) the only finite *perfect* MV -algebra is $\{0, 1\}$.

Definition 2.19. [7] An SBL -algebra is a BL -algebra L verifying this further condition (strict axiom): $(x \odot y)^- = (x^-) \vee (y^-)$

From now on, in this paper $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ (or simply) L is a BL -algebra, unless otherwise state.

3. Integral BL -algebras

In this section we study a class of BL -algebra that called integral BL -algebra and we give some related results.

Definition 3.1. L is called an *integral* BL -algebra, if $x \odot y = 0$, then $x = 0$ or $y = 0$, for all $x, y \in L$.

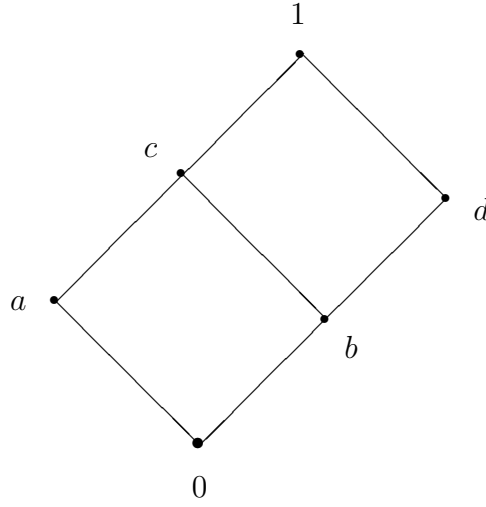
Example 3.2. [10] Let $L = \{0, a, b, c, 1\}$. Define $\wedge, \vee, \odot$ and $\rightarrow$ on L as follows:

$\vee$	0	c	a	b	1
0	0	c	a	b	1
c	c	c	a	b	1
a	a	a	a	1	1
b	b	b	1	b	1
1	1	1	1	1	1
$\rightarrow$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

$\wedge$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1
$\odot$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

Then $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is an integral BL -algebra.

Example 3.3. [10] Let $L = \{0, a, b, c, d, 1\}$. Then L by the following diagram is a bounded lattice.



Now, let $\rightarrow$ and $\odot$ are defined on L as follows:

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

Then $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a BL -algebra, which is not an integral BL -algebra. Since $a \odot b = 0$ for $a, b \neq 0$.

Theorem 3.4. Let P be a filter of L and L/P be an integral BL -algebra. Then P is a primary filter.

Proof. Let $(x \odot y)^- \in P$, for $x, y \in L$. Then $(x \odot y) \rightarrow 0 \in P$. Since by (BL7), $0 \rightarrow (x \odot y) = 1 \in P$, then $[x \odot y] = 0$ and so $[x] \cdot [y] = [x \odot y] = [0]$. Now, since L/P is an integral BL -algebra then $[x] = [0]$ or $[y] = [0]$. Hence, $x^- = x \rightarrow 0 \in P$ or $y^- = y \rightarrow 0 \in P$ and so P is a primary filter. ■

The following example shows that, the converse of Theorem 3.4, is not correct, in general.

Example 3.5. Let $F = \{1, d\}$, in Example 3.3. It is easy to check that F is a primary filter, but L/F is not an integral BL -algebra. Since $a \rightarrow 0 = d \in F$, then $[a] = [0]$ and so $[c] \cdot [c] = [c \odot c] = [a] = [0]$. But $[c] \neq [0]$. Therefore, L/F is not an integral BL -algebra.

Theorem 3.6. Let L be an integral BL -algebra. Then

- (i) L is a local and directly indecomposable BL -algebra and $B(L) = \{0, 1\}$.
- (ii) $M(L) = L \setminus \{0\}$.
- (iii) L is a perfect BL -algebra and $\text{ord}(x) = \infty$ when $0 \neq x \in L$.
- (iv) If L is an Boolean algebra, then $L = \{0, 1\}$.

Proof. (i) Since $\{1\}$ is a filter of L , then by Theorem 2.6, $L/\{1\}$ is well-defined. Now, let $f : L \rightarrow L/\{1\}$ be canonical epimorphism with $[x] = [y]$, for $x, y \in L$. Then $x \rightarrow y \in \{1\}$, $y \rightarrow x \in \{1\}$ and so by (BL6), $x \leq y$ and $y \leq x$. Hence, $x = y$ and so f is isomorphism. Now, since $L \cong L/\{1\}$ and L is an integral BL -algebra, then $L/\{1\}$ is an integral BL -algebra and by Theorem 3.4, $\{1\}$ is a primary filter. Hence, by Theorem 2.11, $L/\{1\}$ is a local BL -algebra and so L is a local BL -algebra. Thus, by Theorem 2.14, L is directly indecomposable BL -algebra and $B(L) = \{0, 1\}$.

(ii) Let $0 \neq x \in L \setminus M(L)$. Then there exists minimal element $n \in \mathbb{N} \cup \{0\}$, such that $x^n = 0$ and $x^m \neq 0$, for any $m < n$. Now, since $x^{n-1} \odot x = 0$ and L is an integral BL -algebra, then $x^{n-1} = 0$ or $x = 0$, which is impossible. Hence, $M(L) \subseteq L \setminus \{0\} \subseteq M(L)$ and so $M(L) = L \setminus \{0\}$.

(iii) Since L is an integral BL -algebra, then by (i), L is a local BL -algebra. Hence, by (ii) and Theorem 2.10, for all $0 \neq x \in L$, $\text{ord}(x) = \infty$. Moreover, if $x = 0$, then $\text{ord}(x^-) = \infty$. Hence, L is a perfect BL -algebra.

(iv) Since L is an integral BL -algebra, then by (i), L is a local BL -algebra and $B(L) = \{0, 1\}$. Since L is a Boolean algebra, then $B(L) = L$. Therefore, $L = \{0, 1\}$. ■

Note. In Example 3.5, since $F = \{1, d\}$ is a primary filter, then by Theorem 2.11, L/F is a local BL -algebra, but it is not an integral BL -algebra. Therefore, the converse of Theorem 3.6(i), is not true in general.

Theorem 3.7. *If P is a maximal and implicative filter of L , then L/P is an integral BL -algebra and so P is a primary filter.*

Proof. Let P be a maximal and implicative filter of L and $[x] \cdot [y] = [0]$, for $[x], [y] \in L/P$. Then by (BL12) and (BL13), $x \rightarrow (y \rightarrow 0) = (x \odot y) \rightarrow 0 \in P$ and $y \rightarrow (x \rightarrow 0) = (x \odot y) \rightarrow 0 \in P$. If $x \in P$, then $y \rightarrow 0 \in P$ and so $[y] = 0$. Similarly, if $y \in P$, then $x \rightarrow 0 \in P$ and so $[x] = 0$. Now, let $x, y \in P$. Then $0 \in P$, that is impossible. If $x, y \notin P$, since P is a maximal and implicative filter, then $x \rightarrow y \in P$ and $y \rightarrow x \in P$. Since P is an implicative filter, then $x \rightarrow 0 \in P$, $y \rightarrow 0 \in P$ and so $[x] = 0$ and $[y] = 0$. Hence, L/P is an integral BL -algebra and so by Theorem 3.4, P is a primary filter. ■

Lemma 3.8. *Let P be a primary filter of L and $[x] \cdot [y] = [0]$, for $[x], [y] \in L/P$. Then $[x]$ or $[y]$ is nilpotent.*

Proof. Let $[x] \cdot [y] = [0]$, for $[x], [y] \in L/P$. Then $(x \odot y)^- \in P$. Since P is a primary filter, then $(x^n)^- \in P$ or $(y^n)^- \in P$, for some $n \in \mathbb{N} \cup \{0\}$. Therefore, $[x]^n = [0]$ or $[y]^n = [0]$ and so $[x]$ or $[y]$ is nilpotent. ■

Theorem 3.9. *Let L be a Godel algebra L and P be a filter of L . Then*

- (i) *P is a primary filter of L if and only if L/P is an integral BL -algebra.*
- (ii) *L is an integral BL -algebra if and only if L is a Local BL -algebra.*

Proof. (i) Let P be a primary filter of L and $[x] \cdot [y] = [0]$, for $[x], [y] \in L/P$. Then by Lemma 3.8, there exist $m, n \in \mathbb{N}$, such that $[x]^n = [0]$ or $[y]^m = [0]$ and so $[x^n] = [0]$ or $[y^m] = [0]$. Now, since L is a Godel algebra, then $x^n = x$ and $y^m = y$ and so $[x] = [0]$ or $[y] = [0]$. Conversely, if L/P is an integral BL -algebra, then by Theorem 3.4, P is a primary filter of L .

(ii) Let L be an integral BL -algebra. Then by Theorem 3.6(i), L is a local BL -algebra. Conversely, let L be a local BL -algebra. Then by $L \cong L/\{1\}$, $L/\{1\}$ is a local BL -algebra and by Theorem 2.11, $\{1\}$ is a primary filter. Therefore, by (i), $L/\{1\}$ is an integral BL -algebra and so L is an integral BL -algebra. ■

Note. In [7], it is proved that, a linearly ordered BL -algebra L is a SBL -algebra if and only if the negation $\neg x = (x \rightarrow 0)$ is a Godel negation.

Definition 3.10. L is called a BL -algebra with Godel negation, if $\xrightarrow{St_r}(\{0\}) = L \setminus \{0\}$, where $\xrightarrow{St_r}(\{0\}) = \{a \in L \mid a \rightarrow 0 = 0\}$.

In [2], it is proved that $\xrightarrow{St_r}(\{0\})$ is a filter of L .

Lemma 3.11. *Let L be a BL -algebra with Godel negation. Then $x \rightarrow 0 = 0$ or $x \rightarrow 0 = 1$, for all $x \in L$.*

Proof. If $x = 0$, then $x \rightarrow 0 = 1$ and if $0 \neq x \in L$, then $x \in \xrightarrow{St_r}(\{0\})$ and so $x \rightarrow 0 = 0$, for all $x \in L$. ■

Example 3.12. BL -algebra in the Example 3.2, is a BL -algebra with Godel negation.

Example 3.13. Let $L = \{0, a, b, 1\}$ be a chain such that $0 < a < b < 1$ and operations $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$, $\odot$ and $\rightarrow$ are defined on L as the following tables:

$\odot$	0	a	b	1	$\rightarrow$	0	a	b	1
0	0	0	0	0	0	1	1	1	1
a	0	0	a	a	a	a	1	1	1
b	0	a	b	b	b	0	a	1	1
1	0	a	b	1	1	0	a	b	1

Then $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a BL -algebra but it is not a BL -algebra with Godel negation.

Theorem 3.14. *Let L be a BL -algebra with Godel negation and F be a proper filter of L . Then*

- (i) if F is a fantastic filter of L , then $F = L \setminus \{0\}$,
- (ii) if F is a positive implicative(Boolea) filter of L , then $F = L \setminus \{0\}$.
- (ii) if F is a obstinate filter of L , then $F = L \setminus \{0\}$.

Proof. (i) Let $0 \neq x \in L$ and F be a proper fantastic filter of L . Then, by Theorem 2.5, $((x \rightarrow 0) \rightarrow 0) \rightarrow x \in F$. Since L is a BL -algebra with Godel negation, then $x \rightarrow 0 = 0$, and so by (BL7), we have

$$((x \rightarrow 0) \rightarrow 0) \rightarrow x = (0 \rightarrow 0) \rightarrow x = 1 \rightarrow x = x \in F$$

. Hence, $L \setminus \{0\} \subseteq F$ and so $F = L \setminus \{0\}$.

- (ii) Since every positive implicative filter is a fantastic filter, then $F = L \setminus \{0\}$.
- (iii) Since every obstinate filter is a fantastic filter, then $F = L \setminus \{0\}$. ■

Corollary 3.15. *In any BL -algebra L with Godel negation, the following hold:*

- (i) every proper fantastic filter is a maximal filter,
- (ii) every proper positive implicative filter is a maximal filter,
- (iii) every proper obstinate filter is a maximal filter.

Theorem 3.16. *Let L be a BL -algebra L with Godel negation. Then the proper positive implicative filters, the proper obstinate filters and the proper fantastic filters are equal and they are exactly $L \setminus \{0\}$.*

Proof. By Theorem 3.14 and Corollary 3.15, the proof is clear. ■

4. Integral filters in BL -algebras

Definition 4.1. A proper filter P of L is called an *integral filter*, if for all $x, y \in L$,

$$(x \odot y)^- \in P \text{ implies } x^- \in P \text{ or } y^- \in P$$

Example 4.2. Let $F = \{1, a, c\}$, in the Example 3.3. Then F is an integral filter. Since, if $(x \odot y)^- \in F$, then $(x \odot y)^- = a$ or c or 1 , and so $(x \odot y) = b$ or d or 0 . If $(x \odot y) = b$, then $x = c$ or b and $y = d$. If $x = c$, then $c^- = b \in F$ and if $x = b$, $b^- \notin F$, then $d^- = a \in F$. By the similar way for $(x \odot y) = d$ or 0 , we conclude that F is an integral filter.

Theorem 4.3. *Every integral filter is a primary filter.*

Proof. Let $n = 1$ in Definition 2.3(iv). Then the proof is clear. ■

The following example shows that the converse of Theorem 4.3 is not true in general.

Example 4.4. Let $F = \{1, d\}$, in Example 3.3. Then it is easy to check that F is a primary filter. Now, since $(c \odot c)^- = a^- = d \in F$ and $c^- = b \notin F$. Therefore, F is not an integral filter.

Theorem 4.5. *Let $F \subseteq G$, where F and G be filters of L and F be an integral filter of L . Then G is an integral filter, too.*

Proof. Let $(x \odot y)^- \in G$, for $x, y \in L$. Then, by (BL9) and (BL11), $((x \odot y) \odot (x \odot y)^-)^- = 0^- = 1 \in F$. Since F is an integral filter, then we get $(x \odot y)^- \in F$ or $(x \odot y)^{- -} \in F$. If $(x \odot y)^- \in F$, then $x^- \in F \subseteq G$ or $y^- \in F \subseteq G$ and so G is an integral filter. If $(x \odot y)^{- -} \in F \subseteq G$, then by $(x \odot y)^- \in G$ and by (BL11), we have $(x \odot y)^{- -} \odot (x \odot y)^- = 0 \in G$. Therefore, $G = L$ and so G is an integral filter. ■

Theorem 4.6. *Let P be a proper filter of L . Then P is an integral filter if and only if L/P is an integral BL-algebra.*

Proof. Let P be an integral filter and $[x] \cdot [y] = [0]$, for $[x], [y] \in L/P$. Then $(x \odot y)^- \in P$, and so $x^- \in P$ or $y^- \in P$. Hence, $[x] = 0$ or $[y] = 0$. Conversely, let $(x \odot y)^- \in P$, for $x, y \in L$. Then $[x \odot y] = [x] \cdot [y] = [0]$. Since L/P is an integral BL-algebra, then $[x] = 0$ or $[y] = 0$. Therefore, $x^- \in P$ or $y^- \in P$. ■

The following theorem describes the relationship between integral filters and integral BL-algebras.

Theorem 4.7. *The following conditions are equivalent on L :*

- (i) $\{1\}$ is an integral filter of L ,
- (ii) any filter of L is an integral filter,
- (iii) L is an integral BL-algebra.

Proof. (i) $\Leftrightarrow$ (ii): By Theorem 4.5, the proof is clear.

(i) $\Rightarrow$ (iii): Since $L \cong L/\{1\}$ and $\{1\}$ is an integral filter, then by Theorem 4.6, L is an integral BL-algebra.

(iii) $\Rightarrow$ (i): By Theorem 4.6, the proof is clear. ■

Theorem 4.8. *L is an integral BL-algebra if and only if L has the Godel negation.*

Proof. Let L be an integral BL-algebra. Then by (BL7), $0 \rightarrow 0 = 1$, and so $0 \notin \xrightarrow{St_r}(\{0\})$. Hence, $\xrightarrow{St_r}(\{0\}) \subseteq L \setminus \{0\}$. Now, let $x \in L \setminus \{0\}$. Then by (BL11), $x \odot x^- = 0$ and so $(x \odot x^-)^- = 1$. Since by Theorem 4.7, $\{1\}$ is an integral filter, then $x^- = 1$ or $(x^-)^- = 1$. If $x^- = 1$, then $x \odot x^- = x \odot 1 = x$ and so $x = 0$, that it is a contradiction. Hence, $(x^-)^- = 1$ and so by (BL10), $x^- = ((x^-)^-)^- = 0$. Hence, $x \in \xrightarrow{St_r}(\{0\})$ and so L is a BL-algebra with Godel negation. Conversely, let $(x \odot y) = 0$, for $x, y \in L$. Then $(x \odot y) \notin \xrightarrow{St_r}(\{0\}) = L \setminus \{0\}$. Since $\xrightarrow{St_r}(\{0\})$ is a proper filter, then $x \notin \xrightarrow{St_r}(\{0\})$ or $y \notin \xrightarrow{St_r}(\{0\})$. Therefore, $x = 0$ or $y = 0$ and so L is an integral BL-algebra. ■

Theorem 4.9. *Every integral BL-algebra is an SBL-algebra.*

Proof. Let L be an integral BL -algebra. We will prove that $(x \odot y)^- = (x^-) \vee (y^-)$, for all $x, y \in L$. If $x \odot y = 0$, then $x = 0$ or $y = 0$ and by Lemma 3.11 and Theorem 4.8, $(x \odot y)^- = 1$ and $(x^-) \vee (y^-) = 1$. Let $x \odot y \neq 0$. Then $x \neq 0$ and $y \neq 0$, hence by Lemma 3.11 and Theorem 4.8, $(x \odot y)^- = 0$ and $(x^-) \vee (y^-) = 0$. Therefore, $(x \odot y)^- = (x^-) \vee (y^-) = 0$. Thus, L is a SBL -algebra. ■

Corollary 4.10. *Any linearly ordered BL -algebra L is a SBL -algebra if and only if L is an integral BL -algebra.*

Proof. By Theorem 4.9, every integral BL -algebra is an SBL -algebra. Conversely, a linearly ordered BL -algebra L is a SBL -algebra if L is with Godel negation and so by Theorem 4.8, L is an integral BL -algebra. ■

Theorem 4.11. *Let L be a BL -algebra with Godel negation. Then*

- (i) $MV(L) = B(L) = \{0, 1\}$.
- (ii) *If L be an MV -algebra, then $L = \{0, 1\}$.*

Proof. (i) It is clear that $0 \in MV(L)$. Now, let $0 \neq x \in MV(L)$. Since $x^- = 0$, then $x = x^{--} = 1$ and so $x = 1$. Hence, $MV(L) = \{0, 1\}$. Now, by Theorems 4.8 and 3.6(ii), $B(L) = \{0, 1\}$ and so $MV(L) = B(L) = \{0, 1\}$.

(ii) Since L is a MV -algebra, then $MV(L) = L$ and so by (i), $L = \{0, 1\}$. ■

In the following theorem we describe the relationship between integral filters and perfect filters.

Theorem 4.12. *Let P be an integral filter of L . Then P is a perfect filter.*

Proof. Let P be an integral filter. Then, by Theorem 4.6, L/P is an integral BL -algebra and so by Theorem 3.6(iii), L/P is perfect BL -algebra. Hence, by Theorem 2.16, P is a perfect filter. ■

Open problem. Under what suitable conditions the converse of Theorem 4.12, is correct in general? We prove the converse of Theorem 4.12, in finite BL -algebra and Godel algebra.

Lemma 4.13. *Let L be finite. Then L is an integral BL -algebra if and only if $MV(L)$ is an integral BL -algebra, where $MV(L) = \{x \in L \mid x^{--} = x\}$.*

Proof. Let L be an integral BL -algebra and $x \odot y = 0$, for $x, y \in MV(L)$. Since $MV(L) \subseteq L$, then $x = 0$ or $y = 0$ and so $MV(L)$ is an integral BL -algebra. Conversely, let $MV(L)$ is an integral BL -algebra. Since L is a finite BL -algebra and $MV(L) \subseteq L$, then $MV(L)$ is finite integral BL -algebra and so by Theorem 3.6(iii), $MV(L)$ is a finite perfect MV -algebra. Therefore, by Theorem 2.18(ii), $MV(L) = \{0, 1\}$. Now, let $x \odot y = 0$, for all $x, y \in L$. Then by (BL16), $x \leq y^-$ and by (BL10), $y^- \in MV(L)$. Thus, $y^- = 0$ or $y^- = 1$ and so $y = 1$ or $y = 0$. If $y = 1$, since $x \odot y = 0$, then $x = 0$ and otherwise $y = 0$. Therefore, L is an integral BL -algebra. ■

Theorem 4.14. *Let L be finite. Then every perfect filter of L is an integral filter.*

Proof. Let P be a perfect filter of finite BL -algebra L . Then, by Theorem 2.16, L/P is a finite perfect BL -algebra. Also, by Theorem 2.18(i), $MV(L/P)$ is a finite perfect MV -algebra. Then, by Theorem 2.18(ii), $MV(L/P) = \{0, 1\}$. Hence, $MV(L/P)$ is an integral BL -algebra and by Lemma 4.13, L/P is an integral BL -algebra. Therefore, by Theorem 4.6, P is an integral filter. ■

Theorem 4.15. *Let L be a Gödel algebra. Then the concept of integral filter, perfect filter and primary filter are coincide.*

Proof. Let P be an integral filter. Then, by Theorem 4.12, P is a perfect filter. Conversely, let P is a perfect filter. Then, by Theorems 2.17 and 3.9(i), L/P is an integral BL -algebra and so by Theorem 4.6, P is an integral filter. Now, let P be a perfect filter. Then, by Theorem 2.17, P is primary filter. Conversely, let P be a primary filter. Then, by Theorem 3.9(i), L/P is an integral BL -algebra and so by Theorem 4.6, P is an integral filter. Hence, by Theorem 4.12, P is a perfect filter. ■

Theorem 4.16. *Let L be an MV -algebra and F be an integral filter of L . Then F is a prime filter.*

Proof. Let $x \vee y \in F$, for $x, y \in L$. Since by (BL8), $x^- \odot y^- \leq x^- \wedge y^-$, then by (BL14), $(x^- \wedge y^-)^- \leq (x^- \odot y^-)^-$ and by (BL15), $(x^- \wedge y^-)^- = x^{--} \vee y^{--} = x \vee y \in F$. Now, we have $(x^- \odot y^-)^- \in F$, $[(x^- \odot y^-)^-] = [1]$ and $[(x^- \odot y^-)]^{--} = [(x^- \odot y^-)^-] = [0]$. Since L is an MV -algebra, then $[x^-] \cdot [y^-] = [x^- \odot y^-] = [0]$. Since by Theorem 4.6, L/F is an integral BL -algebra, then $[x^-] = [0]$ or $[y^-] = [0]$ and so $[x] = [1]$ or $[y] = [1]$. Hence, $x \in F$ or $y \in F$, and so F is a prime filter. ■

Note. In Theorem 4.16, L must be an MV -algebra and it is a necessary condition. Since in the Example 3.2, L is a BL -algebra, which is not a MV -algebra and $F = \{1\}$ is an integral filter of L . But it is not a prime filter, since $a \vee b = 1 \in F$ and $a, b \notin F$.

Corollary 4.17. *Every integral MV -algebra is a BL -chain.*

Proof. Let L be an integral MV -algebra. Then $\{1\}$ is an integral filter and so by Theorem 4.16, it is a prime filter. Therefore, by Theorem 2.7, $L/\{1\}$ is a BL -chain and since $L \cong L/\{1\}$, then L is a BL -chain. ■

In what follows, we study the relations among integral filters, obstinate filters and fantastic filters. The following theorem shows that every obstinate filter is an integral filter but the converse is not true.

Theorem 4.18. *Let F be a proper obstinate filter of L . Then F is an integral filter of L .*

Proof. Let F be a proper obstinate filter, $(x \odot y)^- \in F$ but $x^- \notin F$ and $y^- \notin F$, for $x, y \in L$, by a contrary. Since F is an obstinate filter, then $(x^-)^- \in F$ and by Theorem 2.5(ii), F is a fantastic filter. Now, by Theorem 2.5(i), $((x^-)^- \rightarrow x) \in F$, and so $x \in F$. By the similar way, $y \in F$. Hence, $(x \odot y) \in F$ and so $0 \in F$, which is a contradiction. Therefore, $x^- \in F$ or $y^- \in F$ and so F is an integral filter. ■

The following example shows that the converse of Theorem 4.18, is not correct in general.

Example 4.19. Let $F = \{1, a\}$ in Example 3.2. Since L is an integral BL -algebra, then F is an integral filter, but it is not an obstinate filter. Because $b \notin F$ and $0 = b^- \notin F$. Also, since $((b \rightarrow 0) \rightarrow 0) \rightarrow b = b \notin F$, then by Theorem 2.5, F is not a fantastic filter.

Theorem 4.20. *Let F be an integral and fantastic filter of L . Then F is an obstinate filter of L .*

Proof. Let $x, y \notin F$, for $x, y \in L$. We will show that $x \rightarrow y \in F$ and $y \rightarrow x \in F$. Since $x \notin F$, then $x^- \in F$. Since, if $x^- \notin F$, by (BL11), $(x \odot x^-)^- = 1 \in F$ and since F is an integral filter, then $(x^-)^- \in F$. Now, since F is a fantastic filter, then by Theorem 2.5(i), $((x^-)^- \rightarrow x) \in F$ and so $x \in F$ which is a contradiction. Now, by (BL14), $x^- \leq x \rightarrow y$ and since F is a filter and $x^- \in F$, we get that $x \rightarrow y \in F$. By the similar way we can prove that, $y \rightarrow x \in F$. Therefore, F is an obstinate filter. ■

The following example show that there is a fantastic filter that, is not an obstinate filter and integral filter.

Example 4.21. Let $F = \{1, a\}$ in Example 3.3. Since L is an MV -algebra, then F is an fantastic filter, which is not an obstinate filter. Because $c \notin F$ and $b = c^- \notin F$. Also, since $(b \odot b)^- = 1 \in F$ and $b^- = c \notin F$, then F is not an integral filter.

Now, by the above theorems and Lemma 3.14 [12], we conclude the following theorem:

Theorem 4.22. *Let F be a filter of L . Then the following conditions are equivalent:*

- (i) F is maximal and positive implicative filter,
- (ii) F is maximal and implicative filter,
- (iii) F is an obstinate filter,
- (iv) F is an integral and fantastic filter.

5. Conclusion

The results of this paper will be devoted to study the local BL -algebras, perfect BL -algebras and SBL -algebras which are different extension of Basic Logic. In this paper we introduced the notions of integral BL -algebras and integral filters and we prove these filters are perfect and primary but there is still an open problem: under what suitable conditions a perfect filter is an integral filter? This could be a topic of further research.

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HAAR WAVELET METHOD FOR NUMERICAL SOLUTION OF TELEGRAPH EQUATIONS

Naresh Berwal

*Department of Mathematics
IES, IPS Academy
Indore 452010
India
e-mail: nareshberwal.019@gmail.com*

Dinesh Panchal

*Department of Mathematics DAVV
Indore, 452001
India
e-mail: dkuma_panc@yahoo.co.in*

C.L. Parihar

*Indian Academy of Mathematics
kaushaliya puri
Indore, 452001
India
e-mail: profparihar@hotmail.com*

Abstract. In this paper we modified the result given by Hariharan [18] on the solution of Fisher's equation. We are giving the solution of second-order linear hyperbolic telegraph equation in one - space dimension. The telegraph equation is solved numerically by Haar wavelet method. Two numerical examples show the accuracy of the method. The present method is very simple, small computation costs and flexible.

Keywords: Haar wavelet, second-order hyperbolic telegraph equation, Matlab.

2010 Mathematical Subject Classification: 65T60, 35L20.

1. Introduction

Wavelets have been applied extensively in many engineering field. In this paper we use Haar wavelet method to solve the telegraph equations. The telegraph equations appeared in many engineering field, such as modeling of anomalous diffusive and wave propagation phenomenon, modeling of anomalous diffusion and sub-diffusive systems, Continuous-time random walks.

In this paper we use second-order linear hyperbolic telegraph equation in one-space dimension, given by

$$(1.1) \quad \frac{\delta^2 u}{\delta t^2} + 2\alpha \frac{\delta u}{\delta t} + \beta^2 u = \frac{\delta^2 u}{\delta x^2} + f(x, t), \quad a \leq x \leq b, \quad t \geq 0.$$

subject to the initial conditions

$$\begin{aligned} u(x, 0) &= f(x) \quad a \leq x \leq b, \\ \dot{u}(x, 0) &= f_1(x) \quad a \leq x \leq b \end{aligned}$$

and the Dirichlet boundary conditions

$$U(a, t) = g_0(t), \quad U(b, t) = g_1(t), \quad t \geq 0,$$

where α and β are known constant coefficients, for $\alpha > 0$, $\beta = 0$ equation (1.1) represents a damped wave equation and for $\alpha > \beta > 0$, it is called telegraph equation. We assume that $f(x)$, $f_1(x)$ and their derivatives are continuous functions of x , and $g_0(t)$, $g_1(t)$ and their derivatives are continuous function of t . both the electric voltage and the current in a double conductor satisfy the telegraph equation, where x is distance and t is time.

The hyperbolic partial differential equations model the vibrations of structures (e.g., buildings, beam and machines) and are basis for fundamental equations of atomic physics. Equations of the form equation (1.1) arise in the study of propagation of electrical signals in a cable of transmission line and wave phenomena. Interaction between convection and diffusion or reciprocal action of reaction and diffusion describes a number of nonlinear phenomena in physical, chemical and biological process [8], [9], [15], [18]. In fact the telegraph equation is more suitable than ordinary diffusion equation in modeling reaction diffusion for such branches of sciences. For example biologists encounter these equation in the study of pulsate blood flow in arteries and in one-dimensional random motion of bugs along a hedge [16]. Also the propagation of acoustic waves in Darcy-type porous media [17], and parallel flows of viscous Maxwell fluids [1] are just some of the phenomena governed [3], [10] by equation (1.1).

Haar wavelet are makeup of pairs of piecewise constant functions and mathematically the simplest orthonormal wavelets with a constant support. Due to the mathematical simplicity the Haar wavelet method has turned out to be an effective tool for solving differential and integral equations. Lipik [11]-[14] use Haar wavelet to solve differential and integral equations. Fasal-I Hak, Imram Aziz and Siraj-ul-islam [4] have used Haar wavelet numerical method for eight-order boundary value problem. Hariharan, Kannan and Sharma [5], [6] present a method for solving Fisher's and Fitzhugh - Nagmo equations.

In this paper we modified the result given by Hariharan [5], [6] on the solution of Fisher's equation. This method consists of reducing the problem to a set of algebraic equations by first expanding the terms, which has maximum derivative, given in the equation as Haar functions with unknown coefficients. The operational matrix of integration and product operational matrix are utilized to evaluate the

coefficients of the Haar functions. Since the differentiation of the Haar wavelet results into impulse functions, this approach is avoided and instead, method of integration is preferred. One main advantage of this method is that, we don't need to solve it manually it is fully computer supported.

2. Haar wavelet

Haar functions have been used from 1910 when they were introduced by the Hungarian mathematician Alfred Haar [7]. Haar wavelets are the simplest wavelets among various types of wavelets. They are step functions on the real line that can take only three values 1, -1 and 0. Haar wavelets, like the well-known Walsh functions, form an orthogonal and complete set of functions representing discretized functions and piecewise constant functions.

Haar wavelet is defined for $x \in [0, 1]$

$$(2.1) \quad \psi(x) = \begin{cases} 1 & 0 \leq x < \frac{1}{2} \\ -1 & \frac{1}{2} \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Haar wavelet family for $x \in [0, 1]$ is defined as

$$(2.2) \quad h_i(x) = \begin{cases} 1 & \text{for } x \in [\eta_1, \eta_2) \\ -1 & \text{for } x \in [\eta_2, \eta_3) \\ 0 & \text{otherwise} \end{cases}$$

where $\eta_1 = \frac{K}{m}$, $\eta_2 = \frac{K+0.5}{m}$, $\eta_3 = \frac{K+1}{m}$. The integer $m = 2^j$ ($j = 0, 1, \dots, J$) indicates the level of the wavelet; $k = 0, 1, \dots, m-1$ is the translation parameter. The maximal level of relation is J . The index i is calculated according to the formula $i = m + k + 1$; In the case of minimal values $m = 1, k = 0$, we have $i = 2$. The maximum value of i is $i = 2^{J+1} = M$. It is assumed that the value $i = 1$ corresponding to the scaling function for which $h_1 = 1$ for $x \in [0, 1]$. The interval $[A, B]$ will be divided into M subintervals, hence $\Delta x = \frac{B-A}{M}$ and the matrices are in the dimension of $M \times M$.

We introduce the following notations

$$(2.3) \quad p_{i,1}(x) = \int_0^x h_i(x) dx,$$

$$(2.4) \quad p_{i,v}(x) = \int_0^x p_{i,v-1}(x) dx, \quad v = 2, 3, \dots$$

These integrals can be evaluated by using equation (2.2) and the first two of them are given by

$$(2.5) \quad p_{i,1}(x) = \begin{cases} x - \eta_1 & \text{for } x \in [\eta_1, \eta_2) \\ \eta_3 - x & \text{for } x \in [\eta_2, \eta_3] \\ 0 & \text{elsewhere} \end{cases}$$

$$(2.6) \quad p_{i,2}(x) = \begin{cases} \frac{1}{2}(x - \eta_1)^2 & \text{for } x \in [\eta_1, \eta_2), \\ \frac{1}{4m^2} - \frac{1}{2}(\eta_3 - x)^2 & \text{for } x \in [\eta_2, \eta_3), \\ \frac{1}{4m^2} & \text{for } x \in [\eta_3, 1], \\ 0 & \text{elsewhere.} \end{cases}$$

Similarly, we can find other integrals $p_{i,n}(x)$, $n = 1, 2, \dots$

3. Method for solving telegraph equation

We consider telegraph equation (1.1) with the initial conditions $u(x, 0) = f(x)$, $\dot{u}(x, 0) = f_1(x)$, $0 < x < 1$ and the boundary conditions $u(0, t) = g_0(t)$ and $u(1, t) = g_1(t)$, $t > 0$.

In terms of the Haar wavelet, $\ddot{u}''(x, t)$ can be expanded as

$$(3.1) \quad \ddot{u}''(x, t) = \sum_{i=1}^M a_i h_i(x)$$

where “..” and “..” means differentiation with respect to t and x , respectively, Haar wavelet coefficient is constant in the subinterval $t \in [t_n, t_{n+1}]$.

On twice integration of equation (3.1) with respect to t from t_n to t and with respect to x from 0 to x , the following equations are obtained

$$(3.2) \quad \dot{u}''(x, t) = (t - t_n) \sum_{i=1}^M a_i h_i(x) + \dot{u}''(x, t_n)$$

$$(3.3) \quad u''(x, t) = \frac{1}{2}(t^2 - 2tt_n + t_n^2) \sum_{i=1}^M a_i h_i(x) + (t - t_n) \dot{u}''(x, t_n) + u''(x, t_n)$$

$$(3.4) \quad u'(x, t) = \frac{1}{2}(t^2 - 2tt_n + t_n^2) \sum_{i=1}^M a_i P_{i,1}(x) + (t - t_n) [\dot{u}'(x, t_n) - \dot{u}'(0, t_n)] + u'(x, t_n) - u'(0, t_n) + u'(0, t)$$

$$\begin{aligned}
 (3.5) \quad u(x, t) = & \frac{1}{2} (t^2 - 2 t t_n + t_n^2) \sum_{i=1}^M a_i P_{i,2}(x) \\
 & + (t - t_n) \left[\dot{u}(x, t_n) - \dot{u}(0, t_n) - x \dot{u}'(0, t_n) \right] + u(x, t_n) - u(0, t_n) \\
 & - x \left[u'(0, t_n) - u'(0, t) \right] + u(0, t)
 \end{aligned}$$

$$\begin{aligned}
 (3.6) \quad \dot{u}(x, t) = & (t - t_n) \sum_{i=1}^M a_i P_{i,2}(x) \\
 & + \left[\dot{u}(x, t_n) - \dot{u}(0, t_n) - x \dot{u}'(0, t_n) \right] + x \dot{u}'(0, t) + \dot{u}(0, t)
 \end{aligned}$$

$$(3.7) \quad \ddot{u}(x, t) = \sum_{i=1}^M a_i P_{i,2}(x) + x \ddot{u}'(0, t) + \ddot{u}(0, t)$$

From the initial and boundary conditions, we have the following equations as

$$\begin{aligned}
 u(x, 0) &= f(x), & u(0, t) &= g_0(t), & u(1, t) &= g_1(t), \\
 u(0, t_n) &= g_0(t_n), & u(1, t_n) &= g_1(t_n), & \dot{u}(0, t_n) &= g_0'(t_n), \\
 \dot{u}(1, t_n) &= g_1'(t_n), & \ddot{u}(0, t_n) &= g_0''(t_n), & \ddot{u}(1, t_n) &= g_1''(t_n).
 \end{aligned}$$

At $x = 1$ in the formula (3.5) and (3.7) and by using condition, we have

$$\begin{aligned}
 (3.8) \quad u'(0, t) - u'(0, t_n) = & -\frac{1}{2} (t^2 - 2 t t_n + t_n^2) \sum_{i=1}^M a_i P_{i,2}(1) \\
 & - (t - t_n) [g_1'(t_n) - g_0'(t_n) - \dot{u}'(0, t_n)] \\
 & + g_1(t) - g_1(t_n) + g_0(t_n) - g_0(t)
 \end{aligned}$$

$$(3.9) \quad \ddot{u}'(0, t) = - \sum_{i=1}^M a_i P_{i,2}(1) - g_0''(t) + g_1''(t)$$

If equations (3.8) and (3.9) are substituted into equations (3.3)-(3.5) and the results are discretised by assuming $x \rightarrow x_l$, $t \rightarrow t_{n+1}$, we obtain

$$\begin{aligned}
 (3.10) \quad u''(x_l, t_{n+1}) = & \frac{1}{2} (t_{n+1}^2 - 2 t_{n+1} t_n + t_n^2) \sum_{i=1}^M a_i h_i(x_l) \\
 & + (t_{n+1} - t_n) \ddot{u}''(x_l, t_n) + \ddot{u}''(x_l, t_n)
 \end{aligned}$$

$$\begin{aligned}
u'(x_l, t_{n+1}) &= \frac{1}{2} (t_{n+1}^2 - 2t_{n+1}t_n + t_n^2) \sum_{i=1}^M a_i P_{i,1}(x_l) \\
&\quad + (t_{n+1} - t_n) \dot{u}'(x_l, t_n) \\
(3.11) \quad &\quad + u'(x_l, t_n) - \frac{1}{2} (t_{n+1}^2 - 2t_{n+1}t_n + t_n^2) \sum_{i=1}^M a_i P_{i,2}(1) \\
&\quad - (t_{n+1} - t_n) [g'_1(t_n) - g'_0(t_n)] \\
&\quad + g_1(t_{n+1}) - g_1(t_n) + g_0(t_n) - g_0(t_{n+1})
\end{aligned}$$

$$\begin{aligned}
u(x_l, t_{n+1}) &= \frac{1}{2} (t_{n+1}^2 - 2t_{n+1}t_n + t_n^2) \sum_{i=1}^M a_i P_{i,2}(x_l) \\
&\quad + (t_{n+1} - t_n) [\dot{u}(x_l, t_n) - \dot{u}(0, t_n)] \\
(3.12) \quad &\quad + u(x_l, t_n) - u(0, t_n) \\
&\quad - \frac{x_l}{2} (t_{n+1}^2 - 2t_{n+1}t_n + t_n^2) \sum_{i=1}^M a_i P_{i,2}(1) \\
&\quad - x_l (t_{n+1} - t_n) [g'_1(t_n) - g'_0(t_n)] \\
&\quad - x_l [g_1(t_n) - g_0(t_n) + g_0(t_{n+1}) - g_1(t_{n+1})] + g_0(t_{n+1})
\end{aligned}$$

$$\begin{aligned}
\dot{u}(x_l, t_{n+1}) &= (t_{n+1} - t_n) \sum_{i=1}^M a_i P_{i,2}(x_l) \\
&\quad + [\dot{u}(x_l, t_n) - \dot{u}(0, t_n)] - \\
(3.13) \quad &\quad x_l (t_{n+1} - t_n) \sum_{i=1}^M a_i P_{i,2}(1) \\
&\quad - x_l [g'_1(t_n) - g'_0(t_n)] - x_l [g'_0(t_{n+1}) - g'_1(t_{n+1})] + g'_0(t_{n+1})
\end{aligned}$$

$$\begin{aligned}
\ddot{u}(x_l, t_{n+1}) &= \sum_{i=1}^M a_i [P_{i,2}(x_l) - x_l P_{i,2}(1)] \\
(3.14) \quad &\quad - x_l [g''_0(t_{n+1}) - g''_1(t_{n+1})] + g''_0(t_{n+1})
\end{aligned}$$

From equation (2.6), we obtain

$$(3.15) \quad p_{i,2}(1) = \begin{cases} 0.5 & \text{if } i = 1 \\ \frac{1}{4m^2} & \text{if } i > 1 \end{cases}$$

4. Numerical example

Example1. Consider equation (1.1) with $\alpha = 6, \beta = 2, 0 \leq x \leq 1$ and the following conditions:

$$\begin{aligned} f(x) &= \sin x, \\ f_1(x) &= 0, \\ g_0(x) &= 0, \\ g_1(x) &= \cos(t) \sin(1), \\ f(x, t) &= -2\alpha \sin(t) \sin(x) + \beta^2 \cos(t) \sin(x). \end{aligned}$$

The exact solution of this example is $u(x, t) = \cos(t) \sin(x)$.

After substituting values from equations (3.12)-(3.14) in equation (1.1) and using conditions, we have

$$\begin{aligned} & \sum_{i=1}^M a_i [P_{i,2}(x_l) - x_l P_{i,2}(1) + 12(t_{n+1} - t_n) P_{i,2}(x_l) \\ & - 12x_l(t_{n+1} - t_n) P_{i,2}(1) + 2(t_{n+1}^2 - 2t_{n+1}t_n + t_n^2) P_{i,2}(x_l) \\ (4.1) \quad & - 2x_l(t_{n+1}^2 - 2t_{n+1}t_n + t_n^2) P_{i,2}(1)] \\ & = -\sin x_l - 12 \sin t \sin x_l + 4 \cos t \sin x_l - 12x_l \sin t \sin 1 + 12 \sin t_{n+1} \sin 1 \\ & \quad 4u(x_l, t_n) - 4x_l(t_{n+1} - t_n) \sin t \sin 1 - 4x_l [\cos t \sin 1 - \cos t_{n+1} \sin 1] \\ & \quad + x_l \cos t_{n+1} \sin 1 \end{aligned}$$

Equation (4.1) is the algebraic form of the telegraph equation (1.1). After solving these algebraic equations, we can compute the Haar coefficients a'_i s. Then, from equation (3.12), we obtain the value of u , which is very near to the exact solution. This solution process is started with

$$\begin{aligned} u(x_l, 0) &= f(x_l) \\ u'(x_l, 0) &= f'(x_l) \\ u''(x_l, 0) &= f''(x_l) \end{aligned}$$

In all the results given in the following, J is taken as 3.

Table 1
The absolute error for different values of x & t is

$x / 32$	At $t=0.1$	At $t=0.2$	At $t=0.3$
1	2.66106E-06	2.15080 E-04	4.271600 E-04
3	7.98322E-06	6.44370 E-04	1.279750 E-03
5	1.33055E-05	1.07105 E-03	2.127200 E-03
7	1.86280E-05	1.49340 E-03	2.966080 E-03
9	2.39509E-05	1.90969 E-03	3.793030 E-03
11	2.92741E-05	2.31826 E-03	4.604710 E-03
13	3.45979E-05	2.71743 E-03	5.397840 E-03
15	3.99222E-05	3.10559 E-03	6.169230 E-03
17	4.52472E-05	3.48116 E-03	6.915780 E-03
19	5.05729E-05	3.84261 E-03	7.634440 E-03
21	5.58994E-05	4.18847 E-03	8.322330 E-03
23	6.12269E-05	4.51733 E-03	8.976660 E-03
25	6.65553E-05	4.82783 E-03	9.594750 E-03
27	7.18847E-05	5.11872 E-03	1.017411 E-02
29	7.72152E-05	5.38878 E-03	1.071237 E-02
31	8.25469E-05	5.63691 E-03	1.120731 E-02

The plot of the numerical solution of $u(x, t)$ at different value of t is shown in following figures (1)-(3):

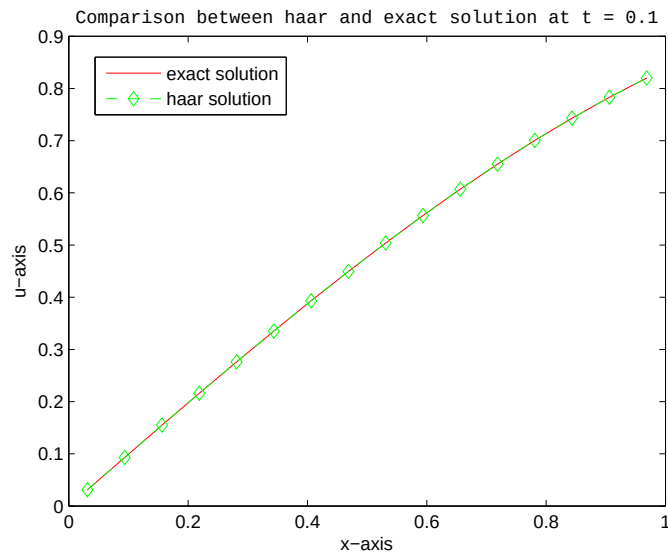


Figure-(1)

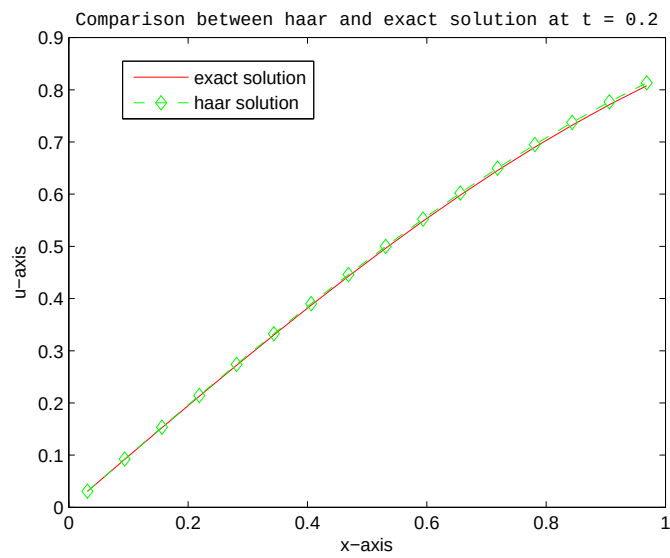


Figure-(2)

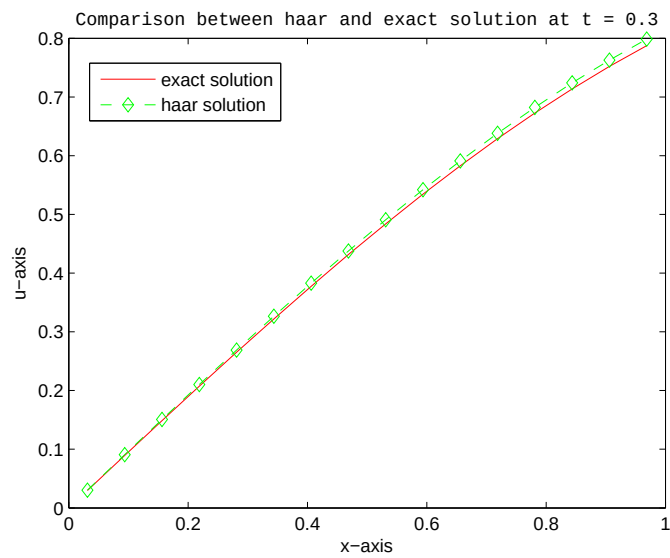


Figure-(3)

Example 2. Consider equation (1.1) with $\alpha = 1$, $\beta = 1$, $0 \leq x \leq 1$ and the following conditions

$$\begin{aligned} f(x) &= x^2, \\ f_1(x) &= 1, \\ g_0(x) &= t, \\ g_1(x) &= 1 + t, \\ f(x, t) &= x^2 + t - 1. \end{aligned}$$

The exact solution of this example is $u(x, t) = x^2 + t$.

Table 2
The absolute error for different values of x & t is

$x / 32$	At $t=0.1$	At $t=0.2$
1	4.589840 E-04	9.135300 E-04
3	4.944960 E-04	9.490410 E-04
5	5.655180 E-04	1.020064 E-03
7	6.720530 E-04	1.126598 E-03
9	8.140980 E-04	1.268643 E-03
11	9.916550 E-04	1.446200 E-03
13	1.204723 E-03	1.659268 E-03
15	1.453303 E-03	1.907848 E-03
17	1.737393 E-03	2.191939 E-03
19	2.056996 E-03	2.511541 E-03
21	2.412109 E-03	2.866655 E-03
23	2.802734 E-03	3.257280 E-03
25	3.228871 E-03	3.683416 E-03
27	3.690518 E-03	4.145064 E-03
29	4.187678 E-03	4.642223 E-03
31	4.589840 E-04	9.135300 E-04

The plot of the numerical solution of $u(x, t)$ at different value of t is shown in following figures (4) & (5) :

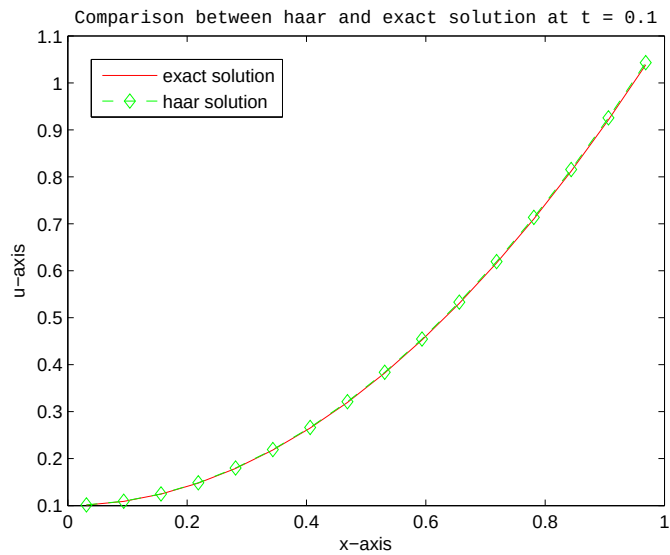


Figure-(4)

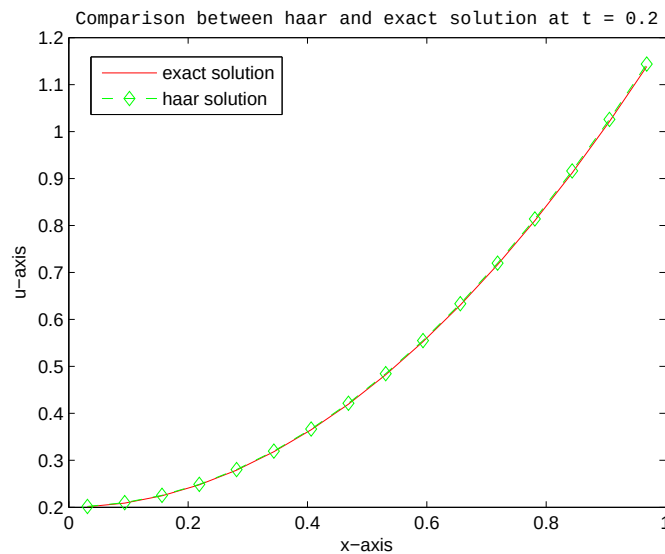


Figure-(5)

Conclusion

In this paper, Haar wavelet method is proposed for the numerical solution for the second-order hyperbolic telegraph equation. Approximate solution of the telegraph equation, obtain by Matlab, are compared with exact solution. This calculation demonstrates the accuracy of the Haar wavelet solution. The main advantage of this method is its simplicity and small computation costs, which is due to the sparsity of the transform matrices and to the small number of significant wavelet coefficient. It is worth mentioning that Haar solution provides excellent results even for small values of M ($M = 16$). For larger values of M , we can obtain the results closer to the real values.

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CAUCHY'S METHOD AND BILATERAL BASIC HYPERGEOMETRIC SERIES

Roselin Antony¹

Hailemariam Fiseha

*Department of Mathematics
College of Natural and Computational Sciences
P.O. Box 231
Mekelle University
Mekelle
Ethiopia*

Abstract. In this paper, we find bilateral basic hypergeometric series adapting Cauchy's method used by Bailey, Slater, Fredric Jouhet and Michael Schlosser.

Key words: bilateral basic hypergeometric series, q -series.

1. Introduction

Schlosser [10] gave a new proof of Ramanujans ${}_1\psi_1$ summation formula [6].

$$(1.1) \quad {}_1\psi_1 \left[\begin{matrix} a; q; z \\ b \end{matrix} \right] = \frac{(q, b/a, az, q/az)_\infty}{(b, q/a, z, b/az)_\infty}$$

valid for $|q| < 1$, $|b/a| < |z| < 1$. This proof was given by a famous method already utilized by Cauchy [5] in the second proof of Jacobi's [8] triple product identity. The same method known as Cauchy's method had also been used by Bailey [3] and Slater [11]. Cauchy's method is used to obtain any bilateral sum from an approximately chosen terminating identity.

Jouhet and Schlosser [9] used Cauchy's method in Jackson's [7] q -Pfaff-Saalschutz summation

$$(1.2) \quad {}_3\phi_2 \left[\begin{matrix} a, b, q^{-n}; q; q \\ c, abq^{1-n/c} \end{matrix} \right] = \frac{(c/a, c/b; q)_n}{(c, c/ab; q)_n}$$

¹Corresponding author. E-mail: roselinmaths@gmail.com

to prove Ramanujan's ${}_1\psi_1$ summation. Also, by considering Bailey's [2] transformation formula, Jouhet and Schlosser proved Bailey's ${}_6\psi_6$ summation formula.

It is necessary to give some standard notations for q -series and basic hypergeometric series. Let q be a complex number, i.e., base with $0 < |q| < 1$. The q -shifted factorial is defined for any complex parameter a by

$$(a)_\infty = (a; q)_\infty = \prod_{j \geq 0} (1 - aq^j)$$

and

$$(a)_k = (a; q)_k = \frac{(a; q)_\infty}{(aq^k; q)_\infty},$$

where k is any integer.

Also,

$$(a_1, a_2, \dots, a_m)_k = (a_1)_k, \dots, (a_m)_k,$$

where k is an integer or infinity.

The basic hypergeometric series is given by

$$(1.3) \quad {}_s\phi_{s-1} \left[\begin{matrix} a_1, \dots, a_s; q; z \\ b_1, \dots, b_{s-1} \end{matrix} \right] = \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_s; q)_k}{(q, b_1, \dots, b_{s-1}; q)_k} z^k$$

and of bilateral basic hypergeometric series,

$$(1.4) \quad {}_s\psi_s \left[\begin{matrix} a_1, \dots, a_s; q; z \\ b_1, \dots, b_s \end{matrix} \right] = \sum_{k=-\infty}^{\infty} \frac{(a_1, \dots, a_s; q)_k}{(b_1, \dots, b_s; q)_k} z^k$$

Also this paper depends on some elementary identities for q -shifted factorials listed by Gasper and Rahman [6]. In this paper, we use Cauchy's method to bilateralize well-known terminating identities of Carlitz [4], Verma and Jain [12] and Agarwal [1].

2. Main results

First, let us consider Carlitz [4] transformation formula, to bilateralize it by Cauchy's method;

$$(2.1) \quad {}_4\phi_3 \left[\begin{matrix} q^{-n}, c, d, \frac{q^{\frac{1}{2}-n}}{cd}; q; q^2 \\ \frac{1}{c} q^{1-n}, \frac{1}{d} q^{1-n}, cd\sqrt{q} \end{matrix} \right] = \frac{(cd; q)_n (c, d, -\sqrt{q}; \sqrt{q})_n}{(cd; \sqrt{q})_n (c, d; q)_n}.$$

First replace n by $2n$ and then shift the index of summation by n such that the new sum runs from $-n$ to n ,

$$\begin{aligned}
& \sum_{k=0}^{2n} \frac{\left(q^{-2n}, c, d, \frac{1}{cd} q^{\frac{1}{2}-2n}; q\right)_k q^2}{\left(q, \frac{1}{c} q^{1-2n}, \frac{1}{d} q^{1-2n}, cd\sqrt{q}; q\right)_k} = \frac{(cd; q)_{2n}(c, d, -\sqrt{q}; \sqrt{q})_{2n}}{(cd; \sqrt{q})_{2n}(c, d; q)_{2n}} \\
& \Rightarrow \sum_{k=-n}^n \frac{\left(q^{-2n}, c, d, \frac{1}{cd} q^{\frac{1}{2}-2n}; q\right)_k q^{2(n+k)}}{\left(q, \frac{1}{c} q^{1-2n}, \frac{1}{d} q^{1-2n}, cd\sqrt{q}; q\right)_k} = \frac{(cd; q)_{2n}(c, d, -\sqrt{q}; \sqrt{q})_{2n}}{(cd; \sqrt{q})_{2n}(c, d; q)_{2n}} \\
& \Rightarrow \frac{\left(q^{-2n}, c, d, \frac{1}{cd} q^{\frac{1}{2}-2n}; q\right)_n q^{2n}}{\left(q, \frac{1}{c} q^{1-2n}, \frac{1}{d} q^{1-2n}, cd\sqrt{q}; q\right)_k} \sum_{k=-n}^n \frac{\left(q^{-n}, cq^n, dq^n, \frac{1}{cd} q^{\frac{1}{2}-n}; q\right)_k q^{2k}}{\left(q^{1+n}, \frac{1}{c} q^{1-n}, \frac{1}{d} q^{1-n}, cdq^{\frac{1}{2}+n}; q\right)_k} \\
& = \frac{(cd; q)_{2n}(c, d, -\sqrt{q}; \sqrt{q})_{2n}}{(cd; \sqrt{q})_{2n}(c, d; q)_{2n}}.
\end{aligned}$$

Take $c = cq^{-n}$ and $d = dq^{-n}$, we get

$$\begin{aligned}
& \sum_{k=-n}^n \frac{\left(q^{-n}, c, d, \frac{1}{cd} q^{\frac{1}{2}+n}; q\right)_k q^{2k}}{\left(q^{1+n}, \frac{q}{c}, \frac{q}{d}, cdq^{\frac{1}{2}-n}; q\right)_k} \\
& = \frac{(cdq^{-2n}; q)_{2n}(cq^{-n}, dq^{-n}, -\sqrt{q}; \sqrt{q})_{2n} \left(q, \frac{1}{c} q^{1-n}, \frac{1}{d} q^{1-n}, cdq^{\frac{1}{2}-2n}; q\right)_n q^{-2n}}{(cdq^{-2n}; \sqrt{q})_{2n}(cq^{-n}, dq^{-n}; q)_{2n} \left(q^{-2n}, cq^{-n}, dq^{-n}, \frac{\sqrt{q}}{cd}; q\right)_n} \\
& \sum_{k=-n}^n \frac{\left(q^{-n}, c, d, \frac{1}{cd} q^{\frac{1}{2}+n}; q\right)_k q^{2k}}{\left(q^{1+n}, \frac{q}{c}, \frac{q}{d}, cdq^{\frac{1}{2}-n}; q\right)_k} \\
& = \frac{(-\sqrt{q}, -q, q, q; q)_n (q/cd; q)_n (\sqrt{q}/c, \sqrt{q}/d; q)_n}{(q/c, q/d, \sqrt{q}/cd; q)_n (q; q)_{2n}}.
\end{aligned}$$

Now, let $n \rightarrow \infty$, and assuming $|q^{3/2}/cd| < 1$, while appealing to Tannery's theorem [1], for interchanging limit and summation, we get

$$\begin{aligned}
(2.2) \quad & \sum_{k=-\infty}^{\infty} \frac{(c, d; q)_k}{\left(\frac{q}{c}, \frac{q}{d}; q\right)_k} \left(\frac{q^{3/2}}{cd}\right)^k = \frac{(-\sqrt{q}, -q, q, q, q/cd; q)_{\infty} (\sqrt{q}/c, \sqrt{q}/d; q)_{\infty}}{(q/c, q/d, \sqrt{q}/cd; q)_{\infty} (q; q)_{\infty}} \\
& \Rightarrow {}_2\psi_2 \left[\begin{matrix} c, d; q; q^{3/2}/cd \\ q/c, q/d \end{matrix} \right] = \frac{(-\sqrt{q}, -q, q, q/cd, \sqrt{q}/c, \sqrt{q}/d; q)_{\infty}}{(q/c, q/d, \sqrt{q}/cd; q)_{\infty}}.
\end{aligned}$$

Next, consider Verma and Jain's summation [12]

$$\begin{aligned}
(2.3) \quad & {}_4\phi_3 \left[\begin{matrix} q^{-n}, c, d, \frac{1}{cd} q^{\frac{3}{2}-n}; q; q \\ \frac{1}{c} q^{1-n}, \frac{1}{d} q^{1-n}, cd\sqrt{q} \end{matrix} \right] \\
& = \frac{(cd/\sqrt{q}; \sqrt{q})_{2n}(c, d; \sqrt{q})_n (q; q)_n}{(cd/\sqrt{q}; \sqrt{q})_n (cd\sqrt{q}; q)_n (c, d; q)_n (\sqrt{q}; \sqrt{q})_n}.
\end{aligned}$$

First replace n by $2n$ and then shift the index of summation by n , such that the new sum runs from $-n$ to n ,

$$\begin{aligned}
& \frac{(cd/\sqrt{q}, cd; q)_{2n}(c, d; \sqrt{q})_{2n}(q; q)_{2n}}{(cd/\sqrt{q}; \sqrt{q})_{2n}(cd\sqrt{q}; q)_{2n}(c, d; q)_{2n}(\sqrt{q}; \sqrt{q})_{2n}} \\
&= \sum_{k=0}^{2n} \frac{(q^{-2n}, c, d, \frac{1}{cd} q^{\frac{3}{2}-2n}; q)_k \cdot q^k}{(q, \frac{1}{c} q^{1-2n}, \frac{1}{d} q^{1-2n}, cd\sqrt{q}; q)_k} \\
&\Rightarrow \sum_{k=-n}^n \frac{(q^{-2n}, c, d, \frac{1}{cd} q^{\frac{3}{2}-2n}; q)_k \cdot q^{n+k}}{(q, \frac{1}{c} q^{1-2n}, \frac{1}{d} q^{1-2n}, cd\sqrt{q}; q)_{n+k}} \\
&= \frac{(q^{-2n}, c, d, \frac{1}{cd} q^{\frac{3}{2}-2n}; q)_n \cdot q^n}{(q, \frac{1}{c} q^{1-2n}, \frac{1}{d} q^{1-2n}, cd\sqrt{q}; q)_n} \sum_{k=-n}^n \frac{(q^{-n}, cq^n, dq^n, \frac{1}{cd} q^{\frac{3}{2}-n}; q)_k \cdot q^k}{(q^{1+n}, \frac{1}{c} q^{1-n}, \frac{1}{d} q^{1-n}, cd\sqrt{q}q^n; q)_k}
\end{aligned}$$

Take $c = cq^{-n}$ and $d = dq^{-n}$, we get

$$\begin{aligned}
& \frac{(cdq^{-2n-\frac{1}{2}}, cdq^{-2n}; q)_{2n}(cq^{-n}, dq^{-n}; \sqrt{q})_{2n}(q; q)_{2n}}{(cdq^{-2n-\frac{1}{2}}; \sqrt{q})_{2n}(cdq^{-2n+\frac{1}{2}}; q)_{2n}(cq^{-n}, dq^{-n}; q)_{2n}(\sqrt{q}; \sqrt{q})_{2n}} \\
&= \frac{(q^{-2n}, cq^{-n}, dq^{-n}, q^{3/2}/cd; q)_n \cdot q^n}{(q, \frac{1}{c} q^{1-n}, \frac{1}{d} q^{1-n}, cdq^{-2n+\frac{1}{2}}; q)_n} \sum_{k=-n}^n \frac{(q^{-n}, c, d, \frac{1}{cd} q^{\frac{3}{2}+n}; q)_k \cdot q^k}{(q^{1+n}, \frac{q}{c}, \frac{q}{d}, cdq^{-n+\frac{1}{2}}; q)_k} \\
&\Rightarrow \sum_{k=-n}^n \frac{(q^{-n}, c, d, \frac{1}{cd} q^{\frac{3}{2}+n}; q)_k \cdot q^k}{(q^{1+n}, \frac{q}{c}, \frac{q}{d}, cdq^{-n+\frac{1}{2}}; q)_k} \\
&= \frac{(cdq^{-2n-\frac{1}{2}}, cdq^{-n-\frac{1}{2}}, cdq^{-2n}, cdq^{-n}; q)_n}{(cdq^{-2n-\frac{1}{2}}, cdq^{-2n}; q)_n (cdq^{-2n+\frac{1}{2}}, cdq^{-n+\frac{1}{2}}; q)_n} \\
&\quad \times \frac{(cq^{-n}, dq^{-n}, cq^{-n+\frac{1}{2}}, dq^{-n+\frac{1}{2}}; q)_n (q; q)_{2n}(q; q)_n}{(cq^{-n}, dq^{-n}, c, d; q)_n (\sqrt{q}, q; q)_n (q^{3/2}/cd; q)_n} \\
&\quad \times \frac{(\frac{1}{c} q^{1-n}, \frac{1}{d} q^{1-n}, cdq^{-2n+\frac{1}{2}}; q)_n \cdot q^{-n}}{(q^{-2n}, cq^{-n}, dq^{-n}; q)_n} \\
&\Rightarrow \sum_{k=-n}^n \frac{(q^{-n}, c, d, \frac{1}{cd} q^{\frac{3}{2}+n}; q)_k \cdot q^k}{(q^{1+n}, \frac{q}{c}, \frac{q}{d}, cdq^{-n+\frac{1}{2}}; q)_k} = \frac{(q/cd, \sqrt{q}/c, \sqrt{q}/d, q; q)_n}{(\sqrt{q}/cd, \sqrt{q}, q/c, q/d; q)_n (q; q)_{2n}}.
\end{aligned}$$

Let $n \rightarrow \infty$, and assuming $|\sqrt{q}/cd| < 1$, while appealing to Tannery's theorem [1], for interchanging limit and summation, we get

$$\begin{aligned}
(2.4) \quad & \sum_{k=-\infty}^{\infty} \frac{(c, d; q)_k}{\left(\frac{q}{c}, \frac{q}{d}; q\right)_k} \left(\frac{\sqrt{q}}{cd}\right)^k = \frac{(q/cd, \sqrt{q}/c, \sqrt{q}/d; q)_{\infty}}{(\sqrt{q}/cd, \sqrt{q}, q/c, q/d; q)_{\infty}} \\
& \implies {}_2\psi_2 \left[\begin{matrix} c, d; q, \sqrt{q}/cd \\ q/c, q/d \end{matrix} \right] = \frac{(q/cd, \sqrt{q}/c, \sqrt{q}/d; q)_{\infty}}{(\sqrt{q}/cd, \sqrt{q}, q/c, q/d; q)_{\infty}}.
\end{aligned}$$

Again consider another summation of Verma and Jain [12];

$$\begin{aligned}
(2.5) \quad & {}_4\phi_3 \left[\begin{matrix} q^{-n}, c, d, q^{\frac{1}{2}-n}/cd; q; q \\ q^{1-n}/c, q^{1-n}/d, cd\sqrt{q} \end{matrix} \right] \\
& = \frac{(cd; q)_n (c, d; \sqrt{q})_n (q; q)_n}{(c, d; q)_n (\sqrt{q}; \sqrt{q})_n (cd/\sqrt{q}; \sqrt{q})_n} \cdot q^{\frac{-n}{2}}.
\end{aligned}$$

First replace n by $2n$ and then shift the index of summation by n , such that the new sum runs from $-n$ to n ,

$$\begin{aligned}
& \frac{(cd; q)_{2n} (c, d; \sqrt{q})_{2n} (q; q)_{2n} \cdot q^{-n}}{(c, d; q)_{2n} (\sqrt{q}; \sqrt{q})_{2n} (cd/\sqrt{q}; \sqrt{q})_{2n}} = \sum_{k=0}^{2n} \frac{(q^{-2n}, c, d, q^{\frac{1}{2}-2n}/cd; q)_k \cdot q^k}{(q, q^{1-2n}/c, q^{1-2n}/d, cd\sqrt{q}; q)_k} \\
& \implies \sum_{k=-n}^n \frac{(q^{-2n}, c, d, q^{\frac{1}{2}-2n}/cd; q)_k \cdot q^{n+k}}{(q, q^{1-2n}/c, q^{1-2n}/d, cd\sqrt{q}; q)_{n+k}} \\
& = \frac{(q^{-2n}, c, d, q^{\frac{1}{2}-2n}/cd; q)_n \cdot q^n}{(q, q^{1-2n}/c, q^{1-2n}/d, cd\sqrt{q}; q)_n} \sum_{k=-n}^n \frac{(q^{-n}, cq^n, dq^n, q^{\frac{1}{2}-n}/cd; q)_k \cdot q^k}{(q^{1+n}, q^{1-n}/c, q^{1-n}/d, cdq^{\frac{1}{2}+n}; q)_k}.
\end{aligned}$$

Take $c = cq^{-n}$ and $d = dq^{-n}$, we get

$$\begin{aligned}
& \sum_{k=-n}^n \frac{(q^{-n}, c, d, \frac{1}{cd} q^{\frac{1}{2}+n}; q)_k \cdot q^k}{(q^{1+n}, \frac{q}{c}, \frac{q}{d}, cdq^{\frac{1}{2}-n}; q)_k} \\
& = \frac{(cdq^{-2n}; q)_{2n} (cq^{-n}, dq^{-n}; \sqrt{q})_{2n} (q; q)_{2n}}{(cq^{-n}, dq^{-n}; q)_{2n} (\sqrt{q}; \sqrt{q})_{2n} (cdq^{-2n-\frac{1}{2}}; \sqrt{q})_{2n}} \cdot q^{-n} \\
& \quad \times \frac{(q, q^{1-n}/c, q^{1-n}/d, cdq^{\frac{1}{2}-2n}; q)_n \cdot q^{-n}}{(q^{-2n}, cq^{-n}, dq^{-n}, \sqrt{q}/cd; q)_n} \\
& = \frac{(cdq^{-2n}, cdq^{-n}; q)_n (cq^{-n}, dq^{-n}, cq^{-n+\frac{1}{2}}, dq^{-n+\frac{1}{2}}; q)_n}{(cq^{-n}, dq^{-n}, c, d; q)_n (\sqrt{q}, q; q)_n (cdq^{-2n-\frac{1}{2}}, cdq^{-2n}; q)_n} \\
& \quad \times \frac{(q; q)_{2n} (q; q)_n (q^{1-n}/c, q^{1-n}/d, cdq^{\frac{1}{2}-2n}; q)_n \cdot q^{-2n}}{(q^{-2n}, cq^{-n}, dq^{-n}, \sqrt{q}/cd; q)_n} \\
& \implies \sum_{k=-n}^n \frac{(q^{-n}, c, d, \frac{1}{cd} q^{\frac{1}{2}+n}; q)_k \cdot q^k}{(q^{1+n}, \frac{q}{c}, \frac{q}{d}, cdq^{\frac{1}{2}-n}; q)_k} = \frac{(q/cd, \sqrt{q}/c, \sqrt{q}/d, q; q)_n}{(\sqrt{q}, \sqrt{q}/cd, q/c, q/d; q)_n}.
\end{aligned}$$

Let $n \rightarrow \infty$, and assuming $|\sqrt{q}/cd| < 1$, while appealing to Tannery's theorem [1], for interchanging limit and summation, we get

$$(2.6) \quad \sum_{k=-\infty}^{\infty} \frac{(c, d; q)_{\infty}}{\left(\frac{q}{c}, \frac{q}{d}; q\right)_{\infty}} = \frac{(q/cd, \sqrt{q}/c, \sqrt{q}/d; q; q)_{\infty}}{(\sqrt{q}, \sqrt{q}/cd, q/c, q/d; q)_{\infty}}$$

$${}_2\psi_2 \left[\begin{matrix} c, d; q; \sqrt{q}/cd \\ q/c, q/d \end{matrix} \right] = \frac{(q/cd, \sqrt{q}/c, \sqrt{q}/d, q; q)_{\infty}}{(\sqrt{q}, \sqrt{q}/cd, q/c, q/d; q)_{\infty}}.$$

Consider Agarwal's summation [1];

$$(2.7) \quad {}_2\phi_1 \left[\begin{matrix} a, b; q, q \\ abq \end{matrix} \right] = \frac{(aq, bq; q)_n}{(q, abq; q)_n}.$$

First replace n by $2n$ and then shift the index of summation by n , such that the new sum runs from $-n$ to n ,

$$\begin{aligned} \sum_{k=0}^{2n} \frac{(a, b; q)_k}{(q, abq; q)_k} \cdot q^k &= \frac{(aq, bq; q)_{2n}}{(q, abq; q)_{2n}} \\ \Rightarrow \sum_{k=-n}^n \frac{(a, b; q)_{n+k}}{(q, abq; q)_{n+k}} \cdot q^{n+k} &= \frac{(aq, bq; q)_{2n}}{(q, abq; q)_{2n}} \\ \Rightarrow \frac{(q, b; q)_n}{(q, abq; q)_n} \cdot q^n \cdot \sum_{k=-n}^n \frac{(aq^n, bq^n; q)_k}{(q^{1+n}, abq^{1+n}; q)_k} \cdot q^k &= \frac{(aq, bq; q)_{2n}}{(q, abq; q)_{2n}} \\ \Rightarrow \sum_{k=-n}^n \frac{(aq^n, bq^n; q)_k}{(q^{1+n}, abq^{1+n}; q)_k} \cdot q^k &= \frac{(aq, bq; q)_{2n}(q, abq; q)_n}{(q, abq; q)_{2n}(a, b; q)_n} \cdot q^{-n}. \end{aligned}$$

Take $a = aq^{-n}$, we get

$$\begin{aligned} \sum_{k=-n}^n \frac{(a, bq^n; q)_k}{(q^{1+n}, abq; q)_k} \cdot q^k &= \frac{(aq^{1-n}, bq; q)_{2n}(q, abq^{1-n}; q)_n}{(q, abq^{1-n}; q)_{2n}(aq^{-n}, b; q)_n} \cdot q^{-n} \\ &= \frac{(bq; q)_{2n}(q, aq, 1/a; q)_n}{(q; q)_{2n}(b, abq, q/a; q)_n}. \end{aligned}$$

Let $n \rightarrow \infty$, while appealing to Tannery's theorem [1], for interchanging limit and summation, we get

$$(2.8) \quad \sum_{k=-\infty}^{\infty} \frac{(a; q)_k}{(abq; q)_k} \cdot q^k = \frac{(bq, aq, q/a; q)_{\infty}}{(b, abq, q/a, q)_{\infty}},$$

$${}_1\psi_1 \left[\begin{matrix} a; q; q \\ abq \end{matrix} \right] = \frac{(bq, aq, 1/a; q)_{\infty}}{(b, abq, q/a; q)_{\infty}}.$$

3. Conclusion

We have established bilateral basic hypergeometric series adapting Cauchy's method used by Bailey, Slater, Fredric Jouhet and Michael Schlosser. It could also be very interesting to find bilateral series for any other terminating basic hypergeometric functions by Cauchy's method.

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SOME CLASSES OF p -VALENT MEROMORPHIC FUNCTIONS DEFINED BY A NEW OPERATOR

M.K. Aouf

A.O. Mostafa

A. Shamandy

E.A. Adwan

Department of Mathematics

Faculty of Science

Mansoura University

Mansoura 35516

Egypt

e-mails: mkaouf127@yahoo.com

adelaeg254@yahoo.com

shamandy16@hotmail.com

eman.a2009@yahoo.com

Abstract. In this paper, we introduce some classes of p -valent meromorphic functions associated with a new operator and to investigate various properties for these subclasses.

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1. Introduction

Let Σ_p denote the class of functions of the form:

$$(1.1) \quad f(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p} z^{n-p} \quad (p \in \mathbb{N} = \{1, 2, \dots\}),$$

which are analytic and p -valent in the punctured unit disc $U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = U \setminus \{0\}$. Let $P_k(\gamma, p)$ ($k \geq 2, 0 \leq \gamma < p, p \in \mathbb{N}$) denote the class of functions

$$(1.2) \quad g(z) = p + \sum_{k=1}^{\infty} c_k z^k$$

which are analytic in U and satisfy for every $r < 1$ ($z = re^{i\theta} \in U$) the conditions

$$(1.3) \quad \begin{aligned} (1) \quad & g(0) = p, \\ (2) \quad & \int_0^{2\pi} \frac{|\operatorname{Re}\{g(z)\} - \gamma|}{(p - \gamma)} d\theta \leq k\pi. \end{aligned}$$

The class $P_k(\gamma, p)$ was introduced and studied by Aouf [1].

We note that:

- (1) $P_k(\gamma, 1) = P_k(\gamma)$ ($k \geq 2, 0 \leq \gamma < 1$) (see Padmanabhan and Parvatham [8]);
- (2) $P_k(0, 1) = P_k$ ($k \geq 2$) (see Pinchuk [9] and Robertson [10]);
- (3) $P_2(\gamma, p) = P(\gamma, p)$ ($0 \leq \gamma < p, p \in \mathbb{N}$), where $P(\gamma, p)$ is the class of functions g of the form (1.2) and satisfy the conditions $g(0)=p$ and $\operatorname{Re}\{g(z)\} > \gamma$ ($0 \leq \gamma < p$) in U ;
- (4) $P_2(0, 1) = P$, where P is the class of functions with positive real part in U ;
- (5) $P_2(\gamma, 1) = P(\gamma)$ ($0 \leq \gamma < 1$), where $h(z) = (1 - \gamma)p(z) + \gamma$, $h(z) \in P(\gamma)$ and $p(z) \in P$.

From (1.2), we have $g(z) \in P_k(\gamma, p)$ if and only if there exists $g_i \in P(\gamma, p)$, $i = 1, 2$ such that (see [1])

$$(1.4) \quad g(z) = \left(\frac{k}{4} + \frac{1}{2}\right) g_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) g_2(z) \quad (z \in U).$$

For analytic functions $f(z) \in \Sigma_p$, given by (1.1) and $\phi(z) \in \Sigma_p$ given by $\phi(z) = z^{-p} + \sum_{n=1}^{\infty} b_{n-p} z^{n-p}$ ($p \in \mathbb{N}$), the Hadamard product (or convolution) of $f(z)$ and $\phi(z)$, is defined by

$$(1.5) \quad (f * \phi)(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p} b_{n-p} z^{n-p} = (\phi * f)(z).$$

Aqlan et al. [4] defined the operator $Q_{\beta, p}^{\alpha} : \Sigma_p \rightarrow \Sigma_p$ by:

$$(1.6) \quad Q_{\beta, p}^{\alpha} f(z) = \begin{cases} z^{-p} + \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} \sum_{n=1}^{\infty} \frac{\Gamma(n+\beta)}{\Gamma(n+\beta+\alpha)} a_{n-p} z^{n-p} & \left(\begin{array}{l} \alpha > 0; \quad \beta > -1; \\ p \in \mathbb{N}; \quad f \in \Sigma_p \end{array} \right) \\ f(z) & (\alpha=0, \beta > -1; p \in \mathbb{N}; f \in \Sigma_p). \end{cases}$$

Mostafa [7] used Aqlan et al. operator and defined the following linear operator $H_{p, \beta, \mu}^{\alpha} : \Sigma_p \rightarrow \Sigma_p$ as follows:

First put

$$(1.7) \quad G_{\beta, p}^{\alpha}(z) = z^{-p} + \frac{\Gamma(\alpha + \beta)}{\Gamma(\beta)} \sum_{n=1}^{\infty} \frac{\Gamma(n + \beta)}{\Gamma(n + \beta + \alpha)} z^{n-p} \quad (p \in \mathbb{N})$$

and let $G_{\beta,p,\mu}^{\alpha*}$ be defined by

$$(1.8) \quad G_{\beta,p}^{\alpha}(z) * G_{\beta,p,\mu}^{\alpha*}(z) = \frac{1}{z^p(1-z)^{\mu}} \quad (\mu > 0; p \in \mathbb{N}).$$

Then

$$(1.9) \quad H_{p,\beta,\mu}^{\alpha}f(z) = G_{\beta,p}^{\alpha*}(z) * f(z) \quad (f \in \Sigma_p).$$

Using (1.7) and (1.9), we have

$$(1.10) \quad H_{p,\beta,\mu}^{\alpha}f(z) = z^{-p} + \frac{\Gamma(\beta)}{\Gamma(\alpha + \beta)} \sum_{n=1}^{\infty} \frac{\Gamma(n + \beta + \alpha)(\mu)_n}{\Gamma(n + \beta)(1)_n} a_{n-p} z^{n-p},$$

where $(\nu)_n$ denotes the Pochhammer symbol given by

$$(\nu)_n = \frac{\Gamma(\nu + n)}{\Gamma(\nu)} = \begin{cases} 1 & (n = 0) \\ \nu(\nu + 1) \dots (\nu + n - 1) & (n \in \mathbb{N}). \end{cases}$$

It is readily verified from (1.10) that (see [7])

$$(1.11) \quad z(H_{p,\beta,\mu}^{\alpha}f(z))' = (\alpha + \beta)H_{p,\beta,\mu}^{\alpha+1}f(z) - (\alpha + \beta + p)H_{p,\beta,\mu}^{\alpha}f(z)$$

and

$$(1.12) \quad z(H_{p,\beta,\mu}^{\alpha}f(z))' = \mu H_{p,\beta,\mu+1}^{\alpha}f(z) - (\mu + p)H_{p,\beta,\mu}^{\alpha}f(z).$$

It is noticed that, putting $\mu = 1$ in (1.10), we obtain the operator

$$(1.13) \quad H_{p,\beta,1}^{\alpha}f(z) = H_{p,\beta}^{\alpha}f(z) = z^{-p} + \frac{\Gamma(\beta)}{\Gamma(\alpha + \beta)} \sum_{n=1}^{\infty} \frac{\Gamma(n + \alpha + \beta)}{\Gamma(n + \beta)} a_{n-p} z^{n-p}.$$

Now we define the following subclasses of the class Σ_p for $0 \leq \gamma, \beta < p, p \in \mathbb{N}$ and $k \geq 2$:

$$(1.14) \quad \sum S_k(p, \gamma) = \left\{ f(z) \in \Sigma_p : -\frac{zf'(z)}{f(z)} \in P_k(p, \gamma), z \in U \right\},$$

$$(1.15) \quad \sum C_k(p, \gamma) = \left\{ f(z) \in \Sigma_p : -\frac{(zf'(z))'}{f'(z)} \in P_k(p, \gamma), z \in U \right\},$$

$$(1.16) \quad \begin{aligned} & \sum V_k(p, \gamma, \zeta) \\ &= \left\{ f(z) \in \Sigma_p, g(z) \in \sum S_2(p, \gamma) : -\frac{zf'(z)}{g(z)} \in P_k(p, \zeta), z \in U \right\} \end{aligned}$$

and

$$(1.17) \quad \sum V_k^*(p, \gamma, \zeta) = \left\{ f(z) \in \sum_p, g(z) \in \sum C_2(p, \gamma) : -\frac{(zf'(z))'}{g'(z)} \in P_k(p, \zeta), z \in U \right\}.$$

We can easily see that:

$$(1.18) \quad f(z) \in \sum C_k(p, \gamma) \iff -\frac{zf'(z)}{p} \in \sum S_k(p, \gamma)$$

and

$$(1.19) \quad f(z) \in \sum V_k^*(p, \gamma, \zeta) \iff -\frac{zf'(z)}{p} \in \sum V_k(p, \gamma, \xi).$$

We note that, for special choices for the parameters k and γ involved in the above classes, we can obtain well-known subclasses

$$\begin{aligned} \sum S_2(p, \gamma) &= \sum S_p^*(\gamma), \sum C_2(p, \gamma) = \sum C_p(\gamma), \\ \sum V_2(p, \gamma, \zeta) &= \sum V_p(\gamma, \zeta) \text{ and } \sum V_2^*(p, \gamma, \zeta) = \sum V_p^*(\gamma, \zeta). \end{aligned}$$

The classes $\sum S_p^*(\gamma)$, $\sum C_p(\gamma)$, $\sum V_p(\gamma, \zeta)$ and $\sum V_p^*(\gamma, \zeta)$ denote the meromorphic p -valent starlike of order γ , meromorphic p -valent convex of order γ , meromorphic p -valent close-to-convex of order γ and type ζ ($0 \leq \gamma$, $\zeta < p$, $p \in \mathbb{N}$) and meromorphic p -valent quasi-convex of order γ and type ζ ($0 \leq \gamma$, $\zeta < p$, $p \in \mathbb{N}$). The classes $\sum S_p^*(\gamma)$ and $\sum C_p(\gamma)$ were studied by Kumar and Shukla [5] and the classes $\sum V_p(\gamma, \zeta)$ and $\sum V_p^*(\gamma, \zeta)$ were introduced by Aouf et al. [2] and Aouf and Xu [3].

Next, by using the linear operator $H_{p,\beta,\mu}^\alpha f(z)$, we introduce the following classes of analytic functions for $0 \leq \gamma, \zeta < p$ and $k \geq 2$

$$(1.20) \quad \sum S_{k,p}(\alpha, \beta, \mu; \gamma) = \left\{ f(z) \in \sum_p : H_{p,\beta,\mu}^\alpha f(z) \in \sum S_k(p, \gamma), z \in U \right\},$$

$$(1.21) \quad \sum C_{k,p}(\alpha, \beta, \mu; \gamma) = \left\{ f(z) \in \sum_p : H_{p,\beta,\mu}^\alpha f(z) \in \sum C_k(p, \gamma), z \in U \right\},$$

$$(1.22) \quad \sum V_{k,p}(\alpha, \beta, \mu; \gamma, \zeta) = \left\{ f(z) \in \sum_p : H_{p,\beta,\mu}^\alpha f(z) \in \sum V_k(p, \zeta), z \in U \right\}$$

and

$$(1.23) \quad \sum V_{k,p}^*(\alpha, \beta, \mu; \gamma, \zeta) = \left\{ f(z) \in \sum_p : H_{p,\beta,\mu}^\alpha f(z) \in \sum V_k^*(p, \zeta), z \in U \right\}.$$

We also note that

$$(1.24) \quad f(z) \in \sum C_{k,p}(\alpha, \beta, \mu; \gamma) \Leftrightarrow -\frac{zf'(z)}{p} \in \sum S_{k,p}(\alpha, \beta, \mu; \gamma)$$

and

$$(1.25) \quad f(z) \in \sum V_{k,p}(\alpha, \beta, \mu; \gamma, \zeta) \Leftrightarrow -\frac{zf'(z)}{p} \in \sum V_{k,p}^*(\alpha, \beta, \mu; \gamma, \zeta).$$

2. Main results

Unless otherwise mentioned, we shall assume in the reminder of this paper that, $\alpha \geq 0$, $\mu > 0$, $\beta > -1$, $0 \leq \gamma$, $\zeta < p$, $k \geq 2$ and $z \in U$.

In order to prove our results, we need the following lemma.

Lemma 1. [6] *Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ and $\Phi(u, v)$ be a complex-valued function satisfying the conditions:*

- (1) $\Phi(u, v)$ is continuous in a domain $D \in \mathbb{C}^2$.
- (2) $(0, 1) \in D$ and $\operatorname{Re} \Phi(1, 0) > 0$.
- (3) $\Re \{\Phi(iu_2, v_1)\} > 0$ where $(iu_2, v_1) \in D$ and $v_1 \leq -\frac{1}{2}(1 + u_2^2)$.

If $h(z) = 1 + c_1z + c_2z^2 + \dots$ is analytic in U such that $(h(z), zh'(z)) \in D$ and $\operatorname{Re} \{\Phi(h(z), zh'(z))\} > 0$ for $z \in U$, then $\operatorname{Re}\{h(z)\} > 0$ in U .

Theorem 1. Let $0 \leq \eta \leq \gamma < p$ and $\eta < \alpha + \beta + p$, then

$$(2.1) \quad \sum S_{k,p}(\alpha + 1, \beta, \mu; \gamma) \subset \sum S_{k,p}(\alpha, \beta, \mu; \eta),$$

where

$$(2.2) \quad \eta = \frac{2[2\gamma(\alpha + \beta + p) + p]}{1 + 2\alpha + 2\beta + 2p + 2\gamma - \sqrt{[2(\alpha + \beta + \gamma + p) + 1]^2 - 8[2\gamma(\alpha + \beta + p) + p]}}.$$

Proof. Let $f(z) \in \sum S_{k,p}(\alpha + 1, \beta, \mu; \gamma)$ and

$$(2.3) \quad -\frac{z(H_{p,\beta,\mu}^\alpha f(z))'}{H_{p,\beta,\mu}^\alpha f(z)} = H(z) = (p - \eta)h(z) + \eta,$$

where

$$(2.4) \quad h(z) = \left(\frac{k}{4} + \frac{1}{2}\right)h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)h_2(z),$$

where $h_i(z)$ ($i = 1, 2$) are analytic in U and $h_i(0) = 1$ ($i = 1, 2$). Using (1.11) in (2.3) and differentiating the resulting equation with respect to z , we have

$$(2.5) \quad -\frac{z(H_{p,\beta,\mu}^{\alpha+1}f(z))'}{H_{p,\beta,\mu}^{\alpha+1}f(z)} - \gamma = \eta - \gamma + (p - \eta)h(z) - \frac{(p - \eta)zh'(z)}{(p - \eta)h(z) + \eta - (\alpha + \beta + p)}.$$

Now, we will show that $H(z) \in P_k(\gamma, p)$ or $h_i(z) \in P$. From (2.4) and (2.5) we have

$$-\frac{z(H_{p,\beta,\mu}^{\alpha+1}f(z))'}{H_{p,\beta,\mu}^{\alpha+1}f(z)} - \gamma = \left(\frac{k}{4} + \frac{1}{2}\right) \left\{ \eta - \gamma + (p - \eta)h_1(z) - \frac{(p - \eta)zh_1'(z)}{(p - \eta)h_1(z) + \eta - (\alpha + \beta + p)} \right\} \\ - \left(\frac{k}{4} - \frac{1}{2}\right) \left\{ \eta - \gamma + (p - \eta)h_2(z) - \frac{(p - \eta)zh_2'(z)}{(p - \eta)h_2(z) + \eta - (\alpha + \beta + p)} \right\},$$

this implies that

$$\operatorname{Re} \left\{ \eta - \gamma + (p - \eta)h_i(z) - \frac{(p - \eta)zh_i'(z)}{(p - \eta)h_i(z) + \eta - (\alpha + \beta + p)} \right\} > 0 \quad (i = 1, 2).$$

We form the functional $\Phi(u, v)$ by taking $u = h_i(z)$, $v = zh_i'(z)$,

$$(2.6) \quad \Phi(u, v) = \eta - \gamma + (p - \eta)u - \frac{(p - \eta)v}{(p - \eta)u + \eta - (\alpha + \beta + p)}.$$

Clearly, the first two conditions of Lemma 1 are satisfied in the domain $D \subseteq \mathbb{C} \setminus \frac{\eta - \alpha + \beta + p}{\eta - p} \times \mathbb{C}$. Now, we verify condition (iii) as follows:

$$\begin{aligned} \operatorname{Re} \{ \Phi(iu_2, v_1) \} &= (\eta - \gamma) + \operatorname{Re} \left\{ -\frac{(p - \eta)v_1}{(p - \eta)iu_2 + \eta - (\alpha + \beta + p)} \right\} \\ &\leq (\eta - \gamma) - \frac{(p - \eta)(\alpha + \beta + p - \eta)(1 + u_2^2)}{2[(p - \eta)^2 u_2^2 + (\eta - \alpha - \beta - p)^2]} \\ &= \frac{A + Bu_2^2}{2C}, \end{aligned}$$

where

$$\begin{aligned} A &= 2(\eta - \gamma)(\eta - \alpha - \beta - p)^2 - (p - \eta)(\alpha + \beta + p - \eta), \\ B &= 2(\eta - \gamma)(p - \eta)^2 - (p - \eta)(\alpha + \beta + p - \eta), \\ C &= (p - \eta)^2 u_2^2 + (\eta - \alpha - \beta - p)^2. \end{aligned}$$

We note that $\operatorname{Re} \{ \Phi(iu_2, v_1) \} < 0$ if and only if $A \leq 0$ and $B < 0$. From η as given by (2.2), we obtain $A \leq 0$ and from $0 \leq \eta \leq \gamma < p$ we have $B < 0$. Therefore applying Lemma 1, $h_i(z) \in P$ ($i = 1, 2$) and consequently $f \in \sum S_{k,p}(\alpha, \beta, \mu; \eta)$.

This completes the proof of Theorem 1.

Theorem 2. Let $0 \leq \eta \leq \gamma < p, \eta < \alpha + \beta + p$ and $k \geq 2$, then

$$(2.7) \quad \sum C_{k,p}(\alpha + 1, \beta, \mu; \gamma) \subset \sum C_{k,p}(\alpha, \beta, \mu; \eta).$$

Proof. Applying (1.24) and Theorem 1, we observe that

$$\begin{aligned} f(z) &\in \sum C_{k,p}(\alpha + 1, \beta, \mu; \gamma) \\ &\iff -\frac{zf'(z)}{p} \in \sum S_{k,p}(\alpha, \beta, \mu; \gamma) \\ &\implies -\frac{zf'(z)}{p} \in \sum S_{k,p}(\alpha, \beta, \mu; \eta) \\ &\iff f(z) \in \sum C_{k,p}(\alpha, \beta, \mu; \eta), \end{aligned}$$

which evidently proves Theorem 2.

Theorem 3. Let $0 \leq \gamma, \zeta < p, \gamma < \alpha + \beta + p$ and $k \geq 2$, then

$$(2.8) \quad \sum V_{k,p}(\alpha + 1, \beta, \mu; \gamma, \zeta) \subset \sum V_{k,p}(\alpha, \beta, \mu; \gamma, \zeta).$$

Proof. Let $f(z) \in \sum V_{k,p}(\alpha + 1, \beta, \mu; \gamma, \zeta)$. Then, in view of the definition of the class $\sum V_{k,p}(\alpha + 1, \beta, \mu; \gamma, \zeta)$, there exists a function $g(z) \in \sum S_{2,p}(\alpha + 1, \beta, \mu; \gamma)$ such that

$$-\frac{z(H_{p,\beta,\mu}^{\alpha+1}f(z))'}{H_{p,\beta,\mu}^{\alpha+1}g(z)} \in P_k(\zeta, p) \quad (z \in U).$$

Now let

$$(2.9) \quad -\frac{z(H_{p,\beta,\mu}^{\alpha}f(z))'}{H_{p,\beta,\mu}^{\alpha}g(z)} = G(z) = (p - \zeta)h(z) + \zeta,$$

where $h(z)$ is given by (2.4). Using (1.11) in (2.9), we have

$$\begin{aligned} (2.10) \quad &(\alpha + \beta)H_{p,\beta,\mu}^{\alpha+1}f(z) - (\alpha + \beta + p)H_{p,\beta,\mu}^{\alpha}f(z) \\ &= -[(p - \zeta)h(z) + \zeta]H_{p,\beta,\mu}^{\alpha}g(z). \end{aligned}$$

Differentiating (2.10) with respect to z and multiplying by z , we obtain

$$\begin{aligned} (2.11) \quad &(\alpha + \beta)z(H_{p,\beta,\mu}^{\alpha+1}f(z))' - (\alpha + \beta + p)z(H_{p,\beta,\mu}^{\alpha}f(z))' \\ &= -(p - \zeta)zh'(z)H_{p,\beta,\mu}^{\alpha}g(z) - [(p - \zeta)h(z) + \zeta]z(H_{p,\beta,\mu}^{\alpha}g(z))'. \end{aligned}$$

Since $g(z) \in \sum S_{2,p}(\alpha + 1, \beta, \mu; \gamma)$, by Theorem 1, $g(z) \in \sum S_{2,p}(\alpha, \beta, \mu; \gamma)$, then we have

$$-\frac{z(H_{p,\beta,\mu}^{\alpha}g(z))'}{H_{p,\beta,\mu}^{\alpha}g(z)} = (p - \gamma)q(z) + \gamma,$$

where $q(z) = 1 + c_1z + c_2z^2 + \dots$ is analytic in U with $q(0) = 1$. Then by using (1.11), we have

$$(2.12) \quad -(\alpha + \beta) \frac{H_{p,\beta,\mu}^{\alpha+1}g(z)}{H_{p,\beta,\mu}^{\alpha}g(z)} = (p - \gamma)q(z) + \gamma - (\alpha + \beta + p).$$

From (2.11) and (2.12), we obtain

$$(2.13) \quad -\frac{z(H_{p,\beta,\mu}^{\alpha+1}f(z))'}{H_{p,\beta,\mu}^{\alpha+1}g(z)} - \zeta = (p - \zeta)h(z) - \frac{(p - \zeta)zh'(z)}{(p - \gamma)q(z) + \gamma - (\alpha + \beta + p)}.$$

Now, we will show that $G(z) \in P_k(\zeta, p)$ or $h_i(z) \in P$, $i = 1, 2$. From (2.4) and (2.13) we have

$$\begin{aligned} & -\frac{z(H_{p,\beta,\mu}^{\alpha+1}f(z))'}{H_{p,\beta,\mu}^{\alpha+1}g(z)} - \zeta \\ &= \left(\frac{k}{4} + \frac{1}{2}\right) \left\{ (p - \zeta)h_1(z) - \frac{(p - \zeta)zh'_1(z)}{(p - \gamma)q(z) + \gamma - (\alpha + \beta + p)} \right\} \\ & - \left(\frac{k}{4} - \frac{1}{2}\right) \left\{ (p - \zeta)h_2(z) - \frac{(p - \zeta)zh'_2(z)}{(p - \gamma)q(z) + \gamma - (\alpha + \beta + p)} \right\}, \end{aligned}$$

this implies that

$$\operatorname{Re} \left\{ (p - \zeta)h_i(z) - \frac{(p - \zeta)zh'_i(z)}{(p - \gamma)q(z) + \gamma - (\alpha + \beta + p)} \right\} > 0 \quad (z \in U; i = 1, 2).$$

We form the functional $\Phi(u, v)$ by choosing $u = h_i(z)$, $v = zh'_i(z)$,

$$\Phi(u, v) = (p - \zeta)u - \frac{(p - \zeta)v}{(p - \gamma)q(z) + \gamma - (\alpha + \beta + p)}.$$

Clearly, the first two conditions of Lemma 1 are satisfied in the domain $D \subseteq \mathbb{C} \setminus Q^* \times \mathbb{C}$, where $Q^* = \left\{ z \in \mathbb{C} \text{ and } \operatorname{Re}(q(z)) = q_1 > \frac{\gamma - (\alpha + \beta + p)}{\gamma - p} \right\}$ and $q(z) = q_1 + iq_2$.

Now, we verify the condition (iii) as follows:

$$\begin{aligned} \operatorname{Re} \{ \Phi(iu_2, v_1) \} &= \operatorname{Re} \left\{ -\frac{(p - \zeta)v_1}{(p - \gamma)(q_1 + iq_2) + \gamma - (\alpha + \beta + p)} \right\} \\ &\leq -\frac{[(\alpha + \beta + p - \gamma) - (p - \gamma)q_1](p - \zeta)(1 + u_2^2)}{2 \{ [(p - \gamma)q_1 + \gamma - \alpha - \beta - p]^2 + [(p - \gamma)q_2]^2 \}} \\ &< 0. \end{aligned}$$

By applying Lemma 1, $h_i(z) \in P$ ($i = 1, 2$) and, consequently,

$$f(z) \in \sum V_{k,p}(\alpha + 1, \beta, \mu; \gamma, \zeta).$$

This completes the proof of Theorem 3.

Theorem 4. Let $0 \leq \gamma, \lambda < p, \gamma < \alpha + \beta + p$ and $k \geq 2$, then

$$(2.14) \quad \sum V_{k,p}^*(\alpha + 1, \beta, \mu; \gamma, \zeta) \subset \sum V_{k,p}^*(\alpha, \beta, \mu; \gamma, \zeta).$$

Proof. Applying (1.25) and Theorem 3, we observe that

$$\begin{aligned} f(z) &\in \sum V_{k,p}^*(\alpha + 1, \beta, \mu; \gamma, \zeta) \\ &\iff -\frac{zf'(z)}{p} \in \sum V_{k,p}(\alpha + 1, \beta, \mu; \gamma, \zeta) \\ &\implies -\frac{zf'(z)}{p} \in \sum V_{k,p}(\alpha, \beta, \mu; \gamma, \zeta) \\ &\iff f(z) \in \sum V_{k,p}^*(\alpha, \beta, \mu; \gamma, \zeta), \end{aligned}$$

which, evidently, proves Theorem 4.

In [5], Kumar and Shukla defined the familiar integral operator $F_{\nu,p}(f)(z)$ as follows:

$$\begin{aligned} F_{\nu,p}(f)(z) &= \frac{\nu}{z^{\nu+p}} \int_0^z t^{\nu+p-1} f(t) dt \quad (\nu > 0) \\ (2.15) \quad &= z^{-p} + \sum_{n=1}^{\infty} \frac{\nu}{\nu+n} a_{n-p} z^{n-p}. \end{aligned}$$

It follows that:

$$(2.16) \quad z(H_{p,\beta,\mu}^\alpha F_{\nu,p}(f)(z))' = \nu H_{p,\beta,\mu}^\alpha f(z) - (\nu + p) H_{p,\beta,\mu}^\alpha F_{\nu,p}(f)(z) \quad (\nu > 0).$$

Theorem 5. If $0 \leq \gamma < p, k \geq 2$ and $f \in \sum S_{k,p}(\alpha, \beta, \mu; \gamma)$, then

$$F_{\nu,p}(f)(z) \in \sum S_{k,p}(\alpha, \beta, \mu; \gamma) \quad (\nu > 0).$$

Proof. Let $f \in \sum S_{k,p}(\alpha, \beta, \mu; \gamma)$ and set

$$(2.17) \quad -\frac{z(H_{p,\beta,\mu}^\alpha F_{\nu,p}(f)(z))'}{H_{p,\beta,\mu}^\alpha F_{\nu,p}(f)(z)} = M(z) = (p - \gamma) h(z) + \gamma,$$

where $h(z)$ is given by (2.4). Using (2.16) and (2.17), we have

$$(2.18) \quad \nu \frac{H_{p,\beta,\mu}^\alpha f(z)}{H_{p,\beta,\mu}^\alpha F_{\nu,p}(f)(z)} = -(p - \gamma) h(z) - \gamma + \nu + p.$$

Taking the logarithmic differentiation on both sides of (2.18) with respect to z and multiplying by z , we have

$$(2.19) \quad -\frac{z(H_{p,\beta,\mu}^\alpha f(z))'}{H_{p,\beta,\mu}^\alpha f(z)} - \gamma = (p-\gamma)h(z) + \frac{(p-\gamma)zh'(z)}{(p-\gamma)h(z) + \gamma - (\nu+p)}.$$

Now, we will show that $M(z) \in P_k(\gamma, p)$ or $h_i(z) \in P$. From (2.4) and (2.19) we have

$$\begin{aligned} -\frac{z(H_{p,\beta,\mu}^\alpha f(z))'}{H_{p,\beta,\mu}^\alpha f(z)} - \gamma &= \left(\frac{k}{4} + \frac{1}{2}\right) \left\{ (p-\gamma)h_1(z) - \frac{(p-\gamma)zh_1'(z)}{(p-\gamma)h_1(z) + \gamma - (\nu+p)} \right\} \\ &\quad - \left(\frac{k}{4} - \frac{1}{2}\right) \left\{ (p-\gamma)h_2(z) - \frac{(p-\gamma)zh_2'(z)}{(p-\gamma)h_2(z) + \gamma - (\nu+p)} \right\}, \end{aligned}$$

this implies that

$$(2.20) \quad \operatorname{Re} \left\{ (p-\gamma)h_i(z) - \frac{(p-\gamma)zh_i'(z)}{(p-\gamma)h_i(z) + \gamma - (\nu+p)} \right\} > 0 \quad (z \in U; i = 1, 2).$$

We form the functional $\Phi(u, v)$ by choosing $u = h_i(z)$, $v = zh_i'(z)$,

$$\Phi(u, v) = (p-\gamma)u - \frac{(p-\gamma)v}{(p-\gamma)u + \gamma - (\nu+p)}.$$

Then, clearly, $\Phi(u, v)$ satisfies all the conditions of Lemma 1. Hence $h_i(z) \in P$ ($i = 1, 2$) and consequently $h(z) \in P_k$ for $z \in U$, which implies that $F_{\nu,p}(f)(z) \in \sum S_{k,p}(\alpha, \beta, \mu; \gamma)$. This completes the proof of Theorem 5.

Next, we derive an inclusion property for the subclass $\sum C_{k,p}(\alpha, \beta, \mu; \gamma)$ involving $F_{\nu,p}(f)(z)$, which is given by the following theorem.

Theorem 6. *If $0 \leq \gamma < p$, $k \geq 2$, $\nu > 0$ and $f \in \sum C_{k,p}(\alpha, \beta, \mu; \gamma)$, then*

$$F_{\nu,p}(f)(z) \in \sum C_{k,p}(\alpha, \beta, \mu; \gamma).$$

Proof. By applying Theorem 5, it follows that

$$\begin{aligned} f &\in \sum C_{k,p}(\alpha, \beta, \mu; \gamma) \\ &\iff -\frac{zf'}{p} \in \sum S_{k,p}(\alpha, \beta, \mu; \gamma) \\ &\implies F_{\nu,p}(f)(z) \left(-\frac{zf'}{p} \right) \in \sum S_{k,p}(\alpha, \beta, \mu; \gamma) \\ &\iff -\frac{z(F_{\nu,p}(f)(z))'}{p} \in \sum S_{k,p}(\alpha, \beta, \mu; \gamma) \\ &\iff F_{\nu,p}(f)(z) \in \sum C_{k,p}(\alpha, \beta, \mu; \gamma). \end{aligned}$$

This completes the proof of Theorem 6.

Using (2.16) instead of (1.11) and the techniques of the proof of Theorems 3 and 4, respectively, we can prove the following theorems, respectively.

Theorem 7. *If $0 \leq \gamma, \zeta < p$, $k \geq 2$, $\nu \geq 0$ and $f \in \sum V_{k,p}(\alpha, \beta, \mu; \gamma, \zeta)$, then*

$$F_{\nu,p}(f)(z) \in \sum V_{k,p}(\alpha, \beta, \mu; \gamma, \zeta).$$

Theorem 8. *If $0 \leq \gamma, \zeta < p$, $k \geq 2$, $\nu \geq 0$ and $f \in \sum V_{k,p}^*(\alpha, \beta, \mu; \gamma, \zeta)$, then*

$$F_{\nu,p}(f)(z) \in \sum V_{k,p}^*(\alpha, \beta, \mu; \gamma, \zeta).$$

Remark 1.

- (i) Using (1.12) instead of (1.11), in our results, we can obtain new results corresponding to the operator $H_{p,\beta,\mu}^\alpha$.
- (ii) Putting $\mu = 1$, in the above results, we obtain the corresponding results for different classes associated with the operator $H_{p,\beta}^\alpha$ defined in (1.13).

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ALGEBRAIC HYPERSTRUCTURES OF SOFT SETS ASSOCIATED WITH TERNARY SEMIHYPERGROUPS

Kostaq Hila

Krisanthi Naka

Department of Mathematics & Computer Science

Faculty of Natural Sciences

University of Gjirokastra

Albania

e-mail: kostaq_hila@yahoo.com

khila@uogj.edu.al

Violeta Leoreanu-Fotea

Faculty of Mathematics

"Al.I. Cuza" University

Iasi

Romania

e-mail: leoreanu2002@yahoo.com

Sabri Sadiku

Faculty of Mining and Metallurgy

University of Prishtina

Kosovo

e-mail: sabsadiku@yahoo.com, sabri.sadiku@uni-pr.edu

Abstract. Molodtsov introduced the concept of soft set, which can be seen and used as a new mathematical tool for dealing with uncertainty. In this paper we introduce and initiate the study of soft ternary semihypergroups by using soft set theory. The notions of soft ternary semihypergroups, soft ternary subsemihypergroups, soft left (right, lateral) hyperideals, soft hyperideals, soft quasi-hyperideals and soft bi-hyperideals are introduced, and several related properties are investigated.

Keywords: ternary semihypergroup, soft set, soft ternary semihypergroup, soft left (lateral, right) hyperideal, soft hyperideal, soft quasi (bi)-hyperideal.

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1. Introduction and preliminaries

The Hyperstructure theory was introduced in 1934, at the eighth Congress of Scandinavian Mathematicians, when F. Marty [1] defined hypergroups based on the notion of hyperoperation, began to analyze their properties and applied them to groups. In the following decades and nowadays, a number of different hyperstructures are widely studied from the theoretical point of view and for their

applications to many subjects of pure and applied mathematics, such as in fuzzy sets and rough set theory, optimization theory, theory of discrete event dynamical systems, cryptography, codes, analysis of computer programs, automata, formal language theory, combinatorics, artificial intelligence, probability, graphs and hypergraphs, geometry, lattices and binary relations (see [2], [3], [4], [5]).

Ternary algebraic operations were considered in the 19th century by several mathematicians. Cayley [6] introduced the notion of "cubic matrix" which was generalized by Kapranov, et al. in 1990 [7]. Ternary structures and their generalization, the so-called n -ary structures, raise certain hopes in view of their possible applications in physics and other sciences. Some significant physical applications in Nambu mechanics, although still hypothetical, in the fractional quantum Hall effect, the non-standard statistics (the anyons), supersymmetric theories, Yang-Baxter equation etc. can be seen in [22], [25], [8], [27], [28], [29]). The notion of an n -ary group was introduced in 1928 by W. Dörnte [15] (under inspiration of Emmy Noether). The idea of investigations of n -ary algebras, i.e., sets with one n -ary operation, seems to be going back to Kasner's lecture [16] at the 53rd annual meeting of the American Association of the Advancement of Science in 1904. Sets with one n -ary operation having different properties were investigated by many authors. Such systems have many applications in different branches. For example, in the theory of automata [24] are used n -ary systems satisfying some associative laws, some others n -ary systems are applied in the theory of quantum groups [14] and combinatorics [30]. Different applications of ternary structures in physics are described by R. Kerner in [8]. In physics there are also used such structures as n -ary Filippov algebras (see [9]) and n -Lie algebras (see [22]). Some n -ary structures induced by hypercubes have applications in error-correcting and error-detecting coding theory, cryptology, as well as in the theory of (t, m, s) -nets (see for example [26]). Ternary semigroups are universal algebras with one associative operation. The theory of ternary algebraic systems was introduced by D. H. Lehmer [17] in 1932. He investigated certain algebraic systems called triplexes which turn out to be commutative ternary groups. The notion of ternary semigroups was introduced by S. Banach (cf. [23]). By an example he showed that a ternary semigroup does not necessary reduce to an ordinary semigroup.

n -ary generalizations of algebraic structures is the most natural way for further development and deeper understanding of their fundamental properties. In [10], Davvaz and Vougiouklis introduced the concept of n -ary hypergroups as a generalization of hypergroups in the sense of Marty. Also, n -ary hypergroups can be seen as an interesting generalization of n -ary groups. Davvaz and et. al. in [11] considered a class of algebraic hypersystems which represent a generalization of semigroups, hypersemigroups and n -ary semigroups. Ternary semihypergroups are algebraic structures with one associative hyperoperation and they are a particular case of an n -ary semihypergroup (n -semihypergroup) for $n = 3$ (cf. [11], [12], [10, 13]). Recently, Hila et. al. [20], [21] introduced and studied some classes of hyperideals in ternary semihypergroups.

Several difficult problems in economics, engineering, environment, social sciences, medicine and many other fields involve uncertain data. We usually come

across the data which is blurred and contains many types of uncertainties. There are several well-known theories developed to overcome difficulties which arise due to uncertainty. For instance, probability theory, fuzzy sets theory [31], rough sets theory [32] and other mathematical tools. But all these theories have their inherited difficulties as pointed out by Molodtsov [33]. In probability theory, we have to make a great deal of experiments in order to check samples. In social sciences and economics, it is not always possible to make so many experiments.

Pawlak [32] introduced the theory of rough sets in 1982. It was a significant approach for modeling vagueness. In this theory, equivalence classes are used to approximate crisp subsets by their upper and lower approximations. This theory has been applied to many problems successfully, yet has its own limitations. It is not always possible to have an appropriate equivalence relation among the elements of a given set, so we cannot have equivalence classes to get upper and lower approximations of a subset. Later, some authors [34] tried to have approximations not with the help of equivalence relation, but by using a relation in general.

Fuzzy set theory was developed by Zadeh [31]. It is the most appropriate approach to deal with uncertainties. However, some authors [33] think that the difficulties in fuzzy set theory are due to the inadequacy of parametrization tools of this theory.

In 1999, Molodtsov introduced the concept of soft sets as a new mathematical tool for dealing with uncertainties that is free from the difficulties affecting existing methods. Soft set theory has rich potential for applications in several directions, few of which had been demonstrated by Molodtsov in his pioneer work [33]. Molodtsov also showed how Soft Set Theory (SST) is free from parametrization inadequacy syndrome of Fuzzy Set Theory (FST), Rough Set Theory (RST), Probability Theory, and Game Theory. SST is a very general framework. Many of the established paradigms appear as special cases of SST. Applications of Soft Set Theory in other disciplines and real life problems are now catching momentum. At present, research in the theory of soft sets is in progress. Theoretical and application aspects of soft set theory are discussed in several papers, such as [35], [36], [37], [38]. Maji et al. [35] applied soft sets to a decision making problem and studied several operations on the theory of soft sets. Pei et al. [36] discussed the relationship between soft sets and information systems. Roy et al. [37] applied fuzzy soft set theory to a decision making problem. Maji et al. [38] studied several operations on the theory of soft sets. Ali et al. [39] also studied some new notions such as the restricted intersection, the restricted union, the restricted difference, and the extended intersection of two soft sets. The algebraic structure of soft sets has been studied by several authors. For example, Aktas and Cagman [40] introduced the basic concepts of soft set theory and compared soft sets to the related concepts of fuzzy sets and rough sets. They also discussed the notion of soft groups and drove their basic properties using Molodtsov's definition of soft sets. Other applications of soft set theory in different algebraic structure can be found in [41], [42], [43], [44], [45], [46], [47], [48], [49] etc. Recently, Leoreanu-Fotea and Corsini [19] and then Davvaz et. al. [50], [51], [52] introduced and analyzed several types of soft hyperstructures: soft hypergroupoids, soft semihypergroups,

soft hypergroups and soft polygroups.

In this paper, we introduce and initiate the study of soft ternary semihypergroups by using soft set theory. The notions of soft ternary semihypergroups, soft ternary subsemihypergroups, soft left (lateral, right) hyperideals, soft hyperideals, soft quasi-hyperideals and soft bi-hyperideals are introduced, and several related properties are investigated.

Recall first the basic terms and definitions from hyperstructure theory and soft set theory.

2. Ternary semihypergroups

Let H be a nonempty set and let $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$ denote the set of all nonempty subsets of H . A map $\circ : H \times H \rightarrow \mathcal{P}^*(H)$ is called a *hyperoperation* or *join operation* on H . A *hypergroupoid* is a pair $(H, \circ)$, where $\circ$ is a hyperoperation on H . A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all $x, y, z \in H$, $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

If $x \in H$ and A, B are non-empty subsets of H then

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, A \circ x = A \circ \{x\} \text{ and } x \circ B = \{x\} \circ B.$$

A nonempty subset B of a semihypergroup H is called a *sub-semihypergroup* of H if $B \circ B \subseteq B$. A semihypergroup $(H, \circ)$ is a *hypergroup* if it satisfies the reproduction axiom: for all $a \in H$, $a \circ H = H \circ a = H$.

A map $f : H \times H \times H \rightarrow \mathcal{P}^*(H)$ is called a *ternary hyperoperation* on H . A *ternary hypergroupoid* is a pair (H, f) where f is a ternary hyperoperation on H .

If A, B, C are non-empty subsets of H , then we define

$$f(A, B, C) = \bigcup_{a \in A, b \in B, c \in C} f(a, b, c).$$

A ternary hypergroupoid (H, f) is called a *ternary semihypergroup* if for all $a_1, a_2, \dots, a_5 \in H$, we have

$$(*) \quad f(f(a_1, a_2, a_3), a_4, a_5) = f(a_1, f(a_2, a_3, a_4), a_5) = f(a_1, a_2, f(a_3, a_4, a_5)).$$

Since we can identify the set $\{x\}$ with the element x , any ternary semigroup [17] is a ternary semihypergroup.

Due to the associative law in a ternary semihypergroup (H, f) , for any elements $x_1, x_2, \dots, x_{2n+1} \in H$ and positive integers m, n with $m \leq n$, one may write

$$\begin{aligned} f(x_1, x_2, \dots, x_{2n+1}) &= f(x_1, \dots, x_m, x_{m+1}, x_{m+2}, \dots, x_{2n+1}) \\ &= f(x_1, \dots, f(f(x_m, x_{m+1}, x_{m+2}), x_{m+3}, x_{m+4}), \dots, x_{2n+1}). \end{aligned}$$

A ternary hypergroupoid (H, f) is *commutative* if for all $a_1, a_2, a_3 \in H$ and for all $\sigma \in \mathbb{S}_3$, $f(a_1, a_2, a_3) = f(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$.

Let (H, f) be a ternary semihypergroup. A nonempty subset T of H is called a *ternary subsemihypergroup* if $f(T, T, T) \subseteq T$.

An element e of a ternary semihypergroup (H, f) is called a *left identity* element of H if for all $a \in H$, $f(e, a, a) = \{a\}$. An element $e \in H$ is called an *identity* element of H if for all $a \in H$, $f(a, a, e) = f(e, a, a) = f(a, e, a) = \{a\}$. Hence $f(e, e, a) = f(e, a, e) = f(a, e, e) = \{a\}$.

A nonempty subset I of a ternary semihypergroup H is called a *left (right, lateral) hyperideal* of H if

$$f(H, H, I) \subseteq I \quad (f(I, H, H) \subseteq I, \quad f(H, I, H) \subseteq I).$$

A nonempty subset I of a ternary semihypergroup H is called a *hyperideal* of H if it is a left, right and lateral hyperideal of H . A nonempty subset I of a ternary semihypergroup H is called *two-sided hyperideal* of H if it is a left and right hyperideal of H .

Different examples can be found in [20], [21].

3. Preliminaries of soft set theory

Let U be an initial universe set and E be a set of parameters. The power set of U is denoted by $\mathcal{P}(U)$ and A is a subset of E .

Definition 3.1. A pair (F, A) is called a *soft set* over U , where $F : A \rightarrow \mathcal{P}(U)$ is a mapping.

In the other words, a soft set over U is a parameterized family of subsets of the universe U . For $a \in A$, $F(a)$ may be considered as the set of a -approximate elements of the soft set (F, A) . Notice that a soft set is not a set, as Molodtsov pointed out in several examples in [33].

Let us consider the following example. We quote it directly from [38].

Example 3.2. [38] Let us consider a soft set (F, E) , which describes the "attractiveness of houses", considered for purchase. Suppose that there are six houses in the universe U , given by $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ and $E = \{e_1, e_2, e_3, e_4, e_5\}$ is a set of decision parameters, where e_i ($i = 1, 2, 3, 4, 5$) stand for the parameters "expensive", "beautiful", "wooden", "cheap" and "in green surroundings", respectively. Consider a mapping $F : E \rightarrow \mathcal{P}(U)$. For instance, suppose that $F(e_1) = \{h_2, h_4\}$, $F(e_2) = \{h_1, h_3\}$, $F(e_3) = \{h_3, h_4, h_5\}$, $F(e_4) = \{h_1, h_3, h_5\}$, $F(e_5) = \{h_1\}$. The soft set (F, E) is a parameterized family $\{F(e_i), i = 1, \dots, 5\}$ of subsets of the set U , and can be viewed as a *collection* of approximations: $(F, E) = \{\text{expensive houses} = \{h_2, h_4\}, \text{beautiful houses} = \{h_1, h_3\}, \text{wooden houses} = \{h_3, h_4, h_5\}, \text{cheap houses} = \{h_1, h_3, h_5\}, \text{in green surroundings houses} = \{h_1\}\}$. Each approximation has two parts: a *predicate* and an *approximate value set*.

In [47], for a soft set (F, A) , the set $Supp(F, A) = \{x \in A | F(x) \neq \emptyset\}$ is called the *support* of the soft set (F, A) . If $Supp(F, A) \neq \emptyset$, then (F, A) is called non-null.

Definition 3.3. Let (F, A) and (G, B) be two soft sets over a common universe U . The *extended intersection* of (F, A) and (G, B) , denoted by $(F, A) \cap_E (G, B)$ is the soft set (K, C) , satisfying the following conditions: (i) $C = A \cup B$; (ii) for all $e \in C$,

$$K(e) = \begin{cases} F(e) & \text{if } e \in A \setminus B, \\ G(e) & \text{if } e \in B \setminus A, \\ F(e) \cup G(e) & \text{if } e \in A \cap B. \end{cases}$$

Definition 3.4. Let (F, A) and (G, B) be two soft sets over a common universe U . The *restricted intersection* of (F, A) and (G, B) , denoted by $(F, A) \cap_R (G, B)$ is the soft set (K, C) , satisfying the following conditions: (i) $C = A \cap B$; (ii) for all $e \in C$, $K(e) = F(e) \cap G(e)$. We write $(F, A) \tilde{\cap} (G, B) = (K, C)$.

Definition 3.5. Let (F, A) and (G, B) be two soft sets over a common universe U . The *bi-intersection* of (F, A) and (G, B) is the soft set (K, C) , where $C = A \cap B$ and $K : C \rightarrow \mathcal{P}(U)$ is a mapping given by $K(x) = F(x) \cap G(x)$ for all $x \in G$. We write $(F, A) \tilde{\cap} (G, B) = (K, C)$.

Definition 3.6. Let (F, A) and (G, B) be two soft sets over a common universe U . The *extended union* of (F, A) and (G, B) , denoted by $(F, A) \cup_E (G, B)$ is the soft set (K, C) , satisfying the following conditions: (i) $C = A \cup B$; (ii) for all $e \in C$,

$$K(e) = \begin{cases} F(e) & \text{if } e \in A \setminus B, \\ G(e) & \text{if } e \in B \setminus A, \\ F(e) \cup G(e) & \text{if } e \in A \cap B. \end{cases}$$

Definition 3.7. Let (F, A) and (G, B) be two soft sets over a common universe U . The *restricted union* of (F, A) and (G, B) , denoted by $(F, A) \cup_R (G, B)$, is the soft set (K, C) satisfying the following conditions: (i) $C = A \cap B$; (ii) for all $e \in C$, $K(e) = F(e) \cup G(e)$. We write $(F, A) \tilde{\cup} (G, B) = (K, C)$.

Definition 3.8. Let $(F_i, A_i)_{i \in I}$ be a nonempty family of soft sets over a common universe U . The union of these soft sets is the soft set (G, B) such that $B = \bigcup_{i \in I} A_i$ and for all $x \in B$, $G(x) = \bigcup_{i \in I(x)} F_i(x)$, where $I(x) = \{i \in I | x \in A_i\}$. We write $\bigcup_{i \in I} (F_i, A_i) = (G, B)$.

Definition 3.9. Let (F, A) and (G, B) be two soft sets over a common universe U . Then (F, A) AND (G, B) denoted by $(F, A) \tilde{\wedge} (G, B)$ is defined by $(F, A) \tilde{\wedge} (G, B) = (K, A \times B)$, where $K(x, y) = F(x) \cap G(y)$ for all $(x, y) \in A \times B$.

Definition 3.10. Let (F, A) and (G, B) be two soft sets over a common universe U . Then (F, A) OR (G, B) denoted by $(F, A) \tilde{\vee} (G, B)$ is defined by $(F, A) \tilde{\vee} (G, B) = (K, A \times B)$, where $K(x, y) = F(x) \cup G(y)$ for all $(x, y) \in A \times B$.

Definition 3.11. Let (F, A) and (G, B) be two soft sets over a common universe U . We say that (F, A) is a *soft subset* of (G, B) , denoted by $(F, A) \widetilde{\subseteq} (G, B)$, if it satisfies: (i) $A \subseteq B$; (ii) $F(a) \subseteq G(a)$ for all $a \in A$.

We say that (F, A) and (G, B) are *soft equal*, denoted by $(F, A) = (G, B)$ if $(F, A) \subseteq (G, B)$ and $(G, B) \subseteq (F, A)$.

4. Soft ternary semihypergroups

Hereafter, we shall consider soft sets over a ternary semihypergroup (H, f) .

Definition 4.1. The *restricted hyperproduct* of soft sets $(F_1, B_1), (F_2, B_2), (F_3, B_3)$ over a ternary semihypergroup H , denoted by $\widehat{f}((F_1, B_1), (F_2, B_2), (F_3, B_3))$, is defined as a soft set

$$(K, D) = \widehat{f}((F_1, B_1), (F_2, B_2), (F_3, B_3)),$$

where $D = \bigcap \{B_i | i = 1, 2, 3\} \neq \emptyset$ and $K : D \rightarrow \mathcal{P}(H)$ defined by

$$K(d) = f(F_1(d), F_2(d), F_3(d)) \text{ for all } d \in D.$$

Definition 4.2. Let (F, A) be a non-null soft set over a ternary semihypergroup (H, f) . Then, (F, A) is called a *soft ternary semihypergroup* over H if $F(x)$ is a ternary subsemihypergroup of H for all $x \in \text{Supp}(F, A)$, i.e.

$$\widehat{f}((F, A), (F, A), (F, A)) \subseteq (F, A).$$

A soft set (H, E) over H is said to be an *absolute soft set* over H , if for all $e \in E$, $H(e) = H$.

Proposition 4.3. A soft set (F, A) over a ternary semihypergroup H is a soft ternary semihypergroup over H if and only if for all $a \in A$, $F(a) \neq \emptyset$ is a ternary subsemihypergroup of H .

Proof. Let (F, A) be a soft ternary hypergroup over H and $a \in A$ be such that $F(a) \neq \emptyset$. By definition

$$\widehat{f}((F, A), (F, A), (F, A)) = (K, A \cap A \cap A) = (K, A)$$

where K is defined by

$$K(a) = f(F(a), F(a), F(a)) \text{ for all } a \in A.$$

Since $\widehat{f}((F, A), (F, A), (F, A)) \subseteq (F, A)$, it follows that $K(a) \subseteq F(a)$ for all $a \in A$, and so $f(F(a), F(a), F(a)) \subseteq F(a)$. This means that $F(a)$ is a ternary subsemihypergroup of H .

Conversely, let $F(a) \neq \emptyset$ be a ternary semihypergroup of H for all $a \in A$. We have $f(F(a), F(a), F(a)) \subseteq F(a)$, that is $K(a) \subseteq F(a)$. Thus, $(K, A) \subseteq (F, A)$, which implies that

$$\widehat{f}((F, A), (F, A), (F, A)) \subseteq (F, A)$$

Hence (F, A) is a ternary semihypergroup over H . ■

Example 4.4. Let $H = \{a, b, c, d, e, g\}$ and $f(x, y, z) = (x*y)*z$ for all $x, y, z \in H$, where $*$ is defined by the table:

$*$	a	b	c	d	e	g
a	a	$\{a, b\}$	c	$\{c, d\}$	e	$\{e, g\}$
b	b	b	d	d	g	g
c	c	$\{c, d\}$	c	$\{c, d\}$	c	$\{c, d\}$
d	d	d	d	d	d	d
e	e	$\{e, g\}$	c	$\{c, d\}$	e	$\{e, g\}$
g	g	g	d	d	g	g

Then (H, f) is a ternary semihypergroup. We define $F : H \rightarrow \mathcal{P}(H)$ by $F(a) = \{a, b\}$, $F(b) = \{b, d, g\}$, $F(c) = \{c, d\}$, $F(d) = \{d\}$, $F(e) = \{a, b, e, g\}$, $F(g) = \{d, g\}$. It is clear that for all $x \in H$, $F(x)$ is a ternary subsemihypergroup of H . Hence, (F, H) is a soft ternary semihypergroup over H . Notice that not every soft set over a ternary semihypergroup H , is a soft ternary semihypergroup over H . Let $G : H \rightarrow \mathcal{P}(H)$ defined by $G(a) = \{a, b, c\}$, $G(b) = \{b, d, g\}$, $G(c) = \{c, d\}$, $G(d) = \{d\}$, $G(e) = \{a, b, e, g\}$, $G(g) = \{d, e, g\}$. Then (G, H) is a soft set over H , but it is not a soft ternary semihypergroup over H , because $G(a)$ and $G(g)$ are not ternary subsemihypergroups of H .

Proposition 4.5. Let (F, A) and (G, B) be two soft ternary semihypergroups over H such that $A \cap B = \emptyset$. Then $(F, A) \cup_E (G, B)$ is a soft ternary semihypergroup over H .

Proof. By definition, $(K, C) = (F, A) \cup_E (G, B)$, where $C = A \cup B$ and $A \cap B = \emptyset$. Then for all $c \in C$, either $c \in A \setminus B$ or $c \in B \setminus A$. If $c \in A \setminus B$, then $K(c) = F(c)$ and if $c \in B \setminus A$, then $K(c) = G(c)$. So in both cases $K(c)$ is a ternary subsemihypergroup of H . Therefore, (K, C) is a soft ternary semihypergroup over H . ■

Proposition 4.6. Let (F, A) and (G, B) be two soft ternary semihypergroups over H such that $A \cap B \neq \emptyset$. Then $(F, A) \cup_E (G, B)$ is a soft ternary semihypergroup over H .

Proof. Let us assume that $A \cap B \neq \emptyset$. Let $(K, C) = (F, A) \cup_E (G, B)$ where

$$K(c) = \begin{cases} F(c) & \text{if } c \in A \setminus B, \\ G(c) & \text{if } c \in B \setminus A, \\ F(c) \cup G(c) & \text{if } c \in A \cap B. \end{cases}$$

To show that (K, C) is a soft ternary semihypergroup over H , we have to show that

$$\widehat{f}((K, C), (K, C), (K, C)) \subseteq (K, C).$$

We have

$$\widehat{f}((K, C), (K, C), (K, C)) = (P, C).$$

and

$$P(c) = f(K(c), K(c), K(c)) \text{ for all } c \in C.$$

For $c \in C$, we have

$$\begin{aligned} P(c) &= f(K(c), K(c), K(c)) \\ &= f((F, A) \cup_E (G, B), (F, A) \cup_E (G, B), (F, A) \cup_E (G, B)) \\ &\subseteq \begin{cases} F(c) & \text{if } c \in A \setminus B, \\ G(c) & \text{if } c \in B \setminus A, \\ F(c) \cap G(c) & \text{if } c \in A \cap B. \end{cases} \end{aligned}$$

Then $P(c) \subseteq K(c)$ for all $c \in C$. Therefore, $(F, A) \cup_E (G, B)$ is a soft ternary semihypergroup over H . ■

Proposition 4.7. *Let (F, A) and (G, B) be two soft ternary semihypergroups over H such that $A \cap B \neq \emptyset$. Then $(F, A) \cap_R (G, B)$ is a soft ternary semihypergroup over H .*

Proof. Let us assume that $A \cap B \neq \emptyset$. Let $(K, C) = (F, A) \cap_R (G, B)$ where $C = A \cap B \neq \emptyset$ and $K(c) = F(c) \cap G(c)$ for all $c \in C$. To show that (K, C) is a soft ternary semihypergroup over H , we have to show that

$$\widehat{f}((K, C), (K, C), (K, C)) \subseteq (K, C).$$

We have

$$\widehat{f}((K, C), (K, C), (K, C)) = (P, C).$$

and

$$P(c) = f(K(c), K(c), K(c)) \text{ for all } c \in C.$$

For $c \in C$, we have

$$\begin{aligned} P(c) &= f(K(c), K(c), K(c)) \\ &= f(F(c) \cap G(c), F(c) \cap G(c), F(c) \cap G(c)) \\ &\subseteq F(c) \cap G(c) \\ &= K(c). \end{aligned}$$

Then $P \subseteq K$. Therefore, $(F, A) \cap_R (G, B)$ is a soft ternary semihypergroup over H . ■

Proposition 4.8. *Let (F, A) and (G, B) be two soft ternary semihypergroups over H . Then $(F, A) \wedge (G, B)$ is a soft ternary semihypergroup over H .*

Proof. Let $(F, A) \wedge (G, B) = (K, A \times B) = (K, C)$ where $K(a, b) = F(a) \cap G(b)$ for all $(a, b) \in A \times B$. We have to show that $(K, A \times B)$ is a soft ternary semihypergroup over H . We have

$$(P, C) = \widehat{f}((K, A \times B), (K, A \times B), (K, A \times B))$$

where $C = A \times B$ and $P(a, b) = f(K(a, b), K(a, b), K(a, b))$ for all $(a, b) \in C$. For $(a, b) \in C$, we have

$$\begin{aligned} P(a, b) &= f(K(a, b), K(a, b), K(a, b)) \\ &= f(F(a) \cap G(b), F(a) \cap G(b), F(a) \cap G(b)) \\ &\subseteq F(a) \cap G(b) \\ &= K(a, b). \end{aligned}$$

Then $P \subseteq K$. Therefore, $(F, A) \wedge (G, B)$ is a soft ternary semihypergroup over H . ■

Definition 4.9. Let $(F_1, A_1), (F_2, A_2), (F_3, A_3)$ be three soft sets over a ternary semihypergroup (H, f) . Define

$$f^*((F_1, A_1), (F_2, A_2), (F_3, A_3)) = (K, A_1 \times A_2 \times A_3)$$

be a soft set where $K(a_1, a_2, a_3) = f((F_1(a_1), F_2(a_2), F_3(a_3)))$.

Proposition 4.10. Let (H, f) be a commutative ternary semihypergroup. If $(F_1, A_1), (F_2, A_2), (F_3, A_3)$ are soft ternary semihypergroup over H , then $f^*((F_1, A_1), (F_2, A_2), (F_3, A_3))$ is a soft ternary semihypergroup over H .

Proof. Let $f^*((F_1, A_1), (F_2, A_2), (F_3, A_3)) = (K, A_1 \times A_2 \times A_3)$ where

$$K(a_1, a_2, a_3) = f(F_1(a_1), F_2(a_2), F_3(a_3))$$

for all $(a_1, a_2, a_3) \in A_1 \times A_2 \times A_3$. We have

$$(P, C) = \widehat{f}((K, A_1 \times A_2 \times A_3), \dots, (K, A_1 \times A_2 \times A_3))$$

where $C = A_1 \times A_2 \times A_3$ and

$$P(a_1, a_2, a_3) = f(K(a_1, a_2, a_3), K(a_1, a_2, a_3), K(a_1, a_2, a_3))$$

for all $(a_1, a_2, a_3) \in C$. For $(a_1, a_2, a_3) \in C$, since H is commutative, we have

$$\begin{aligned} P(a_1, a_2, a_3) &= f(K(a_1, a_2, a_3), K(a_1, a_2, a_3), K(a_1, a_2, a_3)) \\ &= f(f(F_1(a_1), F_2(a_2), F_3(a_3)), f(F_1(a_1), F_2(a_2), F_3(a_3))) \\ &\subseteq f(F_1(a_1), F_2(a_2), F_3(a_3)) \\ &= K(a_1, a_2, a_3). \end{aligned}$$

Then $P \subseteq K$. This completes the proof. ■

5. Soft ternary subsemihypergroups and soft hyperideals

Definition 5.1. Let (F, A) and (G, B) be two soft sets over H such that (F, A) is a ternary semihypergroup and $(G, B) \subseteq (F, A)$. Then (G, B) is called a *soft ternary subsemihypergroup (hyperideal)* of (F, A) if $G(b)$ is a ternary subsemihypergroup (hyperideal) of $F(b)$ for all $b \in B$.

Proposition 5.2. Let (F, A) be a soft ternary semihypergroup over H and let $\{(K_i, A_i) | i \in I\}$ be a non-empty family of soft ternary subsemihypergroups of (F, A) . Then the following hold:

1. $\cap_{i \in I} (K_i, A_i)$ is a soft ternary subsemihypergroup of (F, A) .
2. $\wedge_{i \in I} (K_i, A_i)$ is a soft ternary subsemihypergroup of $\wedge_{i \in I} (F, A)$.
3. $\cup_{i \in I} (K_i, A_i)$ is a soft ternary subsemihypergroup of (F, A) if the set $\{A_i | i \in I\}$ is pairwise disjoint.

Proof. Straightforward. ■

Definition 5.3. A soft set (F, A) over a ternary semihypergroup H is called a *soft left (right, lateral) hyperideal* over H , if $\widehat{f}((H, E), (H, E), (F, A)) \subseteq (F, A)$. A soft set (F, A) over H is called a *soft hyperideal* over H , if it is a soft left, soft right and a soft lateral hyperideal over H .

Proposition 5.4. A soft set (F, A) over H is a soft left (right, lateral) hyperideal over H if and only if for all $a \in A$, $F(a) \neq \emptyset$ is a left (right, lateral) hyperideal of H .

Proof. Let us suppose that (F, A) is a soft left hyperideal over H . We show that $F(a) \neq \emptyset$ is a left hyperideal of H . By the definition we have

$$\widehat{f}((H, E), (H, E), (F, A)) = (K, E \cap E \cap A) = (K, A).$$

where $f(H(a), H(a), F(a)) = K(a)$, for all $a \in A$. That is, $f(H, H, F(a)) = K(a)$. Since $\widehat{f}((H, E), (H, E), (F, A)) \subseteq (F, A)$, where $(K, A) \subseteq (F, A)$. Thus we have, $(K(a) \subseteq F(a)$ for all $a \in A$. Therefore, $f(H, H, F(a)) \subseteq F(a)$. This shows that $F(a)$ is a left hyperideal of H .

Conversely, let us assume that $F(a) \neq \emptyset$ is a left hyperideal of H . We will show that (F, A) is a soft left hyperideal over H . By the definition

$$\widehat{f}((H, E), (H, E), (F, A)) = (K, E \cap E \cap A) = (K, A)$$

where $f(H(a), H(a), F(a)) = K(a)$, for all $a \in A$. That is, $f(H, H, F(a)) = K(a)$. But $f(H, H, F(a)) \subseteq F(a)$. This implies $K(a) \subseteq F(a)$ and so $(K, A) \subseteq (F, A)$. Thus $\widehat{f}((H, E), (H, E), (F, A)) \subseteq (F, A)$. Hence, (F, A) is a soft left hyperideal over H . ■

The soft hyperideal over H defined above is different from the soft hyperideal of a soft ternary semihypergroup. The following example shows this.

Example 5.5. Let $H = \{a, b, c, d, e, f\}$ and $f(x, y, z) = (x \circ y) \circ z$ for all $x, y, z \in H$ with the hyperoperation $\circ$ given by the following table:

$\circ$	0	a	b	c	d	e	f
0	0	0	0	0	0	0	0
a	0	a	$\{a, b\}$	c	$\{c, d\}$	e	$\{e, f\}$
b	0	b	b	d	d	f	f
c	0	c	$\{c, d\}$	c	$\{c, d\}$	c	$\{c, d\}$
d	0	d	d	d	d	d	d
e	0	e	$\{e, f\}$	c	$\{c, d\}$	e	$\{e, f\}$
f	0	f	f	d	d	f	f

Then (H, f) is a ternary semihypergroup. Let we consider the soft set (F, H) , where $F : H \rightarrow \mathcal{P}(H)$ is defined as $F(0) = \{0\}$, $F(a) = \{0, d\}$, $F(b) = \{0, b, d, f\}$, $F(c) = \{0, c, d\}$, $F(d) = \{0, d, f\}$, $F(e) = \{0, e, f\}$, $F(f) = \{0, a, b\}$. It is clear that (F, H) is a soft ternary semihypergroup over H . Let we consider now the soft set $(G, \{b\})$ in which $G : \{b\} \rightarrow \mathcal{P}(H)$ is defined as $G(b) = \{0, d\}$. As $\{b\} \subseteq H$ and $G(b)$ is a hyperideal of $F(b)$, therefore $(G, \{b\})$ is a soft hyperideal of (F, H) . But $G(b) = \{0, d\}$ is not a hyperideal over H , so $(G, \{b\})$ is not a soft hyperideal over H .

Proposition 5.6. *Let (F, A) and (G, B) be any two soft hyperideals over a ternary semihypergroup H , with $A \cap B \neq \emptyset$. Then $(F, A) \cap_R (G, B)$ is a soft hyperideal over H contained in both (F, A) and (G, B) .*

Proof. Straightforward. ■

Proposition 5.7. *Let (F, A) and (G, B) be any two soft hyperideals over a ternary semihypergroup H . Then $(F, A) \cup_E (G, B)$ is a soft hyperideal over H containing both (F, A) and (G, B) .*

Proof. By the definition, $(K, C) = (F, A) \cup_E (G, B)$, where $C = A \cup B$ for all $c \in C = A \cup B$, either $c \in A \setminus B$ or $c \in B \setminus A$ or $c \in A \cap B$. If $c \in A \setminus B$, then $K(c) = F(c)$ if $c \in B \setminus A$, then $K(c) = G(c)$, and if $c \in A \cap B$, then $K(c) = F(c) \cup G(c)$, in all the cases $K(c)$ is a hyperideal of H . Hence, (K, C) is soft hyperideal over H . Since $A \subseteq A \cup B$, $B \subseteq A \cup B$ and $F(c) \subseteq K(c)$, $G(c) \subseteq K(c)$ for all $c \in C$. Therefore, by the definition of soft subsets $(F, A) \subseteq (K, C)$ and $(G, B) \subseteq (K, C)$. ■

Proposition 5.8. *Let (F, A) and (G, B) be any two soft hyperideals over a ternary semihypergroup H . Then $(F, A) \wedge (G, B)$ is a soft hyperideal over H .*

Proof. Straightforward. ■

Proposition 5.9. *Let (F, A) and (G, B) be any two soft hyperideals over a ternary semihypergroup H . Then $(F, A) \vee (G, B)$ is a soft hyperideal over H .*

Proof. Straightforward. ■

Proposition 5.10. *Let (F, A) be a soft ternary semihypergroup over H and let $\{(K_i, A_i) | i \in I\}$ be a non-empty family of soft hyperideals of (F, A) . Then the following hold:*

1. $\cap_{R_{i \in I}}(K_i, A_i)$ is a soft hyperideal of (F, A) .
2. $\wedge_{i \in I}(K_i, A_i)$ is a soft hyperideal of $\wedge_{i \in I}(F, A)$.
3. $\cup_{R_{i \in I}}(K_i, A_i)$ is a soft hyperideal of (F, A) .
4. $\vee_{i \in I}(K_i, A_i)$ is a soft hyperideal of $\vee_{i \in I}(F, A)$.

Proof. Straightforward. ■

6. Soft quasi-hyperideals over ternary semihypergroups

Definition 6.1. A soft set (F, A) over a ternary semihypergroup H is called a *soft quasi-hyperideal* over H if

1. $\widehat{f}((F, A), (H, E), (H, E)) \cap_R \widehat{f}((H, E), (F, A), (H, E))$
 $\cap_R \widehat{f}((H, E), (H, E), (F, A)) \subseteq (F, A)$.
2. $\widehat{f}((F, A), (H, E), (H, E)) \cap_R \widehat{f}((H, E), (H, E), (F, A), (H, E), (H, E))$
 $\cap_R \widehat{f}((H, E), (H, E), (F, A)) \subseteq (F, A)$.

where (H, E) is the absolute soft set over H .

Proposition 6.2. *A soft set (F, A) over a ternary semihypergroup H is a soft quasi-hyperideal over H if and only if for all $a \in A$, $F(a) \neq \emptyset$ is a quasi-hyperideal of H .*

Proof. Let us suppose that a soft set (F, A) over H is a soft quasi-hyperideal over H . We show that $F(a)$ is a quasi-hyperideal of H . By the definition of restricted product,

- (1) $\widehat{f}((F, A), (H, E), (H, E)) = (G, A \cap E \cap E) = (G, A)$
- (2) $\widehat{f}((H, E), (F, A), (H, E)) = (S, E \cap A \cap E) = (S, A)$
- (3) $\widehat{f}((H, E), (H, E), (F, A)) = (I, E \cap E \cap A) = (I, A)$
- (4) $\widehat{f}((H, E), (H, E), (F, A), (H, E), (H, E)) = (J, E \cap E \cap A \cap E \cap E) = (J, A)$

Equations (1), (2), (3) imply that

- (5) $\widehat{f}((F, A), (H, E), (H, E)) \cap_R \widehat{f}((H, E), (F, A), (H, E))$
 $\cap_R \widehat{f}((H, E), (H, E), (F, A)) = (G, A) \cap_R (S, A) \cap_R (I, A)$
- (6) $\widehat{f}((F, A), (H, E), (H, E)) \cap_R \widehat{f}((H, E), (F, A), (H, E))$
 $\cap_R \widehat{f}((H, E), (H, E), (F, A)) = (K, A)$

But (F, A) is a soft quasi-hyperideal over H . Thus $\widehat{f}((F, A), (H, E), (H, E)) \cap_R \widehat{f}((H, E), (F, A), (H, E)) \cap_R \widehat{f}((H, E), (H, E), (F, A)) \subseteq (F, A)$. From equation (6), $(K, A) \subseteq (F, A)$, that is, $K(a) \subseteq F(a)$ for all $a \in A$. Again, equation (6) implies that, $\forall a \in A$,

$$f(F(a), H(a), H(a)) \cap f(H(a), F(a), H(a)) \cap f(H(a), H(a), F(a)) = K(a).$$

This implies that, $\forall a \in A$,

$$(7) \quad f(F(a), H(a), H(a)) \cap f(H(a), F(a), H(a)) \cap f(H(a), H(a), F(a)) \subseteq F(a).$$

Similarly, from equations (1), (3), (4), we have

$$(8) \quad \widehat{f}((F, A), (H, E), (H, E)) \cap_R \widehat{f}((H, E), (H, E), (F, A), (H, E), (H, E)) \cap_R \widehat{f}((H, E), (H, E), (F, A)) = (P, A).$$

Since (F, A) is a soft quasi-hyperideal over H , we have

$$\widehat{f}((F, A), (H, E), (H, E)) \cap_R \widehat{f}((H, E), (H, E), (F, A), (H, E), (H, E)) \cap_R \widehat{f}((H, E), (H, E), (F, A)) \subseteq (F, A).$$

Therefore, $(P, A) \subseteq (F, A)$, that is, for all $a \in A$, $P(a) \subseteq F(a)$. From equation (8), we have for all $a \in A$,

$$f(F(a), H(a), H(a)) \cap f(H(a), H(a), F(a), H(a), H(a)) \cap f(H(a), H(a), F(a)) = P(a).$$

This implies that

$$(9) \quad f(F(a), H(a), H(a)) \cap f(H(a), H(a), F(a), H(a), H(a)) \cap f(H(a), H(a), F(a)) \subseteq F(a).$$

From equations (7) and (9), it is clear that $F(a)$ is a quasi-hyperideal of H .

Conversely, let $F(a) \neq \emptyset$ be a quasi-hyperideal of H for all $a \in A$. We will show that (F, A) is a soft quasi-hyperideal over H . From equations (1), (2), (3) we have

$$\widehat{f}((F, A), (H, E), (H, E)) \cap_R \widehat{f}((H, E), (F, A), (H, E)) \cap_R \widehat{f}((H, E), (H, E), (F, A)) = (G, A) \cap_R (S, A) \cap_R (I, A) = (K, A).$$

By the definition, $\forall a \in A$, we have

$$f(F(a), H(a), H(a)) \cap f(H(a), F(a), H(a)) \cap f(H(a), H(a), F(a)) = K(a).$$

But $F(a)$ is a quasi-hyperideal of H . Therefore,

$$K(a) = f(F(a), H(a), H(a)) \cap f(H(a), F(a), H(a)) \cap f(H(a), H(a), F(a)) \subseteq F(a),$$

and so, $(K, A) \subseteq (F, A)$. Thus

$$(10) \quad \widehat{f}((F, A), (H, E), (H, E)) \cap_R \widehat{f}((H, E), (F, A), (H, E)) \cap_R \widehat{f}((H, E), (H, E), (F, A)) \subseteq (F, A).$$

From equations (1), (3), and (4) we have for, all $a \in A$,

$$\begin{aligned} & \widehat{f}((F, A), (H, E), (H, E)) \cap_R \widehat{f}((H, E), (H, E), (F, A), (H, E), (H, E)) \\ & \cap_R \widehat{f}((H, E), (H, E), (F, A)) = (G, A) \cap_R (I, A) \cap_R (J, A) = (P, A) \end{aligned}$$

and also

$$\begin{aligned} & f(F(a), H(a), H(a)) \cap f(H(a), H(a), F(a), H(a), H(a)) \\ & \cap f(H(a), H(a), F(a)) = P(a). \end{aligned}$$

Since $F(a)$ is a quasi-hyperideal of H , so we have

$$\begin{aligned} P(a) &= f(F(a), H(a), H(a)) \cap f(H(a), H(a), F(a), H(a), H(a)) \\ &\cap f(H(a), H(a), F(a)) \subseteq F(a) \end{aligned}$$

and thus $(P, A) \subseteq (F, A)$. Hence we have

$$\begin{aligned} (11) \quad & \widehat{f}((F, A), (H, E), (H, E)) \cap_R \widehat{f}((H, E), (H, E), (F, A), (H, E), (H, E)) \\ & \cap_R \widehat{f}((H, E), (H, E), (F, A)) \subseteq (F, A). \end{aligned}$$

From equations (10) and (11), (F, A) is a soft quasi-hyperideal over H . ■

Proposition 6.3. *Let (R, A) , (L, B) and (M, C) be soft right, soft left and soft lateral hyperideals over H , respectively. Then $(R, A) \cap_R (M, C) \cap_R (L, B)$ is a soft quasi-hyperideal over H .*

Proof. It is straightforward. ■

Proposition 6.4. *Let (R, A) , (L, B) and (M, C) be soft right, soft left and soft lateral hyperideals over H , respectively, such that $A \cap B \cap C = \emptyset$. Then $(R, A) \cap_E (M, C) \cap_E (L, B)$ is a soft quasi-hyperideal over H .*

Proof. By definition, $(S, D) = (R, A) \cap_E (M, C) \cap_E (L, B)$, where $D = A \cup B \cup C$, $A \cap B \cap C = \emptyset$, and

$$S(d) = \begin{cases} R(d) & \text{if } d \in A \setminus B \cap C \\ M(d) & \text{if } d \in C \setminus A \cap B, \\ L(d) & \text{if } d \in B \setminus A \cap C. \end{cases}$$

for any $d \in D$. In each case, $S(d)$ is a quasi-hyperideal of H . Since every left, right and lateral hyperideal of a ternary semihypergroup H is a quasi-hyperideal of H , thus, by definition, $(S, D) = (R, A) \cap_E (M, C) \cap_E (L, B)$ is a soft quasi-hyperideal over H . ■

Proposition 6.5. *Every soft left (right, lateral) hyperideal over a ternary semihypergroup H is a soft quasi-hyperideal over H .*

Proof. Let (L, A) be a soft left hyperideal over H . Then $L(a)$ is a left hyperideal of H . Since each left hyperideal of H is a quasi-hyperideal of H , therefore $L(a)$ is a quasi-hyperideal of H . Hence (L, A) is a soft quasi-hyperideal over H . ■

The converse of the above proposition is not true in general as the following example shows.

Example 6.6. Let we consider the ternary semihypergroup (H, f) of the Example 5.5. Let we consider $A = \{b\}$ and $G : \{b\} \rightarrow \mathcal{P}(H)$ defined as $G(b) = \{0, b, d, f\}$. Then it can be easily verified that (G, B) is a soft quasi-hyperideal over H . But it is not a soft left and a soft lateral hyperideal over H .

Proposition 6.7. *Every soft left (right, lateral) hyperideal over a ternary semihypergroup H is a soft ternary semihypergroup over H .*

Proof. It is straightforward. ■

Proposition 6.8. *Every soft quasi-hyperideal is a soft ternary semihypergroup over H .*

Proof. It is straightforward. ■

Proposition 6.9. *Let (R, A) , (L, B) and (M, C) be soft right, soft left and soft lateral hyperideals over H , respectively. Then $(R, A) \wedge (M, C) \wedge (L, B)$ is a soft quasi-hyperideal over H .*

Proof. By definition, $(S, D) = (R, A) \wedge (M, C) \wedge (L, B)$, where $D = A \times C \times B$, and for any $(a, c, b) \in A \times C \times B$, $S(a, c, b) = R(a) \cap M(c) \cap L(b)$ is a quasi-hyperideal of H . Since the intersection of a left, right and a lateral hyperideal is a quasi-hyperideal of H , then $(R, A) \wedge (M, C) \wedge (L, B)$ is a soft quasi-hyperideal over H . ■

Proposition 6.10. *Let (F, A) and (G, B) be two soft quasi-hyperideals over a ternary semihypergroup H . Then the following hold:*

1. $(F, A) \cap_R (G, B)$ is a soft quasi-hyperideal over H .
2. $(F, A) \cap_E (G, B)$ is a soft quasi-hyperideal over H .
3. $(F, A) \wedge (G, B)$ is a soft quasi-hyperideal over H .
4. $(F, A) \cup_E (G, B)$ is a soft quasi-hyperideal over H , whenever $A \cap B = \emptyset$.

Proof. It is straightforward. ■

Proposition 6.11. *Let (F, A) be a soft quasi-hyperideal and (G, B) a soft ternary semihypergroup H . Then $(F, A) \cap_R (G, B)$ is a soft quasi-hyperideal of (G, B) .*

Proof. By definition, $(S, C) = (F, A) \cap_R (G, B)$, where $C = A \cap B \neq \emptyset$ and $S(c) = F(c) \cap G(c)$ for all $c \in C$, since $S(c) \subseteq F(c)$ and $S(c) \subseteq G(c)$. We show that $S(c)$ is a quasi-hyperideal of $G(c)$. Since $S(c) \subseteq G(c)$,

$$\begin{aligned} & f(S(c), G(c), G(c)) \cap f(G(c), S(c), G(c), G(c)) \cap f(G(c), G(c), S(c)) \\ & \subseteq f(G(c), G(c), G(c)) \cap f(G(c), G(c), G(c), G(c)) \cap f(G(c), G(c), G(c)) \\ & \subseteq f(G(c), G(c), G(c)) \subseteq G(c) \end{aligned}$$

because $G(c)$ is a ternary subsemihypergroup of H . This implies that

$$(1) \quad f(S(c), G(c), G(c)) \cap f(G(c), S(c), G(c), G(c)) \cap f(G(c), G(c), S(c)) \subseteq G(c).$$

Also $S(c) \subseteq F(c)$. So

$$\begin{aligned} & f(S(c), G(c), G(c)) \cap f(G(c), S(c), G(c), G(c)) \cap f(G(c), G(c), S(c)) \\ & \subseteq f(F(c), G(c), G(c)) \cap f(G(c), F(c), G(c), G(c)) \cap f(G(c), G(c), F(c)) \\ & \subseteq f(F(c), H(c), H(c)) \\ & \cap f(H(c), F(c), H(c), H(c)) \cap f(H(c), H(c), F(c)) \subseteq F(c) \end{aligned}$$

because $F(c)$ is a quasi-hyperideal of H . Thus

$$(2) \quad f(S(c), G(c), G(c)) \cap f(G(c), S(c), G(c), G(c)) \cap f(G(c), G(c), S(c)) \subseteq F(c).$$

From equations (1) and (2), we have

$$(3) \quad f(S(c), G(c), G(c)) \cap f(G(c), S(c), G(c), G(c)) \cap f(G(c), G(c), S(c)) \subseteq F(c) \cap G(c) = S(c).$$

Similarly, we can show that

$$(4) \quad f(S(c), G(c), G(c)) \cap f(G(c), G(c), S(c), G(c), G(c)) \cap f(G(c), G(c), S(c)) \subseteq S(c).$$

From equation (3) and (4), $S(c)$ is a quasi-hyperideal of $G(c)$. Thus $(F, A) \cap_R (G, B)$ is a soft quasi-hyperideal of (G, B) . ■

7. Soft bi-hyperideals over ternary semihypergroups

Definition 7.1. A soft set (F, A) over a ternary semihypergroup H is called a *soft bi-hyperidea* over H if

1. (F, A) is a soft ternary semihypergroup over H .
2. $\widehat{f}((F, A), (H, E), (F, A), (H, E), (F, A)) \subseteq (F, A)$ where (H, E) is the absolute soft set over H .

Proposition 7.2. A soft set (F, A) over a ternary semihypergroup H is a soft bi-hyperideal over H if and only if for all $a \in A$, $F(a) \neq \emptyset$ is a bi-hyperideal of H .

Proof. Let (F, A) be a soft bi-hyperideal over a ternary semihypergroup H . Then by definition, (F, A) is a soft ternary semihypergroup over H . By Proposition 4.3, for any $a \in A$, $F(a) \neq \emptyset$ is a ternary subsemihypergroup of H . Moreover, since (F, A) is a soft bi-hyperideal over H , we have

$\widehat{f}((F, A), (H, E), (F, A), (H, E), (F, A)) \subseteq (F, A)$ where (H, E) is the absolute soft set over H . It follows that $\widehat{f}(F(a), H, F(a), H, F(a)) \subseteq F(a)$, which shows that $F(a)$ is a bi-hyperideal of H .

Conversely, let us suppose that (F, A) is a soft set over H such that for all $a \in A$, $F(a)$ is a bi-hyperideal of H , whenever $F(a) \neq \emptyset$. Then it is clear that

each $F(a) \neq \emptyset$ is a ternary subsemihypergroup of H . Hence, by Proposition 4.3, (F, A) is a soft ternary semihypergroup over H . Furthermore, since $F(a) \neq \emptyset$ is a bi-hyperideal of H , then for all $a \in A$, $\widehat{f}(F(a), H, F(a), H, F(a)) \subseteq F(a)$. Hence, we conclude that $\widehat{f}((F, A), (H, E), (F, A), (H, E), (F, A)) \subseteq (F, A)$. This shows that (F, A) is a soft bi-hyperideal over H . ■

Proposition 7.3. *Every soft quasi-hyperideal over a ternary semihypergroup H is a soft bi-hyperideal over H .*

Proof. It is straightforward. ■

Proposition 7.4. *Let (F, A) be a soft bi-hyperideal over H and (G, B) a soft ternary semihypergroup over H . Then $(F, A) \cap_R (G, B)$ is a soft bi-hyperideal of (G, B) .*

Proof. By the definition, $(S, C) = (F, A) \cap_R (G, B)$ where $C = A \cap B \neq \emptyset$, and S is defined by $S(c) = F(c) \cap G(c)$ for all $c \in C$. We will show that (S, C) is a soft bi-hyperideal of (G, B) . We have

$$\begin{aligned} & \widehat{f}((S, C), (S, C), (S, C)) \\ &= \widehat{f}(((F, A) \cap_R (G, B)), ((F, A) \cap_R (G, B)), ((F, A) \cap_R (G, B))) \\ &\subseteq \widehat{f}((F, A), (F, A), (F, A)) \subseteq (F, A), \end{aligned}$$

because (F, A) is a soft bi-hyperideal over H . This implies that

$$(1) \quad \widehat{f}((S, C), (S, C), (S, C)) \subseteq (F, A)$$

Also

$$\begin{aligned} & \widehat{f}((S, C), (S, C), (S, c)) \\ &= \widehat{f}(((F, A) \cap_R (G, B)), ((F, A) \cap_R (G, B)), ((F, A) \cap_R (G, B))) \\ &\subseteq \widehat{f}((G, B), (G, B), (G, B)) \subseteq (G, B), \end{aligned}$$

because (G, B) is a soft ternary semihypergroup over H . This implies that

$$(2) \quad \widehat{f}((S, C), (S, C), (S, C)) \subseteq (G, B).$$

From equations (1) and (2), we have

$$\widehat{f}((S, C), (S, C), (S, C)) \subseteq (F, A) \cap_R (G, B) = (S, C).$$

This implies that (S, C) is a soft ternary semihypergroup over H . Also

$$\begin{aligned} & \widehat{f}((S, C), (G, B), (S, C), (S, C)) \\ &= \widehat{f}(((F, A) \cap_R (G, B)), (G, B), ((F, A) \cap_R (G, B)), (G, B), ((F, A) \cap_R (G, B))) \\ &\subseteq \widehat{f}((G, B), (G, B), (G, B)) \\ &\subseteq \widehat{f}((G, B), (G, B), (G, B)) \subseteq (G, B), \end{aligned}$$

and so

$$(3) \quad \widehat{f}((S, C), (G, B), (S, C), (S, C)) \subseteq (G, B).$$

Again

$$\begin{aligned}
 & \widehat{f}((S, C), (G, B), (S, C), (S, C)) \\
 &= \widehat{f}(((F, A) \cap_R (G, B)), (G, B), ((F, A) \\
 & \cap_R (G, B)), (G, B), ((F, A) \cap_R (G, B))) \\
 &\subseteq \widehat{f}((F, A), (H, E), (F, A)) \\
 &\subseteq (F, A),
 \end{aligned}$$

because (F, A) is a soft bi-hyperideal over H . This implies that

$$(4) \quad \widehat{f}((S, C), (G, B), (S, C), (S, C)) \subseteq (F, A).$$

From equations (3) and (4), we have

$$\widehat{f}((S, C), (G, B), (S, C), (S, C)) \subseteq (F, A) \cap_R (G, B) = (S, C).$$

Hence (S, C) is a soft bi-hyperideal of (G, B) . ■

Definition 7.5. A soft hyperideal (F, A) over a ternary semihypergroup H is soft idempotent if $\widehat{f}((F, A), (F, A), (F, A)) = (F, A)$.

Theorem 7.6. Let (F, A) be a soft bi-hyperideal over H and (G, B) a soft bi-hyperideal of (F, A) such that $\widehat{f}((G, B), (G, B), (G, B)) = (G, B)$. Then (G, B) is a soft bi-hyperideal over H .

Proof. By the condition, we have $\widehat{f}((G, B), (G, B), (G, B)) \subseteq (G, B)$. This implies that (G, B) is a soft ternary semihypergroup over H . Since $(G, B) \subseteq (F, A)$ and $\widehat{f}((G, B), (G, B), (G, B)) = (G, B)$, we have

$$\begin{aligned}
 & \widehat{f}((G, B), (H, E), (G, B), (H, E), (G, B)) \\
 &= \widehat{f}(\widehat{f}((G, B), (G, B), (G, B)), (H, E), (G, B), (H, E), \widehat{f}((G, B), (G, B), \dots, (G, B))) \\
 &= \widehat{f}((G, B), (G, B), \widehat{f}((G, B)), (H, E), (G, B), (H, E), (G, B)), (G, B), (G, B)) \\
 &\subseteq \widehat{f}((G, B), (G, B), \widehat{f}((F, A)), (H, E), (F, A), (H, E), (F, A)), (G, B), (G, B)) \\
 &\subseteq \widehat{f}((G, B), (G, B), (F, A), (G, B), (G, B)) \text{ since } (F, A) \text{ is a soft bi-hyperideal of } H \\
 &= \widehat{f}((G, B), (G, B), \dots, (F, A), (G, B), \widehat{f}((G, B), (G, B), (G, B))) \\
 &= \widehat{f}((G, B), \widehat{f}((G, B), (F, A), (G, B), (G, B), (G, B), (G, B)), (G, B)) \\
 &\subseteq \widehat{f}((G, B), \widehat{f}((G, B), (F, A), (G, B), (G, B)), (G, B)) \\
 &\subseteq \widehat{f}((G, B), (G, B), (G, B)) = (G, B)
 \end{aligned}$$

because (G, B) is a soft bi-hyperideal of (F, A) . This implies that

$$\widehat{f}((G, B), (H, E), (G, B), (H, E), (G, B)) \subseteq (G, B).$$

Hence, (G, B) is a soft bi-hyperideal over H . ■

Proposition 7.7. *Let (F, A) be a non-empty soft subset over a ternary semihypergroup H . If $(G_i, B_i), i = 1, 2, 3$ are soft left, lateral and right hyperideals respectively, over H such that*

$$\widehat{f}((G_1, B_1), (G_2, B_2), (G_3, B_3)) \subseteq (F, A) \subseteq (G_1, B_1) \cap_R (G_2, B_2) \cap_R (G_3, B_3),$$

then (F, A) is a soft bi-hyperideal over H .

Proof. By definition,

$$\begin{aligned} \widehat{f}((F, A), (F, A), (F, A)) &\subseteq \widehat{f}((G_1, B_1) \cap_R (G_2, B_2) \cap_R (G_3, B_3), (G_1, B_1) \\ &\cap_R (G_2, B_2) \cap_R (G_3, B_3), (G_1, B_1) \cap_R (G_2, B_2) \cap_R (G_3, B_3)) \\ &\subseteq \widehat{f}((G_1, B_1), (G_2, B_2), (G_3, B_3)) \subseteq (F, A). \end{aligned}$$

This implies that $\widehat{f}((F, A), (F, A), (F, A)) \subseteq (F, A)$. Thus (F, A) is a soft ternary semihypergroup over H . Let us consider again

$$\begin{aligned} \widehat{f}((F, A), (H, E), (F, A), (H, E), (F, A)) &\subseteq \widehat{f}((G_1, B_1) \cap_R (G_2, B_2) \cap_R (G_3, B_3), \\ &(H, E), (G_1, B_1) \cap_R (G_2, B_2) \cap_R (G_3, B_3), (H, E), (G_1, B_1) \cap_R (G_2, B_2) \cap_R (G_3, B_3)) \\ &\subseteq \widehat{f}((G_1, B_1), (H, E), (G_2, B_2), (H, E), (G_3, B_3)) \\ &\subseteq \widehat{f}((G_1, B_1), (G_2, B_2), (G_3, B_3)) \subseteq (F, A) \end{aligned}$$

because (G_2, B_2) is a soft lateral hyperideal over H . This implies that

$$\widehat{f}((F, A), (H, E), (F, A), (H, E), (F, A)) \subseteq (F, A).$$

Hence, (F, A) is a soft bi-hyperideal over H . ■

Proposition 7.8. *Let (F, A) and (G, B) be two non-empty soft sets over a ternary semihypergroup H . Then the soft set $(S, C) = \widehat{f}((F, A), (H, E), (G, B))$ is a soft bi-hyperideal over H .*

Proof. By definition, we have

$$\begin{aligned} \widehat{f}((S, C), (S, C), (S, C)) &= \widehat{f}(\widehat{f}((F, A), (H, E), (G, B)), \widehat{f}((F, A), (H, E), (G, B)), \widehat{f}((F, A), (H, E), (G, B))) \\ &= \widehat{f}((F, A), \widehat{f}((H, E), (G, B), (F, A)), \widehat{f}((H, E), (G, B), (F, A)), (H, E), (G, B)) \\ &\subseteq \widehat{f}((F, A), \widehat{f}((H, E)), (H, E), (H, E)), \widehat{f}((H, E)), (H, E), (H, E)), (H, E), (G, B)) \\ &\subseteq \widehat{f}((F, A), \widehat{f}((H, E)), (H, E), (H, E)), (G, B)) \subseteq \widehat{f}((F, A), (H, E), (G, B)) = (S, C). \end{aligned}$$

This implies that $\widehat{f}((S, C), (S, C), (S, C)) \subseteq (S, C)$. Thus, (S, C) is a soft ternary semihypergroup over H . Also, we have

$$\begin{aligned} \widehat{f}((S, C), (H, E), (S, C), (H, E), (S, C)) &= \widehat{f}(\widehat{f}((F, A), (H, E), (G, B)), (H, E), \widehat{f}((F, A), (H, E), (G, B)), (H, E), \\ &\widehat{f}((F, A), (H, E), (G, B))) \\ &= \widehat{f}((F, A), \widehat{f}((H, E), (G, B), (H, E)), \widehat{f}((F, A), (H, E), (G, B)), \\ &\widehat{f}((H, E), (F, A), (H, E)), (G, B)) \end{aligned}$$

$$\begin{aligned}
&\subseteq \widehat{f}((F, A), \widehat{f}((H, E)), (H, E), (H, E)), \widehat{f}((H, E)), (H, E), (H, E)), \\
&\quad \widehat{f}((H, E)), (H, E), (H, E)), (G, B)) \\
&\subseteq \widehat{f}((F, A), \widehat{f}((H, E)), (H, E), (H, E)), (G, B)) \\
&\subseteq \widehat{f}((F, A)), (H, E), (G, B)) = (S, C).
\end{aligned}$$

This implies that $\widehat{f}((S, C)), (H, E), (S, C), (H, E), (S, C)) \subseteq (S, C)$. Hence (S, C) is a soft bi-hyperideal over H . ■

8. Conclusion

Molodtsov introduced the concept of soft set theory, which can be seen and used as a new mathematical tool for dealing with uncertainty. In the present paper, we introduce and initiate the study of soft ternary semihypergroups by using soft sets. The notions of soft ternary semihypergroups, soft ternary subsemihypergroups, soft left (lateral, right) hyperideals, soft hyperideals, soft quasi-hyperideals and soft bi-hyperideals are introduced here, and several related properties are investigated. Based on these results, we could apply soft sets to other types of hyperideals in ternary semihypergroups and do some further work on the properties of soft ternary semihypergroups. Moreover, one may characterize several classes of ternary semihypergroups such as regular ternary semihypergroups by the properties of soft hyperideals (quasi and bi)-hyperideals. Also, by using soft set theory, one may consider other soft algebraic hyperstructures such as soft ternary semihyperrings etc.

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A NEW CHARACTERIZATION OF SPORADIC SIMPLE GROUPS¹**Li-Guan He**

*School of Mathematics
Chongqing Normal University
400047, Chongqing
P.R. China*

Gui-Yun Chen²**Hai-Jing Xu**

*School of Mathematics and Statistics
Southwest University
400715, Chongqing
P.R. China*

Abstract. Let G be a finite group, $k_1(G)$ denote the largest element order of G , and $k_2(G)$, the second largest element order. In this paper, we show that each sporadic simple group G can be uniquely determined by the order of G and $k_i(G)$, where $i \leq 2$.

Key words: finite group, the largest element order, the second largest element order, sporadic simple group, characterization.

AMS Subject Classification: 20D05, 20D08, 20D60

1. Introduction

It is a well-known topic to characterize a finite simple group by using two quantities, the order of G and $\pi_e(G)$ in the past 30 years, where $\pi_e(G)$ denotes the set of orders of elements in G . W.J. Shi characterized some finite simple groups by using $\pi_e(G)$ and $|G|$, for example, see [1]–[6]. Recently, this topic has been finished by V.D. Mazurov, et al. (See [7]). Now the authors will try to characterize some finite simple groups by using less quantities and have successfully characterized simple K_3 -groups, some K_4 -groups by using three numbers: the order of a group, the largest and the second largest element orders. In this paper, we characterize sporadic simple groups via the order of a group and the largest and the second largest element orders.

Notations. The groups mentioned are all finite groups, the number in bracket “()” behind a group is the order of the group, e.g., $L_2(7)(2^3 \cdot 3 \cdot 7)$ means that

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²Corresponding author. E-mail address: guanlihe@126.com and gychen1963@163.com.

$L_2(7)$ is of order $2^3 \cdot 3 \cdot 7$. Let $\pi_e(G)$ denote the set of orders of elements in G , $\pi(G)$ denote the set of all prime divisors of $|G|$, $k_1(G)$, the largest element order of G , and $k_2(G)$, the second largest element order. S_p -subgroup is a Sylow p -subgroup of G . We denote by $\Gamma(G)$ the prime graph of G and $t(G)$ is the number of connected components of $\Gamma(G)$. And we also denote the sets of vertex of the connected components of the prime graph by $\{\pi_i, i = 1, \dots, t(G)\}$, for convenience we call Pi_i ($1 \leq i \leq t(G)$) the connected components and if the order of G is even, denote the component containing 2 by π_1 (see [8]).

In this paper, we come to the following theorems:

Theorem 1. *Let G be a group and A be one of the following sporadic simple groups: M_{11} , M_{12} , J_1 , M_{22} , M_{23} , HS , J_3 , M_{24} , He , Ru , Suz , $O'N$, Co_3 , Co_2 , Ly , Th , J_4 , B , M . Then $G \cong A$ if and only if*

- (i) $k_1(G) = k_1(A)$;
- (ii) $|G| = |A|$.

For other sporadic simple groups, we have

Theorem 2. *Let G be a group and A be one of the following sporadic simple groups: J_2 , McL , Fi_{22} , HN , Fi_{23} , Co_1 , Fi_{24} . Then $G \cong A$ if and only if*

- (i) $k_i(G) = k_i(A)$, where $i = 1, 2$;
- (ii) $|G| = |A|$.

2. Preliminary results

Lemma 1. *Suppose that G has more than one prime graph component. Then one of the following holds:*

- (1) G is a Frobenius group or a 2 – Frobenius group;
- (2) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1 -groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |\text{Out}(K/H)|$.

Proof. The lemma follows from theorem A and Lemma 3 in [8].

Remark. A group G will be called a 2 – Frobenius group provided G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that G/H and K are Frobenius groups with K/H and H as their Frobenius kernels respectively.

Lemma 2. *If G is a Frobenius group of even order with K the Frobenius kernel and H the Frobenius complement, then $t(G) = 2$ and $\Gamma(G) = \{\pi(H), \pi(K)\}$ (see [9]).*

Lemma 3. *If G is a 2-Frobenius group of even order, then $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that $\pi(K/H) = \pi_2$, and $\pi(G/K) \cup \pi(H) = \pi_1$. Moreover, G/K and K/H are cyclic groups satisfying that $|G/K| \mid |\text{Aut}(K/H)|$, $(|G/K|, |K/H|) = 1$, and $|G/K| < |K/H|$. Particularly, G is solvable (see [9]).*

Lemma 4. *Let A be a π' -group of automorphisms of the π -group G , and suppose G or A is solvable. Then for each prime p in π , A leaves invariant some S_p -subgroup of G (See [10], Theorem 6.22).*

Lemma 5. *Let k be a positive integer, and $\pi(k)$ denote the set of prime divisors of k . Suppose that t is a positive integer satisfying that $22 \leq t \leq 46$. Then $\pi(2^{2t} - 1) \not\subseteq \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}$.*

Lemma 6. *Let G be a sporadic simple group. Then $|G|$, $k_1(G)$ and $k_2(G)$ are as in Table 1:*

Table 1

G	$ G $	$k_1(G)$	$k_2(G)$
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	11	8
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	11	10
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	11	8
M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	23	15
M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	23	21
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	15	12
Suz	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	24	21
HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	20	15
McL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	30	15
Co_3	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	30	24
Co_2	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	30	28
Co_1	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$	60	42
He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$	28	21
Fi_{22}	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	30	24
Fi_{23}	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$	60	42
Fi_{24}	$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$	60	45
HN	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$	40	35
Th	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$	39	36
B	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$	70	66
M	$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$	119	110
J_1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	19	15
$O'N$	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$	31	28
J_3	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$	19	17
Ly	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$	67	42
Ru	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$	29	26
J_4	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$	66	44

Proof. The lemma follows from [11].

Lemma 7. *Let G be a finite group and A be one of the following sporadic simple groups: M_{11} , M_{12} , J_1 , M_{22} , M_{23} , HS , J_3 , M_{24} , He , Ru , Suz , $O'N$, Co_3 , Co_2 , Ly , Th , J_4 , B , M . Suppose that*

- (i) $k_1(G) = k_1(A)$;
- (ii) $|G| = |A|$.

Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1 -groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$.

Proof. We only need to prove the cases $A = M_{11}$, M_{12} . For the other cases, we can prove them similarly.

1. Assume that $A = M_{11}$ ($2^4 \cdot 3^2 \cdot 5 \cdot 11$). In such case, $|G| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$ and $k_1(G) = 11$. Because $k_1(G) = 11$, it follows that 11 is an isolated point of $\Gamma(G)$, and therefore $t(G) \geq 2$. By Lemma 1, we know that G is either a Frobenius group or a 2-Frobenius group, or has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1 -groups and K/H is a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. Therefore, we only need to prove that G is neither a Frobenius group nor a 2-Frobenius group.

First, we suppose that G is a Frobenius group. Then by Lemma 2 we get that $t(G) = 2$ and $\Gamma(G) = \{ \pi(H), \pi(K) \}$, where K is the Frobenius kernel and H the Frobenius complement. Since $k_1(G) = 11$, K is either a $\{2, 3, 5\}$ -Hall subgroup or a Sylow 11-subgroup of G . Since K is nilpotent, let S be a Sylow subgroup of K , one has that $|H| \mid (|S| - 1)$. We can find a suitable Sylow subgroup of K such that $|H| \nmid (|S| - 1)$, then we get a contradiction. For this reason, K can't be a Sylow 11-subgroup of G . Hence K is a $\{2, 3, 5\}$ -Hall subgroup. Consider the Sylow 5-subgroup of K , we get $11 \nmid 4$, a contradiction. Therefore, G is not a Frobenius group.

Second, we suppose that G is a 2-Frobenius group. By Lemma 3, we know that $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(K/H) = \pi_2$ and $\pi(G/K) \cup \pi(H) = \pi_1$. Moreover, G/K and K/H are cyclic groups satisfying $|G/K| \mid |Aut(K/H)|$, and $|G/K| < |K/H|$. As 11 is an isolated point of $\Gamma(G)$, $\pi_2(G) = \{11\}$. Therefore, $\pi(G/K) \cup \pi(H) = \{2, 3, 5\}$ and $|K/H| = 11$. Since $|G/K| \mid |Aut(K/H)| = 10$, we know that $3 \mid |H|$. Consider the action on H by the element of order 11. By Lemma 4, there exists a Sylow 3-subgroup L of H fixed by this action. Since $|L| = 3^2$, we have $11 \nmid |Aut(L)|$, which means that such action on L is trivial. Therefore, G has an element of order 33, a contradiction. So G is not a 2-Frobenius group.

Remark. This approach can be used to prove that G is not 2-Frobenius group for the most of other cases. For a few exceptions, we only need to consider $\Omega_1(Z(L))$ of some special Sylow subgroup L to lead to a contradiction. The process can be seen in the case $A = M_{12}$.

2. Assume that $A = M_{12}$ ($2^6 \cdot 3^3 \cdot 5 \cdot 11$). In such case, $|G| = 2^6 \cdot 3^3 \cdot 5 \cdot 11$ and $k_1(G) = 11$. Because $k_1(G) = 11$, it follows that 11 is an isolated point of $\Gamma(G)$, and therefore $t(G) \geq 2$. By Lemma 1, we only need to prove that G is neither a Frobenius group nor a 2-Frobenius group.

Using the similar arguments in case $A = M_{11}$, we can easily show that G is not a Frobenius group. Now we assert that G is not a 2-Frobenius group. Assume the contrary. Let G be a 2-Frobenius group. By Lemma 3, $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(K/H) = \pi_2$ and $\pi(G/K) \cup \pi(H) = \pi_1$. Moreover, G/K and K/H are cyclic groups satisfying $|G/K| \mid |Aut(K/H)|$, and $|G/K| < |K/H|$. As 11 is an isolated point of $\Gamma(G)$, $\pi_2(G) = \{11\}$. Therefore, $\pi(G/K) \cup \pi(H) = \{2, 3, 5\}$ and $|K/H| = 11$. Since $|G/K| \mid |Aut(K/H)| = 10$, we know that $3 \mid |H|$. Consider the action on H by the element y of order 11. Again by Lemma 4, there exists a Sylow 3-subgroup L of H fixed by this action. Obviously, $|L| = 3^3$. Clearly, $\Omega_1(Z(L))$ is an elementary abelian 3-group, and $|\Omega_1(Z(L))| \mid 3^3$. Because $\Omega_1(Z(L))$ is characteristic in L , y fixes $\Omega_1(Z(L))$ too. As $11 \nmid |Aut(\Omega_1(Z(L)))|$, the action on $\Omega_1(Z(L))$ by y is trivial, which implies that G has an element of order 33, a contradiction. So G is not a 2-Frobenius group.

Lemma 8. *Let G be a finite group and A be one of the following sporadic simple groups: J_2 , McL , Fi_{22} , HN , Fi_{23} , Co_1 , Fi_{24} . Suppose that*

- (i) $k_i(G) = k_i(A)$, where $i = 1, 2$;
- (ii) $|G| = |A|$.

Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1 -groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$.

Proof. We only need to prove the case $A = J_2$, and for the other cases, they can be proved similarly.

Assume that $A = J_2$ ($2^7 \cdot 3^3 \cdot 5^2 \cdot 7$). In such case, $|G| = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7$, $k_1(G) = 15$ and $k_2(G) = 12$. Because $k_1(G) = 15$, $k_2(G) = 12$, we have 7 is an isolated point of $\Gamma(G)$, and therefore $t(G) \geq 2$. Using the similar arguments in Lemma 7, we can prove that G is neither a Frobenius group nor a 2-Frobenius group. Therefore, the lemma follows from part 2 of Lemma 1.

From [11], we know all the simple groups of order less than 10^{25} , except that the $L_2(q)$, $L_3(q)$, $U_3(q)$, $L_4(q)$, $U_4(q)$, $S_4(q)$, $G_2(q)$ are stopped at orders 10^6 ($q \leq 125$), 10^{12} ($q \leq 31$), 10^{12} ($q \leq 32$), 10^{16} ($q \leq 11$), 10^{16} ($q \leq 11$), 10^{16} ($q \leq 41$), 10^{20} ($q \leq 25$) respectively.

Let $B(q)$ be one of the following Lie-type simple groups: $L_2(q)$, $L_3(q)$, $U_3(q)$, $L_4(q)$, $U_4(q)$, $S_4(q)$, $G_2(q)$. For the need of discussion, we list the orders of $B(q)$ in Table 2.

Table 2

$B(q)$	$L_2(q)$	$L_3(q)$
$ B(q) $	$q(q^2 - 1)/(q - 1, 2)$	$q^3(q^3 - 1)(q^2 - 1)/(q - 1, 3)$
$B(q)$	$L_4(q)$	$U_4(q)$
$ B(q) $	$q^6(q^4 - 1)(q^3 - 1)(q^2 - 1)/(q - 1, 4)$	$q^6(q^4 - 1)(q^3 + 1)(q^2 - 1)/(q + 1, 4)$
$B(q)$	$G_2(q)$	$U_3(q)$
$ B(q) $	$q^6(q^6 - 1)(q^2 - 1)$	$q^3(q^3 + 1)(q^2 - 1)/(q + 1, 3)$
$B(q)$	$S_4(q)$	
$ B(q) $	$q^4(q^4 - 1)(q^2 - 1)/(q - 1, 2)$	

And for convenient discussion, we list the orders of some $L_2(q)$ in Table 3.

Table 3

$L_2(q)$	$ L_2(q) $	$q^2 - 1$
$L_2(2^5)$	$2^5 \cdot 3 \cdot 11 \cdot 31$	$3 \cdot 11 \cdot 31$
$L_2(2^6)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	$3^2 \cdot 5 \cdot 7 \cdot 13$
$L_2(2^7)$	$2^7 \cdot 3 \cdot 43 \cdot 127$	$3 \cdot 43 \cdot 127$
$L_2(2^8)$	$2^8 \cdot 3 \cdot 5 \cdot 17 \cdot 257$	$3 \cdot 5 \cdot 17 \cdot 257$
$L_2(2^9)$	$2^9 \cdot 3^3 \cdot 7 \cdot 19 \cdot 73$	$3^3 \cdot 7 \cdot 19 \cdot 73$
$L_2(2^{10})$	$2^{10} \cdot 3 \cdot 5^2 \cdot 11 \cdot 31 \cdot 41$	$3 \cdot 5^2 \cdot 11 \cdot 31 \cdot 41$
$L_2(2^{11})$	$2^{11} \cdot 3 \cdot 23 \cdot 89 \cdot 683$	$3 \cdot 23 \cdot 89 \cdot 683$
$L_2(2^{12})$	$2^{12} \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 241$	$3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 241$
$L_2(2^{13})$	$2^{13} \cdot 3 \cdot 2731 \cdot 8191$	$3 \cdot 2731 \cdot 8191$
$L_2(2^{14})$	$2^{14} \cdot 3 \cdot 5 \cdot 29 \cdot 43 \cdot 113 \cdot 127$	$3 \cdot 5 \cdot 29 \cdot 43 \cdot 113 \cdot 127$
$L_2(2^{15})$	$2^{15} \cdot 3^2 \cdot 7 \cdot 11 \cdot 31 \cdot 151 \cdot 331$	$3^2 \cdot 7 \cdot 11 \cdot 31 \cdot 151 \cdot 331$
$L_2(2^{16})$	$2^{16} \cdot 3 \cdot 5 \cdot 17 \cdot 257 \cdot 65537$	$3 \cdot 5 \cdot 17 \cdot 257 \cdot 65537$
$L_2(2^{17})$	$2^{17} \cdot 3 \cdot 43691 \cdot 131071$	$3 \cdot 43691 \cdot 131071$
$L_2(2^{18})$	$2^{18} \cdot 3^3 \cdot 5 \cdot 7 \cdot 13 \cdot 19 \cdot 37 \cdot 73 \cdot 109$	$3^3 \cdot 5 \cdot 7 \cdot 13 \cdot 19 \cdot 37 \cdot 73 \cdot 109$
$L_2(2^{19})$	$2^{19} \cdot 3 \cdot 174763 \cdot 524287$	$3 \cdot 174763 \cdot 524287$
$L_2(2^{20})$	$2^{20} \cdot 3 \cdot 5^2 \cdot 11 \cdot 17 \cdot 31 \cdot 41 \cdot 61681$	$3 \cdot 5^2 \cdot 11 \cdot 17 \cdot 31 \cdot 41 \cdot 61681$
$L_2(2^{21})$	$2^{21} \cdot 3^2 \cdot 7^2 \cdot 43 \cdot 127 \cdot 337 \cdot 5419$	$3^2 \cdot 7^2 \cdot 43 \cdot 127 \cdot 337 \cdot 5419$
$L_2(3^5)$	$2^2 \cdot 3^5 \cdot 11^2 \cdot 61$	$2^3 \cdot 11^2 \cdot 61$
$L_2(3^6)$	$2^3 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13 \cdot 73$	$2^4 \cdot 5 \cdot 7 \cdot 13 \cdot 73$
$L_2(3^7)$	$2^2 \cdot 3^7 \cdot 547 \cdot 1093$	$2^3 \cdot 547 \cdot 1093$
$L_2(3^8)$	$2^5 \cdot 3^8 \cdot 5 \cdot 17 \cdot 41 \cdot 193$	$2^6 \cdot 5 \cdot 17 \cdot 41 \cdot 193$
$L_2(3^9)$	$2^2 \cdot 3^9 \cdot 7 \cdot 13 \cdot 19 \cdot 37 \cdot 757$	$2^3 \cdot 7 \cdot 13 \cdot 19 \cdot 37 \cdot 757$
$L_2(3^{10})$	$3^{10} \cdot 2^3 \cdot 5^2 \cdot 11^2 \cdot 61 \cdot 1181$	$2^4 \cdot 5^2 \cdot 11^2 \cdot 61 \cdot 1181$
$L_2(3^{11})$	$3^{11} \cdot 2^2 \cdot 23 \cdot 67 \cdot 661 \cdot 3851$	$2^3 \cdot 23 \cdot 67 \cdot 661 \cdot 3851$
$L_2(3^{12})$	$3^{12} \cdot 2^4 \cdot 5 \cdot 7 \cdot 13 \cdot 41 \cdot 73 \cdot 6481$	$2^5 \cdot 5 \cdot 7 \cdot 13 \cdot 41 \cdot 73 \cdot 6481$
$L_2(3^{13})$	$3^{13} \cdot 2^2 \cdot 398581 \cdot 797261$	$2^3 \cdot 398581 \cdot 797261$
$L_2(3^{14})$	$3^{14} \cdot 2^3 \cdot 5 \cdot 29 \cdot 547 \cdot 1093 \cdot 16493$	$2^4 \cdot 5 \cdot 29 \cdot 547 \cdot 1093 \cdot 16493$
$L_2(3^{15})$	$3^{15} \cdot 2^2 \cdot 7 \cdot 11^2 \cdot 13 \cdot 31 \cdot 61 \cdot 271 \cdot 4561$	$2^3 \cdot 7 \cdot 11^2 \cdot 13 \cdot 31 \cdot 61 \cdot 271 \cdot 4561$
$L_2(3^{16})$	$3^{16} \cdot 2^6 \cdot 5 \cdot 17 \cdot 41 \cdot 193 \cdot 21523361$	$2^7 \cdot 5 \cdot 17 \cdot 41 \cdot 193 \cdot 21523361$

Table 3

$L_2(q)$	$ L_2(q) $	$q^2 - 1$
$L_2(3^{17})$	$3^{17} \cdot 2^2 \cdot 32285041 \cdot 64570081$	$2^3 \cdot 32285041 \cdot 64570081$
$L_2(3^{18})$	$3^{18} \cdot 2^3 \cdot 5 \cdot 7 \cdot 13 \cdot 19 \cdot 37 \cdot 73 \cdot 757 \cdot 530713$	$2^4 \cdot 5 \cdot 7 \cdot 13 \cdot 19 \cdot 37 \cdot 73 \cdot 757 \cdot 530713$
$L_2(3^{19})$	$3^{19} \cdot 2^2 \cdot 290565367 \cdot 581130733$	$2^3 \cdot 290565367 \cdot 581130733$
$L_2(3^{20})$	$3^{20} \cdot 2^4 \cdot 5^2 \cdot 11^2 \cdot 41 \cdot 61 \cdot 1182 \cdot 42521761$	$2^5 \cdot 5^2 \cdot 11^2 \cdot 41 \cdot 61 \cdot 1182 \cdot 42521761$
$L_2(5^4)$	$5^4 \cdot 2^4 \cdot 3 \cdot 13 \cdot 313$	$2^5 \cdot 3 \cdot 13 \cdot 313$
$L_2(5^5)$	$5^5 \cdot 2^2 \cdot 3 \cdot 11 \cdot 71 \cdot 521$	$2^3 \cdot 3 \cdot 11 \cdot 71 \cdot 521$
$L_2(5^6)$	$5^6 \cdot 2^3 \cdot 3^2 \cdot 7 \cdot 13 \cdot 31 \cdot 601$	$2^4 \cdot 3^2 \cdot 7 \cdot 13 \cdot 31 \cdot 601$
$L_2(5^7)$	$5^7 \cdot 2^2 \cdot 3 \cdot 29 \cdot 449 \cdot 19531$	$2^3 \cdot 3 \cdot 29 \cdot 449 \cdot 19531$
$L_2(5^8)$	$5^8 \cdot 2^5 \cdot 3 \cdot 13 \cdot 17 \cdot 313 \cdot 11489$	$2^6 \cdot 3 \cdot 13 \cdot 17 \cdot 313 \cdot 11489$
$L_2(5^9)$	$5^9 \cdot 2^2 \cdot 3^3 \cdot 7 \cdot 5176 \cdot 488281$	$2^3 \cdot 3^3 \cdot 7 \cdot 5176 \cdot 488281$
$L_2(7^3)$	$7^3 \cdot 2^3 \cdot 3^2 \cdot 19 \cdot 43$	$2^4 \cdot 3^2 \cdot 19 \cdot 43$
$L_2(7^4)$	$7^4 \cdot 2^5 \cdot 3 \cdot 5^2 \cdot 1201$	$2^6 \cdot 3 \cdot 5^2 \cdot 1201$
$L_2(7^5)$	$7^5 \cdot 2^3 \cdot 3 \cdot 11 \cdot 191 \cdot 2801$	$2^4 \cdot 3 \cdot 11 \cdot 191 \cdot 2801$
$L_2(7^6)$	$7^6 \cdot 2^4 \cdot 3^2 \cdot 5^2 \cdot 13 \cdot 19 \cdot 43 \cdot 181$	$2^5 \cdot 3^2 \cdot 5^2 \cdot 13 \cdot 19 \cdot 43 \cdot 181$
$L_2(11^3)$	$11^3 \cdot 2^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 19 \cdot 37$	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 19 \cdot 37$
$L_2(13^2)$	$13^2 \cdot 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 17$	$2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 17$
$L_2(13^3)$	$13^3 \cdot 2^2 \cdot 3^2 \cdot 7 \cdot 61 \cdot 157$	$2^3 \cdot 3^2 \cdot 7 \cdot 61 \cdot 157$

3. Proofs of theorems

Proof of Theorem 1. We only need to prove the sufficiency. And the proof will be made through a case by case analysis.

Case 1.1. Assume that $A = M_{11}$ ($2^4 \cdot 3^2 \cdot 5 \cdot 11$). In this case, $|G| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$, and $k_1(G) = 11$. Since $k_1(G) = 11$, we have $t(G) \geq 2$. By Lemma 7, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1 -groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5\}$ and $11 \in \pi(K/H)$. From [11] we can suppose that $K/H \cong L_2(11)$ ($2^2 \cdot 3 \cdot 5 \cdot 11$) or M_{11} ($2^4 \cdot 3^2 \cdot 5 \cdot 11$).

Suppose that $K/H \cong L_2(11)$ ($2^2 \cdot 3 \cdot 5 \cdot 11$). From [11] we know that $|Out(L_2(11))| = 2$, so we can get that $3 \mid |H|$ by comparing the order of G . Let L be a Sylow 3-subgroup of H . Then $|L| = 3$. As H is a nilpotent group, we have L is characteristic in H , and thus $L \trianglelefteq G$. Consider the action on L by the element of order 11. Clearly, this action is trivial. It implies that G has an element of order 33, a contradiction.

Therefore, we have $K/H \cong M_{11}$ ($2^4 \cdot 3^2 \cdot 5 \cdot 11$). So $H = 1$, $K = G$, and therefore $G \cong M_{11}$.

Case 1.2. Assume that $A = M_{12}$ ($2^6 \cdot 3^3 \cdot 5 \cdot 11$). In this case, $|G| = 2^6 \cdot 3^3 \cdot 5 \cdot 11$, and $k_1(G) = 11$. Since $k_1(G) = 11$, we have $t(G) \geq 2$. By Lemma 7,

G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1 -groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. Therefore, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5\}$ and $11 \in \pi(K/H)$. From [11] we can suppose that $K/H \cong L_2(11) (2^2 \cdot 3 \cdot 5 \cdot 11)$, $M_{11} (2^4 \cdot 3^2 \cdot 5 \cdot 11)$ or $M_{12} (2^6 \cdot 3^3 \cdot 5 \cdot 11)$.

Suppose that $K/H \cong L_2(11) (2^2 \cdot 3 \cdot 5 \cdot 11)$ or $M_{11} (2^4 \cdot 3^2 \cdot 5 \cdot 11)$. From [11] we get that $3 \nmid |Out(K/H)| = 2$, and thus $3 \mid |H|$. Let L be a Sylow 3-subgroup of H . Then $|L| \mid 3^2$. As H is a nilpotent group, we have L is characteristic in H , and thus $L \trianglelefteq G$. Consider the action on L by the element of order 11. Clearly, this action is trivial, which implies that G has an element of order 33, a contradiction.

Therefore, we have $K/H \cong M_{12} (2^6 \cdot 3^3 \cdot 5 \cdot 11)$. So $H = 1$, $K = G$, and therefore $G \cong M_{12}$.

Case 1.3. Assume that $A = J_1 (2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19)$. In this case, $|G| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$, and $k_1(G) = 19$. Since $k_1(G) = 19$, we have $t(G) \geq 3$. By Lemma 2 and Lemma 3, we know that G is neither a Frobenius group, nor a 2-Frobenius group. So by Lemma 1, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1 -groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11\}$ and $19 \in \pi(K/H)$. From [11] we can know that K/H is isomorphic only to $J_1 (2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19)$. So $H = 1$, $K = G$, and therefore $G \cong J_1$.

Case 1.4. Assume that $A = M_{22} (2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11)$. In this case, $|G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$, and $k_1(G) = 11$. Since $k_1(G) = 11$, we have $t(G) \geq 3$. By Lemma 2 and Lemma 3, we know that G is neither a Frobenius group, nor a 2-Frobenius group. So by Lemma 1, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1 -groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. Therefore, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7\}$ and $11 \in \pi(K/H)$. From [11] we can suppose that K/H is isomorphic to one of the following simple groups: $L_2(11) (2^2 \cdot 3 \cdot 5 \cdot 11)$, $M_{11} (2^4 \cdot 3^2 \cdot 5 \cdot 11)$ and $M_{22} (2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11)$.

Suppose that $K/H \cong L_2(11) (2^2 \cdot 3 \cdot 5 \cdot 11)$ or $M_{11} (2^4 \cdot 3^2 \cdot 5 \cdot 11)$. From [11] we have $7 \nmid |Out(K/H)| = 2$, and thus $7 \mid |H|$. Let L be a Sylow 7-subgroup of H . Then $L \trianglelefteq G$. Consider the action on L by the element of order 5. Clearly, this action is also trivial, which implies that G has an element of order 35, a contradiction.

Therefore, we have $K/H \cong M_{22} (2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11)$. So $H = 1$, $K = G$, and therefore $G \cong M_{22}$.

Case 1.5. Assume that $A = M_{23} (2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$. In this case, $|G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$, and $k_1(G) = 23$. Since $k_1(G) = 23$, we have $t(G) \geq 2$. By Lemma 7, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1 -groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11\}$ and $23 \in \pi(K/H)$. From [11]

we can suppose that K/H is isomorphic to one of the following simple groups: $L_2(23)$ ($2^3 \cdot 3 \cdot 11 \cdot 23$), $L_2(2^7)$ and M_{23} ($2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$).

Suppose that $K/H \cong L_2(23)$ ($2^3 \cdot 3 \cdot 11 \cdot 23$). Then $7 \nmid |Out(K/H)|$, and thus $7 \mid |H|$. Let L be a Sylow 7-subgroup of H . Then $L \trianglelefteq G$. Consider the action on L by the element of order 11. Clearly, this action is trivial, which implies that G has an element of order 77, a contradiction.

Suppose that $K/H \cong L_2(2^7)$. From Table 3 we can get that $|K/H| \nmid |G|$, also a contradiction.

Therefore, we have $K/H \cong M_{22}$ ($2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$). So $H = 1$, $K = G$, and therefore $G \cong M_{23}$.

Case 1.6. Assume that $A = HS$ ($2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$). In this case, $|G| = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$, and $k_1(G) = 20$. Since $k_1(G) = 20$, we have $t(G) \geq 2$. By Lemma 7, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1 -groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. Therefore, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7\}$ and $11 \in \pi(K/H)$. From [11] we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(11)$ ($2^2 \cdot 3 \cdot 5 \cdot 11$), M_{11} ($2^4 \cdot 3^2 \cdot 5 \cdot 11$), M_{22} ($2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$), $L_2(2^t)$ ($7 \leq t \leq 9$) and HS ($2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$).

Suppose that $K/H \cong L_2(11)$ ($2^2 \cdot 3 \cdot 5 \cdot 11$), M_{11} ($2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$) or M_{22} ($2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$). Then $5 \nmid |Out(K/H)|$, and thus $5 \mid |H|$. Let L be a Sylow 5-subgroup of H . Then $L \trianglelefteq G$ and $|L| \mid 5^2$. Consider the action on L by the element of order 11. Clearly, this action is trivial, which implies that G has an element of order 55, a contradiction.

Suppose that $K/H \cong L_2(2^t)$ ($7 \leq t \leq 9$). From Table 3 we can get that $|K/H| \nmid |G|$, also a contradiction.

Therefore, we have $K/H \cong HS$ ($2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$). So $H = 1$, $K = G$, and therefore $G \cong HS$.

Case 1.7. Assume that $A = J_3$ ($2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$). In this case, $|G| = 2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$, and $k_1(G) = 19$. Since $k_1(G) = 19$, we have $t(G) \geq 3$. By Lemma 2 and Lemma 3, we know that G is neither a Frobenius group, nor a 2-Frobenius group. Therefore, by Lemma 1, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1 -groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 17\}$ and $19 \in \pi(K/H)$. From [11] we can suppose that K/H is isomorphic to one of the following simple groups: $L_2(19)$ ($2^2 \cdot 3^2 \cdot 5 \cdot 19$), $L_2(2^7)$, $L_2(3^5)$ and J_3 ($2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$).

Suppose that $K/H \cong L_2(19)$ ($2^2 \cdot 3^2 \cdot 5 \cdot 19$). Then $17 \nmid |Out(K/H)|$, and thus $17 \mid |H|$. Let L be a Sylow 17-subgroup of H . Then $L \trianglelefteq G$. Consider the action on L by the element of order 3. Clearly, this action is trivial, which implies that G has an element of order 51, a contradiction.

Suppose that $K/H \cong L_2(2^7)$ or $L_2(3^5)$. From Table 3 we can get that $|K/H| \nmid |G|$, still a contradiction.

Therefore, we have $K/H \cong J_3$ ($2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$). So $H = 1$, $K = G$, and therefore $G \cong J_3$.

Case 1.8. Assume that $A = M_{24}$ ($2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$). In this case, $|G| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$, and $k_1(G) = 23$. Since $k_1(G) = 23$, we have $t(G) \geq 2$. By Lemma 7, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1 -groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11\}$ and $23 \in \pi(K/H)$. From [11] we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(23)$ ($2^3 \cdot 3 \cdot 11 \cdot 23$), M_{23} ($2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$), $L_2(2^t)$ ($7 \leq t \leq 10$) and M_{24} ($2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$).

Suppose that $K/H \cong L_2(23)$ ($2^2 \cdot 3 \cdot 11 \cdot 23$) or M_{23} ($2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$). Then $3 \nmid |Out(K/H)|$, and thus $3 \mid |H|$. Let L be a Sylow 3-subgroup of H . Then $L \trianglelefteq G$ and $|L| \mid 3^2$. Consider the action on L by the element of order 11. Clearly, this action is trivial, which implies that G has an element of order 33, a contradiction.

Suppose that $K/H \cong L_2(2^t)$ ($7 \leq t \leq 10$). From Table 3 we can get that $|K/H| \nmid |G|$, also a contradiction.

Therefore, we have $K/H \cong M_{24}$ ($2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$). So $H = 1$, $K = G$, and therefore $G \cong M_{24}$.

Case 1.9. Assume that $A = He$ ($2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$). In this case, $|G| = 2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$, and $k_1(G) = 28$. Since $k_1(G) = 28$, we have $t(G) \geq 2$. By Lemma 7, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1 -groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7\}$ and $17 \in \pi(K/H)$. From [11] we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(17)$ ($2^4 \cdot 3^2 \cdot 17$), $L_2(16)$ ($2^4 \cdot 3 \cdot 5 \cdot 17$), $S_4(4)$ ($2^8 \cdot 3^2 \cdot 5^2 \cdot 17$), $L_2(2^t)$ ($7 \leq t \leq 10$), $L_2(7^3)$ and He ($2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$).

Suppose that $K/H \cong L_2(17)$ ($2^4 \cdot 3^2 \cdot 17$) or $L_2(16)$ ($2^4 \cdot 3 \cdot 5 \cdot 17$). Then $5 \nmid |Out(K/H)|$, and thus $5 \mid |H|$. Let L be a Sylow 5-subgroup of H . Then $L \trianglelefteq G$ and $|L| \mid 5^2$. Consider the action on L by the element of order 17. Clearly, this action is trivial, which implies that G has an element of order 85, a contradiction.

Suppose that $K/H \cong S_4(4)$ ($2^8 \cdot 3^2 \cdot 5^2 \cdot 17$). Then $3 \nmid |Out(K/H)|$, and thus $3 \mid |H|$. Let L be a Sylow 3-subgroup of H . Then $L \trianglelefteq G$ and $|L| = 3$. Consider the action on L by the element of order 17. Clearly, this action is also trivial, which implies that G has an element of order 51, a contradiction too.

Suppose that $K/H \cong L_2(2^t)$ ($7 \leq t \leq 10$) or $L_2(7^3)$. From Table 3 we can know that $|K/H| \nmid |G|$, still a contradiction.

Therefore, we have $K/H \cong He$ ($2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$). So $H = 1$, $K = G$, and therefore $G \cong He$.

Case 1.10. Assume that $A = Ru$ ($2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$). In this case, $|G| = 2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$, and $k_1(G) = 29$. Since $k_1(G) = 29$, we have $t(G) \geq 2$. By

Lemma 7, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1 -groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 13\}$ and $29 \in \pi(K/H)$. From [11] we can suppose that K/H is isomorphic to one of the following simple groups: $L_2(29)$ ($2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 29$), $L_2(2^t)$ ($7 \leq t \leq 14$) and Ru ($2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$).

Suppose that $K/H \cong L_2(29)$ ($2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 29$). From [11] we know that $13 \nmid |Out(K/H)|$, and thus $13 \mid |H|$. Let L be a Sylow 13-subgroup of H . Then $L \trianglelefteq G$. Consider the action on L by the element of order 5. Clearly, this action is trivial, which implies that G has an element of order 65, a contradiction.

Suppose that $K/H \cong L_2(2^t)$ ($7 \leq t \leq 14$). From Table 3 we can know that $|K/H| \nmid |G|$, also a contradiction.

Therefore, we have $K/H \cong Ru$ ($2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$). So $H = 1$, $K = G$, and therefore $G \cong Ru$.

Case 1.11. Assume that $A = Suz$ ($2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$). In this case, $|G| = 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$, and $k_1(G) = 24$. Since $k_1(G) = 24$, we have $t(G) \geq 2$. By Lemma 7, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1 -groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11\}$ and $13 \in \pi(K/H)$. From [11] we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(13)$ ($2^2 \cdot 3 \cdot 7 \cdot 13$), $L_3(3)$ ($2^4 \cdot 3^3 \cdot 13$), $L_2(25)$ ($2^3 \cdot 3 \cdot 5^2 \cdot 13$), $L_2(27)$ ($2^2 \cdot 3^3 \cdot 7 \cdot 13$), $Sz(8)$ ($2^6 \cdot 5 \cdot 7 \cdot 13$), $U_3(4)$ ($2^6 \cdot 3 \cdot 5^2 \cdot 13$), $L_2(64)$ ($2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$), $G_2(3)$ ($2^6 \cdot 3^6 \cdot 7 \cdot 13$), $L_4(3)$ ($2^7 \cdot 3^6 \cdot 5 \cdot 13$), ${}^2F_4(2)'$ ($2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$), $L_3(9)$ ($2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13$), $G_2(4)$ ($2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$), A_{13} ($2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$), $L_2(2^t)$ ($7 \leq t \leq 13$), $L_2(3^s)$ ($5 \leq s \leq 7$) and Suz ($2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$).

Suppose that K/H is isomorphic to one of the following simple groups:

$L_2(13)$ ($2^2 \cdot 3 \cdot 7 \cdot 13$), $L_3(3)$ ($2^4 \cdot 3^3 \cdot 13$), $L_2(25)$ ($2^3 \cdot 3 \cdot 5^2 \cdot 13$), $L_2(27)$ ($2^2 \cdot 3^3 \cdot 7 \cdot 13$), $Sz(8)$ ($2^6 \cdot 5 \cdot 7 \cdot 13$), $U_3(4)$ ($2^6 \cdot 3 \cdot 5^2 \cdot 13$), $L_2(64)$ ($2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$), $G_2(3)$ ($2^6 \cdot 3^6 \cdot 7 \cdot 13$), $L_4(3)$ ($2^7 \cdot 3^6 \cdot 5 \cdot 13$), ${}^2F_4(2)'$ ($2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$), $L_3(9)$ ($2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13$) and $G_2(4)$ ($2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$). From [11] we get that $11 \nmid |Out(K/H)|$, and thus $11 \mid |H|$. Let L be a Sylow 11-subgroup of H . Then $L \trianglelefteq G$. Consider the action on L by the element of order 3. Clearly, this action is trivial, which implies that G has an element of order 33, a contradiction.

Suppose that $K/H \cong A_{13}$ ($2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$). In such case, $|Out(K/H)| = 2$. So $|G/K| \mid 2$ and thus $3 \mid |H|$. Let L be a Sylow 3-subgroup of H . Then $L \trianglelefteq G$ and $|L| = 3^2$. Consider the action on L by the element of order 11. Clearly, this action is also trivial, which implies that G has an element of order 33, a contradiction too.

Suppose that $K/H \cong L_2(2^t)$ ($7 \leq t \leq 13$) or $L_2(3^s)$ ($5 \leq s \leq 7$). From Table 3 we can know that $|K/H| \nmid |G|$, also a contradiction.

Therefore, we have $K/H \cong Suz$ ($2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$). So $H = 1$, $K = G$, and therefore $G \cong Suz$.

Case 1.12. Assume that $A = O'N$ ($2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$). In this case,

$|G| = 2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$, and $k_1(G) = 31$. Since $k_1(G) = 31$, we have $t(G) \geq 3$. By Lemma 7, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1 -groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11, 19\}$ and $31 \in \pi(K/H)$. From [11] we can suppose that K/H is isomorphic to one of the following simple groups: $L_2(31)$ ($2^5 \cdot 3 \cdot 5 \cdot 31$), $L_2(32)$ ($2^5 \cdot 3 \cdot 11 \cdot 31$), $L_2(2^t)$ ($7 \leq t \leq 9$), $L_2(7^3)$ and $O'N(2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31)$.

Suppose that $K/H \cong L_2(31)$ ($2^5 \cdot 3 \cdot 5 \cdot 31$) or $L_2(32)$ ($2^5 \cdot 3 \cdot 11 \cdot 31$). Then $19 \nmid |Out(K/H)|$, and thus $19 \mid |H|$. Let L be a Sylow 19-subgroup of H . Then $L \trianglelefteq G$. Consider the action on L by the element of order 5. Clearly, this action is trivial, which implies that G has an element of order 95, a contradiction.

Suppose that $K/H \cong L_2(2^t)$ ($7 \leq t \leq 9$) or $L_2(7^3)$. From Table 3 we can know that $|K/H| \nmid |G|$, also a contradiction.

Therefore, we have $K/H \cong O'N(2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31)$. So $H = 1$, $K = G$, and therefore $G \cong O'N$.

Case 1.13. Assume that $A = Co_3$ ($2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$). In this case, $|G| = 2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$, and $k_1(G) = 30$. Since $k_1(G) = 30$, we have $t(G) \geq 2$. By Lemma 7, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1 -groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11\}$ and $23 \in \pi(K/H)$. From [11] we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(23)$ ($2^3 \cdot 3 \cdot 11 \cdot 23$), M_{23} ($2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$), M_{24} ($2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$), $L_2(2^t)$ ($7 \leq t \leq 10$), $L_2(3^s)$ ($5 \leq s \leq 7$) and Co_3 ($2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$).

Suppose that $K/H \cong L_2(23)$ ($2^3 \cdot 3 \cdot 11 \cdot 23$). Then $7 \nmid |Out(K/H)|$, and thus $7 \mid |H|$. Let L be a Sylow 7-subgroup of H . Then $L \trianglelefteq G$. Consider the action on L by the element of order 5. Clearly, this action is trivial, which implies that G has an element of order 35, a contradiction.

Suppose that $K/H \cong M_{23}$ ($2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$) or M_{24} ($2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$). In such case, $5 \nmid |Out(K/H)|$ and therefore $5 \mid |H|$. Let L be a Sylow 5-subgroup of H . Then $L \trianglelefteq G$ and $|L| = 5^2$. Consider the action on L by the element of order 11. Clearly, this action is also trivial, which implies that G has an element of order 55, a contradiction too.

Suppose that $K/H \cong L_2(2^t)$ ($7 \leq t \leq 10$) or $L_2(3^s)$ ($5 \leq s \leq 7$). From Table 3 we can know that $|K/H| \nmid |G|$, also a contradiction.

Therefore, we have $K/H \cong Co_3$ ($2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$). So $H = 1$, $K = G$, and therefore $G \cong Co_3$.

Case 1.14. Assume that $A = Co_2$ ($2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$). In this case, $|G| = 2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$, and $k_1(G) = 30$. Since $k_1(G) = 30$, we have $t(G) \geq 2$. By Lemma 7, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1 -groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$.

So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11, \}$ and $23 \in \pi(K/H)$. From [11] and Table 2, we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(23) (2^3 \cdot 3 \cdot 11 \cdot 23)$, $M_{23} (2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$, $M_{24} (2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$, $Co_2 (2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23)$ and $B(q)$, where $B(q)$ is one of the following simple groups: $L_2(2^t) (7 \leq t \leq 18)$, $L_2(3^s) (5 \leq s \leq 6)$, $L_3(2^t) (5 \leq t \leq 6)$ and $U_3(2^s) (5 \leq s \leq 6)$.

Suppose that $K/H \cong L_2(23) (2^3 \cdot 3 \cdot 11 \cdot 23)$. Then $7 \nmid |Out(K/H)|$, and thus $7 \mid |H|$. Let L be a Sylow 7-subgroup of H . Then $L \trianglelefteq G$. Consider the action on L by the element of order 5. Clearly, this action is also trivial, which implies that G has an element of order 35, a contradiction.

Suppose that $K/H \cong M_{23} (2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$ or $M_{24} (2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$. In such case, $5 \nmid |Out(K/H)|$, and therefore $5 \mid |H|$. Let L be a Sylow 5-subgroup of H . Then $L \trianglelefteq G$ and $|L| = 5^2$. Consider the action on L by the element of order 11. Clearly, this action is also trivial, which implies that G has an element of order 55, a contradiction too.

Suppose that $K/H \cong B(q)$. From Table 2 we know that $(q^2 - 1) \mid |B(q)|$, which implies $(q^2 - 1) \mid |G|$. We can see it is impossible from Table 3.

Therefore, we have $K/H \cong Co_2 (2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23)$. So $H = 1$, $K = G$, and therefore $G \cong Co_2$.

Case 1.15. Assume that $A = Ly (2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67)$. In this case, $|G| = 2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$, and $k_1(G) = 67$. Since $k_1(G) = 67$, we have $t(G) \geq 3$. By Lemma 7, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1 -groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11, 31, 37\}$ and $67 \in \pi(K/H)$. From [11] and Table 2, we can suppose that K/H is isomorphic to $Ly (2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67)$ or $B(q)$, where $B(q)$ is one of the following simple groups: $L_2(2^t) (7 \leq t \leq 8)$, $L_2(3^s) (5 \leq s \leq 7)$ and $L_2(5^r) (4 \leq r \leq 6)$.

Suppose that $K/H \cong B(q)$. From Table 2 we know that $(q^2 - 1) \mid |B(q)|$, which means $(q^2 - 1) \mid |G|$. From Table 3, we can see it is impossible.

Therefore, we have $K/H \cong Ly (2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67)$. So $H = 1$, $K = G$, and therefore $G \cong Ly$.

Case 1.16. Assume that $A = Th (2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31)$. In this case, $|G| = 2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$, and $k_1(G) = 39$. Since $k_1(G) = 39$, we have $t(G) \geq 2$. By Lemma 7, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1 -groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 13, 19\}$ and $31 \in \pi(K/H)$. From [11] and Table 2, we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(31) (2^5 \cdot 3 \cdot 5 \cdot 31)$, $L_2(32) (2^5 \cdot 3 \cdot 11 \cdot 31)$, $L_3(5) (2^5 \cdot 3 \cdot 5^3 \cdot 31)$, $L_2(125) (2^2 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 31)$, $L_5(2) (2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31)$, $L_6(2) (2^{15} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 31)$, $Th (2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31)$

and $B(q)$, where $B(q)$ is one of the following simple groups: $L_2(2^t)$ ($7 \leq t \leq 15$), $L_2(3^s)$ ($5 \leq s \leq 10$), $L_3(2^5)$ and $U_3(2^5)$.

Suppose that K/H is isomorphic to one of the following simple groups: $L_2(31)$ ($2^5 \cdot 3 \cdot 5 \cdot 31$), $L_2(32)$ ($2^5 \cdot 3 \cdot 11 \cdot 31$), $L_3(5)$ ($2^5 \cdot 3 \cdot 5^3 \cdot 31$), $L_2(125)$ ($2^2 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 31$), $L_5(2)$ ($2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$), $L_6(2)$ ($2^{15} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 31$). Then $19 \nmid |Out(K/H)|$, and thus $19 \mid |H|$. Let L be a Sylow 19-subgroup of H . Then $L \trianglelefteq G$. Consider the action on L by the element of order 5. Clearly, this action is also trivial, which implies that G has an element of order 95, a contradiction.

Suppose that $K/H \cong B(q)$. From Table 2 we know that $(q^2 - 1) \mid |B(q)|$, which means $(q^2 - 1) \mid |G|$. We can see it is impossible from Table 3.

Therefore, we have $K/H \cong Th$ ($2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$). So $H = 1$, $K = G$, and therefore $G \cong Th$.

Case 1.17. Assume that $A = J_4$ ($2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$). In this case, $|G| = 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$, and $k_1(G) = 66$. Since $k_1(G) = 66$, we have $t(G) \geq 3$. By Lemma 7, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1 -groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11, 23, 29, 31, 37\}$ and $43 \in \pi(K/H)$. From [11] and Table 2, we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(43)$ ($2^2 \cdot 3 \cdot 7 \cdot 11 \cdot 43$), J_4 ($2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$) and $B(q)$, where $B(q)$ is one of the following simple groups: $L_2(2^t)$ ($7 \leq t \leq 21$), $L_2(11^3)$, $L_3(2^s)$ ($5 \leq s \leq 7$) and $U_3(2^r)$ ($5 \leq r \leq 7$).

Suppose that $K/H \cong L_2(43)$ ($2^2 \cdot 3 \cdot 7 \cdot 11 \cdot 43$). Then $37 \nmid |Out(K/H)|$, and thus $37 \mid |H|$. Let L be a Sylow 37-subgroup of H . Then $L \trianglelefteq G$. Consider the action on L by the element of order 5. Clearly, this action is trivial, which implies that G has an element of order 185, a contradiction.

Suppose that $K/H \cong B(q)$. From Table 2 we know that $(q^2 - 1) \mid |B(q)|$, which means $(q^2 - 1) \mid |G|$. If $B(q)$ is $L_2(2^t)$ ($7 \leq t \leq 21$), $L_2(11^3)$, $L_3(2^s)$ ($6 \leq s \leq 7$) or $U_3(2^r)$ ($6 \leq r \leq 7$), then $(q^2 - 1) \nmid |G|$ by Table 3, which is a contradiction. If $B(q)$ is $L_3(2^5)$ or $U_3(2^5)$, we can easily know that $|B(q)| \nmid |G|$, also a contradiction.

Therefore, we have $K/H \cong J_4$ ($2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$). So $H = 1$, $K = G$, and therefore $G \cong J_4$.

Case 1.18. Assume that $A = B$ ($2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$). In this case, $|G| = 2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$, and $k_1(G) = 70$. Since $k_1(G) = 70$, we have $t(G) \geq 2$. By Lemma 7, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1 -groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11, 13, 17, 19, 23, 31\}$ and $47 \in \pi(K/H)$. From [11] and Table 2, we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(47)$ ($2^4 \cdot 3 \cdot 23 \cdot 47$), B ($2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$) and $B(q)$, where $B(q)$ is one of the following simple groups:

$L_2(2^t)$ ($7 \leq t \leq 41$), $L_2(3^s)$ ($5 \leq s \leq 13$), $L_2(5^r)$ ($4 \leq r \leq 6$), $L_3(2^t)$ ($5 \leq t \leq 13$), $L_3(3^4)$, $U_3(2^t)$ ($5 \leq t \leq 13$), $U_3(3^4)$, $L_4(2^t)$ ($4 \leq t \leq 6$), $U_4(2^t)$ ($4 \leq t \leq 6$), $S_4(2^t)$ ($6 \leq t \leq 10$) and $G_4(2^t)$ ($5 \leq t \leq 6$).

Suppose that $K/H \cong L_2(47)$ ($2^4 \cdot 3 \cdot 23 \cdot 47$). Then $31 \nmid |Out(K/H)|$, and thus $31 \mid |H|$. Let L be a Sylow 31-subgroup of H . Then $L \trianglelefteq G$. Consider the action on L by the element of order 7. Clearly, this action is also trivial, which implies that G has an element of order 217, a contradiction.

Suppose that $K/H \cong B(q)$. From Table 2 we know that $(q^2 - 1) \mid |B(q)|$, which means $(q^2 - 1) \mid |G|$. We can easily know that $|B(q)| \nmid |G|$ by Lemma 5, Table 2 and Table 3, also a contradiction.

Therefore, we have $K/H \cong B$ ($2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$). So $H = 1$, $K = G$, and therefore $G \cong B$.

Case 1.19. Assume that $A = M$ ($2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$). In this case, $|G| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$, and $k_1(G) = 119$. Since $k_1(G) = 119$, we have $t(G) \geq 2$. By Lemma 7, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1 -groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59\}$ and $71 \in \pi(K/H)$. From [11] and Table 2, we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(71)$ ($2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 71$), $M(2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71)$ and $B(q)$, where $B(q)$ is one of the following simple groups:

$L_2(2^t)$ ($7 \leq t \leq 46$), $L_2(3^s)$ ($5 \leq s \leq 20$), $L_2(5^r)$ ($4 \leq r \leq 9$), $L_2(7^x)$ ($3 \leq x \leq 6$), $L_2(13^y)$ ($2 \leq y \leq 3$), $L_3(2^t)$ ($5 \leq t \leq 15$), $L_3(3^s)$ ($4 \leq s \leq 6$), $L_3(5^3)$, $L_3(7^2)$, $U_3(2^t)$ ($5 \leq t \leq 15$), $U_3(3^s)$ ($4 \leq t \leq 6$), $U_3(5^3)$, $U_3(7^2)$, $L_4(2^t)$ ($4 \leq t \leq 7$), $L_4(3^3)$, $U_4(2^t)$ ($4 \leq t \leq 7$), $U_4(3^3)$, $S_4(2^t)$ ($6 \leq t \leq 11$), $S_4(3^s)$ ($4 \leq s \leq 5$), $G_4(2^t)$ ($5 \leq t \leq 7$) and $G_2(3^3)$.

Suppose that $K/H \cong L_2(71)$ ($2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 71$). Then $31 \nmid |Out(K/H)|$, and thus $31 \mid |H|$. Let L be a Sylow 31-subgroup of H . Then $L \trianglelefteq G$. Consider the action on L by the element of order 7. Clearly, this action is also trivial, which implies that G has an element of order 217, a contradiction.

Suppose that $K/H \cong B(q)$. From Table 2 we know that $(q^2 - 1) \mid |B(q)|$, and therefore $(q^2 - 1) \mid |G|$. If $B(q)$ is not $L_2(2^{10})$ and $L_2(13^2)$, then we can easily know that $|B(q)| \nmid |G|$ by Lemma 5, Table 2 and Table 3, a contradiction. We now assume that $B(q)$ is $L_2(2^{10})$ or $L_2(13^2)$. From Table 3, we have $71 \nmid |B(q)|$, also a contradiction.

Therefore, we have $K/H \cong M$ ($2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$). So $H = 1$, $K = G$, and therefore $G \cong M$.

Now Theorem 1 follows from Case 1.1 to Case 1.19.

Proof of Theorem 2. It is enough to prove the sufficiency.

Case 2.1. Assume that $A = J_2$ ($2^7 \cdot 3^3 \cdot 5^2 \cdot 7$). In this case, $|G| = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7$, $k_1(G) = 15$ and $k_2(G) = 12$. Since $k_1(G) = 15$, $k_2(G) = 12$, we have $t(G) \geq 2$.

By Lemma 8, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1 -groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5\}$ and $7 \in \pi(K/H)$. From [11] we can suppose that K/H is isomorphic to one of the following simple groups: $L_2(7)$ ($2^3 \cdot 3 \cdot 7$), $L_2(8)$ ($2^3 \cdot 3^2 \cdot 7$), A_7 ($2^3 \cdot 3^2 \cdot 5 \cdot 7$), $U_3(3)$ ($2^5 \cdot 3^3 \cdot 7$), $L_4(2)$ ($2^6 \cdot 3^2 \cdot 5 \cdot 7$), $L_3(4)$ ($2^6 \cdot 3^2 \cdot 5 \cdot 7$) and $J_2(2^7 \cdot 3^3 \cdot 5^2 \cdot 7)$.

Suppose that K/H is isomorphic to one of the following simple groups: $L_2(7)$ ($2^3 \cdot 3 \cdot 7$), $L_2(8)$ ($2^3 \cdot 3^2 \cdot 7$), A_7 ($2^3 \cdot 3^2 \cdot 5 \cdot 7$), $U_3(3)$ ($2^5 \cdot 3^3 \cdot 7$), $L_4(2)$ ($2^6 \cdot 3^2 \cdot 5 \cdot 7$) and $L_3(4)$ ($2^6 \cdot 3^2 \cdot 5 \cdot 7$). By [11] we know that $5 \nmid |Out(K/H)|$, and thus $5 \mid |H|$. Let L be a Sylow 5-subgroup of H . We know that $L \trianglelefteq G$, and $|L| \mid 5^2$. Consider the action on L by the element of order 7. Clearly, this action is trivial. It implies that G has an element of order 35, a contradiction.

Therefore, we have $K/H \cong J_2(2^7 \cdot 3^3 \cdot 5^2 \cdot 7)$. So $H = 1$, $K = G$, and therefore $G \cong J_2$.

Case 2.2. Assume that $A = McL$ ($2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$). In this case, $|G| = 2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$, $k_1(G) = 30$ and $k_2(G) = 15$. Since $k_1(G) = 30$ and $k_2(G) = 15$, we have $t(G) \geq 2$. By Lemma 8, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1 -groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7\}$ and $11 \in \pi(K/H)$. From [11] we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(11)$ ($2^2 \cdot 3 \cdot 5 \cdot 11$), M_{11} ($2^4 \cdot 3^2 \cdot 5 \cdot 11$), M_{12} ($2^6 \cdot 3^3 \cdot 5 \cdot 11$), M_{22} ($2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$), A_{11} ($2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$), McL ($2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$), $L_2(2^7)$ and $L_2(3^s)$ ($5 \leq s \leq 6$).

Suppose that K/H is isomorphic to one of the following simple groups: $L_2(11)$ ($2^2 \cdot 3 \cdot 5 \cdot 11$), M_{11} ($2^4 \cdot 3^2 \cdot 5 \cdot 11$), M_{12} ($2^6 \cdot 3^3 \cdot 5 \cdot 11$), M_{22} ($2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$) and A_{11} ($2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$). By [11] we know that $5 \nmid |Out(K/H)|$, and thus $5 \mid |H|$. Let L be a Sylow 5-subgroup of H . We know that $L \trianglelefteq G$, and $|L| \mid 5^2$. Consider the action on L by the element of order 11. Clearly, this action is trivial. It implies that G has an element of order 55, which is a contradiction.

Suppose that $K/H \cong L_2(2^7)$ or $L_2(3^s)$ ($5 \leq s \leq 6$). From Table 3, we have $|K/H| \nmid |G|$, still a contradiction.

Therefore, we have $K/H \cong McL$ ($2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$). So $H = 1$, $K = G$, and therefore $G \cong McL$.

Case 2.3. Assume that $A = Fi_{22}$ ($2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$). In this case, $|G| = 2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$, $k_1(G) = 30$ and $k_2(G) = 24$. Since $k_1(G) = 30$ and $k_2(G) = 24$, we have $t(G) \geq 2$. By Lemma 8, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1 -groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11\}$ and $13 \in \pi(K/H)$. From [11] and Table 2 we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(13)$ ($2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 13$), $L_3(3)$ ($2^4 \cdot 3^3 \cdot 13$), $L_2(25)$ ($2^3 \cdot 3 \cdot 5^2 \cdot 13$), $L_2(27)$ ($2^2 \cdot 3^3 \cdot 7 \cdot 13$), $Sz(8)$ ($2^6 \cdot 5 \cdot 7 \cdot 13$), $U_3(4)$ ($2^6 \cdot 3 \cdot 5^2 \cdot 13$), $L_2(64)$ ($2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$),

$G_2(3) (2^6 \cdot 3^6 \cdot 7 \cdot 13)$, $L_4(3) (2^7 \cdot 3^6 \cdot 5 \cdot 13)$, ${}^2F_4(2)' (2^{11} \cdot 3^3 \cdot 5^2 \cdot 13)$, $L_3(9) (2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13)$, $G_2(4) (2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13)$, $A_{13} (2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13)$, $S_6(3) (2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13)$, $O_7(3) (2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13)$, $Suz (2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13)$, $Fi_{22} (2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13)$ and $B(q)$, where $B(q)$ is one of the following simple groups: $L_2(2^t) (7 \leq t \leq 17)$, $L_2(3^s) (5 \leq s \leq 9)$, $L_3(2^t) (5 \leq t \leq 6)$ and $U_3(2^5)$.

Suppose that K/H is isomorphic to one of the following simple groups: $L_2(13) (2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 13)$, $L_3(3) (2^4 \cdot 3^3 \cdot 13)$, $L_2(25) (2^3 \cdot 3 \cdot 5^2 \cdot 13)$, $L_2(27) (2^2 \cdot 3^3 \cdot 7 \cdot 13)$, $Sz(8) (2^6 \cdot 5 \cdot 7 \cdot 13)$, $U_3(4) (2^6 \cdot 3 \cdot 5^2 \cdot 13)$, $L_2(64) (2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13)$, $G_2(3) (2^6 \cdot 3^6 \cdot 7 \cdot 13)$, $L_4(3) (2^7 \cdot 3^6 \cdot 5 \cdot 13)$, ${}^2F_4(2)' (2^{11} \cdot 3^3 \cdot 5^2 \cdot 13)$, $L_3(9) (2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13)$, $G_2(4) (2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13)$, $S_6(3) (2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13)$ and $O_7(3) (2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13)$. By [11] we know that $11 \nmid |Out(K/H)|$, and thus $11 \mid |H|$. Let L be a Sylow 11-subgroup of H . We know that $L \trianglelefteq G$. Consider the action on L by the element of order 7. Clearly, this action is trivial. It implies that G has an element of order 77, which is a contradiction.

Suppose that $K/H \cong A_{13} (2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13)$. By [11] we know that $|Out(K/H)| = 2$, and thus $3 \mid |H|$. Let L be a Sylow 3-subgroup of H . We know that $L \trianglelefteq G$ and $|L| \mid 3^4$. Clearly, $\Omega_1(Z(L))$ is an elementary abelian 3-group, and $|\Omega_1(Z(L))| \mid 3^4$. Since $\Omega_1(Z(L))$ is characteristic in L , we have $\Omega_1(Z(L)) \trianglelefteq G$ for $L \trianglelefteq G$. Consider the action on $\Omega_1(Z(L))$ by the element of order 11. Since $11 \nmid |Aut(\Omega_1(Z(L)))|$, this action is trivial. It implies that G has an element of order 33, also a contradiction.

Suppose that $K/H \cong Suz (2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13)$. By [11] we know that $|Out(K/H)| = 2$, and therefore $3 \mid |H|$. Let L be a Sylow 3-subgroup of H . We know that $L \trianglelefteq G$, and $|L| \mid 3^2$. Consider the action on L by the element of order 13. Clearly, this action is trivial, which implies that G has an element of order 39, a contradiction too.

Suppose that $K/H \cong B(q)$. From Table 2, we know that $(q^2 - 1) \mid |B(q)|$ and therefore $(q^2 - 1) \mid |G|$. But Table 2 and Table 3 show that $|B(q)| \nmid |G|$, still a contradiction.

Therefore, we have $K/H \cong Fi_{22} (2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13)$. So $H = 1$, $K = G$, and therefore $G \cong Fi_{22}$.

Case 2.4. Assume that $A = HN (2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19)$. In this case, $|G| = 2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$, $k_1(G) = 40$ and $k_2(G) = 35$. Since $k_1(G) = 40$ and $k_2(G) = 35$, we have $t(G) \geq 2$. By Lemma 8, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1 -groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11\}$ and $19 \in \pi(K/H)$. From [11] and Table 2 we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(19) (2^2 \cdot 3^2 \cdot 5 \cdot 19)$, $J_1 (2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19)$, $U_3(8) (2^9 \cdot 3^4 \cdot 7 \cdot 19)$, $HN (2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19)$ and $B(q)$, where $B(q)$ is one of the following simple groups: $L_2(2^t) (7 \leq t \leq 14)$, $L_2(3^s) (5 \leq s \leq 6)$, $L_2(5^r) (4 \leq r \leq 6)$ and $U_3(2^5)$.

Suppose that $K/H \cong L_2(19) (2^2 \cdot 3^2 \cdot 5 \cdot 19)$ or $U_3(8) (2^9 \cdot 3^4 \cdot 7 \cdot 19)$. By [11] we know that $11 \nmid |Out(K/H)|$, and thus $11 \mid |H|$. Let L be a Sylow 11-subgroup

of H . We know that $L \trianglelefteq G$. Consider the action on L by the element of order 7. Clearly, this action is trivial. It implies that G has an element of order 77, a contradiction.

Suppose that $K/H \cong J_1 (2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19)$. By [11] we know that $|Out(K/H)| = 1$, and $3 \mid |H|$. Let L be a Sylow 3-subgroup of H . We know that $L \trianglelefteq G$ and $|L| \mid 3^5$. Consider $\Omega_1(Z(L))$. Clearly, $\Omega_1(Z(L))$ is an elementary abelian 3-group and $\Omega_1(Z(L)) \trianglelefteq G$. Because $|\Omega_1(Z(L))| \mid 3^5$, we have $19 \nmid |Aut(\Omega_1(Z(L)))|$. Consider the action on $\Omega_1(Z(L))$ by the element of order 19. One can see that this action is trivial. It implies that G has an element of order 57, also a contradiction.

Suppose that $K/H \cong B(q)$. From Table 2 and Table 3, we can know that $|B(q)| \nmid |G|$, which is a contradiction.

Therefore, we have $K/H \cong HN (2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19)$. So $H = 1$, $K = G$, and therefore $G \cong HN$.

Case 2.5. Assume that $A = Fi_{23} (2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23)$. In this case, $|G| = 2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$, $k_1(G) = 60$ and $k_2(G) = 42$. Since $k_1(G) = 60$ and $k_2(G) = 42$, we have $t(G) \geq 2$. By Lemma 8, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1 -groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11, 13, 17\}$ and $23 \in \pi(K/H)$. From [11] and Table 2 we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(23) (2^3 \cdot 3 \cdot 11 \cdot 23)$, $M_{23} (2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$, $M_{24} (2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$, $Co_3 (2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23)$, $Fi_{23} (2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23)$ and $B(q)$, where $B(q)$ is one of the following simple groups: $L_2(2^t) (7 \leq t \leq 18)$, $L_2(3^s) (5 \leq s \leq 13)$, $L_3(2^t) (5 \leq t \leq 6)$, $L_3(3^4)$, $U_3(2^t) (5 \leq t \leq 6)$ and $U_3(3^4)$.

Suppose that K/H is isomorphic to one of the following simple groups: $L_2(23) (2^3 \cdot 3 \cdot 11 \cdot 23)$, $M_{23} (2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$, $M_{24} (2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23)$ and $Co_3 (2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23)$. By [11] we know that $17 \nmid |Out(K/H)|$, and thus $17 \mid |H|$. Let L be a Sylow 17-subgroup of H . We know that $L \trianglelefteq G$. Consider the action on L by the element of order 5. Clearly, this action is trivial. It implies that G has an element of order 85, a contradiction.

Suppose that $K/H \cong B(q)$. From Table 2 and Table 3, we know that $|B(q)| \nmid |G|$, also a contradiction.

Therefore, we have $K/H \cong Fi_{23} (2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23)$. So $H = 1$, $K = G$, and therefore $G \cong Fi_{23}$.

Case 2.6. Assume that $A = Co_1 (2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23)$. In this case, $|G| = 2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$, $k_1(G) = 60$ and $k_2(G) = 42$. Since $k_1(G) = 60$ and $k_2(G) = 42$, we have $t(G) \geq 2$. By Lemma 8, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1 -groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11, 13\}$ and $23 \in \pi(K/H)$. From [11] and Table 2 we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(23)$ ($2^3 \cdot 3 \cdot 11 \cdot 23$), M_{23} ($2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$), M_{24} ($2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$), Co_3 ($2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$), Co_2 ($2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$), Co_1 ($2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$) and $B(q)$, where $B(q)$ is one of the following simple groups: $L_2(2^t)$ ($7 \leq t \leq 21$), $L_2(3^s)$ ($5 \leq s \leq 9$), $L_2(5^4)$, $L_3(2^t)$ ($5 \leq t \leq 7$) and $U_3(2^t)$ ($5 \leq t \leq 7$).

Suppose that K/H is isomorphic to one of the following simple groups: $L_2(23)$ ($2^3 \cdot 3 \cdot 11 \cdot 23$), M_{23} ($2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$), M_{24} ($2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$), Co_3 ($2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$) and Co_2 ($2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$). By [11] we know that $13 \nmid |Out(K/H)|$, and thus $13 \mid |H|$. Let L be a Sylow 13-subgroup of H . We know that $L \trianglelefteq G$. Consider the action on L by the element of order 5. Clearly, this action is trivial. It implies that G has an element of order 65, which is a contradiction.

Suppose that $K/H \cong B(q)$. From Table 2 and Table 3, we know that $|B(q)| \nmid |G|$, also a contradiction.

Therefore, we have $K/H \cong Co_1$ ($2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$). So $H = 1$, $K = G$, and therefore $G \cong Co_1$.

Case 2.7. Assume that $A = Fi_{24}$ ($2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$). In this case, $|G| = 2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$, $k_1(G) = 60$ and $k_2(G) = 45$. Since $k_1(G) = 60$ and $k_2(G) = 45$, we have $t(G) \geq 2$. By Lemma 8, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1 -groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11, 13, 17, 23\}$ and $29 \in \pi(K/H)$. From [11] and Table 2 we can suppose that K/H is isomorphic to one of the following simple groups:

$L_2(29)$ ($2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 29$), Fi_{24} ($2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$) and $B(q)$, where $B(q)$ is one of the following simple groups: $L_2(2^t)$ ($7 \leq t \leq 21$), $L_2(3^s)$ ($5 \leq s \leq 16$), $L_2(7^3)$, $L_3(2^t)$ ($5 \leq t \leq 7$), $L_3(3^s)$ ($4 \leq s \leq 5$), $U_3(2^t)$ ($5 \leq t \leq 7$) and $U_3(3^s)$ ($4 \leq s \leq 5$).

Suppose that $K/H \cong L_2(29)$ ($2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 29$). By [11] we know that $23 \nmid |Out(K/H)|$, and thus $23 \mid |H|$. Let L be a Sylow 23-subgroup of H . We know that $L \trianglelefteq G$. Consider the action on L by the element of order 3. Clearly, this action is trivial. It implies that G has an element of order 69, a contradiction.

Suppose that $K/H \cong B(q)$. From Table 2 and Table 3, we know that $|B(q)| \nmid |G|$, a contradiction too.

Therefore, we have $K/H \cong Fi_{24}$ ($2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$). So $H = 1$, $K = G$, and therefore $G \cong Fi_{24}$.

Now Theorem 2 follows from Case 2.1 to Case 2.7.

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MODIFIED (G'/G) -EXPANSION METHOD WITH GENERALIZED RICCATI EQUATION TO THE SIXTH-ORDER BOUSSINESQ EQUATION

Muhammad Shakeel

Syed Tauseef Mohyud-Din¹

*Department of Mathematics
Faculty of Sciences
HITEC University Taxila Cantt
Pakistan*

Abstract. In this article, abundant traveling wave solutions of the sixth-order Boussinesq equation have been obtained in a uniform way by using the alternative (G'/G) -expansion method wherein the generalized Riccati equation is used. It is shown that the alternative (G'/G) -expansion method together with the generalized Riccati equation provides advance mathematical tool for solving nonlinear partial differential equations. Numerical results coupled with the graphical representation explicitly reveal the complete reliability and high efficiency of the proposed algorithm.

Keyword: (G'/G) -expansion method, sixth-order Boussinesq equation, traveling wave solutions, nonlinear evolution equations.

1. Introduction

It is well known that nonlinear evolution equations (NLEEs) are widely used as models to describe many important complex physical phenomena in various fields of science, such as, plasma physics, nonlinear optics, solid state physics, chemical kinematics, chemistry, biology and many others [1]-[51]. For better understanding of nonlinear phenomena as well as further applications in practical life, it is more significant to establish exact traveling wave solutions. In the recent past, a wide range of methods have been developed to generate analytical solutions by a diverse group of scientists. For instance, the Backlund transformation method [1], the inverse scattering method [2], the truncated Painleve expansion method [3], the Hirota's bilinear transformation method [4], the Jacobi elliptic function expansion method [5], the generalized Riccati equation method [6], the tanh-coth method [7], [8], the F-expansion method [9,10], the variational iteration method [11], [12],

¹Corresponding author. E-mail: syedtauseefs@hotmail.com

the direct algebraic method [13], the homotopy perturbation method [14]–[16], the Exp-function method [17]–[21] and others [22]–[28].

Another important method was introduced by Wang et al. [29] introduced a direct and concise method, called the (G'/G) -expansion method to look for traveling wave solutions of nonlinear partial differential equations, where $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$; λ and μ are arbitrary constants. The (G'/G) -expansion method is one of the most powerful methods to solve nonlinear problems and several scientists investigated different kind of NLEEs to construct traveling wave solutions via this method and can be found in the articles [30]–[34] for better understanding.

In order to establish the effectiveness and reliability of the (G'/G) -expansion method and to expand the possibility of its application, further research has been carried out by several researchers. For instance, Zhang et al. [35] presented an improved (G'/G) -expansion method to seek more general traveling wave solutions. Zayed [36] presented a new approach of the (G'/G) -expansion method where $G(\xi)$ satisfies the Jacobi elliptic equation, $[G'(\xi)]^2 = e_2 G^4(\xi) + e_1 G^2(\xi) + e_0$, e_2, e_1, e_0 are arbitrary constants, and obtained new exact solutions. Zayed [37] again presented an alternative approach of this method in which $G(\xi)$ satisfies the Riccati equation $G'(\xi) = AG + BG^2(\xi)$ where A and B are arbitrary constants. Consequently, several researchers studied various nonlinear PDEs to generate traveling wave solutions via the improved (G'/G) -expansion method and can be found [38]–[41].

It is significant to observe that there exist some fundamental relationships among numerous complex nonlinear partial differential equations and some basic and soluble nonlinear ordinary differential equations (ODEs), such as the sine-Gordon equation, the sinh-Gordon equation, the Riccati equation, the Weierstrass elliptic equation etc. Therefore, it is natural to use the solutions of these nonlinear ODEs to construct exact solutions of various intricate nonlinear partial differential equations. Based on the relationships of complex nonlinear partial differential equations and ODEs, a number of methods, such as, the sinh-Gordon equation expansion method [42], the generalized F-expansion method [43], [44], the projective Riccati equation method [45], [46], the algebraic method [47] etc. have been developed.

In the present article, we combine the generalized Riccati equation with the (G'/G) -expansion method, called alternative (G'/G) -expansion method introduced recently by Akbar et al. [48] to find the exact traveling wave solutions of the sixth-order Boussinesq equation.

2. The Alternative (G'/G) -Expansion Method

Suppose we have the following nonlinear partial differential equation,

$$(1) \quad P(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \dots) = 0.$$

where $u = u(x, t)$ is an unknown function, P is a polynomial in $u = u(x, t)$ and its partial derivatives in which the highest order derivatives and the nonlinear

terms are involved. The main steps of the alternative (G'/G) -expansion method are:

Step 1. The traveling wave variable,

$$(2) \quad u(x, t) = u(\xi), \xi = x - Vt,$$

where V is the speed of the traveling wave, which converts the Eq. (1) into an ODE in the form,

$$(3) \quad Q(u, u', u'', u''', \dots) = 0,$$

where prime denotes the derivative with respect to ξ .

Step 2. If Eq. (3) is integrable, integrate it term by term one or more times, yields constant(s) of integration.

Step 3. Suppose that the solution of the Eq. (3) can be expressed by means of a polynomial in (G'/G) as follows:

$$(4) \quad u(\xi) = \sum_{i=0}^n a_i \left(\frac{G'}{G}\right)^i, a_n \neq 0$$

where $G = G(\xi)$ satisfies the generalized Riccati equation,

$$(5) \quad G' = r + pG + qG^2,$$

where a_i ($i = 1, 2, 3, \dots, n$), p , q and r are random constants to be determined later.

The generalized Riccati Eq. (5) has the following twenty seven solutions [49].

Family 1.

When $p^2 - 4qr < 0$ and $pq \neq 0$ (or $qr \neq 0$), the solutions of Eq. (5) are,

$$\begin{aligned} G_1 &= \frac{1}{2q} \left[-p + \sqrt{4qr - p^2} \tan \left(\frac{1}{2} \sqrt{4qr - p^2} \xi \right) \right], \\ G_2 &= -\frac{1}{2q} \left[p + \sqrt{4qr - p^2} \cot \left(\frac{1}{2} \sqrt{4qr - p^2} \xi \right) \right], \\ G_3 &= \frac{1}{2q} \left[p + \sqrt{4qr - p^2} \left(\tan \left(\sqrt{4qr - p^2} \xi \right) \pm \sec \left(\sqrt{4qr - p^2} \xi \right) \right) \right], \\ G_4 &= -\frac{1}{2q} \left[p + \sqrt{4qr - p^2} \left(\cot \left(\sqrt{4qr - p^2} \xi \right) \pm \csc \left(\sqrt{4qr - p^2} \xi \right) \right) \right], \\ G_5 &= \frac{1}{4q} \left[-2p + \sqrt{4qr - p^2} \left(\tan \left(\frac{1}{4} \sqrt{4qr - p^2} \xi \right) - \cot \left(\frac{1}{4} \sqrt{4qr - p^2} \xi \right) \right) \right], \\ G_6 &= \frac{1}{2q} \left[-p + \frac{\sqrt{(A^2 - B^2)(4qr - p^2)} - A\sqrt{4qr - p^2} \cos(\sqrt{4qr - p^2} \xi)}{A \sin(\sqrt{4qr - p^2} \xi) + B} \right], \\ G_7 &= \frac{1}{2q} \left[-p + \frac{\sqrt{(A^2 - B^2)(4qr - p^2)} + A\sqrt{4qr - p^2} \cos(\sqrt{4qr - p^2} \xi)}{A \sin(\sqrt{4qr - p^2} \xi) + B} \right], \end{aligned}$$

where A and B are two non-zero real constants and satisfies the condition $A^2 - B^2 > 0$.

$$\begin{aligned}
G_8 &= \frac{-2r \cos\left(\frac{1}{2}\sqrt{4qr-p^2}\xi\right)}{\sqrt{4qr-p^2} \sin\left(\frac{1}{2}\sqrt{4qr-p^2}\xi\right) + p \cos\left(\frac{1}{2}\sqrt{4qr-p^2}\xi\right)}, \\
G_9 &= \frac{2r \sin\left(\frac{1}{2}\sqrt{4qr-p^2}\xi\right)}{-p \sin\left(\frac{1}{2}\sqrt{4qr-p^2}\xi\right) + \sqrt{(4qr-p^2)} \cos\left(\frac{1}{2}\sqrt{4qr-p^2}\xi\right)}, \\
G_{10} &= \frac{-2r \cos\left(\sqrt{4qr-p^2}\xi\right)}{\sqrt{(4qr-p^2)} \sin\left(\sqrt{4qr-p^2}\xi\right) + p \cos\left(\sqrt{4qr-p^2}\xi\right) \pm \sqrt{(4qr-p^2)}}, \\
G_{11} &= \frac{2r \sin\left(\sqrt{4qr-p^2}\xi\right)}{-p \sin\left(\sqrt{4qr-p^2}\xi\right) + \sqrt{(4qr-p^2)} \cos\left(\sqrt{4qr-p^2}\xi\right) \pm \sqrt{(4qr-p^2)}}, \\
G_{12} &= \frac{4r \sin\left(\frac{1}{4}\sqrt{4qr-p^2}\xi\right) \cos\left(\frac{1}{4}\sqrt{4qr-p^2}\xi\right)}{-2p \sin\left(\frac{1}{4}\sqrt{4qr-p^2}\xi\right) \cos\left(\frac{1}{4}\sqrt{4qr-p^2}\xi\right) + 2\sqrt{(4qr-p^2)} \cos^2\left(\frac{1}{4}\sqrt{4qr-p^2}\xi\right) - \sqrt{(4qr-p^2)}}.
\end{aligned}$$

Family 2.

When $p^2 - 4qr > 0$ and $pq \neq 0$ (or $qr \neq 0$), the solutions of Eq. (5) are,

$$\begin{aligned}
G_{13} &= -\frac{1}{2q} \left[p + \sqrt{p^2 - 4qr} \tanh\left(\frac{1}{2}\sqrt{p^2 - 4qr}\xi\right) \right], \\
G_{14} &= -\frac{1}{2q} \left[p + \sqrt{p^2 - 4qr} \coth\left(\frac{1}{2}\sqrt{p^2 - 4qr}\xi\right) \right], \\
G_{15} &= -\frac{1}{2q} \left[p + \sqrt{p^2 - 4qr} \left(\tanh\left(\sqrt{p^2 - 4qr}\xi\right) \pm i \operatorname{sech}\left(\sqrt{p^2 - 4qr}\xi\right) \right) \right], \\
G_{16} &= -\frac{1}{2q} \left[p + \sqrt{p^2 - 4qr} \left(\coth\left(\sqrt{p^2 - 4qr}\xi\right) \pm \operatorname{csc h}\left(\sqrt{p^2 - 4qr}\xi\right) \right) \right], \\
G_{17} &= -\frac{1}{4q} \left[2p + \sqrt{p^2 - 4qr} \left(\tanh\left(\frac{1}{4}\sqrt{p^2 - 4qr}\xi\right) + \coth\left(\frac{1}{4}\sqrt{p^2 - 4qr}\xi\right) \right) \right], \\
G_{18} &= \frac{1}{2q} \left[-p + \frac{\sqrt{(A^2+B^2)(p^2-4qr)} - A\sqrt{p^2-4qr} \cosh\left(\sqrt{p^2-4qr}\xi\right)}{A \sinh\left(\sqrt{p^2-4qr}\xi\right) + B} \right], \\
G_{19} &= \frac{1}{2q} \left[-p - \frac{\sqrt{(B^2-A^2)(p^2-4qr)} + A\sqrt{p^2-4qr} \cosh\left(\sqrt{p^2-4qr}\xi\right)}{A \sinh\left(\sqrt{p^2-4qr}\xi\right) + B} \right],
\end{aligned}$$

where A and B are two non-zero real constants and satisfies the condition $B^2 - A^2 > 0$.

$$\begin{aligned}
G_{20} &= \frac{2r \cosh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)}{\sqrt{p^2-4qr} \sinh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right) - p \cosh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)}, \\
G_{21} &= \frac{2r \sinh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)}{\sqrt{p^2-4qr} r \cosh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right) - p \sinh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)}, \\
G_{22} &= \frac{2r \cosh\left(\sqrt{p^2-4qr}\xi\right)}{\sqrt{p^2-4qr} \sinh\left(\sqrt{p^2-4qr}\xi\right) - p \cosh\left(\sqrt{p^2-4qr}\xi\right) \pm i\sqrt{p^2-4qr}}, \\
G_{23} &= \frac{2r \sinh\left(\sqrt{p^2-4qr}\xi\right)}{-p \sinh\left(\sqrt{p^2-4qr}\xi\right) + \sqrt{p^2-4qr} \cosh\left(\sqrt{p^2-4qr}\xi\right) \pm \sqrt{p^2-4qr}}, \\
G_{24} &= \frac{4r \sinh\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right) \cosh\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right)}{-2p \sinh\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right) \cosh\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right) + 2\sqrt{p^2-4qr} \cosh^2\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right) - \sqrt{p^2-4qr}}.
\end{aligned}$$

Family 3.

When $r = 0$ and $pq \neq 0$, the solutions of Eq. (5) are,

$$\begin{aligned}
G_{25} &= \frac{-pd}{q[d + \cosh(p\xi) - \sinh(p\xi)]}, \\
G_{26} &= -\frac{p[\cosh(p\xi) + \sinh(p\xi)]}{q[d + \cosh(p\xi) + \sinh(p\xi)]},
\end{aligned}$$

where d is an arbitrary constant.

Family 4.

When $q \neq 0$ and $r = p = 0$, the solution of Eq. (5) is,

$$G_{27} = -\frac{1}{q\xi + d_1},$$

where d_1 is an arbitrary constant.

Step 4. In Eq. (4), n is a positive integer which is usually obtained by balancing the highest order nonlinear term(s) with the linear term(s) of the highest order come out in Eq. (3).

Step 5. Substituting Eq. (4) into Eq. (3) and utilizing Eq. (5), we obtain polynomials in G^i and G^{-i} ($i = 0, 1, 2, 3, \dots$). Vanishing each coefficient of the resulted polynomials to zero, yields a set of algebraic equations for a_n , p , q , r , V and constant(s) of integration, if applicable. Suppose with the aid of symbolic computation software such as Maple, the unknown constants a_n , p , q , r and V can be found by solving these set of algebraic equations and substituting these values into Eq. (4), we obtain new exact traveling wave solutions of the nonlinear partial differential equation (1).

3. Some new traveling wave solutions of the sixth-order Boussinesq equation

In this section, the alternating (G'/G) -expansion method together with the generalized Riccati equation is employed to construct some new traveling wave solutions for the sixth-order Boussinesq equation [50], [51]:

$$(1) \quad u_{ttt} - u_{xxx} - [15u u_{4x} + 30u_x u_{3x} + 15(u_{2x})^2 + 45u^2 u_{2x} + 90u u_x^2 + u_{6x}] = 0,$$

Now, using the traveling wave variable (2) in Eq. (1), we have

$$(2) \quad V^2 u'' - u'' - [15u u^{(4)} + 30u' u''' + 15(u'')^2 + 45u^2 u'' + 90u (u')^2 + u^{(6)}] = 0,$$

where $u^{(4)}$ and $u^{(6)}$ denotes the fourth and sixth derivative of u with respect to ξ . Integrating Eq. (2) once, we obtain:

$$(3) \quad C - (1 + 45u^2 - V^2) u' - 15u' u'' - 15u u''' - u^{(5)} = 0.$$

where C is constant of integration. According to Step 3, the solution of Eq. (3) can be expressed by a polynomial in (G'/G) as follows:

$$(4) \quad u(\xi) = a_0 + a_1 \left(\frac{G'}{G}\right) + a_2 \left(\frac{G'}{G}\right)^2 + \dots + a_n \left(\frac{G'}{G}\right)^n, \quad a_n \neq 0$$

where a_i , ($i = 0, 1, 2, 3, \dots, n$) are constant to be determined and $G = G(\xi)$ satisfies the generalized Riccati Eq. (5). Considering the homogeneous balance between the highest order derivative and the nonlinear terms in Eq. (3), we obtain $n = 2$.

Therefore, the solution of Eq. (4) takes the form:

$$(5) \quad u(\xi) = a_0 + a_1 \left(\frac{G'}{G}\right) + a_2 \left(\frac{G'}{G}\right)^2, \quad a_2 \neq 0$$

Using Eq. (5), Eq. (5) can be rewritten as:

$$(6) \quad u(\xi) = a_0 + a_1 (p + r G^{-1} + q G) + a_2 (p + r G^{-1} + q G)^2.$$

Substituting Eq. (6) into Eq. (3), we obtain the following polynomials in G^i and G^{-i} , ($i = 0, 1, 2, 3, \dots$). Setting each coefficient of these resulted polynomial to zero, we obtain a set of simultaneous algebraic equations for a_0, a_1, a_2, p, q, r and V as follows:

$$\begin{aligned} & 540 a_2^2 q^7 + 720 a_2 q^7 + 90 a_2^3 q^7 = 0, \\ & 600 a_1 a_2 q^6 + 25 a_1 a_2^2 q^6 + 540 a_2^3 p q^6 + 2490 a_2^2 p q^6 + 2640 a_2 p q^6 \\ & + 120 a_1 q^6 = 0, \\ & 2220 a_1 a_2 p q^5 + 4590 a_2^2 p^2 q^5 + 1350 a_2^3 p^2 q^5 + 120 a_1^2 q^5 + 360 a_0 a_2 q^5 \\ & + 1740 a_2^2 r q^6 + 180 a_0 a_2^2 q^5 + 1125 a_1 a_2^2 p q^5 + 180 a_2 a_1^2 q^5 + 450 a_2^3 r q^6 \\ & + 3720 a_2 p^2 q^5 + 1680 a_2 r q^6 + 360 a_1 p q^5 = 0, \\ & 3105 a_1 a_2 p^2 q^4 + 345 a_1^2 p q^4 + 240 a_1 r q^5 + 6120 a_2^2 r p q^5 + 720 a_0 a_2^2 p q^4 \\ & + 2160 a_2^3 r p q^5 + 990 a_0 a_2 p q^4 + 900 a_1 a_2^2 r q^5 + 270 a_0 a_1 a_2 q^4 + 1800 a_2^3 p^3 q^4 \\ & + 1500 a_1 a_2 r q^5 + 2250 a_1 a_2^2 p^2 q^4 + 4440 a_2 r p q^5 + 390 a_1 p^2 q^4 + 4260 a_2^2 p^3 q^4 \\ & + 720 a_2 a_1^2 p q^4 + 45 a_1^3 q^4 + 90 a_0 a_1 q^4 + 2490 a_2 p^3 q^4 = 0, \\ & 782 a_2 p^4 q^3 + 180 a_1 p^3 q^3 + 1232 a_2 r^2 q^5 + 90 a_0 a_1^2 q^3 + 90 a_2 a_0^2 q^3 \\ & + 135 a_1^3 p q^3 + 810 a_2^3 r^2 q^5 + 1350 a_2^3 p^4 q^3 + 240 a_1^2 r q^3 + 345 a_1^2 p^2 q^3 \\ & + 3960 a_1 a_2 r p q^4 + 3375 a_1 a_2^2 r p q^4 + 810 a_0 a_1 a_2 p q^3 + 1995 a_1 a_2 p^3 q^3 \\ & + 540 a_0 a_2^2 r q^4 + 1080 a_0 a_2^2 p^2 q^3 + 540 a_1^2 a_2 r q^4 + 1080 a_1^2 a_2 p^2 q^3 \\ & + 2250 a_1 a_2^2 p^3 q^3 + 4050 a_2^3 r p^2 q^4 + 180 a_0 a_1 p q^3 + 930 a_0 a_2 p^2 q^3 \\ & + 600 a_0 a_2 r q^4 + 8010 a_2^2 r p^2 q^4 + 4064 a_2 r p^2 q^4 + 480 a_1 r p q^4 \\ & + 1980 a_2^2 r^2 q^5 + 2040 a_2^2 p^4 q^3 + 2 a_2 q^3 - 2 V^2 a_2 q^3 = 0, \\ & 45 a_1 a_0^2 q^2 + 136 a_1 r^2 q^4 + 31 a_1 p^4 q^2 + 94 a_2 p^5 q^2 + 90 a_1^3 r q^3 \\ & + 135 a_1^3 p^2 q^2 + 540 a_2^3 p^5 q^2 + 135 a_1^2 p^3 q^2 + 3510 a_1 a_2 r p^2 q^3 \\ & + 1020 a_0 a_2 r p q^3 + 4500 a_1 a_2^2 r p^2 q^3 + 540 a_0 a_1 a_2 r q^3 \\ & + 810 a_0 a_1 a_2 p^2 q^2 + 1440 a_0 a_2^2 r p q^3 + 1440 a_2 a_1^2 r p q^3 \\ & + 1200 a_1 a_2 r^2 q^4 + 450 a_1^2 r p q^3 + 555 a_1 a_2 p^4 q^2 + 180 a_0^2 a_2 p q^2 \\ & + 720 a_0 a_2^2 p^3 q^2 + 180 a_0 a_1^2 p q^2 + 1125 a_1 a_2^2 r^2 q^4 - V^2 a_1 q^2 \\ & + 720 a_2 a_1^2 p^3 q^2 + 1125 a_1 a_2^2 p^4 q^2 + 2700 a_2^3 r^2 p q^4 + 3600 a_2^3 r p^3 q^3 \\ & + 120 a_0 a_1 r q^3 + 105 a_0 a_1 p^2 q^2 + 330 a_0 a_2 p^3 q^2 + 4770 a_2^2 r^2 p q^4 \\ & + 4680 a_2^2 r p^3 q^3 + 292 a_1 r p^2 q^3 + 1984 a_2 r^2 p q^4 + 1468 a_2 r p^3 q^3 \\ & + 450 a_2^2 p^5 q^2 + a_1 q^2 + 4 a_2 p q^2 - 4 V^2 a_2 p q^2 = 0, \\ & 272 a_2 r^3 q^4 + a_1 p^5 q + 810 a_0 a_1 a_2 r p q^2 + 2 a_2 p^6 q + 45 a_1^3 p^3 q \\ & + 90 a_2^3 p^6 q + 1080 a_1^2 a_2 r p^2 q^2 + 2250 a_2^2 a_1 p^3 q^2 + 1740 a_1 a_2 r^2 p q^3 \\ & + 1095 a_1 a_2 r p^3 q^2 + 120 a_0 a_1 r p q^2 + 450 a_0 a_2 r p^2 q^2 + 15 a_1^2 p^4 q \\ & + 30 a_2^2 p^6 q + 780 a_2^2 r^3 q^4 + 1080 a_0 a_2^2 r p^2 q^2 + 270 a_0 a_1 a_2 p^3 q \\ & + a_1 p q - V^2 a_1 p q + 2 a_2 r q^2 + 450 a_2^3 r^3 q^4 + 120 a_1^2 r^2 q^3 + 2 a_2 p^2 q \\ & + 2250 a_1 a_2^2 r^2 p q^3 + 90 a_0 a_1^2 q - 2 V^2 a_2 p^2 q + 15 a_0 a_1 p^3 q + 30 a_0 a_2 p^4 q \\ & + 240 a_0 a_2 r^2 q^3 + 3420 a_2^2 r^2 p^2 q^3 + 1080 a_2^2 r p^4 q^2 + 135 a_1^3 r p q^2 + 90 a_0 a_1^2 r q^2 \\ & + 360 a_0 a_2^2 r^2 q^3 + 2700 a_2^3 r^2 p^2 q^3 + 136 a_1 r^2 p q^3 + 52 a_1 r p^3 q^2 + 856 a_2 r^2 p^2 q^3 \\ & + 45 a_1 a_2 p^5 q + 225 a_1^2 r p^2 q^2 + 360 a_2 a_1^2 r^2 q^3 + 45 a_1 a_0^2 p q + 90 a_2 a_0^2 r q^2 \\ & + 90 a_2 a_0^2 p^2 q + 180 a_2 a_1^2 p^4 q + 225 a_1 a_2^2 p^5 q + 1350 a_2^3 r p^4 q^2 + 166 a_2 r p^4 q^2 \\ & + 180 a_0 a_2^2 p^4 q - 2 V^2 a_2 r q^2 = 0, \end{aligned}$$

$$\begin{aligned}
& 94a_2r^2p^5 + 45a_1a_0^2r^2 + 31a_1r^2p^4 + 136a_1r^4q^2 + 135a_1^3r^2p^2 \\
& + a_1r^2 + 540a_2^3r^2p^5 + a_0a_1a_2r^3q + 3510a_1a_2r^3p^2q \\
& + 1020a_0a_2r^3pq + 810a_0a_1a_2r^2p^2 + 90a_1^3r^3q + 450a_2^2r^2p^5 \\
& + 1040a_0a_2^2r^3pq + 1040a_2a_1^2r^3pq + 4500a_2^2a_1r^3p^2q + 135a_1^2r^2p^3 \\
& + 4a_2r^2p + 120a_0a_1r^3q + 105a_0a_1r^2p^2 + 330a_0a_2r^2p^3 + 3600a_2^3r^3p^3q \\
& + 292a_1r^3p^2q + 4680a_2^2r^3p^3q + 4770a_2^2r^4pq^2 + 1200a_1r^4a_2q^2 \\
& + 450a_1^2r^3pq + 555a_1r^2a_2p^4 + 180a_2a_0^2r^2p + 180a_0a_1^2r^2p + 1984a_2r^4pq^2 \\
& + 1468a_2r^3p^3q + 1125a_2^2a_1r^4q^2 + 2700a_2^3r^4pq^2 + 720a_0a_2^2r^2p^3 \\
& + 720a_2a_1^2r^2p^3 + 1125a_2^2a_1r^2p^4 - 4V^2a_2r^2p - V^2a_1r^2 = 0, \\
& 135a_1^3r^3p + 180a_0a_1r^3p + 1080a_0a_2^2r^3p^2 + 1350a_2^3r^3p^4 + 4064a_2r^4p^2q \\
& + 3375a_1r^4a_2^2pq + 8010r^4a_2^2p^2q + 180a_1r^3p^3 + 930a_0a_2r^3p^2 + 810a_0a_1a_2r^3p \\
& + 782a_2r^3p^4 + 3960a_1a_2r^4pq + 90a_0a_1^2r^3 + 480a_1r^4pq + 4050a_2^3r^4p^2q \\
& + 2040a_2^2r^3p^4 + 600a_0a_2r^4q + 240a_1^2r^4q + 1080a_2a_1^2r^3p^2 + 1232a_2r^5q^2 \\
& + 810a_2^3r^5q^2 + 90a_2a_0^2r^3 + 2a_2r^3 + 1980a_2^2r^5q^2 + 345a_1^2r^3p^2 + 540a_0a_2^2r^4q \\
& + 1995a_1a_2r^3p^3 + 2250a_1a_2^2r^3p^3 + 540a_2a_1^2r^4q - 2V^2a_2r^3 = 0, \\
& 720a_0a_2^2r^4p + 4440a_2r^5pq + 6120a_2^2r^5pq + 240a_1r^5q + 1800a_2^3r^4p^3 \\
& + 45a_1^3r^4 + 4260a_2^2r^4p^3 + 390a_1r^4p^2 + 720a_2a_1^2r^4p + 345a_1^2r^4p \\
& + 3105a_1a_2r^4p^2 + 2160a_2^3r^5pq + 990a_0a_2r^4p + 2250a_1a_2^2r^4p^2 \\
& + 270a_0a_1a_2r^4 + 90a_0a_1r^4 + 2490a_2r^4p^3 + 900a_1a_2^2r^5q + 1500a_1a_2r^5q = 0, \\
& 4590a_2^2r^5p^2 + 360a_1r^5p + 3720a_2r^5p^2 + 2220a_1a_2r^5p + 1125a_1a_2^2r^5p \\
& + 1350a_2^3r^5p^2 + 120a_1^2r^5 + 180a_1^2a_2r^5 + 180a_0a_2^2r^5 + 360a_0a_2r^5 \\
& + 1680a_2r^6q + 1740a_2^2r^6q + 450a_2^3r^6q = 0, \\
(7) \quad & 540a_2^3r^6p + 2490a_2^2r^6p + 2640a_2r^6p + 600a_1a_2r^6 + 120a_1r^6 \\
& + 225a_1a_2^2r^6 = 0,
\end{aligned}$$

$$540a_2^2r^7 + 90a_2^3r^7 + 720a_2r^7 = 0,$$

$$C = 0.$$

Solving the above set of algebraic equations by using the symbolic computation software, such as, Maple, we obtain

$$\begin{aligned}
(8) \quad a_0 &= -\frac{1}{3}p^2 + \frac{4}{3}rq, \quad a_1 = 2p, \quad a_2 = -2, \\
V &= \sqrt{p^4 - 8p^2qr + 16q^2r^2 + 1}, \quad C = 0,
\end{aligned}$$

where p , q and r are arbitrary constants.

Now, on the basis of the solutions of Eq. (5), we obtain the following families of solutions of Eq. (1).

Family 1.

When $p^2 - 4qr < 0$ and $pq \neq 0$ (or $qr \neq 0$), the periodic form solutions of Eq. (1) are,

$$u_1 = -\frac{1}{3}p^2 + \frac{4}{3}rq + 2p \left(\frac{2\Psi^2 \sec^2(\Psi\xi)}{-p+2\Psi \tan(\Psi\xi)} \right) - 2 \left(\frac{2\Psi^2 \sec^2(\Psi\xi)}{-p+2\Psi \tan(\Psi\xi)} \right)^2,$$

where $\Psi = \frac{1}{2}\sqrt{4qr - p^2}$, $\xi = x - \sqrt{p^4 - 8p^2qr + 16q^2r^2 + 1}t$, p , q and r are arbitrary constants.

$$u_2 = -\frac{1}{3}p^2 + \frac{4}{3}rq - 2p \left(\frac{2\Psi^2 \csc^2(\Psi\xi)}{p+2\Psi \cot(\Psi\xi)} \right) - 2 \left(\frac{2\Psi^2 \csc^2(\Psi\xi)}{p+2\Psi \cot(\Psi\xi)} \right)^2,$$

$$u_3 = -\frac{1}{3}p^2 + \frac{4}{3}rq + 2p \left(\frac{4\Psi^2 \sec(2\Psi\xi)(1 \pm \sin(2\Psi\xi))}{-p \cos(2\Psi\xi) + 2\Psi \sin(2\Psi\xi) \pm 2\Psi} \right) - 2 \left(\frac{4\Psi^2 \sec(2\Psi\xi)(1 \pm \sin(2\Psi\xi))}{-p \cos(2\Psi\xi) + 2\Psi \sin(2\Psi\xi) \pm 2\Psi} \right)^2,$$

$$u_4 = -\frac{1}{3}p^2 + \frac{4}{3}rq - 2p \left(\frac{4\Psi^2 \csc(2\Psi\xi)(1 \pm \cos(2\Psi\xi))}{p \sin(2\Psi\xi) + 2\Psi \cos(2\Psi\xi) \pm 2\Psi} \right) - 2 \left(\frac{4\Psi^2 \csc(2\Psi\xi)(1 \pm \cos(2\Psi\xi))}{p \sin(2\Psi\xi) + 2\Psi \cos(2\Psi\xi) \pm 2\Psi} \right)^2,$$

$$u_5 = -\frac{1}{3}p^2 + \frac{4}{3}rq - 2p \left(\frac{2\Psi^2 \csc(\Psi\xi)}{p \sin(\Psi\xi) + 2\Psi \cos(2\Psi\xi)} \right) - 2 \left(\frac{2\Psi^2 \csc(\Psi\xi)}{p \sin(\Psi\xi) + 2\Psi \cos(2\Psi\xi)} \right)^2,$$

$$u_6 = -2p \left(\frac{4A\Psi^2 \{ \sqrt{A^2 - B^2} \cos(2\Psi\xi) - B \sin(2\Psi\xi) - A \} \{ A \sin(2\Psi\xi) + B \}}{\{ A^2 \cos^2(2\Psi\xi) - A^2 - B^2 - 2AB \sin(2\Psi\xi) \} \{ pA \sin(2\Psi\xi) + 2A\Psi \cos(2\Psi\xi) + pB - 2\Psi \sqrt{A^2 - B^2} \}} \right) - 2 \left(\frac{4A\Psi^2 \{ \sqrt{A^2 - B^2} \cos(2\Psi\xi) - B \sin(2\Psi\xi) - A \} \{ A \sin(2\Psi\xi) + B \}}{\{ A^2 \cos^2(2\Psi\xi) - A^2 - B^2 - 2AB \sin(2\Psi\xi) \} \{ pA \sin(2\Psi\xi) + 2A\Psi \cos(2\Psi\xi) + pB - 2\Psi \sqrt{A^2 - B^2} \}} \right)^2 - \frac{1}{3}p^2 + \frac{4}{3}rq,$$

$$u_7 = -2p \left(\frac{4A\Psi^2 \{ \sqrt{A^2 - B^2} \cos(2\Psi\xi) + B \sin(2\Psi\xi) + A \} \{ A \sin(2\Psi\xi) + B \}}{\{ A^2 \cos^2(2\Psi\xi) - A^2 - B^2 - 2AB \sin(2\Psi\xi) \} \{ pA \sin(2\Psi\xi) - 2A\Psi \cos(2\Psi\xi) + pB - 2\Psi \sqrt{A^2 - B^2} \}} \right) - 2 \left(\frac{4A\Psi^2 \{ \sqrt{A^2 - B^2} \cos(2\Psi\xi) + B \sin(2\Psi\xi) + A \} \{ A \sin(2\Psi\xi) + B \}}{\{ A^2 \cos^2(2\Psi\xi) - A^2 - B^2 - 2AB \sin(2\Psi\xi) \} \{ pA \sin(2\Psi\xi) - 2A\Psi \cos(2\Psi\xi) + pB - 2\Psi \sqrt{A^2 - B^2} \}} \right)^2 - \frac{1}{3}p^2 + \frac{4}{3}rq,$$

where A and B are two non-zero real constants and satisfies the condition $A^2 - B^2 > 0$.

$$u_8 = -\frac{1}{3}p^2 + \frac{4}{3}rq - 2p \left(\frac{2\Psi^2 \sec(\Psi\xi) \{ p \cos(\Psi\xi) + 2\Psi \sin(\Psi\xi) \}}{2(p^2 - 2rq) \cos^2(\Psi\xi) + 4p\Psi \sin(\Psi\xi) \cos(\Psi\xi) + 4\Psi^2} \right) - 2 \left(\frac{2\Psi^2 \sec(\Psi\xi) \{ p \cos(\Psi\xi) + 2\Psi \sin(\Psi\xi) \}}{2(p^2 - 2rq) \cos^2(\Psi\xi) + 4p\Psi \sin(\Psi\xi) \cos(\Psi\xi) + 4\Psi^2} \right)^2,$$

$$u_9 = -\frac{1}{3}p^2 + \frac{4}{3}rq + 2p \left(\frac{2\Psi^2 \csc(\Psi\xi) \{ p \sin(\Psi\xi) - 2\Psi \cos(\Psi\xi) \}}{2(p^2 - 2rq) \cos^2(\Psi\xi) + 4p\Psi \sin(\Psi\xi) \cos(\Psi\xi) - p^2} \right) - 2 \left(\frac{2\Psi^2 \csc(\Psi\xi) \{ p \sin(\Psi\xi) - 2\Psi \cos(\Psi\xi) \}}{2(p^2 - 2rq) \cos^2(\Psi\xi) + 4p\Psi \sin(\Psi\xi) \cos(\Psi\xi) - p^2} \right)^2,$$

$$u_{10} = -\frac{1}{3}p^2 + \frac{4}{3}rq - 2p \left(\frac{2\Psi^2 \sec(2\Psi\xi) \{ 1 \pm \sin(2\Psi\xi) \} \{ p \cos(2\Psi\xi) + 2\Psi \sin(2\Psi\xi) \pm 2\Psi \}}{(p^2 - 2rq) \cos^2(2\Psi\xi) + 2\Psi \{ 1 \pm \sin(2\Psi\xi) \} \{ 2\Psi \pm p \cos(2\Psi\xi) \}} \right) - 2 \left(\frac{2\Psi^2 \sec(2\Psi\xi) \{ 1 \pm \sin(2\Psi\xi) \} \{ p \cos(2\Psi\xi) + 2\Psi \sin(2\Psi\xi) \pm 2\Psi \}}{(p^2 - 2rq) \cos^2(2\Psi\xi) + 2\Psi \{ 1 \pm \sin(2\Psi\xi) \} \{ 2\Psi \pm p \cos(2\Psi\xi) \}} \right)^2,$$

$$u_{11} = -\frac{1}{3}p^2 + \frac{4}{3}rq \pm 2p \left(\frac{2\Psi^2 \csc(2\Psi\xi) \{ -p \sin(2\Psi\xi) + 2\Psi \cos(2\Psi\xi) \pm 2\Psi \}}{(2rq - p^2) \cos(2\Psi\xi) - 2p \{ 1 \pm \sin(2\Psi\xi) \} \{ 2p\Psi \sin(2\Psi\xi) \pm 2rq \}} \right) - 2 \left(\frac{2\Psi^2 \csc(2\Psi\xi) \{ -p \sin(2\Psi\xi) + 2\Psi \cos(2\Psi\xi) \pm 2\Psi \}}{(2rq - p^2) \cos(2\Psi\xi) - 2p \{ 1 \pm \sin(2\Psi\xi) \} \{ 2p\Psi \sin(2\Psi\xi) \pm 2rq \}} \right)^2,$$

$$u_{12} = -\frac{1}{3}p^2 + \frac{4}{3}rq + 2p \left(\frac{2\Psi^2 \csc(\Psi\xi) \{ p \sin(\Psi\xi) - 2\Psi \cos(\Psi\xi) \}}{2(p^2 - 2rq) \cos^2(\Psi\xi) + 4p\Psi \sin(\Psi\xi) \cos(\Psi\xi) - p^2} \right) - 2 \left(\frac{2\Psi^2 \csc(\Psi\xi) \{ p \sin(\Psi\xi) - 2\Psi \cos(\Psi\xi) \}}{2(p^2 - 2rq) \cos^2(\Psi\xi) + 4p\Psi \sin(\Psi\xi) \cos(\Psi\xi) - p^2} \right)^2.$$

Family 2.

When $p^2 - 4qr > 0$ and $pq \neq 0$ (or $qr \neq 0$), the soliton and soliton-like solutions of Eq. (1) are,

$$u_{13} = -\frac{1}{3}p^2 + \frac{4}{3}rq + 2p \left(\frac{2\Delta^2 \sec h^2(\Delta\xi)}{p+2\Delta \tanh(\Delta\xi)} \right) - 2 \left(\frac{2\Delta^2 \sec h^2(\Delta\xi)}{p+2\Delta \tanh(\Delta\xi)} \right)^2,$$

where $\Delta = \frac{1}{2}\sqrt{p^2 - 4qr}$, $\xi = x - \sqrt{p^4 - 8p^2qr + 16q^2r^2 + 1}t$, p , q and r are arbitrary constants.

$$\begin{aligned}
 u_{14} &= -\frac{1}{3}p^2 + \frac{4}{3}rq - 2p \left(\frac{2\Delta^2 \csc h^2(\Delta\xi)}{p+2\Delta \coth(\Delta\xi)} \right) - 2 \left(\frac{2\Delta^2 \csc h^2(\Delta\xi)}{p+2\Delta \coth(\Delta\xi)} \right)^2, \\
 u_{15} &= -\frac{1}{3}p^2 + \frac{4}{3}rq + 2p \left(\frac{4\Delta^2 \sec h(2\Delta\xi)(1 \mp i \sinh(2\Delta\xi))}{p \cosh(2\Delta\xi) + 2\Delta \sinh(2\Delta\xi) \pm i 2\Delta} \right) - 2 \left(\frac{4\Delta^2 \sec h(2\Delta\xi)(1 \mp i \sinh(2\Delta\xi))}{p \cosh(2\Delta\xi) + 2\Delta \sinh(2\Delta\xi) \pm i 2\Delta} \right)^2, \\
 u_{16} &= -\frac{1}{3}p^2 + \frac{4}{3}rq - 2p \left(\frac{4\Delta^2 \csc h(2\Delta\xi)(1 \pm \cosh(2\Delta\xi))}{p \sinh(2\Delta\xi) + 2\Delta \cosh(2\Delta\xi) \pm 2\Delta} \right) - 2 \left(\frac{4\Delta^2 \csc h(2\Delta\xi)(1 \pm \cosh(2\Delta\xi))}{p \sinh(2\Delta\xi) + 2\Delta \cosh(2\Delta\xi) \pm 2\Delta} \right)^2, \\
 u_{17} &= -\frac{1}{3}p^2 + \frac{4}{3}rq - 2p \left(\frac{\Delta^2 \sec h^2(\Delta\xi/2)}{2 \{ \cosh^2(\Delta\xi/2) - 1 \} \{ p + \Delta (\tanh(\Delta\xi/2) + \coth(\Delta\xi/2)) \}} \right) \\
 &\quad - 2 \left(\frac{\Delta^2 \sec h^2(\Delta\xi/2)}{2 \{ \cosh^2(\Delta\xi/2) - 1 \} \{ p + \Delta (\tanh(\Delta\xi/2) + \coth(\Delta\xi/2)) \}} \right)^2, \\
 u_{18} &= -\frac{1}{3}p^2 + \frac{4}{3}rq - 2p \left(\frac{4A\Delta^2 \{ A - B \sinh(2\Delta\xi) - \sqrt{A^2 + B^2} \cosh(2\Delta\xi) \}}{(A \sin(2\Delta\xi) + B) \{ pA \sinh(2\Delta\xi) + pB - 2\Delta\sqrt{A^2 + B^2} + 2A\Delta \cosh(2\Delta\xi) \}} \right) \\
 &\quad - 2 \left(\frac{4A\Delta^2 \{ A - B \sinh(2\Delta\xi) - \sqrt{A^2 + B^2} \cosh(2\Delta\xi) \}}{(A \sin(2\Delta\xi) + B) \{ pA \sinh(2\Delta\xi) + pB - 2\Delta\sqrt{A^2 + B^2} + 2A\Delta \cosh(2\Delta\xi) \}} \right)^2, \\
 u_{19} &= -\frac{1}{3}p^2 + \frac{4}{3}rq - 2p \left(\frac{4A\Delta^2 \{ A - B \sinh(2\Delta\xi) + \sqrt{A^2 + B^2} \cosh(2\Delta\xi) \}}{(A \sin(2\Delta\xi) + B) \{ pA \sinh(2\Delta\xi) + pB + 2\Delta\sqrt{A^2 + B^2} + 2A\Delta \cosh(2\Delta\xi) \}} \right) \\
 &\quad - 2 \left(\frac{4A\Delta^2 \{ A - B \sinh(2\Delta\xi) + \sqrt{A^2 + B^2} \cosh(2\Delta\xi) \}}{(A \sin(2\Delta\xi) + B) \{ pA \sinh(2\Delta\xi) + pB + 2\Delta\sqrt{A^2 + B^2} + 2A\Delta \cosh(2\Delta\xi) \}} \right)^2,
 \end{aligned}$$

where A and B are two non-zero real constants and satisfies the condition $A^2 - B^2 < 0$.

$$\begin{aligned}
 u_{20} &= -\frac{1}{3}p^2 + \frac{4}{3}rq - 2p \left(\frac{2\Delta^2 \sec h(\Delta\xi)}{2\Delta \sinh(\Delta\xi) - p \cosh(\Delta\xi)} \right) - 2 \left(\frac{2\Delta^2 \sec h(\Delta\xi)}{2\Delta \sinh(\Delta\xi) - p \cosh(\Delta\xi)} \right)^2, \\
 u_{21} &= -\frac{1}{3}p^2 + \frac{4}{3}rq + 2p \left(\frac{2\Delta^2 \csc h(\Delta\xi)}{2\Delta \cosh(\Delta\xi) - p \sinh(\Delta\xi)} \right) - 2 \left(\frac{2\Delta^2 \csc h(\Delta\xi)}{2\Delta \cosh(\Delta\xi) - p \sinh(\Delta\xi)} \right)^2, \\
 u_{22} &= -\frac{1}{3}p^2 + \frac{4}{3}rq + 2p \left(\frac{4\Delta^2 \sec h(2\Delta\xi)(1 \mp i \sinh(2\Delta\xi))}{p \cosh(2\Delta\xi) - 2\Delta \sinh(2\Delta\xi) \mp i 2\Delta} \right) - 2 \left(\frac{4\Delta^2 \sec h(2\Delta\xi)(1 \mp i \sinh(2\Delta\xi))}{p \cosh(2\Delta\xi) - 2\Delta \sinh(2\Delta\xi) \mp i 2\Delta} \right)^2, \\
 u_{23} &= -\frac{1}{3}p^2 + \frac{4}{3}rq + 2p \left(\frac{4\Delta^2 \csc h(2\Delta\xi)(1 \pm \cosh(2\Delta\xi))}{-p \sinh(2\Delta\xi) + 2\Delta \cosh(2\Delta\xi) \pm 2\Delta} \right) - 2 \left(\frac{4\Delta^2 \csc h(2\Delta\xi)(1 \pm \cosh(2\Delta\xi))}{-p \sinh(2\Delta\xi) + 2\Delta \cosh(2\Delta\xi) \pm 2\Delta} \right)^2, \\
 u_{24} &= -\frac{1}{3}p^2 + \frac{4}{3}rq + 2p \left(\frac{2\Delta^2 \csc h(2\xi)}{2\Delta \cosh(\Delta\xi) - p \sinh(\Delta\xi)} \right) - 2 \left(\frac{2\Delta^2 \csc h(2\xi)}{2\Delta \cosh(\Delta\xi) - p \sinh(\Delta\xi)} \right)^2.
 \end{aligned}$$

Family 3.

When $r = 0$ and $pq \neq 0$, the solutions of Eq. (1) are,

$$\begin{aligned}
 u_{25} &= -\frac{1}{3}p^2 + \frac{4}{3}rq + 2p \left(\frac{p(\cosh(p\xi) - \sinh(p\xi))}{d + \cosh(p\xi) - p \sinh(p\xi)} \right) - 2 \left(\frac{p(\cosh(p\xi) - \sinh(p\xi))}{d + \cosh(p\xi) - p \sinh(p\xi)} \right)^2, \\
 u_{26} &= -\frac{1}{3}p^2 + \frac{4}{3}rq + 2p \left(\frac{pd}{d + \cosh(p\xi) + p \sinh(p\xi)} \right) - 2 \left(\frac{pd}{d + \cosh(p\xi) + p \sinh(p\xi)} \right)^2.
 \end{aligned}$$

where d is an arbitrary constant.

Family 4.

When $q \neq 0$ and $r = p = 0$, the solutions of Eq. (1) are

$$u_{27} = -\frac{1}{3}p^2 + \frac{4}{3}rq - 2p \left(\frac{q}{q\xi + d_1} \right) - 2 \left(\frac{q}{q\xi + d_1} \right)^2.$$

where d_1 is an arbitrary constant.

4. Graphical presentation

Graph is an influential tool for communication and it illustrates clearly the solutions of the problems. Therefore, some graphs of the solutions are given below. The graphs readily have shown the periodic and solitary wave forms of the solutions.

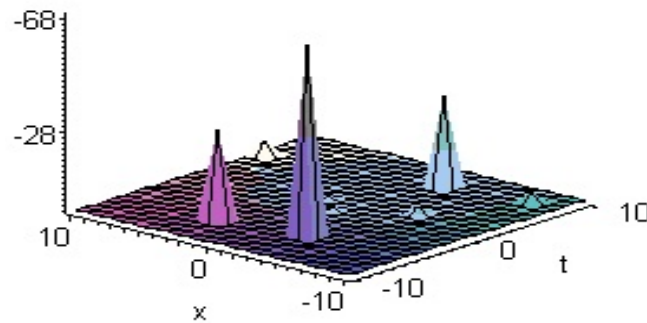


Figure 1: Solitons corresponding to solutions u_1 for $p = q = r = 1$.

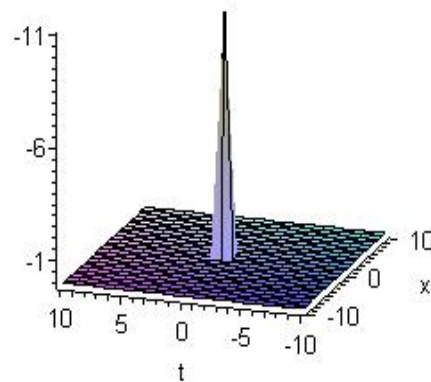


Figure 2: Solitons corresponding to solutions u_2 for $p = q = 1, r = 2$.

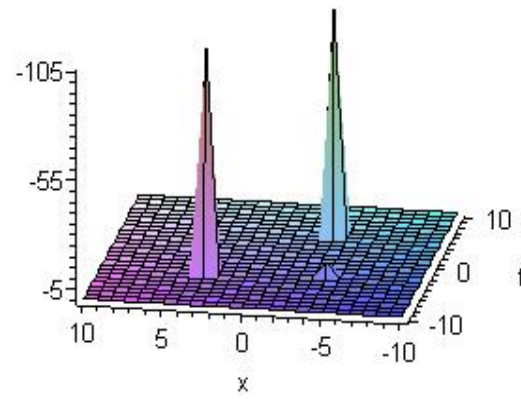


Figure 3: Solitons corresponding to solutions u_8
for $p = q = r = 2$.

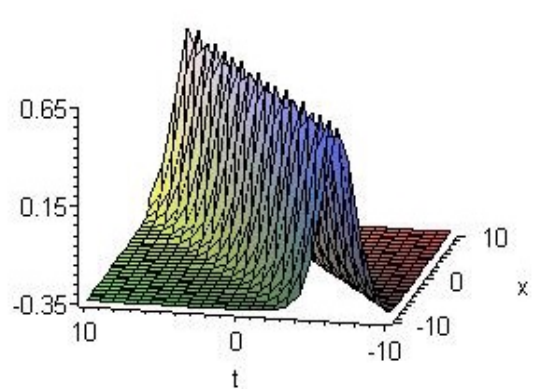


Figure 4: Solitons corresponding to solutions u_{13}
for $p = 3, q = 2, r = 1$.

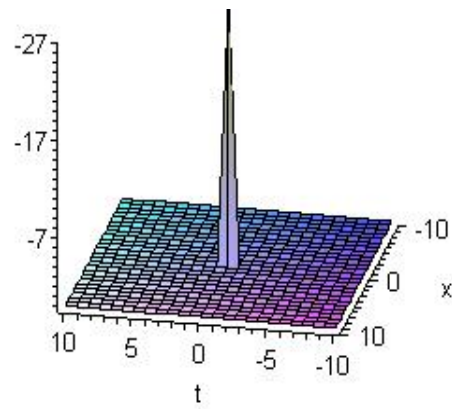


Figure 5: Solitons corresponding to solutions u_{14}
for $p = 2$, $q = 1$, $r = 0.5$.

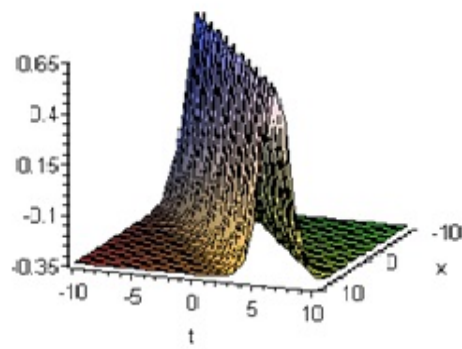


Figure 6: Solitons corresponding to solutions u_{20}
for $p = 3$, $q = 1$, $r = 2$.

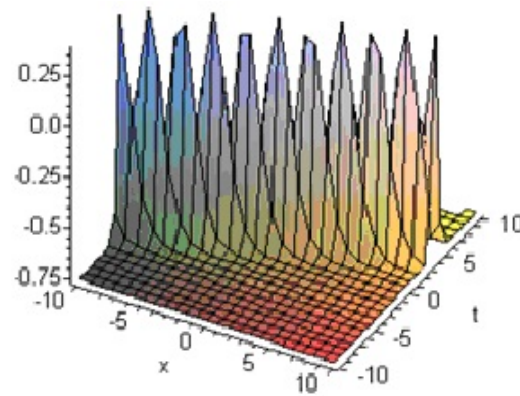


Figure 7: Solitons corresponding to solutions u_{26}
for $p = 1.5$, $q = 1$, $r = 0$.

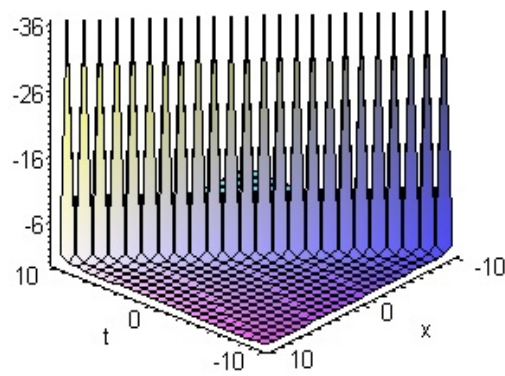


Figure 8: Solitons corresponding to solutions u_{27}
for $p = 0$, $q = 1$, $r = 0$.

5. Conclusion

The (G'/G) -expansion method is an advance mathematical tool for investigating exact solutions of nonlinear partial differential equations associated with complex physical phenomena wherein, in general the second order linear ordinary differential equation is employed as an auxiliary equation. But, in this article, we utilize the generalized Riccati equation as an auxiliary equation; in consequence we obtain further new exact solutions of the sixth-order Boussinesq equation in a unified way. The obtained exact solutions may be important and significant to reveal the internal mechanism of some complicated physical phenomena. The algorithm presented in this article is effective and more powerful and it can be used for other kind of nonlinear evolution equations in mathematical physics.

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CERTAIN PROPERTIES OF MITTAG-LEFFLER FUNCTION WITH ARGUMENT x^α , $\alpha > 0$

Jyotindra C. Prajapati

Department of Mathematical Sciences

Faculty of Applied Sciences

Charotar University of Science and Technology

Changa Anand-388 421

India

e-mail: jyotindra18@rediffmail.com, jyotindraprajapati.maths@charusat.ac.in

Abstract. In this paper, author discusses some interesting properties such as Composition property, Power series expansion, Inverse property, Increasing property, Positivity and Limiting case of Mittag-Leffler function with argument x^α , $\alpha > 0$.

Keywords: Mittag-Leffler function.

2000 Mathematics Subject Classification: 33E12.

1. Introduction

In 1903, a Swedish mathematician Gösta Mittag-Leffler [6] introduced the function $E_\alpha(z)$ in the form:

$$(1.1) \quad E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)},$$

where z is a complex variable, $\alpha > 0$ and $\Gamma(\cdot)$ is the well-known gamma function.

The Mittag-Leffler function (1.1) is an entire function of order $(\operatorname{Re} \alpha)^{-1}$ and is also direct generalization of the exponential function to which it reduces when $\alpha = 1$, or in other words, the Mittag-Leffler function is the parameterized exponential function. If $0 < \alpha < 1$, then it interpolates between the pure exponential $\exp(z)$ and a hypergeometric function $\frac{1}{1-z} = {}_1F_0(1; -; z)$. In recent years, the Mittag-Leffler function has caused extensive interest among scientist, engineers and applied mathematicians. The Mittag-Leffler functions naturally occur as the

solution of fractional order differential equation or fractional order integral equations. Some applications of the function (1.1) have already been discussed in [4], [5], [7] and [8].

Few interesting special cases of $E_\alpha(z)$ are as listed below.

$$(1.2) \quad E_0(z) = \frac{1}{1-z}; \quad |z| < 1,$$

$$(1.3) \quad E_{\frac{1}{2}}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma\left(\frac{n}{2} + 1\right)} = \exp(z^2) \operatorname{erfc}(-z)$$

$$(1.4) \quad E_1(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)} = e^z$$

$$(1.5) \quad E_2(z) = \cos h(\sqrt{z})$$

$$(1.6) \quad E_3(z) = \frac{1}{3} \left[\cos\left(z^{\frac{1}{4}}\right) + 2 \exp\left(-\frac{z^{\frac{1}{3}}}{2}\right) \cos\left(\frac{\sqrt{3}}{2} z^{\frac{1}{3}}\right) \right]$$

$$(1.7) \quad E_4(z) = \frac{1}{2} \left[\cos\left(z^{\frac{1}{4}}\right) + \cos h\left(z^{\frac{1}{4}}\right) \right]$$

2. Mittag-Leffler function with argument x^α and its properties

In this section, the author establishes some interesting properties of the Mittag-Leffler function x^α .

The Mittag-Leffler function does not satisfy the composition property, $E_\alpha(x)E_\alpha(y) \neq E_\alpha(x+y)$, but it can be observed that (Jumarie [1], [2], [3]) the function

$$(2.1) \quad E_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{\alpha n}}{\Gamma(\alpha n + 1)}; \quad \alpha > 0,$$

does satisfy the composition property

$$(2.2) \quad E_\alpha(x^\alpha)E_\alpha(y^\alpha) = E_\alpha\{(x+y)^\alpha\}, \quad \forall x \in \mathbb{R}.$$

The function $E_\alpha(x^\alpha)$ defined in (2.1) converges absolutely for

$$|x| < \left(\frac{\Gamma(\alpha n + \alpha + 1)}{\Gamma(\alpha n + 1)} \right)^{\frac{1}{\alpha}}$$

is a Mittag-Leffler function with argument x^α , $\alpha > 0$, and this also can be reduced in the exponential function for $\alpha = 1$.

(i) Power series expansion of $E_\alpha(x^\alpha)$

$$\begin{aligned}
 (2.3) \quad E_\alpha(x^\alpha) &= \sum_{n=0}^{\infty} \frac{x^{\alpha n}}{\Gamma(\alpha n + 1)}; \quad \alpha > 0, \quad \forall x \in \mathbb{R} \\
 &= 1 + \frac{x^\alpha}{\Gamma(\alpha + 1)} + \frac{x^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{x^{3\alpha}}{\Gamma(3\alpha + 1)} + \dots
 \end{aligned}$$

Taking $x = 0$ in (2.3), we get

$$(2.4) \quad E_\alpha(0) = 1.$$

(ii) Inverse Property and its particular cases:

Putting $y = -x$ in (2.2), yields

$$E_\alpha(0) = E_\alpha(x^\alpha) E_\alpha\{(-x)^\alpha\}.$$

Using (2.4), the above equation yields

$$(2.5) \quad E_\alpha\{(-x)^\alpha\} = \frac{1}{E_\alpha(x^\alpha)}.$$

If $\alpha = 1$, then (2.2) and (2.5) becomes

$$\exp(x + y) = \exp(x) \exp(y) \quad \text{and} \quad \exp(-x) = \frac{1}{\exp(x)}.$$

(iii) Increasing property:

If $x > y > 0 \implies -y > -x$ for odd positive integer α , we can write

$$(-y)^\alpha > (-x)^\alpha;$$

this gives

$$E_\alpha\{(-y)^\alpha\} > E_\alpha\{(-x)^\alpha\};$$

using (2.5), this leads to

$$(2.6) \quad \frac{1}{E_\alpha(y^\alpha)} > \frac{1}{E_\alpha(x^\alpha)} \quad \text{i.e.} \quad E_\alpha(x^\alpha) > E_\alpha(y^\alpha).$$

Now, again if $x > y > 0$, then $x^\alpha > y^\alpha > 0$ implies that

$$(2.7) \quad E_\alpha(x^\alpha) > E_\alpha(y^\alpha).$$

Equations (2.6) and (2.7) imply that $E_\alpha(x^\alpha)$ is strictly increasing function for odd positive integer α .

(iv) Positivity:

For $\alpha \in \mathbb{N}$ and $x \geq 0$, we have

$$(2.8) \quad E_\alpha(x^\alpha) > 0$$

again for

$$x < 0 \implies -x > 0.$$

Therefore, for $\alpha \in \mathbb{N}$, we have

$$\begin{aligned} (-x)^\alpha &> 0 \\ \implies E_\alpha\{(-x^\alpha)\} &> E_\alpha(0) = 1 > 0; \end{aligned}$$

using (2.5), this leads to

$$\frac{1}{E_\alpha(x^\alpha)} > 0$$

and hence

$$(2.9) \quad E_\alpha(x^\alpha) > 0.$$

Equations (2.8) and (2.9) show that

$$E_\alpha(x^\alpha) > 0; \quad \alpha \in \mathbb{N} \text{ and } \forall x \in \mathbb{R}.$$

(v) Limiting case:

Equation (2.3), gives

$$(2.10) \quad E_\alpha(x^\alpha) \rightarrow \infty \text{ as } x \rightarrow \infty \text{ for } \alpha > 0.$$

Now, consider

$$\begin{aligned} \lim_{x \rightarrow -\infty} E_\alpha(x^\alpha) &= \lim_{y \rightarrow \infty} E_\alpha\{(-y)^\alpha\} \\ &= \lim_{y \rightarrow \infty} \frac{1}{E_\alpha(y^\alpha)} = 0. \end{aligned}$$

Therefore,

$$(2.11) \quad E_\alpha(x^\alpha) \rightarrow 0 \text{ as } x \rightarrow -\infty \text{ for } \alpha > 0.$$

From (2.5),

$$\begin{aligned}
E_\alpha\{(-x)^\alpha\} &= \frac{1}{E_\alpha(x^\alpha)} \\
&= \frac{1}{1 + \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots + \frac{x^{\alpha(n+1)}}{\Gamma(\alpha n + \alpha + 1)} + \cdots} \\
&< \frac{\Gamma(\alpha n + \alpha + 1)}{x^{\alpha n + \alpha}}.
\end{aligned}$$

Therefore,

$$\lim_{x \rightarrow \infty} x^{\alpha n} E_\alpha\{(-x)^\alpha\} < \lim_{x \rightarrow \infty} \frac{\Gamma(\alpha n + \alpha + 1)}{x^\alpha} = 0, \quad \alpha > 0,$$

hence

$$(2.12) \quad x^{\alpha n} E_\alpha\{(-x)^\alpha\} \rightarrow 0 \quad \text{as } x \rightarrow \infty \quad \text{for } \alpha > 0.$$

3. Concluding remarks

The results established in this paper seem to be new and stimulate the scope of further research and other computational aspects.

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LOCALIZED NEARLY m -EMBEDDED PROPERTY OF SOME SUBGROUPS OF FINITE GROUPS¹

Yong Xu

*School of Mathematics and Statistics
Henan University of Science and Technology
Luoyang, Henan 471003
China
e-mail: xuy_2011@163.com*

Abstract. Let A be a subgroup of a finite group G and $\Sigma : G_0 \leq G_1 \leq \cdots \leq G_n$ some subgroup series of G . Suppose that for each pair (K, H) such that K is a maximal subgroup of H and $G_{i-1} \leq K < H \leq G_i$, for some i , either $A \cap H = A \cap K$ or $AH = AK$. Then A is said to be Σ -embedded in G ; A is said to be nearly m -embedded in G if G has a subgroup T and a $\{1 \leq G\}$ -embedded subgroup C in G such that $G = AT$ and $T \cap A \leq C \leq A$. In this paper, we localize the above conditions in the G -normalizer of Sylow subgroups of the group G . Some new characterizations of some classes of finite groups are given.

Keywords: nearly m -embedded subgroup; p -nilpotency; formation.

2000 Mathematics Subject Classification: 20D10, 20D15.

1. Introduction

All groups considered in this paper will be finite and G stands for a finite group. Let $\pi(G)$ stand for the set of all prime divisors of the order of G . Let $\mathcal{F}$ denote a formation, $\mathcal{U}$ denote the class of supersolvable groups. The other notation and terminology are standard (see [6]).

Let A be a subgroup of G , $K \leq H \leq G$ and p a prime. Then we say:

- (i) A covers the pair (K, H) if $AH = AK$;
- (ii) A avoids (K, H) if $A \cap H = A \cap K$.

(K, H) is said to be maximal pair of G if K is a maximal subgroup of H . Now, the authors in [4] introduced the following concepts.

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Definition 1.1 Let A be a subgroup of G and $\Sigma = \{G_0 \leq G_1 \leq \cdots \leq G_n\}$ some subgroup series of G . Then we say that A is Σ -embedded in G if A either covers or avoids every maximal pair (K, H) such that $G_{i-1} \leq K < H \leq G_i$, for some i .

Definition 1.2 Let A be a subgroup of G . We say that

- (1) A is m -embedded in G if G has a subnormal subgroup T and a $\{1 \leq G\}$ -embedded subgroup C in G such that $G = AT$ and $T \cap A \leq C \leq A$.
- (2) A is nearly m -embedded in G if G has a subgroup T and a $\{1 \leq G\}$ -embedded subgroup C in G such that $G = AT$ and $T \cap A \leq C \leq A$.

In finite groups, the localized subgroups of a group play an important role in the structure of groups. A question of particular interest is the influence of the properties of the normalizers of the Sylow subgroups on the structure of the group. A nice example is Burnside's Theorem.

Theorem 1.1 (Burnside) *Let P be a Sylow p -subgroup of G . If $N_G(P) = C_G(P)$, then G is p -nilpotent.*

The following extension of Burnside's Theorem due to Hall is also interesting.

Theorem 1.2 ([5]) *Let P be a Sylow p -subgroup of G . If p' -elements of $N_G(P)$ commute with the elements of P and the nilpotency class of P is less than p , then G is p -nilpotent.*

Wielandt, Ballester-Bolinches and Esteban-Romero proved the following results respectively.

Theorem 1.3 ([7]) *A group G is p -nilpotent if it has a regular Sylow p -subgroup whose G -normalizer is p -nilpotent.*

Theorem 1.4 ([1]) *A group G is p -nilpotent if it has a modular Sylow p -subgroup whose G -normalizer is p -nilpotent.*

The main purpose of the present paper is to investigate the structure of the group G that maximal subgroups of Sylow subgroups are nearly m -embedded in G -normalizer.

2. Preliminary results

Lemma 2.1 ([4, Lemma 2.3]) *Let $M \leq G$, N and R be normal subgroups of G . Then*

- (a) *If $E \leq V$ and M is $\{E \leq G\}$ -embedded in G , then $M \cap V$ is $\{E \leq V\}$ -embedded in V .*
- (b) *If $R \leq N$ and M is $\{R \leq G\}$ -embedded in G , then NM is $\{R \leq G\}$ -embedded in G and NM/N is $\{1 \leq G/N\}$ -embedded in G/N .*

Lemma 2.2 ([4, Lemma 2.13]) *Let U be a nearly m -embedded subgroup of G and N a normal subgroup of G . Then:*

- (1) *If $U \leq H \leq G$, then U is nearly m -embedded in H .*
- (2) *If $N \leq U$, then U/N is nearly m -embedded in G/N .*
- (3) *Let π be a set of primes, U a π -subgroup and N a π' -subgroup. Then UN/N is nearly m -embedded in G/N .*

Lemma 2.3 ([9, Theorem 3.1]) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, and G a group with a normal subgroup N such that $G/N \in \mathcal{F}$. If all Sylow subgroups of $F^*(N)$ are cyclic, then $G \in \mathcal{F}$.*

Lemma 2.4 ([4, Theorem 4.2]) *Let G be a group and p a prime dividing of G such that $(|G|, p-1) = 1$. If every maximal subgroup of a Sylow p -subgroup P of G is nearly m -embedded in G , then G is p -nilpotent.*

Lemma 2.5 ([4, Lemma 2.14]) *Let P be a normal non-identity p -subgroup of G with $|P| = p^n$ and $P \cap \Phi(G) = 1$. Suppose that either every maximal subgroup of P is nearly m -embedded in G or there is an integer k such that $1 \leq k < n$ and the subgroups of P of order p^k are m -embedded in G . Then some maximal subgroup of P is normal in G .*

3. Main results

Theorem 3.1 *Let G be a group and p a prime dividing the order of G with $(|G|, p-1) = 1$. If there exists a Sylow p -subgroup P of G such that every maximal subgroup of P is nearly m -embedded in $N_G(P)$ and P' is $\{1 \leq G\}$ -embedded in G , then G is p -nilpotent.*

Proof. Assume that the result is false and let G be a counterexample of minimal order.

(1) Let L be a normal subgroup of G contained in P . Then G/L satisfies the hypothesis.

It is clear that $(|G/L|, p-1) = 1$. For any maximal subgroup P_1/L of P/L , $p = |P/L : P_1/L| = |P : P_1|$, so P_1 is a maximal subgroup of P . By the hypothesis, P_1 is nearly m -embedded in $N_G(P)$ and P' is $\{1 \leq G\}$ -embedded in G . By Lemma 2.2 (2), P_1/L is nearly m -embedded in $N_G(P)/L = N_{G/L}(P/L)$, and $(P/L)' = P'L/L$ is $\{1 \leq G/L\}$ -embedded in G/L by Lemma 2.1 (b). The choice of G implies that (1) is true.

(2) $1 \neq P' \leq O_p(G)$ and G is solvable.

For any $Q \in \text{Syl}_q(N_G(P))$, where $q \neq p$. It is easy to see that every maximal subgroup of P is nearly m -embedded in PQ by Lemma 2.2 (1). Thus PQ satisfies the hypothesis of Lemma 2.4, so PQ is p -nilpotent, hence $Q \leq C_G(P)$. Assume

that P is abelian, then $N_G(P) = C_G(P)$, hence G is p -nilpotent by Burnside's Theorem, a contradiction. So $P' \neq 1$. By the hypothesis, P' is $\{1 \leq G\}$ -embedded in G . By [4, Lemma 2.3], P' is subnormal in G . Hence $1 \neq P' \leq O_p(G)$. By (1), $G/O_p(G)$ is p -nilpotent. Since $(|G|, p-1) = 1$, we conclude that $G/O_p(G)$ is solvable, thus G is solvable.

(3) $G = PQ$, where Q is a Sylow q -subgroup of G with $q \neq p$; $L = O_p(G)$ is a unique minimal normal subgroup of G , and $\Phi(G) = 1$.

Let $\{G_r \mid r \in \pi(G)\}$ be a Sylow system of G and $H = G_p G_r$ for any $r \in \pi(G)$ with $r \neq p$. By Lemma 2.1(a) and Lemma 2.2 (1), the hypothesis is still true for H . If $|\pi(G)| > 2$, then $G_r \leq H$, which implies that G_p normalizes G_r for any $r \in \pi(G)$, hence G is p -nilpotent, a contradiction. Thus we may assume that $|G| = p^a q^b$.

Let L be a minimal normal subgroup of G . By Lemma 2.1(b), $P'L/L$ is $\{1 \leq G/L\}$ -embedded in G/L . If L is a q -group, then we consider the quotient group G/L . Evidently, $PL/L \in \text{Syl}_p(G/L)$. For any maximal subgroup T/L of PL/L , we have $p = |(PL/L) : (T/L)|$, and $T = PL \cap T = (P \cap T)L$. Let $P_1 = P \cap T$. Then $P_1 \cap L = P \cap T \cap L = P \cap L$, so

$$p = |PL : T| = |PL : (P \cap T)L| = |P : P \cap T| = |P : P_1|.$$

Thus P_1 is a maximal subgroup of P . By the hypothesis, P_1 is nearly m -embedded in $N_G(P)$. Then by Lemma 2.2 (3), it is easy to see P_1L/L is also nearly m -embedded in $N_G(P)L/L = N_{G/L}(PL/L)$. By the minimality of G , G/L is p -nilpotent and so is G , a contradiction. Hence L is a p -group and $L \leq P$. By (1), G/L is p -nilpotent. Similarly, if N is another minimal normal subgroup of G , then $N \leq P$ and so G/N is also p -nilpotent. Now it follows that $G \cong G/N \cap L$ is p -nilpotent, a contradiction. Thus L must be the unique minimal normal subgroup of G . Since the class of p -nilpotent groups is a saturated formation, $L \not\leq \Phi(G)$, and $\Phi(G) = 1$. By [8, Theorem 5.3], we get $O_p(G) = F(G) = L$.

(4) The final contradiction.

By (3), $\Phi(G) = 1$. Then G has a maximal subgroup M such that $G = ML$ and $M \cap L = 1$. So $M \cong G/L$ is p -nilpotent. Let $M_{p'}$ be a normal p -complement of M , then $M_{p'} \triangleleft M$. Thus $M \leq N_G(M_{p'}) \leq G$. The maximality of M implies that either $M = N_G(M_{p'})$ or $N_G(M_{p'}) = G$. If the latter holds, then $M_{p'} \triangleleft G$, $M_{p'}$ is actually the normal p -complement of G , a contradiction. Hence $M = N_G(M_{p'})$. Clearly, $P = L(P \cap M)$. Since $P \cap M < P$, there exists a maximal subgroup P_1 of P such that $P \cap M \leq P_1$. By the hypothesis, there are a subgroup T of G and a $\{1 \leq G\}$ -embedded subgroup C of G such that $G = P_1T$ and $P_1 \cap T \leq C \leq P_1$. Thus C either covers or avoids (M, G) . But $CM \leq P_1M \neq G$, hence $C \cap M = C$, that is, $C \leq M$. By [4, Lemma 2.3], C is subnormal in G . Then $C \leq O_p(G) = L$. Hence, $C \leq M \cap L = 1$, and then $|T|_p = p$, where $T_p \in \text{Syl}_p(T)$. Since $N_T(T_p)/C_T(T_p)$ is isomorphic to a subgroup of $\text{Aut}(T_p)$ and $|\text{Aut}(T_p)| = p-1$. By $(|G|, p-1) = 1$, we get $|N_T(T_p)/C_T(T_p)| = 1$, that is, $N_T(T_p) = C_T(T_p)$. So T is p -nilpotent by Burnside's Theorem. Let $T_{p'}$ be a normal p -complement of T , then $T_{p'} \triangleleft T$. Clearly, both $T_{p'}$ and $M_{p'}$ are Hall p' -subgroup of G with

odd order. By applying Groos's result in [3, main Theorem], there exists $g \in G$ such that $T_{p'}^g = M_{p'}$, and $T_{p'}$ is normalized by T , then we may assume that $g \in P_1$. Thus $T^g \leq N_G(T_{p'}^g) \leq N_G(M_{p'}) = M$, hence $G = P_1 T^g = P_1 M$ and $P = P \cap P_1 M = P_1(P \cap M) = P_1$, a contradiction. This contradiction completes the proof of this theorem. ■

Theorem 3.2 *Let G be a group and p a prime dividing the order of G with $(|G|, p-1) = 1$. Suppose that H is a normal subgroup of G such that G/H is p -nilpotent. If there exists a Sylow p -subgroup P of H such that every maximal subgroup of P is nearly m -embedded in $N_G(P)$ and P' is $\{1 \leq G\}$ -embedded in G , then G is p -nilpotent.*

Proof. Assume that the result is false. Let (G, H) be a counterexample with $|G| + |H|$ minimal.

By Lemma 2.1 (a) and Lemma 2.2 (1), it is easy to see that every maximal subgroup of P is nearly m -embedded in $N_H(P)$ and P' is $\{1 \leq H\}$ -embedded in H , then H is p -nilpotent by Theorem 3.1. Let M be a normal p -complement of H . Then $M \trianglelefteq G$. Assume that $M \neq 1$. We consider the quotient group G/M . Similar to the proof of (3) in Theorem 3.1, it is easy to see that the hypothesis is still true for $(G/M, H/M)$, hence G/M is p -nilpotent and so is G , a contradiction. Thus we conclude that $M = 1$. Now $H = P$ is a p -subgroup. Let T/P be a normal p -complement of G/P . It is clear that every maximal subgroup of P is nearly m -embedded in $N_T(P)$ and P' is $\{1 \leq T\}$ -embedded in T by Lemma 2.1 (a) and Lemma 2.2 (1), then T is p -nilpotent by Theorem 3.1, so $T_{p'} \trianglelefteq T \trianglelefteq G$ and $T_{p'}$ is also a Hall p' -subgroup of G , thus $T_{p'} \trianglelefteq G$, hence G is p -nilpotent, a contradiction.

This completes the proof. ■

In [2], the authors introduced the c -supplement subgroup. A subgroup A of G is said to be c -supplement in G if G has a subgroup T such that $TA = G$ and $T \cap A \leq A_G$. Since a normal subgroup is always $\{1 \leq G\}$ -embedded in G , every c -supplement subgroup is nearly m -embedded. Then the following corollary is clear by Theorem 3.2.

Corollary 3.1 *Let G be a group and p a prime dividing the order of G with $(|G|, p-1) = 1$. Suppose that H is a normal subgroup of G such that G/H is p -nilpotent. If there exists a Sylow p -subgroup P of H such that every maximal subgroup of P is a c -supplemented subgroup of $N_G(P)$ and P' is $\{1 \leq G\}$ -embedded in G , then G is p -nilpotent.*

Theorem 3.3 *Let G be a group. For any prime factor p of $|G|$, there exists a Sylow p -subgroup P of G such that every maximal subgroup of P is nearly m -embedded in $N_G(P)$ and P' is $\{1 \leq G\}$ -embedded in G , then G is supersolvable.*

Proof. Assume that the result is false and let G be a counterexample of minimal order.

By Theorem 3.1, G is a Sylow tower group of supersolvable type, so G is solvable. Let L be a minimal normal subgroup of G . Then L is an elementary r -subgroup, where $r \in \pi(G)$. By Lemma 2.1 and Lemma 2.2, G/L satisfies the hypothesis, thus G/L is supersolvable by the minimal choice of G . Since the class of supersolvable subgroups is a saturated formation, we may assume that L is a unique minimal normal subgroup of G and $L \not\leq \Phi(G)$. Hence there exists a maximal subgroup M of G such that $G = LM$ and $L \cap M = 1$. Let $q = \max \pi(G)$ and $Q \in \text{Syl}_q(G)$. Then $Q \trianglelefteq G$, thus $L \leq Q$ by the unique minimal normality of L . Since $Q = O_q(G) \leq F(G) \leq C_G(L)$, L and M normalize $Q \cap M$, thus $Q \cap M \triangleleft G$. So $Q \cap M = 1$ or $L \leq Q \cap M$. If the later happens, then $L \leq M$, that is, $G = LM = M$, a contradiction. So $Q \cap M = 1$. This implies that $|Q| = |G : M| = |L|$, hence $L = Q$. By the hypothesis, every maximal subgroup of Q is nearly m -embedded in $N_G(Q) = G$. Then by Lemma 2.5, there is a maximal subgroup of Q is normal in G . The minimal normality of Q in G implies that $|Q| = |L| = q$. By Lemma 2.3, G is supersolvable, a contradiction. This final contradiction completes our proof. ■

Theorem 3.4 *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a normal subgroup H such that $G/H \in \mathcal{F}$. For any prime factor p of $|H|$, there exists a Sylow p -subgroup P of H such that every maximal subgroup of P is nearly m -embedded in $N_G(P)$ and P' is $\{1 \leq G\}$ -embedded in G , then $G \in \mathcal{F}$.*

Proof. Assume that the result is false. Let (G, H) be a counterexample with $|G| + |H|$ minimal.

By Lemma 2.1 (a) and Lemma 2.2 (1), H satisfies the hypothesis of Theorem 3.3, hence H is supersolvable. Let $p = \max \pi(H)$ and $P \in \text{Syl}_p(H)$. Then $P \trianglelefteq G$. We consider the quotient group G/P , then $G/H \cong (G/P)/(H/P) \in \mathcal{F}$. By Lemma 2.1 and Lemma 2.2, $(G/P, H/P)$ satisfies the hypothesis, thus $G/P \in \mathcal{F}$. Hence we may assume that $H = P$.

Let N be a minimal normal subgroup of G contained in P . By Lemma 2.1 (b) and Lemma 2.2 (2), the hypothesis is still true for $(G/N, P/N)$, then $G/N \in \mathcal{F}$. Since $\mathcal{F}$ is a saturated formation, $N \not\leq \Phi(G)$. So there exists a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. On the other hand, we can conclude that $\Phi(P) = 1$. Otherwise, we have $G/\Phi(P) \in \mathcal{F}$, then $G/\Phi(G) \cong (G/\Phi(P))/(\Phi(G)/\Phi(P)) \in \mathcal{F}$, so $G \in \mathcal{F}$, a contradiction. Hence P is an elementary abelian subgroup. By $N \leq P$, we get $G = NM = PM$ and $P \cap M \triangleleft G$. If $P \cap M \neq 1$, then $N \leq P \cap M$, $N \leq M$, so $G = NM = M$, a contradiction. Thus $P \cap M = 1$, hence $P = N$ is a minimal normal subgroup of G . By the hypothesis, every maximal subgroup of P is nearly m -embedded in $N_G(P) = G$. Then by Lemma 2.5, there is a maximal subgroup of P is normal in G . The minimal normality of P in G implies that $|P| = p$. By Lemma 2.3, $G \in \mathcal{F}$, a contradiction. This final contradiction completes our proof. ■

Recall that a group G was called an A -group if all of its Sylow subgroups are abelian. Let G be an A -group. Then for any $P \in \text{Syl}_p(G)$, we have $P' = 1$, of course, it is $\{1 \leq G\}$ -embedded in G , so we have the following corollary.

Corollary 3.2 *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a normal A -subgroup H such that $G/H \in \mathcal{F}$. If every maximal subgroup of each Sylow subgroup P of H is a c -supplemented subgroup of $N_G(P)$, then $G \in \mathcal{F}$.*

4. Remarks

Remark 4.1 The following example illustrates that the hypothesis in Theorem 3.1 that “ P' is $\{1 \leq G\}$ -embedded in G ” can not be removed.

Example. Let $G = PSL_2(q)$, where $q \equiv \pm 1 \pmod{8}$. Let P be a Sylow 2-subgroup of G . By [6, II, Theorem 8.27], we have the Sylow 2-subgroup of $PSL_2(q)$ is selfnormalizing in $PSL_2(q)$. Evidently, every maximal subgroup of P is normal in $N_G(P) = P$, so every maximal subgroup of P is nearly m -embedded in $N_G(P)$. However, G is not 2-nilpotent.

Remark 4.2 Even if G is a solvable group and p is an odd prime, the hypothesis in Theorem 3.1 that “ P' is $\{1 \leq G\}$ -embedded in G ” could not be omitted, either.

Example. Let $H = Z_3 \times Z_3 \times Z_3$ be an elementary abelian 3-group. It is clear that $\text{Aut}(H)$ has a subgroup $Z_{13} \rtimes Z_3$. Now suppose that

$$G = (Z_3 \times Z_3 \times Z_3) \rtimes (Z_{13} \rtimes Z_3)$$

Let P_3 be a Sylow 3-subgroup of G . It is clear that $N_G(P_3) = P_3$, so every maximal subgroup of P_3 is nearly m -embedded in $N_G(P_3) = P_3$. However, G is not 3-nilpotent.

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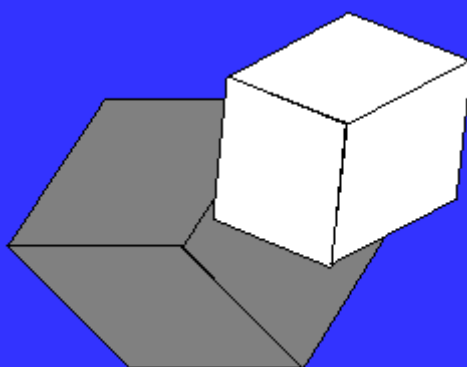
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