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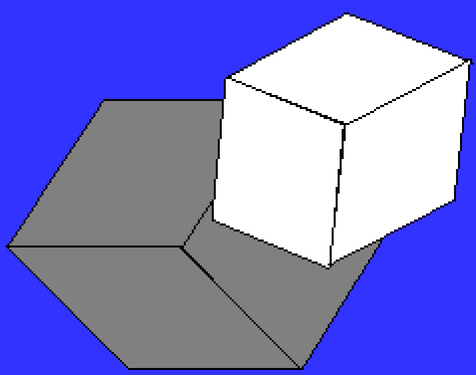

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Via Palladio 8 - 33100 Udine
Tel: +39-0432-26001, Fax: +39-0432-296756
forum@forumeditrice.it

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A NEW PREDICTOR-CORRECTOR ALGORITHM FOR SDP WITH POLYNOMIAL CONVERGENCE[1]

Feixiang Chen
Yuming Feng

School of Mathematics and Statistics
Chongqing Three Gorges University
Wanzhou, Chongqing, 404000
P.R. China
e-mail: cfx2002@126.com

Abstract. We establish the polynomiality of primal-dual interior-point algorithms for SDP based on the direction of the M-Z family of search directions. We show that the polynomial iteration-complexity bounds of the well known algorithms for linear programming, namely, the predictor-corrector algorithm, carry over to the context of SDP.

Keywords: interior-point algorithm; polynomial complexity; path-following methods; semidefinite programming problems.

1. Introduction

Several authors have discussed generalizations of interior-point algorithms for linear programming (LP) to the context of semidefinite programming (SDP). The landmark work in this direction is due to Nesterov and Nemirovskii [1], where a general approach for using interior-point algorithms for solving convex programs is proposed, based on the notion of self-concordant functions. They show that the problem of minimizing a linear function over a convex set can be solved in 'polynomial time' as long as a self-concordant barrier function for the convex set is known. On the other hand, Alizadeh [2] extends Ye's projective potential reduction algorithm for LP to SDP and argues that many known interior point algorithms for LP can also be transformed into algorithms for SDP in a mechanical way. Since then many authors have proposed interior-point algorithms for solving the SDP, including Alizadeh, Haeberly and Overton [3], Kojima, Shida [4]and Shindoh, Kojima, and Hara [5], Monteiro [6], [7], Monteiro and Zhang [8], [9], and Zhang [10]. Most of these more recent works are concentrated on primal-dual algorithms.

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Notation and terminology

The set of all symmetric $n \times n$ matrices is denoted by S^n. For $Q \in S^n$, $Q \succeq 0$ means Q is positive semidefinite and $Q \succ 0$ means Q is positive definite, respectively. The trace of a matrix $Q \in R^{n\times n}$ is denoted by Tr $Q \equiv \sum_{i=1}^{n} Q_{ii}$. The inner product between P and Q in $R^{m\times n}$ is defined as $P \bullet Q \equiv$ Tr $P^T Q$. The Euclidean norm and its associated operator norm are both denoted by $\| \circ \|$; hence, $\|Q\| \equiv \max_{\|u\|=1} \|Qu\|$ for any $Q \in R^{n\times n}$. The Frobenius norm of $Q \in R^{n\times n}$ is $\|Q\|_F \equiv (Q \bullet Q)^{1/2}$. We frequently use the inequalities S^n_+ and S^n_{++} denote the set of all matrices in S^n which are positive semidefinite and positive definite, respectively. It is known that for each $V \in S^n_+$, there exists a unique $U \in S^n_+$, such that $U^2 = V$. The matrix U is called the square root of V and is denoted by $V^{1/2}$.

2. The SDP problem and preliminary discussion

In this section, we describe the SDP in symmetric matrices considered in this paper, state our assumptions, and derive the Newton direction for the central path equation. We also give some existence results for this Newton direction and state a generic path-following algorithm based on it.

Given $C \in S^n$ and $(A_i, b_i) \in S^n \times R$ for $i = 1, ..., m$, a primal-dual pair of SDP problems is defined as

$$(P) \qquad \min\{C \bullet X : A_i \bullet X = b_i, i = 1, ..., m, X \succeq 0\}, \tag{2.1}$$

$$(D) \qquad \max\left\{ b^T y : \sum_{i=1}^{m} y_i A_i + S = C, S \succeq 0 \right\}, \tag{2.2}$$

where $b \equiv (b_1, ..., b_m)^T$.

The set of interior feasible solutions of (1) and (2) are

$$F^0(P) \equiv \{X \in S : X : A_i \bullet X = b_i,\ i = 1, ..., m,\ X \succ 0\}$$
$$F^0(D) \equiv \{(S, y) \in S \times R^m : \textstyle\sum_{i=1}^{m} y_i A_i + S = C,\ S \succ 0\}$$

respectively. Throughout this paper, we assume that $F^0(P) \times F^0(D) \neq \phi$ and that the matrices $A_i, i = 1, \ldots, m$, are linearly independent. Under the first assumption, it is well known that both (1) and (2) have optimal solutions X^* and (S^*, y^*) such that $C \bullet X^* = b^T y^*$, i.e., the optimal values of (1) and (2) coincide. The last condition, called the strong duality, can be alternatively expressed as $X^* \bullet S^* = 0$ or $X^* S^* = 0$. Hence, the set of primal and dual optimal solutions consists of all the solutions $(X, S, y) \in S^n_+ \times S^n_+ \times R^m$ to the following optimality system:

$$XS = 0, \tag{2.3}$$

$$\sum_{i=1}^{m} y_i A_i + S = C, \tag{2.4}$$

$$A_i \bullet X = b_i, i = 1, \ldots, m. \tag{2.5}$$

It is known that for every $\mu > 0$, the perturbed system

$$XS = \mu I, \tag{2.6}$$

$$\sum_{i=1}^{m} y_i A_i + S = C, \tag{2.7}$$

$$A_i \bullet X = b_i, i = 1, \ldots, m. \tag{2.8}$$

has a unique solution, denoted by (X_μ, S_μ), for every $\mu > 0$, and that the limit $\lim\limits_{\mu \to 0}$ exists and is a solution of (1). The set $\{(X_\mu, S_\mu) : \mu > 0\}$ is called the central path associated with (1) and plays a fundamental role in the development of interior point algorithms for solving SDP. Using the square root $X^{1/2}$, (6) can also be alternatively expressed in the following symmetric form:

$$X^{1/2} S X^{1/2} = \mu I.$$

Path following algorithms for solving (1) are based on the idea of approximately tracing the central path. Application of Newton method for computing the solution of (2) with $\mu = \hat{\mu}$ leads to the Newton search direction $(\widehat{\Delta X}, \widehat{\Delta S})$ which solves the linear system

$$X\widehat{\Delta S} + \widehat{\Delta X} S = \hat{\mu} I - XS, \qquad (X + \widehat{\Delta X}, S + \widehat{\Delta S}) \in S^n_+ \times S^n_+. \tag{2.9}$$

This system does not always have a solution. To overcome this bottleneck, if we adapt the M-Z search directions to the monotone SDP, we can describe it as a solution of the system of equations:

$$\begin{aligned} X^{-1/2}(X\Delta S + \Delta X S)X^{1/2} + X^{1/2}(\Delta S X + S\Delta X)X^{-1/2} \\ = 2(\hat{\mu} I - X^{1/2} S X^{1/2}). \end{aligned} \tag{2.10}$$

$$A_i \bullet \Delta X = 0, i = 1, \ldots, m. \tag{2.11}$$

$$\sum_{i=1}^{m} \Delta y_i A_i + \Delta S = 0, \tag{2.12}$$

It was shown in [11] that system (10), (11), (12) has a unique solution. The symmetric component $(\Delta X, \Delta S)$ of this solution is then used as a search direction to generate the next point.

Theorem 2.1 *System* (10), (11), (12) *has a unique solution.*

Lemma 2.1 *Let* $X \in F^0(P)$ *and* $(S, y) \in F^0(D)$ *be given and suppose that* $(\Delta X, \Delta S)$ *is a solution of system* (10), (11), (12) *with* $\hat{\mu} = \sigma\mu$*, then the following statements holds:*

(1) $\Delta S \bullet \Delta X = 0$,

(2) $(X + \alpha\Delta X) \bullet (S + \alpha\Delta S) = (1 - \alpha + \sigma\alpha)(X \bullet S),\ \forall\alpha \in R$.

Proof. Using (11) and (12), we obtain

$$\Delta S \bullet \Delta X = -\left(\sum_{i=1}^{m} \Delta y_i A_i\right) \bullet \Delta X = \sum_{i=1}^{m} \Delta y_i (A_i \Delta X) = 0.$$

and hence (1) follows. In view of (10) , we have

$$\begin{aligned} 2Tr(\sigma\mu I - XS) &= Tr[X\Delta S + \Delta XS] + Tr[\Delta SX + S\Delta X] \\ &= Tr[X\Delta S + \Delta XS + \Delta SX + S\Delta X] \\ &= 2Tr[X\Delta S + S\Delta X] \\ &= 2[X \bullet \Delta S + S \bullet \Delta X] \end{aligned}$$

Using the fact that $Tr(XS) = X \bullet S = n\mu$, we obtain

$$\begin{aligned} (X + \alpha\Delta X) \bullet (S + \alpha\Delta S) &= X \bullet S + \alpha(X \bullet \Delta S + S \bullet \Delta X) + \alpha^2 \Delta S \bullet \Delta X \\ &= X \bullet S + \alpha Tr(\sigma\mu I - XS) \\ &= X \bullet S + \alpha(\sigma n\mu - X \bullet S) \\ &= (1 - \alpha + \sigma\alpha)(X \bullet S) \end{aligned}$$

for every $\alpha \in R$. Hence, (2) holds. ∎

Lemma 2.2 *For all* $Q \in R^{n\times n}$*, the following relations hold:*

$$\sum_{i=1}^{n} |\lambda_i(A)|^2 \le \|A\|_F^2 = \|A^T\|_F^2;$$

Lemma 2.3 *Suppose that* $W \in R^{n\times n}$ *is a nonsingular matrix, then for any* $E \in S^n$*, the following relations hold:*

$$\|E\|_F \le \frac{1}{2}\|WEW^{-1} + (WEW^{-1})^T\|_F. \tag{2.13}$$

Lemma 2.4 *Suppose that* $A_1, A_2 \in R^{n\times n}$*, then the following relations hold:*

$$\|A_1 A_2\|_F \le \|A_1\|A_2\|_F.$$

For a nonsingular matrix $P \in R^{n\times n}$, consider the following operator $H_P : R^{n\times n} \to S^n$ defined as

$$H_P(M) \equiv \frac{1}{2}[PMP^{-1} + (PMP^{-1})^T, \quad \forall M \in R^{n\times n}.$$

The operator H_P has been used by Zhang [10] to characterize the central path of SDP problems.

Lemma 2.5 *Let* $(X, S) \in S^n_{++}$ *be such that* $\|X^{1/2}SX^{1/2} - \mu I\| \leq \mu\gamma$ *for some* $\gamma \in [0, 1)$ *and* $\mu > 0$. *Suppose that* $(\Delta X, \Delta S) \in S^{n\times n} \times S^{n\times n}$ *is a solution of* (4) *for* $\mathcal{W} \in R^{n\times n}$, *where* $\mathcal{W} = \sigma\mu I - X^{1/2}SX^{1/2}$. *Let* $\delta_x = \mu\|X^{-1/2}\Delta X X^{-1/2}\|_F$ *and* $\delta_s = \|X^{1/2}\Delta S X^{1/2}\|_F$. *Then,*

$$\delta_x\delta_s \leq \frac{1}{2}(\delta_x^2 + \delta_s^2) \leq \frac{\|\mathcal{W}\|_F^2}{2(1-\gamma)^2}.$$

Proof. We let $\mathcal{W} = H_{X^{-1/2}}[X\Delta S + \Delta X S]$. Using (4) and simple algebraic manipulation, we can obtain

$$\begin{aligned}\mathcal{W} &= X^{1/2}\Delta S X^{1/2} + \mu X^{-1/2}\Delta X X^{-1/2} + \frac{1}{2}X^{-1/2}\Delta X X^{-1/2}(X^{1/2}SX^{1/2} - \mu I)\\ &\quad + \frac{1}{2}(X^{1/2}SX^{1/2} - \mu I)X^{-1/2}\Delta X X^{-1/2},\end{aligned}$$

from which it follows that

$$\begin{aligned}\|\mathcal{W}\|_F &\geq \|X^{1/2}\Delta S X^{1/2} + \mu X^{-1/2}\Delta X X^{-1/2}\|_F\\ &\quad - \|X^{1/2}SX^{1/2} - \mu I\|\|X^{-1/2}\Delta X X^{-1/2}\|_F\\ &\geq (\|X^{1/2}\Delta S X^{1/2}\|_F^2 + \mu^2\|X^{-1/2}\Delta X X^{-1/2}\|_F^2)^{1/2} - \gamma\mu\delta_x/\mu\\ &\geq \sqrt{\delta_x^2 + \delta_s^2} - \gamma\delta_x\\ &\geq (1-\gamma)\sqrt{\delta_x^2 + \delta_s^2},\end{aligned}$$

where the second inequality follows from the assumption that $\|X^{1/2}SX^{1/2} - \mu I\| \leq \mu\gamma$ and the fact that $(X^{-1/2}\Delta X X^{-1/2}) \bullet (X^{1/2}\Delta S X^{1/2}) = \Delta X \bullet \Delta S \geq 0$, due to the monotonicity of $\mathcal{F}$. The result now follows trivially from the last inequality. ∎

Lemma 2.6 *With the notations above, we have*

$$\|H_{X^{-1/2}}[\Delta X \Delta S]\| \leq \frac{nK^2\mu}{2(1-\gamma)^2}.$$

where $K = max[|-\gamma + \sigma + 1|], |\gamma + \sigma - 1|] \leq 2.$

Proof. Follows immediately the assumption that $(X, S) \in N(\mu, \gamma)$ and Lemma 2.3, we can obtain

$$\begin{aligned}\|H_{X^{-1/2}}[\Delta X\Delta S]\| &\leq \|X^{-1/2}\Delta X\Delta S X^{-1/2}]\| \leq \|X^{-1/2}\Delta X\Delta S X^{-1/2}]\|_F\\ &\leq \|X^{-1/2}\Delta X X^{-1/2}\|_F\|X^{1/2}\Delta S X^{1/2}\|_F\\ &\leq \frac{\|\sigma\mu I - X^{1/2}SX^{1/2}\|_F^2}{2(1-\gamma)^2\mu} \leq \frac{n\|\sigma\mu I - X^{1/2}SX^{1/2}\|^2}{2(1-\gamma)^2\mu}\\ &\leq \frac{nK^2\mu}{2(1-\gamma)^2}.\end{aligned}$$

Hence, the relation holds. ∎

3. The predictor-corrector algorithm

In this section, we give the polynomial convergence analysis of a predictor-corrector algorithm for SDP which is a director extension of the LP predictor-corrector algorithm studied by Mizuno, Todd and Ye.

The algorithm considered in this subsection is as follows.

Algorithm-I

Choose constants $0 < \tau < 1/2$ satisfying the conditions of Theorem 3.1 below. let $\epsilon \in (0,1)$ and $(X^0, S^0) \in F^0(P) \times F^0(D)$ and $\mu_0 = X^0 \bullet S^0/n$ be such that $(X^0, S^0) \in N_F(\mu_0, \tau)$ and set $k = 0$.

Repeat until $\mu_k \leq \varepsilon\mu_0$, do

(1) Compute the solution $(\Delta X_P^k, \Delta S_P^k)$ of system (10), (11), (12) with $(X, S) = (X^k, S^k)$ and $\hat{\mu} = 0$;

(2) Let $\alpha_k \equiv max\{\alpha \in [0,1] : (X^k(\alpha'), S^k(\alpha')) \in N_F((1-\alpha')\mu_k, 2\tau),$ $\forall \alpha' \in [0, \alpha]\}$, where $X^k(\alpha) = X^k + \alpha\Delta X_P^k, S^k(\alpha) = S^k + \alpha\Delta S_P^k$;

(3) Let $(\widehat{X}^k, \widehat{S}^k) \equiv (X^k, S^k) + \alpha_k(\Delta X_P^k, \Delta S_P^k)$ and $\mu_{k+1} = (1-\alpha_k)\mu_k$;

(4) Compute the solution $(\Delta X_C^k, \Delta S_C^k)$ of system (10), (11), (12) with $(X, S) = (\widehat{X}^k, \widehat{S}^k)$ and $\hat{\mu} = \mu_{k+1}$;

(5) Set $(X^{k+1}, S^{k+1}) \equiv (\widehat{X}^k, \widehat{S}^k) + (\Delta X_C^k, \Delta S_C^k)$;

(6) Increment k by 1.

End

The proof of the next lemma is straightforward and, therefore, we omit the details.

Lemma 3.7 *With the notations above, the following relations hold:*

(1) $H_{X^{-1/2}}(X(\alpha)Z(\alpha)) = (1-\alpha)H_{X^{-1/2}}(XZ) + \alpha\gamma\mu I + \alpha^2 H_{X^{-1/2}}(\Delta X \Delta Z)$;

(2) $\mu(\alpha) = (1-\alpha)\mu + \gamma\alpha\mu$;

(3) $H_{X^{-1/2}}(X(\alpha)Z(\alpha)) - \mu(\alpha)I = (1-\alpha)[H_{X^{-1/2}}(XZ) - \mu I] + \alpha\gamma\mu I$ $+ \alpha^2 H_{X^{-1/2}}(\Delta X \Delta Z)$.

By Lemma 3.1, we can obtain that the improvement of the objective value depends on the size of α, so we wish to bound α from below.

Theorem 3.1 *With the notations above, we let*

$$\hat{\alpha} = \max\{\alpha \in (0,1], (X(\alpha), Z(\alpha)) \in N(2\eta)\},$$

then

$$\hat{\alpha} \geq \frac{2}{1 + \sqrt{1 + 16\|H_{X^{-1/2}}(\Delta X \Delta Z)/\mu\|_F}}.$$

Proof. Using Lemma 2.4, we have the following inequality:

$$\begin{aligned}
&\|H_{X^{-1/2}}(X(\alpha)Z(\alpha)) - \mu(\alpha)I\|_F \\
&\quad = \|(1-\alpha)(H_{X^{-1/2}}(XZ) - \mu I) + \alpha^2 H_{X^{-1/2}}(\Delta X \Delta Z)\|_F \\
&\quad \leq \|(1-\alpha)H_{X^{-1/2}}(XZ - \mu I)\|_F + \alpha^2 \|H_{X^{-1/2}}(\Delta X \Delta Z)\|_F \\
&\quad \leq (1-\alpha)\eta\mu + \alpha^2 \|H_{X^{-1/2}}(\Delta X \Delta Z)\|_F.
\end{aligned}$$

We see that for

$$0 \leq \hat{\alpha} \leq \frac{2}{1 + \sqrt{1 + 16\|H_{X^{-1/2}}\Delta X \Delta Z/\mu\|_F}};$$

$$\begin{aligned}
\|H_{X^{-1/2}}(X(\alpha)Z(\alpha)) - \mu(\alpha)I\|_F &\leq (1-\alpha)\eta\mu + \alpha^2\|H_{X^{-1/2}}(\Delta X \Delta Z)\|_F \\
&\leq 2\eta(1-\alpha).
\end{aligned}$$

This because the quadratic term in θ:

$$\|H_{X^{-1/2}}(\Delta X \Delta Z)/\mu\|_F \alpha^2 + \eta\alpha - \eta \leq 0$$

for α between zero and the root

$$\frac{2}{1 + \sqrt{1 + 16\|H_{X^{-1/2}}(\Delta X \Delta Z)/\mu\|_F}}.$$

Thus, $\|H_{X^{-1/2}}(X(\theta)Z(\theta)) - \mu(\alpha)I\|_F \leq 2\eta(1-\alpha)\mu = 2\eta\mu(\alpha)$, then we complete the proof. ∎

Theorem 3.2 *Algorithm-I terminates in at most $\mathcal{O}(\sqrt{n}\log\varepsilon^{-1})$ iterations.*

Proof. The proof follows immediately from Theorem 3.2 and Lemma 2.7 and a simple induction argument. ∎

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PSEUDO-D-LATTICES AND TOPOLOGIES GENERATED BY MEASURES

Anna Avallone

Paolo Vitolo

Dipartimento di Matematica, Informatica ed Economia
Universitá della Basilicata
Viale dell'Ateneo Lucano
85100 Potenza
Italy
e-mail: anna.avallone@unibas.it
paolo.vitolo@unibas.it

Abstract. We prove that every modular measure on a pseudo-D-lattice L generates on L a lattice uniformity which makes uniformly continuous the pseudo-D-lattice operations. As an application, we obtain a uniqueness theorem for modular measures on pseudo D-lattices.

Keywords: modular measures, pseudo-effect algebras, uniqueness theorem.

AMS Classification: 28B05, 06C15.

1. Introduction

Effect algebras (alias D-posets) have been independently introduced in 1994 by D.J. Foulis and M.K. Bennett in [9] and by F. Chovanek and F. Kopka in [11] for modelling unsharp measurement in a quantum mechanical system. They are a generalization of many structures which arise in Quantum Physics (see [14]) and in Mathematical Economics (see [18], [22], [10]), in particular of orthomodular posets and MV-algebras. After 1994, a great number of papers concerning effect algebras have been published.

In 2001, G. Georgescu and A. Jogulescu in [20] introduced the concept of a pseudo-MV-algebra, which is a non-commutative generalization of an MV-algebra, and A. Dvurecenskij and T. Vetterlein in [15] introduced the more general structure of a pseudo-effect algebra, which is a non-commutative generalization of effect algebra. The study of these structures is motivated by the non-commutative nature of certain psychological processes and quantum mechanical experiments (see [13]) and there even exists a programming language based on a non-commutative logic (see [7]). For a study, see for example [15], [16], [12], [13], [23], [26] and many others.

In the study of modular measures on lattice ordered effect algebras, essential tools are topological methods based on the theory of uniform lattices introduced by H. Weber in [24] (see for example [6], [4], [2], [5], [3]). In particular, a starting point for these topological methods was the result that the lattice uniformity generated by a modular measure on a lattice-ordered effect algebra E makes uniformly continuous the effect algebra operations of E (see [4]).

The aim of this paper is to set up the basis for topological methods in the more general study of modular measures on pseudo-D-lattices (i.e. lattice-ordered pseudo-effect algebras), for future development of a measure theory in pseudo-D-lattices.

Thus, in the first part of this paper, we prove that every modular measure on a pseudo-D-lattice L generates on L a D-uniformity, i.e. a lattice uniformity which makes uniformly continuous the pseudo-effect algebra operations, and we study D-uniformities on L.

In the second part, we first prove a uniqueness theorem for measures on pseudo-effect algebras which extends previous results of [21] for a particular case of effect algebra and of [4] in arbitrary effect algebras; then we give a first example of application of the results of the first part, proving that, for modular measures on pseudo-D-lattices, the above uniqueness theorem holds without completeness assumptions on L.

1. Preliminaries

A partial algebra $(E, +, 0, 1)$, where $+$ is a partial binary operation and $0, 1$ are constants, is called a *pseudo-effect algebra* if, for all $a, b, c \in E$, the following properties hold:

(1) $a + b$ and $(a + b) + c$ exist if and only if $b + c$ and $a + (b + c)$ exist and in this case $(a + b) + c = a + (b + c)$.

(2) For any $a \in E$, there exist exactly one $d \in E$ and exactly one $e \in E$ such that $a + d = e + a = 1$.

(3) If $a + b$ exists, there are $d, e \in E$ such that $a + b = d + a = b + e$.

(4) If $1 + a$ or $a + 1$ exists, then $a = 0$.

We note that, if $+$ is commutative, then E becomes an effect algebra.

If we define $a \leq b$ if and only if there exists $c \in E$ such that $a + c = b$, then $\leq$ is a partial ordering on E such that $0 \leq a \leq 1$ for any $a \in E$. If E is a lattice with respect to this order, then we say that E is a *lattice pseudo-effect algebra* or a *pseudo-D-lattice*.

If E is a pseudo-effect algebra, we can define two partial binary operations on E such that, for $a, b \in E$, a/b is defined if and only if $b\backslash a$ is defined if and only if $a \leq b$ and in this case we have $(b \backslash a) + a = a + (a/b) = b$. In particular, we set ${}^{\perp}a = 1 \backslash a$ and $a^{\perp} = a/1$.

In the sequel E is a *pseudo-effect algebra*, L is a *pseudo-D-lattice* and $(G, +)$ is a *topological Abelian group*.

For basic properties of pseudo-effect algebras we refer to [15], [13] and [26]. In particular, we need the following (see 1.4 and 1.6 of [15], 2.7, 2.9, 2.10 and 2.11 of [26] and [13], pp. 32 and 33).

Proposition 1.1. *Let $a, b, c \in E$. Then:*

(1) $a + 0 = 0 + a = a$.
(2) ${}^{\perp}(a^{\perp}) = ({}^{\perp}a)^{\perp} = a$.
(3) $a + b = c$ *if and only if* $a = {}^{\perp}(b + c^{\perp})$ *if and only if* $b = ({}^{\perp}c + a)^{\perp}$.
(4) $a + b = a + c$ *implies* $b = c$; $b + a = c + a$ *implies* $b = c$.
(5) $a + b$ *exists if and only if* $a \leq {}^{\perp}b$ *if and only if* $b \leq a^{\perp}$.
(6) $a \leq b$ *if and only if* ${}^{\perp}b \leq {}^{\perp}a$ *if and only if* $b^{\perp} \leq a^{\perp}$.
(7) *If* $b + c$ *exists, then* $a \leq b$ *if and only if (*$a + c$ *exists and)* $a + c \leq b + c$.
(8) *If* $c + b$ *exists, then* $a \leq b$ *if and only if (*$c + a$ *exists and)* $c + a \leq c + b$.
(9) *If* $a \leq b \leq c$, *then* $c \backslash b \leq c \backslash a$ *and* $b/c \leq a/c$.
(10) *If* $a \leq b \leq c$, *then* $b \backslash a \leq c \backslash a$ *and* $a/b \leq a/c$.
(11) *If* $a \leq b \leq c$, *then* $(c \backslash b)/(c \backslash a) = b \backslash a$ *and* $(a/c) \backslash (b/c) = a/b$.
(12) *If* $a \leq b \leq c$, *then* $(c \backslash a) \backslash (b \backslash a) = c \backslash b$ *and* $(a/b)/(a/c) = b/c$.

Proposition 1.2. *Let $a, b, c \in L$. Then:*

(1) *If* $a \leq c$ *and* $b \leq c$, *then* $c \backslash (a \wedge b) = (c \backslash a) \vee (c \backslash b)$ *and* $(a \wedge b)/c = (a/c) \vee (b/c)$.
(2) *If* $a \leq c$ *and* $b \leq c$, *then* $c \backslash (a \vee b) = (c \backslash a) \wedge (c \backslash b)$ *and* $(a \vee b)/c = (a/c) \wedge (b/c)$.
(3) *If* $c \leq a$ *and* $c \leq b$, *then* $(a \wedge b) \backslash c = (a \backslash c) \wedge (b \backslash c)$ *and* $c/(a \wedge b) = (c/a) \wedge (b/c)$.
(4) *If* $c \leq a$ *and* $c \leq b$, *then* $(a \vee b) \backslash c = (a \backslash c) \vee (b \backslash c)$ *and* $c/(a \vee b) = (c/a) \vee (c/b)$.
(5) $((a \vee b) \backslash a) \wedge ((a \vee b) \backslash b) = 0$ *and* $(a/(a \vee b)) \wedge (b/(a \vee b)) = 0$.

In the sequel, we set $\Delta = \{(a, b) \in E \times E : a = b\}$.

If $a \leq b$, we set $[a, b] = \{c \in E : a \leq c \leq b\}$.

If $(a_n)_{n \in \mathbb{N}}$ is a sequence in E and $a \in E$, we write $a_n \uparrow a$ (respectively, $a_n \downarrow a$) if $(a_n)_{n \in \mathbb{N}}$ is increasing and $a = \sup_n a_n$ (respectively, $(a_n)_{n \in \mathbb{N}}$ is decreasing and $a = \inf_n a_n$).

If $a_1, \ldots, a_n \in E$, we inductively define $a_1 + \cdots + a_n = (a_1 + \cdots + a_{n-1}) + a_n$, provided that the right hand side exists. We say that the finite sequence $(a_1, ..., a_n)$ of E is *orthogonal* if $a_1 + \cdots + a_n$ exists. Given an infinite sequence $(a_n)_{n \in \mathbb{N}}$, we say that it is *orthogonal* if, for every positive integer n, $a_1 + \cdots + a_n$ exists. If, moreover, $\sup_{n \in \mathbb{N}}(a_1 + \cdots + a_n)$ exists, we set $\sum_{n=1}^{\infty} a_n = \sup_{n \in \mathbb{N}}(a_1 + \cdots + a_n)$.

We say that E is *Archimedean* if, for every $a \in E$ with $a \neq 0$, there exists an integer $k > 0$ such that ka exists and $(k+1)a$ does not exist, where $ka = a + \cdots + a$ k times.

We say that E is *σ-complete* if, for every orthogonal sequence (a_n), $\sum_{n=1}^{\infty} a_n$ exists.

If E is a pseudo-D-lattice, we set $a \,{}^*\!\triangle\, b = (a \vee b)\backslash(a \wedge b)$ and $a \triangle^* b = (a \wedge b)/(a \vee b)$.

A function $\mu : E \to G$ is said to be a *measure* if, for every $a, b \in E$ with $a \leq b$, $\mu(b) - \mu(a) = \mu(b\backslash a) = \mu(a/b)$. It is easy to see that μ is a measure if and only if, for every $a, b \in E$ such that $a + b$ exists, $\mu(a + b) = \mu(a) + \mu(b)$. We say that μ is *σ-additive* if, for every orthogonal sequence (a_n) in E such that $a = \sum_{n=1}^{\infty} a_n$ exists, $\mu(a) = \sum_{n=1}^{\infty} \mu(a_n)$. If $\mu : L \to G$, we say that μ is *modular* if, for every $a, b \in L$, $\mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b)$.

A uniformity $\mathcal{U}$ on L is said to be a *lattice uniformity* if the lattice operations $\vee$ and $\wedge$ are uniformly continuous with respect to $\mathcal{U}$. For a study, see [24].

As proved in [19], if L_1 is a lattice, every modular function $\mu : L_1 \to G$ generates on L_1 a lattice uniformity $\mathcal{U}(\mu)$, called *μ-uniformity,* which is the weakest lattice uniformity which makes μ uniformly continuous and a basis of $\mathcal{U}(\mu)$ is the family consisting of the sets

$$U_W = \{(a, b) \in L_1 \times L_1 : \mu(c) - \mu(d) \in W \ \forall\ c, d \in [a \wedge b, a \vee b], d \leq c\},$$

where W is a neighbourhood of 0 in G.

A lattice uniformity $\mathcal{U}$ on L_1 is said to be *exhaustive* if every monotone sequence in L_1 is a Cauchy sequence in $\mathcal{U}$, *σ-order-continuous* (σ-o.c.) if $a_n \uparrow a$ or $a_n \downarrow a$ in L_1 implies that $\{a_n\}$ converges to a in $\mathcal{U}$, and *order-continuous* (o.c.) if the same condition holds for nets.

If $\mu : L_1 \to G$ is a modular function, μ is said to be *exhaustive* (respectively, *σ-o.c.* or *o.c.*) if $\mathcal{U}(\mu)$ is exhaustive (respectively, σ-o.c. or o.c.).

By 3.5 and 3.6 of [25], we have that a modular function μ is exhaustive if and only if $\mu(a_{n+1}) - \mu(a_n)$ converges to 0 for every monotone sequence $(a_n)_{n\in\mathbb{N}}$ in L_1, μ is σ-o.c. if and only if $a_n \uparrow a$ or $a_n \downarrow a$ imply that $(\mu(a_n))$ converges to $\mu(a)$, and μ is o.c. if and only if, for every monotone net $(a_\alpha)_{\alpha\in J}$ order convergent to a, $(\mu(a_\alpha))$ converges to $\mu(a)$.

2. D-uniformities and modular measures

In this section we introduce the concept of D-uniformity on L, which arises in a natural way from the study of modular measures since, as we will see in Theorem 2.9, every G-valued modular measure μ on L generates on L a D-uniformity.

First we need some preliminaries.

Lemma 2.1. *Let $a, b, c \in E$.*

(1) *If $a+b$ exists and $a+b \leq c$, then $c\backslash(a+b) = (c\backslash b)\backslash a$ and $(a+b)/c = b/(a/c)$.*

(2) *If $a + b$ exists, then $a + b = ({}^{\perp}b\backslash a)^{\perp} = {}^{\perp}(b/a^{\perp})$.*

(3) *If $a \leq b$, then $b\backslash a = {}^{\perp}(a + b^{\perp})$ and $a/b = ({}^{\perp}b + a)^{\perp}$.*

Proof. (1) Set $d = c\backslash(a+b)$. Then $c = d+(a+b) = (d+a)+b$, whence $d+a = c\backslash b$. Therefore $d = (c\backslash b)\backslash a$.

In a similar way, setting $e = (a+b)/c$, we have $b+e = a/c$ and therefore $e = b/(a/c)$.

(2) Setting $c = 1$ in (1), we have ${}^{\perp}(a+b) = {}^{\perp}b\backslash a$ and $(a+b)^{\perp} = b/a^{\perp}$. Therefore, by Proposition 1.1-(2), we obtain $a+b = ({}^{\perp}b\backslash a)^{\perp}$ and $a+b = {}^{\perp}(b/a^{\perp})$.

(3) By Proposition 1.1-(5), $a \leq b$ implies that $a+b^{\perp}$ exists. Then (3) follows from (2) and Proposition 1.1-(2). ∎

Lemma 2.2. *Let $a, b \in E$, with $a \leq b$. Then*

$$a/b = a^{\perp}\backslash b^{\perp} \text{ and } b\backslash a = {}^{\perp}b/{}^{\perp}a.$$

Proof. It is sufficient to set $c = 1$ in Proposition 1.1-(11). ∎

Lemma 2.3. *If $a, b \in L$, then*

$$a \mathbin{\triangle^{*}} b = a^{\perp} \mathbin{{}^{*}\triangle} b^{\perp} \ (\text{and } a \mathbin{{}^{*}\triangle} b = {}^{\perp}a \mathbin{{}^{*}\triangle} {}^{\perp}b).$$

Proof. By Lemma 2.2, we have $a \mathbin{\triangle^{*}} b = (a \wedge b)/(a \vee b) = (a \wedge b)^{\perp}\backslash(a \vee b)^{\perp} = (a^{\perp} \vee b^{\perp})\backslash(a^{\perp} \wedge b^{\perp}) = a^{\perp} \mathbin{{}^{*}\triangle} b^{\perp}$. The other equality can be proved in a similar way. ∎

Lemma 2.4. *Let $c, d \in E$ be such that $c \leq d$. Set*

$$I_{c,d} = \{a \in E : \exists\, r, s \in [c, d] : r \leq s, a = s\backslash r\}$$

and

$$J_{c,d} = \{a \in E : \exists\, r, s \in [c, d] : r \leq s, a = r/s\}.$$

Then $I_{c,d} = [0, d\backslash c]$ and $J_{c,d} = [0, c/d]$.

Proof. We prove the first equality. The other equality can be proved in a similar way. Let $a \in I_{c,d}$ and choose $r, s \in E$ such that $c \leq r \leq s \leq d$ and $a = s\backslash r$. By Proposition 1.1-(9) and (10), we obtain $a \leq d\backslash r \leq d\backslash c$. Conversely, let $a \in [0, d\backslash c]$. Then, by Proposition 1.1-(8), $a + c$ exists and $a + c \leq d$. Set $s = a + c$ and $r = c$. Then $c = r \leq s \leq d$ and $a = (a + c)\backslash c = s\backslash r$. ∎

Proposition 2.5. *Let $\mathcal{U}$ be a uniformity on E. Set $E_1 = \{(a, b) \in E \times E : b \leq a\}$ and $E_2 = \{(a, b) \in E \times E : a + b \text{ exists}\}$ $(= \{(a, b) \in E \times E : b \leq a^{\perp}\})$. Then the following conditions are equivalent:*

1. *The operations $(a, b) \in E_2 \to a + b \in E$, $a \in E \to {}^{\perp}a \in E$ and $a \in E \to a^{\perp} \in E$ are uniformly continuous with respect to $\mathcal{U}$.*
2. *The operations $(a, b) \in E_1 \to a\backslash b \in E$ and $(a, b) \in E_1 \to b/a \in E$ are uniformly continuous with respect to $\mathcal{U}$.*

(3) *The operations* $(a,b) \in E_1 \to a\backslash b \in E$ *and* $a \in E \to a^\perp \in E$ *are uniformly continuous with respect to* $\mathcal{U}$.

(4) *The operations* $(a,b) \in E_1 \to b/a \in E$ *and* $a \in E \to {}^\perp a \in E$ *are uniformly continuous with respect to* $\mathcal{U}$.

(5) *The operation* $(a,b) \in E_2 \to a^\perp\backslash b \; (= a/{}^\perp b) \in E$ *is uniformly continuous with respect to* $\mathcal{U}$.

Moreover, if E *is a pseudo D-lattice and* $\mathcal{U}$ *is a lattice uniformity on* E, *each of the above conditions is equivalent to each of the following:*

(6) *The operations* $(a,b) \in E \times E \to a \mathbin{{}^*\!\triangle} b \in E$ *and* $a \in E \to a^\perp \in E$ *are uniformly continuous with respect to* $\mathcal{U}$.

(7) *The operations* $(a,b) \in E \times E \to a \mathbin{\triangle^*} b \in E$ *and* $a \in E \to {}^\perp a \in E$ *are uniformly continuous with respect to* $\mathcal{U}$.

(8) *The operation* $(a,b) \in E \times E \to a^\perp \mathbin{{}^*\!\triangle} b \; (= a \mathbin{\triangle^*} {}^\perp b) \in E$ *is uniformly continuous with respect to* $\mathcal{U}$.

Proof. (1) $\Rightarrow$ (2) The uniform continuity of $\backslash$ and $/$ follows from Lemma 2.1-(3).

(2) $\Rightarrow$ (3) it is trivial.

(2) $\Rightarrow$ (4) it is trivial.

(3) $\Rightarrow$ (1) It is clear that the operation $a \in E \to {}^\perp a$ is uniformly continuous. The uniform continuity of $+$ follows from Lemma 2.1-(2).

(4) $\Rightarrow$ (1) is similar to the proof of (3) $\Rightarrow$ (1).

(3) $\Rightarrow$ (5) is trivial.

(5) $\Rightarrow$ (3) Set $a * b = a^\perp\backslash b$. Then, since $a * 0 = a^\perp$ and $0 * a = {}^\perp a$, we obtain $a\backslash b = ({}^\perp a)^\perp\backslash b = {}^\perp a * b = (0 * a) * b$ and therefore $\backslash$ is uniformly continuous.

(6) $\Rightarrow$ (3), (7) $\Rightarrow$ (4), (8) $\Rightarrow$ (5) and (6) $\Rightarrow$ (8) are trivial.

(7) $\Rightarrow$ (6) follows from Lemma 2.3 and the equality $a^\perp = a \mathbin{\triangle^*} 1$.

(4) $\Rightarrow$ (7) follows from the definition of $a \mathbin{\triangle^*} b$. ∎

Definition 2.6. We say that a lattice uniformity $\mathcal{U}$ on L is a D-uniformity if it satisfies one of the conditions in the above proposition (and hence all).

Therefore, if L is a lattice-ordered effect algebra, a D-uniformity in the sense of Definition 2.6 is a D-uniformity according to [4].

If we set, for subsets U and V of $L \times L$,

$$U\backslash V = \{(a\backslash c, b\backslash d) : c \leq a, d \leq b, (a,b) \in U, (c,d) \in V\},$$

$$U/V = \{(c/a, d/b) : c \leq a, d \leq b, (a,b) \in U, (c,d) \in V\},$$

$$U^\perp = \{(a^\perp, b^\perp) \in L \times L : (a,b) \in U\},$$

$$^\perp U = \{({}^\perp a, {}^\perp b) \in L \times L : (a,b) \in U\}$$

it is clear that a lattice uniformity $\mathcal{U}$ on L is a D-uniformity if and only if, for every $U \in \mathcal{U}$, there exist $V, W \in \mathcal{U}$ such that $V\backslash V \subseteq U$ and $W^\perp \subseteq U$ if and only if, for every $U \in \mathcal{U}$, there exist $V, W \in \mathcal{U}$ such that $V/V \subseteq U$ and ${}^\perp W \subseteq U$.

Moreover the following result holds.

Proposition 2.7. *Let $\mathcal{U}$ be a lattice uniformity on L. Then $\mathcal{U}$ is a D-uniformity if and only if, for every $U \in \mathcal{U}$, there exists $V, W \in \mathcal{U}$ such that $V \backslash \Delta \subseteq U$, $\Delta \backslash V \subseteq U$ and $W^{\perp} \subseteq U$.*

Proof. Suppose that the above conditions are satisfied and we prove that $\mathcal{U}$ is a lattice uniformity.

Since it follows by assumption that $\perp$ is uniformly continuous, we have only to prove that, for every $U \in \mathcal{U}$, there exists $V_2 \in \mathcal{U}$ such that $V_2 \backslash V_2 \subseteq U$.

Let $U \in \mathcal{U}$ and choose $V, V_1, V_2 \in \mathcal{U}$ such that $V \circ V \circ V \subseteq U$, $V_1 \backslash \Delta \subseteq V$, $\Delta / V_1 \subseteq V$ and $V_2 \wedge V_2 \subseteq V_1$.

We prove that $V_2 \backslash V_2 \subseteq U$. Let $(a, b), (c, d)$ be in V_2 such that $c \leq a$ and $d \leq b$. We prove that $(a \backslash c, b \backslash d) \in U$. Indeed, since $(c, c \wedge d) \in \Delta \wedge V_2 \subseteq V_2 \wedge V_2 \subseteq V_1$, we have

$$(*) \qquad (a \backslash c, a \backslash (c \wedge d)) \in \Delta \backslash V_1 \subseteq V.$$

Moreover, from $V_2 \subseteq V_1$, we get

$$(**) \qquad (a \backslash (c \wedge d), b \backslash (c \wedge d)) \in V_1 \backslash \Delta \subseteq V.$$

Finally, since $(c \wedge d, d) \in V_2 \wedge V_2 \subseteq V_1$, we have

$$(***) \qquad (b \backslash (c \wedge d), b \backslash d) \in \Delta \backslash V_1 \subseteq V.$$

From $(*)$, $(**)$ and $(***)$, we obtain $(a \backslash c, b \backslash d) \in V \circ V \circ V \subseteq U$. ■

In the sequel, if $\mu : L \to G$ is a function and W is a neighbourhood of 0 in G, we set

$$\begin{aligned} U_W &= \{(a, b) \in L \times L : \mu(s) - \mu(r) \in W \ \forall\, r, s \in [a \wedge b, a \vee b],\ r \leq s\}, \\ A_W &= \{(a, b) \in L \times L : \mu(c) \in W \ \forall\, c \leq a \mathbin{{}^*\!\triangle} b\}, \\ B_W &= \{(a, b) \in L \times L : \mu(c) \in W \ \forall\, c \leq a \mathbin{\triangle^*} b\}. \end{aligned}$$

Lemma 2.8. *Let $\mu : L \to G$ be a measure and W a neighbourhood of 0 in G. Then $U_W = A_W = B_W$.*

Proof. We use the notations of Lemma 2.4. Applying Lemma 2.4, we have

$$\begin{aligned} A_W &= \{(a, b) \in L \times L :\ \forall\, c \in [0, a \mathbin{{}^*\!\triangle} b],\ \mu(c) \in W\} \\ &= \{(a, b) \in L \times L :\ \forall\, c \in I_{a \wedge b, a \vee b},\ \mu(c) \in W\} \\ &= \{(a, b) \in L \times L :\ \forall\, r, s \in [a \wedge b, a \vee b],\ r \leq s,\ \mu(s \backslash r) \in W\} = U_W \\ &= \{(a, b) \in L \times L :\ \forall\, r, s \in [a \wedge b, a \vee b],\ r \leq s,\ \mu(r / s) \in W\} \\ &= \{(a, b) \in L \times L :\ \forall\, c \in J_{a \wedge b, a \vee b},\ \ \mu(c) \in W\} \\ &= \{(a, b) \in L \times L :\ \forall\, c \in [0, a \mathbin{\triangle^*} b],\ \ \mu(c) \in W\} = B_W. \end{aligned}$$

■

Theorem 2.9. *Let $\mu : L \to G$ be a modular measure. Then the μ-uniformity $\mathcal{U}(\mu)$ is a D-uniformity on L and a base of $\mathcal{U}(\mu)$ is the family consisting of the sets A_W, where W is a neighbourhood of 0 in G.*

Proof. By Lemma 2.8, a base of $\mathcal{U}(\mu)$ is the family consisting of the sets A_W, where W is a neighbourhood of 0 in G. Then, by Proposition (2.7), it is sufficient to prove the following conditions:

(1) $A_W^{\perp} = A_W$.

(2) $A_W \backslash \Delta \subseteq A_W$.

(3) $\Delta / A_W \subseteq A_W$.

Proof of (1). By Lemma 2.3 and 2.8, we have

$$\begin{aligned} A_W^{\perp} &= B_W^{\perp} = \{(a^{\perp}, b^{\perp}) : (a,b) \in B_W\} = \{(a^{\perp}, b^{\perp}) : \ \forall\, c \le a \mathbin{\triangle^{*}} b,\ \mu(c) \in W\} \\ &= \{(a^{\perp}, b^{\perp}) : \ \forall\, c \le a^{\perp} \mathbin{{}^{*}\!\triangle} b^{\perp},\ \mu(c) \in W\} = A_W. \end{aligned}$$

Proof of (2). It is sufficient to prove that, for every a, b, c in L with $c \le a$ and $c \le b$, $(a \backslash c) \mathbin{{}^{*}\!\triangle} (b \backslash c) = a \mathbin{{}^{*}\!\triangle} b$.

Set $d = a \backslash c$ and $e = b \backslash c$. By Proposition 1.2-(3) and (4), we have $d \vee e = (a \vee b) \backslash c$ and $d \wedge e = (a \wedge b) \backslash c$. Therefore, by Proposition 1.1-(12), we obtain $d \mathbin{{}^{*}\!\triangle} e = ((a \vee b) \backslash c) \backslash ((a \wedge b) \backslash c) = (a \vee b) \backslash (a \wedge b) = a \mathbin{{}^{*}\!\triangle} b$.

Proof of (3). Since $A_W = B_W$ by Lemma 2.8, it is sufficient to prove that, for every a, b, c in L with $a \le c$ and $b \le c$, $(c \backslash a) \mathbin{\triangle^{*}} (c \backslash b) = a \mathbin{{}^{*}\!\triangle} b$.

Set $d = c \backslash a$ and $e = c \backslash b$. By Proposition 1.2-(1) and (2), we obtain $d \vee e = c \backslash (a \wedge b)$ and $d \wedge e = c \backslash (a \vee b)$. By Proposition 1.1-(11), we get $d \mathbin{\triangle^{*}} e = (d \wedge e)/(d \vee e) = (c \backslash (a \vee b))/(c \backslash (a \wedge b)) = (a \vee b) \backslash (a \wedge b) = a \mathbin{{}^{*}\!\triangle} b$. ∎

Definition 2.10. A DV-congruence on L (after A. Dvurecenskij and T. Vetterlein) is an equivalence relation N which satisfies the following conditions:

(a) For every $a, b \in L$, if $(a, c) \in N$, $(b, d) \in N$, $a + b$ and $c + d$ exist, then $(a + b, c + d) \in N$.

(b) If $a + b$ exists, then, for every $c \in L$ such that $(c, a) \in N$, there exists $d \in L$ such that $(d, b) \in N$ and $c + d$ exists; and, for every $h \in L$ such that $(h, b) \in N$, there exists $k \in L$ such that $(k, a) \in N$ and $k + h$ exists.

Proposition 2.11. *Let N be a DV-congruence on E. Define the operation $+$ on the quotient E/N in the following way: For every $\hat{a}, \hat{b} \in E/N$, $\hat{a} + \hat{b} = \hat{c}$ if and only if there exist $a' \in \hat{a}$, $b' \in \hat{b}$ and $c' \in \hat{c}$ such that $a' + b' = c'$ in E. Then:*

(1) *$+$ is well defined on E/N and $(E/N, +, \hat{0}, \hat{1})$ is a pseudo-effect algebra.*

(2) *If $c \ge b$, then $\widehat{c \backslash b} = \hat{c} \backslash \hat{b}$ and $\widehat{b / c} = \hat{b}/\hat{c}$.*

Proof. (1) has been proved in 3.3 of [16].

(2) follows from the definition of $+$ in E/N, since, if we set $a = c \backslash b$, we have $\hat{a} + \hat{b} = \hat{c}$, whence $\hat{a} = \hat{c} \backslash \hat{b}$. In a similar way we obtain the other equality. ∎

The aim of the next two theorems is to obtain also in pseudo D-lattices a technique based on the "completion method" of H. Weber (see [24] and [25]) which allowed in many cases to reduce the study of exhaustive modular measures on D-lattices to the study of o.c. modular measures on complete D-lattices (see for example [1]–[5]).

Theorem 2.12. *Let $\mathcal{U}$ be a D-uniformity on L. Then the following properties hold:*

(1) $N(\mathcal{U}) = \bigcap\{U : U \in \mathcal{U}\}$ *is a DV-congruence and a lattice congruence.*

(2) *The quotient* $\hat{L} = L/N(\mathcal{U})$ *is a pseudo-D-lattice.*

(3) *Setting, for* $U \in \mathcal{U}$, $\widehat{U} = \{(\hat{a}, \hat{b}) \in \hat{L} \times \hat{L} : (a, b) \in U\}$, *the quotient uniformity* $\widehat{\mathcal{U}} = \{\widehat{U} : U \in \mathcal{U}\}$ *is a Hausdorff D-uniformity on* $\hat{L}$.

(4) *If G is Hausdorff and $\mu : L \to G$ is a modular measure which is uniformly continuous with respect to $\mathcal{U}$, then the function $\hat{\mu} : \hat{L} \to G$ defined as $\hat{\mu}(\hat{a}) = \mu(a)$ for $a \in \hat{a} \in \hat{L}$ is a well defined modular measure on $\hat{L}$ and the D-uniformity generated by $\hat{\mu}$ coincides with $\widehat{\mathcal{U}}$.*

Proof. (1) $N(\mathcal{U})$ is a lattice congruence by 1.2.2 of [24]. Moreover, it is clear that $N(\mathcal{U})$ satisfies condition (a) of Definition 2.10. Indeed, it is sufficient to observe that, since $\mathcal{U}$ is a D-uniformity, then, for every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V + V \subseteq U$, where

$$V + V = \{(a + c, b + d) : (a, b) \in V, (c, d) \in V, a + c \text{ and } b + d \text{ exist}\}.$$

Now we prove condition (b) of Definition 2.10. Since $\mathcal{U}$ is a D-uniformity, it is clear that $(a, b) \in N(\mathcal{U})$ implies $(a^\perp, b^\perp) \in N(\mathcal{U})$ and $({}^\perp a, {}^\perp b) \in N(\mathcal{U})$.

Now, suppose that $a + b$ exists and let $c \in L$ be such that $(c, a) \in N(\mathcal{U})$. We prove that there exists $d \in L$ such that $(d, b) \in N(\mathcal{U})$ and $c + d$ exists. Set $d = c^\perp \wedge b$. By Proposition 1.1-(5), $c+d$ exists since $d \le c^\perp$. Let $U \in \mathcal{U}$ and choose $V \in \mathcal{U}$ such that $V \wedge \Delta \subseteq U$. Since $a+b$ exists, by Proposition 1.1-(5) we have that $b \le a^\perp$. Therefore, we get $(d, b) = (c^\perp \wedge b, a^\perp \wedge b) = (c^\perp, a^\perp) \wedge (b, b) \in V \wedge \Delta \subseteq U$. Hence, $(d, b) \in N(\mathcal{U})$.

In a similar way, let $h \in L$ be such that $(h, b) \in N(\mathcal{U})$. We prove that there exists $k \in L$ such that $(k, a) \in N(\mathcal{U})$ and $k + h$ exists. Set $k = {}^\perp h \wedge a$. Since $k \le {}^\perp h$, by Proposition 1.1-(5) we have that $h + k$ exists. Moreover, since $a + b$ exists, we have $a \le {}^\perp b$. Let $U \in \mathcal{U}$ and choose $V \in \mathcal{U}$ such that $\Delta \wedge V \subseteq U$. Then we obtain $(k, a) = (a \wedge {}^\perp h, a \wedge {}^\perp b) = (a, a) \wedge ({}^\perp h, {}^\perp b) \in \Delta \wedge V \subseteq U$.

(2) By Proposition 2.11, $\hat{L}$ is a pseudo-effect-algebra. It remains to prove that $\hat{L}$ is a pseudo-D-lattice. By (3) of [16] (page 7), we have that $\hat{a} \le \hat{b}$ if and only if there exists $h \in L$ such that $\hat{a} + \hat{h} = \hat{b}$ if and only if there exist $c, d \in L$ such that $(c, a) \in N(\mathcal{U}), (d, h) \in N(\mathcal{U}), c + d$ exists and $(c + d, b) \in N(\mathcal{U})$.

Moreover, by 1.2.3 of [24], $\hat{L}$ is a lattice with respect to the following order: $\hat{a} \le' \hat{b}$ if and only if there exist $c, k \in L$ such that $(c, a) \in N(\mathcal{U}), (k, b) \in N(\mathcal{U})$ and $c \le k$. Therefore, it is sufficient to observe that $\le$ and $\le'$ coincide.

(3) It is known by Proposition 1.2.4 of [24] that $\widehat{\mathcal{U}}$ is a Hausdorff lattice uniformity. To prove that $\widehat{\mathcal{U}}$ is a D-uniformity, we apply Proposition 2.7. Let $\widehat{U} \in \widehat{\mathcal{U}}$ and choose $W \in \mathcal{U}$ such that $W^{\perp} \subseteq U$. Applying Proposition 2.11-(3), we obtain that $\widehat{W}^{\perp} = \widehat{W^{\perp}}$ and therefore $\widehat{W}^{\perp} \subseteq \widehat{U}$.

Now, choose $V \in \mathcal{U}$ closed such that $V \backslash V \subseteq U$. We prove that $\widehat{V} \backslash \widehat{\Delta} \subseteq \widehat{U}$. Let $a, b, c \in L$ be such that $(\hat{a}, \hat{b}) \in \widehat{V}$, $\hat{c} \leq \hat{a}$ and $\hat{c} \leq \hat{b}$. We prove that $(\hat{a} \backslash \hat{c}, \hat{b} \backslash \hat{c}) \in \widehat{U}$. Since $\hat{c} \leq \hat{a}$ and $\hat{c} \leq \hat{b}$, by 1.2.3 of [24] we can find $d, e, r, s \in L$ such that $\hat{d} = \hat{c}$, $\hat{e} = \hat{a}$, $\hat{r} = \hat{c}$, $\hat{s} = \hat{b}$, $d \leq e$ and $r \leq s$. Since $(\hat{e}, \hat{s}) = (\hat{a}, \hat{b}) \in \widehat{V}$, by Proposition 1.2.4 of [24] we obtain that $(e, s) \in V$. Moreover, since $\hat{d} = \hat{r}$, we have that $(d, r) \in N(\mathcal{U})$. Therefore, we get $(e \backslash d, s \backslash r) \in V \backslash V \subseteq U$. Set $h = e \backslash d$ and $k = s \backslash r$. Hence we have $(\hat{h}, \hat{k}) \in \widehat{U}$. Now it is sufficient to observe that $(\hat{h}, \hat{k}) = (\hat{a} \backslash \hat{c}, \hat{b} \backslash \hat{c})$ by Proposition 2.11-(2).

The other condition of Proposition 2.7 can be proved in a similar way.

(4) is known by the theory of uniform lattices (see [25]). Indeed, since $\mathcal{U}(\mu)$ is the weakest lattice uniformity which makes μ uniformly continuous, we have $\mathcal{U}(\mu) \leq \mathcal{U}$, from which $N(\mathcal{U}) \subseteq N(\mathcal{U}(\mu))$. By Propositions 2.5 and 3.1 of [25], $(a, b) \in N(\mathcal{U}(\mu))$ if and only if μ is constant on the interval $[a \wedge b, a \vee b]$. Then, if $\hat{a} = \hat{b}$, we have $\mu(a) = \mu(b)$ and therefore $\hat{\mu}$ is well defined on $\hat{L}$. It is also known that $\hat{\mu}$ is a modular function, too, and $\mathcal{U}(\hat{\mu}) = \widehat{\mathcal{U}}$. Here we have only to observe that, because of the definition of $+$ in $\hat{L}$ (see Proposition 2.11), $\hat{\mu}$ is a measure, too. ∎

Theorem 2.13. *Let $\mathcal{U}$ be a Hausdorff D-uniformity on L and let $(\tilde{L}, \widetilde{\mathcal{U}})$ be the uniform completion of $(L, \mathcal{U})$. Then the following properties hold:*

1. *The lattice operations $\vee$ and $\wedge$ and the pseudo-D-lattice operations $\backslash$ and $/$ can be extended in a unique way such that $\tilde{L}$ becomes a pseudo-D-lattice.*
2. *$\widetilde{\mathcal{U}}$ is a D-uniformity on $\tilde{L}$.*
3. *If $\mathcal{U}$ is exhaustive, then $\tilde{L}$ is complete as lattice and $\widetilde{\mathcal{U}}$ is o.c.*
4. *If G is complete and Hausdorff and $\mu : L \to G$ is a modular measure which is uniformly continuous with respect to $\mathcal{U}$, then μ can be extended in a unique way to a modular measure $\tilde{\mu} : \tilde{L} \to G$ which is uniformly continuous with respect to $\widetilde{\mathcal{U}}$ and o.c. and $\mu(L)$ is dense in $\tilde{\mu}(\tilde{L})$.*

Proof. (1) By Proposition 1.3.1 of [24], it is known that the lattice operations $\vee$ and $\wedge$ can be extended in a unique way such that $\tilde{L}$ becomes a lattice and $\widetilde{\mathcal{U}}$ is a lattice uniformity.

Then the set $\tilde{L}' = \{(a, b) \in \tilde{L} \times \tilde{L} : b \leq a\}$ coincides with the closure in $(\tilde{L}, \widetilde{\mathcal{U}})$ of the set $\{(a, b) \in L \times L : b \leq a\}$.

Denote again by $/$ and $\backslash$ the uniformly continuous extensions, respectively, of $/$ and $\backslash$ to $\tilde{L}'$.

To prove that $\tilde{L}$ is a pseudo-D-lattice, it is sufficient, by Theorem 2.7 of [26], to prove that $\backslash$ and $/$ have the following properties:

(a) If $a \leq b \leq c$, then $b/c \leq a/c$, $c\backslash b \leq c\backslash a$, $(a/c)\backslash(b/c) = a/b$, $(c\backslash b)/(c\backslash a) = b\backslash a$.

(b) For every $a \in L$, $a\backslash 0 = 0/a = a$.

(a) Let a, b, c in $\tilde{L}$ such that $a \leq b \leq c$. Choose nets (a_α), (b_α) and (c_α) in L convergent, respectively, to a, b and c in $(\tilde{L}, \widetilde{\mathcal{U}})$. Without loss of generality, we may assume that they are indexed in the same way. Moreover, we may suppose that $a_\alpha \leq b_\alpha \leq c_\alpha$ for each α, since (a_α) can be replaced by $(a_\alpha \wedge b_\alpha)$ and (b_α) by $(b_\alpha \wedge c_\alpha)$. Therefore, by the definition of $\backslash$ and $/$, we obtain $b/c = \lim_\alpha(b_\alpha/c_\alpha) \leq \lim_\alpha(a_\alpha/c_\alpha) = a/c$ and $c\backslash b = \lim_\alpha(c_\alpha\backslash b_\alpha) \leq \lim_\alpha(c_\alpha\backslash a_\alpha) = c\backslash a$. Moreover, since $b_\alpha/c_\alpha \leq a_\alpha/c_\alpha$, we have $(a/c)\backslash(b/c) = \lim_\alpha((a_\alpha/c_\alpha)\backslash(b_\alpha/c_\alpha)) = \lim_\alpha(a_\alpha/b_\alpha) = a/b$. In a similar way, since $c_\alpha\backslash b_\alpha \leq c_\alpha\backslash a_\alpha$, we obtain that $(c\backslash b)/(c\backslash a) = b\backslash a$.

In the same way we obtain (b).

(2) It is known that a base of $\widetilde{\mathcal{U}}$ consists of the sets $\{\overline{U} : U \in \mathcal{U}\}$, where $\overline{U}$ is the closure of U in $\widetilde{\mathcal{U}}$. Then, to prove that $\widetilde{\mathcal{U}}$ is a D-uniformity, it is sufficient to prove that, for every $U \in \mathcal{U}$, there exist $V, W \in \mathcal{U}$ such that $\overline{V}\backslash\overline{V} \subseteq U$ and $\overline{W}/\overline{W} \subseteq U$.

Let $U \in \mathcal{U}$ and choose $V \in \mathcal{U}$ such that $V\backslash V \subseteq U$. Let $(a,b) \in \overline{V}$ and $(c,d) \in \overline{V}$ be such that $c \leq a$ and $d \leq b$. Choose nets $((a_\alpha, b_\alpha))$ and $((c_\alpha, d_\alpha))$ in V convergent, respectively, to (a,b) and (c,d) in $\widetilde{\mathcal{U}} \times \widetilde{\mathcal{U}}$. We may suppose that, for each α, $c_\alpha \leq a_\alpha$ and $d_\alpha \leq b_\alpha$. Then $((a_\alpha\backslash c_\alpha, b_\alpha\backslash d_\alpha)) \in V\backslash V \subseteq U$ and, by the definition of $\backslash$, converges to $(a\backslash c, b\backslash d)$. Therefore we get $(a\backslash c, b\backslash d) \in \overline{U}$.

In a similar way we prove that there exists $W \in \mathcal{U}$ such that $\overline{W}/\overline{W} \subseteq \overline{U}$.

(3) and (4) have been proved in Proposition 3.7 of [25]. We have only to observe that the continuity of $\tilde{\mu}$ and the definition of $\backslash$ and $/$ on $\tilde{L}$ imply that $\tilde{\mu}$ is a measure. ∎

Remark. In Theorem 4.6 of [17], it is proved that every Archimedean (and, therefore, every σ-complete) pseudo-MV-algebra is commutative. This is not true if L is an Archimedean pseudo-effect algebra, as the next examples prove.

Let $E = \{0, 1, a, b, c\}$, where a, b, and c are not comparable. Define $a + b = b + c = c + a = 1$, while $b + a$, $c + b$ and $a + c$ are undefined. Then, E is a complete modular pseudo-D-lattice which is not commutative.

Moreover let $\mu : E \to [0,1]$ be defined as $\mu(a) = \mu(b) = \mu(c) = 1/2$, $\mu(0) = 0$ and $\mu(1) = 1$. Then μ is a modular measure on E with $N(\mathcal{U}(\mu)) = \Delta$ and therefore $\tilde{E} = E/N(\mathcal{U}(\mu)) = E$.

Now we obtain an infinite example considering the set F of all sequences with values in E, in which we define $(a_n) + (b_n)$ if and only if, for each $n \in \mathbb{N}$, $a_n + b_n$ exists and in this case $(a_n) + (b_n) = (a_n + b_n)$.

It is easy to see that, since E is finite, F is a complete pseudo-D-lattice. Moreover, if we define $\lambda : F \to [0,1]$ as $\lambda(a) = \sum_{n=1}^{\infty} \mu(a_n)/2^n$ for $a = (a_n) \in F$, we obtain a modular measure on F with $\lambda(a) > 0$ for every $a \in F$ with $a \neq 0$. Then $\tilde{F} = F/N(\mathcal{U}(\lambda)) = F$.

3. Uniqueness theorems

In this section we prove a uniqueness theorem for measures on pseudo-effect-algebras and we apply the results of the previous section to prove that, for modular measures on pseudo D-lattices, the uniqueness theorem holds without completeness assumptions.

We say that E *has the interpolation property* if, for every sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in E, with $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$ for each n, there exists $a \in E$ such that, for each n, $a_n \leq a \leq b_n$.

It is clear that, if E is σ-complete, then E has the interpolation property.

If $\mu : E \to G$ is a measure, we say that:

- E is *μ-chained* if, for every neighbourhood W of 0 in G and every $a \in E$, there exist $a_0, a_1, \ldots, a_r$ in E such that $0 = a_0 \leq a_1 \leq \ldots \leq a_r = a$ and $\mu(c) - \mu(d) \in W$ whenever $c, d \in [a_{i-1}, a_i]$ for some $i \in \{1, \ldots, r\}$.
- μ is *strongly continuous* if, for every neighbourhood W of 0 in G and every $a \in E$, there exists an orthogonal finite family $(b_1, \ldots, b_r)$ in E such that $b_1 + \ldots + b_r = a$ and $\mu(b) \in W$ whenever $b \leq b_i$ for some $i \leq r$.
- If G is a linear space, μ is *convex-ranged* if, for every $a \in E$, $\mu([0, a])$ is convex.

Lemma 3.1. *Let a, b, c in E.*

(1) *If $c \leq a$ and $a + b$ exists, then $(c/a) + b$ exists and $c/(a + b) = (c/a) + b$.*

(2) *If $c \leq a$ and $b + a$ exists, then $b + (a\backslash c)$ exists and $(b + a)\backslash c = b + (a\backslash c)$.*

(3) *If $a \leq b \leq c$, then $(a/b) + (b/c)$ exists and $(a/b) + (b/c) = a/c$.*

(4) *If $a \leq b \leq c$, then $(c\backslash b) + (b\backslash a)$ exists and $(c\backslash b) + (b\backslash a) = c\backslash a$.*

Proof. (1) Since $a + b$ and $c + (c/a) = a$ exist and $+$ is associative, then $d = (c/a) + b$ and $c + d$ exist and we have

$$c/(a + b) = c/((c + (c/a)) + b) = c/(c + d) = d = (c/a) + b.$$

(2) can be proved as (1).

(3) Since $b + (b/c) = c$ exists, by (1) we obtain that $(a/b) + (b/c)$ exists and $(a/b) + (b/c) = a/(b + (b/c)) = a/c$.

(4) In a similar way as (3), we obtain (4) applying (2). ∎

Lemma 3.2. *Let h, k, r, s and a, b, c, d be in E. Then:*

(1) *If $h+k$ and $r+s$ exist and $h+k \leq r+s$, then $k \leq h/(r+s)$ and $h \leq (r+s)\backslash k$.*

(2) *If $b \leq a$ and $c \leq a\backslash b$, then $c + b$ exists, $c + b \leq a$ and $b \leq c/a$.*

(3) *If $b \leq a$ and $c \leq b/a$, then $b + c$ exists, $b + c \leq a$ and $b \leq a\backslash c$.*

Proof. (1) We first apply Proposition 1.1-(8) with $a = k$, $b = h/(r+s)$ and $c = h$. Indeed, by assumption, $r+s = h+(h/(r+s)) = c+b$ and $h+k = c+a$ exist, and $c+a \leq c+b$. Therefore, we have $k = a \leq b = h/(r+s)$.

Now, we apply Proposition 1.1-(7) with $a = h$, $c = k$ and $b = (r+s)\backslash k$. By assumption, we have that $r+s = ((r+s)\backslash k)+k = b+c$ and $h+k = a+c$ exist, and $a+c \leq b+c$. Therefore, $h = a \leq b = (r+s)\backslash k$.

(2) Since $(a\backslash b)+b = a$ exists and $c \leq a\backslash b$ by assumption, then by Proposition 1.1-(7) we have that $c+b$ exists and $c+b \leq a$. By (1), we get $b \leq c/a$.

(3) Since $b+(b/a) = a$ exists and $c \leq b/a$ by assumption, by Proposition 1.1-(8) we have that $b+c$ exists and $b+c \leq a$. By (1), we get $b \leq a\backslash c$. ■

Lemma 3.3. *Let $a_0, a_1, ..., a_n$ be in E such that $a_0 \leq a_1 \leq \ldots \leq a_n$ and, for every $i \in \{1, ..., n\}$, let $b_i = a_{i-1}/a_i$. Then $(b_1, ..., b_n)$ is orthogonal and $b_1 + \cdots + b_n = a_0/a_n$.*

Proof. By Lemma 3.1-(3), we get that $b_1 + b_2 = (a_0/a_1) + (a_1/a_2)$ exists and it is equal to a_0/a_2. By induction, suppose that $b_1 + \cdots b_{n-1}$ exists and it is equal to a_0/a_{n-1}. Then, by Lemma 3.1-(3), we obtain that $b_1 + \cdots b_{n-1} + b_n = (a_0/a_{n-1}) + (a_{n-1}/a_n)$ exists and it is equal to a_0/a_n. ■

Proposition 3.4. *The following conditions are equivalent*

(1) *E is σ-complete.*

(2) *For every increasing sequence $(a_n)_{n\in\mathbb{N}}$ in E, $\sup_n a_n$ exists.*

(3) *For every decreasing sequence $(a_n)_{n\in\mathbb{N}}$ in E, $\inf_n a_n$ exists.*

Proof. The equivalence of (2) and (3) is trivial by Proposition 1.1-(6).

(1) $\Rightarrow$ (2) Let $(a_n)_{n\in\mathbb{N}}$ be an increasing sequence in E and set $b_n = a_{n-1}/a_n$ (where $a_0 = 0$). By Lemma 3.3, (b_n) is an orthogonal sequence and $b_1 + \cdots + b_n = a_n$. Hence, by (1), $a = \sup_n \sum_{i\leq n} b_i = \sup_n a_n$ exists.

(2) $\Rightarrow$ (1) Let $(a_n)_{n\in\mathbb{N}}$ be an orthogonal sequence in E. Set $b_n = a_1 + \ldots + a_n$. Since (b_n) is an increasing sequence, we have that $\sum_{n\in\mathbb{N}} a_n = \sup_n b_n$ exists. ■

Proposition 3.5. *Let $\mu : E \to G$ be a measure. Then the following conditions are equivalent:*

(1) *μ is σ-additive.*

(2) *For every sequence $(a_n)_{n\in\mathbb{N}}$ in E, $a_n \uparrow a \Rightarrow \mu(a) = \lim_n \mu(a_n)$.*

(3) *For every sequence $(a_n)_{n\in\mathbb{N}}$ in E, $a_n \downarrow a \Rightarrow \mu(a) = \lim_n \mu(a_n)$.*

(4) *For every sequence $(a_n)_{n\in\mathbb{N}}$ in E, $a_n \downarrow 0 \Rightarrow \lim_n \mu(a_n) = 0$.*

Proof. (1) $\Rightarrow$ (2) Let (a_n) be such that $a_n \uparrow a$. For each $n \in \mathbb{N}$, set $b_n = a_{n-1}/a_n$, where $a_0 = 0$. By Lemma 3.3, (b_n) is orthogonal and, for each $n \in \mathbb{N}$, $b_1+\cdots+b_n = 0/a_n = a_n$. Therefore, $a = \sup_n a_n = \sum_{n\in\mathbb{N}} b_n$. Since μ is σ-additive, we obtain $\mu(a) = \sum_{n=1}^{\infty} \mu(b_n) = \lim_n \sum_{k=1}^{n} \mu(b_k) = \lim_n \mu(b_1 + \cdots + b_n) = \lim_\mu(a_n)$.

(2) $\Rightarrow$ (3) Let $(a_n)_{n\in\mathbb{N}}$ be such that $a_n \downarrow a$. By Proposition 1.1-(6), we get that $a_n^\perp \uparrow a^\perp$. By (2), we obtain $\mu(a^\perp) = \lim_n \mu(a_n^\perp)$, from which $\mu(a) = \lim_n \mu(a_n)$.

(3) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (1) Let (a_n) be an orthogonal sequence in E such that $a = \sum_{n\in\mathbb{N}} a_n$ exists. Since $a_1 + \cdots + a_n \leq a$, then, for each $n \in \mathbb{N}$, $b_n = a\backslash(a_1 + \cdots + a_n)$ exists. By Proposition (1.1)-9, (b_n) is a decreasing sequence. Moreover we have that $\inf_n b_n = 0$. Indeed, if $c \leq b_n$ for each n, by Lemma 3.2-(2) we obtain that, for each n, $a_1+\ldots+a_n \leq c/a$, from which we get $a = \sup_n(a_1+\ldots+a_n) \leq c/a$. By Lemma 3.2-(3), we have $c \leq a\backslash a = 0$. Now, since $b_n \downarrow 0$, by (4) we have $\lim_n \mu(b_n) = 0$. Since $\lim_n \mu(b_n) = \lim_n(\mu(a) - \mu(a_1 + \cdots + a_n)) = \mu(a) - \sum_{n=1}^{\infty} \mu(a_n)$, we obtain $\mu(a) = \sum_{n=1}^{\infty} \mu(a_n)$. ∎

Proposition 3.6. *Let $\mu : E \to G$ be a measure. Then E is μ-chained if and only if μ is strongly continuous.*

Proof. Suppose that E is μ-chained. Let W be a neighbourhood of 0 in G and $a \in E$. Choose $a_0, a_1, \cdots, a_r$ in E such that $0 = a_0 \leq a_1 \leq \cdots \leq a_r = a$ and $\mu(h) - \mu(k) \in W$ whenever $h, k \in [a_{i-1}, a_i]$ for some $i \in \{1, ..., r\}$. Set $b_i = a_{i-1}/a_i$ for each $i \in \{1, ..., r\}$. By Lemma 3.3, $(b_1, ..., b_r)$ is orthogonal and $b_1 + \cdots + b_r = a$. Let $i \leq r$ and choose $b \leq b_i$. Since $a_{i-1} + b_i$ exists, by Proposition 1.1-(8) $a_{i-1} + b$ exists and $a_{i-1} \leq a_{i-1} + b \leq a_{i-1} + b_i = a_i$. Therefore, we obtain $\mu(b) = \mu(a_{i-1} + b) - \mu(a_{i-1}) \in W$.

Now, suppose that μ is strongly continuous. Let W and V be neighbourhoods of 0 in G with $V - V \subseteq W$ and $a \in E$. Choose an orthogonal family $(b_1, \ldots, b_r)$ in E such that $b_1 + \cdots + b_r = a$ and $\mu(b) \in V$ whenever $b \leq b_i$ for some $i \leq r$. Set $a_0 = 0$ and $a_i = b_1 + \ldots + b_i$ for every $i \leq r$. Then, we have $0 = a_0 \leq a_1 \leq \cdots \leq a_r = a$. Let $i \leq r$ and choose $h, k \in [a_{i-1}, a_i]$. By Proposition 1.1-(10), we have $a_{i-1}/h \leq a_{i-1}/a_i = b_i$ and $a_{i-1}/k \leq a_{i-1}/a_i = b_i$. Therefore, $\mu(h) - \mu(k) = \mu(a_{i-1}/h) - \mu(a_{i-1}/k) \in V - V \subseteq W$. ∎

The following result can be derived by Theorems 4.2 and 4.4 of [8].

Theorem 3.7. *Suppose that E has the interpolation property and let $\mu : E \to R^n$ be a strongly continuous measure. Then, if μ has nonnegative components, $\mu(E)$ is star-shaped with respect to 0. Moreover, if E is a lattice and μ is modular, then μ is convex-ranged.*

Proof. By Theorem 4.2 of [8], $\mu(E')$ is star-shaped with respect to 0 if E' is a μ-chained poset with smallest element 0 and greatest element 1, with a binary relation $\perp$ and a partially defined binary operation $\oplus$ satisfying the following properties:

(a) $a \perp b$ if and only if $a \oplus b$ exists.

(b) $a \oplus 0 = 0 \oplus a = a$.

(c) If $a \leq b$, then there exists c in E' with $a \perp c$ and $a \oplus c = b$.

(d) If $a \perp b$, $c \leq a$ and $d \leq b$, then $c \perp d$ and $c \oplus d \leq a \oplus b$.

Moreover, by Theorem 4.4 of [8], $\mu(E')$ is convex if E' satisfies the additional condition:

(e) If $a \leq c \leq a \oplus b$, then there exists $d \leq b$ such that $a \oplus d = c$.

Now, observe that a pseudo-effect algebra satisfies all the above conditions if we define $a \oplus b = a + b$ and $a \perp b$ if and only if $a + b$ exists. Indeed (a), (b) and (c) are trivial, (d) follows from Proposition 1.1-(7) and (8), and (e) follows from Proposition 1.1-(8). Moreover it is easy to see that, if $d \in E$, the interval $[0, d]$ is a pseudo-effect-algebra if we define, for every $a, b \in E$, $a + b = c$ if and only if $a + b = c$ in E and $c \leq d$. The assumptions on E imply that $[0, d]$ has the interpolation property and it is μ-chained by Proposition 3.6. Therefore, we can apply to $[0, d]$ Theorems 4.2 and 4.4 of [8]. ∎

Now, using the results of Section 3 instead of the corresponding results of [4], it is possible to prove the following Uniqueness theorem for measures on pseudo-effect algebras, proved in [21] for measures on particular effect algebras and in [4] for measures on arbitrary effect algebras.

Theorem 3.8. *Let μ and ν be $[0, +\infty[$-valued measures on E which satisfy the following conditions:*

(a) *μ is convex-ranged.*

(b) *There exist $\alpha \in]0, \mu(1)[$ and $\beta \in]0, \nu(1)[$ such that, for every $a \in E$, $\mu(a) = \alpha$ implies $\nu(a) = \beta$.*

Moreover suppose that one of the following conditions is satisfied:

(1) *E is σ-complete and ν is σ-additive.*

(2) *E has the interpolation property and, for every $a \in E$, $\nu(a) = 0$ implies $\mu(a) = 0$.*

(3) *E has the interpolation property and, for every $a \in E$, $\mu(a) = \alpha$ if and only if $\nu(a) = \beta$.*

Then $\mu = \lambda\nu$, where $\lambda = \frac{\mu(1)}{\nu(1)}$.

Proof. The proof is similar to the proof of Theorem 3.1 of [4]. ∎

Now, we apply the results of Section 2 to prove that, if μ is a modular measure on L, then the Uniqueness theorem holds without completeness assumptions on L.

First, we need the following result.

Proposition 3.9. *Let $\mu : L \to [0, +\infty[$ be a modular measure. Then the following conditions are equivalent:*

(1) *μ is strongly continuous.*

(2) *For every $\varepsilon > 0$, there exists an orthogonal family $(a_1, ..., a_r)$ in L such that $a_1 + \cdots + a_r = 1$ and $\mu(a_i) < \varepsilon$ for every $i \leq n$.*

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1) Let $a \in L$ and $\varepsilon > 0$. Choose an orthogonal family $(a_1, ..., a_r)$ in L such that $a_1 + \cdots + a_r = 1$ and $\mu(a_i) < \varepsilon$ for every $i \leq r$. Set $b_0 = 0$ and $b_i = a_1 + \cdots + a_i$ for every $i \leq r$. Then we have $0 = b_0 \leq b_1 \leq ... \leq b_n = 1$. Since $b_i = b_{i-1} + a_i$, we have $a_i = b_{i-1}/b_i$, from which we obtain $\mu(b_i) - \mu(b_{i-1}) = \mu(a_i) < \varepsilon$ for every $i \leq n$. Setting $c_i = b_i \wedge a$, we can see as in (2.3) of [1], that $0 = c_0 \leq c_1 \leq \cdots \leq c_r = a$ and $\mu(c_i) - \mu(c_{i-1}) < \varepsilon$ for each $i \leq r$. Set $d_0 = 0$ and $d_i = c_{i-1}/c_i$ for $i \leq r$. Then $\mu(d_i) < \varepsilon$ for each $i \leq r$ and, by Lemma 3.3, $d_1 + \ldots + d_r = a$. ∎

Theorem 3.10. *Let $\mu, \nu : L \to [0, +\infty[$ be modular measures with the following properties:*

(1) *μ is strongly continuous.*
(2) *There exist $\alpha \in]0, \mu(1)[$ and $\beta \in]0, \nu(1)[$ such that, if (a_n) is a sequence in L with $\lim_n \mu(a_n) = \alpha$, then $\lim_n \nu(a_n) = \beta$.*

Then $\mu = \lambda\nu$, where $\lambda = \frac{\mu(1)}{\nu(1)}$.

Proof. Denote by $\mathcal{U}$ the supremum of the D-uniformities generated by μ and ν (see Theorem 2.9). It is clear that $\mathcal{U}$ is a D-uniformity. Moreover μ and ν are obviously exhaustive, since they are monotone real-valued. Then $\mathcal{U}$ is exhaustive, too.

Set $\hat{L} = L/N(\mathcal{U})$, $\hat{\mu}(a) = \mu(a)$ and $\hat{\nu}(a) = \nu(a)$ for $a \in \hat{a} \in \hat{L}$, and denote by $\tilde{\mu}$ and $\tilde{\nu}$ the uniformly continuous extensions, respectively, of $\hat{\mu}$ and $\hat{\nu}$ to the uniform completion $(\tilde{L}, \tilde{\mathcal{U}})$ of $\hat{L}$ (see Theorems 2.12 and 2.13).

By Theorem 2.13, $\tilde{L}$ is a complete D-lattice and $\tilde{\mu}$, $\tilde{\nu}$ are o.c. modular measures and therefore σ-additive by Proposition 3.5.

By Proposition 3.9, it is clear that $\tilde{\mu}$ is strongly continuous, too, since $\widehat{1_L} = 1_{\tilde{L}}$. Therefore, by Theorem 3.7, $\tilde{\mu}$ is convex-ranged.

Now let $a \in \tilde{L}$ such that $\tilde{\mu}(a) = \alpha$. Choose (a_n) in $\hat{L}$ which converges to a in $(\tilde{L}, \tilde{\mathcal{U}})$. By the continuity of $\tilde{\mu}$ and $\tilde{\nu}$, we get $\lim_n \hat{\mu}(a_n) = \tilde{\mu}(a) = \alpha$ and $\lim_n \hat{\nu}(a_n) = \tilde{\nu}(a)$. By (2), we get $\tilde{\nu}(a) = \beta$. Then $\tilde{\mu}$ and $\tilde{\nu}$ verify the assumptions of Theorem 3.8.

By Theorem 3.8, we get $\tilde{\mu} = \lambda\tilde{\nu}$, from which $\mu = \lambda\nu$. ∎

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ON INJECTIVITY OF PROJECTION AND SEPARATED PROJECTION ALGEBRAS

M.M. Ebrahimi

M. Mahmoudi

Department of Mathematics
Shahid Beheshti University, G.C.
Tehran
Iran
e-mail: m-ebrahimi@sbu.ac.ir
m-mahmoudi@sbu.ac.ir

Abstract. Projection spaces (algebras) were first introduced by Ehrig et. al. as an algebraic version of ultrametric spaces, and then studied by Giuli, Ebrahimi, Mahmoudi. Computer scientists use projection algebras for algebraic specification of process algebras. A kind of *injectivity* of separated projection algebras have been studied by Giuli. In this paper, we extend this notion to all projection algebras, and introduce some other kind of injectivity, so called m and p-injectivity, and show, among other things, that injectivity, s-injectivity, and m-injectivity coincide, and so we get some more Baer criteria for injectivity.

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1. Introduction and preliminaries

The notion of a projection space (algebra) was first introduced by Ehrig et. al. as an algebraic version of an ultrametric space ([10]). Computer Scientists use this notion for a formal description of parallel concurrent systems. One of the main problem in this scope is the specification of infinite objects (processes) which can not be denoted by finite terms. So, they use projection algebras as a convenient means for algebraic specification of process algebras (see [10], [11] and their references). Projection algebras have also, naturally, been studied by mathematicians, for example in [7], [8], [9], [12], [14].

In this paper we consider three types of closure operators on projection algebras to get the classes of dense and closed monomorphisms naturally arising from them. Then we study injectivity with respect to these classes of monomorphisms in the categories of projection and separated projection algebras arising from these closure operators (see also, [8], [12], [14]).

We now officially recall the category **PRO** of projection algebras.

A *projection space* (considering it as a kind of universal algebra we prefer to call it a *projection algebra*) is in fact a (right) M-set (or M-act) for the monoid $M = \mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$ with the binary operation $m.n = min\{m, n\}$, where $\mathbb{N}$ is the set of natural numbers and $n < \infty, \forall n \in \mathbb{N}$. In other words, it is a set A together with a family $(\lambda_n)_{n \in \mathbb{N}^\infty}$ of unary operations $\lambda_n : A \to A$ (called projections) such that

$$\lambda_m \circ \lambda_n = \lambda_{n.m} \text{ and } \lambda_\infty = id_A$$

for every $m, n \in \mathbb{N}^\infty$. We denote $\lambda_n(a)$ by na, for $n \in \mathbb{N}^\infty,\ a \in A$.

A *projection morphism* between projection algebras is also called an *equivariant map.* In fact, a projection morphism between projection algebras $(A, (\lambda_n)_{n \in \mathbb{N}^\infty})$ and $(B, (\eta_n)_{n \in \mathbb{N}^\infty})$ is a function $f : A \to B$ satisfying $f \circ \lambda_n = \eta_n \circ f$, for every $n \in \mathbb{N}^\infty$, that is, $f(na) = nf(a)$, for every $n \in \mathbb{N}^\infty$ and $a \in A$.

Thus, the category **PRO** of projection algebras is a special kind of the category **MSet** of M-sets (or **MAct** of acts over M, as in [13]), taking the monoid $M = \mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$ with $m.n = min\{m, n\}$.

The category **PRO** has free objects. In fact, for each set X, $F(X) = \mathbb{N}^\infty \times X$ with actions given by $s(n, x) = (sn, x)$, for $s, n \in \mathbb{N}^\infty$ and $x \in X$, is the free projection algebra generated by X. Also, for each set X, the cofree projection algebra generated by X is the set $H(X) = X^{\mathbb{N}^\infty}$, of all functions from $\mathbb{N}^\infty$ to X, with actions given by $(sf)(n) = f(ns)$, for $f \in X^{\mathbb{N}^\infty}$ and $s, n \in \mathbb{N}^\infty$. In other words, the underlying set functor U : **PRO** $\to$ **Set** has a left adjoint F and a right adjoint H (see [6]).

The following proposition is in fact a consequence of the existence of cofree and free projection algebras.

Lemma 1.1. *In the category* **PRO** *we have:*

(1) *Epimorphisms are exactly surjective projection morphisms.*

(2) *Monomorphism are exactly one-one projection morphisms.*

(3) *Isomorphisms are exactly surjective and injective projection morphisms.*

Proof. (1) Suppose $f : A \to B$ is an epimorphism in **PRO**. Consider the Rees factor projection algebra B/Imf, that is $[b] = Imf$ for $b \in Imf$ and $[b] = \{b\}$ for $b \in B - Imf$. Define $g : B \to B/Imf$ by $g(b) = Imf$, for all $b \in B$. Also consider the natural epimorphism $\gamma : B \to B/Imf$ by $\gamma(b) = [b]$, for $b \in B$. Then we have $\gamma f = gf$, and hence $\gamma = g$, since f is an epimorphism. Thus for all $b \in B$, $[b] = Imf$, that is, $b \in Imf$. Therefore $Imf = B$.

(2) If $f : A \to B$ is a non injective monomorphism in **PRO**, then there exist a, b in A such that $f(a) = f(b)$ and $a \neq b$. Define $g, h : \mathbb{N}^\infty \to A$ by $g(n) = na,\ h(n) = nb$ for $n \in \mathbb{N}^\infty$. Then we have $fg = fh$ while $g \neq h$ because $g(\infty) = a \neq b = h(\infty)$. This contradicts the fact that f is a monomorphism.

(3) is a corollary of (1) and (2). ∎

Thus, we can consider monomorphisms $A \to B$ in **PRO** as inclusions and denote it by $A \leq B$. A projection algebra B containing (a copy of) a projection algebra A as a subalgebra is called an *extension* of A. The algebra A is said to be a *retract* of B if there exists a homomorphism $f : B \to A$ such that $f\mid_A = id_A$, in which case f is said to be a *retraction*. A is called *absolute retract* if it is a retract of each of its extensions.

Note that each non-empty projection algebra A has a *zero* (*fixed*) element, that is an element a_0 with $sa_0 = a_0$, for each $s \in \mathbb{N}$. In fact, for each $a \in A$, $1a$ is a zero element of A.

Finally, we mention the following from general results of [2] or [5], which will be needed in this paper.

Proposition 1.2. *The category* **PRO** *has enough injectives, and consequently injectives and absolute retracts coincide in* **PRO**.

Now, we use projection algebras to introduce projection specifications which are useful for computer scientists. But we will not use this version of projection algebras in this paper.

Definition 1.3. A *signature* is a pair $SIG = (S, OP)$, where S is a set of sorts (set symbols) and OP is a set of constants and operation symbols.

An algebra of a signature $SIG = (S, OP)$ or a *SIG-algebra* is a pair $A = (S_A, OP_A)$ where S_A is a family $(A_s)_{s\in S}$ of sets called base sets or domain of A, and OP_A is a family $(N_A)_{N\in OP}$ of elements of A_s for all constant symbols $N : \to s$ and $s \in S$, called constants of A, or functions $N_A : A_{s1} \times A_{s2} \times A_{sn} \to A_s$ for all operation symbols $N \in OP_{s1...sn,s}$ and $s1...sn \in S^* \setminus \{\lambda\}$, $s \in S$, called operations of A.

A *specification* $SPEC = (S, OP, E)$ consists of a signature $SIG = (S, OP)$ and a set E of equations with respect to SIG.

An algebra of a specification $SPEC = (S, OP, E)$ or a *SPEC-algebra* is an algebra A of the signature SIG which satisfies all the equations in E.

Example 1.4. This is the specification $nat1$ of the natural numbers starting with 1.

$$
\begin{array}{rcl}
nat1 & = & \\
sorts & : & nat1 \\
opns & : & 1 : \ \to nat1 \\
 & & succ : nat1 \to nat1 \\
 & & min : nat1 \ \ nat1 \to nat1 \\
eqns & : & \text{for all} m, n \text{ in } nat1 : \\
 & & min(n, 1) = 1 \\
 & & min(1, n) = 1 \\
 & & min(succ(n), succ(m)) = succ(min(n, m))
\end{array}
$$

Definition 1.5. A *projection specification* $SPEC = (S, OP, E)$ is an algebraic specification with

(i) $nat1 \subseteq SPEC$,

(ii) For all $s \in S$ there is an operation symbol $ps : nat1\ s \to s \in OP$, and $pnat1(n, k) = k$, for all $n, k \in nat1$,

(iii) $nat1 \notin rang(OP - (\{pnat1 : nat1\ nat1 \to nat1\} \cup OP((nat1))))$ That is there is no operation symbols $N : s1...sn \to nat1$ in OP except the operation $pnat1$ and the operation symbols from $nat1$,

(iv) if $t1 = t2 \in E - E(nat1)$ then $sort(t1) \neq nat1$, that is there is no additional equations between $nat1$-terms.

A *projection-Spec-algebra* is an algebra of the specification $SPEC$ with the additional properties

(i) For all $s \in S$, (A_s, ps_A) is a projection space,

(ii) the equations N_A are projection compatible, that is $ps_A(k, N_A(a1, ..., an)) = ps_A(k, N_A(ps1_A(k, a1), ..., psn_A(k, an)))$, for all $N : s1...sn \to s$, for all $k \geq 1$, for all $a1 \in A_{s1}, ..., an \in A_{sn}$,

(iii) $A_{nat1} \simeq N_1$.

2. Some closure operators on PRO

In this section we give three kinds of closure operators on the category **PRO**, two of which have already been introduced in [12], [3]. We use these closure operators to define separated projection algebras in the next section. Although some of the results are consequences of the general results of [4], we give direct proofs.

Definition 2.6. For a projection algebra B and each subalgebra A of B define:

(1) The *m-closure* of A in B, for $m \in \mathbb{N}$, by $C_m(A \leq B) := \{b \in B : kb \in A, \forall k \leq m\}$.

(2) the *s-closure* of A in B by $C_s(A \leq B) := \{b \in B : nb \in A,\ \forall n \in \mathbb{N}\}$.

(3) the *p-closure* of A in B by $C_p(A \leq B) := \{b \in B : \exists a \in A, na = nb, \forall n \in \mathbb{N}\}$.

If there is no confusion, we write $C(A)$ instead of $C(A \leq B)$, for $C \in \{C_m, C_s, C_p\}$.

Note 2.7. It is easily seen that for $m \in \mathbb{N}$ and $A \leq B$ in **PRO**,

$$C_m(A) = \{b \in B : mb \in A\} = \{b \in B : mb \in mA\}$$

Lemma 2.8. *Each $C \in \{C_m, C_s, C_p\}$ is an idempotent, hereditary, weakly hereditary, additive, and grounded closure operator on the category* **PRO***, in the sense of* [4]*. That is, for each projection algebra B, we have the following:*

(cl 1) $A \leq C(A)$, *for* $A \leq B$,

(cl 2) $A \leq A' \leq B \Rightarrow C(A \leq B) \leq C(A' \leq B)$, (cl 3) $f(C(A \leq B)) \leq C(f(A) \leq B')$, *for every projection map* $f : B \to B'$,

(idem) $C(C(A)) = C(A)$, *for* $A \leq B$,

(hered) $C(A' \leq A) = C(A' \leq B) \cap A$, *for* $A' \leq A \leq B$,

(w-hered) $C(A \leq C(A)) = C(A)$, *for* $A \leq B$,

(add) $C(A \cup A') = C(A) \cup C(A')$, *for* $A, A' \leq B$,

(ground) $C(\emptyset) = \emptyset$.

Proof. The only part which may be a little tricky is additivity of C_s. In fact, this follows from the fact that for $b \in B$, if $nb \in A \cup A'$, for all $n \in \mathbb{N}$, then two cases may occur: $nb \in A$, for all $n \in \mathbb{N}$, or $nb \in A'$, for all $n \in \mathbb{N}$. This is because, if for some $m \in \mathbb{N}$, $mb \notin A$ (and so $mb \in A'$), then for all $n \in \mathbb{N}$, $nb \in A'$. This is because, if $n \leq m$ then $nb = (nm)b = n(mb) \in A'$, and if $n > m$ then again $nb \in A'$ (because if $nb \in A$ then $mb \in A$ which is contradiction). ∎

For closure operators C and D on **PRO**, define $C \leq D$ if for each $A \leq B$ in **PRO**, $C(A) \leq D(A)$. Then, we have the following strict inequalities.

Lemma 2.9. $C_p < C_s < ... < C_m < ...C_2 < C_1$

Proof. It is enough to show that all the above inclusions are proper. To see this, take $A =\downarrow k$ and $B = \mathbb{N}^\infty$, then $C_k(A) = \mathbb{N}$ but $C_{k+1}(A) =\downarrow k$. Also, taking $A = \mathbb{N}$ and $B = \mathbb{N}^\infty$, we have $C_s(A) = \mathbb{N}^\infty$ but $C_p(A) = \mathbb{N}$. ∎

Definition 2.10. For a closure operator $C \in \{C_m, C_s, C_p\}$ on **PRO**, a subalgebra A of a projection algebra B is called

(1) *C-closed* (respectively, *m-closed, s-closed, p-closed*) if $C(A) = A$,

(2) *C-dense* (respectively, *m-dense, s-dense, p-dense*) if C(A)=B.

A projection map $f : A \to B$ is called *C-dense* (*C-closed*) if $f(A)$ is a C-dense (C-closed) subalgebra of B.

As a corollary of Lemma 2.9, we have

Lemma 2.11. *For a projection algebra B and a subalgebra A of B,*

(1) *If A is m-closed then it is k-closed, for all $k \leq m$, and A is s-closed. Also, the latter implies A is p-closed.*

(2) *If A is p-dense then it is s-dense. Also, the latter implies A is m-dense. And, if A is m-dense then it is k-dense, for all $k \geq m$.*

Notice that, the above lemma holds for morphisms, too.

Theorem 2.12. *For each $C \in \{C_m, C_s, C_p\}$, every projection map has (C-dense morphism, C-closed monomorphism) factorization.*

Proof. Let $A \to B$ be a projection map. Take $D = C(f(A) \leq B)$, $g = f : A \to D$ and $h = \imath : D \to B$. Then $f = hg$ is a (C-dense, C-closed) factorization of f. ∎

3. C-separated projection algebras

Closure operators have been used to generalize the well-known fact that a topological space X is Hausdorff if and only if the diagonal Δ_X is C-closed in $X \times X$. Here, we study the full subcategories

$$\Delta(C) = \{A : \Delta_A \text{ is } C\text{−closed in } A \times A\}$$

for $C \in \{C_m, C_s, C_p\}$.

Definition 3.13. For $C \in \{C_m, C_s, C_p\}$, the subcategories $\Delta(C)$ of **PRO** are denoted by $\mathbf{PRO_m}$, $\mathbf{PRO_s}$, $\mathbf{PRO_p}$, respectively. The elements of these categories are called m (respectively, s, or p)-*separated* projection algebras.

Example 3.14. Clearly, every projection algebra A with identity actions $\lambda_n = id_A$ is a m-separated; $\mathbf{N}^\infty$ as a projection algebra is s-separated; $A = \{0, 1\}$ in which 0, 1 are zero elements is m-separated as well as s-separated.

Using the definitions of closure operators C_m and C_s, we easily get the following.

Lemma 3.15. *A projection algebra A is*

(1) *m-separated if and only if $mx = my$ implies that $x = y$, for $x, y \in A$,*

(2) *m-separated if and only if ma=a, for all $a \in A$; if and only if $mA = A$,*

(3) *s-separated if and only if $nx = ny$, for all $n \in \mathbb{N}$, implies $x = y$.*

Proof. We just prove (2), the rest are straightforward. Let A be m-separated and $ma = a'$, for some $a \in A$. Then $mma = ma'$ and hence $ma = ma'$. So $a = a'$, by (1), since A is m-separated. Conversely, let $ma = a$, for all $a \in A$. If $ma = ma'$ then $a = ma = ma' = a'$. Also, it is clear that the second condition is equivalent to $mA = A$. ∎

Now, we easily get the following strict inclusions.

Lemma 3.16. *We have the following strict inclusions*

$$\mathbf{PRO_1} \subset \mathbf{PRO_2} \subset \cdots \subset \mathbf{PRO}_s$$

Proof. The inclusions are clearly true. To show that they are strict, take $A = \{a, b\}$ with actions given by $na = a$, for $n \in \mathbb{N}^\infty$, and $kb = a$, for $k \leq m$, $kb = b$, for $k \geq m+1$. Then $A \in \mathbf{PRO_{m+1}}$ but $A \notin \mathbf{PRO_m}$. Also, $\mathbb{N}^\infty$ is s-separated but it is not m-separated, for $m \in \mathbb{N}$. ∎

We also have the following equality.

Lemma 3.17. $\mathbf{PRO_s} = \mathbf{PRO_p}$.

Proof. By Lemma 2.11, for a projection algebra A, if Δ is s-closed in $A \times A$ then it is p-closed in $A \times A$. So, $\mathbf{PRO_s} \subseteq \mathbf{PRO_p}$. Conversely, let Δ be p-closed in $A \times A$ and $nx = ny$, for all $n \in \mathbb{N}$, and some $x, y \in A$. Then taking $z = x$, we have $n(z, z) = (nx, ny)$, for all $n \in \mathbb{N}$. Hence, $(x, y) \in C_p(\Delta) = \Delta$, that is $x = y$. ∎

The following results will be used in the study of injectivity in the next section.

Lemma 3.18.

(1) *In* $\mathbf{PRO_m}$*, all morphisms are m-closed and hence s-closed and p-closed. But, here the only m-dense (s-dense, or p-dense) monomorphisms are isomorphisms.*

(2) *In* $\mathbf{PRO_s}$*, all morphisms are p-closed, but the only p-dense monomorphisms are isomorphisms.*

Proof. (1) Let $A \leq B \in \mathbf{PRO_m}$. For $b \in C_m(A)$, $mb \in A$ and so, $mb = mmb \in mA$, but by Lemma 3.15(2), $mA = A$. Thus, $b \in A$ and $C_m(A) = A$. Since $C_m \geq C_s \geq C_p$, $A = C_s(A) = C_p(A)$. Further, if A is m-dense in B then $C_m(A) = B$ and so $A = B$.

(2) Let $A \leq B \in \mathbf{PRO_s}$. For $b \in C_p(A)$, there exists $a \in A$ such that $nb = na$, for $n \in \mathbb{N}$. Then $b = a \in A$, since A is s-separated. So $C_p(A) = A$. If A is s-dense in B then $B = C_p(A) = A$. ∎

Lemma 3.19.

(1) *Each projection algebra A has a proper p-dense, and hence s-dense, extension.*

(2) *Each s-separated projection algebra A has a proper m-dense extension in* $\mathbf{PRO_s}$.

Proof. (1) Take the extension $B = A \cup \{*\}$, $* \notin A$, of A with actions $n* = a_0$, for a zero element a_0 in A and $n \in \mathbb{N}$, also $\infty * = *$. Then B is a proper p-dense (s-dense) extension of A.

(2) Take the set B as defined in (a) with actions $k* = a_0$, for $k \leq m$, and $n* = *$, for $n \geq m + 1$. Then B is a proper m-dense extension of A in $\mathbf{PRO_s}$. ∎

Theorem 3.20. *The categories* $\mathbf{PRO_m}$ *and* $\mathbf{PRO_s} = \mathbf{PRO_p}$ *are reflective subcategories of* $\mathbf{PRO}$.

Proof. Define the congruence relations $\sim_m$, $\sim_s$ on a projection algebra A by

$$a \sim_m b \Leftrightarrow ma = mb; \quad a \sim_s b \Leftrightarrow na = nb, \ \forall n \in \mathbb{N}$$

Then the natural quotient maps $\gamma_m : A \to A/\sim_m$, and $\gamma_s : A \to A/\sim_s$ which take $a \in A$ to $[a]$ are reflection arrows from $\mathbf{PRO}$ to $\mathbf{PRO_m}$ and $\mathbf{PRO_s}$, respectively.

More precisely, if $f : A \to B$ is a projection map, where B is an m-separated projection algebra, then (by Decomposition Theorem of maps) there exists a unique projection map $\overline{f} : A/\sim_m \to B$, $\overline{f}([a]) = f(a)$, with the property that $\overline{f}\gamma_m = f$. Similarly, γ_s is a reflection arrow. ∎

The following may also be used to study projectivity, which we will not be studying in this paper.

Theorem 3.21.

(1) *In* $\mathbf{PRO_m}$, $\mathbf{PRO_s}$, *the monomorphisms are exactly one-one projection maps.*

(2) *In* $\mathbf{PRO_m}$, *the epimorphisms, onto projection maps, and m-dense (s-dense, p-dense) projection maps are the same.*

(3) *In* $\mathbf{PRO_s}$, *the epimorphisms are exactly s-dense morphisms and the onto projection maps are exactly p-dense morphisms.*

Proof. (1) Follows from Theorem 3.20 and the fact that in **PRO** the monomorphisms are exactly one-one projection maps. More precisely, if $f : A \to B$ is a monomorphism in $\mathbf{PRO_m}$ then it is a monomorphism in **PRO** because if $g, h : C \to A$ are projection maps with $fg = fh$ then, by Lemma 3.20, $C/\sim_m$ is m-separated and there are projection maps $\overline{g}, \overline{h} : C/\sim \to A$ with $\overline{g}\gamma_m = g$ and $\overline{h}\gamma_m = h$. Now, $f\overline{g}\gamma_m = f\overline{h}\gamma_m$, and hence $\overline{g}\gamma_m = \overline{h}\gamma_m$, since f is a monomorphism in $\mathbf{PRO_m}$. Thus, $g = h$, and f is a monomorphism in **PRO** and hence one-one. A similar argument is true for $\mathbf{PRO_s}$.

(2) To show that epimorphisms in $\mathbf{PRO}_m$ are onto, apply the same proof as Lemma 1.1 for epimorphisms in **PRO**. Further, m-dense maps in $\mathbf{PRO_m}$ are onto (see Lemma 3.18). So s-dense and p-dense maps are also onto here. Also, it is clear that onto maps are m (respectively, s and p)-dense.

(3) Let $f : A \to B$ be an s-dense map in $\mathbf{PRO_s}$. If $g, h : B \to C$ are morphisms in $\mathbf{PRO_s}$ such that $hf = gf$, then for $b \in B = C_s(f(A))$ we have $nh(b) = h(nb) = g(nb) = ng(b)$, for every $n \in \mathbb{N}$. Now the fact that C is s-separated, implies $h(b) = g(b)$. So, $h = g$ and f is epic. Conversely, let $f : A \to B$ be epic. Let $f = h \circ e$ be an (s-dense, s-closed) factorization of f. By Corollary 4.24 in the next section, there is a retraction h' such that $h'h = id$. Hence, $(hh')h = h(h'h) = h$. But h is epic, since f is so. Thus $hh' = id$ and so h is an isomorphism. Then f, being a composition of an s-dense map and an isomorphism, is s-dense. The second part follows from Lemma 3.18. ∎

4. Injectivity of projection and separated projection algebras

In this final section, the behaviour of injectivity with respect to C-dense (C-closed) monomorphisms, for $C = C_m, C_s, C_p$, and ordinary injectivity is investigated. The results extends [8].

First recall the following injectivity definition.

Definition 4.22. A projection algebra A is called *m-dense injective* (*s-dense injective, p-dense injective*) if it is injective with respect to m-dense (respectively, s-dense, p-dense) monomorphisms. That is, $Hom(-, A)$ maps dense monomorphisms in **PRO** to epimorphisms in **Set**.

It is clear that injectivity implies m-injectivity, this implies s-injectivity, and the latter implies p-injectivity.

To study these injectivities, first we recall the following result.

Theorem 4.23. [8] *A projection algebra A is a retract of its extension B if and only if $C_p(A) = C_s(A)$.*

Proof. Let A be a retract of B. So, there is a projection map $f : B \to A$ such that $f\mid_A = id_A$. By Lemma 2.9, $C_p(A) \subseteq C_s(A)$. Let $b \in C_s(A)$. Then, $nb \in A$, for all $n \in \mathbb{N}$. So, $nf(b) = f(nb) = nb$, for all $n \in \mathbb{N}$. Since $f(b) \in A$, this shows that $b \in C_p(A)$. Conversely, let $C_p(A) = C_s(A)$. Define $g : B \to A$ by $g(b) = b$, for $b \in A$, and $g(b) = a$, for $b \in C_s(A) \setminus A$, where $a \in A$ is chosen in A such that $nb = na$, for all $n \in \mathbb{N}$, which exists since $C_s(A) = C_p(A)$. Also, for $b \in B \setminus C_s(A)$, define $g(b) = a_0$, where a_0 is a zero element of A, if $\mathbb{N} \cap A = \emptyset$, and define $g(b) = (k-1)b$, where k is the least natural number with $kb \notin A$, if $\mathbb{N} \cap A \neq \emptyset$. Then g is a projection map, and $g\mid_A = id_A$. ∎

Corollary 4.24. *The s-closed (m-closed) one-one projection maps are retractable, but not conversely. Also, the p-closed one-one projection maps are not necessarily retractable*

Proof. Let A, B be projection algebras with $A \leq B$, and A be s-closed in B. Applying Theorem 4.23, we show that $C_s(A) = C_p(A)$. But, $C_s(A) = A \subseteq C_p(A)$, since A is s-closed. And the other inclusion is always true. To see that the converses are not true, consider an injective projection algebra A (for example take $A = \mathbb{N}^\infty$). Using Lemma 3.19, A has a proper s-dense extension, say B. Then, A being injective is a retract of B, but it is not s-closed in B, since otherwise A being s-closed and s-dense in B, is equal to B, a contradiction.

For the last part, consider the inclusion map $\mathbb{N} \hookrightarrow \mathbb{N}^\infty$ (see the proof of Lemma 2.9). ∎

The situation for the separated projection algebras is as follows.

Corollary 4.25.

(1) *In $\mathbf{PRO_s}$, s-closed monomorphisms are exactly retractable ones.*

(2) *In $\mathbf{PRO_m}$, all monomorphisms are retractable.*

Proof. (1) Let A be a retract of its extension B. By Theorem 4.23, $C_s(A) = C_p(A)$. Then for $b \in C_s(A)$, there exists $a \in A$ such that $nb = na$, for all $n \in \mathbb{N}$. Since B is s-separated, this implies that $b = a \in A$. So, A is s-closed in B. The converse is true by Corollary 4.24.

(2) Let $A \leq B$ in $\mathbf{PRO_m}$. By Lemma 3.18, A is s-closed in B. So, by Corollary 4.24, A is a retract of B. ∎

Theorem 4.26. *In* $\mathbf{PRO_m}$*, all objects are* m *(respectively,* s *and* p*)-dense injective, as well as injective.*

Proof. Applying the above corollary, all objects in $\mathbf{PRO_m}$ are injective. Also, by Lemma 3.18, m (s or p)-dense monomorphisms in this category are isomorphisms and have inverses. So, all objects are also m (s or p)-dense injective. ∎

Also, by Lemma 3.18, the only p-dense monomorphisms in $\mathbf{PRO_s}$ are isomorphisms. So,

Theorem 4.27. *In* $\mathbf{PRO_s}$*, all objects are* p*-dense injective.*

For p-closure, we have

Lemma 4.28. *In* $\mathbf{PRO}$*,* p*-dense monomorphisms are retractable.*

Proof. We apply Theorem 4.23. Let $A \leq B$ be projection algebras and A be p-dense in B. Then $C_p(A) = B$ and so $C_s(A) \subseteq C_p(A)$. The other inclusion always holds. ∎

Notice that, the converse of the above lemma does not hold. For example, consider $\downarrow k \hookrightarrow \mathbb{N}$, for $k \neq \infty$.

The above lemma implies that

Theorem 4.29. *In* $\mathbf{PRO}$*, all objects are* p*-dense injective.*

Now we characterize injectivity in $\mathbf{PRO_s}$.

Theorem 4.30. *For an* s*-separated projection algebra* A*, the following are equivalent:*

(1) *A is m-dense injective in* $\mathbf{PRO_s}$.

(2) *A is s-dense injective in* $\mathbf{PRO_s}$.

(3) *A is injective in* $\mathbf{PRO_s}$.

(4) *A is injective in* $\mathbf{PRO}$.

Proof. (2)⇒(3): Consider a monomorphism $h : B \to C$ and a morphism $f : B \to A$ in $\mathbf{PRO_s}$. Let $h = lg : B \to D \to C$ be an (s-dense, s-closed) factorization of h. Then, g is monic and since A is s-dense injective, there exists a projection map $g' : D \to A$ such that $g'g = f$. Also, since l is s-closed, by Corollary 4.25, there exists a projection map $l' : C \to D$ such that $l'l = id_D$. Now, $g'l' : C \to A$ is a projection map with $g'l'h = g'l'lg = g'g = f$.

(3)⇒(4) proved in [14].

The other parts are clearly true. ∎

Lemma 4.31. *In* $\mathbf{PRO}$*, for a projection algebra* A *the following are equivalent:*

(1) *A is a retract of each of its m-dense extension.*

(2) *A is a retract of each of its s-dense extension.*

(3) *A is a retract of each of its extension.*

Proof. (1)⇒(2) is true, because s-dense maps are m-dense.

(2)⇒(3): Let B be an extension of A. Consider the (s-dense, s-closed) factorization fg of the inclusion map $\imath_A : A \hookrightarrow B$:

$$\begin{tikzcd} A \arrow[rr, hook] \arrow[dr, "g"'] & & B \\ & C_s(A) \arrow[ur, "f"'] & \end{tikzcd}$$

By Corollary 4.25, there exists a retraction $f' : B \to C_s(A)$, and by (2) there is a retraction $g' : C_s(A) \to A$. Then $g'f'$ is the required retraction.

(3)⇒ (1) is trivial. ∎

Theorem 4.32. *In* **PRO***, for a projection algebra A the following are equivalent:*

(1) *A is m-dense injective.*

(2) *A is s-dense injective.*

(3) *A is injective.*

Proof. It is enough to prove (2)⇒(3). Let A be s-dense injective. Then A is a retract of each of its s-dense extension. So, by the above lemma, A is absolute retract. Hence, A is injective (see Proposition 1.2). ∎

We close the paper by the following remarks.

Remark 4.33. Notice that, for each closure operator C on an equational category $\mathcal{A}$ of algebras with enough injectives, defining C-injectives as injectives with respect to C-dense monomorphisms, it can be shown, in a similar way as given above, that C-injectivity and injectivity coincide, whenever in $\mathcal{A}$ each morphism f has a (D,R) factorization $f = hg$, where D is the class of C-dense maps and R is the class of retractable monomorphisms.

Remark 4.34. Define a *divisible* (*m-divisible*) projection algebra to be a projection algebra A such that $nA = A$, for all $n \in \mathbb{N}$ ($mA = A$). By Lemma 3.15(2), A is m-divisible if and only if $A \in \mathbf{PRO_m}$. Also, A is divisible if and only if $A \in \mathbf{PRO_m}$, for all $m \in \mathbb{N}$, if and only if $A \in \mathbf{PRO_1}$. Moreover, in **PRO**, m-divisibility implies injectivity, and in $\mathbf{PRO_m}$, m-divisibility and injectivity are equivalent.

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QUASI-PERMUTATION REPRESENTATIONS OF SOME MINIMAL NON-ABELIAN p-GROUPS

Mohammad Hassan Abbaspour

Islamic Azad University
Khoy Branch
Iran
e-mail: m.abbaspour@iaukhoy.ac.ir

Houshang Behravesh

Department of Mathematics
University of Urmia
Iran
e-mail: h.behravesh@urmia.ac.ir

Abstract. In [1], $c(G)$, $q(G)$ and $p(G)$ are defined for a finite group G. In this paper, we will calculate $c(G)$, $q(G)$ and $p(G)$ for the following minimal non abelian p-groups:

$$G = \langle a, b \mid a^{p^m} = b^p = c^p = 1, [a, b] = c, [a, c] = [b, c] = 1\rangle,$$

and will show that

$$c(G) = q(G) = p(G) = p\, c(Z(G)) = p^m + p^2.$$

Keywords: Quasi-permutation representations, p-groups, Character theory.

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1. Introduction

By a quasi-permutation matrix we mean a square matrix over the complex field $\mathbb{C}$ with non-negative integral trace. Thus every permutation matrix over $\mathbb{C}$ is a quasi-permutation matrix. For a given finite group G, let $p(G)$ denote the minimal degree of a faithful permutation representation of G (or of a faithful representation of G by permutation matrices), let $q(G)$ denote the minimal degree of a faithful representation of G by quasi-permutation matrices over the rational field $\mathbb{Q}$, and let $c(G)$ denote the minimal degree of a faithful representation of G by complex quasi-permutation matrices. See [4]. It is easy to see that

$$c(G) \leq q(G) \leq p(G)$$

where G is a finite group.

Let G be a non abelian group. G is called a minimal non abelian group, if all its proper subgroups are abelian groups. In [6], all minimal non abelian p-groups are determined as the next Lemma.

Lemma 1.1 *Let G be a minimal non abelian p-group. Then $G = \langle a, b \rangle$, is one of the following groups:*

(1) $G = Q_8$,

(2) $G = \left\langle a, b \mid a^{p^m} = b^{p^n} = 1, a^b = a^{1+p^{m-1}} \right\rangle \quad (m > 1)$,

(3) $G = \left\langle a, b \mid a^{p^m} = b^{p^n} = c^p = 1, [a, b] = c, [a, c] = [b, c] = 1 \right\rangle$.

Furthermore, the last group is not metacyclic and in the case $p = 2$, we have $m \geq n$, $m \geq 2$. Also, $|G| = p^{m+n+1}$, $G' = \langle c \rangle$ and $Z(G) = \langle a^p \rangle \times \langle b^p \rangle \times \langle c \rangle$.

The groups (1) and (2) in the above Lemma are metacyclic and the quasi permutatation representation of such groups has calculated in [2], and [3]. Therefore, to determine the quasi permutation representation of minimal non abelian p-groups it is enough to consider only the group

$$G = \left\langle a, b \mid a^{p^m} = b^{p^n} = c^p = 1,\ [a, b] = c,\ [a, c] = [b, c] = 1 \right\rangle,$$

and determine $p(G), q(G)$ and $c(G)$. In This paper we consider a special case of this group, that is $n = 1$.

$$G = \left\langle a, b \mid a^{p^m} = b^p = c^p = 1,\ [a, b] = c,\ [a, c] = [b, c] = 1 \right\rangle,$$

2. Comparing the characters of G and $Z(G)$

In this section we state some relations between the characters of G and $Z(G)$, and construct the character table of G. In the next section, these connections will help us to compare the Galois conjugacy classes of irreducible characters of G and $Z(G)$, and conclude that $c(G) \geq p\, c(Z(G))$. Since $Z(G)$ is abelian, so computing $c(Z(G))$ is immediate (see [4]).

Lemma 2.1 *Let*

$$G = \left\langle a, b \mid a^{p^m} = b^p = c^p = 1,\ [a, b] = c,\ [a, c] = [b, c] = 1 \right\rangle,$$

where $m \geq 2$ in the case $p = 2$. Then

a) $|G| = p^{m+2}$,

b) $Z(G) = \langle a^p \rangle \langle c \rangle \cong C_{p^{m-1}} \times C_p$ *and* $|Z(G)| = p^m$,

c) $G' = \langle c \rangle$, *and* $|G'| = p$,

d) $G/G' = \langle aG' \rangle \langle bG' \rangle \cong C_{p^m} \times C_p$ *and* $|G/G'| = p^{m+1}$,

e) $\mathrm{cd}(G) = \{1, p\}$,

f) $|\mathrm{Lin}(G)| = p^{m+1}$ *and* $|\mathrm{Irr}(G|G')| = (p-1)p^{m-1}$, *where by* $\mathrm{Irr}(G|G')$ *we mean the set of non-linear irreducible characters of* G.

g) *The conjugacy classes of* G *are* $A_{i,j}$*'s* $(0 \le i < p^m,\ 0 \le j < p$, *and* $p \nmid i$ *or* $j \neq 0)$ *and the central classes , where*

$$A_{i,j} = a^i b^j \langle c \rangle .$$

Proof. a), b), c) and d) are clear by Lemma 1.1.

e) For each $\chi \in \mathrm{Irr}(G)$, we have $\chi(1)^2 \le |G : Z(G)|$ by [[5], Corollary (2.30)]. So the result follows from (b).

f) It is clear from (d), (e) and the fact that $|G| = \Sigma\chi(1)^2$ where χ runs over $\mathrm{Irr}(G)$.

g) Let x be an arbitrary non central element of G. For each element y in G we have

$$x^y = x[x,y] \in xG' = \{x, xc, xc^2, \ldots, xc^{p-1}\}.$$

Since the order of each conjugacy classes of G divides the order of G, so the result follows. ∎

Lemma 2.2 *Let* ψ *be a linear character of* G. *Denote the restriction of* ψ *to* $Z(G)$, *by* ψ_Z. *Then* ψ_Z *is an irreducible character* χ *of* $Z(G)$ *such that* $\chi(c) = 1$. *Furthermore, for a given* $\chi \in \mathrm{Irr}(Z(G))$ *with* $\chi(c) = 1$, *there are exactly* p^2 *linear character* ψ *of* G *such that* $\psi_Z = \chi$.

Proof. Let $\omega = e^{(2\pi i/p^{m-1})}$, $\mu = e^{(2\pi i/p)}$ and $u = e^{(2\pi i/p^m)}$. Let $\chi_{\upsilon,r}$ be the characters of $Z(G)$, where $0 \le \upsilon < p^{m-1},\ 0 \le r < p$ and

$$\chi_{\upsilon,r}(a^p) = \omega^{\upsilon},\ \chi_{\upsilon,r}(c) = \mu^r.$$

Also let $\psi_{s,t}$ be the linear characters of G, where $0 \le s < p^m,\ 0 \le t < p$ and

$$\psi_{s,t}(a) = u^s,\ \psi_{s,t}(b) = \mu^t.$$

Now, since $\omega = u^p$, so for each $0 \le \upsilon < p^{m-1}$ we have

$$\psi'_{\upsilon+\alpha p^{m-1},\beta} = \chi_{\upsilon,o} \qquad (1)$$

where $0 \le \alpha, \beta < p$ and by ψ' we mean ψ_Z. Note that the characters $\chi_{\upsilon,o}$ are exactly the characters of $Z(G)$ such that they have value 1 on c. ∎

The characters $\chi_{\upsilon,r}$ $(0 < r < p)$ of $Z(G)$ have not the value 1 on c. In the next lemma we connect these characters to the non-linear irreducible characters of G.

Lemma 2.3 *The non-linear irreducible characters of G are exactly the characters $p\overline{\chi}_{\upsilon,r}$, $(0 \leq \upsilon < p^{m-1}, 0 < r < p)$ defined as*

$$\overline{\chi}_{\upsilon,r}(x) = \begin{cases} \chi_{\upsilon,r}(x) & \text{if } x \in Z(G) \\ 0 & \text{if } x \notin Z(G) \end{cases}.$$

Proof. By [[5], Lemma (2.27) part (c) and Corollary (2.30)], every non-linear irreducible character ψ of G vanishes on $G - Z(G)$ and $\psi_Z = p\chi$ for some irreducible character χ of $Z(G)$.

Now, to complete the proof, it is enough to show that, for $0 \leq \upsilon < p^{m-1}$, $p\overline{\chi}_{\upsilon,o}$, can not be an irreducible non-linear character of G. To prove this, let us compute the inner product of $p\overline{\chi}_{\upsilon,o}$ and $\psi_{\upsilon,o}$:

$$\begin{aligned}
\langle p\overline{\chi}_{\upsilon,o}, \psi_{\upsilon,o}\rangle &= \frac{1}{|G|}\sum_{g\in G} p\overline{\chi}_{\upsilon,o}(g)\psi_{\upsilon,o}(g^{-1}) \\
&= \frac{1}{|G|}\sum_{g\in Z(G)} p\chi_{\upsilon,o}(g)\psi'_{\upsilon,o}(g^{-1}) \\
&= \frac{1}{|G|}\sum_{g\in Z(G)} p\chi_{\upsilon,o}(g)\chi_{\upsilon,o}(g^{-1}) \\
&= p\ \frac{|Z(G)|}{|G|}\frac{1}{|Z(G)|}\sum_{g\in Z(G)} \chi_{\upsilon,o}(g)\chi_{\upsilon,o}(g^{-1}) \\
&= p\ \frac{|Z(G)|}{|G|}\ \langle \chi_{\upsilon,o}, \chi_{\upsilon,o}\rangle \\
&= p\ \frac{|Z(G)|}{|G|} = \frac{1}{p} \neq 0.
\end{aligned}$$

Note that $p\overline{\chi}_{\upsilon,o}$ vanishes on $G - Z(G)$ and $\psi' = \psi_Z$, so the second equality holds. For the third equality, put $\alpha = \beta = 0$ in the relation (1). Hence $p\overline{\chi}_{\upsilon,o}$, is not even a character of G by [[5], Corollary (2.17)]. ∎

Theorem 2.4 *Let*

$$G = \left\langle a, b \mid a^{p^m} = b^p = c^p = 1,\ [a,b] = c,\ [a,c] = [b,c] = 1 \right\rangle.$$

Then the character table of G has the form

	$Z(G)$	$a^i b^j$
$\psi_{\upsilon+\alpha p^{m-1},\beta}$	$\chi_{\upsilon,0}$	$u^{i\upsilon}\mu^{i\alpha+j\beta}$
$p\overline{\chi}_{\upsilon,r}$	$p\chi_{\upsilon,r}$	0

where $(0 \leq i < p^m,\ 0 \leq j < p,\ p \nmid i$ *or* $j \neq 0)$ *and* $(0 \leq \upsilon < p^{m-1},\ 0 < r < p,$ $0 \leq \alpha, \beta < p)$.

Proof. This table follows by Lemmas 2.1, 2.2 and 2.3. Note that, for $0 \leq s < p^m$, $0 \leq t < p$, we have

$$\psi_{s,t}(a^i b^j) = u^{si} \mu^{tj}.$$

Also, note that

$$u^{p^{m-1}} = \mu. \qquad \blacksquare$$

3. Computing $c(G)$, $q(G)$ and $p(G)$

First, we state some notations and algorithms from [1]. Let G be a finite group. Let $\mathcal{C}_i$ for $0 \leq i \leq r$ be the Galois conjugacy classes of irreducible complex characters of the group G over the rational field $\mathbb{Q}$, for $0 \leq i \leq r$, suppose that ψ_i is a representative of the class $\mathcal{C}_i$ with $\psi_0 = 1_G$. Write $\Psi_i = \sum \mathcal{C}_i$, and $K_i = \ker \psi_i$. Clearly, $K_i = \ker \Psi_i$. For $I \subseteq \{0, 1, ..., r\}$, put

$$K_I = \bigcap_{i \in I} K_i.$$

Also, if $I \subseteq \{1, ..., r\}$, then we will use the notation $m(\chi)$, where $\chi = \sum_{i \in I} \Psi_i$ and $m(\chi) = -\min\left\{\sum_{i \in I} \Psi_i(g) : g \in G\right\}$. Moreover let $H_G = \bigcap_{g \in G} H^g$ be the core of $H \leq G$.

Theorem 3.1 *Let G be a finite group. Then in the above notation*

$$c(G) = \min\Big\{\chi(1) + m(\chi) : \chi = \sum_{i \in I} \Psi_i, K_I = 1,\ I \subseteq \{1, ..., r\}, K_J \neq 1, \text{if } J \subset I\Big\}$$

$$p(G) = \min\Big\{\sum_{i=1}^{n} |G : H_i| : H_i \leq G \text{ for } i = 1, 2, ..., n \text{ and } \bigcap_{i=1}^{n} (H_i)_G = 1\Big\}.$$

Proof. See [[1], Theorems (2.2) and the Proof of (3.6)]. $\blacksquare$

Lemma 3.2 *Let G be as in Lemma* 2.1 *and let we have the same notations as in Lemma* 2.2 *for irreducible characters of $Z(G)$. Then the Galois conjugacy classes of* $\mathrm{Irr}(Z(G))$ *are* $\mathcal{C}_0 = \{\chi_{0,0}\}$ *and*

$$\mathcal{C}_{p^j} = \{\chi_{ip^j,0} : 1 \leq i \leq p^{m-1-j} - 1, (i, p^{m-1}) = 1\},$$

where $0 \leq j \leq m-2$, *and the Galois conjugacy classes of the characters* $\chi_{\upsilon,r}$, $r \neq 0$.

Proof. First, note that no $\chi_{v,r}$ can be a Galois conjugate to some $\chi_{v',0}$ when $r \neq 0$, because of their different values on c. So, we consider only the characters $\chi_{v,0}$. Since $Z(G) = \langle a^2 \rangle \langle c \rangle$ and $\chi_{v,0}(\langle c \rangle) = 1$, so there is a one to one corresponding between the set of characters $\chi_{v,0}$ of $Z(G)$ and the set of the characters of the cyclic group $\langle a^2 \rangle \cong C_{p^{m-1}}$ via the map $\chi_{v,0} \mapsto \chi_v$. Clearly, this map preserves Galois conjugates. Now let $\Gamma(\chi)$, denote the Galois group of $\mathbb{Q}(\chi)$ over $\mathbb{Q}$. Then, for $0 \leq j \leq m-2$, we have

$$\Gamma(\chi_{p^j,0}) = \Gamma(\chi_{p^j}) = \mathrm{Gal}(\mathbb{Q}(\chi^{p^j})/\mathbb{Q}) \text{ and}$$

$$\Gamma(\chi_{p^j}) = \{\sigma_i : \sigma_i \text{ is an } \mathbb{Q}-\text{automorphism of } \mathbb{Q}(\omega^{p^j}) \text{ and } \sigma_i(\omega^{p^j}) = \omega^{ip^j}\},$$

where $1 \leq i \leq p^{m-1-j} - 1, (i, p^{m-1}) = 1$. This shows that $\mathcal{C}_{p^j}$, is the Galois conjugacy class of the character $\chi_{p^j,0}$. The order of the class $\mathcal{C}_{p^j}$ is equal to $(p-1)p^{m-2-j}$. These classes are all different and counting their elements shows that they are all Galois conjugacy classes of $Z(G)$. ∎

Lemma 3.3 *Let G be as in Lemma 2.1 and let we have the same notations as in Lemma 2.2 and 2.3 for irreducible characters of G. Then the Galois conjugacy classes of $\mathrm{Irr}(G)$ are*

(1) *The Galois conjugacy classes of the characters $\psi_{\alpha p^{m-1},\beta}$ where $0 \leq \alpha, \beta < p$,*

(2) *The Galois conjugacy classes*

$$\mathcal{C}_{p^j}^{(\beta)} = \{\psi_{ip^j+\alpha p^{m-1},\overline{i\beta}} : 1 \leq i \leq p^{m-1-j} - 1, (i,p) = 1, 0 \leq \alpha < p\},$$

where $0 \leq j \leq m-2$, $0 \leq \beta < p$ and by $\overline{i\beta}$ we mean $i\beta$ module p,

(3) *The Galois conjugacy classes of the characters $p\overline{\chi}_{v,r}, r \neq 0$.*

Proof. Each character in (1) is 1 on $Z(G)$, so it can not be a Galois conjugate to any character in (2) or (3). The characters in (3) have degree p, so these characters can not be Galois conjugate to (linear) characters in (1) or (2). Thus, we consider only the characters in (2). We show that, for $0 \leq j \leq m-2$, $\mathcal{C}_{p^j}^{(\beta)}$ is the Galois conjugacy class of the character $\psi_{p^j,\beta}$. Let $\psi_{ip^j+\alpha p^{m-1},\overline{i\beta}} \in \mathcal{C}_{p^j}^{(\beta)}$. Let $\tau \in \Gamma(\psi_{p^j,\beta}) = \mathrm{Gal}(\mathbb{Q}(u^{p^j}, \mu^\beta)/\mathbb{Q})$ and $(u^{p^j})^\tau = (u^{p^j})^{i+\alpha p^{m-1-j}}$. Then $(u^{p^j})^\tau = (u^{p^j})^i \mu^\alpha$ and $(\mu^\beta)^\tau = \mu^{i\beta}$. Therefore, $\psi_{p^j,\beta}^\tau = \psi_{ip^j+\alpha p^{m-1},\overline{i\beta}}$. On the other hand, $|\Gamma(\psi_{p^j,\beta})| = \varphi(p^{m-j}) = (p-1)p^{m-j-1} = |\mathcal{C}_{p^j}^{(\beta)}|$ where φ is the Euler function. Therefore, $\mathcal{C}_{p^j}^{(\beta)}$ is the Galois conjugacy class of the character $\psi_{p^j,\beta}$. Counting the elements of these classes shows that they are, in addition to classes of (1) and (3), all Galois conjugacy classes of G. ∎

Lemma 3.4 *Let G be a finite group and $\chi \in \mathrm{Irr}(G)$. Then $\Sigma_{\alpha \in \Gamma(\chi)} \chi^\alpha$ is a rational valued character of G.*

Proof. By [[1], Corollary 3.7]. ∎

Now, as the notations introduced before Lemma 3.1, let $\Psi_{p^j} = \sum \mathcal{C}_{p^j}$, $\Psi_{p^j}^{(\beta)} = \sum \mathcal{C}_{p^j}^{(\beta)}$ where $0 \leq j < m-2$. Let Υ_i's be the sum's of Galois conjugacy classes of characters (1) in the Lemma 3.3. Also, let Φ_i's be the sums of Galois conjugacy classes of characters $\chi_{\upsilon,r}, r \neq 0$ and Φ_i''s be the sums of Galois conjugacy classes of characters $p\overline{\chi}_{\upsilon,r}$. Note that these sums are rational valued characters by Lemma 3.4 and $\Psi_{p^j}^{(\beta)} = \sum \mathcal{C}_{p^j}^{(\beta)} = p \sum \mathcal{C}_{p^j} = p\Psi_{p^j}$ on $Z(G)$. Also $\Phi_i' = p\,\Phi_i$ on $Z(G)$ and $\Phi_i' = 0$ on $G - Z(G)$.

Theorem 3.5 *Let G be as in Lemma* 2.1. *Then, in the algorithm given in Theorem* 3.1, *the classes Υ_i are not used for computing $c(G)$. Therefore,*

$$c(G) = q(G) = p(G) = p\,c(Z(G)) = p^m + p^2.$$

Proof. Let $\mathbb{S}$ be a subset of the set of class sums Υ_i that has used in computing $c(G)$ with a set $\mathbb{T}$ of other class sums. Since $Z(G) \subseteq \bigcap_{S\in\mathbb{S}} \ker S$, so by the algorithm of $c(G)$, given in Theorem 3.1, $\mathbb{T} \neq \varnothing$. Now there is an element T_i of $\mathbb{T}$ such that T_i is vanishes on $G - Z(G)$, because otherwise, $c \in \bigcap_{T\in\mathbb{T}} \ker T$, that is a contradiction. Therefore the kernels of the elements of $\mathbb{T}$ have no intersection in $G - Z(G)$, and clearly no non-trivial intersection in $Z(G)$. Thus

$$\bigcap_{T\in\mathbb{T}} \ker T = 1.$$

This is a contradiction to the choice of the sets $\mathbb{S}$ and $\mathbb{T}$. Therefore, $\mathbb{S} = \varnothing$.

Now, by the algorithm of $c(G)$ given in Theorem 3.1 and the argument after Lemma 3.4, we conclude that $p\,c(Z(G)) \leq c(G)$. In other hand, let $H_1 = \langle a \rangle$ and $H_2 = \langle b, c \rangle$ then, by Theorem 3.1,

$$p(G) \leq |G : H_1| + |G : H_2| = p^2 + p^m.$$

Since $p\,c(Z(G)) = p^2 + p^m$ by [[4], Theorem A], so

$$c(G) = q(G) = p(G) = p\,c(Z(G)) = p^m + p^2,$$

and the proof is complete. ∎

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AN ANALYTICAL SOLUTION OF FLUID FLOW THROUGH NARROWING SYSTEMS

A.D. Patel
I.A. Salehbhai
Department of Mathematics
S.V. National Institute of Technology
Surat, Gujarat
India

J. Banerjee
Department of Mechanical Engineering
S.V. National Institute of Technology
Surat, Gujarat
India

V.K. Katiyar
Department of Mathematics
Indian Institute of Technology
Roorkee
India

A.K. Shukla
Department of Mathematics
S.V. National Institute of Technology
Surat, Gujarat
India
e-mail: ajayshukla2@rediffmail.com

Abstract. Narrowing of pipeline network is an important aspect in drinking water distribution systems, sewage system and in oil-well techniques. In the proposed problem, a flow equation in simple pipeline network has been studied to solve the velocity flow. The deposition causing narrowing has been replaced by using sinusoidal model with axial velocity. In this paper, we used MAPLE 11.02 for plotting the graphs.

Keywords: narrowing systems, finite Hankel transforms, Laplace transforms, Bessel functions.
AMS Classification: 76D05, 65R10, 44A10, 44A15, 33C10.

Nomenclature

- D Substantial derivative,
- V Velocity vector,
- t Time,
- ρ Density,
- p Pressure,
- μ Dynamic viscosity,
- ν Kinematic viscosity

1. Introduction and preliminaries

The continuity and Navier-Stokes equations (Murlidhar and Biswas [1]) for incompressible flow are:

$$\nabla \cdot V = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0 \tag{1.1}$$

$$\rho\left(\frac{DV}{Dt}\right) = -\nabla p + \mu \nabla^2 V, \tag{1.2}$$

Equation (1.2) can be easily reduce to r, θ and z directions follows as

$$\begin{aligned}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} \\ = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\mu}{\rho}\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (ru)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta}\right)\end{aligned} \tag{1.3}$$

$$\begin{aligned}\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r} \\ = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{\mu}{\rho}\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (rv)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 v}{\partial z^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta}\right)\end{aligned} \tag{1.4}$$

$$\begin{aligned}\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z} \\ = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 w}{\partial z^2}\right)\end{aligned} \tag{1.5}$$

This equation plays a very important role in mathematical modeling of real world problems and can also be reduce to different form of equation by applying specific conditions. In the present paper, our aim is to construct a mathematical model for the study of fluid flow in narrowing systems by using (1.2) with the help of equation of continuity and applying the Laplace and finite Hankel transform techniques, which yields the analytical solution.

Several bio-mathematicians (Verma et. al [2], [3], Ponalagusamy [4], Chaturani et. al. [5], [6], [7]) applied the concept of the narrowing system in the study of blood flow through a stenosised artery by using different mathematical tools.

2. Used integral transforms and special functions

The Laplace Transform (Debnath [8]) is defined as,

$$L\{f(x)\} = \int_0^\infty e^{-st} f(t)\, dt \tag{2.6}$$

The zero order Bessel function $J_0(x)$ (Rainville [9]) is defined as

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{x}{2}\right)^{2m} \tag{2.7}$$

The zero order finite Hankel transform (Debnath [8]) is defined as,

$$H_0\{f(r)\} = \tilde{f}_0(\lambda_n) = \int_0^R r\, f(r)\, J_0(r\lambda_n)\, dr, \tag{2.8}$$

where λ_n are the roots of the equation $J_0(R\lambda_n) = 0$.

In 1903, Mittag-Leffler [10] introduced the function $E_\alpha(z)$, defined as

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \tag{2.9}$$

where z is a complex variable and $\Gamma(s)$ is a gamma function, $\alpha \geq 0$.

In 1905, Wiman [11] introduced the generalization of $E_\alpha(z)$ as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} = \frac{1}{2\alpha\pi\, i} \int \frac{e^{\xi^{\frac{1}{\alpha}}} \xi^{\frac{1-\beta}{\alpha}}}{\xi - z} d\xi, \tag{2.10}$$
$$(\alpha, \beta \in \mathrm{C};\ Re(\alpha) > 0,\ Re(\beta) > 0).$$

Shukla and Prajapati [12] also derived the following integral

$$\int_0^{\infty} e^{-st} t^{\beta-1} \frac{d^k}{dz^k} E_{\alpha,\beta}(yt^\alpha)\, dt = \frac{k!\, s^{\alpha-\beta}}{(s^\alpha - y)^{k+1}}. \tag{2.11}$$

3. Mathematical formulation of the problem

Let a long circular cylinder in which fluid is at rest initially and a constant pressure gradient is imposed along the axis of the cylinder, due to the pressure gradient fluid is set into the motion (constant ρ and μ). Let Z as the direction of the axis of cylinder along which the flow takes place and let r be the radial direction outward from the Z-axis, consider the flow is fully developed and axially symmetric. Here, we assume that there are some depositions of thickness δ on the wall of the cylinder which causes the narrowing the system, which satisfies the following equation of the thickness due to deposition:

$$R = R_0 - \frac{\delta}{2}\left(1 + \cos\frac{\pi z}{z_o}\right),$$

where δ is the deposition thickness, R_0 is the distance from axis of the cylindrical boundary and z is the distance from $z = 0$ to the point of calculation P.

If $z = 0$ then $R = R_0 - \delta$ and if $z = z_0$ then $R = R_0$.

Since velocity u and v are zero, pressure depends on z then we arrive at the conclusion by using equations (1.3), (1.4) and (1.5):

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

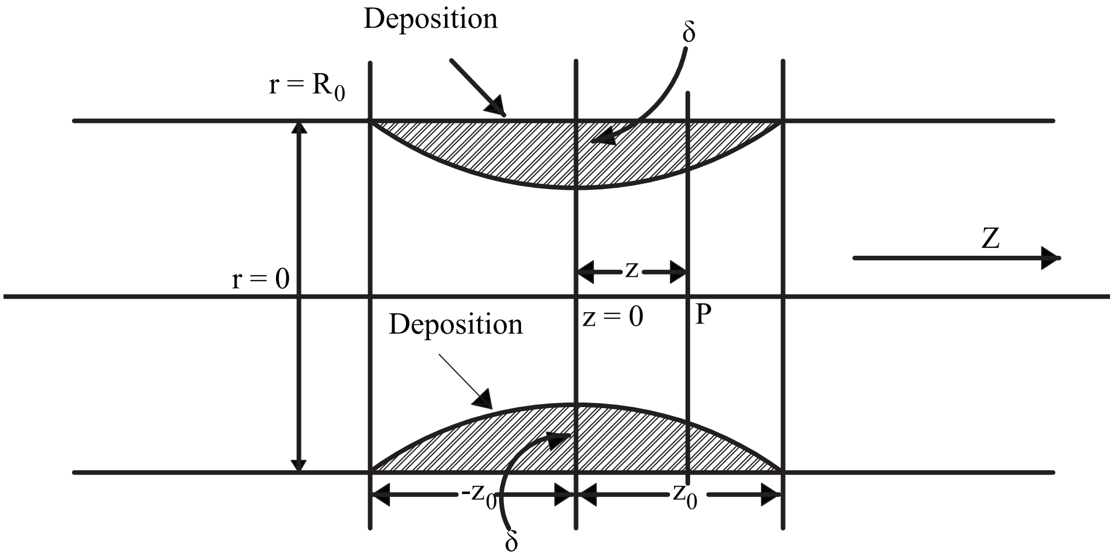


Figure 1: Schematic diagram of narrowing system

Here we consider the velocity component is invariant in the θ and z directions, then above equation reduces to:

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right)$$

Now, applying the equation of continuity

$$\frac{\partial w}{\partial z} = 0,$$

the Z-momentum equation can be written in a simplified form as:

$$\rho\frac{\partial w}{\partial t} = P + \mu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) \tag{3.12}$$

where μ is dynamic viscosity and

$$P = -\frac{\partial p}{\partial z}.$$

Initial condition and boundary conditions are considered as:

$$\begin{cases} w\left(r,0\right) = 0 \\ w\left(R,t\right) = 0 \\ w\left(0,t\right) \text{ is finite} \end{cases} \tag{3.13}$$

4. Solution of the problem

The method of integral transform is used to obtain the solution of the problem. Let

$$\bar{w}(\lambda_n, t) = H_0(w(r,t)) = \int_0^R r\,w(r,t)\,J_0(r\lambda_n)\,dr \tag{4.14}$$

where λ_n are the roots of the equation $J_0(R\lambda_n) = 0$. Also, by using the recurrence relation of the Bessel function, we have

$$\int_0^R r\,J_0(r\lambda_n)\,dr = \frac{R}{\lambda_n} J_1((R\lambda_n)\,. \tag{4.15}$$

By taking the zero order finite Hankel transform (2.8) of (3.12) and using (4.14), (4.15) & (3,13) , yields

$$\rho\frac{\partial \bar{w}}{\partial t} = \frac{PRJ_1(R\lambda_n)}{\lambda_n} - \mu\lambda_n^2\bar{w}. \tag{4.16}$$

Let

$$\tilde{\bar{w}}(\lambda_n, s) = L\{\bar{w}(\lambda_n, t)\} = \int_0^\infty e^{-st}\bar{w}(\lambda_n, t)\,dt. \tag{4.17}$$

Also,

$$\int_0^\infty e^{-st}dt = \frac{1}{s}. \tag{4.18}$$

By taking the Laplace transform (2.6) of (4.16) and using (4.17), (4.18) & (3.13), we get

$$\rho s\tilde{\bar{w}}(s) = \frac{PRJ_1(R\lambda_n)}{\lambda_n s} - \mu\lambda_n^2\tilde{\bar{w}}.$$

Further simplification gives

$$\tilde{\bar{w}} = \frac{PRJ_1(R\lambda_n)}{(\rho s + \mu\lambda_n^2)\,\lambda_n s}. \tag{4.19}$$

Now, taking the inverse Laplace transform of this equation, gives

$$\bar{w} = \frac{PRJ_1(R\lambda_n)}{\lambda_n\rho} L^{-1}\left\{\frac{1}{s\left(s + \frac{\mu}{\rho}\lambda_n^2\right)}\right\}.$$

Now, using Convolution theorem (Debnath [8]), we have

$$\bar{w} = \frac{PRJ_1(R\lambda_n)}{\lambda_n\rho}\int_0^t L^{-1}\left\{\frac{1}{\left(s + \frac{\mu}{\rho}\lambda_n^2\right)}, u\right\} du,$$

and using (2.11), we get:

$$\bar{w} = \frac{PRJ_1\left(R\lambda_n\right)t}{\lambda_n\rho}E_{1,2}\left(-\lambda_n^2\frac{\mu}{\rho}t\right). \tag{4.20}$$

We can easily verify that

$$tE_{1,2}\left(mt\right) = \frac{1}{m}\left[e^{mt} - 1\right]$$

and putting this result in (4.20) afterwards taking the inverse finite Hankel transform yields

$$w = -\frac{2}{R^2}\sum_{n=1}^{\infty}\left\{\frac{PRJ_1\left(R\lambda_n\right)}{\lambda_n\rho}\frac{1}{\lambda_n^2\frac{\mu}{\rho}}\left[e^{-\lambda_n^2\frac{\mu}{\rho}t} - 1\right]\right\}\frac{J_0\left(r\lambda_n\right)}{J_1^2\left(R\lambda_n\right)}.$$

Further simplification of this result becomes in following form,

$$w\left(r,t\right) = \frac{P}{4\mu}\left(R^2 - r^2\right) - \frac{2P}{\mu R}\sum_{n=1}^{\infty}\frac{J_0\left(\lambda_n r\right)}{\lambda_n^3 J_1\left(\lambda_n R\right)}e^{\left(-\frac{\mu}{\rho}\lambda_n^2 t\right)}. \tag{4.21}$$

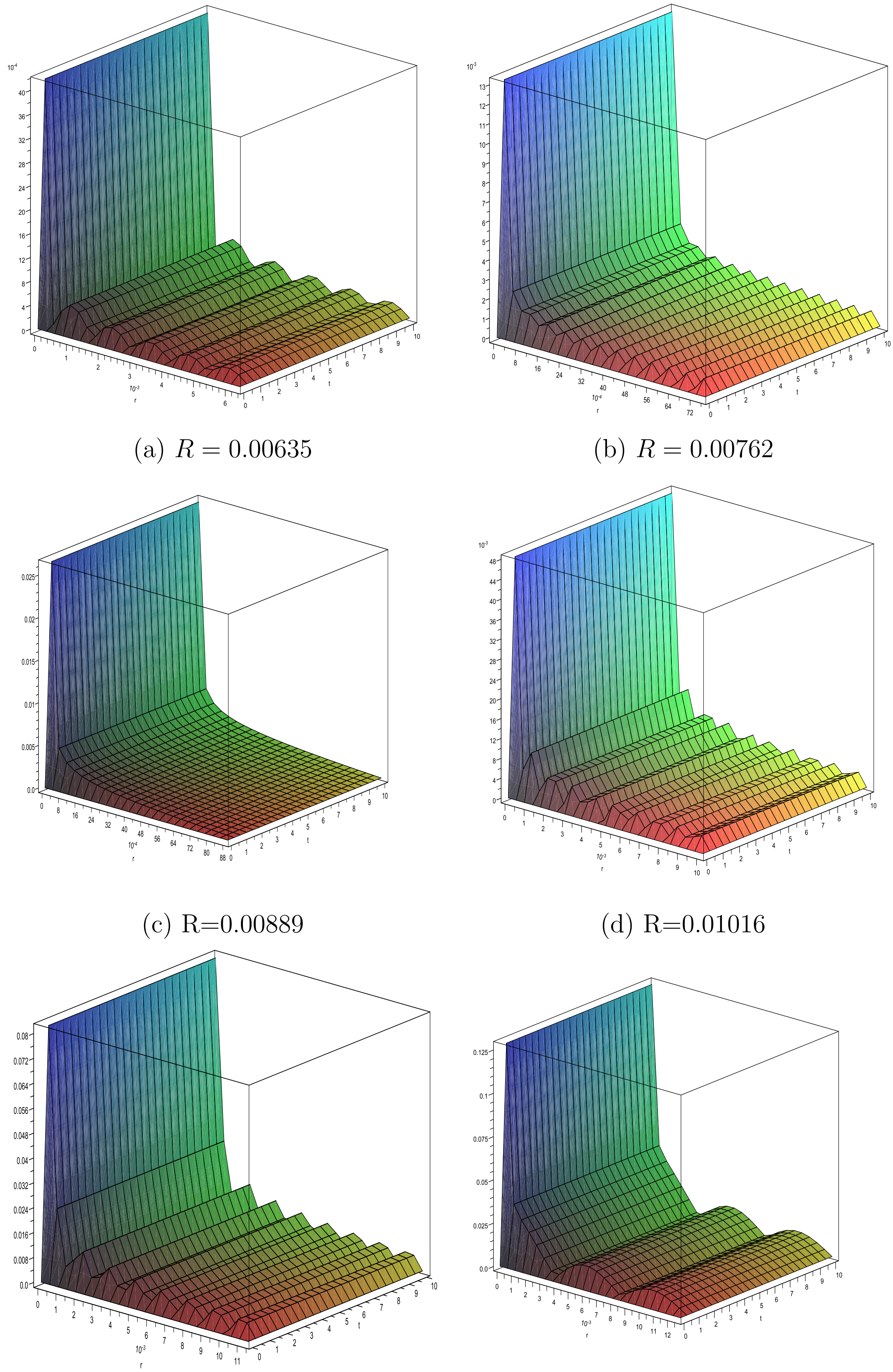
If $t \to \infty$, then $w\left(r,t\right) = \dfrac{P}{4\mu}\left(R^2 - r^2\right)$.

5. Conclusion

In this paper, we obtained the analytic solution of fluid flow through narrow system in the terms of Bessel function and Mittag-Leffler function by applying Laplace transform and finite Hankel transform techniques. The behavior of the flow has also been shown in the graphs for different values of operational radius R.

By using $P = 101325$ Pa, $\rho = 1000$ Ns/m^2, $\mu = 0.0010020$ kg/m^3 and $R_0 = 0.0127$ m in

$$w\left(r,t\right) = \frac{P}{4\mu}\left(R^2 - r^2\right) - \frac{2P}{\mu R}\sum_{n=1}^{\infty}\frac{J_0\left(\lambda_n r\right)}{\lambda_n^3 J_1\left(\lambda_n R\right)}e^{\left(-\frac{\mu}{\rho}\lambda_n^2 t\right)}.$$



(a) $R = 0.00635$ (b) $R = 0.00762$

(c) R=0.00889 (d) R=0.01016

(e) R=0.01143 (f) R=0.0127

Figure 2: Velocity profile for different operational radius R.

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ON GENERALIZED HILBERT ALGEBRAS

R.A. Borzooei

Department of Mathematics
University of Shahid Beheshti
Tehran
Iran
e-mail: borzooei@sbu.ac.ir

J. Shohani

Department of Mathematics
Sistan and Balushestan University
Zahedan
Iran

Abstract. In this paper by considering the notion of generalized Hilbert algebra which is named g-Hilbert algebra, we obtain some properties of it. Moreover, we show that for all $n \geq 3$ there exist at least one proper g-Hilbert algebra of order n. Because g-Hilbert algebra is not a Boolean algebra we define the concept of branch in g-Hilbert algebras and we prove that any branch in commutative g-Hilbert algebras is a Boolean algebra.

Keywords: generalized Hilbert algebra, implication algebras, complemented lattice, distributive lattice, Boolean algebra.

AMS Subject Classification (2000): 06F35, 03G25, 08A30.

1. Introduction

Hilbert algebras [7] represent the algebraic counterpart of the implicative fragment of Intuitionistic Propositional Logic. In [7] Diego gives a topological representation for Hilbert algebras and he proves that every Hilbert algebra is isomorphic to a subalgebra of the implicative reduct of a Heyting algebra generated by a certain topological space. Also, Hilbert algebras, or positive implication algebras [14], are the duals of Henkin algebras called by him implicative models in [9]. Positive implicative BCK-algebras [11] are actually another version of Henkin algebras. As a matter of fact, these algebras are an algebraic counterpart of positive implicational calculus. Various expansions of Hilbert algebras by a conjunction-like operation have also been studied in the literature. The most extensively investigated among them are implicative semilattices, which are known also as Brouwerian semilattices.

Now, in this paper we give a generalization of positive implicative BCK-algebras and Hilbert algebras which is called a generalized Hilbert algebra that it is in form of variety. In follow, we obtain some properties of generalized Hilbert algebra and we show that any branch in commutative generalized Hilbert algebras is a Boolean algebra.

2. Generalized Hilbert algebras

Definition 2.1. [7] A *Hilbert algebra* is a triplet $(H, \rightarrow, 1)$ of type (2,0), where H is a nonempty set, "$\rightarrow$" is a binary operation which satisfies the following axioms:

(H1) $x \rightarrow (y \rightarrow x) = 1$,

(H2) $(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1$,

(H3) $x \rightarrow y = 1$ and $y \rightarrow x = 1$ imply $x = y$,

for all $x, y, z \in H$.

Proposition 2.2. *[8] If* $(H, \rightarrow, 1)$ *be a Hilbert algebra, then,*

(i) $x \rightarrow x = 1$,

(ii) $1 \rightarrow x = x$,

(iii) $x \rightarrow 1 = 1$,

(iv) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,

(v) $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$,

for all $x, y, z \in H$.

Definition 2.3. A *generalized Hilbert algebra* (or briefly, *g-Hilbert algebra*) is an algebra $(G_H, \rightarrow, 1)$ of type (2,0) which satisfies the following axioms;

(GH1) $1 \rightarrow x = x$,

(GH2) $x \rightarrow x = 1$,

(GH3) $z \rightarrow (y \rightarrow x) = y \rightarrow (z \rightarrow x)$,

(GH4) $z \rightarrow (y \rightarrow x) = (z \rightarrow y) \rightarrow (z \rightarrow x)$,

for all $x, y, z \in G_H$.

Example 2.4. Let $(X, \leq, 1)$ be a unital poset and implication " $\rightarrow$ " on X is defined as follows:

$$x \rightarrow y = \begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{otherwise.} \end{cases}$$

Then $(X, \rightarrow, 1)$ is a g-Hilbert algebra.

Example 2.5. Let $(X, \leq)$ be a poset. Then $Y \subseteq X$ is called *increasing subset* if it is closed under $\leq$, i.e for every $x \in Y$ and every $y \in X$ if $x \leq y$ then $y \in Y$. Now, let $\mathcal{P}_i(X)$ be the set of all increasing subset of X and for any $y \in X$, $[y) = \{x \in X : y \leq x\}$. Then it is easy to see that $(\mathcal{P}_i(X), \rightarrow, X)$ is a g-Hilbert algebra where the implication " $\rightarrow$ " is defined by

$$U \rightarrow V = \{x \in X : [x) \cap U \subseteq V\}$$

for $U, V \in \mathcal{P}_i(X)$.

Theorem 2.6. *Any Hilbert algebra is a g-Hilbert algebra.*

Proof. The proof is clear by Proposition 2.2. ■

Note. The converse of Theorem 2.6 is not correct in general.

Example 2.7. Let G_H={a,b,1} and operation $\rightarrow$ on G_H is defined as follows

$\rightarrow$	a	b	1
a	1	1	1
b	1	1	1
1	a	b	1

It is routine to check G_H={a,b,1} is a g-Hilbert algebra but it is not a Hilbert algebra, since $a \rightarrow b = b \rightarrow a = 1$ but $a \neq b$.

Proposition 2.8. *Let* $(G_H, \rightarrow, 1)$ *be a g-Hilbert algebra. Then:*

(i) $x \rightarrow 1 = 1$,

(ii) $(y \rightarrow z) \rightarrow ((z \rightarrow x) \rightarrow (y \rightarrow x)) = 1$,

(iii) $(z \rightarrow x) \rightarrow ((y \rightarrow z) \rightarrow (y \rightarrow x)) = 1$,

(iv) $(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1$,

(v) $x \rightarrow (x \rightarrow y) = x \rightarrow y$,

(vi) $x \rightarrow (y \rightarrow x) = 1$,

(vii) $y \rightarrow ((y \rightarrow x) \rightarrow x) = 1$.

for all $x, y, z \in G_H$.

Proof. (i) Let $x \in G_H$. Then, by (GH2) and (GH4),

$$1 = 1 \rightarrow 1 = (x \rightarrow x) \rightarrow (x \rightarrow x) = x \rightarrow (x \rightarrow x) = x \rightarrow 1.$$

(ii) Let $x, y, z \in G_H$. Then,

$$\begin{aligned}
&(y \to z) \to ((z \to x) \to (y \to x)) \\
&\qquad = (z \to x) \to ((y \to z) \to (y \to x)), \quad \text{(by (GH3))} \\
&\qquad = (z \to x) \to (y \to (z \to x)), \quad \text{(by (GH4))} \\
&\qquad = y \to ((z \to x) \to (z \to x)), \quad \text{(by (GH3))} \\
&\qquad = y \to 1, \quad \text{(by (GH2))} \\
&\qquad = 1 \ \ \text{(by (i))}.
\end{aligned}$$

(iii) Let $x, y, z \in G_H$. Then by (GH3) and (ii);

$$(z \to x) \to ((y \to z) \to (y \to x)) = (y \to z) \to ((z \to x) \to (y \to x)) = 1.$$

(iv) Let $x, y, z \in G_H$. Then by (GH4) and (GH2);

$$\begin{aligned}
&(x \to (y \to z)) \to ((x \to y) \to (x \to z)) \\
&\qquad = ((x \to y) \to (x \to z)) \to ((x \to y) \to (x \to z)) = 1.
\end{aligned}$$

(v) Let $x, y, z \in G_H$. Then by (GH4) and (GH1);

$$x \to (x \to y) = (x \to x) \to (x \to y) = 1 \to (x \to y) = x \to y.$$

(vi) Let $x, y, z \in G_H$. Then by (GH4) and (i);

$$x \to (y \to x) = (x \to y) \to (x \to x) = (x \to y) \to 1 = 1.$$

(vii) Let $x, y, z \in G_H$. Then by (GH3) and (GH2);

$$y \to ((y \to \ x) \to x) = (y \to x) \to (y \to x) = 1.$$ ■

Definition 2.9. Let $(G_H, \to, 1)$ be a g-Hilbert algebra, then, G_H is called a *proper g-Hilbert algebra* if it is not a Hilbert algebra.

Proposition 2.10. *If G_H is a proper g-Hilbert algebra of order n, then $n \geq 3$.*

Proof. By Definition 2.3, Proposition 2.8 and Example 2.7, the proof is clear. ■

Theorem 2.11. *Let $(G_H, \to, 1)$ be a proper g-Hilbert algebra and $a \notin G_H$. Then $G_H^{'} = G_H \cup \{a\}$ with the following operation is a proper g-Hilbert algebra.*

$$x \to y = \begin{cases} x \to y & , \ x, y \in G_H, \\ a & , \ x = 1, y = a, \\ 1 & , \ x \in G_H^{'} - \{1\}, y = a, \\ y & , \ x = a, y \in G_H. \end{cases}$$

Proof. The proof of axioms (GH1), (GH2) and (GH3) are clear. So, we should only prove the axiom (GH4). For this case, we consider the following cases:

Case 1. $x, y \in G_H$ and $z = a$:

$$\begin{aligned} z \to (x \to y) &= a \to (x \to y) = x \to y \\ &= (a \to x) \to (a \to y) = (z \to x) \to (z \to y). \end{aligned}$$

Case 2. $x, z \in G_H$ and $y = a$:
If $x \neq 1$ and $z \neq 1$, then

$$\begin{aligned} z \to (x \to y) &= z \to (x \to a) = 1 = (z \to x) \to 1 \\ &= (z \to x) \to (z \to a) = (z \to x) \to (z \to y). \end{aligned}$$

If $x \neq 1$ and $z = 1$, then

$$\begin{aligned} z \to (x \to y) &= z \to (x \to a) = 1 = x \to a = x \to y \\ &= (1 \to x) \to (1 \to y) = (z \to x) \to (z \to y). \end{aligned}$$

If $x = 1$ and $z \neq 1$, then

$$z \to (x \to y) = z \to y = 1 \to (z \to y) = (z \to x) \to (z \to y).$$

If $x = 1$ and $z = 1$, then

$$z \to (x \to y) = y = 1 \to y = (1 \to 1) \to (1 \to y) = (z \to x) \to (z \to y).$$

Case 3. $y, z \in G_H$ and $x = a$.
The proof is similar to the proof of Case 2, by some modification.

Case 4. $x \in G_H$ and $y = z = a$.
If $x \neq 1$, then

$$\begin{aligned} z \to (x \to y) &= a \to (x \to y) = x \to y \\ &= (a \to x) \to (a \to y) = (z \to x) \to (z \to y). \end{aligned}$$

If $x = 1$, then

$$\begin{aligned} z \to (x \to y) &= a \to (1 \to a) = a \to a = 1 = 1 \to 1 \\ &= (a \to 1) \to (a \to a) = (z \to x) \to (z \to y). \end{aligned}$$

Case 5. $y \in G_H$ and $x = z = a$ or $z \in G$ and $y = x = a$:
The proof is similar to the proof of Case 4, by some modification.

Case 6. $x = y = z = a$:

$$\begin{aligned} z \to (x \to y) &= a \to (a \to a) = a \to 1 = 1 = 1 \to 1 \\ &= (a \to a) \to (a \to a) = (z \to x) \to (z \to y). \end{aligned}$$

Hence, $(G'_H, \to, 1)$ is a g-Hilbert algebra. ∎

Corollary 2.12. *For any natural number $n \geq 3$, there exist at least one proper g-Hilbert algebra of order n.*

Definition 2.13. Let $(G_H, \rightarrow, 1)$ be a g-Hilbert algebra and $a \in G_H$. Then, the set $B(a) = \{x \in G_H | a \rightarrow x = 1\}$ is called a *branch* of X.

It is clear that $1, a \in B(a)$ and so $B(a) \neq \emptyset$.

Theorem 2.14. *Let $(G_H, \rightarrow, 1)$ be a g-Hilbert algebra such that for all $x, y \in G_H$, $B(x) \cap B(y) = \{1\}$ and $x \rightarrow y \neq y$. Then G_H is a proper g-Hilbert algebra.*

Proof. Let G_H be a Hilbert algebra, by contrary. By Proposition 2.2(iv) and (i), $y \rightarrow ((y \rightarrow x) \rightarrow x) = 1$ and so $(y \rightarrow x) \rightarrow x \in B(y)$. Now, let $z = y \rightarrow x$. Then, by Proposition 2.2(iv), (i) and (iii),

$$x \rightarrow (z \rightarrow x) = z \rightarrow (x \rightarrow x) = z \rightarrow 1 = 1$$

and so, $(y \rightarrow x) \rightarrow x = z \rightarrow x \in B(x)$. Hence,

$$(y \rightarrow x) \rightarrow x \in B(x) \cap B(y) = \{1\}$$

and so, $(y \rightarrow x) \rightarrow x = 1$. On the other hand, by (GH4) and (GH2) and Proposition 2.2(iii),

$$x \rightarrow (y \rightarrow x) = (x \rightarrow y) \rightarrow (x \rightarrow x) = (x \rightarrow y) \rightarrow 1 = 1$$

and so, by (H3) we get that, $y \rightarrow x = x$, which is a contradiction. Therefore, G_H is a proper g-Hilbert algebra. ∎

3. Generalized Hilbert algebra induced by a quasi ordered set

From now one in this paper, G_H denote a g-Hilbert algebra, unless otherwise mentioned.

Proposition 3.1. *Let relation $\preceq$ on G_H be defined as follows:*

$$x \preceq y \quad \textit{if and only if} \quad x \rightarrow y = 1$$

Then "$\preceq$" is a quasi order relation.

Proof. Reflexive condition is clear. Now, we should prove the transitive condition. Let $x, y, z \in G_H$. If $x \preceq y$ and $y \preceq z$, then $x \rightarrow y = 1$ and $y \rightarrow z = 1$ and so by $(GH1)$ and $(GH4)$,

$$x \rightarrow z = 1 \rightarrow (x \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z) = x \rightarrow (y \rightarrow z) = x \rightarrow 1 = 1$$

Then $x \preceq z$. ∎

Proposition 3.2. *Let $x \preceq y$, for $x, y \in G_H$. Then, for all $z \in G_H$,*

(i) $y \rightarrow z \preceq x \rightarrow z$,

(ii) $z \to x \preceq z \to y$.

Proof. (i) Since $x \to y = 1$, then

$$\begin{aligned}(y \to z) \to (x \to z) &= 1 \to ((y \to z) \to (x \to z))\\ &= (x \to y) \to ((y \to z) \to (x \to z))\\ &= (y \to z) \to ((x \to y) \to (x \to z))\\ &= (y \to z) \to (x \to (y \to z))\\ &= x \to ((y \to z) \to (y \to z))\\ &= x \to 1 = 1\end{aligned}$$

Hence, $y \to z \preceq x \to z$.

(ii) Since $x \to y = 1$, then by (GH4),

$$(z \to x) \to (z \to y) = z \to (x \to y) = z \to 1 = 1$$

Hence, $z \to x \preceq z \to y$. ∎

We define Θ on G_H as follows:

$$x\Theta y \Longleftrightarrow x \preceq y, y \preceq x$$

Then, Θ is a congruence relation on G_H. It is clear that Θ is an equivalence relation on G_H. Let $x, y, u, v \in G_H$, such that $x\Theta y$ and $u\Theta v$. Then $x \preceq y$, $y \preceq x$, $u \preceq v$ and $v \preceq u$. By Proposition 3.2, we obtain $x \to u \preceq x \to v$ and $x \to v \preceq y \to v$. Now, by transitivity of $\preceq$, we get $x \to u \preceq y \to v$. Similarly, we have $y \to v \preceq x \to u$ and so Θ is a congruence relation on G_H.

Now, let $\frac{G_H}{\Theta} = \{[x]_\Theta | x \in G_H\}$ and $\ll$ on $\frac{G_H}{\Theta}$ is defined as follows:

$$[x] \ll [y] \Longleftrightarrow x\Theta y.$$

It is clear that $(\frac{G_H}{\Theta}, \ll)$ is a poset.

Furthermore, $(\frac{G_H}{\Theta}, \ll, [1]_\Theta)$ is a g-Hilbert algebra with the following operation,

$$[x]_\Theta \to [y]_\Theta = [x \to y]_\Theta$$

Theorem 3.3. *Suppose that (P, θ) is a quasi ordered set, $1 \notin P$ and $G_H = P \cup \{1\}$. Let "$\to$" on G_H is defined as follows:*

$$x \to y = \begin{cases} 1 & , \quad x\theta y,\\ y & , \quad x \not\theta y.\end{cases}$$

Then $(G_H, \to, 1)$ is a g-Hilbert algebra.

Proof. Since θ is reflexive, obviously $x \to x = 1$, for all $x \in G_H$. Since $1 \notin P$, then $1 \not\theta x$ for every $x \in G_H$ and so $1 \to x = x$. Hence we have (GH1) and (GH2). Now, we should prove (GH3). Let $x, y, z \in G_H$. We consider the following cases:

Case 1. $y \not\theta x$ and $z \not\theta x$:

$$z \to (y \to x) = z \to x = x = y \to x = y \to (z \to x).$$

Case 2. $y\theta x$ and $z \not\theta x$:

$$z \to (y \to x) = z \to 1 = 1 = y \to x = y \to (z \to x).$$

Case 3. $y \not\theta x$ and $z\theta x$:

$$z \to (y \to x) = z \to x = 1 = y \to 1 = y \to (z \to x).$$

Case 4. $y\theta x$ and $z\theta x$:

$$z \to (y \to x) = z \to 1 = 1 = y \to 1 = y \to (z \to x).$$

Hence, we have (GH3).

Now, we will prove (GH4). Let $x, y, z \in G_H$. Then, we consider the following cases:

Case 1. $z\theta y$ and $z\theta x$:

If $x\theta y$, then:

$$(z \to y) \to (z \to x) = 1 \to 1 = 1 = z \to 1 = z \to (y \to x).$$

If $x \not\theta y$, then:

$$(z \to y) \to (z \to x) = 1 \to 1 = 1 = z \to x = z \to (y \to x).$$

Case 2. $z \not\theta y$ and $z\theta x$:

If $x\theta y$, then:

$$(z \to y) \to (z \to x) = y \to 1 = 1 = z \to 1 = z \to (y \to x).$$

If $x \not\theta y$, then:

$$(z \to y) \to (z \to x) = y \to 1 = 1 = z \to x = z \to (y \to x).$$

Case 3. $z\theta y$ and $z \not\theta x$:

If $x\theta y$, then by transitive condition $z\theta x$, which is not impossible.

If $x \not\theta y$, then:

$$(z \to y) \to (z \to x) = 1 \to x = x = z \to x = z \to (y \to x).$$

Case 4. $z \not\theta y$ and $z \not\theta x$:

If $x\theta y$, then:

$$(z \to y) \to (z \to x) = y \to x = 1 = z \to 1 = z \to (y \to x).$$

If $x \not\theta y$, then

$$(z \to y) \to (z \to x) = y \to x = x = z \to x = z \to (y \to x).$$

Hence, we have $(GH4)$. Therefore, G_H is a g-Hilbert algebra. ∎

4. Relation between generalized Hilbert algebras and implication algebras

Definition 4.1. [1] An *implication algebra* is a set X with a binary operation "$\to$" which satisfies the following axioms:

(I1) $(x \to y) \to x = x$,

(I2) $(x \to y) \to y = (y \to x) \to x$,

(I3) $x \to (y \to z) = y \to (x \to z)$,

for all $x, y, z \in X$.

In any implication algebra $(X, \to)$, we have

(I4) $x \to (x \to y) = x \to y$,

(I5) $x \to x = y \to y$,

(I6) there exists a unique element 1 in X such that, for all $x \in X$,

- (a) $x \to x = 1$, $1 \to x = x$ and $x \to 1 = 1$,
- (b) if $x \to y = 1$ and $y \to x = 1$ then $x = y$,
- (c) $x \to (y \to x) = 1$

for all $x, y \in X$.

Definition 4.2. G_H is called *commutative* if for all $x, y \in G_H$,

$$(y \to x) \to x = (x \to y) \to y.$$

Lemma 4.3. *Let G_H be commutative. If $x \to y = y \to x = 1$, then $x = y$.*

Proof. Let $x \to y = y \to x = 1$, for $x, y \in G$. Since G_H is commutative, then by (GH1),

$$x = 1 \to x = (y \to x) \to x = (x \to y) \to y = 1 \to y = y.$$ ∎

Lemma 4.4. [1] *Let $(X, \to, 1)$ be an implication algebra. Then,*

(i) $x \le y$, *imply* $y \to z \le x \to z$

(ii) $x \le y$, *imply* $z \to x \le z \to y$

(iii) $x \to y \le (y \to z) \to (x \to z)$ *and* $y \to z \le (x \to y) \to (x \to z)$.

Theorem 4.5. $(X, \to, 1)$ *is an implication algebra if and only if* $(X, \to, 1)$ *is a commutative g-Hilbert algebra.*

Proof. $(\Rightarrow)$ Let $(X, \to, 1)$ be an implication algebra. By Theorem 2.6, it is enough to prove that X is a Hilbert algebra. By (I6)(b) and (c), we have (H1) and (H3). It is enough to prove (H2). Let $x, y, z \in X$. Then, by (I4), (I3) and Lemma 4.4,

$$\begin{aligned}
&(x \to (y \to z)) \to ((x \to y) \to (x \to z)) \\
&\qquad = (x \to (y \to z)) \to ((x \to y) \to (x \to (x \to z))), \\
&\qquad \geq (x \to (y \to z)) \to (y \to (x \to z))), \\
&\qquad = (x \to (y \to z)) \to (x \to (y \to z))), \\
&\qquad = 1.
\end{aligned}$$

Hence, by (I6)(a),(b), we have $(x \to (y \to z)) \to ((x \to y) \to (x \to z))) = 1$, and so (H2) is hold. Hence, $(X, \to, 1)$ is a Hilbert algebra and so, by Theorem 2.6, it is a g-Hilbert algebra. Moreover, by (I2) it is commutative.

$(\Leftarrow)$ Let $(X, \to, 1)$ be a commutative g-Hilbert algebra. Since X is commutative, then we have (I2). Moreover, by (GH3), we have (I3). Now, it is enough to prove that (I1). Let $x, y \in X$. Then, by (GH3), (GH2) and Proposition 2.8(i),

$$x \to ((x \to y) \to x) = (x \to y) \to (x \to x) = (x \to y) \to 1 = 1$$

Hence,

$$x \to ((x \to y) \to x) = 1. \tag{1}$$

Moreover,

$$\begin{aligned}
((x \to y) \to x) \to x &= (x \to (x \to y)) \to (x \to y), \quad \text{Since } X \text{ is commutative} \\
&= ((x \to x) \to (x \to y)) \to (x \to y), \quad \text{by (GH4)} \\
&= (1 \to (x \to y)) \to (x \to y), \quad \text{by (GH2)} \\
&= (x \to y) \to (x \to y) \\
&= 1.
\end{aligned}$$

Hence,

$$((x \to y) \to x) \to x = 1, \tag{2}$$

and so, by (1), (2) and Lemma 4.3 we have (I1). Therefore, $(X, \to, 1)$ is an implication algebra. ∎

Example 4.6. Let $G_H = \{a, b, c, 1\}$ and operation "$\to$" on G_H is defined as follows:

$\to$	a	b	c	1
a	1	1	1	1
b	1	1	1	1
c	1	1	1	1
1	a	b	c	1

Then, $(G_H, \to, 1)$ is a g-Hilbert algebra which is not an implication algebra, since $(b \to c) \to c \neq (c \to b) \to b$. Hence, a commutative condition is necessary in the last theorem.

5. Lattice structure on commutative generalized Hilbert algebras

Proposition 5.1. *If G_H is commutative, then*

(i) $(x \to y) \to x = x$,

(ii) $x \to (x \to y) = x \to y$,

for all $x, y \in G_H$.

Proof. (i) Let $x, y \in G_H$. Then,

$$\begin{aligned} x \to ((x \to y) \to x) &= (x \to (x \to y)) \to (x \to x), \quad \text{(by (GH4))} \\ &= (x \to (x \to y)) \to 1, \quad \text{(by (GH2))} \\ &= 1 \end{aligned}$$

On the other hand,

$$\begin{aligned} ((x \to y) \to x) \to x &= (x \to (x \to y)) \to (x \to y), \quad \text{(since } G_H \text{ is commutative)} \\ &= ((x \to x) \to (x \to y)) \to (x \to y), \quad \text{(by (GH4))} \\ &= (1 \to (x \to y)) \to (x \to y), \quad \text{(by (GH2))} \\ &= (x \to y) \to (x \to y), \quad \text{(by (GH1))} \\ &= 1 \quad \text{(by (GH2))}. \end{aligned}$$

Hence, by Lemma 4.3 we obtain $(x \to y) \to x = x$.

(ii) By using (i) twice, we have

$$x \to (x \to y) = ((x \to y) \to x) \to (x \to y) = x \to y. \qquad \blacksquare$$

Corollary 5.2. *Let G_H be commutative and relation $\preceq$ on G_H, is defined by $x \preceq y$ if and only if $x \to y = 1$. Then, " $\preceq$ " is a partial order on G_H.*

Proof. By (GH2), Proposition 5.1(i) we get that $\preceq$ is reflexive and anti symmetric. Let $x \to y = 1$ and $y \to z = 1$, then

$$x \to z = 1 \to (x \to z) = (x \to y) \to (x \to z) = x \to (y \to z) = x \to 1 = 1.$$

Thus, $\preceq$ is a partial order on G_H. $\blacksquare$

Proposition 5.3. *For any $p \in G_H$, $B(p)$ is a subalgebra of G_H.*

Proof. It is clear that $1 \in B(p)$. Now, let $a, b \in B(p)$. Then, $p \preceq a$ and $p \preceq b$ and so by (GH4)and (GH2),

$$p \to (a \to b) = (p \to a) \to (p \to b) = 1 \to 1 = 1.$$

Hence, $p \preceq (a \to b)$ and so $a \to b \in B(p)$. Therefore, $B(p)$ is a subalgebra of G_H. $\blacksquare$

Theorem 5.4. *If G_H is commutative, then the following are hold:*

(i) $(G_H, \vee)$ *is a* $\vee-$*semi lattice, when* $a \vee b = (a \to b) \to b$, *for any* $a, b \in G_H$,

(ii) *For any* $p \in G_H$, $(B(p), \wedge)$ *is a* $\wedge-$*semi lattice, when* $a \wedge b = ((a \to p) \vee (b \to p)) \to p$, *for any* $a, b \in B(p)$,

(iii) *For any* $p \in G_H$, $(B(p), \wedge, \vee)$ *is a complemented lattice.*

Proof. By Corollary 5.2, $(G_H, \preceq)$ and so $(B(p), \preceq)$, for any $p \in G_H$ is a partial ordered set. (i) Let $a, b \in G_H$. First we should prove that $(a \to b) \to b$ is an upper bound of a, b. By (GH3) and (GH2), $a \to ((a \to b) \to b) = (a \to b) \to (a \to b) = 1$, and so $a \preceq (a \to b) \to b$. Moreover, by (GH3), $b \preceq (a \to b) \to b$. Hence, $(a \to b) \to b$ is an upper bound of a, b. Now, let $c \in G$ such that $a, b \preceq c$. Then $a \to c = 1$ and so by commutative condition and (GH1),

$$(c \to a) \to a = (a \to c) \to c = 1 \to c = c. \tag{1}$$

Hence,

$$\begin{aligned}
&((a \to b) \to b) \to c \\
&\qquad = ((a \to b) \to b) \to ((c \to a) \to a), \quad \text{(by (1))} \\
&\qquad = (c \to a) \to (((a \to b) \to b) \to a), \quad \text{(by (GH3))} \\
&\qquad = (c \to a) \to (((b \to a) \to a) \to a), \quad \text{(by commutative condition)} \\
&\qquad = (c \to a) \to ((a \to (b \to a)) \to (b \to a)), \quad \text{(by commutative condition)} \\
&\qquad = (c \to a) \to ((b \to (a \to a)) \to (b \to a)), \quad \text{(by (GH3))} \\
&\qquad = (c \to a) \to ((b \to 1) \to (b \to a)), \quad \text{(by (GH2))} \\
&\qquad = (c \to a) \to (1 \to (b \to a)), \quad \text{(by Proposition 2.8(i))} \\
&\qquad = (c \to a) \to (b \to a), \quad \text{(by } (GH1)) \\
&\qquad = b \to ((c \to a) \to a), \quad \text{(by (GH3))} \\
&\qquad = b \to c, \quad \text{(by (1))} \\
&\qquad = 1 \quad \text{(since } b \preceq c).
\end{aligned}$$

Therefore, $(a \to b) \to b \preceq c$ and so $a \vee b = (a \to b) \to b$. Hence, $(G_H, \vee)$ is a $\vee-$semi lattice.

(ii) Let $p \in G_H$ and $a, b \in B(p)$. Then $p \preceq a, b$.
Since $a \to p \preceq (a \to p) \vee (b \to p)$ then, by Proposition 3.2,

$$((a \to p) \vee (b \to p)) \to p \preceq (a \to p) \to p = a \vee p = a.$$

Similarly, we can prove that $((a \to p) \vee (b \to p)) \to p \preceq b$. Hence, $((a \to p) \vee (b \to p)) \to p$ is a lower bound of a and b. Now, let $c \in G_H$ such that $c \preceq a, b$. Then, by Proposition 3.2, $a \to p \preceq c \to p$ and $b \to p \preceq c \to p$ and so $((a \to p) \vee (b \to p)) \preceq c \to p$. Hence, by Proposition 3.2, $c \preceq c \vee p = (c \to p) \to$

$p \preceq [a \to p) \vee (b \to p)] \to p$. Therefore, $((a \to p) \vee (b \to p)) \to p$ is a greatest lower bound of a and b and so

$$a \wedge b = ((a \to p) \vee (b \to p)) \to p.$$

Now, since $a \wedge b \in B(p)$, then $(B(p), \wedge)$ is a $\wedge-$semilattice.

(iii) Let $p \in G_H$. Then, for any $a \in B(p)$,

$$\begin{aligned} (a \to p) \vee a &= ((a \to p) \to a) \to a \\ &= (a \to (a \to p)) \to (a \to p), \quad \text{(by commutative condition)} \\ &= (a \to p) \to (a \to p), \quad \text{(by Proposition 2.8(v))} \\ &= 1, \quad \text{(by } (GH2)) \end{aligned}$$

Moreover, by commutative condition,

$$\begin{aligned} a \wedge (a \to p) &= ((a \to p) \vee ((a \to p) \to p)) \to p \\ &= ((a \to p) \vee ((p \to a) \to a)) \to p \\ &= ((a \to p) \vee (p \vee a)) \to p \\ &= (a \to p) \vee a) \to p \\ &= 1 \to p \\ &= p \end{aligned}$$

Therefore, $(B(p), \wedge, \vee)$ is a complemented lattice. ∎

Lemma 5.5. *Let G_H be a commutative g-Hilbert algebra and $p \in G_H$. Then, for any $a, b \in B(p)$,*

$$(a \to p) \vee (b \to p) = (a \wedge b) \to p.$$

Proof. Let $p \in G_H$ and $a, b \in B(p)$. Since $a \wedge b \preceq a$ and $a \wedge b \preceq b$, then, by Proposition 3.2, $a \to p \preceq (a \wedge b) \to p$ and $b \to p \preceq (a \wedge b) \to p$ and so $(a \wedge b) \to p$ is an upper bound of $a \to p$ and $b \to p$. Now, let $u \in B(p)$ be an other upper bound of $a \to p$ and $b \to p$. Then, $a \to p \preceq u$ and $b \to p \preceq u$ and so, by Proposition 3.2, $u \to p \preceq (a \to p) \to p$ and $u \to p \preceq (b \to p) \to p$. Hence, $u \to p \preceq a \vee p = a$ and $u \to p \preceq b \vee p = b$ and so $u \to p \preceq a \wedge b$. Thus,

$$a \wedge b \to p \preceq (u \to p) \to p = u \vee p = u$$

Therefore, $a \wedge b \to p$ is last upper bound of $a \to p$ and $b \to p$, that is,

$$(a \to p) \vee (b \to p) = (a \wedge b) \to p.$$ ∎

Theorem 5.6. *If G_H is commutative, then for any $p \in G_H$, $(B(p), \vee, \wedge)$ is a distributive lattice.*

Proof. Let $p \in G_H$ and $a, b, c \in B(p)$. We have to prove that $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. Since $p \preceq b$, then by Proposition 3.2, $a \to p \preceq a \to b$. Moreover, by Proposition 2.8(vi), $b \preceq a \to b$ and so $(a \to p) \vee b \preceq a \to b$. Let $c = (a \to p) \vee b$. Then

$$(1) \qquad c \preceq a \to b \text{ and } b \preceq c \text{ and so } c = b \vee c = (c \to b) \to b.$$

Moreover, by the proof of Theorem 5.4(iii),

$$(2) \qquad (a \to c) \to c = a \vee c = a \vee (a \to p) \vee b = 1 \vee b = 1.$$

Moreover,

$$\begin{aligned}
(a \to b) \to c &= (a \to b) \to (1 \to c), \quad \text{(by (GH1))} \\
&= (a \to b) \to (((a \to c) \to c) \to c), \quad \text{(by (2))} \\
&= (a \to b) \to ((a \to c) \vee c), \\
&= (a \to b) \to (a \to c), \quad \text{(by Proposition 2.8(vi))} \\
&= (a \to b) \to (a \to ((c \to b) \to b), \quad \text{(by (1))} \\
&= (a \to b) \to ((c \to b) \to (a \to b)), \quad \text{(by (GH3))} \\
&= (c \to b) \to ((a \to b) \to (a \to b)), \quad \text{(by (GH3))} \\
&= (c \to b) \to 1, \quad \text{(by (GH1))} \\
&= 1.
\end{aligned}$$

and this implies that $a \to b \preceq c$. Hence, by (1) and Proposition 5.1(i),

$$(2) \qquad a \to b = c = (a \to p) \vee b.$$

Now, since (2) holds for any $a, b \in B(p)$ and since $B(p)$ is a subalgebra, then $a, b \to p \in B(p)$ and so, by (2),

$$(3) \qquad (a \to (b \to p)) \to p = ((a \to p) \vee (b \to p)) \to p = a \wedge b.$$

Hence,

$$\begin{aligned}
a \to (a \wedge b) &= a \to ((a \to (b \to p)) \to p), \quad \text{(by (3))} \\
&= (a \to (b \to p)) \to (a \to p), \quad \text{(by Proposition 2.8(v))} \\
&= (b \to (a \to p)) \to (a \to p), \quad \text{(by (GH3))} \\
&= b \vee (a \to p), \\
&= a \to b, \quad \text{(by (2))}
\end{aligned}$$

and this implies that

$$(4) \qquad a \to (a \wedge b) = a \to b.$$

Now, let $k = (a \wedge b) \vee (a \wedge c)$. Since, $a \wedge b \preceq k$, then by Proposition 3.2, (GH1), (GH2), (GH3) and (4),

$$1 = a \to 1 = a \to (b \to b) = b \to (a \to b) = b \to (a \to (a \wedge b)) \preceq b \to (a \to k)$$

and so, by Proposition 2.8(i), $b \to (a \to k) = 1$ and this implies that $b \preceq a \to k$. Similarly, $c \preceq a \to k$ and so

$$b \vee c \preceq a \to k. \tag{5}$$

and, by Proposition 3.2,

$$(a \to k) \to k \preceq (b \vee c) \to k. \tag{6}$$

Now, since $a \wedge b \preceq a$ and $a \wedge c \preceq a$, then $k = (a \wedge b) \vee (a \wedge c) \preceq a$ and so $k \to a = 1$. Hence, by (6) and the commutative condition,

$$a = 1 \to a = (k \to a) \to a = (a \to k) \to k \preceq (b \vee c) \to k. \tag{7}$$

Hence, by (5), (7) and the commutative property,

$$(a \to k) \vee ((b \vee c) \to k) \succeq (a \to k) \vee a = ((a \to k) \to a) \to a = a \to a = 1.$$

Now, since $(a \to k) \vee ((b \vee c) \to k) \preceq 1$, then, by Proposition 5.1(i),

$$(a \to k) \vee (b \vee c) \to k) = 1. \tag{8}$$

Moreover, since $a \wedge b \preceq b \vee c$ and $a \wedge c \preceq b \vee c$, then $k = (a \wedge b) \vee (a \wedge c) \preceq b \vee c$. Now, since we proved that $k \preceq a$. Hence by (8),

$$a \wedge (b \vee c) = ((a \to k) \vee ((b \vee c) \to k)) \to k = 1 \to k = k = (a \wedge b) \vee (a \wedge c).$$

Therefore, $(B(p), \wedge, \vee)$ is a distributive Lattice. ∎

Corollary 5.7. *If G_H is commutative, then for any $p \in G_H$, $(B(p), \vee, \wedge)$ is a Boolean lattice.*

Proof. By Theorems 5.4 and 5.6, the proof is clear. ∎

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m^{th} POWER SYMMETRIC n-SIGRAPHS

R. Rangarajan

Department of Studies in Mathematics
University of Mysore
Manasagangotri, Mysore 570 006
India

P. Siva Kota Reddy

Department of Mathematics
Acharya Institute of Technology
Soladevanahalli, Bangalore-560 090
India
e-mail: reddy_math@yahoo.com

N.D. Soner

Department of Studies in Mathematics
University of Mysore
Manasagangotri, Mysore 570 006
India

Abstract. An *n-tuple* $(a_1, a_2, ..., a_n)$ is *symmetric*, if $a_k = a_{n-k+1}$, $1 \leq k \leq n$. A *symmetric n-sigraph* (*symmetric n-marked graph*) is an ordered pair $S_n = (G, \sigma)$ ($S_n = (G, \mu)$), where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\sigma : E \to H_n$ ($\mu : V \to H_n$) is a function. The m^{th} *power graph* of a graph $G = (V, E)$ is a graph $G^m = (V, E')$, with same vertex set as G, and has two vertices u and v are adjacent if their distance in G is m or less. Analogously, one can define the m^{th} *power symmetric n-sigraph* S_n^m of a symmetric n-sigraph $S_n = (G, \sigma)$ as a symmetric n-sigraph, $S_n^m = (G^m, \sigma')$, where G^m is the underlying graph of S_n^m, and for any edge $e = uv$ in S_n^m, $\sigma'(e) = \mu(u)\mu(v)$, where for any $v \in V$,

$$\mu(v) = \prod_{u \in N(v)} \sigma(uv).$$

It is shown that for any symmetric n-sigraph S_n, its m^{th} power symmetric n-sigraph S_n^m is i-balanced. We then give structural characterization of m^{th} power symmetric n-sigraphs. Further, we obtain some switching equivalence relationship between m^{th} power symmetric n-sigraph and line symmetric n-sigraph.

Keywords and phrases: symmetric n-sigraphs, symmetric n-marked graphs, balance, switching, m^{th} power symmetric n-sigraph, line symmetric n-sigraphs, complementary symmetric n-sigraphs, complementation.

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1. Introduction

For standard terminology and notion in graph theory we refer the reader to Harary [2]; the non-standard will be given in this paper as and when required. We treat only finite simple graphs without self loops and isolates.

Let $n \geq 1$ be an integer. An *n-tuple* $(a_1, a_2, ..., a_n)$ is *symmetric* if $a_k = a_{n-k+1}$, $1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, ..., a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. Note that H_n is a group under coordinate wise multiplication, and the order of H_n is 2^m, where $m = \lceil \frac{n}{2} \rceil$.

A *symmetric n-sigraph* (*symmetric n-marked graph*) is an ordered pair $S_n = (G, \sigma)$ ($S_n = (G, \mu)$), where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\sigma : E \to H_n$ ($\mu : V \to H_n$) is a function.

In this paper by an *n-tuple/n-sigraph/n-marked graph* we always mean a symmetric n-tuple/symmetric n-sigraph/symmetric n-marked graph.

An n-tuple $(a_1, a_2, ..., a_n)$ is the *identity n-tuple*, if $a_k = +$, for $1 \leq k \leq n$, otherwise it is a *non-identity n-tuple*. In an n-sigraph $S_n = (G, \sigma)$ an edge labelled with the identity n-tuple is called an *identity edge*, otherwise it is a *non-identity edge*.

Further, in an n-sigraph $S_n = (G, \sigma)$, for any $A \subseteq E(G)$ the *n-tuple* $\sigma(A)$ is the product of the n-tuples on the edges of A.

In [6], the authors defined two notions of balance in n-sigraph $S_n = (G, \sigma)$ as follows (see, also, R. Rangarajan and P. Siva Kota Reddy [3]):

Definition. Let $S_n = (G, \sigma)$ be an n-sigraph. Then,

(i) S_n is *identity balanced* (or *i-balanced*), if product of n-tuples on each cycle of S_n is the identity n-tuple, and

(ii) S_n is *balanced*, if every cycle in S_n contains an even number of non-identity edges.

Note: An i-balanced n-sigraph need not be balanced and conversely.

The following characterization of i-balanced n-sigraphs is obtained in [6].

Proposition 1 (**E. Sampathkumar et al. [6]**) *An n-sigraph $S_n = (G, \sigma)$ is i-balanced if, and only if, it is possible to assign n-tuples to its vertices such that the n-tuple of each edge uv is equal to the product of the n-tuples of u and v.*

In [6], the authors also have defined switching and cycle isomorphism of an n-sigraph $S_n = (G, \sigma)$ as follows:

Let $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$, be two n-sigraphs. Then S_n and S'_n are said to be *isomorphic*, if there exists an isomorphism $\phi : G \to G'$ such that if uv is an edge in S_n with label $(a_1, a_2, ..., a_n)$ then $\phi(u)\phi(v)$ is an edge in S'_n with label $(a_1, a_2, ..., a_n)$.

Given an n-marking μ of an n-sigraph $S_n = (G, \sigma)$, *switching* S_n with respect to μ is the operation of changing the n-tuple of every edge uv of S_n by

$\mu(u)\sigma(uv)\mu(v)$. The n-sigraph obtained in this way is denoted by $\mathcal{S}_\mu(S_n)$ and is called the *μ-switched n-sigraph* or just *switched n-sigraph*.

Further, an n-sigraph S_n *switches* to n-sigraph S'_n (or that they are *switching equivalent* to each other), written as $S_n \sim S'_n$, whenever there exists an n-marking of S_n such that $\mathcal{S}_\mu(S_n) \cong S'_n$.

Two n-sigraphs $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$ are said to be *cycle isomorphic*, if there exists an isomorphism $\phi : G \to G'$ such that the n-tuple $\sigma(C)$ of every cycle C in S_n equals to the n-tuple $\sigma(\phi(C))$ in S'_n.

We make use of the following known result (see [6]).

Proposition 2 (**E. Sampathkumar et al. [6]**) *Given a graph G, any two n-sigraphs with G as underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

2. m^{th} Power n-sigraph

The *m^{th} power graph G^m* of a graph G is defined in [1] as follows: The m^{th} power graph has same vertex set as G, and has two vertices u and v are adjacent if their distance in G is n or less.

In this paper, we introduce a natural extension of the notion of m^{th} power graphs to the realm of n-sigraphs: Consider the n-marking μ on vertices of S_n defined as follows: for each vertex $v \in V$, $\mu(v)$ is the product of the n-tuples on the edges incident at v. The *m^{th} power n-sigraph* of S_n is an n-sigraph $S_n^m = (G^m, \sigma')$, where G^m is the underlying graph of S^m, where for any edge $e = uv \in G^m$, $\sigma'(uv) = \mu(u)\mu(v)$.

Hence, we shall call a given n-sigraph $S_n = (G, \sigma)$ a *m^{th} power n-sigraph* if it is isomorphic to the m^{th} power sigraph $(S'_n)^m = ((G')^m, \sigma')$ of some n-sigraph S'_n.

In the following subsection, we shall present a characterization n-sigraphs which are m^{th} power n-sigraphs.

2.1. Switching invariant m^{th} power n-sigraphs

The following result indicates the limitations of the notion of m^{th} power n-sigraphs as introduced above, since the entire class of i-unbalanced n-sigraphs is forbidden to m^{th} power n-sigraphs.

Proposition 3 *For any n-sigraph $S_n = (G, \sigma)$, its m^{th} power n-sigraph S_n^m is i-balanced.*

Proof. Let σ' denote the labelling of S_n^m. Then by definition of S_n^m, we see that $\sigma'(uv) = \mu(u)\mu(v)$, for every edge uv of S_n^m and hence, by Proposition 1, the result follows. ∎

The following result characterizes n-sigraphs which are m^{th} power n-sigraphs.

Proposition 4 *An n-sigraph $S_n = (G, \sigma)$ is a m^{th} power n-sigraph if, and only if, S_n is i-balanced n-sigraph and its underlying graph G is a m^{th} power graph.*

Proof. Suppose that S_n is i-balanced and G is a m^{th} power graph. Then there exists a graph H such that $H^m \cong G$. Since S_n is i-balanced, by Proposition 1, there exists an n-marking μ of G such that each edge uv in S_n satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the n-sigraph $S'_n = (H, \sigma')$, where for any edge e in H, $\sigma'(e)$ is the n-marking of the corresponding vertex in G. Then clearly, $(S'_n)^m \cong S_n$. Hence S_n is a m^{th} power n-sigraph.

Conversely, suppose that $S_n = (G, \sigma)$ is a m^{th} power n-sigraph. Then there exists an n-sigraph $S'_n = (H, \sigma')$ such that $(S'_n)^m \cong S_n$. Hence G is the m^{th} power graph of H and by Proposition 3, S_n is i-balanced. ∎

For any positive integer k, the k^{th} iterated m^{th} power n-sigraph, $(S_n^m)^k$ of S_n is defined as follows:

$$(S_n^m)^0 = S_n,\ (S_n^m)^k = S_n^m((S_n^m)^{k-1}).$$

Corollary 3.1. *For any n-sigraph $S_n = (G, \sigma)$ and any positive integer k, $(S_n^m)^k$ is i-balanced.*

The *line n-sigraph* $L(S_n)$ of an n-sigraph $S_n = (G, \sigma)$ is defined as follows (See [7]): $L(S_n) = (L(G), \sigma')$, where for any edge ee'in $L(G)$, $\sigma'(ee') = \sigma(e)\sigma(e')$.

Proposition 5 (**E. Sampathkumar et al.** [**7**]) *For any n-sigraph $S_n = (G, \sigma)$, its line n-sigraph $L(S_n)$ is i-balanced.*

For any positive integer k, the k^{th} iterated line n-sigraph, $L^k(S_n)$ of S_n is defined as follows:

$$L^0(S_n) = S_n,\ L^k(S_n) = L(L^{k-1}(S_n)).$$

Corollary 5.1. *For any n-sigraph $S_n = (G, \sigma)$ and for any positive integer k, $L^k(S_n)$ is i-balanced.*

3. Switching equivalence of line n-sigraphs and m^{th} power n-sigraphs

We now characterize n-sigraphs whose line n-sigraphs and its m^{th} power n-sigraphs are switching equivalent. In the case of graphs the following result is due to J. Akiyama et al. [1].

Proposition 6 (**J. Akiyama et al.** [**1**]) *For any $m \geq 2$, the solutions to the equation $L(G) \cong G^m$ are graphs $G = pK_3$, where p is an arbitrary integer.*

Proposition 7 *For any n-sigraph $S_n = (G, \sigma)$, $L(S_n) \sim S_n^m$, where $m \geq 2$ if, and only if, G is pK_3, where p is an arbitrary integer.*

Proof. Suppose $L(S_n) \sim S_n^m$. This implies, $L(G) \cong G^m$ and hence by Proposition-18, we see that the graph G must be isomorphic to pK_3.

Conversely, suppose that G is mK_3. Then $L(G) \cong G^m$ by Proposition-18. Now, if S_n is an n-sigraph with underlying graph as pK_3, by Proposition-5 and 3, $L(S_n)$ and S_n^m are i-balanced and hence, the result follows from Proposition-2. ∎

Note that for $m = 1$, in the above Proposition is reduced to the following result of R. Rangarajan et al. [4].

Proposition 8 (**R. Rangarajan et al.** [**4**]) *For any n-sigraph $S_n = (G, \sigma)$, $L(S_n) \sim S_n$ if, and only if, S_n is an i-balanced n-sigraph which is 2-regular.*

4. Complementation

In this section, we investigate the notion of complementation of a graph whose edges have signs (a *sigraph*) in the more general context of graphs with multiple signs on their edges. We look at two kinds of complementation: complementing some or all of the signs, and reversing the order of the signs on each edge.

For any $t \in H_n$, the *t-complement* of $a = (a_1, a_2, ..., a_n)$ is: $a^t = at$. For any $T \subseteq H_n$, and $t \in H_n$, the *t-complement* of T is $T^t = \{a^t : a \in T\}$.

For any $t \in H_n$, the *t-complement* of an n-sigraph $S_n = (G, \sigma)$, written $(S_n^m)^t$, is the same graph but with each edge label $a = (a_1, a_2, ..., a_n)$ replaced by a^t.

For an n-sigraph $S_n = (G, \sigma)$, the S_n^m is i-balanced (Proposition 3) and $L(S_n)$ is also i-balanced (Proposition 5). We now examine, the conditions under which t-complement of S_n^m and $L(S_n)$ are i-balanced, where for any $t \in H_n$.

Proposition 9 *Let $S_n = (G, \sigma)$ be an n-sigraph. Then, for any $t \in H_n$,*

(i) *If G^m is bipartite then $(S_n^m)^t$ is i-balanced.*

(ii) *If $L(G)$ is bipartite then $(L(S_n))^t$ is i-balanced.*

Proof. (i) Since, by Proposition 3, S_n^m is i-balanced, for each k, $1 \leq k \leq n$, the number of n-tuples on any cycle C in S_n^m whose k^{th} co-ordinate are $-$ is even. Also, since G^m is bipartite, all cycles have even length; thus, for each k, $1 \leq k \leq n$, the number of n-tuples on any cycle C in S_n^m whose k^{th} co-ordinate are $+$ is also even. This implies that the same thing is true in any t-complement, where for any $t \in H_n$. Hence $(S_n^m)^t$ is i-balanced.

Similarly (ii) follows. ∎

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COMMON FIXED POINT THEOREMS FOR FINITE NUMBER OF MAPPINGS WITHOUT CONTINUITY AND COMPATIBILITY ON UNIFORMLY CONVEX BANACH SPACE

Sushil Sharma

Alok Pande

Chetna Kothari

Department of Mathematics
Madhav Science College
Vikram University
Ujjain-456010
India
e-mail: sksharma2005@yahoo.com

Abstract. The purpose of this paper is to prove some common fixed point theorems for finite number of discontinuous, noncompatible mappings on noncomplete uniformly convex Banach space. Our results extend, generalize several known results of fixed point theory in different spaces. We give an example and also give formulas for total number of commutativity conditions for finite number of mappings.

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Keywords: noncompatible mappings, common fixed points, Banach space, weak compatible mappings.

1. Introduction

Husain and Sehgal [2] proved common fixed point theorems for a family of mappings. Khan and Imdad [8] extended result of Husain and sehgal [2] and proved fixed point theorems for a class of mappings. Imdad, Khan and Sessa [3] extended above results and proved common fixed points for three mappings defined on a closed subset of a uniformly convex Banach space.

Rashwan [9] extended result of Imdad, khan and Sessa [3] by employing four compatible mappings of type (A) instead of weakly commuting mappings and by using one continuous mapping as opposed to two.

Sharma and Bamboria [11] improved results of Rashwan [9] by removing the condition of continuity and replacing the compatibility of mappings of type (A) by weak compatibility.

Sharma and Tilwankar [12] proved a common fixed point theorem for four mappings under the condition of weak compatible mappings by using the new property (E.A). For the study of discontinuous and noncompatible mappings in fixed point theory we refer to Sharma and Deshpande [13] and Sharma, Deshpande and Tiwari [14].

Several observations motivated us to prove common fixed point theorem for ten noncompatible, discontinuous mappings in noncomplete uniformly convex Banach space. We also extend our results for finite number of mappings. Our main theorems extend, improve, generalize some known results in uniformly convex Banach space. We give an example to validate our result.

To prove existence of common fixed point for finite number of mappings some commutativity conditions are required. How many commutativity conditions are necessary? We give answer of this question by giving formulas.

Throughout the paper X stands for a Banach space. Let R^+ denote the set of all non-negative real numbers and F be the family of mappings f from $(R^+)^5$ into R^+ such that each f is upper-semicontinuous, non-decreasing in each coordinate variable.

The modulus of convexity of X is a function δ from $(0,2]$ into $(0,1]$ defined by

$$\delta(\varepsilon)=\inf\left\{1-\frac{1}{2}\|x-y\|,\ x,y\in X,\ \|x\|=\|y\|=1,\ \|x-y\|\geq\varepsilon\right\}.$$

Moreover, if X is uniformly convex, then δ is strictly increasing, $\delta(\varepsilon)\longrightarrow 0$ as $\varepsilon\longrightarrow 0,\ \delta(2)=1,\ \eta(t)<2$ when $t<1$ and η is the inverse of δ.

For our theorem we need the following lemma:

Lemma 1.1. ([1]) *Let X be uniformly convex Banach space and B_r, the closed ball in X centered at the origin with radius $r>0$. If $x_1,x_2,x_3\in B_r$ satisfy*

$$\|x_1-x_2\|\geq\|x_2-x_3\|\geq d>0 \text{ and if } \|x_2\|\geq\left(1-\frac{1}{2}\delta\left(\frac{d}{\ell}\right)\right)\ell,$$

then

$$\|x_1-x_3\|\leq\eta\left(1-\frac{1}{2}\delta\left(\frac{d}{\ell}\right)\right)\|x_1-x_2\|.$$

Now, we begin with some known definitions:

Definition 1.1. ([10]) Let S and T be self-mappings on X. Then $\{S,T\}$ is called a *weakly commuting pair* on X if

$$\|STx-TSx\|\leq\|Sx-Tx\| \text{ for all } x\in X.$$

Definition 1.2. ([4]) Let $S,T:X\longrightarrow X$ be mappings. S and T are said to be compatible if

$$\lim_{n\longrightarrow\infty}\|STx_n-TSx_n\|=0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\longrightarrow\infty}Sx_n=\lim_{n\longrightarrow\infty}Tx_n=t \text{ for some } t\in X.$$

Clearly, commuting maps are weakly commuting and weakly commuting maps are compatible. On the other hand, examples are given by Jungck [4], [5], [6] and Sessa [10] to show neither of the above implications are reversible.

Definition 1.3. [7] Two self mappings S and T are said to be weakly compatible if they commute at their coincidence points; i.e., if $Tu = Su$ for some $u \in X$, then $TSu = STu$.

2. Common fixed point theorems

In a paper, Imdad, Khan and Seesa [3] proved the following theorem:

Theorem A. *Let X be uniformly convex and K a non-empty closed subset of X. Let A, S and T be three self-mappings of K satisfying the following conditions:*

(1) *S and T are continuous, $AK \subset SK \cap TK$,*

(2) *$\{A, S\}$ and $\{A, T\}$ are weakly commuting pairs on K,*

(3) *there exists a function $f \in F$ such that for every $x, y \in K$:*

$$\|Ax-Ay\| \leq f(\|Sx-Ty\|,\ \|Sx-Ax\|,\ \|Sx-Ay\|,\ \|Ty-Ax\|,\ \|Ty-Ay\|),$$

where f has the additional requirements:

(a) *for $t > 0, f(t, t, 0, \alpha t, t) \leq \beta t$ and $f(t, t, \alpha t, 0, t) \leq \beta t$ being $\beta < 1$ for $\alpha < 2$ and $\beta = 1$ for $\alpha = 2$, $\alpha, \beta \in R^+$,*

(b) *$f(t, 0, t, t, 0) < t$ for $t > 0$.*

Then, there exists a point u in K such that

(c) *u is the unique common fixed point of A, S and T.*

(d) *For any $x_0 \in K$, the sequence $\{Ax_n\}$ defined by*

$$Tx_{2n} = Ax_{2n-1},\ Sx_{2n+1} = Ax_{2n},\ \text{for } n = 0, 1, 2...,$$

converges strongly to u.

Rashwan [9] extended Theorem A for compatible mappings of type (A) and proved the following:

Theorem B. *Let X and K be as in Theorem A. Let A, B, S and T be mappings on K satisfying the following conditions:*

(1) *one of A, B, S and T is continuous and $AK \subset TK$, $BK \subset SK$,*

(2) *$\{A, S\}$ and $\{B, T\}$ are compatible of type (A),*

(3) *there exists a function $f \in F$ such that for every $x, y \in K$:*

$$\|Ax-By\| \leq f(\|Sx-Ty\|,\ \|Sx-Ax\|,\ \|Sx-By\|,\ \|Ty-Ax\|,\ \|Ty-By\|),$$

where f satisfies the conditions (a) *and* (b) *as in Theorem Arm.*

Then, there exists a point u in K such that

(a) *u is the unique common fixed point of A, B, S and T,*

(b) *for any $x_0 \in K$, the sequence $\{y_n\}$ defined by*

$$y_{2n} = Sx_{2n} = Bx_{2n-1},\ y_{2n+1} = Tx_{2n+1} = Ax_{2n},\ n = 1, 2, 3, ...$$

converges strongly to u.

Sharma and Bamboria [11] proved the following.

Theorem C. *Let X be uniformly convex Banach space and K a non-empty closed subset of X. Let A, B, S and T be mappings on K satisfying the following conditions:*

(1) *$AK \subset TK$ and $BK \subset SK$,*

(2) *there exists a function $f \in F$ such that for every $x, y \in K$:*

$$\|Ax-By\| \leq f(\|Sx-Ty\|,\ \|Sx-Ax\|,\ \|Sx-By\|,\ \|Ty-Ax\|,\ \|Ty-By\|),$$

where f satisfies the conditions (a) *and* (b) *as in Theorem* A,

(3) *one of AK, BK, SK or TK is complete subspace of X, then*

 (a) *A and S have a coincidence point,*
 (b) *B and T have a coincidence point.*

Further if

(4) *the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible, then A, B, S and T have a common fixed point z in K.*

Further, z is the unique common fixed point of A and S and of B and T.

Sharma and Tilwankar [12] proved the following by using (E.A) property.

Theorem D. *Let X be uniformly convex Banach space and K a non-empty closed subset of X. Let A, B, S and T be mappings on K satisfying the following conditions:*

(1) *$AK \subset TK$ and $BK \subset SK$,*

(2) *$\{A, S\}$ or $\{B, T\}$ satisfies the property* (E.A),

(3) *for every* $x, y \in K$:

$$\|Ax - By\| \leq \max(\|Sx - Ty\|, \|Sx - By\|, \|Ty - By\|),$$

(4) *one of AK, BK, SK or TK is closed subset of X, then*

(a) *A and S have a coincidence point,*

(b) *B and T have a coincidence point.*

Further if

(5) *the pairs* $\{A, S\}$ *and* $\{B, T\}$ *are weakly compatible, then*

(c) *A, B, S and T have a common fixed point z in K.*

Further z is the unique common fixed point of A and S and of B and T.

3. Main results

Theorem 3.1. *Let X be uniformly convex Banach space and K a non-empty closed subset of X. Let A, B, S, T, I, J, L, U, P and Q be mappings on K satisfying the following conditions:*

(3.1) $P(K) \subset STJU(K)$ *and* $Q(K) \subset ABIL(K)$,

(3.2) *there exists a function* $f \in F$ *such that for every* $x, y \in K$:

$$\|Px - Qy\| \leq f(\|ABILy - STJUx\|, \|Px - STJUx\|, \|Qy - STJUx\|, \|Px - ABILy\|, \|Qy - ABILy\|),$$

(3.3) *if one of* $P(K), ABIL(K), STJU(K)$ *or* $Q(K)$ *is complete subspace of X, then*

(i) *P and STJU have a coincidence point,*

(ii) *Q and ABIL have a coincidence point,*

(3.4) $AB = BA,\ AI = IA,\ AL = LA,\ BI = IB,\ BL = LB,\ IL = LI,$
$QL = LQ,\ QI = IQ,\ QB = BQ,\ ST = TS,\ SJ = JS,\ SU = US,$
$TJ = JT,\ TU = UT,\ JU = UJ,\ PU = UP,\ PJ = JP,\ PT = TP.$

Further if

(3.5) *the pairs* $\{P, STJU\}$ *and* $\{Q, ABIL\}$ *are weakly compatible, then A, B, S, T, I, J, L, U, P and Q have a common fixed point z in X.*

Here f satisfy the following two conditions.

(a) *for* $t > 0$, $f(t,t,0,\alpha t,t) \le \beta t$ *and* $f(t,t,\alpha t,0,t) \le \beta t$ *being* $\beta < 1$ *for* $\alpha < 2$ *and* $\beta = 1$ *for* $\alpha = 2$, $\alpha,\beta \in R^+$,

(b) $f(t,0,t,t,0) < t$ *or* $f(0,t,0,t,0) < t$ *for* $t > 0$.

Proof. Let $x_0 \in K$, since $P(K) \subset STJU(K)$ and $Q(K) \subset ABIL(K)$, we can always define a sequence $\{y_n\}$ such that

$$\begin{aligned} y_{2n} &= Qx_{2n-1} = ABILx_{2n}, \\ y_{2n+1} &= Px_{2n} = STJUx_{2n+1}, \quad n = 1,2,3,.... \end{aligned}$$

Let $d_n = \|y_n - y_{n+1}\|$, $n = 0,1,2,...$

$$d = \lim_{n \longrightarrow \infty} d_n.$$

Now, for an even n, we have

$$\begin{aligned} d_n &= \|y_n - y_{n+1}\| = \|Px_n - Qx_{n-1}\| \\ &\le f(\|ABILx_{n-1} - STJUx_n\|, \|Px_n - STJUx_n\|, \\ &\|Qx_{n-1} - STJUx_n\|, \|Px_n - ABILx_{n-1}\|, \|Qx_{n-1} - ABILx_{n-1}\|) \\ &= f(\|y_{n-1} - y_n\|, \|y_{n+1} - y_n\|, \|y_n - y_n\|, \|y_{n+1} - y_{n-1}\|, \|y_n - y_{n-1}\|) \\ &= f(\|y_{n-1} - y_n\|, \|y_{n+1} - y_n\|, 0, \|y_{n+1} - y_{n-1}\|, \|y_n - y_{n-1}\|) \\ &\le f(\|y_{n-1} - y_n\|, \|y_{n+1} - y_n\|, 0, \|y_{n+1} - y_n\| + \|y_n - y_{n-1}\|, \|y_n - y_{n-1}\|) \end{aligned} \tag{3.6}$$

which implies

$$d_n = f(d_{n-1}, d_n, 0, d_n + d_{n-1}, d_{n-1}).$$

Similarly. for an odd n, we obtain

$$\begin{aligned} d_n &= \|y_n - y_{n+1}\| = \|Px_{n-1} - Qx_n\| \\ &\le f(\|ABILx_n - STJUx_{n-1}\|, \|Px_{n-1} - STJUx_{n-1}\|, \\ &\|Qx_n - STJUx_{n-1}\|, \|Px_{n-1} - ABILx_n\|, \|Qx_n - ABILx_n\|) \\ &= f(\|y_n - y_{n-1}\|, \|y_n - y_{n-1}\|, \|y_{n+1} - y_{n-1}\|, \|y_n - y_n\|, \|y_{n+1} - y_n\|) \\ &= f(\|y_n - y_{n-1}\|, \|y_n - y_{n-1}\|, \|y_{n+1} - y_{n-1}\|, 0, \|y_{n+1} - y_n\|) \\ &\le f(\|y_n - y_{n-1}\|, \|y_n - y_{n-1}\|, \|y_{n+1} - y_n\| + \|y_n - y_{n-1}\|, 0, \|y_{n+1} - y_n\|) \\ d_n &= f(d_{n-1}, d_{n-1}, d_n + d_{n-1}, 0, d_n) \end{aligned} \tag{3.7}$$

If $d_n > d_{n-1}$, for some $n \ge 1$, then $d_{n-1} + d_n = \alpha d_n$ with $\alpha < 2$, $\alpha \in R$. Since f is nondecreasing in each coordinate variable

$$d_n \le \begin{cases} f(d_n, d_n, 0, \alpha d_n, d_n), & \text{if } n \text{ is even,} \\ f(d_n, d_n, \alpha d_n, 0, d_n), & \text{if } n \text{ is odd.} \end{cases}$$

In both cases, by (a) we get $d_n \le \beta d_n < d_n$, for some $\beta < 1$, $\beta \in R^+$, a contradiction. Thus, $d_{n-1} \ge d_n$ for $n = 1,2,3,...$

Suppose $d > 0$. Without loss of generality, we can postulate that the origin of X belongs to K

$$\lim_{n \longrightarrow \infty} \sup \|y_n\| = \ell' > 0.$$

Let $\ell \in R^+$ be chosen in such a way that $\ell' < 1$ and $eta\left(1 - \frac{1}{2}\delta\left(\frac{d}{\ell}\right)\right) < \ell'$, then there exists a sequence $\{n(k)\}$, $k = 0, 1, 2, ...$, $n(0) \geq 1$, of positive integers such that

$$\|y_{n(k)}\| \geq \left(1 - \frac{1}{2}\delta\left(\frac{d}{\ell}\right)\right),$$

where as it is $\|y_n\| \leq \ell$ for any $n \geq n(0)$.

Since $d_{n(k)-1} \geq d_{n(k)} \geq d > 0,\ k = 0, 1, 2, ...$, from Lemma 1.1 it follows that

$$\|y_{n(k)-1} - y_{n(k)+1}\| \leq \eta\left(\frac{\ell'}{\ell}\right) d_{n(k)-1}, \tag{3.8}$$

where $\eta\left(\frac{\ell'}{\ell}\right) < 2$ being $\frac{\ell'}{\ell} < 1$.

Then, by (3.6), (3.7) and (3.8), we have

$$d_{n(k)} \leq \begin{cases} f\left(d_{n(k)-1}, d_{n(k)-1}, 0, \eta\left(\frac{\ell'}{\ell}\right) d_{n(k)-1}, d_{n(k)-1}\right), & \text{if } n \text{ is even,} \\ f\left(d_{n(k)-1}, d_{n(k)-1}, 0, \eta\left(\frac{\ell'}{\ell}\right) d_{n(k)-1}, d_{n(k)-1}\right), & \text{if } n \text{ is odd.} \end{cases}$$

In both cases, (a) implies

$$d_{n(k)} \leq \beta d_{n(k)-1} \text{ for some } \beta < 1.$$

Observing that β does not depend on k, the foregoing inequality gives, as $n \longrightarrow \infty$, that $d \leq \beta d < d$, a contradiction. This means that $d = 0$.

Now, we wish to prove that $\{y_n\}$ is a Cauchy sequence. Since $\lim_{n \longrightarrow \infty} d_n = 0$, it is sufficient to show that the sequence $\{y_{2n}\}$ is a Cauchy sequence. If not, then there is an $\varepsilon > 0$ such that for every even integer $2k$, $k = 0, 1, 2, ...$, there exists two sequences $\{2_{n(k)}\}, \{2_{m(k)}\}$ with $2k \leq 2_{n(k)} \leq 2_{m(k)}$ for which

$$\|y_{n(k)} - y_{m(k)}\| > \varepsilon. \tag{3.9}$$

For each even integer $2k$, let $2m(k)$ be the least even integer exceeding $n(k)$ and satisfying (3.9). Then

$$\|y_{2n(k)} - y_{2m(k)-2}\| \leq \varepsilon \text{ and } \|y_{2n(k)} - y_{2m(k)}\| > \varepsilon.$$

For each $k = 0, 1, 2, ...$, we have

$$\begin{aligned} \varepsilon &< \|y_{2n(k)} - y_{2m(k)}\| \leq \|y_{2n(k)} - y_{2m(k)-2}\| + \|y_{2m(k)-2} - y_{2m(k)-1}\| \\ &+ \|y_{2m(k)-1} - y_{2m(k)}\| \\ &\leq \varepsilon + d_{2m(k)-2} + d_{2m(k)-1}, \end{aligned}$$

which implies

$$\lim_{k \to \infty} \|y_{2n(k)} - y_{2m(k)}\| = \varepsilon. \tag{3.10}$$

Further, from the triangular inequality, it follows that

$$\big|\|y_{2n(k)} - y_{2m(k)-1}\| - \|y_{2n(k)} - y_{2m(k)}\|\big| \leq d_{2m(k)-1}$$

and

$$\big|\|y_{2n(k)+1} - y_{2m(k)-1}\| - \|y_{2n(k)} - y_{2m(k)}\|\big| \leq d_{2m(k)-1} + d_{2n(k)}.$$

Hence, for $k \longrightarrow \infty$, we find by (3.10) that

$$\|y_{2n(k)} - y_{2m(k)-1}\| \longrightarrow \varepsilon \text{ and } \|y_{2n(k)+1} - y_{2m(k)-1}\| \longrightarrow \varepsilon. \tag{3.11}$$

On the other hand, using (3.2) we deduce that

$$\begin{aligned} \|y_{2n(k)} - y_{2m(k)}\| &\leq d_{2n(k)} + \|y_{2n(k)+1} - y_{2m(k)}\| \\ &\leq d_{2n(k)} + f(\|y_{2m(k)-1} - y_{2n(k)}\|, d_{2n(k)}, \\ &\quad \|y_{2m(k)-1} - y_{2n(k)+1}\|, \|y_{2n(k)} - y_{2m(k)}\|, d_{2n(k)}). \end{aligned} \tag{3.12}$$

By (3.10), (3.11), the upper-semicontinuity and non-decreasing properties of f and condition (b), we have from (3.12), for $k \longrightarrow \infty$, $\varepsilon \leq f(\varepsilon, 0, \varepsilon, \varepsilon, 0) < \varepsilon$, which is a contradiction. Therefore, $\{y_{2n}\}$ is a Cauchy sequence in K and so is $\{y_n\}$. But K is a closed subset of a Banach space X, therefore $\{y_n\}$ converges to a point z in K. On the other hand, the subsequences $\{Px_{2n}\}$, $\{Qx_{2n-1}\}$, $\{STJUx_{2n+1}\}$ and $\{ABILx_{2n}\}$ of $\{y_n\}$ also converges to z.

Now, suppose that $STJU(K)$ is complete. Note that the subsequence $\{y_{2n+1}\}$ is contained in $STJU(K)$ and has a limit in $STJU(K)$ call it z.

Let $u \in (STJU)^{-1}z$. Then $STJUu = z$. By (3.2), we have

$$\begin{aligned} \|Pu - Qx_{2n+1}\| \leq f(&\|ABILx_{2n+1} - STJUu\|, \|Pu - STJUu\|, \\ &\|Qx_{2n+1} - STJUu\|, \|Pu - ABILx_{2n+1}\|, \|Qx_{2n+1} - ABILx_{2n+1}\|). \end{aligned}$$

Taking the limit $n \longrightarrow \infty$, we have

$$\begin{aligned} \|Pu - z\| &\leq f(\|z - z\|, \|Pu - z\|, \|z - z\|, \|Pu - z\|, \|z - z\|) \\ \|Pu - z\| &\leq f(0, \|Pu - z\|, 0, \|Pu - z\|, 0), \end{aligned}$$

which is a contradiction and so $Pu = z$. Therefore, $Pu = z = STJUu$, i.e., u is a coincidence point of P and $STJU$.

Let $v \in (ABIL)^{-1}z$, then $ABILv = z$. By (3.2), we have

$$\begin{aligned} \|Px_{2n} - Qv\| &\leq f(\|ABILv - STJUx_{2n}\|, \|Px_{2n} - STJUx_{2n}\|, \\ &\quad \|Qv - STJUx_{2n}\|, \|Px_{2n} - ABILv\|, \|Qv - ABILv\|). \end{aligned}$$

Taking the limit $n \longrightarrow \infty$, we have

$$\begin{aligned} \|z - Qv\| &\leq f(\|z - z\|, \|z - z\|, \|Qv - z\|, \|z - z\|, \|Qv - z\|) \\ \|z - Qv\| &\leq f(0, 0, \|Qv - z\|, 0, \|Qv - z\|). \end{aligned} \tag{3.13}$$

Let be $\|z-Qv\| > 0$. Being f non-decreasing in each coordinate variable from (3.13), we obtain

$$\|z-Qv\| \leq f(\|z-Qv\|, \|z-Qv\|, \alpha\|z-Qv\|, 0, \|z-Qv\|),$$

where $1 \leq \alpha < 2$. Applying (a), then we deduce for some $\beta < 1$ that

$$\|z-Qv\| \leq \beta\|z-Qv\| < \|z-Qv\|,$$

which is a contradiction and so $Qv = z$. Since $ABILv = z$, thus $ABILv = Qv = z$, i.e., v is a coincidence point of $ABIL$ and Q.

If $P(K)$ is complete, then by (3.1), $z \in P(K) \subset STJU(K)$.

Similarly, if $Q(K)$ is complete, then $z \in Q(K) \subset ABIL(K)$.

Since the pair $\{P, STJU\}$ is weakly compatible, therefore P and $STJU$ commute at their coincidence point, i.e., if $Pu = STJUu$ for some $u \in X$, then

$$P(STJU)u = (STJU)Pu \quad \text{or} \quad Pz = STJUz.$$

Similarly,

$$Q(ABIL)v = (ABIL)Qv \text{ or } Qz = ABILz.$$

Now, we prove $Pz = z$. By (3.2), we have

$$\begin{aligned}\|Pz - Qx_{2n+1}\| \leq f(&\|ABILx_{2n+1} - STJUz\|, \|Pz - STJUz\|,\\ &\|Qx_{2n+1} - STJUz\|, \|Pz - ABILx_{2n+1}\|, \|Qx_{2n+1} - ABILx_{2n+1}\|).\end{aligned}$$

Taking the limit $n \longrightarrow \infty$, we have

$$\begin{aligned}\|Pz - z\| &\leq f(\|z-Pz\|, \|Pz-Pz\|, \|z-Pz\|, \|Pz-z\|, \|z-z\|)\\ &= f(\|z-Pz\|, 0, \|z-Pz\|, \|Pz-z\|, 0)\\ \|Pz-z\| &< \|Pz-z\|,\end{aligned}$$

which is a contradiction and so $Pz = z$ and, therefore, $Pz = z = STJUz$.

Similarly, the pair $\{Q, ABIL\}$ is weakly compatible, therefore Q and $ABIL$ commute at their coincidence point, i.e., if $Qv = ABILv$, for some $v \in X$, then $Q(ABIL)v = (ABIL)Qv$ or $Qz = ABILz$.

Now, we prove $Qz = z$. By (3.2), we have

$$\begin{aligned}\|Px_{2n} - Qz\| \leq f(&\|ABILz - STJUx_{2n}\|, \|Px_{2n} - STJUx_{2n}\|,\\ &\|Qz - STJUx_{2n}\|, \|Px_{2n} - ABILz\|, \|Qz - ABILz\|).\end{aligned}$$

Taking the limit $n \longrightarrow \infty$, we have

$$\begin{aligned}\|z-Qz\| &\leq f(\|Qz-z\|, \|z-z\|, \|Qz-z\|, \|z-Qz\|, \|Qz-Qz\|)\\ \|z-Qz\| &\leq f(\|Qz-z\|, 0, \|Qz-z\|, \|z-Qz\|, 0)\\ \|z-Qz\| &< \|z-Qz\|,\end{aligned}$$

which is a contradiction and so $Qz = z$ and, therefore, $Qz = ABILz = z$.

By (3.2), we have

$$\|Pz - Q(Lz)\| \leq f(\|ABIL(Lz) - STJUz\|, \|Pz - STJUz\|, \|Q(Lz) - STJUz\|, \\ \|Pz - ABIL(Lz)\|, \|Q(Lz) - ABIL(Lz)\|).$$

Taking the limit $n \longrightarrow \infty$, we have

$$\begin{aligned} \|z - Lz\| &\leq f(\|Lz - z\|, \|z - z\|, \|Lz - z\|, \|Lz - z\|, \|Lz - Lz\|) \\ &\leq f(\|Lz - z\|, 0, \|Lz - z\|, \|Lz - z\|, 0) \\ \|Lz - z\| &< \|Lz - z\|, \end{aligned}$$

which is a contradiction and so $Lz = z$. Since $ABILz = z$, we have $ABIz = z$.

By using (3.2) and (3.4), we have

$$\|Pz - Q(Iz)\| \leq f(\|ABIL(Iz) - STJUz\|, \|Pz - STJUz\|, \|Q(Iz) - STJUz\|, \\ \|Pz - ABIL(Iz)\|, \|Q(Iz) - ABIL(Iz)\|).$$

Taking the limit $n \longrightarrow \infty$, we have

$$\begin{aligned} \|z - Iz\| &\leq f(\|Iz - z\|, \|z - z\|, \|Iz - z\|, \|z - Iz\|, \|Iz - Iz|) \\ &\leq f(\|Iz - z\|, 0, \|Iz - z\|, \|z - Iz\|, 0) \\ \|Iz - z\| &< \|Iz - z\|, \end{aligned}$$

which is a contradiction and so $Iz = z$. Since $ABIz = z$, we have $ABz = z$.

Now, we prove $Bz = z$. By putting $x = z$ and $y = Bz$ in (3.2) and (3.4), we have

$$\|Pz - Q(Bz)\| \leq f(\|ABIL(Bz) - STJUz\|, \|Pz - STJUz\|, \\ \|Q(Bz) - STJUz\|, \|Pz - ABIL(Bz)\|, \|Q(Bz) - ABIL(Bz)\|).$$

Taking the limit $n \longrightarrow \infty$, we have

$$\begin{aligned} \|z - Bz\| &\leq f(\|Bz - z\|, \|z - z\|, \|Bz - z\|, \|z - Bz\|, \|Bz - Bz\|) \\ &\leq f(\|Bz - z\|, 0, \|Bz - z\|, \|z - Bz\|, 0) \\ \|Bz - z\| &< \|Bz - z\|, \end{aligned}$$

which is a contradiction and so $Bz = z$. Since $ABz = z$, we have $Az = z$.

Now, we prove $Uz = z$. By using (3.2) and (3.4), we have

$$\|P(Uz) - Qz\| \leq f(\|ABILz - STJU(Uz)\|, \|P(Uz) - STJU(Uz)\|, \\ \|Qz - STJU(Uz)\|, \|P(Uz) - ABILz\|, \|Qz - ABILz\|).$$

Taking the limit $n \longrightarrow \infty$, we have

$$\begin{aligned} \|Uz - z\| &\leq f(\|z - Uz\|, \|Uz - Uz\|, \|z - Uz\|, \|Uz - z\|, \|z - z\|) \\ &\leq f(\|z - Uz\|, 0, \|z - Uz\|, \|Uz - z\|, 0) \\ \|Uz - z\| &< \|Uz - z\|, \end{aligned}$$

which is a contradiction and so $Uz = z$. Since $STJUz = z$, we have $STJz = z$.

Now, we prove $Jz = z$. By using (3.2) and (3.4), we have

$$\begin{aligned}\|P(Uz) - Qz\| \le f(&\|ABILz - STJU(Jz)\|, \|P(Jz) - STJU(Jz)\|, \\ &\|Qz - STJU(Jz)\|, \|P(Jz) - ABILz\|, \|Qz - ABILz\|).\end{aligned}$$

Taking the limit $n \longrightarrow \infty$, we have

$$\begin{aligned}\|Jz - z\| &\le f(\|z - Jz\|, \|Jz - Jz\|, \|z - Jz\|, \|Jz - z\|, \|z - z\|) \\ &\le f(\|z - Jz\|, 0, \|z - Jz\|, \|Jz - z\|, 0) \\ \|Jz - z\| &< \|Jz - z\|,\end{aligned}$$

which is a contradiction and so $Jz = z$. Since $STJz = z$, we have $STz = z$.

Now, we prove $Tz = z$. By using (3.2) and (3.4), we have

$$\begin{aligned}\|P(Tz) - Qz\| \le f(&\|ABILz - STJU(Tz)\|, \|P(Tz) - STJU(Tz)\|, \\ &\|Qz - STJU(Tz)\|, \|P(Tz) - ABILz\|, \|Qz - ABILz\|).\end{aligned}$$

Taking the limit $n \longrightarrow \infty$, we have

$$\begin{aligned}\|Tz - z\| &\le f(\|z - Tz\|, \|Tz - Tz\|, \|z - Tz\|, \|Tz - z\|, \|z - z\|) \\ &\le f(\|z - Tz\|, 0, \|z - Tz\|, \|Tz - z\|, 0) \\ \|Tz - z\| &< \|Tz - z\|,\end{aligned}$$

which is a contradiction and so $Tz = z$. Since $STz = z$, we have $Sz = z$.

By combining the above results, we have

$$Az = Bz = Sz = Tz = Iz = Jz = Lz = Uz = Pz = Qz = z.$$

That is z is a common fixed point of $A, B, S, T, I, J, L, U, P$ and Q.

For the uniqueness of the common fixed point, let w $(w \neq z)$ be another common fixed point of $A, B, S, T, I, J, L, U, P$ and Q. Then, by (3.2), we have

$$\begin{aligned}\|Pz - Qw\| \le f(&\|ABILw - STJUz\|, \|Pz - STJUz\|, \\ &\|Qw - STJUz\|, \|Pz - ABILw\|, \|Qw - ABILw\|).\end{aligned}$$

This gives

$$\begin{aligned}\|z - w\| &\le f(\|w - z\|, \|z - z\|, \|w - z\|, \|z - w\|, \|w - w\|) \\ &\le f(\|w - z\|, 0, \|w - z\|, \|z - w\|, 0) \\ \|w - z\| &< \|w - z\|,\end{aligned}$$

which is a contradiction and so $w = z$.

This completes the proof of the Theorem. ∎

If we put $P = Q$ in Theorem 3.1, we have

Corollary 1. *Let X be uniformly convex Banach space and K a non-empty closed subset of X. Let A, B, S, T, I, J, L, U and P be mappings on K satisfying the following conditions:*

(1) *$P(K) \subset ABIL(K)$ and $P(K) \subset STJU(K)$,*

(2) *there exists a function $f \in F$ such that for every $x, y \in K$:*

$$\begin{aligned}\|Px - \ Py\| \leq f(&\|ABILy - STJUx\|, \|Px - STJUx\|, \\ &\|Py - STJUx\|, \|Px - ABILy\|, \|Py - ABILy\|)\end{aligned}$$

(3) *if one of $P(K), ABIL(K)$ or $STJU(K)$ is a complete subspace of X, then*

(i) *P and STJU have a coincidence point,*

(ii) *P and ABIL have a coincidence point,*

(4) $AB = BA,\ AI = IA,\ AL = LA,\ BI = IB,\ BL = LB,\ IL = LI,$
$PL = LP,\ PI = IP,\ PB = BP,\ ST = TS,\ SJ = JS,\ SU = US,$
$TJ = JT,\ TU = UT,\ JU = UJ,\ PU = UP,\ PJ = JP,\ PT = TP.$

Further, if

(5) *the pairs $\{P, STJU\}$ and $\{P, ABIL\}$ are weakly compatible, then A, B, S, T, I, J, L, U and P have a common fixed point z in X.*

If we put $L = U = Ix$ (The identity map on X) in Theorem 3.1, we have
Corollary 2. *Let X be uniformly convex Banach space and K a non-empty closed subset of X. Let A, B, S, T, I, J, P and Q be mappings on K satisfying the following conditions:*

(1) *$P(K) \subset ABI(K)$ and $Q(K) \subset STJ(K)$,*

(2) *there exists a function $f \in F$ such that for every $x, y \in K$:*

$$\begin{aligned}\|Px - Qy\| \leq f(&\|ABIy - STJx\|, \|Px - STJx\|, \\ &\|Qy - STJx\|, \|Px - ABIy\|, \|Qy - ABIy\|)\end{aligned}$$

(3) *if one of P(K), ABI(K), STJ(K) or Q(K) is a complete subspace of X, then*

(i) *P and STJ have a coincidence point,*

(ii) *Q and ABI have a coincidence point,*

(4) $AB = BA,\ AI = IA,\ BI = IB,\ QI = IQ,\ QB = BQ,\ ST = TS,$
$SJ = JS,\ TJ = JT,\ PJ = JP,\ PT = TP.$

Further, if

(5) *the pairs $\{P, STJ\}$ and $\{Q, ABI\}$ are weakly compatible, then A, B, S, T, I, J, P and Q have a common fixed point z in X.*

If we put $P = Q$ in Corollary 2, we have the following.

Corollary 3. *Let X be uniformly convex Banach space and K a non-empty closed subset of X. Let A, B, S, T, I, J and P be mappings on K satisfying the following conditions:*

(1) $P(K) \subset ABI(K)$ *and* $P(K) \subset STJ(K)$,

(2) *there exists a function $f \in F$ such that for every $x, y \in K$:*

$$\|Px - Py\| \leq f(\|ABIy - STJx\|, \|Px - STJx\|, \|Py - STJx\|, \|Px - ABIy\|, \|Py - ABIy\|)$$

(3) *if one of $P(K), ABI(K)$ or $STJ(K)$ is a complete subspace of X, then*

  (i) *P and STJ have a coincidence point,*

  (ii) *P and ABI have a coincidence point,*

(4) $AB = BA$, $AI = IA$, $BI = IB$, $PI = IP$, $PB = BP$, $ST = TS$, $SJ = JS$, $TJ = JT$, $PJ = JP$, $PT = TP$.

Further, if

(5) *the pairs $\{P, STJ\}$ and $\{P, ABI\}$ are weakly compatible, then A, B, S, T, I, J and P have a common fixed point z in X.*

If we put $I = J = Ix$ (the identity map on X) in Corollary 3, we have the following.

Corollary 4. *Let X be uniformly convex Banach space and K a non-empty closed subset of X. Let A, B, S, T and P be mappings on K satisfying the following conditions:*

(1) $P(K) \subset AB(K)$ *and* $P(K) \subset ST(K)$,

(2) *there exists a function $f \in F$ such that for every $x, y \in K$:*

$$\|Px - Py\| \leq f(\|ABy - STx\|, \|Px - STx\|, \|Py - STx\|, \|Px - ABy\|, \ \|Py - ABy\|).$$

(3) *if one of P(K), AB(K) or ST(K) is complete subspace of X, then*

  (i) *P and ST have a coincidence point,*

  (ii) *P and AB have a coincidence point,*

(4) $AB = BA$, $PB = BP$, $ST = TS$, $PT = TP$.

Further, if

(5) *the pairs* $\{P, ST\}$ *and* $\{P, AB\}$ *are weakly compatible, then* A, B, S, T *and* P *have a common fixed point* z *in* X.

Remark 1. If we put $P = Ix$ (the identity map on X) in Corollary 4, we obtain the results due to Sharma and Bamboria [11], which improves the results of Rashwan [9].

If we put $B = P = Ix$ (the identity map on X) in Corollary 4, we improve results of Imdad, Khan and Sessa [3] in the following way.

Corollary 5. *Let* X *be uniformly convex and* K *a non-empty closed subset of* X. *Let* A, S *and* T *be three self-mappings of* K *satisfying the following conditions:*

(1) $AK \subset SK \cap TK$,

(2) $\{A, S\}$ *and* $\{A, T\}$ *are weakly compatible pairs,*

(3) *there exists a function* $f \in F$ *such that for every* $x, y \in K$:

$$\|Ax - Ay\| \leq f(\|Sx - Ty\|, \|Sx - Ax\|, \|Sx - Ay\|,\ \|Ty - Ax\|,\ \|Ty - Ay\|),$$

where f has the additional requirements:

(a) *for* $t > 0$, $f(t, t, 0, \alpha t, t) \leq \beta t$ *and* $f(t, t, \alpha t, 0, t) \leq \beta t$ *being* $\beta < 1$ *for* $\alpha < 2$ *and* $\beta = 1$ *for* $\alpha = 2$, $\alpha, \beta \in R^+$,

(b) $f(t, 0, t, t, 0) < t$ *for* $t > 0$.

Then, there exists a point z *in* K *such that* z *is the unique common fixed point of* A, S *and* T.

Now, we extend Theorem 3.1 for a finite number of mappings in the following way:

Theorem 3.2. *Let* X *be uniformly convex Banach space and* K *a non-empty closed subset of* X. *Let* $A_1, A_2, ..., A_n, S_1, S_2, ..., S_n$, P *and* Q *be mappings from* X *into itself such that*

(3.14) $P(K) \subset S_1 S_2 ... S_n(K), Q(K) \subset A_1 A_2 ... A_n(K)$,

(3.15) $$\|Px - Qy\| \leq f(\|A_1 A_2 ... A_n y - S_1 S_2 ... S_n x\|, \|Px - S_1 S_2 ... S_n x\|, \|Qy - A_1 A_2 ... A_n y\|, \|Qy - S_1 S_2 ... S_n x\|, \|Px - A_1 A_2 ... A_n y\|)$$ *for all* $x, y \in X$,

(3.16) *if one of* $P(K), A_1 A_2 ... A_n(K), S_1 S_2 ... S_n(K)$ *or* $Q(K)$ *is a complete subspace of* X, *then*

(i) P *and* $S_1 S_2 ... S_n$ *have a coincidence point and*

(ii) Q *and* $A_1 A_2 ... A_n$ *have a coincidence point.*

Further, if

(3.17) A_1 *commutes with* A_2, A_3, ..., A_n,
A_2 *commutes with* A_3, A_4, ..., A_n,
A_3 *commutes with* A_4, A_5,..., A_n,
....
A_{n-1} *commutes with* A_n.

Similarly,

S_1 *commutes with* S_2, S_3, ..., S_n,
S_2 *commutes with* S_3, S_4..., S_n,
S_3 *commutes with* S_4, S_5..., S_n,
....
S_{n-1} *commutes with* S_n,
P *commutes with* S_2, S_3, ..., S_n,
Q *commutes with* A_2, A_3, ..., A_n.

(3.18) *the pairs* $\{P, S_1S_2...S_n\}$ *and* $\{Q, A_1A_2...A_n\}$ *are weakly compatible, then*
(iii) $A_1, A_2, ..., A_n, S_1, S_2, ..., S_n$, P *and* Q *have a unique common fixed point in* X.

Proof. Since $P(K) \subset S_1S_2...S_n(K)$, for any point $x_0 \in X$ there exists a point $x_1 \in X$ such that $Px_0 = S_1S_2...S_nx_1$. Since $Q(K) \subset A_1A_2...A_n(K)$, for this point x_1 we can choose a point $x_2 \in X$ such that $Qx_1 = A_1A_2...A_nx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that for $n = 0, 1, 2, ...$,

$$\begin{aligned} y_{2n} &= Qx_{2n-1} = A_1A_2...A_nx_{2n}, \\ y_{2n+1} &= Px_{2n} = S_1S_2...S_nx_{2n+1}. \end{aligned}$$

By using the method of the proof of Theorem 3.1, we can see that conclusions (i), (ii) and (iii) hold.

Observations. Now, we are giving a formula for commutative conditions:

(i) If the number of mappings are even and finite in above theorems and corollaries then there will be $\frac{n^2-2n-8}{4}$ commutativity conditions, where $n = 4, 6, 8, 10, 12, ...$ up to finite values. For example, if $n = 10$, then 18 commutativity conditions are required. (See (3.4)).

(ii) If the number of mappings are odd and finite in above theorems and corollaries, then there will be $\frac{n^2-9}{4}$ commutativity conditions, where $n = 5, 7, 9, 11...$ up to finite values. For example, if $n = 7$, then 10 commutativity conditions are required. (See (4) in Corollary 3).

(iii) If $n = 1, 2, 3, 4$, then any commutativity condition is not required. (See Theorem C and Corollary 5.)

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ERROR LOCATING CODES DEALING WITH REPEATED BURST ERRORS

Bal Kishan Dass

Department of Mathematics
University of Delhi
Delhi - 110 007
India
e-mail: dassbk@rediffmail.com

Ritu Arora[1]

Department of Mathematics
JDM College (University of Delhi)
Sir Ganga Ram Hospital Marg
New Delhi-110060
India
e-mail: rituaroraind@gmail.com

Abstract. This paper obtains bound for linear codes which are capable to detect and locate errors which occur during the process of transmission. The kind of errors considered are known as repeated burst errors of length b(fixed) introduced by Dass and Garg [10] which has its seeds in the work carried out by Srinivas et al. [15] in connection with models of stroke-induced epilepsy which is an area of mathematical biology. An illustration for such kind of codes has also been provided.

Keywords: block codes, burst of length b(fixed), error detection.
AMS Subject Classification: 94B20, 94B65, 94B25.

1. Introduction

The search for practical coding techniques on error control in digital data transmission has concentrated in two areas: error detection and error correction. A coding technique lying midway between error detection and error correction was introduced by Wolf and Elspas [16]. In this technique the block of received digits is to be regarded as subdivided into mutually exclusive sub-blocks and while decoding it is possible to detect the error and in addition the receiver is able to specify which particular sub-block contains error. Such codes are referred to as Error-Locating codes (EL-codes). They permit the location of digit errors within a sub-block of the received message block without permitting the precise location

[1]Corresponding author.

of the erroneous digit positions. The amount of redundancy required for such codes is not excessive and EL-codes provide an attractive alternative to conventional error detection in decision feed back communication systems. If errors are detected, the receiver requests the transmission of the corrupted block of digits and this process is repeated for each incoming block. The use of EL-codes can soften this compromise between short and long block lengths by providing an additional design parameter. Wolf and Elspas [16] studied binary codes which are capable of detecting and locating a single sub-block containing random errors. Such codes for burst errors were initiated by Dass [6].

Codes developed at the early stages were meant to detect and correct random errors, however it was noticed later that in many kinds of channels the likelihood of the occurrence of errors is more in adjacent positions rather than their occurrence in a random manner. It was in this spirit that the codes correcting single errors and double adjacent errors were developed by Abramson [1]. This idea was generalized and such errors were put in the category of errors called 'burst errors'. A burst of length b is defined as follows:

Definition 1. A burst of length b is a vector whose only non-zero components are among some b consecutive components, the first and last of which is non-zero.

This definition was given by Fire [11] in a research report wherein he called such errors as open-loop burst errors. There is yet another kind of burst errors due to Chien and Tang [4]. It was noted by them that in several channels errors occur in the form of a burst but the end digits of the burst do not get corrupted. Channels due to Alexander *et al.* [2] fall in this category. This prompted Chien and Tang to propose a modification in the definition of a burst and they defined a burst of length b, which shall be called as CT burst of length b, as follows:

Definition 2. A CT burst of length b is a vector whose only non-zero components are confined to some b consecutive positions, the first of which is non-zero.

The nature of burst errors differs from channel to channel depending upon the behaviour of channels or the kind of errors which occur during the process of transmission. This prompted Dass [5] to further modify the definition of CT burst as follows:

Definition 3. A burst of length b(fixed) is an n-tuple whose only non-zero components are confined to b consecutive positions, the first of which is non-zero and the number of its starting positions is among the first $n - b + 1$ components.

Also, in very busy communication channels, errors repeat themselves. So is a situation when errors occur in the form of bursts. So we need to consider repeated bursts. The models studied by Srinivas et al. [15] fall into this category and codes developed pertaining to these may play an important role in subjects like mathematical biology. They studied the changes in the neuronal network properties during epileptiform activity *in vitro* in planar two-dimensional networks cultured on a multielectrode array, using the *in vitro* model of stroke-induced epilepsy.

A 2-repeated burst (open-loop) of length b(refer Berardi et al. [3]) is defined as follows:

Definition 4. A 2-repeated burst of length b is a vector of length n whose only non-zero components are confined to two distinct sets of b consecutive components, the first and the last component of each set being non-zero.

As an illustration, (00104200210500) is a 2-repeated burst of length 4 over GF(5).

Yet another kind of 2-repeated burst error of length b(fixed) has been studied by Dass and Garg [10] in which they defined such an error as follows:

Definition 5. A 2-repeated burst of length b(fixed) is an n-tuple whose only non-zero components are confined to 2 distinct sets of b consecutive digits, the first component of each set is non-zero and the number of its starting positions is among the first $n - 2b + 1$ components.

As an illustration (010010000) is a 2-repeated burst of length up to 3(fixed) whereas (0010000100000) is a 2-repeated burst of length at most 5(fixed) over GF(2).

On the similar lines, an m-repeated burst of length b(fixed) has been defined by Dass *et al.* [8] as follows:

Definition 6. An m-repeated burst of length b(fixed) is an n-tuple whose only non-zero components are confined to m distinct sets of b consecutive digits, the first component of each set is non-zero and the number of its starting positions is among the first $n - mb + 1$ components.

The development of codes locating repeated burst errors will improve the efficiency of the communication channel as it will reduce the number of parity-check digits required in comparison with the codes dealing with the location of the usual burst error locating codes while considering such repeated bursts as single bursts.

This paper mainly presents a study of error-locating codes, in which errors occur in the form of 2-repeated bursts of length b(fixed)

This paper has been organized as follows. In section 2, we derive the necessary condition for the detection and location of 2-repeated bursts of length b(fixed) followed by a sufficient condition for the existence of such a code.

An illustration of a code locating 2-repeated bursts of length 3(fixed) over GF(2) has also been given.

In section 3, necessary condition for the detection and location of m-repeated bursts of length b(fixed) has been given. After that a sufficient condition for the existence of such a code has been given.

In what follows, we shall consider a linear code to be a subspace of n-tuples over GF(q). The block of n digits, consisting of r check digits and $k = n - r$ information digits, is considered to be divided into s mutually exclusive sub-blocks. Each sub-block contains $t = n/s$ digits. The distance between vectors will be considered in the Hamming's sense [12].

2. 2-repeated burst of length b(fixed) error locating codes

In this section, we consider (n, k) linear codes over GF(q) that are capable of detecting and locating all 2-repeated bursts of length b(fixed) within a single sub-block. An EL-code capable of detecting and locating a single sub-block containing an error which is in the form of a 2-repeated burst of length b(fixed) must satisfy the following conditions:

(a) The syndrome resulting from the occurrence of a 2-repeated burst of length b(fixed) within any one sub-block must be distinct from the all zero syndrome.

(b) The syndrome resulting from the occurrence of any 2-repeated burst of length b(fixed) within a single sub-block must be distinct from the syndrome resulting likewise from any 2-repeated burst of length b(fixed) within any other sub-block.

In this section we shall derive two results. The first result gives a lower bound on the number of check digits required for the existence of a linear code over GF(q) capable of detecting and locating a single sub-block containing errors that are 2-repeated bursts of length b(fixed). In the second result, we derive an upper bound on the number of check digits which ensures the existence of such a code.

Since the code is divided into several blocks of length t each and we wish to detect a 2-repeated burst of length b(fixed), we may come across with a situation when the difference between $2b$ and t becomes small. We note that if $t-b+1 < 2b$ and we consider any two 2-repeated bursts x_1 and x_2 of length b(fixed) such that their non-zero components are confined to first $t - b + 1$ positions then their difference $x_1 - x_2$ is again a 2-repeated burst of length b(fixed). However, if we do not restrict ourselves to first $t - b + 1$ positions then we may not get a 2-repeated burst of length b(fixed) as explained with the help of the following examples:

Example 1. Suppose $t = 7, b = 3$. Let $x_1 = (1111110)$ and $x_2 = (1111100)$. Then x_1 and x_2 are 2-repeated bursts of length 3(fixed) whereas $x_1 - x_2$= (0000010) is not a 2-repeated burst of length 3(fixed).

Example 2. Suppose $t = 10, b = 4$. Let x_1= (1111001101) and x_2 = (1001001001). Then x_1 and x_2 are 2-repeated bursts of length 4(fixed) whereas $x_1 - x_2$ =(0110000100) is not a 2-repeated burst of length 4(fixed). So, when $t-b+1 < 2b$ we consider the collection of those vectors in which the non -zero components are confined to first $t - b + 1$ positions whereas when $t - b + 1 \geq 2b$ we consider the collection of those vectors in which the non-zero components are confined to some two (fixed) b consecutive positions so that patterns to be detected are not code words.

Theorem 1. *The number of parity check digits r in an (n, k) linear code subdivided into s sub-blocks of length t each, that locates a single corrupted sub-block containing errors that are 2-repeated bursts of length b(fixed) is at least*

$$(1) \qquad \begin{cases} \log_q\{1+s(q^{t-b+1}-1)\} & \textit{when} \;\; t-b+1<2b \\ \log_q\{1+s(q^{2b}-1)\} & \textit{when} \;\; t-b+1\geq 2b. \end{cases}$$

Proof. Let there be an (n,k) linear code over GF(q) that locates a 2-repeated burst of length b(fixed) within a single corrupted sub-block. Maximum number of distinct syndromes available using r check bits is q^r. The proof proceeds by first counting the number of syndromes that are required to be distinct by condition (a) and (b) and then setting this number less than or equal to q^r. First we consider a sub-block, say i^{th} sub-block of length t.

In view of the observations made before Theorem 1, we discuss the following two cases:

Case 1: when $t-b+1<2b$.
Let X consist of all those vectors in which all the non-zero components are confined to the first $t-b+1$ positions of the i^{th} sub-block. We observe that the syndromes of all the elements of X should be different; else for any x_1, x_2 belonging to X having the same syndrome would imply that the syndrome of x_1-x_2 which is also an element of X and hence a 2-repeated burst of length b(fixed) within the same sub-block becomes zero; in violation of condition (a). Also since the code locates a single sub-block containing errors that are 2-repeated bursts of length b(fixed), the syndromes produced by similar vectors in different sub-blocks must be distinct by condition (b). Thus, the syndromes of vectors which are 2-repeated bursts of length b(fixed) in fixed positions, whether in the same sub-block or in different sub-blocks, must be distinct. (It may be noted that different fixed components may be chosen in different sub-blocks.) As there are $(q^{t-b+1}-1)$ distinct non-zero syndromes corresponding to the vectors in any one sub-block and there are s sub-blocks in all, so we must have at least $(1+s(q^{t-b+1}-1))$ distinct syndromes counting the all zero syndrome. Therefore, we must have

$$q^r \geq \{1+s(q^{t-b+1}-1)\} \text{ when } t-b+1<2b$$

or

$$(2) \qquad r \geq \log_q\{1+s(q^{t-b+1}-1)\} \text{ when } t-b+1<2b.$$

Case 2: when $t-b+1\geq 2b$.
Let X consist of all those vectors in which all the non-zero components are confined to some two fixed b consecutive positions of the i^{th} sub-block. As discussed in case 1, the syndromes of all the elements of X are different. As, in this case, there are $(q^{2b}-1)$ distinct non-zero syndromes corresponding to the vectors in any one sub-block and there are s sub-blocks in all, so we must have at least $(1+s(q^{2b}-1))$ distinct syndromes counting the all zero syndrome. Therefore, we must have

$$q^r \geq (1+s(q^{2b}-1)) \text{ when } \; t-b+1\geq 2b$$

or

$$(3) \qquad r \geq \log_q\{1+s(q^{2b}-1)\} \text{ when } \; t-b+1\geq 2b.$$

From (2) and (3) we get the required result.

In the following result, we derive another bound on the number of check digits required for the existence of such a code. The proof is based on the technique used to establish Varsharmov-Gilbert Sacks bound by constructing a parity check matrix for such a code (refer Sacks [14], Theorem 4.7 Peterson and Weldon [13]). This technique not only ensures the existence of such a code but also gives a method for the construction of the code.

Theorem 2. *An (n, k) linear EL code over GF(q) capable of detecting a 2-repeated burst of length b(fixed) within a single sub-block and of locating that sub-block can always be constructed provided that*

$$
\begin{aligned}
r > (b-1) + \log_q \Bigg[&\{1 + (q-1)q^{b-1}(t-2b+1)\} \\
&\cdot \left\{ 1 + (s-1)\sum_{i=1}^{2} \binom{t-ib+i}{i} (q-1)^i q^{i(b-1)} \right\} \Bigg]
\end{aligned} \tag{4}
$$

where r is the number of check digits.

Proof. In order to prove the existence of such a code, we construct an $(n-k)\times n$ parity check matrix H for such a code by a synthesis procedure. For that we first construct a matrix H_1 from which the requisite parity check matrix H shall be obtained by reversing the order of the columns of each sub-block.

After adding $(s-1)t$ columns appropriately corresponding to the first $(s-1)$ sub-blocks, suppose that we have added the first $j-1$ columns $h_1, h_2, h_3 \ldots h_{j-1}$ of the s-th sub-block to the matrix H_1, out of which the first $b-1$ columns $h_1, h_2, h_3 \ldots h_{b-1}$ may be chosen arbitrarily (non-zero). We now lay down the condition to add the j-th column h_j to H_1 as follows:

According to condition (a), for the detection of 2-repeated burst of length b(fixed) in the s^{th} sub-block h_j should not be a linear combination of immediately preceding $b-1$ columns $h_{j-b+1}, h_{j-b+2} \ldots h_{j-1}$ together with any linear combination of b consecutive columns out of the first $j-b$ columns of the s^{th} sub-block. i.e.,

$$
\begin{aligned}
h_j \neq (&\alpha_1 h_{j-b+1} + \alpha_2 h_{j-b+2} + \ldots + \alpha_{b-1} h_{j-1}) \\
&+ (\beta_1 h_{i+1} + \beta_2 h_{i+2} + \ldots + \beta_b h_{i+b})
\end{aligned} \tag{5}
$$

where $\alpha_i, \beta_i \in GF(q)$ and either all the coefficients β_i's are zero or if the p-th coefficient β_p is the last non-zero coefficient then $b \leq p \leq j-b$.

The number of ways in which the coefficients α_i's can be selected is q^{b-1} and to enumerate the coefficients β_i's is equivalent to enumerate the number of bursts of length b(fixed) in a vector of length $j-b$.

This number (refer Dass [5]), including the vector of all zeros is

$$1 + (j-2b+1)(q-1)q^{b-1}.$$

So, the number of linear combinations on the right hand side of (5) is

$$q^{b-1}[1 + (j-2b+1)(q-1)q^{b-1}]. \tag{6}$$

Now, according to condition (b), for the location of 2-repeated burst of length b(fixed), h_j should not be a linear combination of the immediately preceding $b-1$ columns together with any b consecutive columns out of the remaining $j-b$ columns of the s-th sub-block together with linear combination of any two sets of b consecutive columns out of any one of the previously chosen $s-1$ sub-blocks, the coefficient of the last column of either both or one of the sets being non-zero.

The number of 2-repeated bursts of length b(fixed) in a sub-block of length t (refer Dass *et al.* [9]) is

$$\sum_{i=1}^{2} \binom{t-ib+i}{i} (q-1)^i q^{i(b-1)}.$$

Since there are $(s-1)$ previous sub-blocks, therefore number of such linear combinations becomes

$$(s-1)\left\{ \sum_{i=1}^{2} \binom{t-ib+i}{i} (q-1)^i q^{i(b-1)} \right\}. \tag{7}$$

So, for the location of 2-repeated burst of length b(fixed) the number of linear combinations to which h_j can not be equal to is the product computed in expr.(6) and expr.(7). i.e.,

$$\text{expr.}(6) \times \text{expr.}(7) \tag{8}$$

Thus, the total number of linear combinations that h_j can not be equal to is the sum of linear combinations in (6) and (8).

At worst, all these combinations might yield distinct sum.Therefore, h_j can be added to the s-th sub-block of H_1 provided that

$$\begin{aligned} q^r > q^{b-1}\{1+(q-1)q^{b-1}(j-2b+1)\}\Big[1+(s-1) \\ \cdot\left\{ \sum_{i=1}^{2} \binom{t-ib+i}{i} (q-1)^i q^{i(b-1)} \right\}\Big]. \end{aligned} \tag{9}$$

To obtain the length of the block as t replacing j by t in the above expression we get

$$\begin{aligned} r > (b-1) + \log_q \Big[\{1+(q-1)q^{b-1}(t-2b+1)\} \\ \cdot\Big\{1+(s-1)\Big(\sum_{i=1}^{2} \binom{t-ib+i}{i} (q-1)^i q^{i(b-1)}\Big)\Big\}\Big]. \end{aligned}$$

The required matrix H can be obtained from H_1 by reversing the order of the columns in each sub-block.

Remark 1. It may be noted that it hardly matters whether we reverse the order of columns within the subblock or we reverse the order of the columns of the entire matrix H_1.

Alternate Form 1. *Let B be the largest value of b satisfying the inequality* (4)*. Then for $b = B + 1$, the inequality* (4) *gets reversed and we get*

$$r \leq B + \log_q \left[\{1 + (q-1)q^B (t - 2B - 1)\} \right.$$
$$\left. \left\{ 1 + (s-1) \sum_{i=1}^{2} \binom{t - iB}{i} (q-1)^i q^{iB} \right\} \right]. \tag{10}$$

Example 3. For an $(27, 14)$ linear code over GF(2) we construct the following parity check matrix $H(13 \times 27)$, according to the synthesis procedure given in the proof of Theorem 2 by taking $s = 3$, $t = 9$, $b = 3$.

$$\left[\begin{array}{ccccccccc@{\quad}ccccccccc@{\quad}ccccccccc}
0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&1&1&1&0&1&1&1&0&0&1\\
0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&0&1&0&1&1&0&0&1&0&0&1&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&1&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&1&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&0&1&0&1&1&1&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&1&0&0&0&1&0&1&0&0
\end{array}\right]$$

The null space of this matrix can be used as a code to locate a sub-block of length $t = 9$ containing 2-repeated bursts of length 3(fixed). It may be easily verified that:

1. Syndromes of 2-repeated bursts of length 3(fixed) within any sub-block are all non-zero showing thereby that the code detects all 2-repeated bursts of length 3(fixed) occurring within a sub block.

2. The syndrome of the 2-repeated burst of length 3(fixed) within any sub-block is different from the syndrome of a 2-repeated burst of length 3(fixed) within any other sub-block thereby ensuring that the code locates any 2-repeated burst of length 3(fixed) occurring within single sub-block. (This has been verified through MS-Excel program).

Observation. Syndromes of some of the 2-repeated bursts of length 3(fixed) occurring within the second sub-block turn out to be the same. For coding efficiency it is desired that the syndromes of the error patterns within any sub-block are identical whenever possible.

3. Location of m-repeated burst of length b(fixed)

In this section, the results of the previous section have been extended to the case of m-repeated burst of length b(fixed).

It may be noted that an EL-code capable of detecting and locating a single sub-block containing an error which is in the form of an m-repeated burst of length b(fixed) must satisfy the following conditions:

(c) The syndrome resulting from the occurrence of an m-repeated burst of length b(fixed) within any one sub-block must be distinct from the all zero syndrome.

(d) The syndrome resulting from the occurrence of any m-repeated burst of length b(fixed) within a single sub-block must be distinct from the syndrome resulting likewise from any m-repeated burst of length b(fixed) within any other sub-block.

In this section, we shall derive two results. The first result gives a lower bound on the number of check digits required for the existence of a linear code over GF(q) capable of detecting and locating a single sub-block containing errors that are m-repeated bursts of length b(fixed). In the second result, we derive an upper bound on the number of check digits which ensures the existence of such a code.

Theorem 3. *The number of parity check digits r in an (n, k) linear code subdivided into s sub-blocks of length t each, that locates a single corrupted sub-block containing errors that are m-repeated bursts of length b(fixed) is at least*

$$\begin{cases} \log_q\{1+s(q^{t-b+1}-1)\} & \textit{when } \; t-b+1 < mb \\ \log_q\{1+s(q^{mb}-1)\} & \textit{when } \; t-b+1 \geq mb. \end{cases} \tag{11}$$

Proof. The proof of this result is on the similar lines as that of proof of Theorem 1 so we omit the proof. ∎

Remark 2. For $m = 2$, this result coincides with the Theorem 1 when 2-repeated bursts of length b(fixed) are considered.

For $m = 1$, this result is similar to the result obtained in the Theorem 1 due to Dass and Chand [7] when bursts of length b(fixed) are considered.

In the following result we derive another bound on the number of check digits required for the existence of the code considered in the Theorem 3.

Theorem 4. *A code capable of detecting an m-repeated burst of length b(fixed) within a single sub-block and of locating that sub-block can always be constructed provided that*

$$\begin{aligned} r > (b-1) + \log_q \Bigg[& \left\{ \sum_{i=0}^{m-1} \binom{t-(i+1)b+i}{i} \cdot (q-1)^i q^{i(b-1)} \right\} \\ & \cdot \left\{ 1+(s-1)\sum_{i=1}^{m} \binom{t-ib+i}{i} (q-1)^i q^{i(b-1)} \right\} \Bigg] \end{aligned} \tag{12}$$

where r is the number of check digits.

Proof. As in Theorem 3, we omit the proof because proof of this result is on the similar lines as that of proof of Theorem 2. ∎

Remark 3. For $m = 2$, this result coincides with Theorem 2 when 2-repeated bursts of length b(fixed) are considered.

For $m = 1$, this result coincides with the result obtained in Theorem 2 due to Dass and Chand [7] when bursts of length b(fixed) are considered.

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$\mathcal{M}$-INJECTIVITY IN THE CATEGORY Act-S

Leila Shahbaz

Department of Mathematics
University of Maragheh
Maragheh 55181-83111
Iran
e-mail: leilashahbaz@yahoo.com

Abstract. Injectivity is one of the central notions in many branches of mathematics. Different kinds of injectivity with respect to the class of all monomorphisms and with respect to some special subclasses of monomorphisms in the category **Act-S** of acts over a semigroup S have been studied before. In this paper, we take the category **Act-S** of acts over a semigroup S, and $\mathcal{M}$ as an arbitrary subclass of monomorphisms, and study some kinds of injectivity with respect to $\mathcal{M}$. Also, the behaviour of these notions of injectivity with respect to products, coproducts, and direct sums is studied. As a result we give some characterizations of semigroups.

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1. Introduction and preliminaries

One of the very useful notions in many branches of mathematics as well as in computer science is the notion of acts of a semigroup or a monoid on a set. In the following we first recall some facts about the category **Act-S** needed in this paper.

Let S be a semigroup, A be a set, and

$$\begin{array}{rcl} \mu : A \times S & \longrightarrow & A \\ (a,s) & \longmapsto & as := \mu(a,s), \end{array}$$

be a map. The set A is called a (*right*) S-*act* or a (*right*) *act* over S, if the map μ satisfies $a(st) = (as)t$ for $a \in A$ and $s, t \in S$. In this case, μ is called the action of S on A.

If S is a monoid with 1 as its identity, we usually also require that $a1 = a$ for $a \in A$.

A subset A' of an S-act A is said to be a *subact* of A if $a's \in A'$ for all $s \in S$ and $a' \in A'$; and in this case we write $A' \leq A$.

A *homomorphism* (also called an *equivariant* map or an S*-map*) from an S-act A to an S-act B is a function from A to B such that for each $a \in A, s \in S$, $f(as) = f(a)s$.

Since id_A and the composition of two S-maps are S-maps, we have the category **Act-S** of all right S-acts and S-maps between them.

Note that, the class of S-acts is an equational class, and so the category **Act-S** is complete and cocomplete (has all products, equalizers, pullbacks, coproducts, coequalizers, and pushouts). In fact, limits and colimits in this category are computed as in the category **Set** of sets and equipped with a natural action. Also, monomorphisms (epimorphisms) in **Act-S** are exactly one-one (onto) S-maps. Therefore, we do not distinguish between monomorphisms of acts and inclusions, and call an S-act B containing (an isomorphic copy of) an S-act A an *extension* of A.

An element z of S is called a right zero element if for each $s \in S, sz = z$. Also, an element z of S is called a left zero element if for each $s \in S, zs = z$. An element 0 or θ of S is a zero element if it is right and left zero.

An S-act A is said to be finitely generated if $A = \bigcup_{i=1}^{n} a_i S^1$, for some $a_1, ...,$ $a_n \in A$ and $n \in N$, where S^1 is the semigroup S with an adjoined identity 1. We say that a semigroup S is finitely generated if it is finitely generated as an S-act with its operation as the action.

A semigroup S is said to be *Noetherian* if it satisfies the ascending chain condition on its right ideals.

A semigroup S is said to be *left reversible* if every two right ideals of S have a nonempty intersection. For more information about semigroups and acts see [6], [8] and [9].

An S-act A is said to be *decomposable* if there exist subacts $B, C \subsetneq A$ such that $A = B \cup C$ and $B \cap C = \emptyset$. Otherwise, A is called *indecomposable*.

An act A is called torsion free if for any $x, y \in A$ and for any element $s \in S$ the equality $xs = ys$ implies $x = y$.

Recall that for a category $\mathcal{A}$ and a subclass $\mathcal{M}$ of monomorphisms in $\mathcal{A}$, we say that $\mathcal{A}$ satisfies the $\mathcal{M}$*-transferability property* if any diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$$

with $f \in \mathcal{M}$ can be completed to a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "u"] \\ C \arrow[r, "v"] & D \end{tikzcd}$$

with $v \in \mathcal{M}$.

Note that, since pushouts exist in the category **Act-S**, the above condition is equivalent to "pushout transfer monomorphism"; that is, the pushout map corresponding to a morphism in $\mathcal{M}$ again belongs to $\mathcal{M}$.

Recall that for a family $\{A_i : i \in I\}$ of S-acts with a unique fixed element 0, the *direct sum* $\bigoplus_{i\in I} A_i$ is defined to be the subact of the product $\prod_{i\in I} A_i$ consisting of all $(a_i)_{i\in I}$ such that $a_i = 0$ for all $i \in I$ except a finite number.

2. $\mathcal{M}$-injectivity and some kinds of weak $\mathcal{M}$-injectivity

In this section, we study $\mathcal{M}$-injectivity and some kinds of weak $\mathcal{M}$-injectivity of acts for a subclass $\mathcal{M}$ of monomorphisms. Any map with prefix $\mathcal{M}$ means being in $\mathcal{M}$.

Definition 2.1 We call an S-act E:

(1) $\mathcal{M}$-*injective* if it is injective with respect to $\mathcal{M}$-morphisms. An injective act with respect to all act monomorphisms is simply called *injective.*

(2) $\mathcal{M}$-*absolute retract* if it is a retract of each of its $\mathcal{M}$-extensions; that is for every $\mathcal{M}$-morphism $h : A \to B$ there exists an S-map $g : B \to A$ such that $gh = id_A$.

Clearly, if A is an injective S-act, then it is $\mathcal{M}$-injective for any subclass $\mathcal{M}$ of monomorphisms, but the converse is not necessarily true. For example, take $\mathcal{M}$ as the class of sequentially dense or sequentially pure monomorphisms and see [10] and [2].

The following theorem is one of the important theorems about injectivity of S-acts with respect to any subclass of monomorphisms. This theorem was first proved by P. Berthiaume in [4] for injective acts. H. Barzegar [3] and B. Banaschewski [1] proved it for $\mathcal{M}$-injective acts for any subclass $\mathcal{M}$ of monomorphisms.

Theorem 2.2 *Let S be a semigroup. Then, if pushouts transfer $\mathcal{M}$-morphisms, the following are equivalent for an S-act A:*

(i) *A is $\mathcal{M}$-injective.*

(ii) *A is an $\mathcal{M}$-absolute retract.*

Here we give a criterion like the Skornjakov-Baer criterion [13], for $\mathcal{M}$-injectivity of acts over a semigroup.

Theorem 2.3 *Let for each $\mathcal{M}$-extension A of B, D be an $\mathcal{M}$-extension of B with $D \subseteq A$. Then, for an S-act E, the following are equivalent:*

(i) *E is $\mathcal{M}$-injective.*

(ii) *For every $\mathcal{M}$-morphism $h : B \rightarrowtail B \cup aS^1$ to a singly generated extension of B and every S-map $f : B \to E$ there exists an S-map $g : B \cup aS^1 \to E$ such that $gh = f$.*

Proof. It is clear that (i)⇒(ii).

(ii)⇒(i) Let $h : B \rightarrowtail A$ be an $\mathcal{M}$-morphism and $f : B \to E$ be an S-map. Applying Zorn's Lemma on the poset of all subacts (D_α, g_α) of A which are $\mathcal{M}$-extensions of B, with $h(B) \subseteq D_\alpha$ and such that there exists an S-map $g_\alpha : D_\alpha \to E$ with $g_\alpha h = f$ with the order

$$(D_\alpha, g_\alpha) \leq (D_\beta, g_\beta) \Leftrightarrow D_\alpha \subseteq D_\beta, g_\beta \mid_{D_\alpha} = g_\alpha$$

we get a maximal such subact, say (D, g). If $D = A$ then the proof is complete. Otherwise, there exists $a \in A - D$. Now, since by the hypothesis $D \cup aS^1$ is an $\mathcal{M}$-extension of B we get that $\overline{g} : B \to D \cup aS^1$ is an $\mathcal{M}$-morphism, and by (ii) there is an S-map $\overline{g}$ which extends g. This contradicts the maximality of D, so $A = D$. ■

Now, we define some kinds of weak $\mathcal{M}$-injectivity and then compare them with $\mathcal{M}$-injectivity.

Definition 2.4 A right ideal I of S is called a right $\mathcal{M}$-ideal if the inclusion map from I into S belongs to $\mathcal{M}$.

Definition 2.5 An S-act A is said to be

(1) *ideal $\mathcal{M}$-injective*, if every S-map $f : I \to A$ from a right $\mathcal{M}$-ideal I of S can be represented as λ_a, for some $a \in A$.

(2) *weakly $\mathcal{M}$-injective* if for each right $\mathcal{M}$-ideal I of S, any S-map $f : I \to A$ can be extended to an S-map $g : S \to A$.

(3) *finitely $\mathcal{M}$-injective* (*c$\mathcal{M}$-injective*) if for each $\mathcal{M}$-morphism $h : F \to B$ from a finitely generated (cyclic) act F and for any S-map $f : F \to A$ there exits an S-map $g : B \to A$ such that $gh = f$.

(4) *$F\mathcal{M}$-injective* (*$P\mathcal{M}$-injective)* if every S-map $f : I \to A$ from a finitely generated (principal) right $\mathcal{M}$-ideal I of S can be extended to an S-map $\overline{f} : S \to A$.

(5) A semigroup S is called *completely $\mathcal{M}$-injective* if all right S-acts are $\mathcal{M}$-injective. Similarly for the other types of $\mathcal{M}$-injectivity.

Remark 2.6

(1) Ideal (weakly) $\mathcal{M}$-injective acts need not be $\mathcal{M}$-injective in the usual sense (let $\mathcal{M} = \mathcal{M}$ono and see [9]).

(2) Ideal $\mathcal{M}$-injectivity implies weak $\mathcal{M}$-injectivity but weakly $\mathcal{M}$-injective acts need not be ideal $\mathcal{M}$-injective. For the case where S is a monoid, ideal $\mathcal{M}$-injectivity coincides with weak $\mathcal{M}$-injectivity.

Lemma 2.7 *A retract of any kind of $\mathcal{M}$-injective act is $\mathcal{M}$-injective of that type.*

The following results show when all acts are finitely $\mathcal{M}$-injective.

Lemma 2.8 *If pushouts transfer $\mathcal{M}$-morphisms then every finitely generated finitely $\mathcal{M}$-injective S-act is $\mathcal{M}$-injective.*

Proof. Let A be a finitely generated finitely $\mathcal{M}$-injective S-act. Consider the following diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "id_A"'] & B \\ A & \end{tikzcd}$$

in which B is an $\mathcal{M}$-extension of A. Using that A is finitely generated, there exists $\pi : B \to A$ such that $\pi \circ i = id_A$. This implies that A is an $\mathcal{M}$-absolute retract. Now, by Theorem 2.2, A is $\mathcal{M}$-injective. ∎

Theorem 2.9 *If pushouts transfer $\mathcal{M}$-morphisms then a semigroup S is completely finitely $\mathcal{M}$-injective if and only if all finitely generated S-acts are $\mathcal{M}$-injective.*

Proof. ($\Rightarrow$) The proof is similar to the proof of the above lemma.

($\Leftarrow$) Let A be any S-act and $h : F \to B$ be an $\mathcal{M}$-morphism from a finitely generated act F, and $f : F \to A$ be any S-map. Then, by hypothesis, F is $\mathcal{M}$-injective and so an $\mathcal{M}$-absolute retract, by Theorem 2.2. Thus there exists an S-map $g : B \to F$ such that $gh = id_F$. Then the composite $fg : B \to A$ is an S-map with $(fg)h = f$. So, A is finitely $\mathcal{M}$-injective. ∎

Proposition 2.10 *An S-act A with a zero element is $c\mathcal{M}$-injective if and only if for any $\mathcal{M}$-morphism $h : P \rightarrowtail D$ from a cyclic act P into any indecomposable act D, any S-map $f : P \to A$ can be extended to an S-map $g : D \to A$.*

Proof. ($\Rightarrow$) is clear.

($\Leftarrow$) Let A be an S-act with a zero element and $h : P \to B$ be an $\mathcal{M}$-morphism from a cyclic act P, and $f : P \to A$ be any S-map. Consider the decomposition of $B = \bigsqcup_{i\in I} B_i$ into its indecomposable subacts B_i which exists by Theorem I.5.10 of [9]. Since P is cyclic, there exists $i \in I$ such that $h(P) \subseteq B_i$. Thus by the hypothesis there exists an S-map $g : B_i \to A$ which extends f. Define $\overline{f} : B = \bigsqcup_{i\in I} B_i \to A$ by

$$\overline{f}(b) = \begin{cases} g(b) & \text{if } b \in B_i \\ \theta & \text{if } b \notin B_i \end{cases}$$

where θ is the zero element of A. Then $\overline{f}$ is an S-map which extends f. ∎

The following results show when all acts are $c\mathcal{M}$-injective.

Lemma 2.11 *If pushouts transfer $\mathcal{M}$-morphisms then every cyclic $c\mathcal{M}$-injective S-act is $\mathcal{M}$-injective.*

Proof. It is similar to the proof of Theorem 2.9 by replacing finitely generated acts with cyclic acts. ∎

Theorem 2.12 *If pushouts transfer $\mathcal{M}$-morphisms then a semigroup S is completely $c\mathcal{M}$-injective if and only if all cyclic S-acts are $\mathcal{M}$-injective.*

Proof. The proof is similar to the proof of the above lemma. ∎

The following theorem characterizes semigroups over which all acts are ideal $\mathcal{M}$-injective.

Theorem 2.13 *Every S-act is ideal $\mathcal{M}$-injective if and only if every right $\mathcal{M}$-ideal of the semigroup S is generated by an idempotent.*

Proof. Consider the identity map id_I from a right $\mathcal{M}$-ideal I of S which is of the form λ_a for some element a in I, by hypothesis. Thus, $a = id_I(a) = \lambda_a(a) = aa = a^2$ and so a is an idempotent element. For the converse, let $I = eS$ be a right $\mathcal{M}$-ideal of S, where e is an idempotent element. Consider an S-map $f : I = eS \to A$. Thus $f = \lambda_a$ for $a = f(e)$. Thus A is ideal $\mathcal{M}$-injective. ∎

In the following we characterize semigroups over which all acts are weakly $\mathcal{M}$-injective.

Theorem 2.14 *The following conditions are equivalent:*

(1) *Each right $\mathcal{M}$-ideal of S is a retract of S.*

(2) *S is completely weakly $\mathcal{M}$-injective.*

(3) *Each right $\mathcal{M}$-ideal of S is weakly $\mathcal{M}$-injective.*

Proof. (1) $\Rightarrow$ (2) Let A be an S-act, and $f : I \to A$ be an S-map from a right $\mathcal{M}$-ideal I of S. By the hypothesis, there is a retraction $g : S \to I$. Then $fg : S \to A$ is an S-map which extends f. So, A is weakly $\mathcal{M}$-injective.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1) Let I be a right $\mathcal{M}$-ideal of S. Then there exists an S-map π from I to S with $\pi\mid_I = id_I$, since I is weakly $\mathcal{M}$-injective by (2). So I is a retract of S. ∎

In the following we characterize semigroups over which all acts are $F\mathcal{M}$-injective ($P\mathcal{M}$-injective).

Theorem 2.15 *The following conditions are equivalent:*

(1) *Each finitely generated* (*principal*) *right* $\mathcal{M}$*-ideal of* S *is a retract of* S.

(2) S *is completely* $F\mathcal{M}$*-injective* ($P\mathcal{M}$*-injective*).

(3) *Each finitely generated* (*principal*) *right* $\mathcal{M}$*-ideal of* S *is* $F\mathcal{M}$*-injective* ($P\mathcal{M}$*-injective*).

Proof. (1) $\Rightarrow$ (2) Let A be an S-act, and $f : I \to A$ be an S-map from a finitely generated (principal) right $\mathcal{M}$-ideal I of S. By the hypothesis, there is a retraction $g : S \to I$. Then $fg : S \to A$ is an S-map which extends f. So, A is $F\mathcal{M}$-injective ($P\mathcal{M}$-injective).

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1) Let I be a finitely generated (principal) right $\mathcal{M}$-ideal of S. Then there exists an S-map π from I to S with $\pi\mid_I = id_I$, since I is $F\mathcal{M}$-injective ($P\mathcal{M}$-injective) by (2). So I is a retract of S. ■

Definition 2.16 A right $\mathcal{M}$-ideal I of S is called $\mathcal{M}$*-intersection large* in S if the intersection of I with any nonempty right $\mathcal{M}$-ideal of S is always nonempty.

Theorem 2.17 *Suppose that* S *is a semigroup with a zero element and for each right* $\mathcal{M}$*-ideal* I *of* S, J *is a right* $\mathcal{M}$*-ideal of* S *with* $I \subseteq J$. *Then a right* S*-act* A *is weakly* $\mathcal{M}$*-injective if and only if for any* $\mathcal{M}$*-intersection large right* $\mathcal{M}$*-ideal* I *of* S, *every* S*-map from* I *into* A *can be extended to an* S*-map from* S *into* A.

Proof. The only if part is obvious. To prove the converse, let I be any right $\mathcal{M}$-ideal of S and f be an S-map from I into A. Let P be the set of pairs (J, g) where J is a right $\mathcal{M}$-ideal of S which contains I and g is an S-map from J into A which extends f. Consider P as the ordered set with the order

$$(J_1, g_1) \leq (J_2, g_2) \Leftrightarrow J_1 \subseteq J_2, g_2\mid_{J_1} = g_1.$$

Applying Zorn's lemma, P has a maximal element (K, g_K). If K is not an $\mathcal{M}$-intersection large ideal, there exists a right $\mathcal{M}$-ideal $L \neq \emptyset$ of S such that $L \cap K = \emptyset$. Define a mapping $g^* : K \cup L \to A$ by

$$g^*(s) = \begin{cases} g_K(s) & \text{if } s \in K \\ \theta \text{ (the zero element of A)} & \text{if } s \in L \end{cases}$$

It is easy to see that g^* is an S-map from $K \cup L$ into A, which extends g_K and hence f. Thus $(K \cup L, g^*) \in P$, which contradicts the fact that (K, g_K) is maximal in P. Hence K must be an $\mathcal{M}$-intersection large ideal. Hence by the hypothesis, g_K can be extended to an S-map from S into A, which is therefore an extension of f. ■

Definition 2.18 A semigroup S is called $\mathcal{M}$-Noetherian if it satisfies the ascending chain condition on its right $\mathcal{M}$-ideals.

Remark 2.19 If a semigroup S is $\mathcal{M}$-Noetherian then every right $\mathcal{M}$-ideal I of S is finitely generated. For if, let I be a right $\mathcal{M}$-ideal of S, and assume that $I \neq \emptyset$ is not finitely generated. Thus there exists $x_1 \in I$. Let $\{x_1, x_2, x_3, \ldots\} \subseteq I$ be a countable subset of generating elements of I which are different. Thus

$$\langle x_1 \rangle = I_1 \subseteq \langle x_1, x_2 \rangle = I_2 \subseteq \langle x_1, x_2, x_3 \rangle = I_3 \subseteq ... \subseteq \langle x_1, x_2, ..., x_n \rangle = I_n \subseteq ...$$

is an ascending chain of right $\mathcal{M}$-ideals of S which does not stop and this is a contradiction. Thus I is finitely generated.

Theorem 2.20 *Let S be a left reversible and right $\mathcal{M}$-Noetherian semigroup. Suppose A is a torsion free S-act such that every cyclic subact of A is ideal $\mathcal{M}$-injective. Then A is itself ideal $\mathcal{M}$-injective.*

Proof. Let $I = \bigcup\limits_{i=1}^{n} s_i S^1$ be a right $\mathcal{M}$-ideal of S, and $f : I \to A$ be an S-map. If $f(s_i) = x_i$, then $f\mid_{s_i S^1} \colon s_i S^1 \to x_i S^1$ is an S-map. Since $x_i S^1$ is ideal $\mathcal{M}$-injective by the assumption, there exists $a_i \in x_i S^1$ such that $f(s_i) = a_i s_i, i = 1, 2, \ldots, n$. Since by left reversiblity of S the intersection of all right ideals of S are nonempty, there exists $c_i \in S$ such that $s_1 c_1 = s_2 c_2 = \ldots = s_n c_n$. Then $f(s_1)c_1 = f(s_2)c_2 = \ldots = f(s_n)c_n$ and so $a_1 s_1 c_1 = a_2 s_2 c_2 = \ldots = a_n s_n c_n$. Since A is torsion free we have $a_1 = a_2 = \ldots = a_n = a$. Thus $f(s) = as, s \in I$ and hence A is ideal $\mathcal{M}$-injective. ∎

Remark 2.21 The condition that S is left reversible is necessary in the above theorem. For, take $\mathcal{M}$ as the class of all monomorphisms and see [11], Example 1.3.

Theorem 2.22 *Let S be an $\mathcal{M}$-Noetherian semigroup, A be an S-act which is not finitely generated and each proper subact of A is ideal $\mathcal{M}$-injective. Then A is ideal $\mathcal{M}$-injective.*

Proof. Let $I = \bigcup\limits_{i=1}^{n} s_i S^1$ be a right $\mathcal{M}$-ideal of S, and $f : I \to A$ be an S-map. If $f(I) = A$, then $A = \bigcup\limits_{i=1}^{n} a_i S^1$ where $f(s_i) = a_i \in A$, which implies that A is finitely generated. Hence $f(I) \neq A$. Then $f(I)$ is ideal $\mathcal{M}$-injective by the hypothesis and so $f : I \to f(I)$ is given by $f(s) = as$ for every $s \in I$. Hence A is ideal $\mathcal{M}$-injective. ∎

Remark 2.23 The condition that A is not finitely generated is necessary in the above theorem. For, take $\mathcal{M}$ as the class of all monomorphisms and see [11], Example 1.5.

3. Products, coproducts, and direct sums of different kinds of $\mathcal{M}$-injective acts

In this section we consider the behaviour of different kinds of $\mathcal{M}$-injective acts with respect to products, coproducts, and direct sums.

In the following theorems, it is shown that as usual, the above types of $\mathcal{M}$-injectivity well-behaves with respect to products similar to the case of injectivity using the universal property of products, but not as well with coproducts and direct sums.

Theorem 3.24 *Let $\{A_i : i \in I\}$ be a family of S-acts. Then the product $\prod_{i\in I} A_i$ is $\mathcal{M}$-injective (finitely $\mathcal{M}$-injective, $c\mathcal{M}$-injective, $F\mathcal{M}$-injective, $P\mathcal{M}$-injective) if each A_i is $\mathcal{M}$-injective (finitely $\mathcal{M}$-injective, $c\mathcal{M}$-injective, $F\mathcal{M}$-injective, $P\mathcal{M}$-injective). The converse is true if each A_i has a zero element.*

In the case of weakly $\mathcal{M}$-injective (ideal $\mathcal{M}$-injective) acts we have the following.

Theorem 3.25 *Let $\{A_i : i \in I\}$ be a family of S-acts. Then the product $\prod_{i\in I} A_i$ is weakly $\mathcal{M}$-injective (ideal $\mathcal{M}$-injective) if and only if each A_i is weakly $\mathcal{M}$-injective (ideal $\mathcal{M}$-injective).*

Proof. The fact that the product of weakly $\mathcal{M}$-injective acts is weakly $\mathcal{M}$-injective is proved similar to the case of injectivity using the universal property of products. To prove the converse, let $A = \prod_{i\in I} A_i$ be weakly $\mathcal{M}$-injective, $k \in I$, and define an S-map $f_i : J \to A_i$ by $f_k = f$, and for $i \neq k, f = \lambda_{a_i}$, where a_i is any element of A_i and J is an $\mathcal{M}$-right ideal of S. Then we get an S-map $\overline{f}$ using the universal property of products which extends to an S-map $\overline{\overline{f}} : S \to A$ by weak $\mathcal{M}$-injectivity of A. Now, $p_k\overline{\overline{f}} : S \to A_k$ extends f, where $p_k : A \to A_k$ is the kth projection map. So A is weakly $\mathcal{M}$-injective. ∎

In regard with coproducts, first note that the following is trivially true.

Proposition 3.26 *Let $\{A_i : i \in I\}$ be a family of S-acts with a zero element. If the coproduct $\coprod_{i\in I} A_i$ is $\mathcal{M}$-injective (ideal $\mathcal{M}$-injective, weakly $\mathcal{M}$-injective, finitely $\mathcal{M}$-injective, $c\mathcal{M}$-injective, $F\mathcal{M}$-injective, $P\mathcal{M}$-injective) then each A_i is $\mathcal{M}$-injective (ideal $\mathcal{M}$-injective, weakly $\mathcal{M}$-injective, finitely $\mathcal{M}$-injective, $c\mathcal{M}$-injective, $F\mathcal{M}$-injective, $P\mathcal{M}$-injective).*

The converse of the above theorem is not necessarily true in general (see [10], Theorem 3.4). But in the case of $c\mathcal{M}$-injective ($P\mathcal{M}$-injective) acts the converse is also true.

Proposition 3.27 *Let $\{A_i : i \in I\}$ be a family of $c\mathcal{M}$-injective ($P\mathcal{M}$-injective) S-acts. Then the coproduct $\coprod_{i\in I} A_i$ is $c\mathcal{M}$-injective ($P\mathcal{M}$-injective).*

Proof. Let $\{A_i : i \in I\}$ be a family of $c\mathcal{M}$-injective acts. Notice that for any $\mathcal{M}$-morphism $g : F \to B$ from a cyclic act F and any S-map $f : F \to \coprod A_i$ we have $Imf \subseteq A_i$ for some $i \in I$. Hence f can be extended to an S-map $\overline{f}$, since A_i is $c\mathcal{M}$-injective. ∎

In the case of ideal $\mathcal{M}$-injective (weakly $\mathcal{M}$-injective) acts we have the following.

Theorem 3.28 *Let a semigroup S be left reversible. Then the coproduct $\coprod_{i\in I} A_i$ of each family of ideal $\mathcal{M}$-injective (weakly $\mathcal{M}$-injective) acts is ideal $\mathcal{M}$-injective (weakly $\mathcal{M}$-injective).*

Proof. Let $\{A_i : i \in I\}$ be a family of ideal $\mathcal{M}$-injective acts. Let $f : J \to \coprod_{i\in I} A_i$, which J is an $\mathcal{M}$-right ideal of S, be an S-map. Suppose there exist $i, j \in I, i \neq j$, with $Imf \cap A_i \neq \emptyset$ and $Imf \cap A_j \neq \emptyset$. Then J is a disjoint union of two ideals in contradiction with left reversibility of S. This implies the existence of $i \in I$ such that $Imf \subseteq A_i$. Since A_i is ideal $\mathcal{M}$-injective, f can be written in the form of λ_a for some a in A_i. Thus $\coprod_{i\in I} A_i$ is ideal $\mathcal{M}$-injective. ∎

Theorem 3.29 *If the coproduct $\coprod_{i\in I} A_i$ of each family of ideal $\mathcal{M}$-injective acts is ideal $\mathcal{M}$-injective then any two right $\mathcal{M}$-ideals of S have a nonempty intersection.*

Proof. Let I, J be two $\mathcal{M}$-right ideals of S such that $I \cap J = \emptyset$. By assumption $\Theta \sqcup \Theta = \{b_1, b_2\}$, where Θ is the one element act, is ideal $\mathcal{M}$-injective. The S-map $f : I \sqcup J \to \Theta \sqcup \Theta$ given by

$$f(s) = \begin{cases} b_1 & \text{if } s \in I \\ b_2 & \text{if } s \in J \end{cases}$$

can not be written in the form of λ_a for some $a \in \Theta \sqcup \Theta$ which is a contradiction. Thus each pair of $\mathcal{M}$-right ideals of S have a nonempty intersection. ∎

For the direct sum of different kinds of $\mathcal{M}$-injectivity we first trivially have:

Theorem 3.30 *Let $\{A_i : i \in I\}$ be a family of S-acts with zero such that the direct sum $\bigoplus_{i\in I} A_i$ is $\mathcal{M}$-injective (ideal $\mathcal{M}$-injective, weakly $\mathcal{M}$-injective, finitely $\mathcal{M}$-injective, $c\mathcal{M}$-injective, $F\mathcal{M}$-injective, $P\mathcal{M}$-injective). Then each A_i is $\mathcal{M}$-injective (ideal $\mathcal{M}$-injective, weakly $\mathcal{M}$-injective, finitely $\mathcal{M}$-injective, $c\mathcal{M}$-injective, $F\mathcal{M}$-injective, $P\mathcal{M}$-injective).*

Remark 3.31 The converse of the above theorem is true in the case of finitely $\mathcal{M}$-injective, $c\mathcal{M}$-injective, $F\mathcal{M}$-injective, and $P\mathcal{M}$-injective acts. But it is not true in general, for example take $\mathcal{M}$ as the class of sequentially dense monomorphisms and see [10].

Proposition 3.32 *Each direct sum of finitely $\mathcal{M}$-injective ($c\mathcal{M}$-injective, $F\mathcal{M}$-injective, $P\mathcal{M}$-injective) acts is finitely $\mathcal{M}$-injective ($c\mathcal{M}$-injective, $F\mathcal{M}$-injective, $P\mathcal{M}$-injective).*

Proof. Let $\{A_i\}_{i\in I}$ be a family of finitely $\mathcal{M}$-injective acts. Consider the diagram

$$\begin{tikzcd} X \arrow[r, "\lambda"] \arrow[d, "f"'] & Y \\ \bigoplus_{i\in I} A_i & \end{tikzcd}$$

in which λ is an $\mathcal{M}$-morphism, f is a homomorphism, and X is a finitely generated act. Let $\{x_1, x_2, ..., x_n\}$ be the generating set of X. Then for each j, the element $f(x_j)$ in $\bigoplus_{i\in I} A_i$ has only finitely many nonzero coordinates. Since there are only finitely many x_j, the set $\{f(x_1), f(x_2), ..., f(x_n)\}$ collectively involves only finitely many A_i, say $A_{i_1}, ..., A_{i_n}$. Hence $Imf \subseteq A_{i_1} \bigoplus ... \bigoplus A_{i_n}$ which, being a finite direct sum of finitely $\mathcal{M}$-injective acts (which is in fact a product of finitely $\mathcal{M}$-injective acts), is finitely $\mathcal{M}$-injective, by Theorem 3.25. Hence there is a homomorphism $\bar{f} : Y \to A_{i_1} \bigoplus ... \bigoplus A_{i_n}$ which extends f. We may regard $\bar{f}$ as a homomorphism whose image is in the larger S-act $\bigoplus_{i\in I} A_i$. ∎

Theorem 3.33 *Let a semigroup S be ($\mathcal{M}$-)Noetherian. Then the direct sum $\bigoplus_{i\in I} A_i$ of each family of weakly $\mathcal{M}$-injective acts is weakly $\mathcal{M}$-injective.*

Proof. The proof is similar to the proof of the above theorem by replacing X with every right $\mathcal{M}$-ideal I of S. ∎

About direct sum of ideal $\mathcal{M}$-injective acts we have

Theorem 3.34 *Let S be a semigroup with the zero 0 and for each family $\{I_i : i \in I\}$ of right $\mathcal{M}$-ideals of S, $I = \bigcup_{i\in I} I_i$ be a right $\mathcal{M}$-ideal of S. Then each direct sum of (ideal $\mathcal{M}$-) injective acts is ideal $\mathcal{M}$-injective if and only if S is $\mathcal{M}$-Noetherian.*

Proof. ($\Rightarrow$) Let $\{0\} = I_0 \subseteq I_1 \subseteq \ldots \subseteq I_i \subseteq \ldots$ be an ascending chain of right $\mathcal{M}$-ideals of S, and $I = \bigcup_{i\in I} I_i$. By the hypothesis, I is a right $\mathcal{M}$-ideal of S. Consider the Rees factor acts I/I_i for each i, and let E_i be the injective hull of

I/I_i. Then $E = \bigoplus_i E_i$ is ideal $\mathcal{M}$-injective by the hypothesis. Consider natural epimorphisms $f_i : I \to I/I_i$ and define an S-map $f : I \to E$ by $f(s) = (f_i(s))_i$. Notice that for each $s \in I$ only finitely many components of $f(s)$ are nonzero, because $s \in I_k$ for some k, and so $f_i(s) = 0$ for all $i \geq k$. Now, since E is ideal $\mathcal{M}$-injective by assumption, there exists an element $a \in E, f = \lambda_a$. Since $a \in E = \bigoplus_i E_i, a = (a_1, a_2, \ldots, a_k, \ldots)$, so there is t such that $a_k = 0$ for all $k \geq t$. Then since for any $x \in I, f(x) = ax$, and since $(ax)_t = a_t x = 0$, it follows that $I \subseteq I_t$. Hence $I_{t+1} = I_{t+2} = \ldots = I$. Thus S satisfies the ascending chain condition on its $\mathcal{M}$-right ideals and so is $\mathcal{M}$-Noetherian.

($\Leftarrow$) Let $\{A_i : i \in I\}$ be a family of ideal $\mathcal{M}$-injective acts with zero elements. Let $f : I \to \bigoplus_{i\in I} A_i$ be an S-map from a right $\mathcal{M}$-ideal I of S and assume that I is generated by $\{s_1, s_2, \ldots, s_n\}$, since S is $\mathcal{M}$-Noetherian. Then, since only finitely many components of each $f(s_i)$ are nonzero, we get that Imf is contained in a direct sum of finitely many A_i, say $i_1, i_2, \ldots, i_m$. Then, $A_{i_1} \oplus A_{i_2} \oplus \cdots \oplus A_{i_m}$ which is in fact a product, is ideal $\mathcal{M}$-injective, $f = \lambda_a$, for some $a \in A_{i_1} \oplus A_{i_2} \oplus \cdots \oplus A_{i_m} \subseteq \bigoplus_{i\in I} A_i$. ∎

Proposition 3.35 *Let for each family $\{I_i : i \in I\}$ of right $\mathcal{M}$-ideals of a semigroup S, $I = \bigcup_{i\in I} I_i$ be a right $\mathcal{M}$-ideal of S. If every finitely $\mathcal{M}$-injective act is ideal $\mathcal{M}$-injective then S is $\mathcal{M}$-Noetherian. The converse is true if S is a monoid.*

Proof. Let $\{A_i : i \in I\}$ be a family of ideal $\mathcal{M}$-injective acts. Since each ideal $\mathcal{M}$-injective act is finitely $\mathcal{M}$-injective thus each A_i is finitely $\mathcal{M}$-injective. We know that each direct sum of finitely $\mathcal{M}$-injective acts is finitely $\mathcal{M}$-injective and so by the hypothesis it is ideal $\mathcal{M}$-injective. So, by Theorem 3.34, S is $\mathcal{M}$-Noetherian.

For the converse, let S be a monoid, A be a finitely $\mathcal{M}$-injective act, and $f : I \to A$ be an S-map from a right $\mathcal{M}$-ideal I of S. Since S is $\mathcal{M}$-Noetherian, I is finitely generated. Now, since A is finitely $\mathcal{M}$-injective there exists an S-map $g : S \to A$ which extends f. Then g is of the form λ_a for $a = g(1)$ where 1 is the identity element of the monoid S. Thus f is also of the form λ_a and hence A is ideal $\mathcal{M}$-injective. ∎

We recall the following Theorem from [12].

Theorem 3.36 *Each direct sum of injective S-acts is injective if and only if S is Noetherian.*

Corollary 3.37 *If S is an $\mathcal{M}$-Noetherian semigroup and ideal $\mathcal{M}$-injectivity (weak $\mathcal{M}$-injectivity) for S-acts implies injectivity, then S must be Noetherian.*

Proof. By Theorem 3.36, S is Noetherian if and only if every direct sum of injective S-acts is injective. Now, if $\{A_i : i \in I\}$ is any family of injective acts then they are also ideal $\mathcal{M}$-injective, and so, by Theorem 3.34, their direct sum is ideal $\mathcal{M}$-injective and hence injective by the hypothesis, and so the result. ∎

Definition 3.38 An S-act A is called *countably $\sum$-ideal $\mathcal{M}$-injective* if any countable direct sum of A with itself is ideal $\mathcal{M}$-injective.

Theorem 3.39 *Let for each family $\{I_i : i \in I\}$ of right $\mathcal{M}$-ideals of S, $I = \bigcup_{i\in I} I_i$ be a right $\mathcal{M}$-ideal of S. Then the following are equivalent:*

(1) *Each direct sum of injective acts is ideal $\mathcal{M}$-injective.*

(2) *Each injective act is countably $\sum$-ideal $\mathcal{M}$-injective.*

(3) *S is $\mathcal{M}$-Noetherian.*

Proof. (1)⇒(2) is clear.

(2)⇒(3) Applying the notations of Theorem 3.34, put $A = \prod_{n\in\mathbb{N}} E_n$. Then A is injective, and so by (2), $\bigoplus_{n\in\mathbb{N}} A$ is ideal $\mathcal{M}$-injective. But, $A = \prod_{n\in\mathbb{N}} E_n = E_m \oplus \prod_{n\neq m} E_n$, for each $m \in \mathbb{N}$. Thus $\bigoplus_{m\in\mathbb{N}} A = \bigoplus_{m\in\mathbb{N}} E_m \oplus \bigoplus_{m\in\mathbb{N}} \prod_{n\neq m} E_n = E \oplus \bigoplus_{m\in\mathbb{N}} \prod_{n\neq m} E_n$, which means that E is a direct summand of an ideal $\mathcal{M}$-injective act and hence is ideal $\mathcal{M}$-injective, by Theorem 3.30. The rest of the proof is similar to Theorem 3.34.

(3)⇒(1) is proved similar to Theorem 3.34. ∎

4. Some Baer conditions

The condition that weak injectivity implies injectivity is known as the *Baer Criterion* for injectivity. In this section we give some Baer conditions.

Definition 4.40 An S-act A is called

(i) *quasi injective* if any S-map $f : B \to A$ from a subact B of A can be extended to A.

(ii) *$\sum$-injective* (*$\sum$-quasi injective)* if every direct sum of A with itself is injective (quasi injective).

Theorem 4.41 *The following conditions are equivalent:*

(1) *Each weakly $\mathcal{M}$-injective act is injective and S is $\mathcal{M}$-Noetherian.*

(2) *Each $F\mathcal{M}$-injective act is injective.*

(3) *Each $F\mathcal{M}$-injective act is quasi injective with a zero element.*

Proof. (1)⇒(2) Since S is $\mathcal{M}$-Noetherian, each right $\mathcal{M}$-ideal is finitely generated, so an act A is weakly $\mathcal{M}$-injective if and only if it is $F\mathcal{M}$-injective.

(2)⇒(3) is clear.

(3)⇒(2) Let A be $F\mathcal{M}$-injective and $E(A)$ be the injective hull of A. Then, by Theorem 3.32, $A \oplus E(A)$ is $F\mathcal{M}$-injective and hence by (3) quasi injective. The rest of the proof is quite similar to the proof of (4)⇒(1) of Theorem 4.42.

(2)⇒(1) Since every weakly $\mathcal{M}$-injective act is $F\mathcal{M}$-injective, we get the first part. To see that S is $\mathcal{M}$-Noetherian, using Theorem 3.36 we show that any direct sum of injective acts is injective. This is true because of (2) and since ,by Proposition 3.32, any direct sum of $F\mathcal{M}$-injective acts is $F\mathcal{M}$-injective. ∎

Theorem 4.42 *If S is an $\mathcal{M}$-Noetherian semigroup, then the following are equivalent:*

(1) *Each weakly (ideal) $\mathcal{M}$-injective act is injective.*

(2) *Each weakly (ideal) $\mathcal{M}$-injective act is $\sum$-injective and has a fixed element.*

(3) *Each weakly (ideal) $\mathcal{M}$-injective act is $\sum$-quasi injective and has a fixed element.*

(4) *Each weakly (ideal) $\mathcal{M}$-injective act is quasi injective and has a fixed element.*

Proof. (1)⇒(2) is true using Theorem 3.33. Also recall that each injective act has a fixed element.

(2)⇒(3) and (3)⇒(4) are clear.

(4)⇒(1) Let A be weakly $\mathcal{M}$-injective and $E(A)$ be the injective hull of A. Then $A \oplus E(A)$ is weakly $\mathcal{M}$-injective and hence quasi injective by (4). So, considering the monomorphism $A \hookrightarrow E(A) \xrightarrow{\tau_{E(A)}} A \oplus E(A)$, where $\tau_{E(A)}$ is the injection $x \mapsto (0, x)$, and the S-map $\tau_A : A \to A \oplus E(A)$, there exists an S-map $g : A \oplus E(A) \to A \oplus E(A)$ such that $g\tau_{E(A)} |_A = \tau_A$. This implies that $p_A g \tau_{E(A)} |_A = id_A$ and so A is a retract of the injective act $E(A)$ and hence is injective. ∎

Theorem 4.43 *Let for each family $\{I_i : i \in I\}$ of right $\mathcal{M}$-ideals of S, $I = \bigcup_{i \in I} I_i$ be a right $\mathcal{M}$-ideal of S. Then the following are equivalent:*

(1) *The direct sum of each family of weakly $\mathcal{M}$-injective acts is ideal $\mathcal{M}$-injective.*

(2) *S is $\mathcal{M}$-Noetherian and weak $\mathcal{M}$-injectivity implies ideal $\mathcal{M}$-injectivity.*

(3) *Each $F\mathcal{M}$-injective act is ideal $\mathcal{M}$-injective.*

Proof. (1)⇒(2) Similar to the proof of Theorem 3.34 it is shown that S is $\mathcal{M}$-Noetherian, the rest is clear.

(2)⇒(1) It is concluded from Theorem 3.34.

(2) ⇒ (3) Since S is $\mathcal{M}$-Noetherian, each right $\mathcal{M}$-ideal of S is finitely generated and so weak $\mathcal{M}$-injectivity coincides with $F\mathcal{M}$-injectivity. So (3) holds.

(3) ⇒ (2) Since every weakly $\mathcal{M}$-injective act is $F\mathcal{M}$-injective we get the first part. For the second part, let $\{A_i : i \in I\}$ be a family of injective acts. Since each injective act is $F\mathcal{M}$-injective and since each direct sum of $F\mathcal{M}$-injective acts is $F\mathcal{M}$-injective, thus the direct sum $\bigoplus_{i\in I} A_i$ is $F\mathcal{M}$-injective and so ideal $\mathcal{M}$-injective by the hypothesis. Thus S is $\mathcal{M}$-Noetherian, by Theorem 3.34. ∎

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ON KÖTHE-TOEPLITZ DUALS OF SOME NEW AND GENERALIZED DIFFERENCE SEQUENCE SPACES

A.A. Ansari

V.K. Chaudhry

Department of Mathematics and Statistics
D.D.U. Gorakhpur University
Gorakhpur-273009
India
e-mail: aaansari@in.com
2 vkc_ddumath@yahoo.co.in

Abstract. In this paper we define the sequence spaces $\triangle_{v,r}^m(l_\infty)$, $\triangle_{v,r}^m(c)$ and $\triangle_{v,r}^m(c_0)$, $(m \in N,\ r \in R)$, and have studied some of their topological properties and have computed their Köthe-Toeplitz duals.

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1. Introduction

Let l_∞, c and c_0 be the linear spaces of bounded convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by

$$\|x\|_\infty = \sup_k |x_k|$$

where $k \in N = \{1, 2, 3, ...\}$, the set of positive integers.

In 1981, Kizmaz [8] introduced the concept of difference sequences and have defined $\triangle-$bounded, $\triangle-$ convergent and $\triangle-$ null sequence spaces. Using the concept of difference sequences, Et [4] has defined $\triangle^2-$bounded, $\triangle^2-$ convergent and $\triangle^2-$null sequence spaces. Further, this notion was generalized by Et and Colak [6] and have defined $\triangle^m-$bounded, $\triangle^m-$convergent and $\triangle^m-$null sequence spaces. Later on, Et and Esi [5] have defined $\triangle_v^m-$bounded, $\triangle_v^m-$convergent and $\triangle_v^m-$null sequence spaces where $v = (v_k)$ be any fixed sequence of non-zero complex numbers. Recently, Bektas and Colak [1] have defined the sequence spaces

$$l_\infty(\triangle_r^m) = \{x = (x_k) : (k^r \triangle^m x_k) \in l_\infty\},$$
$$c(\triangle_r^m) = \{x = (x_k) : (k^r \triangle^m x_k) \in c\},$$
$$c_0(\triangle_r^m) = \{x = (x_k) : (k^r \triangle^m x_k) \in c_0\}.$$

where $m \in N$, $r \in R$, $\triangle_r^m x = (\triangle_r^m x_k) = (k^r \triangle^m x_k) = (k^r(\triangle^{m-1}x_k - \triangle^{m-1}x_{k+1}))$ and

$$\triangle^m x_k = \sum_{j=1}^{m} (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} x_{k+j}.$$

These are Banach spaces with norm

$$||x||_{\triangle_r} = \sum_{i=1}^{m} |x_i| + \sup_k k^r |\triangle^m x_k|.$$

It is trivial that $c_0(\triangle_r^m) \subset c_0(\triangle_r^{m+1})$, $c(\triangle_r^m) \subset c(\triangle_r^{m+1})$, $l_\infty(\triangle_r^m) \subset l_\infty(\triangle_r^{m+1})$ and $c_0(\triangle_r^m) \subset c(\triangle_r^m) \subset l_\infty(\triangle_r^m)$ are satisfied and are strict [1].

For convenience, we denote these spaces $\triangle_r^m(l_\infty) = l_\infty(\triangle_r^m)$, $\triangle_r^m(c) = c(\triangle_r^m)$ and $\triangle_r^m(c_0) = c_0(\triangle_r^m)$.

Let $v = (v_k)$ be any fixed sequence of non-zero complex numbers. Now, we define

$$\triangle_{v,r}^m(l_\infty) = \{x = (x_k) \ : \ (k^r \triangle_v^m x_k) \in l_\infty\},$$
$$\triangle_{v,r}^m(c) = \{x = (x_k) \ : \ (k^r \triangle_v^m x_k) \in c\},$$
$$\triangle_{v,r}^m(c_0) = \{x = (x_k) \ : \ (k^r \triangle_v^m x_k) \in c_0\}.$$

where $m \in N$, $r \in R$, $\triangle_{v,r}^m(x) = (k^r \triangle_v^m x_k) = (k^r(\triangle_v^{m-1}x_k - \triangle_v^{m-1}x_{k+1}))$ and

$$\triangle_v^m x_k = \sum_{j=1}^{m} (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} v_{k+j}\, x_{k+j}.$$

Throughout the paper, we write X for l_∞ or c or c_0. $\triangle_{v,r}^m(X)$ is the generalization of several known sequence spaces, for instance, the following classes arise from $\triangle_{v,r}^m(X)$ as the special cases.

(i) If we take $v = (v_k) = (1, 1, ...)$, then $\triangle_{v,r}^m(X) = \triangle_r^m(X)$ [1].

(ii) If we take $r = 0$, then $\triangle_{v,r}^m(X) = \triangle_v^m(X)$ [5].

(iii) If we take $v = (v_k) = (1, 1, ...)$ and $r = 0$, then $\triangle_{v,r}^m(X) = \triangle^m(X)$ [6].

(iv) If we take $v = (v_k) = (1, 1, ...)$, $r = 0$ and $m = 2$, then $\triangle_{v,r}^m(X) = \triangle^2(X)$ [4].

(v) If we take $v = (v_k) = (1, 1, ...)$, $r = 0$ and $m = 1$, then $\triangle_{v,r}^m(X) = \triangle(X)$ [8].

(vi) If we take $r = 0$ and $m = 1$, then $\triangle_{v,r}^m(X) = \triangle_v(X)$ [2].

(vii) If we take $v_k = 1$ for all $k \in N$, $r < 1$ and $m = 1$, then $\triangle_{v,r}^m(X) = \triangle_r(X)$ [10].

2. Main results

Theorem 2.1. *The sequence spaces $\triangle_{v,r}^m(l_\infty)$, $\triangle_{v,r}^m(c)$ and $\triangle_{v,r}^m(c_0)$ are Banach spaces normed by*

$$||x||_{\triangle_{v,r}} = \sum_{i=1}^{m} |v_i x_i| + \sup_k k^r |\triangle_v^m x_k| \tag{2.1}$$

Let us define the operator

$$D : \triangle_{v,r}^m(X) \to \triangle_{v,r}^m(X)$$

by

$$Dx = (0, 0, 0, ...x_{m+1}, x_{m+2}, ...),$$

where $x = (x_1, x_2, x_3, ...)$. It is trivial that D is bounded linear operator on $\triangle_{v,r}^m(X)$. Furthermore, the set

$$D[\triangle_{v,r}^m(X)] = \{x = (x_k) : x \in \triangle_{v,r}^m(X),\ x_1 = x_2 = ... = x_m = 0\}$$

is a subspace of $\triangle_{v,r}^m(X)$ and $\|x\|_{\triangle_{v,r}} = \|\triangle_{v,r}^m(X)\|_\infty$ in $D[\triangle_{v,r}^m(X)]$. $D[\triangle_{v,r}^m(X)]$ and X are equivalent as topological space. Hence

$$\triangle_{v,r}^m : D[\triangle_{v,r}^m(X)] \to X,$$

defined by

$$\triangle_{v,r}^m x = y = (\triangle_{v,r}^m x_k) = (k^r(\triangle_v^m x_k)) \tag{2.2}$$

is a linear homeomorphism [9].

3. Dual spaces

In this section, we give Köthe-Toeplitz duals of $\triangle_{v,r}^m(l_\infty)$, $\triangle_{v,r}^m(c)$ and $\triangle_{v,r}^m(c_0)$. Also, we show that these spaces are not perfect spaces. Further, we show that $\triangle_{v,r}^m(l_\infty)$, and $\triangle_{v,r}^m(c)$ are not normal and not monotone spaces.

Lemma 3.1. $\sup_k k^r |\triangle_v^m x_k| < \infty$ *if and only if*

(i) $\sup_k k^{r-1} |\triangle_v^{m-1} x_k| < \infty$

(ii) $\sup_k k^r |\triangle_v^{m-1} x_k - k(k+1)^{-1} \triangle_v^{m-1} x_{k+1}| < \infty.$

Proof. Let $\sup_k k^r|\triangle_v^m x_k| < \infty$. Then

$$\begin{aligned} |\triangle_v^{m-1} x_1 - \triangle_v^{m-1} x_{k+1}| &= \left|\sum_{j=1}^{k} (\triangle_v^{m-1} x_j - \triangle_v^{m-1} x_{j+1})\right| \\ &\leq \sum_{j=1}^{k} |\triangle_v^m x_j| = O(k^{1-r}). \end{aligned}$$

This implies $\sup_k k^{r-1}|\triangle_v^{m-1} x_k| < \infty$,

$$|\triangle_v^{m-1} x_k - k(k+1)^{-1}\triangle_v^{m-1} x_{k+1}| = |k(k+1)^{-1}\triangle_v^m x_k + (k+1)^{-1}\triangle_v^{m-1} x_k| = O(k^{-r})$$

We have (ii).

Now, suppose (i) and (ii) hold. Then

$$k^r|\triangle_v^{m-1} x_k - k(k+1)^{-1}\triangle_v^{m-1} x_{k+1}| \geq k^{r+1}(k+1)^{-1}|\triangle_v^m x_k| - k^r(k+1)^{-1}|\triangle_v^{m-1} x_k.|$$

This implies $\sup_k k^r|\triangle_v^m x_k| < \infty$.

Lemma 3.2. *$\sup_k k^{r-n}|\triangle_v x_k| < \infty$ implies $\sup_k k^{r-(n+1)}|v_k x_k| < \infty$ for all $n \in N$ with $r < (n+1)$.*

Proof. Let $\sup_k k^{r-n}|\triangle_v x_k| < \infty$. Then

$$\begin{aligned} |v_1 x_1 - v_{k+1} x_{k+1}| &\leq \sum_{i=1}^{k} |v_i x_i - v_{i+1} x_{i+1}| \\ &\leq \sum_{i=1}^{k} |\triangle_v x_i| = O(k^{(n+1)-r}) \end{aligned}$$

This implies $\sup_k k^{r-(n+1)}|v_k x_k| < \infty$.

Lemma 3.3. *$\sup_k k^{r-n}|\triangle_v^{m-n} x_k| < \infty$ implies $\sup_k k^{r-(n+1)}|\triangle_v^{m-(n+1)} x_k| < \infty$ for all $n, m \in N$ and $r \leq n < m$.*

Proof. If we put $\triangle_v^{m-n} x_k$ instead of $\triangle_v x_k$ in Lemma 2.2, the result is immediate.

Corollary 3.4. *$\sup_k k^{r-1}|\triangle_v^{m-1} x_k| < \infty$ implies $\sup_k k^{r-m}|v_k\, x_k| < \infty$.*

Corollary 3.5. *$x \in \triangle_{v,r}^m(l_\infty)$ implies $\sup_k k^{r-m}|v_k\, x_k| < \infty$.*

Lemma 3.6. ([8]) *Let (P_n) be sequence of positive real numbers increasing monotonically to infinity, then*

(i) *If* $\sup_n \left|\sum_{i=1}^{n} P_i a_i\right| < \infty$, *then* $\sup_n \left|P_n \sum_{k=n+1}^{\infty} a_k\right| < \infty$,

(ii) *If* $\sum_{k=1}^{\infty} P_k a_k$ *is convergent, then* $\lim_{n\to\infty} P_n \sum_{k=n+1}^{\infty} a_k = 0$.

Definition 3.7. ([7]) Let X be a sequence space and define

$$\begin{aligned} X^{\alpha} &= \Big\{a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k| < \infty, \text{ for all } x \in X\Big\}, \\ X^{\beta} &= \Big\{a = (a_k) : \sum_{k=1}^{\infty} a_k x_k \text{ is convergent, for all } x \in X\Big\}, \\ X^{\gamma} &= \Big\{a = (a_k) : \sup_n |\sum_{k=1}^{n} a_k x_k| < \infty, \text{ for all } x \in X\Big\}. \end{aligned}$$

Definition 3.8. ([7]) Let X be a sequence space. Then X is called

(i) *perfect* if $X = X^{\alpha\alpha}$,

(ii) *normal* if $y \in X$ whenever $|y_k| \le |x_k|$, $k \ge 1$, for some $x \in X$,

(iii) *monotone* provided X contains the canonical preimages of all its stepspaces.

Lemma 3.9. ([7]) *Let X be a sequence space. Then we have*

(i) *X is perfect $\Rightarrow$ X is normal $\Rightarrow$ X is monotone,*

(ii) *X is normal $\Rightarrow X^{\alpha} = X^{\gamma}$,*

(iii) *X is monotone $\Rightarrow X^{\alpha} = X^{\beta}$.*

Theorem 3.10. *Let m be a positive integer and $r \in R$,*

(a) *We put*

$$M_{\alpha}(v, r) = \Big\{a = (a_k) : \sum_{k=1}^{\infty} k^{m-r} |a_k v_k^{-1}| < \infty\Big\}.$$

Then

$$[\triangle_{v,r}^{m}(l_{\infty})]^{\alpha} = [\triangle_{v,r}^{m}(c)]^{\alpha} = [\triangle_{v,r}^{m}(c_0)]^{\alpha} = M_{\alpha}(v, r) \tag{3.1}$$

(b) *We put*

$$M_{\alpha\alpha}(v,r)=\Big\{a=(a_k):\sup_k k^{r-m}|a_kv_k|<\infty\Big\}.$$

Then

$$[\triangle^m_{v,r}(l_\infty)]^{\alpha\alpha}=[\triangle^m_{v,r}(c)]^{\alpha\alpha}=[\triangle^m_{v,r}(c_0)]^{\alpha\alpha}=M_{\alpha\alpha}(v,r) \tag{3.2}$$

Proof. (a) First, we assume that $a\in M_\alpha(v,r)$. Then

$$\sum_{k=1}^\infty |a_kx_k|=\sum_{k=1}^\infty k^{m-r}|a_kv_k^{-1}|k^{r-m}|x_kv_k|<\infty,$$

for each $x\in\triangle^m_{v,r}(l_\infty)$, by Corollary 3.5.

Thus, we have shown

$$M_\alpha(v,r)\subset[\triangle^m_{v,r}(l_\infty)]^\alpha \tag{3.3}$$

Conversely, let $a\notin M_\alpha(v,r)$. Then, for some k, we have

$$\sum_{k=1}^\infty k^{m-r}|a_kv_k^{-1}|=\infty.$$

So, there is a strictly increasing sequence (n_i) of positive integers n_i such that

$$\sum_{k=n_i+1}^{n_{i+1}} k^{m-r}|a_kv_k^{-1}|>i.$$

We define a sequence $x=(x_k)$ by

$$x_k=\begin{cases}0 & (1\le k\le n_i)\\ \dfrac{v_k^{-1}k^{m-r}}{i} & (n_i+1<k\le n_{i+1};\ i=1,2,...)\end{cases}$$

Then, we see that

$$k^r|\triangle^m v_kx_k|=\frac{m!}{i}\quad (n_i+1<k\le n_{i+1};\ i=1,2,...).$$

Hence, $x\in\triangle^m_{v,r}(c_0)$ and $\displaystyle\sum_{k=1}^\infty|a_kx_k|>\sum_{i=1}^\infty 1=\infty$.

Thus, $a\notin[\triangle^m_{v,r}(c_0)]^\alpha$, and hence we have shown

$$[\triangle^m_{v,r}(c_0)]^\alpha\subset M_\alpha(v,r) \tag{3.4}$$

Since $\triangle_{v,r}^m(c_0) \subset \triangle_{v,r}^m(c) \subset \triangle_{v,r}^m(l_\infty)$ implies $[\triangle_{v,r}^m(l_\infty)]^\alpha \subset [\triangle_{v,r}^m(c)]^\alpha \subset [\triangle_{v,r}^m(c_0)]^\alpha$, (3.1) follows from (3.3) and (3.4).

(b) First, we assume that $a \in M_{\alpha\alpha}(v,r)$. Then

$$\sum_{k=1}^{\infty} |a_k x_k| \le \sup_k k^{r-m} |a_k v_k| \sum_{k=1}^{\infty} k^{m-r} |x_k v_k^{-1}| < \infty,$$

for each $x \in [\triangle_{v,r}^m(c_0)]^\alpha = M_\alpha(v,r)$, by part (a).

Thus, we have shown

$$M_{\alpha\alpha}(v,r) \subset [\triangle_{v,r}^m(c_0)]^{\alpha\alpha} \tag{3.5}$$

Conversely, let $a \notin M_{\alpha\alpha}(v,r)$. Then, we have

$$\sup_k k^{r-m} |a_k v_k| = \infty.$$

Hence, there is a strictly increasing sequence of $(k(i))$ of positive integers $k(i)$ such that

$$[k(i)]^{r-m} \, |a_{k(i)} v_{k(i)}| > i^m.$$

We define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} |a_{k(i)}|^{-1}, & k = k(i) \\ 0, & k \ne k(i). \end{cases}$$

Then, we see that

$$\sum_{k=1}^{\infty} k^{m-r} |x_k v_k^{-1}| = \sum_{i=1}^{\infty} [k(i)]^{m-r} |a_{k(i)} v_{k(i)}|^{-1} \le \sum_{i=1}^{\infty} i^{-m} < \infty.$$

Hence, $x \in [\triangle_{v,r}^m(l_\infty)]^\alpha$ and $\sum\limits_{k=1}^{\infty} |a_k x_k| = \sum\limits_{i=1}^{\infty} 1 = \infty$.

Thus, $a \notin [\triangle_{v,r}^m(l_\infty)]^{\alpha\alpha}$, and hence, we have shown

$$[\triangle_{v,r}^m(l_\infty)]^{\alpha\alpha} \subset M_{\alpha\alpha}(v,r). \tag{3.6}$$

Since $[\triangle_{v,r}^m(l_\infty)]^\alpha \subset [\triangle_{v,r}^m(c)]^\alpha \subset [\triangle_{v,r}^m(c_0)]^\alpha$ implies $[\triangle_{v,r}^m(c_0)]^{\alpha\alpha} \subset [\triangle_{v,r}^m(c)]^{\alpha\alpha} \subset [\triangle_{v,r}^m(l_\infty)]^{\alpha\alpha}$, (3.2) follows from (3.5) and (3.6).

From Theorem 3.10, we have the following corollaries:

Corollary 3.11. *If we take $v_k = (1, 1, ...)$, then we obtain*

(i) $[\triangle_r^m(l_\infty)]^\alpha = [\triangle_r^m(c)]^\alpha = [\triangle_r^m(c_0)]^\alpha = \{a = (a_k) : \sum_{k=1}^{\infty} k^{m-r}|a_k| < \infty\}$

(ii) $[\triangle_r^m(l_\infty)]^{\alpha\alpha} = [\triangle_r^m(c)]^{\alpha\alpha} = [\triangle_r^m(c_0)]^{\alpha\alpha} = \{a = (a_k) : \sup_k k^{r-m}|a_k| < \infty\}$.

Corollary 3.12. *If we take $r = 0$, then we obtain*

(i) $[\triangle_v^m(l_\infty)]^\alpha = [\triangle_v^m(c)]^\alpha = [\triangle_v^m(c_0)]^\alpha = \{a = (a_k) : \sum_{k=1}^{\infty} k^{m}|a_k v_k^{-1}| < \infty\}$ [5]

(ii) $[\triangle_v^m(l_\infty)]^{\alpha\alpha} = [\triangle_v^m(c)]^{\alpha\alpha} = [\triangle_v^m(c_0)]^{\alpha\alpha} = \{a = (a_k) : \sup_k k^{-m}|a_k v_k| < \infty\}$. [5]

Corollary 3.13. *If we take $v_k = k^m$ and $r = 0$, then we obtain*

(i) $[\triangle_v^m(l_\infty)]^\alpha = [\triangle_v^m(c)]^\alpha = [\triangle_v^m(c_0)]^\alpha = l_1$ [5]

(ii) $[\triangle_v^m(l_\infty)]^{\alpha\alpha} = [\triangle_v^m(c)]^{\alpha\alpha} = [\triangle_v^m(c_0)]^{\alpha\alpha} = l_\infty$. [5]

Corollary 3.14. *If we take $v_k = (1, 1, ...)$ and $r = 0$, then we obtain*

(i) $[\triangle^m(l_\infty)]^\alpha = [\triangle^m(c)]^\alpha = [\triangle^m(c_0)]^\alpha = \{a = (a_k) : \sum_{k=1}^{\infty} k^{m}|a_k| < \infty\}$ [6]

(ii) $[\triangle^m(l_\infty)]^{\alpha\alpha} = [\triangle^m(c)]^{\alpha\alpha} = [\triangle^m(c_0)]^{\alpha\alpha} = \{a = (a_k) : \sup_k k^{-m}|a_k| < \infty\}$. [6]

Corollary 3.15. *If we take $v_k = (1, 1, ...)$, $r = 0$ and $m = 2$, then we obtain*

(i) $[\triangle^2(l_\infty)]^\alpha = [\triangle^2(c)]^\alpha = \{a = (a_k) : \sum_{k=1}^{\infty} k^{2}|a_k| < \infty\}$ [4]

(ii) $[\triangle^2(l_\infty)]^{\alpha\alpha} = [\triangle^2(c)]^{\alpha\alpha} = \{a = (a_k) : \sup_k k^{-2}|a_k| < \infty\}$. [4]

Corollary 3.16. *If we take $v_k = (1, 1, ...)$, $r = 0$ and $m = 1$, then we obtain*

(i) $[\triangle(l_\infty)]^\alpha = [\triangle(c)]^\alpha = \{a = (a_k) : \sum_{k=1}^{\infty} k|a_k| < \infty\}$ [8]

(ii) $[\triangle(l_\infty)]^{\alpha\alpha} = [\triangle(c)]^{\alpha\alpha} = \{a = (a_k) : \sup_k k^{-1}|a_k| < \infty\}$. [5]

Corollary 3.17. *If we take $v_k = k^m$, then we obtain*

(i) $[\triangle^m_{v,r}(l_\infty)]^\alpha = [\triangle^m_{v,r}(c)]^\alpha = [\triangle^m_{v,r}(c_0)]^\alpha = \{a = (a_k) : \sum_{k=1}^{\infty} k^{-r}|a_k| < \infty\}$.

(ii) $[\triangle^m_{v,r}(l_\infty)]^{\alpha\alpha} = [\triangle^m_{v,r}(c)]^{\alpha\alpha} = [\triangle^m_{v,r}(c_0)]^{\alpha\alpha} = \{a = (a_k) : \sup_k k^r|a_k| < \infty\}$.

By Lemma 3.8, we also have

Corollary 3.18. *The sequence spaces $\triangle^m_{v,r}(l_\infty)$, $\triangle^m_{v,r}(c)$ and $\triangle^m_{v,r}(c_0)$ are not perfect.*

Lemma 3.19. *Let m be any positive integers and let r be any real number.*

a) *We put*

$$M_\beta(v,r) = \{a = (a_k) : \sum_{k=1}^{\infty} k^{m-r} a_k v_k^{-1} \text{ is convergent, } \sum_{k=1}^{\infty} k^{m-(r+1)}|R_k| < \infty\},$$

where $R_k = \sum_{j=k+1}^{\infty} a_j v_j^{-1}$. Then

$$(D[\triangle^m_{v,r}(l_\infty)])^\beta = M_\beta(v,r).$$

b) *We put*

$$M_\gamma(v,r) = \{a = (a_k) : \sup_n |\sum_{k=1}^{n} k^{m-r} a_k v_k^{-1}| < \infty, \ \sum_{k=1}^{\infty} k^{m-(r+1)}|R_k| < \infty\},$$

where $R_k = \sum_{j=k+1}^{\infty} a_j v_j^{-1}$. Then

$$(D[\triangle^m_{v,r}(l_\infty)])^\gamma = M_\gamma(v,r).$$

Proof. (a) If $x \in D[\triangle^m_{v,r}(l_\infty)]$, then there exist one and only one $y = (y_k) \in l_\infty$ such that

$$\begin{aligned} x_k &= v_k^{-1} \sum_{j=1}^{k-m} (-1)^m \binom{k-j-1}{m-1} j^{-r} y_j \\ &= v_k^{-1} \sum_{j=1}^{k} (-1)^m \binom{k+m-j-1}{m-1} (j-m)^{-r} y_{j-m} \\ y_{1-m} &= y_{2-m} = ... = y_0 = 0, \end{aligned}$$

for sufficiently large k, for instance $k > m$ by (2.2). Let $a \in M_\beta(v,r)$ and suppose that $\binom{-1}{-1} = 1$ (in some literature it is assumed that $\binom{r}{k} = 0$ for $k < 0$). Then, we may write

$$\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n} a_k \Big(v_k^{-1} \sum_{j=1}^{k-m} (-1)^m \begin{pmatrix} k-j-1 \\ m-1 \end{pmatrix} j^{-r} y_j \Big)$$

$$= (-1)^m \sum_{k=1}^{n-m} R_{k+m-1} (k+m-1)^{m-(r+1)} \frac{1}{(k+m-1)^{m-(r+1)}}$$

$$\sum_{j=1}^{k} \begin{pmatrix} k+m-j-2 \\ m-2 \end{pmatrix} j^{-r} y_j - n^{m-r} R_n n^{r-m} x_n v_n \tag{3.7}$$

Since $\sum\limits_{k=1}^{\infty} k^{m-(r+1)} |R_k| < \infty$, the series $\sum\limits_{k=1}^{\infty} (k+m-1)^{m-(r+1)} R_{k+m-1} z_k$ is absolutely convergent, where

$$z = (z_k) = \left(\frac{1}{(k+m-1)^{m-(r+1)}} \sum_{j=1}^{k} \begin{pmatrix} k+m-j-2 \\ m-2 \end{pmatrix} j^{-r} y_j \right).$$

Moreover, we have $n^{m-r} R_n \to 0$ as $n \to \infty$ (Lemma 3.6), $\sup\limits_k n^{r-m} |x_n v_n| < \infty$ (Corollary 3.5), hence $\sum\limits_{k=1}^{\infty} a_k x_k$ is convergent for each $x \in D[\triangle_{v,r}^m (l_\infty)]$, so $a \in (D[\triangle_{v,r}^m (l_\infty)])^\beta$.

Let $a \in (D[\triangle_{v,r}^m (l_\infty)])^\beta$. Then, $\sum\limits_{k=1}^{\infty} a_k x_k$ is convergent for each $x \in D[\triangle_{v,r}^m (l_\infty)]$. For the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} 0, & k \le m, \\ v_k^{-1} k^{m-r}, & k > m, \end{cases}$$

we may write

$$\sum_{k=1}^{\infty} k^{m-r} a_k v_k^{-1} = \sum_{k=1}^{m} k^{m-r} a_k v_k^{-1} + \sum_{k=m+1}^{\infty} a_k x_k < \infty.$$

Thus, the series $\sum\limits_{k=1}^{\infty} k^{m-r} a_k v_k^{-1}$ is convergent. This implies $n^{m-r} R_n \to 0$ as $n \to \infty$ by Lemma 3.6.

Now, let $a \in D[\triangle_{v.r}^m (l_\infty)]^\beta - M_\beta(v, r)$. Then $\sum\limits_{k=1}^{\infty} k^{m-(r+1)} |R_k|$ is divergent, that is $\sum\limits_{k=1}^{\infty} k^{m-(r+1)} |R_k| = \infty$.

We define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} 0, & k \le m \\ v_k^{-1} \sum\limits_{j=1}^{k-1} j^{m-(r+1)} \operatorname{sgn} R_j, & k > m, \end{cases}$$

where $a_k > 0$ for all k or $a_k < 0$ for all k. Since $k^r|\triangle_v^m x_k| = (m-1)!$ for $k > m$, it is trivial that $x = (x_k) \in D[\triangle_{v,r}^m(l_\infty)]$. Then, we may write for $n > m$

$$\sum_{k=1}^{n} a_k x_k = -\sum_{k=1}^{m} R_{k-1}\triangle_v x_{k-1} - \sum_{k=1}^{n-1} R_{k+m-1}\triangle_v x_{k+m-1} - n^{m-r}R_n n^{r-m} x_n v_n$$

Now, letting $n \to \infty$, we get

$$\begin{aligned}\sum_{k=1}^{\infty} a_k x_k &= -\sum_{k=1}^{\infty} R_{k+m-1}\triangle_v x_{k+m-1} \\ &= \sum_{k=1}^{\infty}(k+m-1)^{m-(r+1)}|R_{k+m-1}| = \infty.\end{aligned}$$

This contradicts to $a \in (D[\triangle_{v,r}^m(l_\infty)])^\beta$. Hence, $a \in M_\beta(v,r)$.

(b) can be proved by the same way as above, using Lemma 3.6.

Lemma 3.20. $(D[\triangle_{v,r}^m(l_\infty)])^\eta = (D[\triangle_{v,r}^m(c)])^\eta$ *for* $\eta = \beta$ *or* γ.

Lemma 3.21.

(i) $[\triangle_{v,r}^m(l_\infty)]^\eta = (D[\triangle_{v,r}^m(l_\infty)])^\eta$

(ii) $[\triangle_{v,r}^m(c)]^\eta = (D[\triangle_{v,r}^m(l_\infty)])^\eta$

for $\eta = \beta$ *or* γ.

Proof. (i) We give the proof for $\eta = \beta$ only. It can be proved in a similar way for $\eta = \gamma$. Since $D[\triangle_{v,r}^m(l_\infty)] \subset \triangle_{v,r}^m(l_\infty)$, then $[\triangle_{v,r}^m(l_\infty)]^\beta \subset (D[\triangle_{v,r}^m(l_\infty)])^\beta$. Let $a \in (D[\triangle_{v,r}^m(l_\infty)])^\beta$. If $x = (x_k) \in \triangle_{v,r}^m(l_\infty)$ defined by

$$x_k = \begin{cases} x_k, & k \le m \\ x_k', & k > m, \end{cases}$$

where $x' = (x_k') \in D[\triangle_{v,r}^m(l_\infty)]$, then we may write for $n > m$

$$\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{m} a_k x_k + \sum_{k=m+1}^{n} a_k x_k'.$$

Now, letting $n \to \infty$, we get the series, in the same way as the proof of Lemma 3.19,

$$\sum_{k=1}^{\infty} a_k x_k = \sum_{k=1}^{m} a_k x_k + (-1)^m \sum_{k=1}^{\infty}(k+m-1)^{m-(r+1)} R_{k+m-1} z_k$$

is convergent. This implies that $a \in [\triangle_{v,r}^m(l_\infty)]^\beta$.

(ii) can be proved by the same way as above.

Theorem 3.22. *Let X stands for l_∞ or c. Then*

a) $$[\triangle^m_{v,r}(X)]^\beta = \Big\{a = (a_k) : \sum_{k=1}^{\infty} k^{m-r} a_k v_k^{-1} \text{ is convergent, } \sum_{k=1}^{\infty} k^{m-(r+1)} |R_k| < \infty\Big\},$$

b) $$[\triangle^m_{v,r}(X)]^\gamma = \Big\{a = (a_k) : \sup_n \Big| \sum_{k=1}^{n} k^{m-r} a_k v_k^{-1}\Big| < \infty, \sum_{k=1}^{\infty} k^{m-(r+1)} |R_k| < \infty\Big\},$$

where $R_k = \sum\limits_{j=k+1}^{\infty} a_j v_j^{-1}$.

Proof. The proof follows from Lemma 3.19, Lemma 3.20 and Lemma 3.21.

From Theorem 3.22, we have the following corollaries:

Corollary 3.23. *If we take $v_k = (1, 1, ...)$, then we obtain*

(i) $$[\triangle^m_{r}(X)]^\beta = \Big\{a = (a_k) : \sum_{k=1}^{\infty} k^{m-r} a_k \text{ is convergent, } \sum_{k=1}^{\infty} k^{m-(r+1)} |R_k| < \infty\Big\},$$

(ii) $$[\triangle^m_{r}(X)]^\gamma = \Big\{a = (a_k) : \sup_n | \sum_{k=1}^{n} k^{m-r} a_k| < \infty, \sum_{k=1}^{\infty} k^{m-(r+1)} |R_k| < \infty\Big\},$$

where $R_k = \sum\limits_{j=k+1}^{\infty} a_j$.

Corollary 3.24. *If we take $r = 0$, then we obtain*

(i) $$[\triangle^m_{v}(X)]^\beta = \Big\{a = (a_k) : \sum_{k=1}^{\infty} k^{m} a_k v_k^{-1} \text{ is convergent, } \sum_{k=1}^{\infty} k^{m-1} |R_k| < \infty\Big\},$$

(ii) $[\triangle_v^m(X)]^\gamma = \Big\{a = (a_k) : z \sup_n \Big| \sum_{k=1}^{n} k^m a_k v_k^{-1} \Big| < \infty,$

$$\sum_{k=1}^{\infty} k^{m-1} |R_k| < \infty \Big\},$$

where $R_k = \sum_{j=k+1}^{\infty} a_j v_j^{-1}$.

Corollary 3.25. *If we take* $v_k = (1, 1, ...)$ *and* $r = 0$*, then we obtain*

(i) $[\triangle^m(X)]^\beta = \Big\{a = (a_k) : \sum_{k=1}^{\infty} k^m a_k$ *is convergent,* $\sum_{k=1}^{\infty} k^{m-1}|R_k| < \infty\Big\}$, [3]

(ii) $[\triangle^m(X)]^\gamma = \Big\{a = (a_k) : \sup_n \Big| \sum_{k=1}^{n} k^m a_k \Big| < \infty, \sum_{k=1}^{\infty} k^{m-1}|R_k| < \infty\Big\}$, [3]

where $R_k = \sum_{j=k+1}^{\infty} a_j$.

Corollary 3.26. *If we take* $v_k = (1, 1, ...)$, $r = 0$ *and* $m = 1$*, then we obtain*

(i) $[\triangle(X)]^\beta = \Big\{a = (a_k) : \sum_{k=1}^{\infty} k a_k$ *is convergent,* $\sum_{k=1}^{\infty} |R_k| < \infty\Big\}$, [8]

(ii) $[\triangle(X)]^\beta = \Big\{a = (a_k) : \sup_n \Big| \sum_{k=1}^{n} k a_k \Big| < \infty, \sum_{k=1}^{\infty} |R_k| < \infty\Big\}$, [8]

where $R_k = \sum_{j=k+1}^{\infty} a_j$.

By Lemma 3.9, Theorem 3.10 and Corollary 3.18, we also have

Corollary 3.27.

(i) $\triangle_{v,r}^m(l_\infty), \triangle_{v,r}^m(c)$ *are not normal.*

(ii) $\triangle_{v,r}^m(l_\infty), \triangle_{v,r}^m(c)$ *are not monotone.*

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FUZZY HYPERVECTOR SPACES (REDEFINED)

R. Ameri

School of Mathematics
Statistics and Computer Sciences
University of Tehran
Iran
e-mail: rameri@ut.ac.ir

M. Motameni

Faculty of Mathematical Science
University of Mazandaran
Babolsar
Iran
e-mail: motameni.m@gmail.com

Abstract. In this paper, we introduce and analyze a new type of fuzzy hypervector spaces, as a generalization of fuzzy vector spaces. In this regards, we investigate the basic properties of fuzzy hyper vector spaces and obtain some related results.

Keywords: fuzzy hypervector space, hypervector space, fuzzy vector space, subfuzzy hypervector space.

1. Introduction

The notion of a hypergroup was introduced by F. Marty in 1934 [19]. Since then many researchers have worked on hyperalgebraic structures and developed this theory (for more details see [9], [10]). In 1990, M.S. Tallini introduced the notion of hypervector spaces ([24], [25]) and studied basic properties of them.

As it is well-known the concept of a fuzzy subset of a nonempty set was introduced by Zadeh in 1965 [27] as a function from a nonempty set X into the unit real interval $I = [0, 1]$. Rosenfeld [21] applied this to the group theory and then many researchers developed it in all branches of algebra. The concepts of fuzzy field and fuzzy linear space over a fuzzy field were introduced and discussed by Nanda [20]. In 1977, Katsaras and Liu [15] formulated and studied the notion of fuzzy vector subspaces over the field of real or complex numbers.

Fuzzy set theory has been well developed in the context of hyperalgebraic structure theory. (for example see [1]-[6], [11], [13], [14]). The study of fuzzy hyperstructure is divided into three groups. Crisp hyperoperations defined through fuzzy sets have been initiated by Corsini [8]. Fuzzy hyperalgebras which is a direct extension of the concept of fuzzy algebras. This idea has been extended to fuzzy hypergroups by Zahedi [28]. A completely different approach is an idea defining a fuzzy hypersemigroup considering a fuzzy hyperoperation and a nonempty set that assigns to every pair of elements a fuzzy set. This idea was studied by Corsini and Tofan [12] and then studied by Kehagias, Konstantinidou and Serafimidis [23]. This idea was continued by Sen, Ameri and Chowdhury in [22], where fuzzy semihypergroups are introduced and analyzed. In 2009, Leoreanu and Davvaz [17] introduced the notion of a fuzzy hyperring and then fuzzy hepermodule based on the fuzzy semihypergroup in [22] and made connections.

In [1], Ameri introduced and studied fuzzy hypervector spaces. Now in this paper we introduce and study a new type of a fuzzy hypervector spaces (which is different from that) and obtain some results. We will proceed by giving a connection between fuzzy hypervector spaces and hypervector spaces.

2. Preliminaries

In this section, we present some definitions and simple properties of hypervector spaces and fuzzy subsets, that we need for developing our paper.

A mapping $\circ : H \times H \longrightarrow P^*(H)$ is called a *hyperoperation* (or a join operation), where $P^*(H)$ is the set of all non-empty subsets of H. The join operation is extended to subsets of H in natural way, so that $A \circ B$ is given by

$$A \circ B = \bigcup \{a \circ b : a \in A \text{ and } b \in B\ \}$$

The notations $a \circ A$ and $A \circ a$ are used for $\{a\} \circ A$ and $A \circ \{a\}$, respectively. Generally, the singleton $\{a\}$ is identified by its element a.

Definition 2.1 Let K be a field and $(V, +)$ be an abelian group. We define a *hypervector* space over K to be the quadrupled $(V, +, \circ, K)$, where " $\circ$ " is a mapping

$$\circ : K \times V \quad \longrightarrow \quad P^*(V),$$

such that for all $a, b \in K$ and $x, y \in V$ the following conditions hold:

(H_1) $a \circ (x + y) \subseteq a \circ x + a \circ y$;

(H_2) $(a + b) \circ x \subseteq a \circ x + b \circ x$;

(H_3) $a \circ (b \circ x) = (ab) \circ x$;

(H_4) $a \circ (-x) = (-a) \circ x = -(a \circ x)$;

(H_5) $x \in 1 \circ x$.

Remark.

(i) In the right hand side of the right distributivity law (H_1) the sum is meant in the sense of Frobenius, that is we consider the set of all sums of an element of $a \circ x$ with an element of $a \circ y$. Similarly we have that for left distributivity law (H_2).

(ii) We say $(V, +, \circ, K)$ is *anti-left distributive* if

$$\forall a, b \in K,\ \forall x \in V,\ (a+b) \circ x \supseteq a \circ x + b \circ x,$$

and *strongly left distributive*, if

$$\forall a, b \in K,\ \forall x \in V,\ (a+b) \circ x = a \circ x + b \circ x$$

In a similar way, we define the *anti-right distributive and* strongly right distributive hypervector spaces, respectively. V is called strongly distributive if it is both strongly left and strongly right distributive. (For more details see [25]).

(iii) The left hand side of associativity law (H_3) means the set-theoretical union of all the sets $a \circ y$, where y runs over the set $b \circ x$, i.e.,

$$a \circ (b \circ x) = \bigcup_{y \in b \circ x} a \circ y.$$

(iv) Let $\Omega_V = 0 \circ \underline{0}$, where $\underline{0}$ is the zero of $(V, +)$. It has been shown if V is either strongly right or left distributive, then Ω_V is a subgroup of $(V, +)$. (For more details see [24]).

Example 2.2 [24] In $(\mathbb{R}^2, +)$ we define the product times a scalar in $\mathbb{R}$ by setting:

$$\forall a \in \mathbb{R},\ \forall x \in \mathbb{R}^2 : a \circ x = \begin{cases} \text{line } ox & \text{if } x \neq \underline{0}, \\ \{\underline{0}\} & \text{if } x = \underline{0}, \end{cases}$$

where $\underline{0} = (0, 0)$. Then $(\mathbb{R}^2, +, \circ, \mathbb{R})$ is a strongly left distributive hypervector space.

Definition 2.3 [3] A nonempty subset W of V is a *subhyperspace* if W is itself a hypervector space with the hyperoperation on V, i.e.,

$$\begin{cases} W \neq \varnothing, \\ \forall x, y \in W \Longrightarrow x - y \in W, \\ \forall a \in K,\ \forall x \in W \Longrightarrow a \circ x \subseteq W. \end{cases}$$

In this case, we write $W \leqslant V$.

Definition 2.4

(i) (**Extension principle**) Let $f : X \longrightarrow Y$ be a mapping and $\mu \in FS(X)$ and $\nu \in FS(Y)$. Then we define $f(\mu) \in FS(Y)$ and $f^{-1}(\nu) \in FS(X)$ respectively as follows:

$$f(\mu)(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \varnothing, \\ 0 & \text{otherwise,} \end{cases}$$

(ii) $f^{-1}(\nu)(x) = \nu(f(x)), \quad \forall x \in X.$

Definition 2.5 [22] Let S be a nonempty set. $F^*(S)$ denotes the set of all fuzzy subsets of S. A fuzzy hyperoperation on S is a mapping $\circ : S \times S \longmapsto F^*(S)$ written as $(a, b) \longmapsto a \circ b$. In other words the fuzzy hyperoperation "$\circ$", assigns to every pair (a, b) in H^2 , a nonempty fuzzy subset of H. S together with a fuzzy hyperoperation $\circ$ is called a *fuzzy hypergroupoid*.

Definition 2.6 [22] A fuzzy hypergroupoid $(S, \circ)$ is called a *fuzzy hypersemigroup* if

$$\forall a, b, c \in S, \ (a \circ b) \circ c = a \circ (b \circ c),$$

where for any fuzzy subset μ of S and for all $r \in S$:

(1) $(a \circ \mu)(r) = \bigvee_{t \in S} ((a \circ t)(r) \wedge \mu(t)), \ (\mu \circ a)(r) = \bigvee_{t \in S} ((t \circ a)(r) \wedge \mu(t)),$

(2) If A is a nonempty subset of S and $x \in S$, then for all $t \in S$ we have

$$(x \circ A)(t) = \bigvee_{a \in A} (x \circ a)(t) \text{ and } (A \circ x)(t) = \bigvee_{a \in A} (a \circ x)(t),$$

(3) Let μ, ν be two fuzzy subsets of a fuzzy hypergroupoid $(S, \circ)$ then

$$(\mu \circ \nu)(t) = \bigvee_{p,q \in S} (\mu(p) \wedge (p \circ q)(t) \wedge \nu(q)), \text{ for all } t \in S.$$

3. Fuzzy hypervector space

In this section, we introduce a new type of fuzzy hyper vector spaces dealing with the new definition of fuzzy hyperstrucures [22], and obtain some basic properties of such spaces.

Definition 3.1 Let K be a field and $(V, +)$ an abelian group. A *fuzzy hypervector space* over K is a quadruple $(V, +, \odot, K)$, where "$\odot$" is a fuzzy hyper operation

$$\begin{aligned} \odot : K \times V \quad &\longrightarrow \quad F^*(V) \\ (a, v) \quad &\longmapsto \quad a \odot v \end{aligned}$$

such that for all $\alpha, \beta \in K$ and $a, b \in V$ the followings hold:

$(FH1)$ $\alpha \odot (a + b) \subseteq (\alpha \odot a) + (\alpha \odot b)$;
$(FH2)$ $(\alpha + \beta) \odot a \subseteq (\alpha \odot a) + (\beta \odot a)$;
$(FH3)$ $\alpha \odot (\beta \odot x) = (\alpha\beta) \odot x$;
$(FH4)$ $a \odot (-x) = (-a) \odot x = -(a \odot x)$;
$(FH5)$ $\chi_x \subseteq 1 \odot x$.

Remark.

(i) In the right hand side of the right distributivity law $(FH1)$ the sum is meant in the sense of fuzzy sum, that is for fuzzy subsets μ and ν of V

$$(\mu+\nu)(z)=\bigvee_{z=x+y}(\mu(x)\bigwedge\nu(y)).$$

Similarly, we have for left distributivity law (H_2).

(ii) We say that $(V,+,\odot,K)$ is *anti-left distributive* if

$$\forall a,b\in K,\ \forall x\in V,\ (a+b)\odot x\supseteq a\odot x+b\odot x,$$

and strongly left distributive, if

$$\forall a,b\in K,\ \forall x\in V,\ (a+b)\odot x=a\odot x+b\odot x,$$

(iii) Let $\Omega_V=0\odot\underline{0}$, where $\underline{0}$ is the zero of $(V,+)$. It can be easily shown that if V is either strongly right or left distributive, then Ω_V is a subgroup of $(V,+)$.

Here, we present examples of fuzzy hypervector spaces.

Example 3.2 Let $(V,+)$ be an arbitrary abelian group and K be a field. Define fuzzy hyperoperation: $\odot:K\times V\longrightarrow F^*(V)$ by

$$\forall a\in V, r\in K,\qquad r\odot a=\chi_{\{ra\}}$$

where $\chi_{\{ra\}}$ is the characteristic function. It is easy to verify that $(V,+,\odot,K)$ is a fuzzy hypervector space over the field K. ■

This example shows that every fuzzy hypervector space is a generalization of a classic hypervector space.

Example 3.3 Let $(V,+)$ be a an abelian group and K be a field. Define following fuzzy hyperoperation "$\odot$" by

$$\forall a\in V, r\in K,\qquad (r\odot a)(t)=\frac{1}{2}\qquad \text{if } t\in r\circ a$$

and 0 otherwise. Then $(V,+,\odot,K)$ is a fuzzy hypervector space over the field K. ■

Example 3.4 Let $(V,+)$ be an abelian group and μ be a nonzero fuzzy semigroup of V, then for $a,b\in V$, we define the fuzzy hyperoperation

$$(a\odot b)(t)=\begin{cases}\mu(a)\wedge\mu(b) & \text{if } t=ab,\\ 0 & \text{otherwise,}\end{cases}$$

then $(V,+,\odot)$ is a fuzzy hypervector space over field K. ■

Definition 3.5 A nonempty subset W of V is a *subfuzzy hypervector space* if W is itself a fuzzy hypervector space with the fuzzy hyper operation on V, that is,

$$\begin{cases} W \neq \varnothing, \\ \forall x, y \in W \Longrightarrow x - y \in W, \\ (\forall a \in K,\ \forall x \in W,\ \ (a \odot x)(v) > 0) \Longrightarrow v \in W. \end{cases}$$

Lemma 3.6 *A nonempty subset W of V is a subfuzzy hypervector space if and only if, $\forall a, b \in K, \forall u, v \in W$, we have*

$$(a \odot u + b \odot v)(t) > 0 \Longrightarrow t \in W.$$

Proof. Let W be a subfuzzy hypervector space of V. Suppose that for $a, b \in K$ and $u, v \in W$, we have

$$(a \odot u + b \odot v)(t) > 0.$$

On the other hand,

$$(a \odot u + b \odot v)(t) = \bigvee_{t=t_1+t_2} ((a \odot u)(t_1) \wedge (b \odot v)(t_2))(t) > 0.$$

Then, there exists $u_1, u_2 \in V$ such that $t = u_1 + u_2$ and $(a \odot u)(u_1) > 0$, $(b \odot v)(u_2) > 0$, by Definition 3.5 we obtain $u_1 \in W, u_2 \in W$ and hence $t \in W$.

Conversely, for $u, v \in W$ then by Definition 3.5 we have $\chi_u \subseteq 1 \odot u$ and $\chi_v \subseteq 1 \odot v$, so $(\chi_u + \chi_v)(u + v) \subseteq (1 \odot u + 1 \odot v)(u + v) > 0$ then $u + v \in W$.

Also, if $(a \odot x)(t) > 0$, then $(a \odot x + \chi_0)(t) > 0$, which means $(a \odot x + 1 \odot x)(t) > 0$ and implies that $t \in W$. ∎

Definition 3.7 Let V, W be two fuzzy hypervector spaces over a field K. Then, the mapping $T : V \longrightarrow W$ is called

(i) *weak linear transformation* if
$T(x + y) = T(x) + T(y)$ and $T(a \odot x) \cap a \odot T(x) \neq \phi$.

(ii) *linear transformation* if
$T(x + y) = T(x) + T(y)$ and $T(a \odot x) \subseteq a \odot T(x)$.

(iii) *good linear transformation* if
$T(x + y) = T(x) + T(y)$ and $T(a \odot x) = a \odot T(x)$.

Theorem 3.8 *Let $(V, +, \odot, K)$ be a fuzzy hypervector space over a field K and S be a vector space over the field K. If we consider the mapping $T : V \to S$ which is onto, then $(T(V), +, \overline{\odot}, K)$ is a fuzzy hypervector space where $a \overline{\odot} \nu = T(a \odot \nu)$, $a \in K$, $v \in V$.*

Proof. For $\alpha \in K, a, b \in V$ we have

$$\begin{aligned}
(\alpha\overline{\odot}(a+b))(t) &= T(\alpha \odot (a+b))(t) \\
&= \bigvee_{T(x)=t} (\alpha \odot (a+b))(x) \\
&\subseteq \bigvee_{T(x)=t} ((\alpha \odot a) + (\alpha \odot b))(x)) \\
&= \bigvee_{T(x)=t} (\bigvee_{x=u+v} ((\alpha \odot a)(u) \wedge (\alpha \odot b)(v)) \\
&= \bigvee_{t=T(u+v)=T(u)+T(v)} ((\alpha \odot a)(u) \wedge (\alpha \odot b)(v))
\end{aligned}$$

On the other hand we have:

$$\begin{aligned}
((\alpha\overline{\odot}a) + (\alpha\overline{\odot}b)(t) &= \bigvee_{t=r+s} ((\alpha\overline{\odot}a)(r) \wedge (\alpha\overline{\odot}b)(s)) \\
&= \bigvee_{t=r+s} (T(\alpha \odot a)(r) \wedge T(\alpha \odot b)(s)) \\
&= \bigvee_{t=r+s} (\bigvee_{T(u)=r} (\alpha \odot a)(u)) \wedge (\sup_{T(v)=s} (\alpha \odot b)(v)) \\
&= \bigvee_{t=T(u+v)=T(u)+T(v)} ((\alpha \odot a)(u) \wedge (\alpha \odot b)(v)).
\end{aligned}$$

Similarly, we can prove conditions (FH2), (FH3), (FH4) and (FH5). ■

Let $(V, +, \odot, K)$ be a fuzzy hypervector space (resp. strong left distributive) and W be a subfuzzy hypervector space of V . Let $\pi : V \longrightarrow V/W$ be the projection map. Define the fuzzy hyperoperation "$*$ " on the abelian group $(V/W, +)$ by

$$\begin{aligned}
* : K \times V/W &\longrightarrow F^*(V/W) \\
(a, v + W) &\longmapsto \overline{a \odot v}
\end{aligned}$$

in which $(\overline{a \odot v}) = \pi(a \odot v)$. Note that by Theorem 3.8, $(V/W, +, *, K)$ is a fuzzy hypervector space (resp. strong left distributive).

The next result immediately follows:

Corollary 3.9 *Let $(V, +, \odot)$ be a fuzzy hypervector space over a field K and W be a subfuzzy hypervector space of V. Then $(V/W, +, *, K)$ is a fuzzy hypervector space.*

Definition 3.10 If μ is a nonempty subset of V, then the smallest sub-fuzzy hypervector space of V containing μ is called *fuzzy linear space generated by μ* and is denoted by $\langle\mu\rangle$. In other words, $\langle\mu\rangle = \bigcap_{\mu \subseteq \nu \leq V} \nu$.

Lemma 3.11 *If μ is a nonempty subset of V then*

$$\langle\mu\rangle = \left\{ t \in V : \chi_t \subseteq \sum_{i=1}^{n}(a_i \odot s_i), a_i \in K, s_i \in V, \mu(s_i) > 0, n \in N \right\}.$$

Proof. Let $A = \left\{ t \in V | \chi_t \subseteq \sum_{i=1}^{n}(a_i \odot s_i), a_i \in K, s_i \in V, \mu(s_i) > 0, n \in N \right\}.$

We will show that A is the smallest subfuzzy hypervector space of V containing S. First, we show that A is a subfuzzy hypervector space of V containing S. Let $t_1, t_2 \in A$; then there exists $a_i, \acute{a}_i \in K$, $s_i, \acute{s}_i \in V$ such that

$$\chi_{t_1} \subseteq \bigcup_{i=1}^{n} a_i \odot s_i, \quad \chi_{t_2} \subseteq \bigcup_{i=1}^{m} \acute{a}_i \odot \acute{s}_i.$$

Then,

$$\chi_{t_1 - t_2} = \chi_{t_1} - \chi_{t_2} \subseteq \sum_{i=1}^{n} a_i \odot s_i - \sum_{j=1}^{m} \acute{a}_j \odot \acute{s}_j = \sum_{k=1}^{m+n} b_k \odot l_k,$$

where $b_k = a_k$, $b_{k+j} = a_{\acute{j}}$, $l_k = s_k$ and $l_{k+j} = s_{\acute{j}}$, for $1 \leq k \leq n$, and $1 \leq j \leq m$. Thus, $t_1 - t_2 \in A$.

Also, let us suppose that, for $t \in A$, $k \in K$, we have $(k \odot t)(x) > 0$. We will show that $x \in A$. For this, we have

$$(k \odot \chi_t)(x) = \sup_{s \in V}((k \odot s)(x) \wedge \chi_t(s)) = (k \odot t)(x) > 0.$$

On the other hand, we have

$$\begin{aligned}
0 < (k \odot t)(x) &= (k \odot \chi_t)(x) \subseteq k \odot \left(\sum_{i=1}^{n} a_i \odot s_i \right)(x) \\
&= \sum_{i=1}^{n}((ka_i) \odot s_i)(x) = \sum_{i=1}^{m}(b \odot s_i)(x) > 0 \\
\Longrightarrow & \bigvee_{x = \sum_{i=1}^{m} x_i} ((b \odot s_i) \wedge \ldots \wedge (b \odot s_m))(x_m) > 0 \\
& \Longrightarrow \exists x_1, \ldots, x_m \in W; x = \sum_{i=1}^{n} x_i \text{ and } (b \odot s_i)(x_i) > 0 \text{ for } 1 \leq i \leq m \\
& \Longrightarrow x_i \in A \Longrightarrow x \in A.
\end{aligned}$$

Thus, A is a subfuzzy hypervector space of V.

Now, let θ be a subfuzzy hypervector space of V containing μ and $t \in A$. Then, $\chi_t \subseteq \sum_{i=1}^{n} a_i \odot s_i$, for $a_i \in K, \mu(s_i) > 0, n \in N$. Since θ is a subfuzzy hypervector space containing μ, so for $s_i \in V$, $\theta(s_i) > 0$ we have $\sum_{i=1}^{n} a_i \odot s_i \subseteq \theta$. Thus, $A \leq \theta$. Hence, A is the smallest and for all $s \in V$ such that $\mu(s) > 0$, we have $\chi_s \subseteq 1_k \odot a$ then $s \in A$ and so $\mu \leq A$. ■

Definition 3.12 Let V, W be two fuzzy hypervector space over a field K, and $T : V \longrightarrow W$ be a linear transformation. Then the kernel of T is denoted by $kerT$ and defined by

$$KerT = \{x \in V \mid \chi_{T(x)} \subseteq \Omega_W\}$$

where $\Omega_W = 0_K \odot 0_W$.

Theorem 3.13 *Let U, V be two fuzzy hypervector spaces (resp.strongly left) over K and $T : V \longrightarrow U$ be a linear transformation. Then, $KerT$ is a subfuzzy hypervector space of V.*

Proof. $T(\Omega_V) = T(0 \odot \underline{0}_V) \subseteq 0 \odot T(\underline{0}_V) = 0 \odot \underline{0}_U = \Omega_U$. Therefore, $KerT \neq \phi$. Also, for all $a, b \in K, x, y \in KerT$, we have $\chi_{T(x)} \in \Omega_U$ and $\chi_{T(y)} \in \Omega_U$ so

$$\begin{aligned} \chi_{T(a\odot x+a\odot y)} &= \chi_{T(a\odot x)} + \chi_{T(b\odot y)} \subseteq \chi_{a\odot T(x)} + \chi_{b\odot T(y)} \\ &\subseteq a \odot \chi_{T(x)} + b \odot \chi_{T(x)} \subseteq a \odot \Omega_U + b \odot \Omega_U = \Omega_U. \end{aligned}$$

Now, by Lemma 3.6 since $(a \odot x + b \odot y)(v) > 0$, we have $\chi_{T(v)} \subseteq \Omega_U$. Hence, $v \in KerT$ and so $KerT$ is subfuzzy hypervector space of U. ■

It is easy to see that, if W is a subfuzzy hypervector space of V over a field K, then

$$\begin{aligned} \Pi : V &\longrightarrow V/W \\ x &\longmapsto x + W \end{aligned}$$

is a good linear transformation, such that $\Omega_V \subseteq KerT$ and it is called projection or canonical transformation.

Theorem 3.14 *Let V, U be two fuzzy hypervector spaces and $T : V \longrightarrow U$ be a good linear transformation:*

(i) *if W is a subfuzzy hypervector space of V, then $T(W)$ is a subfuzzy hypervector space of U.*

(ii) *if L is a subfuzzy hypervector space of U, then $T^{-1}(L)$ is a subfuzzy hypervector space of V containing $kerT$.*

Proof. (i) Let $a \in K$ and $x', y' \in T(W)$, such that $x' = T(x), y' = T(y)$ for $x, y \in W$. Then $x + y \in W$ and if $(a \odot x)(t) > 0 \Longrightarrow t \in W$. So, $x' - y' = T(x) - T(y) = T(x - y) \in T(W)$.

Now, let $(a \odot x')(t) > 0$. Then, $(a \odot T(x))(t) > 0$, and hence $T(a \odot x)(t) > 0$. Thus, by extension principle, we have $\sup\limits_{T(z)=t} (a \odot x)(z) > 0$ so, there exists y such that $(a \odot x)(y) > 0, T(y) = t$. Then, $y \in W$, and so $T(y) \in T(W)$, thus $t \in T(W)$, and hence $T(W) \leq U$.

(ii) The first part can be proved in a similar way as in (i). Now, if $x \in KerT$, then $T(x) \in 0_U \subseteq 0 \odot L \subseteq L$. Therefore, $x \in T^{-1}(L)$ and so $KerT \subseteq T^{-1}(L)$. ∎

Theorem 3.15 *Let V and U be two left distributive fuzzy hypervector spaces and $T : V \longrightarrow U$ be a good linear transformation. Then there is an one-to-one correspondence between subfuzzy hypervector spaces of V containing $KerT$ and subfuzzy hypervector spaces of U.*

Proof. Let $\mathbf{A} = \{W | W \leq V, W \supseteq T\}$ and $\mathbf{B} = \{L | L \leq U\}$. We will show that the following map is one-to-one and onto:

$$\begin{array}{rcl} \varphi : \mathbf{A} & \longrightarrow & \mathbf{B} \\ W & \longmapsto & T(W) \end{array}$$

Then, $T(W)$ is an element of $\mathbf{B}$ for all $W \in \mathbf{A}$. Let W_1, W_2 be two elements of $\mathbf{A}$, such that $W_1 \neq W_2$ then there exists $w_1 \in W_1 - W_2$ or $w_2 \in W_2 - W_1$. If $w_1 \in W_1 - W_2$ then $T(w_1) \in T(W_1) - T(W_2)$, and so $T(W_1) \neq T(W_2)$, and if $w_2 \in W_2 - W_1$, similarly $T(W_1) \neq T(W_2)$. Also, for an arbitrary $L \in \mathbf{B}$, suppose $W = T^{-1}(L)$. Then, by Theorem 3.10, $W \in \mathbf{A}$ and $T(W) \in \mathbf{B}$. Hence φ is one-to-one and onto. ∎

The next result follows immediately from Theorem 3.15:

Corollary 3.16 *If V is a left distributive fuzzy hypervector space, then every subfuzzy hypervector space of V/W, is of the form L/W, in which L is a subfuzzy hypervector space V containing W.*

4. Connections between fuzzy hypervector spaces and hypervector spaces

Connections between fuzzy hyperoperations and hyperoperations on fuzzy hypersemigroups, fuzzy hyperrings and fuzzy hypermodules have been studied in [22],[17].

Now, in the next theorem, we establish a similar result for hypervector spaces.

Theorem 4.1 *If $(V, +, \odot)$ is a fuzzy hypervector space over a field K, then $(V, +, \circ)$ is a hypervector space over the field K.*

Proof. For all $x \in V, \alpha \in K$ define a hyperoperation "$\circ$" on V as $\alpha \circ x = \{z \in V \mid (\alpha \odot x)(z) > 0\}$. We have to check the conditions of Definition 2.1. First, for all $x, y \in V$, $\alpha \in K$, we have:

$$t \in \alpha(x + y) \Longleftrightarrow (\alpha \odot (x + y))(t) > 0.$$

This means that

$$(\alpha\odot(x+y))(t) \subseteq ((\alpha\odot x)+(\alpha\odot y))(t) = \bigvee_{t=u+v} ((\alpha\odot x)(u) \wedge (\alpha\odot y)(v)) > 0.$$

Hence, there exists $u, v \in V$ such that $((\alpha\odot x))(u) > 0$, and so $u \in \alpha \circ x$ and $((\alpha\odot y))(v) > 0$. Thus $v \in \alpha \circ y$, and so $t = u + v \in \alpha \circ x + \alpha \circ y$.

Similarly, we can obtain other conditions of Definition 2.1. Therefore, $(V, +, \circ)$ is a hypervector space over field K, as desired. ∎

Hence, the exists a map $\psi : FHV \to HV$ with $\psi((V, +, \odot)) = (V, +, \circ)$, where HV denotes the class of all hypervector spaces and FHV the class of all fuzzy hypervector spaces.

Now, we will obtain a fuzzy hypervector space from a hypervector space $(V, +, \circ)$.

Theorem 4.2 *If $(V, +, \circ)$ is a hypervector space over a field K, then $(V, +, \odot)$ is a fuzzy hypervector space over the field K.*

Proof. We will show that for all $x, y, t \in V$, $\alpha \in K$ we have $\alpha\odot(x+y) \subseteq (\alpha\odot x)+(\alpha\odot y)$. Let $(V, +, \circ)$ is a hypervector space over a field K, then $\forall x \in V, \forall \alpha \in R$ we define the fuzzy hyperoperation: $\alpha\odot x = \chi_{\alpha\circ x}$. Now,

$$\begin{aligned}(\alpha\odot(x+y))(t) &= \chi_{\alpha\circ(x+y)}(t) \subseteq \chi_{\alpha\circ x+\alpha\circ y}(t)\\ &= \begin{cases} 1 & \text{if } t = \alpha\circ x + \alpha\circ y,\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

On the other hand,

$$\begin{aligned}((\alpha\odot x)+(\alpha\odot y))(t) &= \bigvee_{t=u+v} ((\alpha\odot x)(u) \wedge (\alpha\odot y)(v))\\ &= \bigvee_{t=u+v} (\chi_{\alpha\circ x}(u) \wedge \chi_{\alpha\circ y}(v))\\ &= \begin{cases} 1 & \text{if } t = u+v = \alpha\circ x + \alpha\circ y,\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

Similarly, we obtain other conditions of Definition 3.1. ∎

Therefore, there exists a map $\varphi : HV \to FHV$ such that

$$\varphi((V, +, \circ)) = (V, +, \odot).$$

Recall that if V, W are two fuzzy hypervector spaces, the map $f : V \to W$ is called a homomorphism if $T : V \to W$ is a linear transformation and if T is an one to one correspondence then it is called an isomorphism.

The next two theorems will make connections between homomorphisms of fuzzy hypervector spaces and homomorphism of hypervector spaces.

Theorem 4.3 *Let $(V_1, +, \odot_1)$ and $(V_2, +, \odot_2)$ be fuzzy hypervector spaces over a field K and $(V_1, +, \circ_1) = \psi(V_1, +, \odot_1)$, $(V_2, +, \circ_2) = \psi(V_2, +, \odot_2)$ be the associated hypervector spaces over the field K. If $f : V_1 \to V_2$ is a homomorphism of fuzzy hypervector spaces, then f is a homomorphism of hypervector spaces, too.*

Proof. For all $x, y \in V, \alpha \in K$ we have $f(\alpha \odot_1 x) \leq \alpha \odot_2 f(x)$. If $u \in \alpha \circ_1 x$, then $(\alpha \odot_1 x)(u) > 0$. Denote $v = f(u)$. We have

$$(f(\alpha \odot_1 x))(v) = \bigvee_{f(s)=v} (\alpha \odot_1 x)(s) \geq (\alpha \odot_1 x)(u) > 0.$$

Hence, $(\alpha \odot_2 f(x))(v) > 0$ and so $v \in \alpha \circ_2 f(x)$, which means that $f(\alpha \circ_1 x) \subseteq \alpha \circ_2 f(x)$. And obviously, $f(x + y) = f(x) + f(y)$. ∎

Theorem 4.4 *Let $(V_1, +, \circ_1)$ and $(V_2, +, \circ_2)$ be two hypervector spaces over field K and $(V_1, +, \odot_1) = \psi(V_1, +, \circ_1)$, $(V_2, +, \odot_2) = \psi(V_2, +, \circ_2)$ be the associated hypervector spaces over field K. The map $f : V_1 \to V_2$ is a homomorphism of fuzzy hypervector spaces if and only if it is a homomorphism of hypervector spaces.*

Proof. Suppose that f is a homomorphism of hypervector spaces. Let $x \in V$, $\alpha \in K$. For all $t \in Imf$ we have

$$\begin{aligned}
(f(\alpha \odot_1 x))(t) &= \bigvee_{f(r)=t} (\alpha \odot_1 x)(r) = \bigvee_{f(r)=t} \chi_{\alpha \circ_1 x}(r) \\
&= \begin{cases} 1 & \text{if } t \in f(\alpha \circ x_1), \\ 0 & \text{otherwise,} \end{cases} \\
&= \chi_{f(\alpha \circ_1 x)(t)} \leq \chi_{\alpha \circ_2 f(x)}(t) = (\alpha \odot_2 f(x))(t).
\end{aligned}$$

Obviously, $f(x + y) = f(x) + f(y)$.

Conversely, let $x, y \in V_1$, $\alpha \in K$. We have $f(\alpha \odot_1 x) \leq \alpha \odot_2 f(x)$, whence $\chi_f(\alpha \circ_1 x) \leq \chi_{\alpha \circ_2 f(x)}$. This means $f(\alpha \circ_1 x) \subseteq \alpha \circ_2 f(x)$. ∎

The next theorem establishes a connection between subfuzzy hypervector spaces of a fuzzy hypervector spaces and subhypervector spaces of the corresponding hypervector space.

Theorem 4.5 (i) *If $(V', +, \odot)$ is a subfuzzy hypervector space of $(V, +, \odot)$ over a field K, then $(V', +, \circ) = \psi(V', +, \odot)$ is a subhypervector space of $(V, +, \circ) = \psi(V, +, \odot)$ over the field K.*

(ii) *$(V', +, \circ)$ is a subhypervector space of $(V, +, \circ)$ over a field K if and only if $(V', +, \odot) = \varphi(V', +, \circ)$ is a subfuzzy hypervector space of $(V, +, \odot) = \psi(V, +, \circ)$.*

Proof. (i) For all $x \in V', \alpha \in K$ we will show that $\alpha \circ x \subseteq V'$. since $(V', +, \odot)$ is a subfuzzy hypervector space of $(V, +, \odot)$ so if for all $x \in V'$, $\alpha \in K$, $(\alpha \odot x)(t) > 0 \Rightarrow t \in V'$. This means that $t \in \alpha \circ x \Rightarrow t \in V'$. Hence, $\alpha \circ x \subseteq V'$.

(ii) It can be shown by a similar way as in (i). ∎

The above theorem is a connection between subfuzzy hypervector spaces of a fuzzy hypervector spaces and subhypervector spaces of the corresponding hypervector space.

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SUR LES ALGEBRES DE LIE D'UN SYSTEME DE CHAMPS DE VECTEURS PERMUTABLES

H.S.G. Ravelonirina
P. Randriambololondrantomalala
M. Anona

Département de Mathématiques et Informatique
Faculté des Sciences
Université d'Antananarivo
Antananarivo 101, BP 906
Madagascar
e-mail: rhsammy@yahoo.fr
princypcpc@yahoo.fr
mfanona@yahoo.fr

Résumé. Soient M une variété $C^\infty-$ différentiable et S un système de q $C^\infty-$ champs de vecteurs qui commutent deux à deux. Ce système définit une structure de feuilletage généralisé $\mathcal{F}$ sur M. L'algèbre de Lie A_S des champs de vecteurs de M qui commutent avec S est à la fois un module sur l'anneau des $C^\infty-$ fonctions qui sont constantes sur les feuilles de $\mathcal{F}$ et une sous-algèbre de Lie de l'algèbre de Lie des automorphismes infinitésimaux au feuilletage. On détermine toutes les dérivations de l'algèbre de Lie A_S.
Mots clés: algèbre de Lie, champ de vecteurs permutables, feuilletage généralisé, cohomologie locale de Chevalley-Eilenberg, cohomologie de de Rham.

Abstract. Let be M a $C^\infty-$ differentiable manifold and S a system of q $C^\infty-$ vector fields which commute mutually. This system defines a generalized foliation $\mathcal{F}$ on M. The Lie algebra A_S of vector fields in M which commute with S is both a module over the ring of $C^\infty-$ functions that are constant on the leaves of $\mathcal{F}$ and a sub-Lie algebra of the foliation preserving vector fields. We determine all derivations of the Lie algebra A_S.

Keywords: Lie Algebra, commuting vector fields, generalized foliation, local cohomology of Chevalley-Eilenberg, cohomology of de Rham.

AMS Subject Classification: Primary 17B66; 17B56; Secondary 53C12; 47B47.

1. Introduction

Soient M une variété différentiable paracompacte de classe C^∞ et $\mathcal{X}(\mathrm{M})$ l'algèbre de Lie des champs de vecteurs de M. Dans son article [8], Takens a montré que toute dérivation de l'algèbre de Lie $\mathcal{X}(\mathrm{M})$ est une dérivée de Lie par rapport à un champ de vecteurs de M. Dans le cas où l'algèbre de Lie est une sous-algèbre de Lie attachée à un feuilletage régulier sur M, Lichnérowicz cf. [3] a prouvé aussi des résultats analogues. Nous avons étendu ces résultats dans le cas d'une distribution involutive non régulière cf. [5], où l'anneau de base contient toutes les fonctions de classe C^∞ de la variété. Dans [6], nous avons abordé le même problème sur les algèbres de Lie des champs de vecteurs polynomiaux $\mathfrak{P}$ sur $\mathbb{R}^n$ qui contiennent tous les champs constants et le champ d'Euler. Nous avons prouvé que toute dérivation de $\mathfrak{P}$ est une dérivée de Lie par rapport à un champ de vecteurs polynomiaux de $\mathbb{R}^n$. Dans ce papier, nous étudions une sous-algèbre de Lie de $\chi(\mathrm{M})$ dont l'anneau des fonctions de classe C^∞ du module sous-jacent est tronqué. Plus précisement, M est une variété différentiable de dimension $m+q$ et S un système de $q \geq 1$ champs de vecteurs qui commutent deux à deux, et de rang p avec $0 \leq p(x) \leq q$, pour tout $x \in \mathrm{M}$. Il existe une structure de feuilletage généralisé $\mathfrak{F}$ définie par le système S cf. [1]. On note $L_{\mathfrak{F}}$ l'algèbre de Lie des champs des automorphismes infinitésimaux de $\mathfrak{F}$ et, A_S l'algèbre de Lie des champs de vecteurs de M qui commutent avec S. Toutes les feuilles sont supposées régulières. L'algèbre de Lie A_S se décompose en une somme semi-directe d'algèbres de Lie A_S^1 et A_S^2, où A_S^1 (resp. A_S^2) est un module (resp. l'algèbre de Lie engendrée par S) sur l'anneau des fonctions constantes aux feuilles. Ainsi A_S est une sous-algèbre de Lie de $L_{\mathfrak{F}}$. De plus, l'algèbre de Lie A_S^2 est un idéal caractéristique de A_S. Par ailleurs, on donne une condition nécéssaire et suffisante pour que toute dérivation de A_S soit locale; de même pour que l'idéal dérivé de A_S coïncide à A_S. Ainsi, les caractéristiques d'une dérivation non locale de A_S sont obtenues. En étudiant la dérivation locale de A_S dans l'idéal caractéristique A_S^2, on peut déterminer toutes les dérivations locales non intérieures de A_S. Par suite, en utilisant l'algèbre quotient de A_S par A_S^2 et un résultat de [5], on peut décomposer toute dérivation locale de A_S en une somme de dérivation intérieure de A_S et de dérivation locale non intérieure trouvée auparavant. Dans le cas où le rang p de S est constant supérieur ou égal à 1, le premier espace de cohomologie locale de Chevalley-Eilenberg de A_S est isomorphe à $(\mathrm{H}^1_R(\mathrm{B}) \times \mathbb{R})^p \times \mathbb{R}^{p^2}$, où $\mathrm{H}^1_R(\mathrm{B})$ désigne le premier espace de cohomologie de de Rham sur les formes basiques au feuilletage de M. Si le système S est réduit à un champ de Liouville, on retrouve par une méthode différente un résultat de Lecomte dans [4].

2. Préliminaires

Soit M une variété réelle $C^\infty-$ différentiable paracompacte de dimension $m+q$ où $m, q \geq 1$. Tous les objets étudiés sont supposés de classe C^∞. On désigne par $F(\mathrm{M})$ l'anneau des fonctions C^∞ sur M, $\chi(\mathrm{M})$ l'algèbre de Lie des champs de vecteurs sur M, S un système $\{X_1, \ldots, X_q\}$ de rang p de champs de vecteurs, avec $0 \leq p(x) \leq q$ pour tout $x \in \mathrm{M}$. Les éléments de S vérifient $[X_i, X_j] = 0$ pour tous $i, j \in \{1, \ldots, q\}$. On considère l'algèbre de Lie A_S des champs de vecteurs X de M tels que $[X, X_i] = 0$ pour tout $i \in \{1, \ldots, q\}$.

On peut déduire du système S un champ de plans P, qui à tout $x \in \mathrm{M}$ correspond le sous-espace vectoriel engendré par $X_1(x), \ldots, X_q(x)$ de $T_x(\mathrm{M})$. P est un champ de plans de classe C^∞ de système générateur S. Tout champ de vecteurs $X = g^j X_j$ de P avec $g^j \in F(\mathrm{M})$, vérifie pour tout i

$$[X, X_i] = -\left(X_i(g^j)\right) X_j$$

c'est-à-dire, P est invariant par tout champ de vecteurs de P. D'après le théorème de Sussmann cf. [7], il existe un feuilletage généralisé $\mathfrak{F}$ sur M dont la feuille en un point x de M est la variété intégrale maximale $I(x)$ telle que pour tout $y \in I(x)$,

$T_y\left(I(x)\right) = P_y$ cf. [1]. On note $F_0\left(\mathrm{M}\right)$ l'anneau des fonctions sur M constantes aux feuilles. La sous-algèbre de Lie A_S^2 des champs de vecteurs de M engendrée par S sur $F_0\left(\mathrm{M}\right)$ est commutative. De plus A_S^2 est une sous-algèbre de Lie de l'algèbre de Lie L des champs de vecteurs tangents aux feuilles. Par ailleurs, A_S^1 désigne l'ensemble des champs de vecteurs de M tel que A_S^1 et L sont deux sous-modules supplémentaires dans l'algèbre de Lie $L_{\mathfrak{F}}$ des automorphismes infinitésimaux au feuilletage.

On suppose que toutes les feuilles soient régulières, sauf mention expresse. Le théorème de Dazord cf. [1] p.415 assure l'existence d'une carte adaptée (U, x^a, y^i) (resp. (U, x^a)), avec $1 \leq a \leq m+q-p$, $1 \leq i \leq p$ si $p \geq 1$ (resp. $1 \leq a \leq m+q$ si $p = 0$) au voisinage de chaque point x de M où la dimension de $I(x)$ est constante $p(x) = p$. Il existe une permutation ζ de $\{1, \ldots, q\}$ tels que pour $p \geq 1$ (resp. $p = 0$) $\left(X_{\zeta_i} \underset{U}{=} \frac{\partial}{\partial y^i}\right)_{1 \leq i \leq p}$ et $\left(X_{\zeta_l} \underset{U}{=} 0\right)_{p<l\leq q}$ (resp. $\left(X_l \underset{U}{=} 0\right)_{1\leq l\leq q}$). On utilisera de tels ouverts pour les domaines de cartes adaptées au feuilletage. On conviendra dans la suite sauf mention expresse que les indices a, b, c vont de 1 à $m + q - p$ et i, j, l de 1 à p si $p \geq 1$. De même, les indices fixes a_0, a_1, b_0 appartiennent à $\{1, \ldots, m + q - p\}$ et i_0, j_0 à $\{1, \ldots, p\}$ si $p \geq 1$.

L'anneau $F_0\left(U\right) = \left\{f_{|U} \text{ tel que } f \in F_0\left(\mathrm{M}\right)\right\}$ est l'ensemble des fonctions sur U ne dépendant pas des coordonnées y^i. L'algèbre de Lie A_S sur toute carte adaptée U, coïncide au $F_0\left(U\right)$-module des champs sur U engendré par $\frac{\partial}{\partial x^1}, \ldots,$ $\frac{\partial}{\partial x^{m+q-p}}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^p}$ où $p \geq 1$. Le module $A_S\left(U\right)$ se décompose en produit semi-direct

$$A_S\left(U\right) = A_S^1\left(U\right) \oplus A_S^2\left(U\right)$$

où $A_S^1\left(U\right)$ est la sous-algèbre de $A_S\left(U\right)$ engendrée par $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{m+q-p}}$ sur $F_0\left(U\right)$ et, où $A_S^2\left(U\right)$ est l'idéal commutatif de $A_S\left(U\right)$ engendré par $\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^p}$ sur $F_0\left(U\right)$.

Dans le cas où $p = 0$, $F_0\left(U\right) = F\left(U\right)$ et

$$A_S\left(U\right) = A_S^1\left(U\right) \oplus A_S^2\left(U\right)$$

avec $A_S^1\left(U\right) = \mathcal{X}(U)$ et $A_S^2\left(U\right) = \{0\}$.

On s'interesse à l'étude des $\mathbb{R}-$dérivations de l'algèbre de Lie A_S. Le cas trivial où le rang est identiquement nul sur M, est déjà étudié par [8]. Donc, on suppose que $S \neq \{0\}$.

Remarque 2.1 Si la variété M est connexe, le feuilletage défini est régulier d'après une assertion de [1] p.416.

3. Etude des dérivations de A_S

Dans toute la suite $x \in \mathrm{M}$ est un point quelconque, U est une carte adaptée contenant x telle que la dimension de $I(x)$ est une constante égale à p sur U. On utilisera la convention d'Einstein sur la sommation d'indices, sauf mention expresse.

Définition 3.1 Le centralisateur (resp. Le centre) de A_S est l'ensemble des X dans $\mathcal{X}(\mathrm{M})$ (resp. dans A_S) tels que $[X, A_S] = \{0\}$.

Proposition 3.2 *Le centralisateur* $\mathfrak{C}$ *de* A_S *est le* $\mathbb{R}-$*espace vectoriel engendré par* S.

Démonstration. Il est immédiat que le $\mathbb{R}-$espace vectoriel engendré par S est inclus dans $\mathfrak{C}$.

Réciproquement, soit X appartenant à $\mathfrak{C}$. Sur U, si $p = 0$, alors la preuve est donnée par un résultat de [8]. Si $p \geq 1$, soit $X_{|U} = X^a \frac{\partial}{\partial x^a} + X'^i \frac{\partial}{\partial y^i} \in \mathfrak{C}_U$, on a

$$\left[X^a \frac{\partial}{\partial x^a} + X'^i \frac{\partial}{\partial y^i}, \frac{\partial}{\partial x^b}\right] = 0, \ \left[X^a \frac{\partial}{\partial x^a} + X'^i \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right] = 0$$

pour tous b, j. Ainsi, chaque X^a et X'^i sont des constantes réelles en supposant que U est connexe. Par ailleurs,

$$\left[X^a \frac{\partial}{\partial x^a} + X'^i \frac{\partial}{\partial y^i}, x^c \frac{\partial}{\partial x^c}\right] = 0$$

alors on en déduit que chaque $X^a = 0$ pour tous $X \in \mathfrak{C}$ et U adaptée à $\mathfrak{F}$. Donc $\mathfrak{C}$ est contenu dans le $\mathbb{R}-$espace vectoriel engendré par S. D'où le résultat. ∎

Remarque 3.3 Le système S n'est pas en général une base du centre de A_S. Par exemple, sur le tore T^2 avec $S = \{X\}$, où X est un champ de vecteurs invariant ayant une trajectoire dense.

Définition 3.4 Une $\mathbb{R}$-dérivation D d'une sous-algèbre de Lie $\mathfrak{A}$ des champs de vecteurs sur M, est une application $\mathbb{R}-$linéaire de $\mathfrak{A}$ dans $\mathfrak{A}$ telle que

$$\forall X, Y \in \mathfrak{A}, \ D[X, Y] = [D(X), Y] + [X, D(Y)]. \tag{3.1}$$

L'application D est dite dérivation intérieure de $\mathfrak{A}$ si $D = [X, .] = \mathrm{L}_X$, avec L_X la dérivée de Lie par rapport à $X \in \mathfrak{A}$.

Dans cette section, une $\mathbb{R}$-dérivation d'une algèbre de Lie $\mathfrak{A}$ est tout simplement appelée dérivation de $\mathfrak{A}$.

Proposition 3.5 *Soient* D *une dérivation de* A_S *et* U *un domaine d'une carte adaptée de* M *tels qu'il existe* $X \in A_S$ *avec* $X_{|U} \equiv 0$, *alors* $(D(X))_{|A_S^1} \underset{U}{\equiv} 0$

Démonstration. On considère une dérivation D de A_S. On suppose que $X \in A_S$ et $X_{|U} \equiv 0$. On peut écrire $(D(X))_{|A_S^1} \underset{U}{=} D_X^a \frac{\partial}{\partial x^a}$. Si $D(X)$ est non identiquement nul sur $A_S^1(U)$, il existe un point $z \in U$ tel que l'une au moins des composantes correspondantes de $D(X)$ soit non nulle en z. On suppose qu'il existe un entier a_0 tel que $D_X^{a_0}(z) \neq 0$, donc on peut trouver un ouvert V_z contenant z tel que $D_X^{a_0}(y) \neq 0$ pout tout $y \in V_z$. On prend $f \in F_0(\mathrm{M})$ où $f_{|V_z} = x^{a_0}$ avec

$\mathrm{supp}\,(f) \subset U$ et, $Y \in A_S$ tel que $Y_{|V_z} = \frac{\partial}{\partial y^{i_0}}$. De cette façon, $[X, fY]_{|U} \equiv 0$ et $[X, fY]_{|\complement U \subset \complement \mathrm{supp}(f)} \equiv 0$ et $[X, fY] \equiv 0$. Ainsi, la relation suivante

$$D\left([X, fY]\right) = \left[D\left(X\right), fY\right] + \left[X, D\left(fY\right)\right] \tag{3.2}$$

aboutit à une contradiction. D'où le résultat. ∎

Définition 3.6 Soient $\mathfrak{A}$ et $\mathfrak{B}$ deux sous-modules d'une même algèbre de Lie. Le sous-module engendré par tous les crochets de $X \in \mathfrak{A}$ et $Y \in \mathfrak{B}$ est noté par $[\mathfrak{A}, \mathfrak{B}]$. Si $\mathfrak{A} = \mathfrak{B}$ et que $\mathfrak{A}$ est une algèbre de Lie, alors on l'appelle idéal dérivé de $\mathfrak{A}$.

Proposition 3.7 *L'idéal dérivé de A_S^1 est égal à A_S^1, l'idéal dérivé de A_S^2 est nul. Ainsi, l'idéal dérivé de A_S coïncide à la somme directe de module A_S^1 et de l'algèbre de Lie A engendrée par les $[X, Y]$ où $X \in A_S^1$ et $Y \in A_S^2$.*

Démonstration. On peut adapter la preuve de la Proposition 2.9 p.141 de [5] pour avoir $[A_S^1, A_S^1] = A_S^1$. Ainsi, l'idéal dérivé de A_S^1 coïncide à A_S^1. Par ailleurs, il est clair que $[A_S^2, A_S^2]$ est réduit à $\{0\}$. Comme $[A_S, A_S] = [A_S^1 \oplus A_S^2, A_S^1 \oplus A_S^2]$, alors cette dérnière devient $A_S^1 \oplus [A_S^1, A_S^2]$ avec $\oplus$ désigne une somme directe de modules, d'où le résultat. ∎

Dans la suite, on note $A = [A_S^1, A_S^2]$.

Proposition 3.8 *Toute dérivation non locale de A_S est à la fois à valeur dans le centre de A_S, nulle sur A_S^1 et sur $A \subset A_S^2$.*

Démonstration. Soit D une dérivation non locale de A_S, alors on peut trouver $X \in A_S$ et U un ouvert de M tel que $X_{|U} \equiv 0$ avec $D\left(X\right)$ n'est pas nul sur U. Donc il existe un ouvert $W \subset U$ contenant $x \in$ M tel que $D\left(X\right)\left(y\right) \neq 0$ pour tout $y \in W$. D'après la Proposition 3.5, on écrit $D\left(X\right) \underset{U}{=} D_X^i X_i$. Il s'en suit qu'il existe i_0 tel que $D_X^{i_0}(y) \neq 0$ pour tout y dans un ouvert $W' \subset W$ contenant x. Supposons que $D\left(X\right)$ n'appartient pas à $\mathfrak{C}$, alors on peut supposer que $\left(D_X^{i_0}\right)_{|W'}$ est non constante. On prend $f \in F_0\left(\mathrm{M}\right)$ tel que $\mathrm{supp}\,(f) \subset U$ avec $f(x) \neq 0$. Aussi, peut-on trouver $Y \in A_S^1$ tel que $Y_{|U} = \frac{\partial}{\partial x^{a_0}}$ de façon que $\frac{\partial D_X^{i_0}}{\partial x^{a_0}}(x) \neq 0$. Dans ce cas, une relation analogue à celle de (3.2) donne une contradiction au point x. Par conséquent, $D\left(X\right) \in \mathfrak{C}$. On déduit du résultat qui précède et de la propriété (3.1) d'une dérivation que $D\left[A_S, A_S\right] = \{0\}$. Par la Proposition 3.7, on en tire que $D\left(A_S^1\right) = \{0\}$ et $D\left(A\right) = \{0\}$. ∎

Proposition 3.9 *L'algèbre de Lie A_S^2 est stable par toute dérivation locale de A_S.*

Démonstration. Soit D une dérivation locale de A_S, D_U est une dérivation de $A_S\left(U\right)$ en faisant le même raisonnement que celui de [8] p.157. Sur U, si $p = 0$ alors la preuve est évidente. Sur cet ouvert, si $p \geq 1$ alors l'algèbre de Lie A_S s'écrit $A_S\left(U\right) = A_S^1\left(U\right) \oplus A_S^2\left(U\right)$. Or, chaque $\frac{\partial}{\partial y^i}$ est un élément du centre de $A_S\left(U\right)$ et que le centre d'une algèbre de Lie est un idéal caractéristique, alors $D_U\left(\frac{\partial}{\partial y^i}\right)$ appartient au centre qui est contenu dans $A_S^2(U)$ par la Proposition 3.2. Pour chaque a, b, de la relation $\left[\frac{\partial}{\partial x^a}, x^b\frac{\partial}{\partial y^i}\right] = \delta_a^b\frac{\partial}{\partial y^i}$, on obtient $\delta_a^b D_U\left(\frac{\partial}{\partial y^i}\right) = \left[D_U\left(\frac{\partial}{\partial x^a}\right), x^b\frac{\partial}{\partial y^i}\right] + \left[\frac{\partial}{\partial x^a}, D_U\left(x^b\frac{\partial}{\partial y^i}\right)\right]$. Or $A_S^2(U)$ est un idéal de $A_S(U)$, alors $\left[\frac{\partial}{\partial x^a}, D_U\left(x^b\frac{\partial}{\partial y^i}\right)\right] \in A_S^2(U)$. Ainsi, en posant $D_U\left(x^b\frac{\partial}{\partial y^i}\right) = D_{i,b}^c\frac{\partial}{\partial x^c} + D_{i,b}^{j+m+q-p}\frac{\partial}{\partial y^j}$, chaque $D_{i,b}^c$ est constant. D'autre part, en appliquant D_U à l'égalité $x^b\frac{\partial}{\partial y^i} = \left[x^c\frac{\partial}{\partial x^c}, x^b\frac{\partial}{\partial y^i}\right]$, il s'en suit que $D_U\left(x^b\frac{\partial}{\partial y^i}\right) + \left[D_U\left(x^b\frac{\partial}{\partial y^i}\right), x^c\frac{\partial}{\partial x^c}\right]$ appartient à $A_S^2(U)$. Comme chaque $D_{i,b}^c$ est constant, on obtient $D_{i,b}^c = 0$. Autrement dit, $D_U\left(x^b\frac{\partial}{\partial y^i}\right)$ est un élément de $A_S^2(U)$. En dérivant par D_U la relation $\left[f\frac{\partial}{\partial x^a}, x^a\frac{\partial}{\partial y^i}\right] = f\frac{\partial}{\partial y^i}$ pour tout $f \in F_0(U)$, on trouve que $D_U\left(f\frac{\partial}{\partial y^i}\right)$ est encore dans $A_S^2(U)$. Sachant que tout élément de $A_S^2(U)$ est engendré par les $\frac{\partial}{\partial y^i}$ sur les fonctions de $F_0(U)$, toute dérivation D_U de $A_S(U)$ préserve l'idéal $A_S^2(U)$. D'où le résultat. ∎

Proposition 3.10 *L'algèbre de Lie A_S^2 est un idéal caractéristique commutatif de A_S.*

Démonstration. Soit D une dérivation de A_S, D est la somme d'une dérivation locale D_0 et d'une dérivation non locale D_1 de A_S. D'après la Proposition 3.9, on a $D_0(A_S^2) \subset A_S^2$; et de la Proposition 3.8, $D_1(A_S) \subset A_S^2$. En utilisant la $\mathbb{R}$-linéarité de D et $[A_S^2, A_S^2] = \{0\}$, on a le résultat. ∎

Théorème 3.11 *On suppose que pour tout $x \in \mathrm{M}$, $0 < p(x) < q$. Les assertions suivantes sont équivalentes:*

1. *Toute dérivation de A_S est locale.*
2. *Il existe $X \in A_S^1$ et $h \in F_0(\mathrm{M})$ tels que $X(h)$ est partout non nul sur* M.
3. *L'idéal dérivé de A_S est A_S.*

Démonstration. (1.) ⇔ (2.): Soit D une dérivation de A_S, D est la somme d'une dérivation locale et d'une dérivation non locale de A_S. On note alors D_1 cette dérivation non locale. Etant donné un $X \in A_S^2 - \{0\}$, on calcule $D_1(X)$. Par le fait que D_1 soit $\mathbb{R}-$linéaire, on peut supposer seulement qu'il existe $f \in F_0(\mathrm{M})$ et i_0 tels que $X = fX_{i_0}$. On écrit $D_1(X) = D_f^i X_i$; avec les $D_f^i \in \mathbb{R}$ d'après la Proposition 3.8 et la Proposition 3.2. Soient $Y \in A_S^1$, $g \in F_0(\mathrm{M})$ tels que $Y(g)$ est partout non nul, et $h \in F_0(\mathrm{M})$. Comme $D_1[hY, gX_{i_0}] = 0$ d'après la Proposition 3.8, alors chaque $D_{hY(g)}^i = 0$. Or, on peut trouver h tel que $hY(g) = f$ en posant $h = \frac{f}{Y(g)} \in F_0(\mathrm{M})$. Ainsi, $D_1(X) = 0$ pour tout $X \in A_S^2$ et par conséquent, $D_1 = 0$ car $D_{1|A_S^1} = 0$ d'après la Proposition 3.8. C'est-à-dire que toute dérivation D de A_S est locale. Réciproquement, soit D l'application $\mathbb{R}$-linéaire définie par

$$D(X) = \begin{cases} 0 & \text{si } X \in A_S - \mathfrak{C}, \\ \sum\limits_{1\leq j\leq q} \alpha^j \sum\limits_{1\leq k\leq q} X_k & \text{si } X = \alpha^i X_i \text{ où } \alpha^j \in \mathbb{R} \text{ pour tout } j = 1, \ldots, q. \end{cases}$$

En supposant que quel que soit $x \in$ M, $p(x) < q$; il existe i_0 dans $\{1, \ldots, q\}$ et un ouvert U de M, tels que $X_{i_0|U} \equiv 0$. Ainsi, on a $D(X_{i_0}) = \sum_{1 \leq k \leq q} X_k$ tel que, $D\left(X_{i_0}\right)_{|U}$ est non nul, car pour tout $x \in$ U, $p(x) > 0$. Si pour tous $X \in A_S^1$ et $h \in F_0$ (M), il existe $x \in$ M tels que $X\left(h\right)\left(x\right) = 0$, alors $fX\left(h\right) = 1$ est impossible, quel que soit $f \in F_0$ (M). Alors, $[A_S^1, A_S^2]$ ne contient pas d'éléments de $\mathfrak{C} - \{0\}$, et on a $D\left(A\right) = \{0\}$. Ainsi, D est une dérivation non locale de A_S.

(2.) $\Leftrightarrow$ (3.): Si l'idéal dérivé de A_S est A_S, alors toute dérivation non locale de A_S est nulle, d'après la Proposition 3.8. Ainsi, toute dérivation de A_S est locale. D'après (1.) $\Rightarrow$ (2.), on a le résultat. Réciproquement, la deuxième partie de la preuve de (1.) $\Leftrightarrow$ (2.) permet de conclure. ∎

Remarque 3.12 On peut omettre l'hypothèse "pour tout $x \in$ M, $0 < p(x) < q$" en prouvant (2.) $\Rightarrow$ (1.), et (2.) $\Rightarrow$ (3.) du Théorème 3.11.

Remarque 3.13 Si quels que soient $f,\ h \in F_0$ (M) et $X \in A_S^1$, $fX\left(h\right) \neq 1$, la réciproque de la Proposition 3.8 est fausse car la dérivation D de A_S définie par

$$D(X) = \begin{cases} 0 & \text{si } X \in A_S - \mathfrak{C}, \\ \alpha^k X_k & \text{si } X = \alpha^k X_k \text{ où } \alpha^k \in \mathbb{R} \text{ pour tout } k = 1, \ldots, q. \end{cases}$$

est une dérivation locale. Pourtant, D vérifie toutes les conditions nécéssaires de cette proposition.

Dans les trois propositions suivantes, on suppose que $p \geq 1$ sur U.

Proposition 3.14 *Soit D une dérivation locale de A_S dans A_S^2. Si $D_U = \beta^j \otimes \frac{\partial}{\partial y^j}$ où chaque β^i est une forme linéaire de $A_S\left(U\right)$ dans $F_0\left(U\right)$, alors β^i est fermée. De plus, si chaque β^i s'annule sur $A_S^2\left(U\right)$, alors pour tous $X, Y \in A_S(U)$, on a pour tout i*

$$\beta^i\left[X, Y\right] \frac{\partial}{\partial y^i} = \left[\beta^i\left(X\right) \frac{\partial}{\partial y^i}, Y\right] + \left[X, \beta^i\left(Y\right) \frac{\partial}{\partial y^i}\right]. \tag{3.3}$$

Démonstration. On prend $i \in \{1, \ldots, p\}$, β^i est de la forme $\beta^i = \beta_a^i dx^a + \beta'^i_j dy^j$ où chaque $\beta_a^i, \beta'^i_j \in F_0\left(U\right)$. Soient $X, Y \in A_S\left(U\right)$, par la propriété d'une dérivation, on obtient

$$D_U\left[X, Y\right] = \left[D_U\left(X\right), Y\right] + \left[X, D_U\left(Y\right)\right] \tag{3.4}$$

En posant $X = X^a \frac{\partial}{\partial x^a} + X'^j \frac{\partial}{\partial y^j}$ et $Y = Y^a \frac{\partial}{\partial x^a} + Y'^j \frac{\partial}{\partial y^j}$, alors on doit avoir

$$D_U\left[X, Y\right] = \beta_b^i X^a \frac{\partial Y^b}{\partial x^a} \frac{\partial}{\partial y^i} - \beta_b^i Y^a \frac{\partial X^b}{\partial x^a} \frac{\partial}{\partial y^i} + \beta'^i_j X^a \frac{\partial Y'^j}{\partial x^a} \frac{\partial}{\partial y^i} - \beta'^i_j Y^a \frac{\partial X'^j}{\partial x^a} \frac{\partial}{\partial y^i}$$

Le second membre de (3.4) devient

$$
\begin{aligned}
&Y^a\frac{\partial\beta_b^i}{\partial x^a}X^b\frac{\partial}{\partial y^i}-Y^a\frac{\partial X^b}{\partial x^a}\beta_b^i\frac{\partial}{\partial y^i}-Y^a\frac{\partial{\beta'}^i_j}{\partial x^a}X'^j\frac{\partial}{\partial y^i}-Y^a\frac{\partial X'^j}{\partial x^a}{\beta'}^i_j\frac{\partial}{\partial y^i}\\
&\qquad+X^a\frac{\partial{\beta'}^i_j}{\partial x^a}Y'^b\frac{\partial}{\partial y^i}+X^a\frac{\partial Y'^j}{\partial x^a}{\beta'}^i_j\frac{\partial}{\partial y^i}
\end{aligned}
\tag{3.5}
$$

Par identification membre à membre, on a

$$
-Y^a\frac{\partial\beta_b^i}{\partial x^a}X^b\frac{\partial}{\partial y^i}-Y^a\frac{\partial{\beta'}^i_j}{\partial x^a}X'^j\frac{\partial}{\partial y^i}+X^a\frac{\partial\beta_b^i}{\partial x^a}Y^b\frac{\partial}{\partial y^i}+X^a\frac{\partial{\beta'}^i_j}{\partial x^a}Y'^j\frac{\partial}{\partial y^i}=0
\tag{3.6}
$$

Par ailleurs, β^i est fermée si et seulement si

$$
\mathrm{d}\beta^i=\left(\frac{\partial\beta_a^i}{\partial x^b}-\frac{\partial\beta_b^i}{\partial x^a}\right)dx^b\underset{a<b}{\wedge}dx^a+\left(\frac{\partial{\beta'}^i_j}{\partial x^a}\right)dx^a\wedge dy^j=0
$$

où d désigne la différentielle extérieure. C'est-à-dire $\frac{\partial\beta_a^i}{\partial x^b}-\frac{\partial\beta_b^i}{\partial x^a}=0$ et $\frac{\partial{\beta'}^i_j}{\partial x^a}=0$ quels que soient j,a,b.

On prend $a_0,\ b_0$ avec $Y^{a_0}=X^{b_0}=1$, $X'^j=Y'^j=0$ pour tout j, et les autres nuls dans la relation (3.6). Ainsi, $\frac{\partial\beta_{a_0}^i}{\partial x^{b_0}}-\frac{\partial\beta_{b_0}^i}{\partial x^{a_0}}=0$, pour toute valeur arbitraire de a_0,b_0.

Soient $a_1,\ j_0$ avec $Y^{a_1}=X'^{j_0}=1$, tous les autres sont nuls et, $Y'^j=X^a=0$ pour tous j,a dans (3.6). On a alors $\frac{\partial{\beta'}^i_{j_0}}{\partial x^{a_1}}=0$, pour chaque valeur arbitraire de a_1,j_0. D'où la forme β^i est fermée.

Si β^i s'annule sur $A_S^2(U)$, alors $\beta^i=\beta_a^idx^a$, pour tout a. On a

$$
\begin{aligned}
&\beta^i\left[X,Y\right]\frac{\partial}{\partial y^i}\\
&=\beta^i\left(X^a\frac{\partial Y^b}{\partial x^a}\frac{\partial}{\partial x^b}+X^a\frac{\partial Y''^l}{\partial x^a}\frac{\partial}{\partial y^l}-Y^a\frac{\partial X^b}{\partial x^a}\frac{\partial}{\partial x^b}-Y^a\frac{\partial X''^l}{\partial x^a}\frac{\partial}{\partial y^l}\right)\frac{\partial}{\partial y^i}\ .\\
&=\beta_a^iX^b\frac{\partial Y^a}{\partial x^b}\frac{\partial}{\partial y^i}-\beta_a^iY^b\frac{\partial X^a}{\partial x^b}\frac{\partial}{\partial y^i}
\end{aligned}
\tag{3.7}
$$

De plus,

$$
\begin{aligned}
\left[\beta^i\left(X\right)\frac{\partial}{\partial y^i},Y\right]+\left[X,\beta^i\left(Y\right)\frac{\partial}{\partial y^i}\right]&=-Y^a\frac{\partial\beta_a^i}{\partial x^a}X^b\frac{\partial}{\partial y^i}-Y^a\beta_b^i\frac{\partial X^b}{\partial x^a}\frac{\partial}{\partial y^i}\\
&\quad+X^a\frac{\partial\beta_b^i}{\partial x^a}Y^b\frac{\partial}{\partial y^i}+X^a\beta_b^i\frac{\partial Y^b}{\partial x^a}\frac{\partial}{\partial y^i}.
\end{aligned}
\tag{3.8}
$$

Comme la forme β^i est fermée, alors $\frac{\partial\beta_a^i}{\partial x^b}-\frac{\partial\beta_b^i}{\partial x^a}$ est nul quels que soient a,b. Par conséquent, $X^a\frac{\partial\beta_b^i}{\partial x^a}Y^b\frac{\partial}{\partial y^i}=Y^a\frac{\partial\beta_b^i}{\partial x^a}X^b\frac{\partial}{\partial y^i}$. Ainsi, en identifiant (3.7) et (3.8); on obtient le résultat (3.3). ∎

Proposition 3.15 *Soit D une dérivation locale de A_S vers A_S^2. Il existe des $1-$formes différentielles fermées α^i et ω^i dans U, avec $i = 1, \ldots, p$ telles que:*

1. $D_U = (\alpha^j + \omega^j) \otimes \frac{\partial}{\partial y^j}$, *où chaque $ker(\alpha^j)$ contient $A_S^2(U)$ et chaque $ker(\omega^j)$ contient $A_S^1(U)$.*
2. *chaque $\alpha^i [X,Y] = X.\alpha^i (Y) - Y.\alpha^i (X)$, pour tous champs $X, Y \in A_S(U)$.*

On notera $D_U^{\alpha,\omega}$ la dérivation $(\alpha^j + \omega^j) \otimes \frac{\partial}{\partial y^j}$ de $A_S(U)$ vers $A_S^2(U)$.

Démonstration. Soit $D : A_S \longrightarrow A_S^2$ une dérivation locale de l'algèbre de Lie A_S, donc la restriction $D_U : A_S(U) \longrightarrow A_S^2(U)$ l'est aussi. La dérivation D_U étant une application $\mathbb{R}-$linéaire de $A_S(U)$ vers $A_S^2(U)$. D_U doit s'écrire sous la forme

$$D_U = \beta^j \otimes \frac{\partial}{\partial y^j}$$

où les β^i sont des formes linéaires de $A_S(U)$ sur $F_0(U)$.

L'algèbre de Lie A_S^2 étant un idéal caractéristique commutatif de A_S d'après la Proposition 3.9, la restriction de D sur A_S^2 est donc une dérivation de A_S^2. Alors $D_{U|A_S^2} = \omega^j \otimes \frac{\partial}{\partial y^j}$, où ω^i sont des formes linéaires de $A_S(U)$ dans $F_0(U)$. En vertu de la Proposition 3.14, les formes β^i et ω^i sont fermées. Les formes ω^i peuvent se décomposer en $\omega^i = \omega^i_{|A_S^1(U)} + \omega^i_{|A_S^2(U)}$. En posant $\alpha^i = \beta^i - \omega^i$, les formes α^i s'annulent sur $A_S^2(U)$ pour tout i. On peut choisir α^i pour que chaque $\omega^i_{|A_S^1(U)}$ soit nulle. D'où l'assertion 3.15..

Comme $\alpha^i = \beta^i - \omega^i$, alors chaque forme α^i est fermée. Par le fait que les α^i soient fermées, pour tous $X, Y \in A_S(U)$, on a l'égalité suivante pour tout j

$$\alpha^j [X,Y] \frac{\partial}{\partial y^j} = \left[\alpha^j (X) \frac{\partial}{\partial y^j}, Y\right] + \left[X, \alpha^j (Y) \frac{\partial}{\partial y^j}\right]$$

d'après la Proposition 3.14. D'où l'assertion 3.15.

Réciproquement, il est immédiat de constater qu'une application D_U de $A_S(U)$ dans $A_S^2(U)$ vérifiant les assertions 3.15. et 3.15. est une dérivation de $A_S(U)$. ∎

Proposition 3.16 *La dérivation $D_U^{\alpha,\omega}$ de $A_S(U)$ vers $A_S^2(U)$ de la Proposition 3.15, avec $\alpha = (\alpha^1, \ldots, \alpha^p)$ et $\omega = (\omega^1, \ldots, \omega^p)$ est intérieure si et seulement si, pour tout i, $\omega^i \equiv 0$ et α^i sont des formes exactes. Dans ce cas, on a $D_U^{\alpha,\omega} = -\mathrm{L}_{f^i \frac{\partial}{\partial y^i}}$, où chaque $\alpha^i = df^i$ avec f^i sont des fonctions de $F_0(U)$.*

Démonstration. On suppose que $D_U^{\alpha,\omega} = \mathrm{L}_Y$ avec $Y = Y'^i \frac{\partial}{\partial y^i} \in A_S^2(U)$. Pour simplifier les notations, on prend $\alpha^i = \alpha^i_j dx^j$ et $\omega^i = \omega^i_j dy^j$, pour chaque i.

Soit $X = (X^1, \ldots, X^{m+q-p}, X'^1, \ldots, X'^p)$ un élément de $A_S(U)$, or $D_U = (\alpha^i + \omega^i) \otimes \frac{\partial}{\partial y^i}$ donc

$$\begin{aligned} D_U(X) &= \left(\left(\alpha^i + \omega^i\right) \otimes \frac{\partial}{\partial y^i}\right)(X) = \left(X^j \alpha^i_j + X'^j \omega^i_j\right) \frac{\partial}{\partial y^i} \\ &= \left[Y'^i \frac{\partial}{\partial y^i}, X^j \frac{\partial}{\partial x^j} + X'^j \frac{\partial}{\partial y^j}\right] = -X^j \frac{\partial Y'^i}{\partial x^j} \frac{\partial}{\partial y^i}. \end{aligned}$$

On a pour tout i

$$X^j \alpha_j^i + X'^j \omega_j^i = -X^j \frac{\partial Y'^i}{\partial x^j} \tag{3.9}$$

On pose dans (3.9) $X^j = 0$ quel que soit j et, $X'^j = 1$ pour un j fixé, avec $X'^l = 0$ pour $l \neq j$; on obtient $\omega_j^i = 0$ quel que soit i.

Maintenant, on pose dans (3.9) $X^j = 1$ pour j fixé, avec $X^l = 0$ pour $l \neq j$, on a $\alpha_j^i = -\frac{\partial Y'^i}{\partial x^j}$ quel que soit i. Ainsi chaque $\alpha^i = -\frac{\partial Y'^i}{\partial x^j} dx^j = df^i$ et $f^i = -Y'^i \in F_0(U)$. Donc α^i est une 1-forme exacte sur U, pour tout i.

Inversement, d'après l'assertion 3.15. de la Proposition 3.15, $D_U = \alpha^i \otimes \frac{\partial}{\partial y^i}$ car $\omega^j = 0$ quel que soit j. Or les α^i sont des formes exactes, alors $\alpha^i = df^i$ où f^i sont des fonctions de $F_0(U)$.

Soit $X = (X^1, \ldots, X^{m+q-p}, X'^1, \ldots, X'^p) \in A_S(U)$, on obtient

$$\begin{aligned} \alpha^i(X) &= \left(\frac{\partial f^i}{\partial x^1} dx^1 + \frac{\partial f^i}{\partial x^2} dx^2 + \cdots + \frac{\partial f^i}{\partial x^{m+q-p}} dx^{m+q-p} \right)(X) \\ &= X^1 \frac{\partial f^i}{\partial x^1} + X^2 \frac{\partial f^i}{\partial x^2} + \cdots + X^{m+q-p} \frac{\partial f^i}{\partial x^{m+q-p}} \end{aligned}$$

Comme

$$D_U\left(X^1, \ldots, X^{m+q-p}, X'^1, \ldots, X'^p\right) = \left(\alpha^i \otimes \frac{\partial}{\partial y^i}\right)(X)$$

$$= \left(X^1 \frac{\partial f^i}{\partial x^1} + X^2 \frac{\partial f^i}{\partial x^2} + \cdots + X^{m+q-p} \frac{\partial f^i}{\partial x^{m+q-p}}\right) \frac{\partial}{\partial y^i} = -\left[f^i \frac{\partial}{\partial y^i}, X^j \frac{\partial}{\partial x^j}\right]$$

$$= -\left[f^i \frac{\partial}{\partial y^i}, X^j \frac{\partial}{\partial x^j}\right] - \left[f^i \frac{\partial}{\partial y^i}, X'^j \frac{\partial}{\partial y^j}\right] = -\left[f^i \frac{\partial}{\partial y^i}, X^j \frac{\partial}{\partial x^j} + X'^j \frac{\partial}{\partial y^j}\right]$$

car f^i et X'^i ne dépendent pas des y^l.

Alors $D_U^{\alpha,0} = D_U = -\mathrm{L}_{f^i \frac{\partial}{\partial y^i}}$ avec $f^j \frac{\partial}{\partial y^j} \in A_S^2(U)$.

Il en résulte que $D_U^{\alpha,\omega}$ est une dérivation intérieure si et seulement si les $\omega^i \equiv 0$ et $\alpha^i = df^i$, où les f^i sont des fonctions de $F_0(U)$. Dans ce cas, la dérivation $D_U^{\alpha,0} = -\mathrm{L}_{f^i \frac{\partial}{\partial y^i}}$. ∎

On rappelle le résultat classique suivant:

Proposition 3.17 *Soit $\mathfrak{A}$ une sous-algèbre de Lie des champs de vecteurs de* M, *Γ un idéal caractéristique de $\mathfrak{A}$, D une dérivation sur $\mathfrak{A}$, π la projection canonique de $\mathfrak{A}$ sur l'algèbre-quotient $\mathfrak{A}/\Gamma$. En posant $D'_\pi(X) = \pi(D(X))$ pour tout $X \in \mathfrak{A}$, D' définit une dérivation sur $\mathfrak{A}/\Gamma$. En particulier, si $D = \mathrm{L}_X$ alors $D' = \mathrm{L}_{\pi(X)}$.*

Proposition 3.18 *Toute dérivation locale D de l'algèbre de Lie A_S s'écrit d'une manière unique sous la forme $\mathrm{L}_X + D^0$ avec $X \in A_S^1$ et, pour toute carte adaptée U, ${D^0}_{|U} = 0$ si la dimension de U est nulle; ${D^0}_{|U} = D^{\alpha,\omega}$ une dérivation définie par la Proposition 3.15 sinon.*

Démonstration. Soit D une dérivation locale de A_S. Il vient que l'algèbre de Lie quotient $A_S(U)/A_S^2(U)$ est isomorphe à $A_S^1(U)$, et est donc isomorphe à l'algèbre de Lie des champs de vecteurs sur un ouvert de $\mathbb{R}^{m+q-p}$. Or toute dérivation de $\chi(\mathbb{R}^{m+q-p})$ est intérieure d'après un résultat de [5], alors toute dérivation de l'algèbre de Lie $A_S(U)/A_S^2(U)$ est intérieure. En vertu de la Proposition 3.17, toute dérivation D_U de $A_S(U)$ est de la forme $D'_U = \mathrm{L}_{\pi(Y)}$, avec $Y \in A_S^1(U)$, où $\pi : A_S(U) \to A_S(U)/A_S^2(U)$ est la projection canonique. En posant $D^0 = D - \mathrm{L}_X$ où $X \in A_S^1$ tel que $X_{|U} = Y$, la dérivation correspondante ${D^0}'_U$ de l'algèbre-quotient est nulle, $D^0{}_U$ est donc une dérivation de $A_S(U)$ dans $A_S^2(U)$. Si $p = 0$ alors $D^0{}_U = 0$. Si $p > 0$, d'après la Proposition 3.15, sur une carte adaptée au feuilletage; $D^0{}_U$ est de la forme $D_U^{\alpha,\omega}$. D'où la décomposition annoncée. ∎

Théorème 3.19 *Si le rang de S est constant égal à $p \in [1,q]$, le premier espace de cohomologie locale de Chevalley-Eilenberg $\mathrm{H}^1_{loc}(A_S)$ de A_S est isomorphe à $(\mathrm{H}^1_R(\mathrm{B}) \times \mathbb{R})^p \times \mathbb{R}^{p^2}$, où $\mathrm{H}^1_R(\mathrm{B})$ désigne le premier espace de cohomologie de de Rham sur les formes basiques au feuilletage de* M.

Démonstration. Soit D une dérivation locale de A_S, alors la restriction D_U de D à une carte adaptée U au feuilletage est une dérivation de $A_S(U)$. D'après la Proposition 3.18, D_U se décompose en une somme de deux dérivations $D_U = \mathrm{L}_{X_{|U}} + D_U^{\alpha,\omega}$, où $X_{|U} \in A_S^1(U)$ et, où $D_U^{\alpha,\omega}$ est une dérivation définie dans la Proposition 3.15. Si le rang $p \geq 1$ de S est constant, une dérivation $D_U^{\alpha,\omega}$ s'écrit d'une façon unique $D_U^{\alpha,\omega} = D_U^{\alpha,0} + D_U^{0,\omega}$ et l'expression $D_U^{\alpha,0} \circ D_U^{0,\omega}$ est nulle. L'algèbre des dérivations de la forme $D_U^{0,\omega}$ est isomorphe à l'algèbre $\mathrm{gl}(\mathbb{R}^p)$ des endomorphismes de $A_S^2(U)$. D'autre part, en notant $\alpha_U = (\alpha_U^1, \ldots, \alpha_U^p)$, on a la somme des dérivations $D_U^{\alpha,0} = D_U^{\alpha^1,0} + \cdots + D_U^{\alpha^p,0}$ telles que $D_U^{\alpha^i,0} \circ D_U^{\alpha^j,0} = 0$ pour tous i,j. Les α^i, $i = 1,\ldots,p$ sont des tenseurs invariants par transition des cartes adaptées. L'ensemble des α^i s'identifie à $Z^1(\mathrm{B})_{|U} \times \mathbb{R}$ cf. [4], $Z^1(\mathrm{B})_{|U}$ étant l'ensemble des 1-formes basiques et fermées sur U. L'ouvert U est un domaine d'une carte adaptée quelconque de M, d'où le résultat. ∎

Remarque 3.20 On suppose qu'il existe une feuille singulière du feuilletage $\mathfrak{F}$. En travaillant sur l'ensemble ouvert des points réguliers R dense dans M, on trouve sur la variété R le même résultat que celui de la Proposition 3.18. Si le prolongement de X correspondant à D dans cette proposition est dans A_S^1 et que chaque prolongement de α et de ω sont C^∞, alors la Proposition 3.18 reste valable sur M.

Exemple 3.21 Soit $\mathrm{M} = \mathbb{R}^3$ de coordonnées canoniques (x,y,z), $S = \left\{\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\}$. Les éléments de A_S sont de la forme $f(x)\frac{\partial}{\partial x} + g(x)\frac{\partial}{\partial y} + h(x)\frac{\partial}{\partial z}$, pour toutes fonctions C^∞, f, g et h ne dépendant que de x. D'après nos théorèmes, le premier espace de cohomologie de Chevalley-Eilenberg $\mathrm{H}^1(A_S) = \mathrm{H}^1_{loc}(A_S)$ est de dimension six. La Proposition 3.15 donne la construction d'une base des dérivations non intérieures de A_S dont les éléments sont les suivants:

$$D_1 = dy \otimes \frac{\partial}{\partial y} \qquad D_2 = dz \otimes \frac{\partial}{\partial y} \qquad D_3 = dy \otimes \frac{\partial}{\partial z}$$

$$D_4 = dz \otimes \frac{\partial}{\partial z} \qquad D_5 = \psi \otimes \frac{\partial}{\partial y} \qquad D_6 = \psi \otimes \frac{\partial}{\partial z}$$

où ψ désigne l'application $\psi\left(f\left(x\right)\frac{\partial}{\partial x}\right) = \frac{\partial f(x)}{\partial x}$.

Remarque 3.22

1. Si la structure de la variété M feuilletée par $\{X_1, \ldots, X_p\}$ est transversalement orientable, alors chaque forme α^i du Théorème 3.19 s'écrit

$$\alpha^i = \gamma^i + k\varphi$$

   où chaque γ^i est une 1-forme basique fermée, k un nombre réel et φ la divergence de la structure transversale.

2. Si C est le champ de Liouville sur le fibré vectoriel TM de la variété M. On désigne par $A_C = \{X \in \mathcal{X}(T\mathrm{M})$ tel que $[X, C] = 0\}$. Soit $\{0\}$ la section nulle de TM, on pose $S = \{C\}$ dans la variété $\overset{\circ}{T}\mathrm{M} = T\mathrm{M} - \{0\}$. L'algèbre de Lie A_S est égale à l'algèbre de Lie $\overset{\circ}{A}_C$ définie dans [2]. Toute dérivation de $\overset{\circ}{A}_C$ est une dérivation indiquée dans la Proposition 3.18. Ce résultat est prolongeable sur TM, d'où le résultat de [4] sur $\mathrm{H}^1\left(A_C\right)$.

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C-ESSENTIALNESS AND WELL-BEHAVEDNESS OF C-INJECTIVITY IN Act-S

Leila Shahbaz

Department of Mathematics
University of Maragheh
Maragheh 55181-83111
Iran
e-mail: leilashahbaz@yahoo.com

Abstract. An important notion related to injectivity with respect to monomorphisms or any other class $\mathcal{M}$ of morphisms in a category $\mathcal{A}$ is essentialness. In this paper, taking $\mathcal{A}$ to be the category of right acts over a semigroup S, C to be an arbitrary closure operator in the category **Act-S**, and $\mathcal{M}_d$ to be the class of C-dense monomorphisms resulting from a closure operator C, we study the properties of $\mathcal{M}_d$-essential monomorphisms and we show the existence of a maximal $\mathcal{M}_d$-essential extension for any given act. Finally, the behavior of $\mathcal{M}_d$-injectivity in the sense that the three so called Well-behavedness propositions hold is studied. We show that the idempotency and weak hereditariness of a closure operator C are sufficient, but not necessary, conditions for the well-behavedness of $\mathcal{M}_d$-injectivity. The class of sequentially dense monomorphisms resulting from a special closure operator (sequential closure operator) and injectivity with respect to this class of monomorphisms have been studied by Giuli, Ebrahimi, Mahmoudi, Moghaddasi, and the author. Some of these results generalize some of the results about the class of sequentially dense monomorphisms.

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1. Introduction and preliminaries

An important notion related to injectivity with respect to monomorphisms or any other class $\mathcal{M}$ of morphisms in a category $\mathcal{A}$ is essentialness. In fact, injectivity is characterized and injective hulls are defined using essentialness (see, for example, [1], [18], and [6]). Recall that for a subclass $\mathcal{M}$ of the class $\mathcal{M}ono$ of monomorphisms of a category $\mathcal{A}$ and $M\xrightarrow{m}X \in \mathcal{M}$, one usually uses one of the following definitions to say that m is *essential*:

(1) $M\overset{m}{\rightarrowtail}X\xrightarrow{f}Y \in \mathcal{M} \Rightarrow f \in \mathcal{M}$.

(2) $M\overset{m}{\rightarrowtail}X\xrightarrow{f}Y \in \mathcal{M}ono \Rightarrow f \in \mathcal{M}ono$.

(3) $M\overset{m}{\rightarrowtail}X\xrightarrow{f}Y \in \mathcal{M} \Rightarrow f \in \mathcal{M}ono$.

Clearly, condition (3) is weaker than the other two and if $\mathcal{M}$ is taken to be the class $\mathcal{M}ono$ of all monomorphisms (in which case m is said to be an *essential monomorphism*), all the above three conditions are equivalent, but not necessarily otherwise (see, for example, [2], [3], [19]). Definition (1) is usually used for an arbitrary class $\mathcal{M}$ of morphisms of an arbitrary category $\mathcal{A}$ (see [1], [6], and [18]). The second is the one which is used in Universal Algebra, and the third one has been used when $\mathcal{M}$ is an special class of monomorphisms, in particular pure monomorphisms in an equational class of algebras. Further, Banaschewski [1] defines and studies conditions on a category $\mathcal{A}$ and a subclass $\mathcal{M}$ of monomorphisms in $\mathcal{A}$ under which $\mathcal{M}$-injectivity *behaves well* in the sense that the following three propositions hold (the definition of the terms will be given in the sequel):

Proposition 1.1 (First Theorem of Well-Behavedness) *For every $A \in \mathcal{A}$, the following conditions are equivalent:*

(I1) *A is $\mathcal{M}$-injective.*

(I2) *A is an $\mathcal{M}$-absolute retract.*

(I3) *A has no proper $\mathcal{M}$-essential extensions.*

Proposition 1.2 (Second Theorem of Well-Behavedness) *Every $A \in \mathcal{A}$ has an $\mathcal{M}$-injective hull which is unique up to isomorphism.*

Proposition 1.3 (Third Theorem of Well-Behavedness) *For an extension B of A, the following conditions are equivalent:*

(H1) *B is an $\mathcal{M}$-injective hull of A.*

(H2) *B is a maximal $\mathcal{M}$-essential extension of A.*

(H3) *B is a minimal $\mathcal{M}$-injective extension of A.*

Banaschewski [1] gives the following sufficient conditions on the pair $\mathcal{M}$ and $\mathcal{A}$ which ensure the well-behavedness of $\mathcal{M}$-injectivity in $\mathcal{A}$.

Proposition 1.4 *$\mathcal{M}$-injectivity behaves well in $\mathcal{A}$ if the following conditions hold:*

(E1) *$\mathcal{M}$ is transitive (closed under composition).*

(E2) *$\mathcal{M}$ is isomorphism closed.*

(E3) *$\mathcal{A}$ fulfills Banaschewski's $\mathcal{M}$-condition.*

(E4) *$\mathcal{A}$ satisfies the $\mathcal{M}$-transferability property.*

(E5) *$\mathcal{A}$ has $\mathcal{M}$-direct limits.*

(E6) *$\mathcal{A}$ is $\mathcal{M}$-essentially bounded.*

In this paper, we take $\mathcal{A}$ to be the category **Act-S** of acts over a semigroup S, C to be an arbitrary closure operator in the category **Act-S**, and $\mathcal{M}_d$ to be the class of C-dense monomorphisms and study the above notions of essentiality with respect to this class. We will see that the above notions of essentiality are equivalent for this subclass $\mathcal{M}_d$ of $\mathcal{M}ono$, too, and investigate some of the properties of $\mathcal{M}_d$-essential monomorphisms normally needed in the study of $\mathcal{M}_d$-injectivity. Among other things, the existence of a maximal such essential extension for any given act is shown. Finally, the behavior of $\mathcal{M}_d$-injectivity in the sense that the above so called well-behavedness propositions hold is studied. We show that the idempotency and weakly hereditariness of a closure operator C are sufficient, but not necessary, conditions for the well-behavedness of $\mathcal{M}_d$-injectivity. Some of these results generalize some of the results in [8], [11], [12], [14], [15], and [16].

In the following we first recall from [10] and [7] some facts about the category **Act-S** needed in this paper.

Let S be a semigroup, A be a set, and

$$\begin{array}{rcl} \mu : A \times S & \longrightarrow & A \\ (a,s) & \longmapsto & as := \mu(a,s), \end{array}$$

be a map. The set A is called a (*right*) *S-act* or a (*right*) *act* over S, if the map μ satisfies $a(st) = (as)t$ for $a \in A$ and $s, t \in S$. In this case, μ is called the action of S on A.

If S is a monoid with 1 as its identity, we usually also require that $a1 = a$ for $a \in A$.

A subset A' of an S-act A is said to be a *subact* of A if $a's \in A'$ for all $s \in S$ and $a' \in A'$; and in this case we write $A' \leq A$.

A *homomorphism* (also called an *equivariant* map or an *S-map*) from an S-act A to an S-act B is a function from A to B such that for each $a \in A$, $s \in S$, $f(as) = f(a)s$.

Since id_A and the composition of two S-maps are S-maps, we have the category **Act-S** of all right S-acts and S-maps between them.

Note that the class of S-acts is an equational class, and so the category **Act-S** is complete and cocomplete (has all products, equalizers, pullbacks, coproducts, coequalizers, and pushouts). In fact, limits and colimits in this category are computed as in the category **Set** of sets and equipped with a natural action. Also, monomorphisms (epimorphisms) in **Act-S** are exactly one-one (onto) S-maps. Therefore, we do not distinguish between monomorphisms of acts and inclusions, and call an S-act B containing (an isomorphic copy of) an S-act A an *extension* of A.

For an S-act A and $a \in A$ we denote the S-map $f : S \to A$, given by $f(s) = as$ for all $s \in S$, by λ_a.

Recall that an element a of an S-act A is called a *fixed* or a *zero* element if $as = a$ for all $s \in S$.

Also, recall that for a family $\{A_i : i \in I\}$ of S-acts with a unique fixed element 0, the *direct sum* $\bigoplus_{i \in I} A_i$ is defined to be the subact of the product

$\prod_{i\in I} A_i$ consisting of all $(a_i)_{i\in I}$ such that $a_i = 0$ for all $i \in I$ except a finite number.

Denoting the lattice of all subacts of an S-act B by $SubB$, following [5] for the general definition of closure operators on a category, we get:

Definition 1.5 A family $C = (C_B)_{B\in \mathbf{Act-S}}$, with $C_B : SubB \to SubB$, taking $A \leq B$ to $C_B(A)$, is called a *closure operator* on **Act-S** if it satisfies the following laws:

(c_1) (*Extension*) $A \leq C_B(A)$,

(c_2) (*Monotonicity*) $A_1 \leq A_2$ implies $C_B(A_1) \leq C_B(A_2)$,

(c_3) (*Continuity*) $f(C_B(A)) \leq C_D(f(A))$, for all morphisms $f : B \to D$.

Now, one has the usual two classes of monomorphisms related to the notion of a closure operator as follows:

Definition 1.6 Let $A \leq B$ be in **Act-S**. We say that A is C-*closed* in B if $C_B(A) = A$, and it is C-*dense* in B if $C_B(A) = B$. Also, an S-map $f : A \to B$ is said to be C-dense (C-closed) if $f(A)$ is a C-dense (C-closed) subact of B.

We denote the class of all C-dense monomorphisms by $\mathcal{M}_d$ and recall some of the properties of this class from [17].

Definition 1.7 A closure operator C is said to be:

(a) *Weakly hereditary* if for every S-act B and every $A \leq B$, A is C-dense in $C_B(A)$.

(b) *Idempotent* if $C_B(C_B(A)) = C_B(A)$ for all S-acts B and $A \leq B$.

Remark 1.8 Notice that all isomorphisms are C-dense and the composition of an isomorphism with a C-dense monomorphism is C-dense. Also, the composition of a C-dense monomorphism with a surjective morphism is a C-dense morphism.

As the following result of [17] shows, the class of C-dense monomorphisms is not always closed under composition.

Theorem 1.9 *For a semigroup S and a closure operator C, the following are equivalent:*

(i) *The closure operator C is idempotent and weakly hereditary.*

(ii) *The class $\mathcal{M}_d$ is closed under composition and the closure operator C is weakly hereditary.*

(iii) *Each S-map $f : A \to B$ has a (C-dense, C-closed) factorization.*

We recall the following lemma from [9]:

Lemma 1.10 *Pushouts transfer monomorphisms in* **Act-**S.

We recall the following from [17] which is a counterpart of (E4) in [1].

Proposition 1.11 *In* **Act-S**, *pushouts transfer C-dense monomorphisms.*

We recall the following from [17] which is a counterpart of (E5) in [1].

Proposition 1.12 **Act-S** *has $\mathcal{M}_d$-directed colimits.*

Definition 1.13 We call an S-act A, *C-dense injective* or *C-injective* if it is injective with respect to C-dense monomorphisms; that is, for every C-dense monomorphism $h : B \to D$ and every S-map $f : B \to A$ there exists an S-map $g : D \to A$ such that $gh = f$.

We recall the following theorem from [17] which is desirable in the study of any type of injectivity.

Theorem 1.14 *Let S be a semigroup. Then, an S-act A is C-injective if and only if it is C-absolute retract (retract of any of its C-dense extensions).*

2. C-dense essential monomorphisms

Now that we have introduced the class $\mathcal{M}_d$ of C-dense monomorphisms, we begin the study of essentiality with respect to this class. Recall the three different notions of essentiality with respect to a subclass $\mathcal{M}$ of monomorphisms given in the introduction. We also mentioned there that for some classes $\mathcal{M}$, specially for the class $\mathcal{M}ono$, these three notions of essentiality are in fact equivalent. In the following theorem we prove that this is also the case for the class $\mathcal{M}_d$. We then investigate some properties of essentiality, usually needed in the study of injectivity with respect to the class $\mathcal{M}_d$.

Theorem 2.15 *For a C-dense monomorphism $f : A \to B$, the following are equivalent:*

(i) *Any S-map $g : B \to D$ for which gf is a C-dense monomorphism is itself a C-dense monomorphism.*

(ii) *Any S-map $g : B \to D$ for which gf is a C-dense monomorphism is a monomorphism.*

(iii) *Any S-map $g : B \to D$ for which gf is a monomorphism is itself a monomorphism.*

(iv) *For every congruence ρ on B with $\rho \neq \Delta_B$ one has $\rho\,|_A = \rho \cap (A \times A) \neq \Delta_A$.*

Proof. (i)⇒(ii) Let $g : B \to D$ be such that $gf \in \mathcal{M}_d$, then by the assumption $g \in \mathcal{M}_d$. Thus g is a monomorphism.

(ii)⇒(iii) Let $g : B \to D$ be an S-map such that gf is a monomorphism. Then since $gf : A \to g(B)$ is a C-dense monomorphism, and by (ii), we get that $g : B \to g(B)$ is a monomorphism and hence g is a monomorphism.

(iii)⇔(iv) It is obtained using Lemma III.1.15 of [10].

(iv)⇒(i) Let $g : B \to D$ be such that $gf \in \mathcal{M}_d$, by (iii)⇔(iv), we get that g is a monomorphism. Since the class $\mathcal{M}_d$ is right cancellable, g is C-dense. Thus $g \in \mathcal{M}_d$. ∎

Definition 2.16 We call a C-dense monomorphism satisfying one of the equivalent conditions of the above theorem an *$\mathcal{M}_d$-essential* or *C-dense essential* monomorphism.

It follows by the above theorem that:

Corollary 2.17 *A monomorphism f is $\mathcal{M}_d$-essential if and only if it is essential as well as C-dense.*

Remark 2.18

(a) Since the composition of two essential monomorphisms is clearly essential, if the closure operator C is idempotent and weakly hereditary, we get from Corollary 2.17 that the composition of $\mathcal{M}_d$-essential monomorphisms is an $\mathcal{M}_d$-essential monomorphism.

(b) Let the closure operator C be idempotent and weakly hereditary and $A \subseteq A' \subseteq B$. Then A is $\mathcal{M}_d$-essential in B if and only if A is $\mathcal{M}_d$-essential in A' and A' is $\mathcal{M}_d$-essential in B.

(c) If gf is $\mathcal{M}_d$-essential and g is a monomorphism then g is $\mathcal{M}_d$-essential.

(d) Any directed colimit of $\mathcal{M}_d$-essential monomorphisms is an $\mathcal{M}_d$-essential monomorphism.

Definition 2.19 A category $\mathcal{A}$ is called $\mathcal{M}$-*essentially bounded*, for a subclass $\mathcal{M}$ of its monomorphisms, if every $A \in \mathcal{A}$ has only a set of $\mathcal{M}$-essential extensions.

The following is a counterpart of (E6) in [1].

Proposition 2.20 *The category* **Act-S** *is $\mathcal{M}_d$-essentially bounded.*

Proof. By using the fact that each S-act admits only a set of essential extensions and Corollary 2.17, we get that each S-act has only a set of $\mathcal{M}_d$-essential extensions. ∎

Definition 2.21 For a category $\mathcal{A}$, a class $\mathcal{M}$ of monomorphisms is said to satisfy *Banaschewski's $\mathcal{M}$-condition* if for every $\mathcal{M}$-morphism $f : A \to B$ there exists a homomorphism $g : B \to D$ such that gf is an $\mathcal{M}$-essential morphism.

The following is a counterpart of (E3) in [1].

Proposition 2.22 **Act-S** *fulfills Banaschewski's $\mathcal{M}_d$-condition.*

Proof. Let $A \xrightarrow{f} B \in \mathcal{M}_d$. Consider the poset

$$\mathcal{P} = \{\theta \in Con(B) : A \xrightarrow{f} B \xrightarrow{\gamma_\theta} B/\theta \text{ is a C} - \text{dense monomorphism}\}$$

under the usual ordering of relations. Let

$$... \leq \rho_i \leq ...$$

$i \in I$, be a chain in $\mathcal{P}$. Then $\rho = \bigcup_{i\in I} \rho_i$ is also a congruence which is an upper bound of this chain which belongs to $\mathcal{P}$. Indeed, let $x, y \in A$ with $x\rho y$. Then $x\rho_j y$ for some $j \in I$. Since $\gamma_{\rho_j} f$ is a monomorphism we have $x = y$. This means that $\rho \in \mathcal{P}$. Applying Zorn's Lemma, there exists a maximal such a congruence, say θ. Let $g : B \rightarrow B/\theta$. Then maximality of θ implies that $g \circ f : A \rightarrow B/\theta$ is an essential monomorphism. Indeed, suppose $h : B/\theta \to D$ is a homomorphism whose restriction on A is monomorphism. Define a relation σ on B by

$$x\sigma y \Leftrightarrow [x]_\theta (ker f)[y]_\theta$$

for any $x, y \in B$. Then σ is a congruence on B such that $\theta \leq \sigma$ and $\gamma_\sigma f$ is a monomorphism. Hence $\sigma = \theta$ which means that h is a monomorphism. Since g is surjective, it is C-dense and so, by Corollary 2.17, it is $\mathcal{M}_d$-essential. ∎

Lemma 2.23 *Let A be a C-dense subact of B. If A is a proper retract of B ($A \ncong B$) then A is not $\mathcal{M}_d$-essential in B.*

Definition 2.24 Let A be an S-act. Then by a *maximal $\mathcal{M}_d$-essential extension of* A we mean an $\mathcal{M}_d$-essential extension B of A such that every homomorphism $h : B \to D$ from B to an $\mathcal{M}_d$-essential extension D of A for which $h\mid_A$ is the inclusion map, is an isomorphism.

Lemma 2.25 *If B is an $\mathcal{M}_d$-essential extension of A and A is embedded into some $(C-)$ injective act Q, then B can also be embedded into Q.*

Proof. Suppose A is $\mathcal{M}_d$-essential in B and consider the diagram

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "i"'] & B \arrow[dl, "\bar{i}"] \\ Q & \end{tikzcd}$$

where Q is $(C-)$ injective and i is a monomorphism. Since Q is $(C-)$ injective, there exists an S-map $\bar{i}$ such that $\bar{i}\mid_A = i$. Since A is $\mathcal{M}_d$-essential in B, $\bar{i}$ is a monomorphism. ∎

Proposition 2.26 *Every right S-act has a maximal $\mathcal{M}_d$-essential extension.*

Proof. Let A be an arbitrary act and Q be an injective act into which A can be embedded which exists by [4]. By the above Lemma A and all its $\mathcal{M}_d$-essential extensions are subacts of Q. Let $\mathcal{P}$ be the set of all $\mathcal{M}_d$-essential extensions of A. Consider $\mathcal{P}$ as a partially ordered set under inclusion. By Zorn's Lemma, $\mathcal{P}$ has a maximal element, say E. Then E is clearly a maximal $\mathcal{M}_d$-essential extension of A. ∎

3. Well-behavedness of C-dense injectivity

Banaschewski defines and gives some sufficient, but not necessary, conditions on a category $\mathcal{A}$ and a subclass $\mathcal{M}$ of its monomorphisms under which $\mathcal{M}$-injectivity *is well behaved* with respect to the notions such as $\mathcal{M}$-absolute retract and $\mathcal{M}$-essentialness. Recall the three well-behavedness theorems given in the introduction. In this section we study these so called well-behavedness theorems of injectivity for C-injectivity. We show that the idempotency and weakly hereditariness of the closure operator C are sufficient, but not necessary (take C as the sequential closure operator and see [14]), conditions for C-injectivity to be well behaved.

First, applying Proposition 1.4, and the results of former sections about (E1)-(E6) for the class $\mathcal{M}_d$ of C-dense monomorphisms in the category **Act-S**, we get:

Theorem 3.27 *If C is an idempotent and weakly hereditary closure operator then $\mathcal{M}_d$-injectivity behaves well in the category* **Act-S**.

But, we see that the mentioned condition on C is not necessary for the First Theorem of Well-Behavedness.

Theorem 3.28 (First Theorem of Well-Behavedness) *For a semigroup S, a closure operator C, and any S-act A, the following are equivalent:*

(i) *A is C-injective.*

(ii) *A is C-absolute retract.*

(iii) *A has no proper C-essential extension.*

Proof. (i)$\Longleftrightarrow$(ii) is clear by Theorem 1.14.

(ii)$\Longleftrightarrow$(iii) Let A be C-absolute retract and B be a proper C-dense extension of A. By hypothesis, A is a retract of B. Then, by Lemma 2.23, B is not an $\mathcal{M}_d$-essential extension of A. For the converse, let B be a C-dense extension of A. Then, by Proposition 2.22, there is an S-map $g : B \to D$ such that gi is $\mathcal{M}_d$-essential, where $i : A \to B$ is the inclusion map. Then, by hypothesis, gi has to be an isomorphism. Now, $\pi = (gi)^{-1}g : B \to A$ is an epimorphism and $\pi(a) = a$ for all $a \in A$. ∎

Now, giving a definition, we state the Second Theorem of Well-Behavedness of C-injectivity.

Definition 3.29 By a *C-dense injective hull* or *C-injective hull* of an S-act A we mean a C-essential extension of A which is C-injective.

For an S-act A, C-injective hull is unique up to isomorphism (if it exists).

The Second Theorem of Well-Behavedness of C-injectivity is about the existence of C-injective hull, which is proved in the following theorem for S-acts, for an idempotent and weakly hereditary closure operator C.

Theorem 3.30 (Second Theorem of Well-Behavedness) *If C is an idempotent and weakly hereditary closure operator then for each S-act A the C-injective hull of A exists.*

Proof. Take a maximal C-essential extension E of an S-act A which exists by Proposition 2.26. We claim that E is C-injective. To prove this, let $g : B \to D$ be any C-dense monomorphism and $h : B \to E$ be any homomorphism. Form the following pushout

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "h"'] & D \arrow[d, "v"] \\ E \arrow[r, "u"] & P = (E \sqcup D)/\theta \end{tikzcd}$$

by Proposition 1.11, u is a C-dense monomorphism and hence retractable by Theorem 3.28 and Remark 2.18(b). This proves that E is C-injective. ∎

Finally, we give the Third Theorem of Well-Behavedness of C-injectivity, which is about the relation between C-injective hull and C-essential extension.

Definition 3.31 Let A be an S-act. Then, by a *minimal C-injective C-dense extension* of A we mean a C-dense extension B of A such that B is C-injective, and every (C-dense) monomorphism $k : D \to B$ from a C-injective C-dense extension D of A which maps A identically is an isomorphism.

Theorem 3.32 (Third Theorem of Well-Behavedness) *If C is an idempotent and weakly hereditary closure operator then for every extension B of an S-act A, the following are equivalent:*

(i) *B is the C-injective hull of A.*

(ii) *B is a maximal C-essential extension of A.*

(iii) *B is a minimal C-injective C-dense extension of A.*

Proof. (i)⇒(ii) Let D be an extension of B which is a C-essential extension of A. Then applying Remark 2.18 (b), D is a C-essential extension of B. But, by Theorem 3.28, B being C-injective has no proper C-essential extension and so $D = B$.

(ii)⇒(i) If B is a maximal C-essential extension of A then, using Lemma 2.18, it has no proper C-essential extension. So, by Theorem 3.28, B is C-injective and hence the C-injective hull of A.

(i)⇒(iii) Similar to the first part of the proof, if $D \leq B$ is a C-injective extension of A, since A is C-essential in B it is concluded that the same is true for D and then since D is C-injective, applying Theorem 3.28, we get $B = D$.

(iii)⇒(i) Let $E(A)$ be the C-injective hull of A, which exists by Theorem 3.30. Since B is C-injective, there is an S-map $f : E(A) \to B$ such that $f|_A = A \hookrightarrow B$. Since A is essential in $E(A)$, f has to be a monomorphism. So, by (iii), $B \cong E(A)$. ∎

Two other conditions can be added to the equivalent conditions given in the preceding theorem. To give them we need the following definition:

Definition 3.33

(a) By a *smallest C-injective C-dense extension* of an act A we mean a C-dense C-injective extension B of A such that for each C-injective extension D of A there exists a monomorphism $g : B \to D$ such that $g\,|_A$ is the inclusion map.

(b) By a *largest $\mathcal{M}_d$-essential extension* of an act A we mean an $\mathcal{M}_d$-essential extension B of A such that for each $\mathcal{M}_d$-essential extension D of A there exists an S-map $h : D \to B$ such that $h\,|_A$ is the inclusion map.

Theorem 3.34 *The following conditions are equivalent to the conditions of Theorem* 3.32*:*

(iv) *B is a largest C-essential extension of A.*

(v) *B is a smallest C-injective C-dense extension of A.*

Proof. Using the notations of Theorem 3.32, we have:

(iii)⇒(iv) Let $f : A \to B$ be a minimal C-injective extension of A. Consider $h : A \to B'$ as the C-injective hull of A which exists by Theorem 3.30. Then, by maximality of f, we get that the S-map $g : B' \to B$ which exists, since B is C-injective, and is a monomorphism, (since h is C-essential), is an isomorphism. So f is C-essential and evidently is a largest C-essential extension of A.

(iv)⇒(v) Take $E(A)$ to be the C-injective hull of A which exists by Theorem 3.30. Since $E(A)$ is a C-essential extension of A and B is a largest C-essential extension of A, we obtain an S-map $h : E(A) \to B$ such that $h\,|_A$ is the inclusion map. Now, since A is C-essential in $E(A)$, h is a monomorphism and so, since B is a C-essential extension of A, Remark 2.18 (b), implies that h is C-essential. But, $E(A)$ is C-injective, and so, by Theorem 3.28, has no proper C-essential extension. Hence, h is an isomorphism. Therefore, B is C-injective. So, B is evidently a smallest C-injective C-dense extension of A.

(v)$\Rightarrow$(i) Suppose $E(A)$ is the C-injective hull of A which exists by Theorem 3.30. Then, since $E(A)$ is C-injective and B is a smallest C-injective C-dense extension of A, there exists an S-map $g : B \to E(A)$ such that $g\mid_A$ is the inclusion map. Also since A is C-essential in $E(A)$ we get that g is C-essential by Remark 2.18 (b). But, B is C-injective and so has no proper C-essential extension. Thus, g is an isomorphism. Hence, B is a C-essential extension and so it is a C-injective hull of A. ∎

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NUMBERS IN THE n DIMENSIONAL SPACE

Nicola D'Alfonso

Independent Scholar
Milan
Italy
e-mail: nicola.dalfonso@hotmail.com

Abstract. This paper introduces the numbers in the n dimensional space. Namely, if in the first dimension we have the real numbers and in the second the complex numbers, in the next dimensions we have the complete numbers introduced here.

Keywords: complex numbers, complete numbers, real numbers, n dimensional space, extent of the numbers.

1. Introduction

Definition 1.1. We can define real number r(a) as the position of the straight line R that can be reached starting from that unitary through operations of translation of positions.

We can observe, with regard to this, Figure 1.

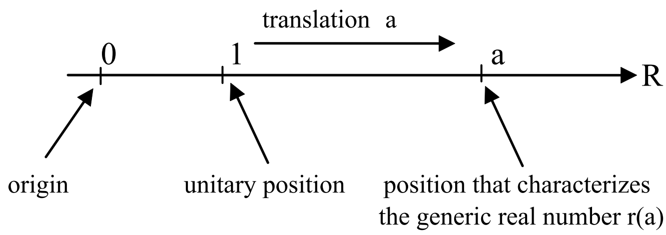

Figure 1: Cartesian representation of the real numbers

The straight line R that appears in the figure is defined line of the real numbers.

Theorem 1.2. *Real numbers can be expressed in the following way:*

$$r(a) = a$$

Proof. The proof is immediate and is a consequence of the bijection between translation operation of value (a) and the positions (a) on the line of the real numbers. ■

For more information on real numbers see [1], Chapter 1.

Definition 1.3. We can define complex number $c(t, \theta)$ as the position of the plane RI that can be reached starting from that unitary through operations of translation of positions and plane rotation of straight lines.

We can observe, with regard to this, Figure 2.

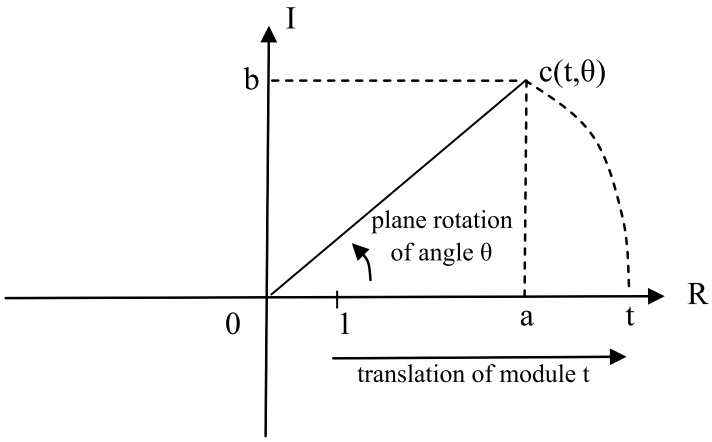

Figure 2: Cartesian representation of the complex numbers

We note that the position $c(t, \theta)$ is reached from that unitary of the line R before translating it of modulus t, and after making line R turn of the angle θ in the plane RI.

The straight line I that appears in the figure is defined line of the imaginary numbers and together with the line R of the real numbers identify the plan RI of the complex numbers.

Theorem 1.4. *Complex numbers can be expressed in the following way:*

$$c(t, \theta) = t \cdot [\cos(\theta) + i \cdot \sin(\theta)]$$

Proof. Making reference to trigonometric relations shown in Figure 3

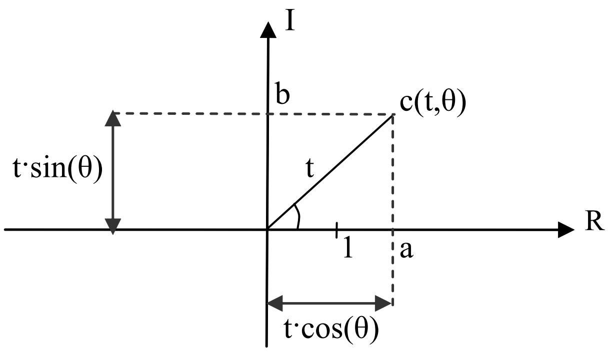

Figure 3: Trigonometric representation of complex numbers

we obtain just the result expected. ∎

Definition 1.5. The symbol t that indicates the distance of a complex number $c(t, \theta)$ from the origin is defined modulus.

Theorem 1.6. *The modulus t has the following property:*

$$t = \sqrt{a^2 + b^2}$$

Proof. By using Pythagoras' theorem on the triangle identified in Figure 3 on the preceding page we can obtain the relation:

$$t^2 = a^2 + b^2$$

from which results the previous one. ∎

Definition 1.7. The symbol θ that expresses the rotation that has to undergo the line R to align itself with the straight line that joins $c(t, \theta)$ to the origin is defined plane phase.

Theorem 1.8. *The plane phase θ has the following property:*

$$\theta = \arctan\left(\frac{b}{a}\right)$$

Proof. Making reference again to the same triangle of Figure 3 on the facing page we obtain the relation:

$$\frac{b}{a} = \tan(\theta)$$

from which results the previous one. ∎

Theorem 1.9. *Complex numbers can be expressed in the following way:*

$$c(t, \theta) = t \cdot [\cos(\theta + k \cdot 360) + i \cdot \sin(\theta + k \cdot 360)] \quad \textit{for } k = 0, \pm 1, \pm 2, \pm 3 \ldots$$

Proof. The proof is immediate and is a consequence of the periodicity of the functions sin() and cos(). ∎

Theorem 1.10. *Complex numbers can be expressed in the following way:*

$$c(t, \theta) = c(a, b) = a + i \cdot b$$

Proof. The proof is immediate and is a consequence of the bijection between translation and rotation operations of values (t, θ) and the positions (a,b) of the plane RI. ∎

For more information on complex numbers see [1], Chapter 3.

The transition from the first dimension of the real numbers to the second dimension of the complex numbers has required an operation of rotation. By further extending this procedure will be possible to introduce the n dimensional numbers and define their operations.

2. Numbers in three dimensional space

2.1. Introduction to the complete numbers

Definition 2.1. We can define complete number $o(t, \theta, \gamma)$ as the position of the space RIU that can be reached starting from that unitary through operations of translation of positions, of plane rotation of straight lines and spatial rotation of planes.

We can observe, with regard to this, Figure 4.

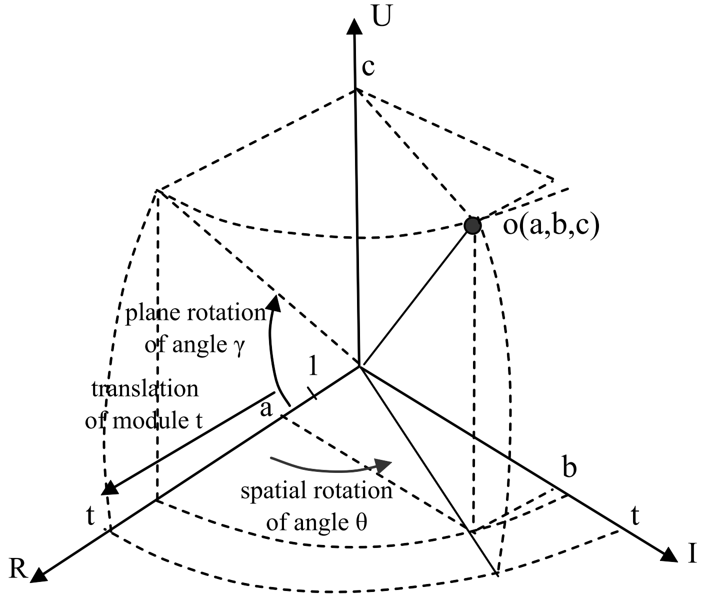

Figure 4: Cartesian representation of the complete numbers

We note that the position $o(t, \theta, \gamma)$ is reached from that unitary of the line R before translating it of modulus t, after making line R turn of the angle γ in the plane RI, and finally making the whole plane RU turn of the angle θ.

The straight line U that appears in the figure is defined line of the outgoing numbers and together with the line R of the real numbers and the line I of the imaginary numbers identify the space RIU of the complete numbers.

Theorem 2.2. *Complete numbers can be expressed in the following way:*

$$o(t, \theta, \gamma) = t \cdot \{[\cos(\gamma) \cdot \cos(\theta)] + i \cdot [\cos(\gamma) \cdot \sin(\theta)] + u \cdot [\sin(\gamma)]\}$$

Proof. Making reference to trigonometric relations shown in Figure 5 we obtain just the result expected. ∎

Definition 2.3. The symbol t that indicates the distance of a complete number $o(t, \theta, \gamma)$ from the origin is defined modulus.

Theorem 2.4. *The modulus t has the following property:*

$$t = \sqrt{a^2 + b^2 + c^2}$$

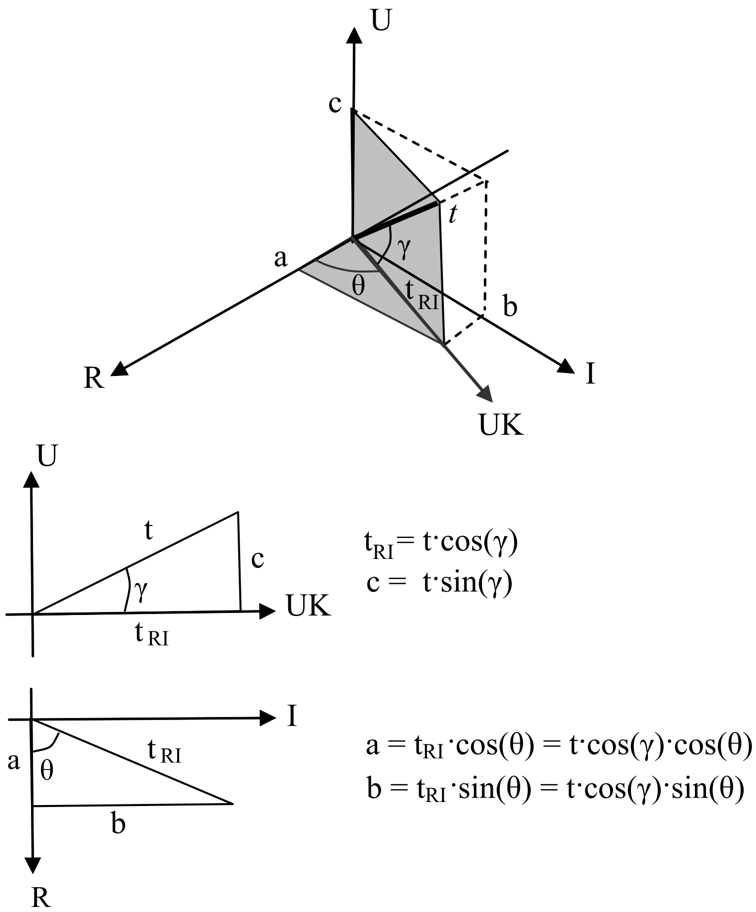

Figure 5: Trigonometric representation of the complete numbers

Proof. By using Pythagoras' theorem on the two triangles identified in Figure 5 we can obtain the following relations:

$$t^2 = t_{RI}^2 + c^2$$

$$t_{RI}^2 = a^2 + b^2$$

from which result the previous one. ■

Definition 2.5. The symbol γ that expresses the rotation that has to undergo the line R to align itself with the projection on the plane RU of the straight line that joins $o(t, \theta, \gamma)$ to the origin is defined plane phase.

Theorem 2.6. *The plane phase γ has the following property:*

$$\gamma = \arctan\left(\frac{c}{\sqrt{a^2 + b^2}}\right)$$

Proof. Making reference to the first triangle in Figure 5 we can write:

$$\gamma = \arctan\left(\frac{c}{t_{RI}}\right)$$

while making reference to the second one, we can write:

$$t_{RI}^2 = a^2 + b^2$$

from which results just the result expected. ∎

Definition 2.7. The symbol θ that expresses the rotation that has to undergo the line R to align itself with the projection on the plane RI of the straight line that joins $o(t, \theta, \gamma)$ to the origin is defined spatial phase.

Theorem 2.8. *The spatial phase θ has the following property:*

$$\theta = \arctan\left(\frac{b}{a}\right)$$

Proof. Making reference to the second triangle in Figure 5 on the preceding page we obtain the following relation:

$$\frac{b}{a} = \tan(\theta)$$

from which results the previous one. ∎

Theorem 2.9. *Complete numbers can be expressed in the following way:*

$$\begin{aligned} o(t,\theta,\gamma) = t \cdot \{&[\cos(\gamma + j \cdot 360) \cdot \cos(\theta + k \cdot 360)] + \\ &+ i \cdot [\cos(\gamma + j \cdot 360) \cdot \sin(\theta + k \cdot 360)] + u \cdot [\sin(\gamma + j \cdot 360)]\} \end{aligned}$$

$$\textit{for} \begin{cases} j = 0, \pm 1, \pm 2, \pm 3 \ldots \\ k = 0, \pm 1, \pm 2, \pm 3 \ldots \end{cases}$$

Proof. The proof is immediate and is a consequence of the periodicity of the functions sin() and cos(). ∎

Definition 2.10. A complete numbers not belonging to the line U can be defined in standard representation if provided with phases θ and γ which satisfy the conventions introduced hereunder.

For positions P(a,b,c) of the half-space R^+IU not belonging to the planes RI, RU, IU the phases of the standard representation will be those shown in Figure 6 on the next page.

The phases shown in the figure can be determined using the formulas:

$$\gamma = \arctan\left(\frac{c}{|\sqrt{a^2 + b^2}|}\right)$$

$$\theta = \arctan\left(\frac{b}{a}\right)$$

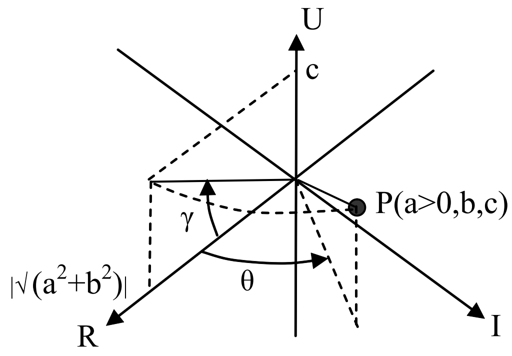

Figure 6: Phases that identify the positions of the half-space R^+IU according to the standard representation

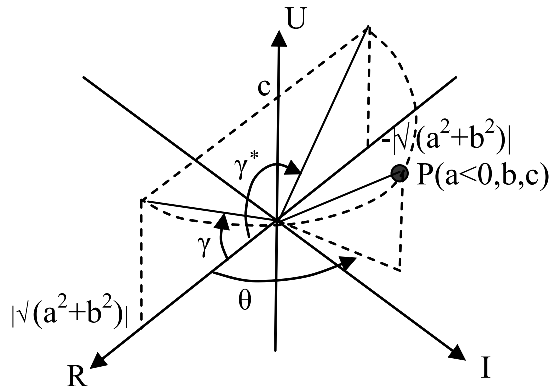

Figure 7: Phases that identify the positions of the half-space R^-IU according to the standard representation

For positions P(a,b,c) of the half-space R^-IU not belonging to the planes RI, RU, IU the phases of the standard representation will be those shown in Figure 7.

The phases shown in the figure can be determined using the formulas:

$$\gamma = \arctan\left(\frac{c}{|\sqrt{a^2+b^2}|}\right)$$
$$\theta = \arctan\left(\frac{b}{a}\right)$$

We note that the plane phase γ is not calculated by the formula:

$$\gamma = \arctan\left(\frac{c}{-|\sqrt{a^2+b^2}|}\right)$$

because it would correspond to the value γ^*.

For positions P(a,b,c) of the plane RI not belonging to the lines R and I the phases of the standard representation will be those shown in Figure 8 on the next page.

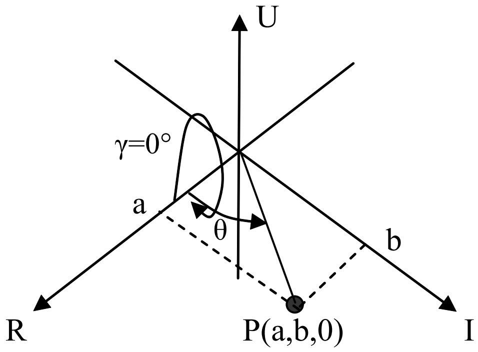

Figure 8: Phases that identify the positions of the plane RI according to the standard representation

The phases shown in the figure can be determined using the formulas:

$$\gamma = \arctan\left(\frac{c}{\sqrt{a^2+b^2}}\right)$$
$$\theta = \arctan\left(\frac{b}{a}\right)$$

For positions P(a,b,c) of the half-plane $R^{+}U$ not belonging to the lines R and U the phases of the standard representation will be those shown in Figure 9.

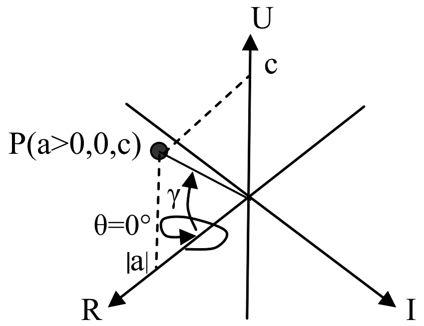

Figure 9: Phases that identify the positions of the half-plane $R^{+}U$ according to the standard representation

The phases shown in the figure can be determined using the formulas:

$$\gamma = \arctan\left(\frac{c}{|a|}\right)$$
$$\theta = 0^\circ$$

For positions P(a,b,c) of the half-plane $R^{-}U$ not belonging to the lines R and U the phases of the standard representation will be those shown in Figure 10 on the next page.

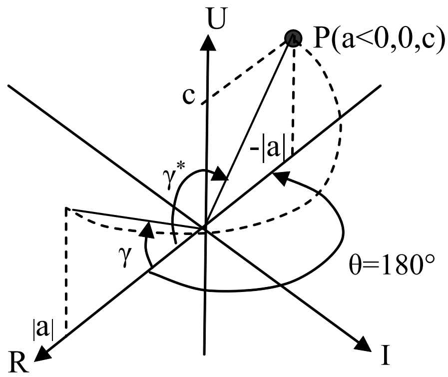

Figure 10: Phases that identify the positions of the half-plane RU according to the standard representation

The phases shown in the figure can be determined using the formulas:

$$\gamma = \arctan\left(\frac{c}{|a|}\right)$$

$$\theta = 180°$$

We note that the plane phase γ is not calculated by the formula:

$$\gamma = \arctan\left(\frac{c}{-|a|}\right)$$

because it would correspond to the value γ^*.

For positions P(a,b,c) of the half-plane I^+U not belonging to the lines I and U the phases of the standard representation will be those shown in Figure 11.

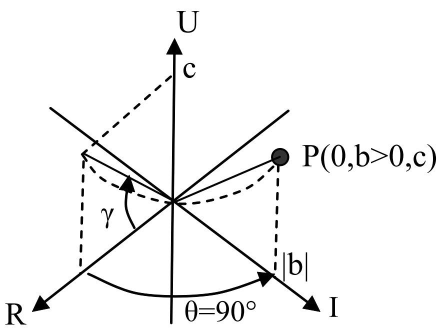

Figure 11: Phases that identify the positions of the half-plane I^+U according to the standard representation

The phases shown in the figure can be determined using the formulas:

$$\gamma = \arctan\left(\frac{c}{|b|}\right)$$

$$\theta = 90°$$

For positions P(a,b,c) of the half-plane $I^{-}U$ not belonging to the lines I and U the phases of the standard representation will be those shown in Figure 12.

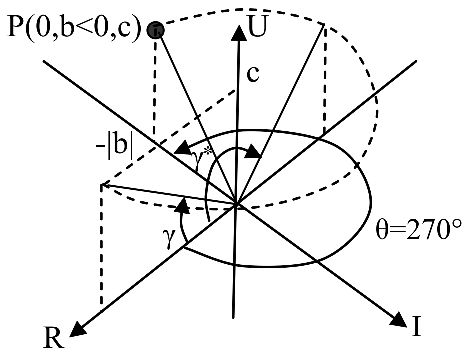

Figure 12: Phases that identify the positions of the half-plane $I^{-}U$ according to the standard representation

The phases shown in the figure can be determined using the formulas:

$$\gamma = \arctan\left(\frac{c}{|b|}\right)$$
$$\theta = 270^\circ$$

We note that the plane phase γ is not calculated by the formula:

$$\gamma = \arctan\left(\frac{c}{-|b|}\right)$$

because it would correspond to the value γ^*.

For positions P(a,b,c) of the half-line R^{+} the phases of the standard representation will be those shown in Figure 13.

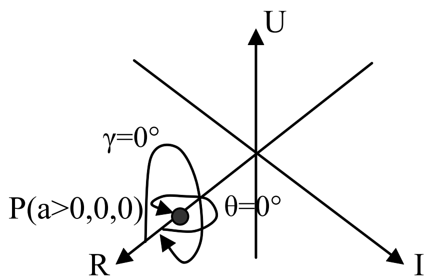

Figure 13: Phases that identify the positions of the half-line R^{+} according to the standard representation

The phases shown in the figure can be determined using the formulas:

$$\gamma = 0^\circ$$
$$\theta = 0^\circ$$

For positions P(a,b,c) of the half-line R^- the phases of the standard representation will be those shown in Figure 14.

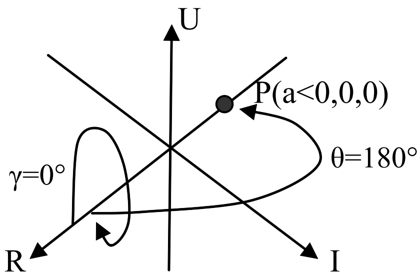

Figure 14: Phases that identify the positions of the half-line R^- according to the standard representation

The phases shown in the figure can be determined using the formulas:

$$\gamma = 0^\circ$$
$$\theta = 180^\circ$$

For positions P(a,b,c) of the half-line I^+ the phases of the standard representation will be those shown in Figure 15.

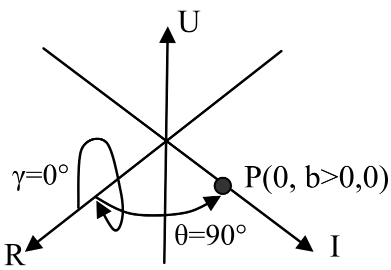

Figure 15: Phases that identify the positions of the half-line I^+ according to the standard representation

The phases shown in the figure can be determined using the formulas:

$$\gamma = 0^\circ$$
$$\theta = 90^\circ$$

For positions P(a,b,c) of the half-line I^- the phases of the standard representation will be those shown in Figure 16 on the next page.

The phases shown in the figure can be determined using the formulas:

$$\gamma = 0^\circ$$
$$\theta = 270^\circ$$

Theorem 2.11. *The standard representation of a complete number of coordinates (a,b,c) not lying on the line U requires to give to the algebraic root* $\sqrt{a^2+b^2}$ *the following positive solution:*

$$\sqrt{a^2+b^2} = \left|\sqrt{a^2+b^2}\,\right|$$

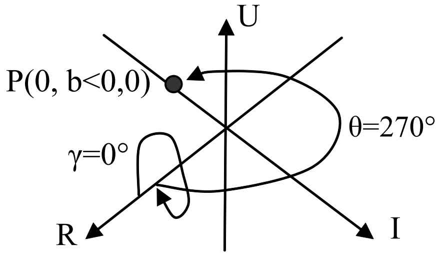

Figure 16: Phases that identify the positions of the half-line I^- according to the standard representation

Proof. In the case of the standard representations previously examined (that cover every region of the space RIU with the exception of the line U) the phase γ assumes the values provided by the formula:

$$\gamma = \arctan\left(\frac{c}{\sqrt{a^2+b^2}}\right)$$

when we give to the algebraic root $\sqrt{a^2+b^2}$ its positive solutions. And this immediately proves the thesis. ∎

Definition 2.12. A complete numbers not belonging to the line U can be defined in complementary representation if provided with phases obtained by the values θ and γ of the standard representation through those substitutions which allow us to identify the same positions.

Theorem 2.13. *If we call θ and γ the phases that allow to a complete number not belonging to the line U and in standard representation to identify any position of the space RIU, an alternative set of phases able to individuate the same position has the following values: $(\theta + 180°)$ and $(180° - \gamma)$.*

Proof. Since the following relations are valid:

$$\begin{aligned}
&\cos(180° - \gamma) \cdot \cos(\theta + 180°) = \cos(\gamma) \cdot \cos(\theta) \\
&\cos(180° - \gamma) \cdot \sin(\theta + 180°) = \cos(\gamma) \cdot \sin(\theta) \\
&\sin(180° - \gamma) = \sin(\gamma)
\end{aligned}$$

we can write:

$o(t, \theta, \gamma) = o(t, \theta{+}180°, 180°{-}\gamma)$

proving the thesis. ∎

Theorem 2.14. *Complete numbers not belonging to the line U are in complementary representation if provided with phases obtained by replacing the values θ and γ of the standard representation with the values $(\theta + 180°)$ and $(180° - \gamma)$.*

Proof. The definition of the complete numbers in complementary representation and the theorem 2.13 directly prove the thesis. ∎

Making reference to what we saw for the standard representation, the conventions adopted for the phases of the complementary representation will be those introduced hereunder.

For positions P(a,b,c) of the half-space R^+IU not belonging to the planes RI, RU, IU the phases of the complementary representation will be those shown in Figure 17.

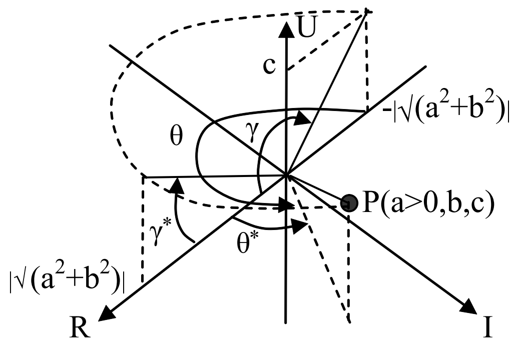

Figure 17: Phases that identify the positions of the half-space R^+IU according to the complementary representation

The phases shown in the figure can be determined using the formulas:

$$\gamma = \arctan\left(\frac{c}{-|\sqrt{a^2+b^2}|}\right)$$
$$\theta = \arctan\left(\frac{-b}{-a}\right)$$

We note that the plane phase γ and the spatial phase θ are not calculated by the formulas:

$$\gamma = \arctan\left(\frac{c}{|\sqrt{a^2+b^2}|}\right)$$
$$\theta = \arctan\left(\frac{b}{a}\right)$$

because they would correspond to the values γ^* and θ^*.

For positions P(a,b,c) of the half-space R^-IU not belonging to the planes RI, RU, IU the phases of the complementary representation will be those shown in Figure 18 on the next page.

The phases shown in the figure can be determined using the formulas:

$$\gamma = \arctan\left(\frac{c}{-|\sqrt{a^2+b^2}|}\right)$$
$$\theta = \arctan\left(\frac{-b}{-a}\right)$$

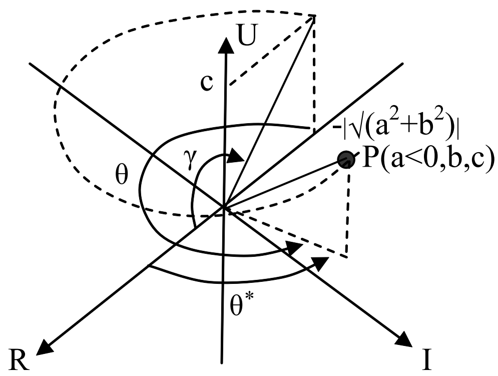

Figure 18: Phases that identify the positions of the half-space $R^{-}IU$ according to the complementary representation

We note that the spatial phase θ is not calculated by the formula:

$$\theta = \arctan\left(\frac{b}{a}\right)$$

because it would correspond to the value θ^*.

For positions P(a,b,c) of the plane RI not belonging to the lines R and I the phases of the complementary representation will be those shown in Figure 19.

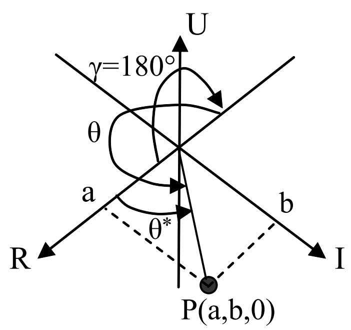

Figure 19: Phases that identify the positions of the plane RI according to the complementary representation

The phases shown in the figure can be determined using the formulas:

$$\gamma = 180^\circ$$
$$\theta = \arctan\left(\frac{-b}{-a}\right)$$

We note that the spatial phase θ is not calculated by the formula:

$$\theta = \arctan\left(\frac{b}{a}\right)$$

because it would correspond to the value θ^*.

For positions P(a,b,c) of the half-plane $R^{+}U$ not belonging to the lines R and U the phases of the complementary representation will be those shown in Figure 20.

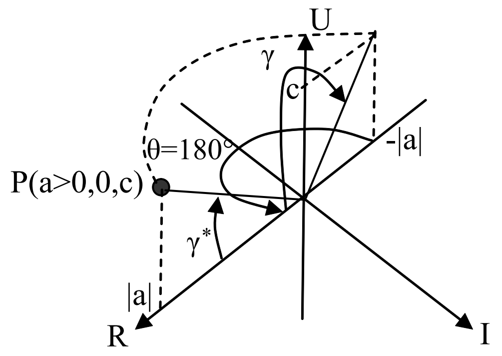

Figure 20: Phases that identify the positions of the half-plane $R^{+}U$ according to the complementary representation

The phases shown in the figure can be determined using the formulas:

$$\gamma = \arctan\left(\frac{c}{-|a|}\right)$$
$$\theta = 180^\circ$$

We note that the plane phase γ is not calculated by the formula:

$$\gamma = \arctan\left(\frac{c}{|a|}\right)$$

because it would correspond to the value γ^{*}.

For positions P(a,b,c) of the half-plane $R^{-}U$ not belonging to the lines R and U the phases of the complementary representation will be those shown in Figure 21.

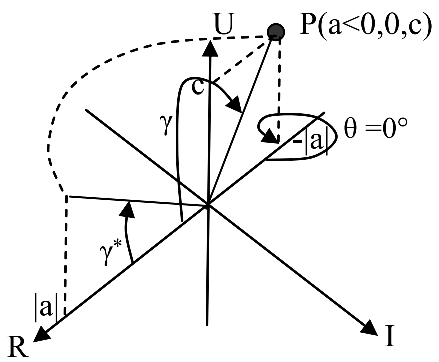

Figure 21: Phases that identify the positions of the half-plane $R^{-}U$ according to the complementary representation

The phases shown in the figure can be determined using the formulas:

$$\gamma = \arctan\left(\frac{c}{-|a|}\right)$$
$$\theta = 0°$$

We note that the plane phase γ is not calculated by the formula:

$$\gamma = \arctan\left(\frac{c}{|a|}\right)$$

because it would correspond to the value γ^*.

For positions P(a,b,c) of the half-plane I^+U not belonging to the lines I and U the phases of the complementary representation will be those shown in Figure 22.

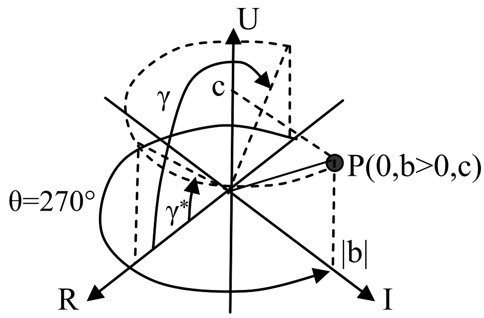

Figure 22: Phases that identify the positions of the half-plane I^+U according to the complementary representation

The phases shown in the figure can be determined using the formulas:

$$\gamma = \arctan\left(\frac{c}{-|b|}\right)$$
$$\theta = 270°$$

We note that the plane phase γ is not calculated by the formula:

$$\gamma = \arctan\left(\frac{c}{|b|}\right)$$

because it would correspond to the value γ^*.

For positions P(a,b,c) of the half-plane I^-U not belonging to the lines I and U the phases of the complementary representation will be those shown in Figure 23 on the facing page.

The phases shown in the figure can be determined using the formulas:

$$\gamma = \arctan\left(\frac{c}{-|b|}\right)$$
$$\theta = 90°$$

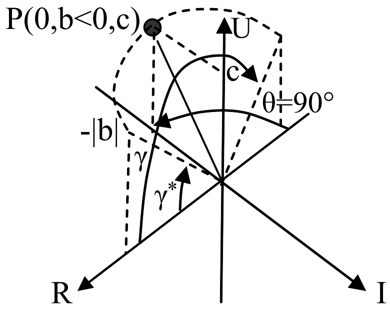

Figure 23: Phases that identify the positions of the half-plane $I^{-}U$ according to the complementary representation

We note that the plane phase γ is not calculated by the formula:

$$\gamma = \arctan\left(\frac{c}{|b|}\right)$$

because it would correspond to the value γ^{*}.

For positions P(a,b,c) of the half-line R^{+} the phases of the complementary representation will be those shown in Figure 24.

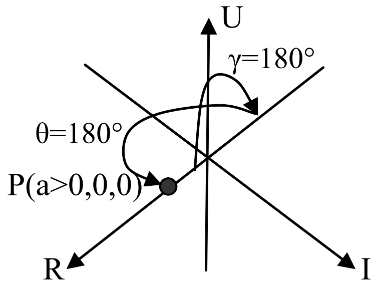

Figure 24: Phases that identify the positions of the half-line R^{+} according to the complementary representation

The phases shown in the figure can be determined using the formulas:

$$\gamma = 180^{\circ}$$
$$\theta = 180^{\circ}$$

For positions P(a,b,c) of the half-line R^{-} the phases of the complementary representation will be those shown in Figure 25 on the following page.

The phases shown in the figure can be determined using the formulas:

$$\gamma = 180^{\circ}$$
$$\theta = 0^{\circ}$$

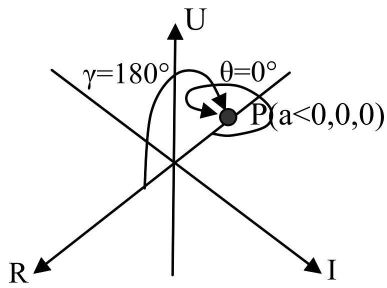

Figure 25: Phases that identify the positions of the half-line R^- according to the complementary representation

For positions P(a,b,c) of the half-line I^+ the phases of the complementary representation will be those shown in Figure 26.

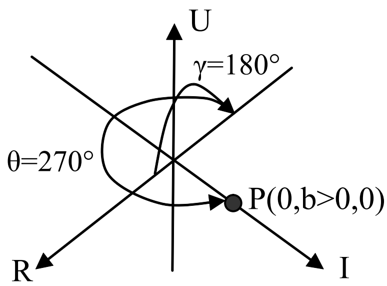

Figure 26: Phases that identify the positions of the half-line I^+ according to the complementary representation

The phases shown in the figure can be determined using the formulas:

$$\gamma = 180°$$
$$\theta = 270°$$

For positions P(a,b,c) of the half-line I^- the phases of the complementary representation will be those shown in Figure 27.

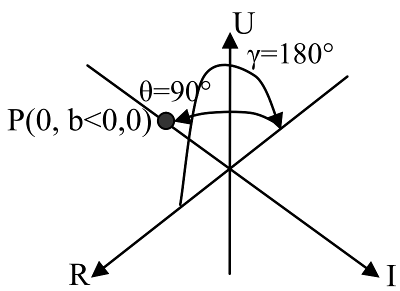

Figure 27: Phases that identify the positions of the half-line I^- according to the complementary representation

The phases shown in the figure can be determined using the formulas:

$$\gamma = 180°$$
$$\theta = 90°$$

Theorem 2.15. *The complementary representation of a complete number of coordinates (a,b,c) not lying on the line U requires to give to the algebraic root* $\sqrt{a^2+b^2}$ *the following negative solution:*

$$\sqrt{a^2+b^2} = -\left|\sqrt{a^2+b^2}\,\right|$$

Proof. In the case of the complementary representations previously examined (that cover every region of the space RIU with the exception of the line U) the phase γ assumes the values provided by the formula:

$$\gamma = \arctan\left(\frac{c}{\sqrt{a^2+b^2}}\right)$$

when we give to the algebraic root $\sqrt{a^2+b^2}$ its negative solutions. And this immediately proves the thesis. ∎

Theorem 2.16. *Each position of the line U corresponds to a complete number for each value assigned to the spatial phase θ.*

Proof. By assigning at the expression of the complete numbers the values $\gamma = \pm 90°$ that characterize the outgoing numbers of the line U:

$$o(t,\theta,\pm 90°) = t \cdot \{[\cos(\pm 90°) \cdot \cos(\theta)] + i \cdot [\cos(\pm 90°) \cdot \sin(\theta)] + u \cdot [\sin(\pm 90°)]\}$$

we obtain the same result regardless of the value of the spatial phase θ:

$$o(t,\theta,\pm 90°) = t \cdot u \cdot [\sin(\pm 90°)] = \pm t \cdot u$$

proving the thesis. ∎

Definition 2.17. A complete numbers belonging to the line U can be defined in standard representation if provided with spatial phase θ equal to zero.

Definition 2.18. A complete numbers belonging to the line U can be defined in complementary representation if provided with spatial phase θ different from zero.

Since the non zero values of the spatial phase are unlimited, unlimited will also be the complementary representation related to the complete numbers belonging to the line U.

Theorem 2.19. *Complex numbers cannot be expressed in the following way:*

$$o(a,b,c) = a + i \cdot b + u \cdot c$$

namely:

$$o(t,\theta,\gamma) \neq o(a,b,c) = a + i \cdot b + u \cdot c$$

Proof. The proof comes from the absence of bijection between translation and rotation operations of values (t, θ, γ) and the positions (a,b,c) of the space RIU, as confirmed by the existence of the complementary representation (Theorem 2.14). ∎

Since it is impossible to associate the complete numbers to the individual positions of the space, we can always express them in terms of their coordinates (a,b,c), provided that we make explicit the phases involved as well.

In other words we should use the following notation:

$$o(a,b,c)_{(t,\theta,\gamma)} = a_{(t)} + i \cdot b_{(\theta)} + u \cdot c_{(\gamma)}$$

where the values of t, θ, γ, if not yet given, should be reported to those which characterize the standard representation.

However it is even possible to introduce a more concise notation by indicating what representation is associate to the coordinates (a,b,c) or, in the case of the outgoing numbers, the value of the spatial phase θ. In practice for the standard representation we have:

$$o(a,b,c)_{(S)} = (a + i \cdot b + u \cdot c)_{(S)}$$

for the complementary representation:

$$o(a,b,c)_{(C)} = (a + i \cdot b + u \cdot c)_{(C)}$$

and finally for the outgoing numbers:

$$o(a,b,c)_{(\theta)} = u \cdot c_{(\theta)}$$

While any other notation of the following type:

$$o(a,b,c) = a + i \cdot b + u \cdot c$$

that is devoid of sufficient information to trace the values of the phases θ and γ, will be able to represent the positions of the space RIU, but not the complete numbers.

2.2. Addition

Definition 2.20. In the space RIU we can define addition between two positions $o_1(a_1, b_1, c_1)$ and $o_2(a_2, b_2, c_2)$ as the position $o_{1+2}(a_{1+2}, b_{1+2}, c_{1+2})$ represented also with the symbol $o_1(a_1, b_1, c_1) + o_2(a_2, b_2, c_2)$ that satisfies the following condition:

$$o_{1+2}(a_{1+2}, b_{1+2}, c_{1+2}) = o_{1+2}(a_1 + a_2, b_1 + b_2, c_1 + c_2)$$

This condition is equivalent to take the position of the space RIU provided with the following coordinates:

$$\begin{aligned} a_{1+2} &= a_1 + a_2 \\ b_{1+2} &= b_1 + b_2 \\ c_{1+2} &= c_1 + c_2 \end{aligned}$$

We can observe, with regard to this, Figure 28.

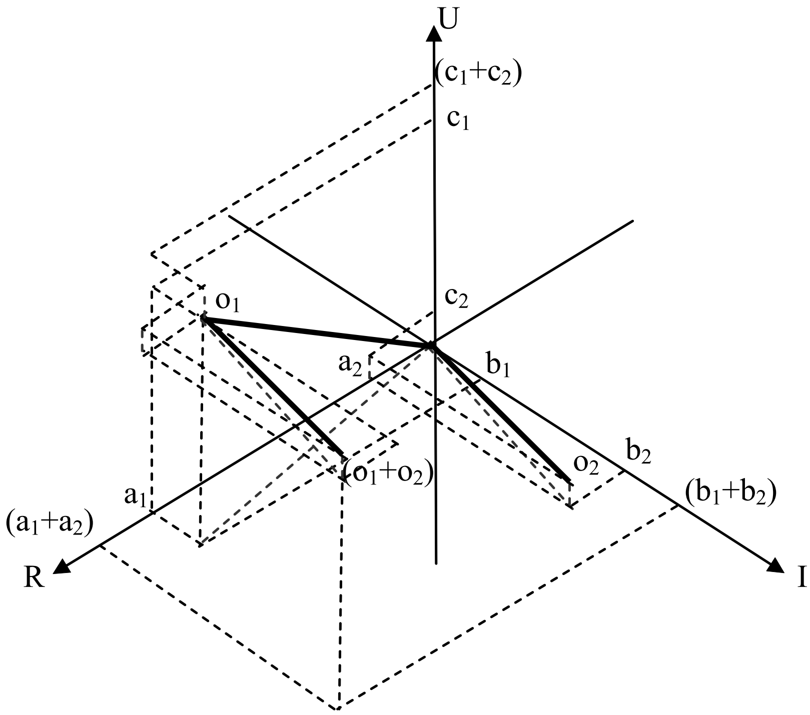

Figure 28: Representation of the addition between two complete numbers

It should be emphasized that the addition is not defined in terms of translations and rotations, and this means that it must be considered an operation that works on the positions and not on the complete numbers. If in one or two dimensions this does not happen is due to the fact that in such contexts there is a bijection between positions and numbers.

Since the addition works on the positions, the notation to use for the various terms involved will be the following:

$$o(a, b, c) = a + i \cdot b + u \cdot c$$

To integrate the operation of addition, working on the positions, with the others, working on the complete numbers, will be enough making reference to the complete number that we can obtain assigning to the sum the phases of the standard representation.

Theorem 2.21. *For the operation of addition is defined neuter the position 0, namely for:*

$$o_2(a_2, b_2, c_2) = 0$$

we have:

$$o_1(a_1, b_1, c_1) + o_2(a_2, b_2, c_2) = o_1(a_1, b_1, c_1)$$

Proof. a_1,b_1,c_1,a_2,b_2,c_2 being real numbers, we can write:

$$\begin{aligned}a_{1+2} &= a_1 + a_2 = a_1 + 0 = a_1\\ b_{1+2} &= b_1 + b_2 = b_1 + 0 = b_1\\ c_{1+2} &= c_1 + a_2 = c_1 + 0 = c_1\end{aligned}$$

proving the thesis. ∎

Theorem 2.22. *For the operation of addition is defined opposite the position symmetric with respect to the origin, namely for:*

$$o_2(a_2, b_2, c_2) = o_2(-a_1, -b_1, -c_1)$$

we have:

$$o_1(a_1, b_1, c_1) + o_2(a_2, b_2, c_2) = 0$$

Proof. a_1,b_1,c_1,a_2,b_2,c_2 being real numbers, we can write:

$$\begin{aligned}a_{1+2} &= a_1 + a_2 = a_1 - a_1 = 0\\ b_{1+2} &= b_1 + b_2 = b_1 - b_1 = 0\\ c_{1+2} &= c_1 + a_2 = c_1 - c_1 = 0\end{aligned}$$

proving the thesis. ∎

Theorem 2.23. *For the operation of addition is valid the commutative property, namely:*

$$o_1(a_1, b_1, c_1) + o_2(a_2, b_2, c_2) = o_2(a_2, b_2, c_2) + o_1(a_1, b_1, c_1)$$

Proof. a_1,b_1,c_1,a_2,b_2,c_2 being real numbers, we can write:

$$\begin{aligned}a_{1+2} &= a_1 + a_2\\ b_{1+2} &= b_1 + b_2\\ c_{1+2} &= c_1 + c_2\end{aligned}$$

$$\begin{aligned}a_{2+1} &= a_2 + a_1 = a_1 + a_2\\ b_{2+1} &= b_2 + b_1 = b_1 + b_2\\ c_{2+1} &= c_2 + c_1 = c_1 + c_2\end{aligned}$$

proving the thesis. ∎

Theorem 2.24. *For the operation of addition are valid the associative and dissociative properties, namely for:*

$$o_2(a_2, b_2, c_2) = o_3(a_3, b_3, c_3) + o_4(a_4, b_4, c_4)$$

we have:

$$\begin{aligned}o_1(a_1, b_1, c_1) + o_2(a_2, b_2, c_2) &= [o_1(a_1, b_1, c_1) + o_3(a_3, b_3, c_3)] + o_4(a_4, b_4, c_4)\\ [o_1(a_1, b_1, c_1) + o_3(a_3, b_3, c_3)] + o_4(a_4, b_4, c_4) &= o_1(a_1, b_1, c_1) + o_2(a_2, b_2, c_2)\end{aligned}$$

Proof. a_1,b_1,c_1,a_2,b_2,c_2,a_3,b_3,c_3,a_4,b_4,c_4 being real numbers, we can write:

$$a_{1+2} = a_1 + a_2 = a_1 + (a_3 + a_4) = (a_1 + a_3) + a_4 = a_{(1+3)+4}$$
$$b_{1+2} = b_1 + b_2 = b_1 + (b_3 + b_4) = (b_1 + b_3) + b_4 = b_{(1+3)+4}$$
$$c_{1+2} = c_1 + c_2 = c_1 + (c_3 + c_4) = (c_1 + c_3) + c_4 = c_{(1+3)+4}$$

$$a_{(1+3)+4} = (a_1 + a_3) + a_4 = a_1 + (a_3 + a_4) = a_1 + a_2 = a_{1+2}$$
$$b_{(1+3)+4} = (b_1 + b_3) + b_4 = b_1 + (b_3 + b_4) = b_1 + b_2 = b_{1+2}$$
$$c_{(1+3)+4} = (c_1 + c_3) + c_4 = c_1 + (c_3 + c_4) = c_1 + a_2 = c_{1+2}$$

proving the thesis. ∎

2.3. Subtraction

Definition 2.25. In the space RIU we can define subtraction between two positions $o_1(a_1, b_1, c_1)$ and $o_2(a_2, b_2, c_2)$ as the position $o_{1-2}(a_{1-2}, b_{1-2}, c_{1-2})$ represented also with the symbol $o_1(a_1, b_1, c_1) - o_2(a_2, b_2, c_2)$ that satisfies the following condition:

$$o_{1-2}(a_{1-2}, b_{1-2}, c_{1-2}) + o_2(a_2, b_2, c_2) = o_1(a_1, b_1, c_1)$$

This condition defines the subtraction as the inverse operation of addition, and it is equivalent to require:

$$a_{1-2} = a_1 - a_2$$
$$b_{1-2} = b_1 - b_2$$
$$c_{1-2} = c_1 - c_2$$

It should be emphasized that the subtraction is not defined in terms of translations and rotations, and this means that it must be considered an operation that works on the positions and not on the complete numbers. If in one or two dimensions this does not happen is due to the fact that in such contexts there is a bijection between positions and numbers.

Since the subtraction works on the positions, the notation to use for the various terms involved will be the following:

$$o(a, b, c) = a + i \cdot b + u \cdot c$$

To integrate the operation of subtraction, working on the positions, with the others, working on the complete numbers, will be enough making reference to the complete number that we can obtain assigning to the difference the phases of the standard representation.

Theorem 2.26. *For the operation of subtraction is defined neuter the position 0, namely for:*

$$o_2(a_2, b_2, c_2) = 0$$

we have:

$$o_1(a_1, b_1, c_1) - o_2(a_2, b_2, c_2) = o_1(a_1, b_1, c_1)$$

Proof. a_1,b_1,c_1,a_2,b_2,c_2 being real numbers, we can write:

$$\begin{aligned}
a_{1-2} &= a_1 - a_2 = a_1 - 0 = a_1 \\
b_{1-2} &= b_1 - b_2 = b_1 - 0 = b_1 \\
c_{1-2} &= c_1 - a_2 = c_1 - 0 = c_1
\end{aligned}$$

proving the thesis. ∎

Theorem 2.27. *For the operation of subtraction is defined identical, the same position with respect to the origin, namely for:*

$$o_2(a_2, b_2, c_2) = o_2(a_1, b_1, c_1)$$

we have:

$$o_1(a_1, b_1, c_1) - o_2(a_2, b_2, c_2) = 0$$

Proof. a_1,b_1,c_1,a_2,b_2,c_2 being real numbers, we can write:

$$\begin{aligned}
a_{1-2} &= a_1 - a_2 = a_1 - a_1 = 0 \\
b_{1-2} &= b_1 - b_2 = b_1 - b_1 = 0 \\
c_{1-2} &= c_1 - a_2 = c_1 - c_1 = 0
\end{aligned}$$

proving the thesis. ∎

Theorem 2.28. *For the operation of subtraction is valid the invariantive property, namely:*

$$\begin{aligned}
o_1(a_1, b_1, c_1) - o_2(a_2, b_2, c_2) =&[o_1(a_1, b_1, c_1) + o_3(a_3, b_3, c_3)]+ \\
&- [o_2(a_2, b_2, c_2) + o_3(a_3, b_3, c_3)] \\
o_1(a_1, b_1, c_1) - o_2(a_2, b_2, c_2) =&[o_1(a_1, b_1, c_1) - o_3(a_3, b_3, c_3)]+ \\
&- [o_2(a_2, b_2, c_2) - o_3(a_3, b_3, c_3)]
\end{aligned}$$

Proof. a_1,b_1,c_1,a_2,b_2,c_2,a_3,b_3,c_3 being real numbers, we can write:

$$\begin{aligned}
a_{1-2} &= a_1 - a_2 \\
b_{1-2} &= b_1 - b_2 \\
c_{1-2} &= c_1 - c_2
\end{aligned}$$

$$\begin{aligned}
a_{(1+3)-(2+3)} &= (a_1 + a_3) - (a_2 + a_3) = a_1 + a_3 - a_2 - a_3 = a_1 - a_2 \\
b_{(1+3)-(2+3)} &= (b_1 + b_3) - (b_2 + b_3) = b_1 + b_3 - b_2 - b_3 = b_1 - b_2 \\
c_{(1+3)-(2+3)} &= (c_1 + c_3) - (c_2 + c_3) = c_1 + c_3 - c_2 - c_3 = c_1 - c_2
\end{aligned}$$

$$\begin{aligned}
a_{(1-3)-(2-3)} &= (a_1 - a_3) - (a_2 - a_3) = a_1 - a_3 - a_2 + a_3 = a_1 - a_2 \\
b_{(1-3)-(2-3)} &= (b_1 - b_3) - (b_2 - b_3) = b_1 - b_3 - b_2 + b_3 = b_1 - b_2 \\
c_{(1-3)-(2-3)} &= (c_1 - c_3) - (c_2 - c_3) = c_1 - c_3 - c_2 + c_3 = c_1 - c_2
\end{aligned}$$

proving the thesis. ∎

Theorem 2.29. *It is valid the equivalence between addition and subtraction, namely:*

$$o_1(a_1,b_1,c_1)+o_2(a_2,b_2,c_2)=o_1(a_1,b_1,c_1)-[-o_2(a_2,b_2,c_2)]$$
$$o_1(a_1,b_1,c_1)-o_2(a_2,b_2,c_2)=o_1(a_1,b_1,c_1)+[-o_2(a_2,b_2,c_2)]$$

Proof. a_1,b_1,c_1,a_2,b_2,c_2 being real numbers, we can write:

$$\begin{aligned} a_{1+2} &= a_1+a_2 \\ b_{1+2} &= b_1+b_2 \\ c_{1+2} &= c_1+c_2 \end{aligned}$$

$$\begin{aligned} a_{1-(-2)} &= a_1-(-a_2)=a_1+a_2 \\ b_{1-(-2)} &= b_1-(-b_2)=b_1+b_2 \\ c_{1-(-2)} &= c_1-(-c_2)=c_1+c_2 \end{aligned}$$

$$\begin{aligned} a_{1-2} &= a_1-a_2 \\ b_{1-2} &= b_1-b_2 \\ c_{1-2} &= c_1-c_2 \end{aligned}$$

$$\begin{aligned} a_{1+(-2)} &= a_1+(-a_2)=a_1-a_2 \\ b_{1+(-2)} &= b_1+(-b_2)=b_1-b_2 \\ c_{1+(-2)} &= c_1+(-c_2)=c_1-c_2 \end{aligned}$$

proving the thesis. ∎

2.4. Multiplication

Definition 2.30. In the space RIU we can define multiplication between two complete numbers $o_1(t_1,\theta_1,\gamma_1)$ and $o_2(t_2,\theta_2,\gamma_2)$ as the number $o_{1\cdot2}(t_{1\cdot2},\theta_{1\cdot2},\gamma_{1\cdot2})$ represented also with the symbol $o_1(t_1,\theta_1,\gamma_1)\cdot o_2(t_2,\theta_2,\gamma_2)$ that satisfies the following condition:

$$o_{1\cdot2}(t_{1\cdot2},\theta_{1\cdot2},\gamma_{1\cdot2})=o_{1\cdot2}(t_1\cdot t_2,\theta_1+\theta_2,\gamma_1+\gamma_2)$$

This condition defines the multiplication and it is equivalent to require:

$$\begin{aligned} t_{1\cdot2} &= t_1\cdot t_2 \\ \theta_{1\cdot2} &= \theta_1+\theta_2 \\ \gamma_{1\cdot2} &= \gamma_1+\gamma_2 \end{aligned}$$

We can observe, with regard to this, Figure 29 on the following page.

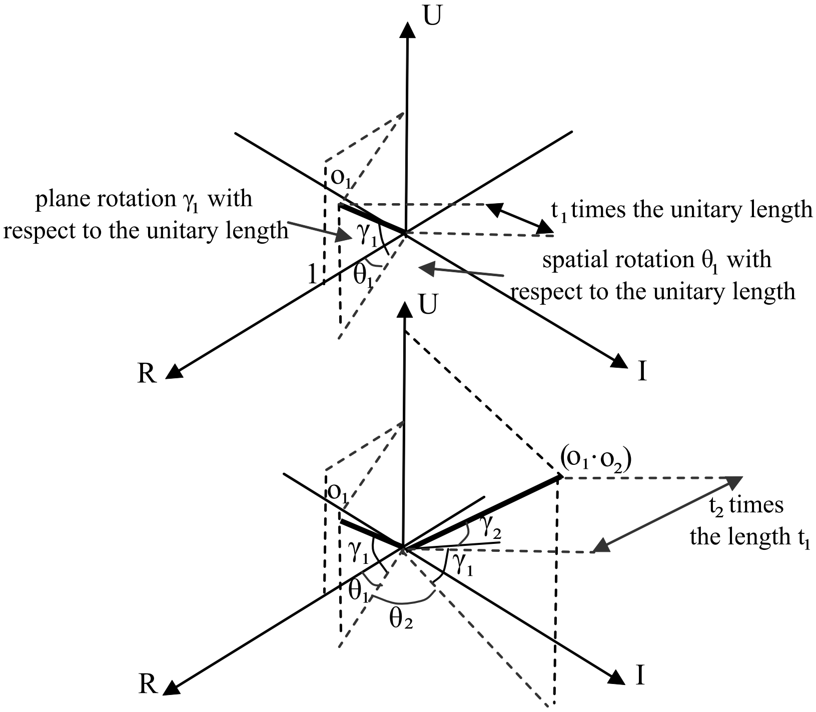

Figure 29: Representation of the multiplication between two complete numbers

Theorem 2.31. *With $o_1(t_1, \theta_1, \gamma_1)$ and $o_2(t_2, \theta_2, \gamma_2)$ in standard representation, and both not belonging to the line U, their multiplication may be expressed in the following way:*

$$o_{1\cdot 2}(a_{1\cdot 2}, b_{1\cdot 2}, c_{1\cdot 2})_{(t_{1\cdot 2}, \theta_{1\cdot 2}, \gamma_{1\cdot 2})} = a_{1\cdot 2(t_1 \cdot t_2)} + i \cdot b_{1\cdot 2(\theta_1+\theta_2)} + u \cdot c_{1\cdot 2(\gamma_1+\gamma_2)}$$

where:

$$a_{1\cdot 2} = (a_1 \cdot a_2 - b_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2+b_1^2}\right| \cdot \left|\sqrt{a_2^2+b_2^2}\right|}\right)$$
$$b_{1\cdot 2} = (b_1 \cdot a_2 + a_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2+b_1^2}\right| \cdot \left|\sqrt{a_2^2+b_2^2}\right|}\right)$$
$$c_{1\cdot 2} = c_1 \cdot \left|\sqrt{a_2^2+b_2^2}\right| + c_2 \cdot \left|\sqrt{a_1^2+b_1^2}\right|$$

Proof. The multiplication between two complete numbers, as we know, satisfies the following formula:

$$o_{1\cdot 2}(t_{1\cdot 2}, \theta_{1\cdot 2}, \gamma_{1\cdot 2}) = t_1 \cdot t_2 \cdot \{[\cos(\gamma_1+\gamma_2) \cdot \cos(\theta_1+\theta_2)] +$$
$$+ i \cdot [\cos(\gamma_1+\gamma_2) \cdot \sin(\theta_1+\theta_2)] + u \cdot [\sin(\gamma_1+\gamma_2)]\}$$

For the moduli and the phases involved will be valid the following relation as well:

$$t = \sqrt{a^2+b^2+c^2}$$
$$\gamma = \arctan\left(\frac{c}{\sqrt{a^2+b^2}}\right)$$
$$\theta = \arctan\left(\frac{b}{a}\right)$$

This means that we can write the coordinates sought in the following way:

$$a_{1\cdot 2} = \sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}\cdot\cos\left[\arctan\left(\frac{c_1}{\sqrt{a_1^2+b_1^2}}\right)+ + \arctan\left(\frac{c_2}{\sqrt{a_2^2+b_2^2}}\right)\right]\cdot\cos\left[\arctan\left(\frac{b_1}{a_1}\right)+\arctan\left(\frac{b_2}{a_2}\right)\right]$$

$$b_{1\cdot 2} = \sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}\cdot\cos\left[\arctan\left(\frac{c_1}{\sqrt{a_1^2+b_1^2}}\right)+ + \arctan\left(\frac{c_2}{\sqrt{a_2^2+b_2^2}}\right)\right]\cdot\sin\left[\arctan\left(\frac{b_1}{a_1}\right)+\arctan\left(\frac{b_2}{a_2}\right)\right]$$

$$c_{1\cdot 2} = \sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}\cdot\sin\left[\arctan\left(\frac{c_1}{\sqrt{a_1^2+b_1^2}}\right)+ + \arctan\left(\frac{c_2}{\sqrt{a_2^2+b_2^2}}\right)\right]$$

To continue with the proof, we have to use the following trigonometric relations:

$$\cos(x+y) = \cos(x)\cdot\cos(y) - \sin(x)\cdot\sin(y)$$
$$\sin(x+y) = \sin(x)\cdot\cos(y) + \cos(x)\cdot\sin(y)$$
$$\cos\left[\arctan\left(\frac{c}{\sqrt{a^2+b^2}}\right)\right] = \sqrt{\frac{a^2+b^2}{a^2+b^2+c^2}}$$
$$\sin\left[\arctan\left(\frac{c}{\sqrt{a^2+b^2}}\right)\right] = \sqrt{\frac{c^2}{a^2+b^2+c^2}}$$
$$\cos\left[\arctan\left(\frac{b}{a}\right)\right] = \sqrt{\frac{a^2}{a^2+b^2}}$$
$$\sin\left[\arctan\left(\frac{b}{a}\right)\right] = \sqrt{\frac{b^2}{a^2+b^2}}$$

To determine the value of the coordinate $a_{1\cdot 2}$ the steps to perform will be the following:

$$
\begin{aligned}
a_{1\cdot 2} =&\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}\cdot\left(\sqrt{\frac{a_1^2+b_1^2}{a_1^2+b_1^2+c_1^2}}\cdot\sqrt{\frac{a_2^2+b_2^2}{a_2^2+b_2^2+c_2^2}}+\right.\\
&\left.-\sqrt{\frac{c_1^2}{a_1^2+b_1^2+c_1^2}}\cdot\sqrt{\frac{c_2^2}{a_2^2+b_2^2+c_2^2}}\right)\cdot\left(\sqrt{\frac{a_1^2}{a_1^2+b_1^2}}\cdot\sqrt{\frac{a_2^2}{a_2^2+b_2^2}}+\right.\\
&\left.-\sqrt{\frac{b_1^2}{a_1^2+b_1^2}}\cdot\sqrt{\frac{b_2^2}{a_2^2+b_2^2}}\right)=\\
=&\left(\sqrt{a_1^2+b_1^2}\cdot\sqrt{a_2^2+b_2^2}-\sqrt{c_1^2}\cdot\sqrt{c_2^2}\right)\cdot\left(\frac{\sqrt{a_1^2}\cdot\sqrt{a_2^2}-\sqrt{b_1^2}\cdot\sqrt{b_2^2}}{\sqrt{a_1^2+b_1^2}\cdot\sqrt{a_2^2+b_2^2}}\right)=\\
=&\left(\sqrt{a_1^2}\cdot\sqrt{a_2^2}-\sqrt{b_1^2}\cdot\sqrt{b_2^2}\right)-\sqrt{c_1^2}\cdot\sqrt{c_2^2}\cdot\left(\frac{\sqrt{a_1^2}\cdot\sqrt{a_2^2}-\sqrt{b_1^2}\cdot\sqrt{b_2^2}}{\sqrt{a_1^2+b_1^2}\cdot\sqrt{a_2^2+b_2^2}}\right)=\\
=&\left(\sqrt{a_1^2}\cdot\sqrt{a_2^2}-\sqrt{b_1^2}\cdot\sqrt{b_2^2}\right)\cdot\left(1-\frac{\sqrt{c_1^2}\cdot\sqrt{c_2^2}}{\sqrt{a_1^2+b_1^2}\cdot\sqrt{a_2^2+b_2^2}}\right)
\end{aligned}
$$

To determine the value of the coordinate $b_{1\cdot 2}$ the steps to perform will be the following:

$$
\begin{aligned}
b_{1\cdot 2} =&\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}\cdot\left(\sqrt{\frac{a_1^2+b_1^2}{a_1^2+b_1^2+c_1^2}}\cdot\sqrt{\frac{a_2^2+b_2^2}{a_2^2+b_2^2+c_2^2}}+\right.\\
&\left.-\sqrt{\frac{c_1^2}{a_1^2+b_1^2+c_1^2}}\cdot\sqrt{\frac{c_2^2}{a_2^2+b_2^2+c_2^2}}\right)\cdot\left(\sqrt{\frac{b_1^2}{a_1^2+b_1^2}}\cdot\sqrt{\frac{a_2^2}{a_2^2+b_2^2}}+\right.\\
&\left.+\sqrt{\frac{a_1^2}{a_1^2+b_1^2}}\cdot\sqrt{\frac{b_2^2}{a_2^2+b_2^2}}\right)=\\
=&\left(\sqrt{a_1^2+b_1^2}\cdot\sqrt{a_2^2+b_2^2}-\sqrt{c_1^2}\cdot\sqrt{c_2^2}\right)\cdot\left(\frac{\sqrt{b_1^2}\cdot\sqrt{a_2^2}+\sqrt{a_1^2}\cdot\sqrt{b_2^2}}{\sqrt{a_1^2+b_1^2}\cdot\sqrt{a_2^2+b_2^2}}\right)=\\
=&\left(\sqrt{b_1^2}\cdot\sqrt{a_2^2}+\sqrt{a_1^2}\cdot\sqrt{b_2^2}\right)-\sqrt{c_1^2}\cdot\sqrt{c_2^2}\cdot\left(\frac{\sqrt{b_1^2}\cdot\sqrt{a_2^2}+\sqrt{a_1^2}\cdot\sqrt{b_2^2}}{\sqrt{a_1^2+b_1^2}\cdot\sqrt{a_2^2+b_2^2}}\right)=\\
=&\left(\sqrt{b_1^2}\cdot\sqrt{a_2^2}+\sqrt{a_1^2}\cdot\sqrt{b_2^2}\right)\cdot\left(1-\frac{\sqrt{c_1^2}\cdot\sqrt{c_2^2}}{\sqrt{a_1^2+b_1^2}\cdot\sqrt{a_2^2+b_2^2}}\right)
\end{aligned}
$$

To determine the value of the coordinate $c_{1\cdot 2}$ the steps to perform will be the following:

$$
\begin{aligned}
c_{1\cdot 2} =&\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}\cdot\left(\sqrt{\frac{c_1^2}{a_1^2+b_1^2+c_1^2}}\cdot\sqrt{\frac{a_2^2+b_2^2}{a_2^2+b_2^2+c_2^2}}+\right.\\
&\left.+\sqrt{\frac{a_1^2+b_1^2}{a_1^2+b_1^2+c_1^2}}\cdot\sqrt{\frac{c_2^2}{a_2^2+b_2^2+c_2^2}}\right)=\sqrt{c_1^2}\cdot\sqrt{a_2^2+b_2^2}+\sqrt{c_2^2}\cdot\sqrt{a_1^2+b_1^2}
\end{aligned}
$$

These relations are valid in general, in the precise sense that they are also able to include cases where the coefficients a,b,c are zero (provided that we work with complete numbers not belonging in the line U). But their main peculiarity is that to contain many roots of the form $\sqrt{x^2}$.

Since the radicand x^2 is always positive we know that the operation of algebraic root considered here is permitted, and therefore it will be able to take as result two opposite values: one positive and one negative. This means that from mathematical point of view we obtain a relation able to satisfies the multiplication rule for each possible combination of signs attributable to the roots involved.

For example if we adopt the convention of attributing to the roots always the positive value, we obtain the following result:

$$\begin{aligned}\sqrt{a^2} &= |a| \\ \sqrt{b^2} &= |b| \\ \sqrt{c^2} &= |c|\end{aligned}$$

to which correspond relations able to satisfy the multiplication role as a function of the modulus of the coordinates involved. This means that distinct complete numbers will be able to give the same result of the multiplication if their coordinates will have the same modulus.

Wanting to find relations that satisfy the multiplication rule as a function of the effective coordinates of the complete numbers involved, we must assign to the roots the same sign of the coefficient located within them:

$$\begin{aligned}\sqrt{a^2} &= a \\ \sqrt{b^2} &= b \\ \sqrt{c^2} &= c\end{aligned}$$

The relations obtained will be the following:

$$\begin{aligned}a_{1\cdot 2} &= (a_1 \cdot a_2 - b_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{\sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2}}\right) \\ b_{1\cdot 2} &= (b_1 \cdot a_2 + a_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{\sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2}}\right) \\ c_{1\cdot 2} &= c_1 \cdot \sqrt{a_2^2 + b_2^2} + c_2 \cdot \sqrt{a_1^2 + b_1^2}\end{aligned} \tag{2.1}$$

Since the complete numbers involved are in standard representation, as determined by the theorem 2.11 we must consider the following relations:

$$\begin{aligned}\sqrt{a_1^2 + b_1^2} &= \left|\sqrt{a_1^2 + b_1^2}\,\right| \\ \sqrt{a_2^2 + b_2^2} &= \left|\sqrt{a_2^2 + b_2^2}\,\right|\end{aligned}$$

that combined with those indicated by the formulas (2.1), proving the thesis. ∎

As an example of the theorem just proved, suppose you have to multiply the complete numbers in standard representation provided with coordinates:

$$a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 1.$$

Their modulus may be calculated in the following way:

$$t_1 = t_2 = \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

For their phases we should refer to the formulas related to the standard representation:

$$\begin{aligned}\gamma_1 = \gamma_2 &= \arctan\left(\frac{c_1}{\left|\sqrt{a_1^2 + b_1^2}\right|}\right) = \arctan\left(\frac{c_2}{\left|\sqrt{a_2^2 + b_2^2}\right|}\right) = \\ &= \arctan\left(\frac{1}{|\sqrt{2}|}\right) \simeq 35.26^\circ \\ \theta_1 = \theta_2 &= \arctan\left(\frac{b_1}{a_1}\right) = \arctan\left(\frac{b_2}{a_2}\right) = \arctan\left(\frac{1}{1}\right) = 45^\circ\end{aligned}$$

By applying the multiplication rule we obtain as result the complete number provided with the following values of modulus and phases:

$$\begin{aligned}t_{1\cdot 2} &= t_1 \cdot t_2 = 3 \\ \gamma_{1\cdot 2} &= \gamma_1 + \gamma_2 \simeq 70.52^\circ \\ \theta_{1\cdot 2} &= \theta_1 + \theta_2 = 90^\circ\end{aligned}$$

and the following coordinates:

$$\begin{aligned}a_{1\cdot 2} &= t_{1\cdot 2} \cdot \cos(\gamma_{1\cdot 2}) \cdot \cos(\theta_{1\cdot 2}) = 3 \cdot \cos(\simeq 70.52^\circ) \cdot \cos(90^\circ) = 0 \\ b_{1\cdot 2} &= t_{1\cdot 2} \cdot \cos(\gamma_{1\cdot 2}) \cdot \sin(\theta_{1\cdot 2}) = 3 \cdot \cos(\simeq 70.52^\circ) \cdot \sin(90^\circ) = 1 \\ c_{1\cdot 2} &= t_{1\cdot 2} \cdot \sin(\gamma_{1\cdot 2}) = 3 \cdot \sin(\simeq 70.52^\circ) = 2 \cdot \sqrt{2}\end{aligned}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$\begin{aligned}a_{1\cdot 2} =& (a_1 \cdot a_2 - b_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2 + b_1^2}\right| \cdot \left|\sqrt{a_2^2 + b_2^2}\right|}\right) = \\ =& (1 - 1) \cdot \left(1 - \frac{1}{2}\right) = 0 \\ b_{1\cdot 2} =& (b_1 \cdot a_2 + a_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2 + b_1^2}\right| \cdot \left|\sqrt{a_2^2 + b_2^2}\right|}\right) = \\ =& (1 + 1) \cdot \left(1 - \frac{1}{2}\right) = 1 \\ c_{1\cdot 2} =& c_1 \cdot \left|\sqrt{a_2^2 + b_2^2}\right| + c_2 \cdot \left|\sqrt{a_1^2 + b_1^2}\right| = 1 \cdot \sqrt{2} + 1 \cdot \sqrt{2} = 2 \cdot \sqrt{2}\end{aligned}$$

Theorem 2.32. *With $o_1(t_1, \theta_1, \gamma_1)$ and $o_2(t_2, \theta_2, \gamma_2)$ in complementary representation, and both not belonging to the line U, their multiplication may be expressed in the following way:*

$$o_{1\cdot 2}(a_{1\cdot 2}, b_{1\cdot 2}, c_{1\cdot 2})_{(t_{1\cdot 2}, \theta_{1\cdot 2}, \gamma_{1\cdot 2})} = a_{1\cdot 2(t_1\cdot t_2)} + i \cdot b_{1\cdot 2(\theta_1+\theta_2)} + u \cdot c_{1\cdot 2(\gamma_1+\gamma_2)}$$

where:

$$a_{1\cdot 2} = (a_1 \cdot a_2 - b_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2 + b_1^2}\,\right| \cdot \left|\sqrt{a_2^2 + b_2^2}\,\right|}\right)$$

$$b_{1\cdot 2} = (b_1 \cdot a_2 + a_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2 + b_1^2}\,\right| \cdot \left|\sqrt{a_2^2 + b_2^2}\,\right|}\right)$$

$$c_{1\cdot 2} = -c_1 \cdot \left|\sqrt{a_2^2 + b_2^2}\,\right| - c_2 \cdot \left|\sqrt{a_1^2 + b_1^2}\,\right|$$

Proof. Since the complete numbers involved are in complementary representation, as determined by the theorem 2.15 we must consider the following relations:

$$\sqrt{a_1^2 + b_1^2} = -\left|\sqrt{a_1^2 + b_1^2}\,\right|$$
$$\sqrt{a_2^2 + b_2^2} = -\left|\sqrt{a_2^2 + b_2^2}\,\right|$$

that combined with those indicated by the formulas (2.1), proving the thesis. ∎

As an example of the theorem just proved, suppose you have to multiply the complete numbers in complementary representation provided with coordinates: $a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 1$.

Their modulus may be calculated in the following way:

$$t_1 = t_2 = \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

For their phases we should refer to the formulas related to the complementary representation:

$$\gamma_1 = \gamma_2 = \arctan\left(\frac{c_1}{-\left|\sqrt{a_1^2 + b_1^2}\,\right|}\right) = \arctan\left(\frac{c_2}{-\left|\sqrt{a_2^2 + b_2^2}\,\right|}\right) =$$
$$= \arctan\left(\frac{1}{-|\sqrt{2}|}\right) \simeq 144.73^\circ$$
$$\theta_1 = \theta_2 = \arctan\left(\frac{-b_1}{-a_1}\right) = \arctan\left(\frac{-b_2}{-a_2}\right) = \arctan\left(\frac{-1}{-1}\right) = 225^\circ$$

By applying the multiplication rule we obtain as result the complete number provided with the following values of modulus and phases:

$$t_{1\cdot 2} = t_1 \cdot t_2 = 3$$
$$\gamma_{1\cdot 2} = \gamma_1 + \gamma_2 \simeq 289.46^\circ$$
$$\theta_{1\cdot 2} = \theta_1 + \theta_2 = 450^\circ = 90^\circ$$

and the following coordinates:

$$a_{1\cdot 2} = t_{1\cdot 2} \cdot \cos(\gamma_{1\cdot 2}) \cdot \cos(\theta_{1\cdot 2}) = 3 \cdot \cos(\simeq 289.46°) \cdot \cos(90°) = 0$$
$$b_{1\cdot 2} = t_{1\cdot 2} \cdot \cos(\gamma_{1\cdot 2}) \cdot \sin(\theta_{1\cdot 2}) = 3 \cdot \cos(\simeq 289.46°) \cdot \sin(90°) = 1$$
$$c_{1\cdot 2} = t_{1\cdot 2} \cdot \sin(\gamma_{1\cdot 2}) = 3 \cdot \sin(\simeq 289.46°) = -2 \cdot \sqrt{2}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$a_{1\cdot 2} = (a_1 \cdot a_2 - b_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2 + b_1^2}\right| \cdot \left|\sqrt{a_2^2 + b_2^2}\right|}\right) =$$
$$= (1-1) \cdot \left(1 - \frac{1}{2}\right) = 0$$
$$b_{1\cdot 2} = (b_1 \cdot a_2 + a_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2 + b_1^2}\right| \cdot \left|\sqrt{a_2^2 + b_2^2}\right|}\right) =$$
$$= (1+1) \cdot \left(1 - \frac{1}{2}\right) = 1$$
$$c_{1\cdot 2} = -c_1 \cdot \left|\sqrt{a_2^2 + b_2^2}\right| - c_2 \cdot \left|\sqrt{a_1^2 + b_1^2}\right| = -1 \cdot \sqrt{2} - 1 \cdot \sqrt{2} = -2 \cdot \sqrt{2}$$

Theorem 2.33. *With $o_1(t_1, \theta_1, \gamma_1)$ in standard representation and $o_2(t_2, \theta_2, \gamma_2)$ in complementary representation, and both not belonging to the line U, their multiplication may be expressed in the following way:*

$$o_{1\cdot 2}(a_{1\cdot 2}, b_{1\cdot 2}, c_{1\cdot 2})_{(t_{1\cdot 2}, \theta_{1\cdot 2}, \gamma_{1\cdot 2})} = a_{1\cdot 2(t_1 \cdot t_2)} + i \cdot b_{1\cdot 2(\theta_1 + \theta_2)} + u \cdot c_{1\cdot 2(\gamma_1 + \gamma_2)}$$

where:

$$a_{1\cdot 2} = (a_1 \cdot a_2 - b_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2 + b_1^2}\right| \cdot \left|\sqrt{a_2^2 + b_2^2}\right|}\right)$$
$$b_{1\cdot 2} = (b_1 \cdot a_2 + a_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2 + b_1^2}\right| \cdot \left|\sqrt{a_2^2 + b_2^2}\right|}\right)$$
$$c_{1\cdot 2} = c_2 \cdot \left|\sqrt{a_1^2 + b_1^2}\right| - c_1 \cdot \left|\sqrt{a_2^2 + b_2^2}\right|$$

Proof. Since the first factor is in standard representation, as determined by the theorem 2.11 we must consider the following relation:

$$\sqrt{a_1^2 + b_1^2} = \left|\sqrt{a_1^2 + b_1^2}\right|$$

while being the second factor in complementary representation, as determined by the theorem 2.15 we must consider the following relation:

$$\sqrt{a_2^2 + b_2^2} = -\left|\sqrt{a_2^2 + b_2^2}\right|$$

that combined with those indicated by the formulas (2.1), proving the thesis. ∎

As an example of the theorem just proved, suppose you have to multiply the complete number in standard representation provided with coordinates $a_1 = b_1 = c_1 = 1$ by that in complementary representation provided with the same coordinates coordinates: $a_2 = b_2 = c_2 = 1$.

Their modulus may be calculated in the following way:

$$t_1 = t_2 = \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

For their phases we should refer to the formulas related to the standard and complementary representations:

$$\begin{aligned}
\gamma_1 &= \arctan\left(\frac{c_1}{\left|\sqrt{a_1^2 + b_1^2}\,\right|}\right) = \arctan\left(\frac{1}{|\sqrt{2}|}\right) \simeq 35.26^\circ \\
\gamma_2 &= \arctan\left(\frac{c_2}{-\left|\sqrt{a_2^2 + b_2^2}\,\right|}\right) = \arctan\left(\frac{1}{-|\sqrt{2}|}\right) \simeq 144.73^\circ \\
\theta_1 &= \arctan\left(\frac{b_1}{a_1}\right) = \arctan\left(\frac{1}{1}\right) = 45^\circ \\
\theta_2 &= \arctan\left(\frac{-b_2}{-a_2}\right) = \arctan\left(\frac{-1}{-1}\right) = 225^\circ
\end{aligned}$$

By applying the multiplication rule we obtain as result the complete number provided with the following values of modulus and phases:

$$\begin{aligned}
t_{1\cdot 2} &= t_1 \cdot t_2 = 3 \\
\gamma_{1\cdot 2} &= \gamma_1 + \gamma_2 = 180^\circ \\
\theta_{1\cdot 2} &= \theta_1 + \theta_2 = 270^\circ
\end{aligned}$$

and the following coordinates:

$$\begin{aligned}
a_{1\cdot 2} &= t_{1\cdot 2} \cdot \cos(\gamma_{1\cdot 2}) \cdot \cos(\theta_{1\cdot 2}) = 3 \cdot \cos(180^\circ) \cdot \cos(270^\circ) = 0 \\
b_{1\cdot 2} &= t_{1\cdot 2} \cdot \cos(\gamma_{1\cdot 2}) \cdot \sin(\theta_{1\cdot 2}) = 3 \cdot \cos(180^\circ) \cdot \sin(270^\circ) = 3 \\
c_{1\cdot 2} &= t_{1\cdot 2} \cdot \sin(\gamma_{1\cdot 2}) = 3 \cdot \sin(180^\circ) = 0
\end{aligned}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$\begin{aligned}
a_{1\cdot 2} &= (a_1 \cdot a_2 - b_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2 + b_1^2}\,\right| \cdot \left|\sqrt{a_2^2 + b_2^2}\,\right|}\right) \\
&= (1 - 1) \cdot \left(1 + \frac{1}{2}\right) = 0 \\
b_{1\cdot 2} &= (b_1 \cdot a_2 + a_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2 + b_1^2}\,\right| \cdot \left|\sqrt{a_2^2 + b_2^2}\,\right|}\right) \\
&= (1 + 1) \cdot \left(1 + \frac{1}{2}\right) = 3 \\
c_{1\cdot 2} &= c_2 \cdot \left|\sqrt{a_1^2 + b_1^2}\,\right| - c_1 \cdot \left|\sqrt{a_2^2 + b_2^2}\,\right| = 1 \cdot \sqrt{2} - 1 \cdot \sqrt{2} = 0
\end{aligned}$$

Theorem 2.34. *With $o_1(t_1, \theta_1, \gamma_1)$ in complementary representation and $o_2(t_2, \theta_2, \gamma_2)$ in standard representation, and both not belonging to the line U, their multiplication may be expressed in the following way:*

$$o_{1\cdot 2}(a_{1\cdot 2}, b_{1\cdot 2}, c_{1\cdot 2})_{(t_{1\cdot 2}, \theta_{1\cdot 2}, \gamma_{1\cdot 2})} = a_{1\cdot 2(t_1 \cdot t_2)} + i \cdot b_{1\cdot 2(\theta_1+\theta_2)} + u \cdot c_{1\cdot 2(\gamma_1+\gamma_2)}$$

where:

$$\begin{aligned}
a_{1\cdot 2} &= (a_1 \cdot a_2 - b_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2+b_1^2}\,\right| \cdot \left|\sqrt{a_2^2+b_2^2}\,\right|}\right) \\
b_{1\cdot 2} &= (b_1 \cdot a_2 + a_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2+b_1^2}\,\right| \cdot \left|\sqrt{a_2^2+b_2^2}\,\right|}\right) \\
c_{1\cdot 2} &= c_1 \cdot \left|\sqrt{a_2^2+b_2^2}\,\right| - c_2 \cdot \left|\sqrt{a_1^2+b_1^2}\,\right|
\end{aligned}$$

Proof. Since the first factor is in complementary representation, as determined by the theorem 2.15 we must consider the following relation:

$$\sqrt{a_1^2+b_1^2} = -\left|\sqrt{a_1^2+b_1^2}\,\right|$$

while being the second factor in standard representation, as determined by the theorem 2.11 we must consider the following relation:

$$\sqrt{a_2^2+b_2^2} = \left|\sqrt{a_2^2+b_2^2}\,\right|$$

that combined with those indicated by the formulas (2.1), proving the thesis. ∎

As an example of the theorem just proved, suppose you have to multiply the complete number in complementary representation provided with coordinates $a_1 = b_1 = c_1 = 1$ by that in standard representation provided with the same coordinates coordinates: $a_2 = b_2 = c_2 = 1$.

Their modulus may be calculated in the following way:

$$t_1 = t_2 = \sqrt{a_1^2+b_1^2+c_1^2} = \sqrt{a_2^2+b_2^2+c_2^2} = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

For their phases we should refer to the formulas related to the complementary and standard representations:

$$\begin{aligned}
\gamma_1 &= \arctan\left(\frac{c_1}{-\left|\sqrt{a_1^2+b_1^2}\,\right|}\right) = \arctan\left(\frac{1}{-|\sqrt{2}|}\right) \simeq 144.73^\circ \\
\gamma_2 &= \arctan\left(\frac{c_2}{\left|\sqrt{a_2^2+b_2^2}\,\right|}\right) = \arctan\left(\frac{1}{|\sqrt{2}|}\right) \simeq 35.26^\circ \\
\theta_1 &= \arctan\left(\frac{-b_1}{-a_1}\right) = \arctan\left(\frac{-1}{-1}\right) = 225^\circ \\
\theta_2 &= \arctan\left(\frac{b_2}{a_2}\right) = \arctan\left(\frac{1}{1}\right) = 45^\circ
\end{aligned}$$

By applying the multiplication rule we obtain as result the complete number provided with the following values of modulus and phases:

$$\begin{aligned} t_{1\cdot 2} &= t_1 \cdot t_2 = 3 \\ \gamma_{1\cdot 2} &= \gamma_1 + \gamma_2 = 180^\circ \\ \theta_{1\cdot 2} &= \theta_1 + \theta_2 = 270^\circ \end{aligned}$$

and the following coordinates:

$$\begin{aligned} a_{1\cdot 2} &= t_{1\cdot 2} \cdot \cos(\gamma_{1\cdot 2}) \cdot \cos(\theta_{1\cdot 2}) = 3 \cdot \cos(180^\circ) \cdot \cos(270^\circ) = 0 \\ b_{1\cdot 2} &= t_{1\cdot 2} \cdot \cos(\gamma_{1\cdot 2}) \cdot \sin(\theta_{1\cdot 2}) = 3 \cdot \cos(180^\circ) \cdot \sin(270^\circ) = 3 \\ c_{1\cdot 2} &= t_{1\cdot 2} \cdot \sin(\gamma_{1\cdot 2}) = 3 \cdot \sin(180^\circ) = 0 \end{aligned}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$\begin{aligned} a_{1\cdot 2} &= (a_1 \cdot a_2 - b_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2 + b_1^2}\,\right| \cdot \left|\sqrt{a_2^2 + b_2^2}\,\right|}\right) \\ &= (1 - 1) \cdot \left(1 + \frac{1}{2}\right) = 0 \\ b_{1\cdot 2} &= (b_1 \cdot a_2 + a_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2 + b_1^2}\,\right| \cdot \left|\sqrt{a_2^2 + b_2^2}\,\right|}\right) \\ &= (1 + 1) \cdot \left(1 + \frac{1}{2}\right) = 3 \\ c_{1\cdot 2} &= c_1 \cdot \left|\sqrt{a_2^2 + b_2^2}\,\right| - c_2 \cdot \left|\sqrt{a_1^2 + b_1^2}\,\right| = 1 \cdot \sqrt{2} - 1 \cdot \sqrt{2} = 0 \end{aligned}$$

Theorem 2.35. *With only $o_1(t_1, \theta_1, \gamma_1)$ belonging to the line U and $o_2(t_2, \theta_2, \gamma_2)$ in standard representation, their multiplication may be expressed in the following way:*

$$o_{1\cdot 2}(a_{1\cdot 2}, b_{1\cdot 2}, c_{1\cdot 2})_{(t_{1\cdot 2}, \theta_{1\cdot 2}, \gamma_{1\cdot 2})} = a_{1\cdot 2(t_1 \cdot t_2)} + i \cdot b_{1\cdot 2(\theta_1 + \theta_2)} + u \cdot c_{1\cdot 2(\gamma_1 + \gamma_2)}$$

where:

$$\begin{aligned} a_{1\cdot 2} &= -(c_1 \cdot c_2) \cdot \frac{a_2 \cdot \cos(\theta_1) - b_2 \cdot \sin(\theta_1)}{\left|\sqrt{a_2^2 + b_2^2}\,\right|} \\ b_{1\cdot 2} &= -(c_1 \cdot c_2) \cdot \frac{a_2 \cdot \sin(\theta_1) + b_2 \cdot \cos(\theta_1)}{\left|\sqrt{a_2^2 + b_2^2}\,\right|} \\ c_{1\cdot 2} &= c_1 \cdot \left|\sqrt{a_2^2 + b_2^2}\,\right| \end{aligned}$$

Proof. The multiplication between two complete numbers, as we know, satisfies the following formula:

$$\begin{aligned} o_{1\cdot 2}(t_{1\cdot 2}, \theta_{1\cdot 2}, \gamma_{1\cdot 2}) = t_1 \cdot t_2 \cdot \{&[\cos(\gamma_1 + \gamma_2) \cdot \cos(\theta_1 + \theta_2)] + \\ &+ i \cdot [\cos(\gamma_1 + \gamma_2) \cdot \sin(\theta_1 + \theta_2)] + u \cdot [\sin(\gamma_1 + \gamma_2)]\} \end{aligned}$$

Since $o_1(t_1, \theta_1, \gamma_1)$ belongs to the line U will be provided with the following values of modulus and phases:

$$t_1 = \sqrt{c_1^2}$$
$$\gamma_1 = \text{ sign } (c_1) \cdot 90^\circ$$
$$\theta_1 \text{ known } \neq \arctan\left(\frac{b_1}{a_1}\right)$$

unlike $o_2(t_2, \theta_2, \gamma_2)$ that will be provided with the following values:

$$t_2 = \sqrt{a_2^2 + b_2^2 + c_2^2}$$
$$\gamma_2 = \arctan\left(\frac{c_2}{\sqrt{a_2^2 + b_2^2}}\right)$$
$$\theta_2 = \arctan\left(\frac{b_2}{a_2}\right)$$

This means that we can write the coordinates sought in the following way:

$$a_{1\cdot 2} = \sqrt{c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2} \cdot \cos\left[\text{ sign } (c_1) \cdot 90^\circ + \arctan\left(\frac{c_2}{\sqrt{a_2^2 + b_2^2}}\right)\right] \cdot$$
$$\cdot \cos\left[\theta_1 + \arctan\left(\frac{b_2}{a_2}\right)\right]$$
$$b_{1\cdot 2} = \sqrt{c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2} \cdot \cos\left[\text{ sign } (c_1) \cdot 90^\circ + \arctan\left(\frac{c_2}{\sqrt{a_2^2 + b_2^2}}\right)\right] \cdot$$
$$\cdot \sin\left[\theta_1 + \arctan\left(\frac{b_2}{a_2}\right)\right]$$
$$c_{1\cdot 2} = \sqrt{c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2} \cdot \sin\left[\text{ sign } (c_1) \cdot 90^\circ + \arctan\left(\frac{c_2}{\sqrt{a_2^2 + b_2^2}}\right)\right]$$

To continue with the proof, we have to use the following trigonometric relations:

$$\cos(x + y) = \cos(x) \cdot \cos(y) - \sin(x) \cdot \sin(y)$$
$$\sin(x + y) = \sin(x) \cdot \cos(y) + \cos(x) \cdot \sin(y)$$
$$\cos\left[\arctan\left(\frac{c}{\sqrt{a^2 + b^2}}\right)\right] = \sqrt{\frac{a^2 + b^2}{a^2 + b^2 + c^2}}$$
$$\sin\left[\arctan\left(\frac{c}{\sqrt{a^2 + b^2}}\right)\right] = \sqrt{\frac{c^2}{a^2 + b^2 + c^2}}$$
$$\cos\left[\arctan\left(\frac{b}{a}\right)\right] = \sqrt{\frac{a^2}{a^2 + b^2}}$$
$$\sin\left[\arctan\left(\frac{b}{a}\right)\right] = \sqrt{\frac{b^2}{a^2 + b^2}}$$
$$\cos\left[\text{ sign } (x) \cdot 90^\circ + y\right] = -\text{ sign } (x) \cdot sin(y)$$
$$\sin\left[\text{ sign } (x) \cdot 90^\circ + y\right] = \text{ sign } (x) \cdot cos(y)$$

To determine the value of the coordinate $a_{1\cdot 2}$ the steps to perform will be the following:

$$
\begin{aligned}
a_{1\cdot 2} = &- \text{ sign}\,(c_1) \cdot \sqrt{c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2} \cdot \sqrt{\frac{c_2^2}{a_2^2 + b_2^2 + c_2^2}} \cdot \\
&\cdot \left[\sqrt{\frac{a_2^2}{a_2^2 + b_2^2}} \cdot \cos(\theta_1) - \sqrt{\frac{b_2^2}{a_2^2 + b_2^2}} \cdot \sin(\theta_1)\right] = \\
= &- \text{ sign}\,(c_1) \cdot \sqrt{c_1^2} \cdot \sqrt{c_2^2} \cdot \frac{\sqrt{a_2^2} \cdot \cos(\theta_1) - \sqrt{b_2^2} \cdot \sin(\theta_1)}{\sqrt{a_2^2 + b_2^2}}
\end{aligned}
$$

To determine the value of the coordinate $b_{1\cdot 2}$ the steps to perform will be the following:

$$
\begin{aligned}
b_{1\cdot 2} = &- \text{ sign}\,(c_1) \cdot \sqrt{c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2} \cdot \sqrt{\frac{c_2^2}{a_2^2 + b_2^2 + c_2^2}} \cdot \\
&\cdot \left[\sqrt{\frac{a_2^2}{a_2^2 + b_2^2}} \cdot \sin(\theta_1) + \sqrt{\frac{b_2^2}{a_2^2 + b_2^2}} \cdot \cos(\theta_1)\right] = \\
= &- \text{ sign}\,(c_1) \cdot \sqrt{c_1^2} \cdot \sqrt{c_2^2} \cdot \frac{\sqrt{a_2^2} \cdot \sin(\theta_1) + \sqrt{b_2^2} \cdot \cos(\theta_1)}{\sqrt{a_2^2 + b_2^2}}
\end{aligned}
$$

To determine the value of the coordinate $c_{1\cdot 2}$ the steps to perform will be the following:

$$
c_{1\cdot 2} = \text{ sign}\,(c_1) \cdot \sqrt{c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2} \cdot \sqrt{\frac{a_2^2 + b_2^2}{a_2^2 + b_2^2 + c_2^2}} = \text{ sign}\,(c_1) \cdot \sqrt{c_1^2} \cdot \sqrt{a_2^2 + b_2^2}
$$

These relations are valid in general, in the precise sense that they are also able to include cases where the coefficients a_2,b_2,c_2 are zero (provided that $o_2(a_2, b_2, c_2)$ remains in the context of the complete numbers not belonging in the line U).

Wanting to find relations that satisfy the multiplication rule as a function of the effective coordinates of the complete numbers involved, we must adopt for the coefficients a,b,c the convention $\sqrt{x^2} = x$, with the exception of c_1 for which we should adopt the convention $\sqrt{x^2} = |x|$. The reason is simple because if we adopt for c_1 the usual convention, we will have:

$$
\text{sign}\,(c_1) \cdot \sqrt{c_1^2} = |c_1|
$$

and therefore a result of the multiplication that depends on the modulus of the coordinate c_1. While adopting $\sqrt{x^2} = |x|$ we will have:

$$
\text{sign}\,(c_1) \cdot \sqrt{c_1^2} = c_1
$$

and therefore a result of the multiplication that depends on the effective value of this coordinate.

The relations obtained will be the following:

$$
(2.2)\qquad
\begin{aligned}
a_{1\cdot 2} &= -(c_1 \cdot c_2) \cdot \frac{a_2 \cdot \cos(\theta_1) - b_2 \cdot \sin(\theta_1)}{\sqrt{a_2^2 + b_2^2}} \\
b_{1\cdot 2} &= -(c_1 \cdot c_2) \cdot \frac{a_2 \cdot \sin(\theta_1) + b_2 \cdot \cos(\theta_1)}{\sqrt{a_2^2 + b_2^2}} \\
c_{1\cdot 2} &= c_1 \cdot \sqrt{a_2^2 + b_2^2}
\end{aligned}
$$

Since the number $o_2(a_2, b_2, c_2)$ is in standard representation, as determined by the Theorem 2.11 we must consider the following relation:

$$\sqrt{a_2^2 + b_2^2} = \left|\sqrt{a_2^2 + b_2^2}\,\right|$$

that combined with those indicated by formulas (2.2), proving the thesis. ■

As an example of the theorem just proved, suppose you have to multiply the outgoing numbers of coordinate: $c_1 = 1$ and phase $\theta_1 = 30°$ by a complete number in standard representation provided with coordinates: $a_2 = 1$, $b_2 = -1$, $c_2 = 1$.

Their modulus may be calculated in the following way:

$$
\begin{aligned}
t_1 &= \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{c_1^2} = \sqrt{1} = 1 \\
t_2 &= \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}
\end{aligned}
$$

For their phases in the case of the outgoing number we have:

$$
\begin{aligned}
\gamma_1 &= \text{ sign}\,(c_1) \cdot 90° = 90° \\
\theta_1 &= 30°
\end{aligned}
$$

while in the case of the complete number we should refer to the formulas related to the standard representation:

$$
\begin{aligned}
\gamma_2 &= \arctan\left(\frac{c_2}{\left|\sqrt{a_2^2 + b_2^2}\,\right|}\right) = \arctan\left(\frac{1}{|\sqrt{2}|}\right) \simeq 35.26° \\
\theta_2 &= \arctan\left(\frac{b_2}{a_2}\right) = \arctan\left(\frac{-1}{1}\right) = -45°
\end{aligned}
$$

By applying the multiplication rule we obtain as result the complete number provided with the following values of modulus and phases:

$$
\begin{aligned}
t_{1\cdot 2} &= t_1 \cdot t_2 = \sqrt{3} \\
\gamma_{1\cdot 2} &= \gamma_1 + \gamma_2 \simeq 125.26° \\
\theta_{1\cdot 2} &= \theta_1 + \theta_2 = -15°
\end{aligned}
$$

and the following coordinates:

$$a_{1\cdot 2} = t_{1\cdot 2} \cdot \cos(\gamma_{1\cdot 2}) \cdot \cos(\theta_{1\cdot 2}) = \sqrt{3} \cdot \cos(\simeq 125.26°) \cdot \cos(-15°) \simeq -0.97$$
$$b_{1\cdot 2} = t_{1\cdot 2} \cdot \cos(\gamma_{1\cdot 2}) \cdot \sin(\theta_{1\cdot 2}) = \sqrt{3} \cdot \cos(\simeq 125.26°) \cdot \sin(-15°) \simeq 0.26$$
$$c_{1\cdot 2} = t_{1\cdot 2} \cdot \sin(\gamma_{1\cdot 2}) = \sqrt{3} \cdot \sin(\simeq 125.26°) = \sqrt{2}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$a_{1\cdot 2} = -(c_1 \cdot c_2) \cdot \frac{a_2 \cdot \cos(\theta_1) - b_2 \cdot \sin(\theta_1)}{\left|\sqrt{a_2^2 + b_2^2}\right|} = -\frac{\cos(30°) + \sin(30°)}{\left|\sqrt{2}\right|} \simeq -0.97$$
$$b_{1\cdot 2} = -(c_1 \cdot c_2) \cdot \frac{a_2 \cdot \sin(\theta_1) + b_2 \cdot \cos(\theta_1)}{\left|\sqrt{a_2^2 + b_2^2}\right|} = -\frac{\sin(30°) - \cos(30°)}{\left|\sqrt{2}\right|} \simeq 0.26$$
$$c_{1\cdot 2} = c_1 \cdot \left|\sqrt{a_2^2 + b_2^2}\right| = \sqrt{2}$$

Theorem 2.36. *With only $o_1(t_1, \theta_1, \gamma_1)$ belonging to the line U and $o_2(t_2, \theta_2, \gamma_2)$ in complementary representation, their multiplication may be expressed in the following way:*

$$o_{1\cdot 2}(a_{1\cdot 2}, b_{1\cdot 2}, c_{1\cdot 2})_{(t_{1\cdot 2}, \theta_{1\cdot 2}, \gamma_{1\cdot 2})} = a_{1\cdot 2(t_1 \cdot t_2)} + i \cdot b_{1\cdot 2(\theta_1 + \theta_2)} + u \cdot c_{1\cdot 2(\gamma_1 + \gamma_2)}$$

where:

$$a_{1\cdot 2} = (c_1 \cdot c_2) \cdot \frac{a_2 \cdot \cos(\theta_1) - b_2 \cdot \sin(\theta_1)}{\left|\sqrt{a_2^2 + b_2^2}\right|}$$
$$b_{1\cdot 2} = (c_1 \cdot c_2) \cdot \frac{a_2 \cdot \sin(\theta_1) + b_2 \cdot \cos(\theta_1)}{\left|\sqrt{a_2^2 + b_2^2}\right|}$$
$$c_{1\cdot 2} = -c_1 \cdot \left|\sqrt{a_2^2 + b_2^2}\right|$$

Proof. Since the number $o_2(a_2, b_2, c_2)$ is in complementary representation, as determined by Theorem 2.15, we must consider the following relation:

$$\sqrt{a_2^2 + b_2^2} = -\left|\sqrt{a_2^2 + b_2^2}\right|$$

that combined with those indicated by the formulas (2.2), proving the thesis. ∎

As an example of the theorem just proved, suppose you have to multiply the outgoing numbers of coordinate: $c_1 = 1$ and phase $\theta_1 = 30°$ by a complete number in complementary representation provided with coordinates: $a_2 = 1$, $b_2 = -1$, $c_2 = 1$.

Their modulus may be calculated in the following way:

$$t_1 = \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{c_1^2} = \sqrt{1} = 1$$
$$t_2 = \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$$

For their phases in the case of the outgoing number we have:

$$\gamma_1 = \text{ sign } (c_1) \cdot 90^\circ = 90^\circ$$
$$\theta_1 = 30^\circ$$

while in the case of the complete number we should refer to the formulas related to the complementary representation:

$$\gamma_2 = \arctan\left(\frac{c_2}{-\left|\sqrt{a_2^2 + b_2^2}\,\right|}\right) = \arctan\left(\frac{1}{-|\sqrt{2}|}\right) \simeq 144.74^\circ$$
$$\theta_2 = \arctan\left(\frac{-b_2}{-a_2}\right) = \arctan\left(\frac{1}{-1}\right) = 135^\circ$$

By applying the multiplication rule we obtain as result the complete number provided with the following values of modulus and phases:

$$t_{1\cdot 2} = t_1 \cdot t_2 = \sqrt{3}$$
$$\gamma_{1\cdot 2} = \gamma_1 + \gamma_2 \simeq 234.74^\circ$$
$$\theta_{1\cdot 2} = \theta_1 + \theta_2 = 165^\circ$$

and the following coordinates:

$$a_{1\cdot 2} = t_{1\cdot 2} \cdot \cos(\gamma_{1\cdot 2}) \cdot \cos(\theta_{1\cdot 2}) = \sqrt{3} \cdot \cos(\simeq 234.74^\circ) \cdot \cos(165^\circ) \simeq 0.97$$
$$b_{1\cdot 2} = t_{1\cdot 2} \cdot \cos(\gamma_{1\cdot 2}) \cdot \sin(\theta_{1\cdot 2}) = \sqrt{3} \cdot \cos(\simeq 234.74^\circ) \cdot \sin(165^\circ) \simeq -0.26$$
$$c_{1\cdot 2} = t_{1\cdot 2} \cdot \sin(\gamma_{1\cdot 2}) = \sqrt{3} \cdot \sin(\simeq 234.74^\circ) = -\sqrt{2}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$a_{1\cdot 2} = (c_1 \cdot c_2) \cdot \frac{a_2 \cdot \cos(\theta_1) - b_2 \cdot \sin(\theta_1)}{\left|\sqrt{a_2^2 + b_2^2}\right|} = \frac{\cos(30^\circ) + \sin(30^\circ)}{\left|\sqrt{2}\right|} \simeq 0.97$$
$$b_{1\cdot 2} = (c_1 \cdot c_2) \cdot \frac{a_2 \cdot \sin(\theta_1) + b_2 \cdot \cos(\theta_1)}{\left|\sqrt{a_2^2 + b_2^2}\right|} = \frac{\sin(30^\circ) - \cos(30^\circ)}{\left|\sqrt{2}\right|} \simeq -0.26$$
$$c_{1\cdot 2} = -c_1 \cdot \left|\sqrt{a_2^2 + b_2^2}\right| = -\sqrt{2}$$

Theorem 2.37. *With only $o_2(t_2, \theta_2, \gamma_2)$ belonging to the line U and $o_1(t_1, \theta_1, \gamma_1)$ in standard representation, their multiplication may be expressed in the following way:*

$$o_{1\cdot 2}(a_{1\cdot 2}, b_{1\cdot 2}, c_{1\cdot 2})_{(t_{1\cdot 2}, \theta_{1\cdot 2}, \gamma_{1\cdot 2})} = a_{1\cdot 2(t_1 \cdot t_2)} + i \cdot b_{1\cdot 2(\theta_1 + \theta_2)} + u \cdot c_{1\cdot 2(\gamma_1 + \gamma_2)}$$

where:

$$\begin{aligned} a_{1\cdot 2} &= -(c_1 \cdot c_2) \cdot \frac{a_1 \cdot \cos(\theta_2) - b_1 \cdot \sin(\theta_2)}{|\sqrt{a_1^2 + b_1^2}\,|} \\ b_{1\cdot 2} &= -(c_1 \cdot c_2) \cdot \frac{a_1 \cdot \sin(\theta_2) + b_1 \cdot \cos(\theta_2)}{|\sqrt{a_1^2 + b_1^2}\,|} \\ c_{1\cdot 2} &= c_2 \cdot \left|\sqrt{a_1^2 + b_1^2}\right| \end{aligned}$$

Proof. The multiplication between two complete numbers, as we know, satisfies the following formula:

$$\begin{gathered} o_{1\cdot 2}(t_{1\cdot 2}, \theta_{1\cdot 2}, \gamma_{1\cdot 2}) = t_1 \cdot t_2 \cdot \{[\cos(\gamma_1 + \gamma_2) \cdot \cos(\theta_1 + \theta_2)] + \\ + i \cdot [\cos(\gamma_1 + \gamma_2) \cdot \sin(\theta_1 + \theta_2)] + u \cdot [\sin(\gamma_1 + \gamma_2)]\} \end{gathered}$$

Since $o_2(t_2, \theta_2, \gamma_2)$ belongs to the line U will be provided with the following values of modulus and phases:

$$\begin{aligned} t_2 &= \sqrt{c_2^2} \\ \gamma_2 &= \text{ sign } (c_2) \cdot 90^\circ \\ \theta_2 & \text{ known } \neq \arctan\left(\frac{b_2}{a_2}\right) \end{aligned}$$

unlike $o_1(t_1, \theta_1, \gamma_1)$ that will be provided with the following values:

$$\begin{aligned} t_1 &= \sqrt{a_1^2 + b_1^2 + c_1^2} \\ \gamma_1 &= \arctan\left(\frac{c_1}{\sqrt{a_1^2 + b_1^2}}\right) \\ \theta_1 &= \arctan\left(\frac{b_1}{a_1}\right) \end{aligned}$$

This means that we can write the coordinates sought in the following way:

$$\begin{aligned} a_{1\cdot 2} =& \sqrt{c_1^2 + b_1^2 + c_1^2} \cdot \sqrt{c_2^2} \cdot \cos\left[\arctan\left(\frac{c_1}{\sqrt{a_1^2 + b_1^2}}\right) + \text{ sign } (c_2) \cdot 90^\circ\right] \cdot \\ & \cdot \cos\left[\arctan\left(\frac{b_1}{a_1}\right) + \theta_2\right] \\ b_{1\cdot 2} =& \sqrt{c_1^2 + b_1^2 + c_1^2} \cdot \sqrt{c_2^2} \cdot \cos\left[\arctan\left(\frac{c_1}{\sqrt{a_1^2 + b_1^2}}\right) + \text{ sign } (c_2) \cdot 90^\circ\right] \cdot \\ & \cdot \sin\left[\arctan\left(\frac{b_1}{a_1}\right) + \theta_2\right] \\ c_{1\cdot 2} =& \sqrt{c_1^2 + b_1^2 + c_1^2} \cdot \sqrt{c_2^2} \cdot \sin\left[\arctan\left(\frac{c_1}{\sqrt{a_1^2 + b_1^2}}\right) + \text{ sign } (c_2) \cdot 90^\circ\right] \end{aligned}$$

To continue with the proof, we have to use the following trigonometric relations:

$$\cos(x+y) = \cos(x)\cdot\cos(y) - \sin(x)\cdot\sin(y)$$
$$\sin(x+y) = \sin(x)\cdot\cos(y) + \cos(x)\cdot\sin(y)$$
$$\cos\left[\arctan\left(\frac{c}{\sqrt{a^2+b^2}}\right)\right] = \sqrt{\frac{a^2+b^2}{a^2+b^2+c^2}}$$
$$\sin\left[\arctan\left(\frac{c}{\sqrt{a^2+b^2}}\right)\right] = \sqrt{\frac{c^2}{a^2+b^2+c^2}}$$
$$\cos\left[\arctan\left(\frac{b}{a}\right)\right] = \sqrt{\frac{a^2}{a^2+b^2}}$$
$$\sin\left[\arctan\left(\frac{b}{a}\right)\right] = \sqrt{\frac{b^2}{a^2+b^2}}$$
$$\cos[x + \text{ sign }(y)\cdot 90°] = -\text{ sign }(y)\cdot sin(x)$$
$$\sin[x + \text{ sign }(y)\cdot 90°] = \text{ sign }(y)\cdot cos(x)$$

To determine the value of the coordinate $a_{1\cdot 2}$ the steps to perform will be the following:

$$\begin{aligned} a_{1\cdot 2} = &-\text{ sign }(c_2)\cdot\sqrt{c_1^2+b_1^2+c_1^2}\cdot\sqrt{c_2^2}\cdot\sqrt{\frac{c_1^2}{a_1^2+b_1^2+c_1^2}}\cdot \\ &\cdot\left[\sqrt{\frac{a_1^2}{a_1^2+b_1^2}}\cdot\cos(\theta_2) - \sqrt{\frac{b_1^2}{a_1^2+b_1^2}}\cdot\sin(\theta_2)\right] = \\ = &-\text{ sign }(c_2)\cdot\sqrt{c_1^2}\cdot\sqrt{c_2^2}\cdot\frac{\sqrt{a_1^2}\cdot\cos(\theta_2) - \sqrt{b_1^2}\cdot\sin(\theta_2)}{\sqrt{a_1^2+b_1^2}} \end{aligned}$$

To determine the value of the coordinate $b_{1\cdot 2}$ the steps to perform will be the following:

$$\begin{aligned} b_{1\cdot 2} = &-\text{ sign }(c_2)\cdot\sqrt{c_1^2+b_1^2+c_1^2}\cdot\sqrt{c_2^2}\cdot\sqrt{\frac{c_1^2}{a_1^2+b_1^2+c_1^2}}\cdot \\ &\cdot\left[\sqrt{\frac{b_1^2}{a_1^2+b_1^2}}\cdot\cos(\theta_2) + \sqrt{\frac{a_1^2}{a_1^2+b_1^2}}\cdot\sin(\theta_2)\right] = \\ = &-\text{ sign }(c_2)\cdot\sqrt{c_1^2}\cdot\sqrt{c_2^2}\cdot\frac{\sqrt{b_1^2}\cdot\cos(\theta_2) + \sqrt{a_1^2}\cdot\sin(\theta_2)}{\sqrt{a_1^2+b_1^2}} \end{aligned}$$

To determine the value of the coordinate $c_{1\cdot 2}$ the steps to perform will be the following:

$$c_{1\cdot 2} = \text{ sign }(c_2)\cdot\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{c_2^2}\cdot\sqrt{\frac{a_1^2+b_1^2}{a_1^2+b_1^2+c_1^2}} = \text{ sign }(c_2)\cdot\sqrt{c_2^2}\cdot\sqrt{a_1^2+b_1^2}$$

These relations are valid in general, in the precise sense that they are also able to include cases where the coefficients a_1,b_1,c_1 are zero (provided that $o_1(a_1, b_1, c_1)$ remains in the context of the complete numbers not belonging in the line U).

Wanting to find relations that satisfy the multiplication rule as a function of the effective coordinates of the complete numbers involved, we must adopt for the coefficients a,b,c the convention $\sqrt{x^2} = x$, with the exception of c_2 for which we should adopt the convention $\sqrt{x^2} = |x|$. The reason is simple because if we adopt for c_2 the usual convention, we will have:

$$\operatorname{sign}(c_2) \cdot \sqrt{c_2^2} = |c_2|$$

and therefore a result of the multiplication that depends on the modulus of the coordinate c_2. While adopting $\sqrt{x^2} = |x|$ we will have:

$$\operatorname{sign}(c_2) \cdot \sqrt{c_2^2} = c_2$$

and therefore a result of the multiplication that depends on the effective value of this coordinate.

The relations obtained will be the following:

$$\begin{aligned}
a_{1\cdot 2} &= -(c_1 \cdot c_2) \cdot \frac{a_1 \cdot \cos(\theta_2) - b_1 \cdot \sin(\theta_2)}{\sqrt{a_1^2 + b_1^2}} \\
b_{1\cdot 2} &= -(c_1 \cdot c_2) \cdot \frac{a_1 \cdot \sin(\theta_2) + b_1 \cdot \cos(\theta_2)}{\sqrt{a_1^2 + b_1^2}} \\
c_{1\cdot 2} &= c_2 \cdot \sqrt{a_1^2 + b_1^2}
\end{aligned} \tag{2.3}$$

Since the number $o_1(a_1, b_1, c_1)$ is in standard representation, as determined by the theorem 2.11 we must consider the following relation:

$$\sqrt{a_1^2 + b_1^2} = \left|\sqrt{a_1^2 + b_1^2}\,\right|$$

that combined with those indicated by the formulas (2.3), proving the thesis. ∎

As an example of the theorem just proved, suppose you have to multiply the complete number in standard representation provided with coordinates: $a_1 = 1$, $b_1 = -1$, $c_1 = 1$ by an outgoing numbers of coordinate: $c_2 = 1$ and phase $\theta_2 = 30°$.

Their modulus may be calculated in the following way:

$$\begin{aligned}
t_1 &= \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3} \\
t_2 &= \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{c_2^2} = \sqrt{1} = 1
\end{aligned}$$

For their phases in the case of the outgoing number we have:

$$\begin{aligned}
\gamma_2 &= \operatorname{sign}(c_2) \cdot 90° = 90° \\
\theta_2 &= 30°
\end{aligned}$$

while in the case of the complete number we should refer to the formulas related to the standard representation:

$$\gamma_1 = \arctan\left(\frac{c_1}{|\sqrt{a_1^2+b_1^2}\,|}\right) = \arctan\left(\frac{1}{|\sqrt{2}|}\right) \simeq 35.26^\circ$$
$$\theta_1 = \arctan\left(\frac{b_1}{a_1}\right) = \arctan\left(\frac{-1}{1}\right) = -45^\circ$$

By applying the multiplication rule we obtain as result the complete number provided with the following values of modulus and phases:

$$t_{1\cdot 2} = t_1 \cdot t_2 = \sqrt{3}$$
$$\gamma_{1\cdot 2} = \gamma_1 + \gamma_2 \simeq 125.26^\circ$$
$$\theta_{1\cdot 2} = \theta_1 + \theta_2 = -15^\circ$$

and the following coordinates:

$$a_{1\cdot 2} = t_{1\cdot 2} \cdot \cos(\gamma_{1\cdot 2}) \cdot \cos(\theta_{1\cdot 2}) = \sqrt{3} \cdot \cos(\simeq 125.26^\circ) \cdot \cos(-15^\circ) \simeq -0.97$$
$$b_{1\cdot 2} = t_{1\cdot 2} \cdot \cos(\gamma_{1\cdot 2}) \cdot \sin(\theta_{1\cdot 2}) = \sqrt{3} \cdot \cos(\simeq 125.26^\circ) \cdot \sin(-15^\circ) \simeq 0.26$$
$$c_{1\cdot 2} = t_{1\cdot 2} \cdot \sin(\gamma_{1\cdot 2}) = \sqrt{3} \cdot \sin(\simeq 125.26^\circ) = \sqrt{2}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$a_{1\cdot 2} = -(c_1 \cdot c_2) \cdot \frac{a_1 \cdot \cos(\theta_2) - b_1 \cdot \sin(\theta_2)}{|\sqrt{a_1^2+b_1^2}\,|} = -\frac{\cos(30^\circ) + \sin(30^\circ)}{|\sqrt{2}\,|} \simeq -0.97$$
$$b_{1\cdot 2} = -(c_1 \cdot c_2) \cdot \frac{a_1 \cdot \sin(\theta_2) + b_1 \cdot \cos(\theta_2)}{|\sqrt{a_1^2+b_1^2}\,|} = -\frac{\sin(30^\circ) - \cos(30^\circ)}{|\sqrt{2}\,|} \simeq 0.26$$
$$c_{1\cdot 2} = c_2 \cdot \left|\sqrt{a_1^2+b_1^2}\right| = \sqrt{2}$$

Theorem 2.38. *With only $o_2(t_2, \theta_2, \gamma_2)$ belonging to the line U and $o_1(t_1, \theta_1, \gamma_1)$ in complementary representation, their multiplication may be expressed in the following way:*

$$o_{1\cdot 2}(a_{1\cdot 2}, b_{1\cdot 2}, c_{1\cdot 2})_{(t_{1\cdot 2}, \theta_{1\cdot 2}, \gamma_{1\cdot 2})} = a_{1\cdot 2(t_1 \cdot t_2)} + i \cdot b_{1\cdot 2(\theta_1+\theta_2)} + u \cdot c_{1\cdot 2(\gamma_1+\gamma_2)}$$

where

$$a_{1\cdot 2} = (c_1 \cdot c_2) \cdot \frac{a_1 \cdot \cos(\theta_2) - b_1 \cdot \sin(\theta_2)}{|\sqrt{a_1^2+b_1^2}\,|}$$
$$b_{1\cdot 2} = (c_1 \cdot c_2) \cdot \frac{a_1 \cdot \sin(\theta_2) + b_1 \cdot \cos(\theta_2)}{|\sqrt{a_1^2+b_1^2}\,|}$$
$$c_{1\cdot 2} = -c_2 \cdot \left|\sqrt{a_1^2+b_1^2}\right|$$

Proof. Since the number $o_1(a_1, b_1, c_1)$ is in complementary representation, as determined by the theorem 2.15 we must consider the following relation:

$$\sqrt{a_1^2 + b_1^2} = -\left|\sqrt{a_1^2 + b_1^2}\,\right|$$

that combined with those indicated by the formulas (2.3), proving the thesis. ∎

As an example of the theorem just proved, suppose you have to multiply a complete number in complementary representation provided with coordinates: $a_2 = 1$, $b_2 = -1$, $c_2 = 1$ by the outgoing numbers of coordinate: $c_1 = 1$ and phase $\theta_1 = 30°$.

Their modulus may be calculated in the following way:

$$t_1 = \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$$
$$t_2 = \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{c_2^2} = \sqrt{1} = 1$$

For their phases in the case of the outgoing number we have:

$$\gamma_2 = \ \operatorname{sign}(c_2) \cdot 90° = 90°$$
$$\theta_2 = 30°$$

while in the case of the complete number we should refer to the formulas related to the complementary representation:

$$\gamma_2 = \arctan\left(\frac{c_1}{-\left|\sqrt{a_1^2 + b_1^2}\,\right|}\right) = \arctan\left(\frac{1}{-|\sqrt{2}|}\right) \simeq 144.74°$$
$$\theta_2 = \arctan\left(\frac{-b_1}{-a_1}\right) = \arctan\left(\frac{1}{-1}\right) = 135°$$

By applying the multiplication rule we obtain as result the complete number provided with the following values of modulus and phases:

$$t_{1\cdot 2} = t_1 \cdot t_2 = \sqrt{3}$$
$$\gamma_{1\cdot 2} = \gamma_1 + \gamma_2 \simeq 234.74°$$
$$\theta_{1\cdot 2} = \theta_1 + \theta_2 = 165°$$

and the following coordinates:

$$a_{1\cdot 2} = t_{1\cdot 2} \cdot \cos(\gamma_{1\cdot 2}) \cdot \cos(\theta_{1\cdot 2}) = \sqrt{3} \cdot \cos(\simeq 234.74°) \cdot \cos(165°) \simeq 0.97$$
$$b_{1\cdot 2} = t_{1\cdot 2} \cdot \cos(\gamma_{1\cdot 2}) \cdot \sin(\theta_{1\cdot 2}) = \sqrt{3} \cdot \cos(\simeq 234.74°) \cdot \sin(165°) \simeq -0.26$$
$$c_{1\cdot 2} = t_{1\cdot 2} \cdot \sin(\gamma_{1\cdot 2}) = \sqrt{3} \cdot \sin(\simeq 234.74°) = -\sqrt{2}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$
\begin{aligned}
a_{1\cdot 2} &= (c_1 \cdot c_2) \cdot \frac{a_1 \cdot \cos(\theta_2) - b_1 \cdot \sin(\theta_2)}{|\sqrt{a_1^2 + b_1^2}\,|} = \frac{\cos(30°) + \sin(30°)}{|\sqrt{2}\,|} \simeq 0.97 \\
b_{1\cdot 2} &= (c_1 \cdot c_2) \cdot \frac{a_1 \cdot \sin(\theta_2) + b_1 \cdot \cos(\theta_2)}{|\sqrt{a_1^2 + b_1^2}\,|} = \frac{\sin(30°) - \cos(30°)}{|\sqrt{2}\,|} \simeq -0.26 \\
c_{1\cdot 2} &= -c_2 \cdot \left|\sqrt{a_1^2 + b_1^2}\right| = -\sqrt{2}
\end{aligned}
$$

Theorem 2.39. *With $o_1(t_1, \theta_1, \gamma_1)$ and $o_2(t_2, \theta_2, \gamma_2)$ both belonging to the line U, their multiplication may be expressed in the following way:*

$$o_{1\cdot 2}(a_{1\cdot 2}, b_{1\cdot 2}, c_{1\cdot 2})_{(t_{1\cdot 2}, \theta_{1\cdot 2}, \gamma_{1\cdot 2})} = a_{1\cdot 2(t_1 \cdot t_2)} + i \cdot b_{1\cdot 2(\theta_1 + \theta_2)} + u \cdot c_{1\cdot 2(\gamma_1 + \gamma_2)}$$

where

$$
\begin{aligned}
a_{1\cdot 2} &= -(c_1 \cdot c_2) \cdot \cos(\theta_1 + \theta_2) \\
b_{1\cdot 2} &= -(c_1 \cdot c_2) \cdot \sin(\theta_1 + \theta_2) \\
c_{1\cdot 2} &= 0
\end{aligned}
$$

Proof. The multiplication between two complete numbers, as we know, satisfies the following formula:

$$
\begin{aligned}
o_{1\cdot 2}(t_{1\cdot 2}, \theta_{1\cdot 2}, \gamma_{1\cdot 2}) = t_1 \cdot t_2 \cdot \{&[\cos(\gamma_1 + \gamma_2) \cdot \cos(\theta_1 + \theta_2)] + \\
+ i \cdot [\cos(\gamma_1 + \gamma_2) \cdot \sin(\theta_1 + \theta_2)] &+ u \cdot [\sin(\gamma_1 + \gamma_2)]\}
\end{aligned}
$$

Since $o_1(t_1, \theta_1, \gamma_1)$ and $o_2(t_2, \theta_2, \gamma_2)$ belong to the line U will be provided with the following values of modulus and phases:

$$
\begin{aligned}
t_1 &= \sqrt{c_1^2} \\
t_2 &= \sqrt{c_2^2} \\
\gamma_1 &= \operatorname{sign}(c_1) \cdot 90° \\
\gamma_2 &= \operatorname{sign}(c_2) \cdot 90° \\
\theta_1 &\text{ known } \neq \arctan\left(\frac{b_1}{a_1}\right) \\
\theta_2 &\text{ known } \neq \arctan\left(\frac{b_2}{a_2}\right)
\end{aligned}
$$

This means that we can write the coordinates sought in the following way:

$$
\begin{aligned}
a_{1\cdot 2} &= \sqrt{c_1^2} \cdot \sqrt{c_2^2} \cdot \cos[\operatorname{sign}(c_1) \cdot 90° + \operatorname{sign}(c_2) \cdot 90°] \cdot \cos(\theta_1 + \theta_2) \\
b_{1\cdot 2} &= \sqrt{c_1^2} \cdot \sqrt{c_2^2} \cdot \cos[\operatorname{sign}(c_1) \cdot 90° + \operatorname{sign}(c_2) \cdot 90°] \cdot \sin(\theta_1 + \theta_2) \\
c_{1\cdot 2} &= \sqrt{c_1^2} \cdot \sqrt{c_2^2} \cdot \sin[\operatorname{sign}(c_1) \cdot 90° + \operatorname{sign}(c_2) \cdot 90°]
\end{aligned}
$$

Considering that when c_1 and c_2 have the same sign we obtained:

$$\cos\left[\operatorname{sign}(c_1)\cdot 90^\circ + \operatorname{sign}(c_2)\cdot 90^\circ\right] = \cos(\pm 180^\circ) = -1 = -\operatorname{sign}(c_1)\cdot\operatorname{sign}(c_2)$$
$$\sin\left[\operatorname{sign}(c_1)\cdot 90^\circ + \operatorname{sign}(c_2)\cdot 90^\circ\right] = \sin(\pm 180^\circ) = 0$$

and that when they have the opposite sign we obtained:

$$\cos\left[\operatorname{sign}(c_1)\cdot 90^\circ + \operatorname{sign}(c_2)\cdot 90^\circ\right] = \cos(\pm 0^\circ) = 1 = -\operatorname{sign}(c_1)\cdot\operatorname{sign}(c_2)$$
$$\sin\left[\operatorname{sign}(c_1)\cdot 90^\circ + \operatorname{sign}(c_2)\cdot 90^\circ\right] = \sin(\pm 0^\circ) = 0$$

we can write:

$$\begin{aligned}
a_{1\cdot 2} &= -\operatorname{sign}(c_1)\cdot\operatorname{sign}(c_2)\cdot\sqrt{c_1^2}\cdot\sqrt{c_2^2}\cdot\cos(\theta_1+\theta_2)\\
b_{1\cdot 2} &= -\operatorname{sign}(c_1)\cdot\operatorname{sign}(c_2)\cdot\sqrt{c_1^2}\cdot\sqrt{c_2^2}\cdot\sin(\theta_1+\theta_2)\\
c_{1\cdot 2} &= 0
\end{aligned}$$

Wanting to find relations that satisfy the multiplication rule as a function of the effective coordinates of the complete numbers involved, we must adopt for the coefficients c_1,c_2 the convention $\sqrt{x^2} = |x|$. In fact in this way we obtain:

$$\begin{aligned}
\operatorname{sign}(c_1)\cdot\sqrt{c_1^2} &= c_1\\
\operatorname{sign}(c_2)\cdot\sqrt{c_2^2} &= c_2
\end{aligned}$$

and therefore a result of the multiplication that depends on the effective value of this coordinate. The relation that we obtain following these conventions proves the thesis. ∎

As an example of the theorem just proved, suppose you have to multiply the outgoing numbers of coordinate: $c_1 = 1$ and phase $\theta_1 = 30^\circ$ by the outgoing number of coordinate: $c_2 = 1$ and phase $\theta_2 = 30^\circ$.

Their modulus may be calculated in the following way:

$$\begin{aligned}
t_1 &= \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{c_1^2} = \sqrt{1} = 1\\
t_2 &= \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{c_2^2} = \sqrt{1} = 1
\end{aligned}$$

For their phases we have:

$$\begin{aligned}
\gamma_1 &= \operatorname{sign}(c_1)\cdot 90^\circ = 90^\circ\\
\gamma_2 &= \operatorname{sign}(c_2)\cdot 90^\circ = 90^\circ\\
\theta_1 &= 30^\circ\\
\theta_2 &= 30^\circ
\end{aligned}$$

By applying the multiplication rule we obtain as result the complete number provided with the following values of modulus and phases:

$$\begin{aligned}
t_{1\cdot 2} &= t_1\cdot t_2 = 1\\
\gamma_{1\cdot 2} &= \gamma_1 + \gamma_2 = 180^\circ\\
\theta_{1\cdot 2} &= \theta_1 + \theta_2 = 60^\circ
\end{aligned}$$

and the following coordinates:

$$a_{1\cdot 2} = t_{1\cdot 2} \cdot \cos(\gamma_{1\cdot 2}) \cdot \cos(\theta_{1\cdot 2}) = 1 \cdot \cos(180^\circ) \cdot \cos(60^\circ) = -\frac{1}{2}$$
$$b_{1\cdot 2} = t_{1\cdot 2} \cdot \cos(\gamma_{1\cdot 2}) \cdot \sin(\theta_{1\cdot 2}) = 1 \cdot \cos(180^\circ) \cdot \sin(60^\circ) = -\frac{\sqrt{3}}{2}$$
$$c_{1\cdot 2} = t_{1\cdot 2} \cdot \sin(\gamma_{1\cdot 2}) = 1 \cdot \sin(180^\circ) = 0$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$a_{1\cdot 2} = -(c_1 \cdot c_2) \cdot \cos(\theta_1 + \theta_2) = -\cos(60^\circ) = -\frac{1}{2}$$
$$b_{1\cdot 2} = -(c_1 \cdot c_2) \cdot \sin(\theta_1 + \theta_2) = -\sin(60^\circ) = -\frac{\sqrt{3}}{2}$$
$$c_{1\cdot 2} = 0$$

Theorem 2.40. *For the operation of multiplication is defined null the complete number 0, namely for:*

$$o_2(t_2, \theta_2, \gamma_2) = 0$$

we have:

$$o_1(t_1, \theta_1, \gamma_1) \cdot o_2(t_2, \theta_2, \gamma_2) = 0$$

Proof. t_1,θ_1,γ_1,t_2,θ_2,γ_2 being real numbers, we can write:

$$t_{1\cdot 2} = t_1 \cdot t_2 = t_1 \cdot 0 = 0$$
$$\theta_{1\cdot 2} = \theta_1 + \theta_2 = \theta_1 + indeterminate = indeterminate$$
$$\gamma_{1\cdot 2} = \gamma_1 + \gamma_2 = \gamma_1 + indeterminate = indeterminate$$

proving the thesis. ■

Theorem 2.41. *For the operation of multiplication is defined neuter the complete number $1_{(S)}$, namely for:*

$$o_2(a_2, b_2, c_2)_{(S)} = 1_{(S)}$$

we have:

$$o_1(a_1, b_1, c_1)_{(t_1,\theta_1,\gamma_1)} \cdot o_2(a_2, b_2, c_2)_{(S)} = o_1(a_1, b_1, c_1)_{(t_1,\theta_1,\gamma_1)}$$

Proof. t_1,θ_1,γ_1,t_2,θ_2,γ_2 being real numbers, we can write:

$$t_{1\cdot 2} = t_1 \cdot t_2 = t_1 \cdot 1 = t_1$$
$$\theta_{1\cdot 2} = \theta_1 + \theta_2 = \theta_1 + 0 = \theta_1$$
$$\gamma_{1\cdot 2} = \gamma_1 + \gamma_2 = \gamma_1 + 0 = \gamma_1$$

proving the thesis. ■

Theorem 2.42. *For the operation of multiplication is defined inverse the complete number that identifies the inverse position with respect the origin, namely for:*

$$o_2(t_2,\theta_2,\gamma_2) = o_2(\frac{1}{t_1},-\theta_1,-\gamma_1)$$

we have:

$$o_1(t_1,\theta_1,\gamma_1)\cdot o_2(t_2,\theta_2,\gamma_2) = 1_{(S)}$$

Proof. t_1,θ_1,γ_1,t_2,θ_2,γ_2 being real numbers, we can write:

$$\begin{aligned}
t_{1\cdot 2} &= t_1\cdot t_2 = t_1\cdot\frac{1}{t_1} = 1\\
\theta_{1\cdot 2} &= \theta_1+\theta_2 = \theta_1-\theta_1 = 0\\
\gamma_{1\cdot 2} &= \gamma_1+\gamma_2 = \gamma_1-\gamma_1 = 0
\end{aligned}$$

proving the thesis. ∎

Theorem 2.43. *For the operation of multiplication is valid the commutative property, namely:*

$$o_1(t_1,\theta_1,\gamma_1)\cdot o_2(t_2,\theta_2,\gamma_2) = o_2(t_2,\theta_2,\gamma_2)\cdot o_1(t_1,\theta_1,\gamma_1)$$

Proof. t_1,θ_1,γ_1,t_2,θ_2,γ_2 being real numbers, we can write:

$$\begin{aligned}
t_{1\cdot 2} &= t_1\cdot t_2\\
\theta_{1\cdot 2} &= \theta_1+\theta_2\\
\gamma_{1\cdot 2} &= \gamma_1+\gamma_2
\end{aligned}$$

$$\begin{aligned}
t_{2\cdot 1} &= t_2\cdot t_1 = t_1\cdot t_2\\
\theta_{2\cdot 1} &= \theta_2+\theta_1 = \theta_1+\theta_2\\
\gamma_{2\cdot 1} &= \gamma_2+\gamma_1 = \gamma_1+\gamma_2
\end{aligned}$$

proving the thesis. ∎

Theorem 2.44. *For the operation of multiplication are valid the associative and dissociative properties, namely for:*

$$o_2(t_2,\theta_2,\gamma_2) = o_3(t_3,\theta_3,\gamma_3)+o_4(t_4,\theta_4,\gamma_4)$$

we have:

$$\begin{aligned}
[o_1(t_1,\theta_1,\gamma_1)\cdot o_3(t_3,\theta_3,\gamma_3)]\cdot o_4(t_4,\theta_4,\gamma_4) &= o_1(t_1,\theta_1,\gamma_1)\cdot o_2(t_2,\theta_2,\gamma_2)\\
o_1(t_1,\theta_1,\gamma_1)\cdot o_2(t_2,\theta_2,\gamma_2) &= [o_1(t_1,\theta_1,\gamma_1)\cdot o_3(t_3,\theta_3,\gamma_3)]\cdot o_4(t_4,\theta_4,\gamma_4)
\end{aligned}$$

Proof. t_1,θ_1,γ_1,t_2,θ_2,γ_2,t_3,θ_3,γ_3,t_4,θ_4,γ_4 being real numbers, we can write:

$$\begin{aligned}
t_{(1\cdot 3)\cdot 4} &= (t_1\cdot t_3)\cdot t_4 = t_1\cdot(t_3\cdot t_4)\\
\theta_{(1\cdot 3)\cdot 4} &= (\theta_1+\theta_3)+\theta_4 = \theta_1+(\theta_3+\theta_4)\\
\gamma_{(1\cdot 3)\cdot 4} &= (\gamma_1+\gamma_3)+\gamma_4 = \gamma_1+(\gamma_3+\gamma_4)
\end{aligned}$$

$$t_{1\cdot 2} = t_1 \cdot t_2 = t_1 \cdot (t_3 \cdot t_4)$$
$$\theta_{1\cdot 2} = \theta_1 + \theta_2 = \theta_1 + (\theta_3 + \theta_4)$$
$$\gamma_{1\cdot 2} = \gamma_1 + \gamma_2 = \gamma_1 + (\gamma_3 + \gamma_4)$$

proving the thesis. ∎

Theorem 2.45. *It is not valid the distributive property of multiplication over addition, namely for:*

$$o_2(t_2, \theta_2, \gamma_2) = o_3(t_3, \theta_3, \gamma_3) + o_4(t_4, \theta_4, \gamma_4)$$

we have:

$$o_1(t_1, \theta_1, \gamma_1) \cdot o_2(t_2, \theta_2, \gamma_2) \neq [o_1(t_1, \theta_1, \gamma_1) \cdot o_3(t_3, \theta_3, \gamma_3)] + [o_1(t_1, \theta_1, \gamma_1) \cdot o_4(t_4, \theta_4, \gamma_4)]$$

Proof. Referring to the situation described by theorem 2.31 and considering that a_1,b_1,c_1,a_2,b_2,c_2, a_3,b_3,c_3,a_4,b_4,c_4 are real numbers, we can write:

$$\begin{aligned}
c_{1\cdot 2} &= c_1 \cdot \left|\sqrt{(a_2^2 + b_2^2)}\right| + c2 \cdot \left|\sqrt{(a_1^2 + b_1^2)}\right| \\
c_{(1\cdot 3)+(1\cdot 4)} &= \left[c_1 \cdot \left|\sqrt{(a_3^2 + b_3^2)}\right| + c_3 \cdot \left|\sqrt{(a_1^2 + b_1^2)}\right|\right] \\
&\quad + \left[c_1 \cdot \left|\sqrt{(a_4^2 + b_4^2)}\right| + c_4 \cdot \left|\sqrt{(a_1^2 + b_1^2)}\right|\right] \\
&= c_1 \cdot \left[\left|\sqrt{(a_3^2 + b_3^2)}\right| + \left|\sqrt{(a_4^2 + b_4^2)}\right|\right] + (c_3 + c_4) \cdot \left|\sqrt{(a_1^2 + b_1^2)}\right| \\
&= c_1 \cdot \left[\left|\sqrt{(a_3^2 + b_3^2)}\right| + \left|\sqrt{(a_4^2 + b_4^2)}\right|\right] + c_2 \cdot \left|\sqrt{(a_1^2 + b_1^2)}\right| \neq c_{1\cdot 2}
\end{aligned}$$

proving the thesis. ∎

Theorem 2.46. *It is not valid the distributive property of multiplication over subtraction, namely for:*

$$o_2(t_2, \theta_2, \gamma_2) = o_3(t_3, \theta_3, \gamma_3) - o_4(t_4, \theta_4, \gamma_4)$$

we have:

$$o_1(t_1, \theta_1, \gamma_1) \cdot o_2(t_2, \theta_2, \gamma_2) \neq [o_1(t_1, \theta_1, \gamma_1) \cdot o_3(t_3, \theta_3, \gamma_3)] - [o_1(t_1, \theta_1, \gamma_1) \cdot o_4(t_4, \theta_4, \gamma_4)]$$

Proof. Referring to the situation described by theorem 2.31 and considering that a_1,b_1,c_1,a_2,b_2,c_2, a_3,b_3,c_3,a_4,b_4,c_4 are real numbers, we can write:

$$\begin{aligned}
c_{1\cdot 2} &= c_1 \cdot \left|\sqrt{(a_2^2 + b_2^2)}\right| + c2 \cdot \left|\sqrt{(a_1^2 + b_1^2)}\right| \\
c_{(1\cdot 3)-(1\cdot 4)} &= \left[c_1 \cdot \left|\sqrt{(a_3^2 + b_3^2)}\right| + c3 \cdot \left|\sqrt{(a_1^2 + b_1^2)}\right|\right] \\
&\quad - \left[c_1 \cdot \left|\sqrt{(a_4^2 + b_4^2)}\right| + c4 \cdot \left|\sqrt{(a_1^2 + b_1^2)}\right|\right] \\
&= c_1 \cdot \left[\left|\sqrt{(a_3^2 + b_3^2)}\right| - \left|\sqrt{(a_4^2 + b_4^2)}\right|\right] + (c_3 - c_4) \cdot \left|\sqrt{(a_1^2 + b_1^2)}\right| \\
&= c_1 \cdot \left[\left|\sqrt{(a_3^2 + b_3^2)}\right| - \left|\sqrt{(a_4^2 + b_4^2)}\right|\right] + c_2 \cdot \left|\sqrt{(a_1^2 + b_1^2)}\right| \neq c_{1\cdot 2}
\end{aligned}$$

proving the thesis. ∎

2.5. Divisiton

Definition 2.47. In the space RIU we can define division between two complete numbers $o_1(t_1, \theta_1, \gamma_1)$ and $o_2(t_2, \theta_2, \gamma_2)$ as the number $o_{\frac{1}{2}}(a_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}})$ represented also with the symbol $\frac{o_1(t_1,\theta_1,\gamma_1)}{o_2(t_2,\theta_2,\gamma_2)}$ that satisfies the following conditions:

1. $o_{\frac{1}{2}}(t_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}}) \cdot o_2(t_2, \theta_2, \gamma_2) = o_1(t_1, \theta_1, \gamma_1)$
2. $o_2(t_2, \theta_2, \gamma_2) \neq 0$

The first condition defines the division as the inverse operation of multiplication, and it is equivalent to require that:

$$\begin{aligned} t_{\frac{1}{2}} &= \frac{t_1}{t_2} \\ \theta_{\frac{1}{2}} &= \theta_1 - \theta_2 \\ \gamma_{\frac{1}{2}} &= \gamma_1 - \gamma_2 \end{aligned}$$

The second condition gets its own justification by the necessity of defining the divisions in an univocal way. In fact when that condition is not valid, the expression:

$$o_{\frac{1}{2}}(t_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}}) \cdot 0 = 0$$

besides to require a zero dividend $o_1(t_1, \theta_1, \gamma_1)$ as well, would be satisfied by more values of $o_{\frac{1}{2}}(t_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}})$.

Theorem 2.48. *With $o_1(t_1, \theta_1, \gamma_1)$ and $o_2(t_2, \theta_2, \gamma_2)$ in standard representation, and both not belonging to the line U, their division may be expressed in the following way:*

$$o_{\frac{1}{2}}(a_{\frac{1}{2}}, b_{\frac{1}{2}}, c_{\frac{1}{2}})_{(t_{\frac{1}{2}},\theta_{\frac{1}{2}},\gamma_{\frac{1}{2}})} = a_{\frac{1}{2}(\frac{t_1}{t_2})} + i \cdot b_{\frac{1}{2}(\theta_1-\theta_2)} + u \cdot c_{\frac{1}{2}(\gamma_1-\gamma_2)}$$

where

$$\begin{aligned} a_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (a_1 \cdot a_2 + b_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2 + b_1^2}\right| \cdot \left|\sqrt{a_2^2 + b_2^2}\right|}\right) \\ b_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (b_1 \cdot a_2 - a_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2 + b_1^2}\right| \cdot \left|\sqrt{a_2^2 + b_2^2}\right|}\right) \\ c_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left[c_1 \cdot \left|\sqrt{a_2^2 + b_2^2}\right| - c_2 \cdot \left|\sqrt{a_1^2 + b_1^2}\right|\right] \end{aligned}$$

Proof. The division between two complete numbers, as we know, satisfies the following formula:

$$\begin{aligned} o_{\frac{1}{2}}(t_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}}) = \tfrac{t_1}{t_2} \cdot \{&[\cos(\gamma_1 - \gamma_2) \cdot \cos(\theta_1 - \theta_2)] \\ +i \cdot [\cos(\gamma_1 - \gamma_2) \cdot \sin(\theta_1 - \theta_2)] &+ u \cdot [\sin(\gamma_1 - \gamma_2)]\} \end{aligned}$$

For the moduli and the phases involved will be valid the following relation as well:

$$
\begin{aligned}
t &= \sqrt{a^2+b^2+c^2} \\
\gamma &= \arctan\left(\frac{c}{\sqrt{a^2+b^2}}\right) \\
\theta &= \arctan\left(\frac{b}{a}\right)
\end{aligned}
$$

This means that we can write the coordinates sought in the following way:

$$
\begin{aligned}
a_{\frac{1}{2}} =& \frac{\sqrt{a_1^2+b_1^2+c_1^2}}{\sqrt{a_2^2+b_2^2+c_2^2}} \cdot \cos\left[\arctan\left(\frac{c_1}{\sqrt{a_1^2+b_1^2}}\right) - \arctan\left(\frac{c_2}{\sqrt{a_2^2+b_2^2}}\right)\right] \cdot \\
&\cdot \cos\left[\arctan\left(\frac{b_1}{a_1}\right) - \arctan\left(\frac{b_2}{a_2}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
b_{\frac{1}{2}} =& \frac{\sqrt{a_1^2+b_1^2+c_1^2}}{\sqrt{a_2^2+b_2^2+c_2^2}} \cdot \cos\left[\arctan\left(\frac{c_1}{\sqrt{a_1^2+b_1^2}}\right) - \arctan\left(\frac{c_2}{\sqrt{a_2^2+b_2^2}}\right)\right] \cdot \\
&\cdot \sin\left[\arctan\left(\frac{b_1}{a_1}\right) - \arctan\left(\frac{b_2}{a_2}\right)\right]
\end{aligned}
$$

$$
c_{\frac{1}{2}} = \frac{\sqrt{a_1^2+b_1^2+c_1^2}}{\sqrt{a_2^2+b_2^2+c_2^2}} \cdot \sin\left[\arctan\left(\frac{c_1}{\sqrt{a_1^2+b_1^2}}\right) - \arctan\left(\frac{c_2}{\sqrt{a_2^2+b_2^2}}\right)\right]
$$

To continue with the proof, we have to use the following trigonometric relations:

$$
\begin{aligned}
&\cos(x-y) = \cos(x)\cdot\cos(y) + \sin(x)\cdot\sin(y) \\
&\sin(x-y) = \sin(x)\cdot\cos(y) - \cos(x)\cdot\sin(y) \\
&\cos\left[\arctan\left(\frac{c}{\sqrt{a^2+b^2}}\right)\right] = \sqrt{\frac{a^2+b^2}{a^2+b^2+c^2}} \\
&\sin\left[\arctan\left(\frac{c}{\sqrt{a^2+b^2}}\right)\right] = \sqrt{\frac{c^2}{a^2+b^2+c^2}} \\
&\cos\left[\arctan\left(\frac{b}{a}\right)\right] = \sqrt{\frac{a^2}{a^2+b^2}} \\
&\sin\left[\arctan\left(\frac{b}{a}\right)\right] = \sqrt{\frac{b^2}{a^2+b^2}}
\end{aligned}
$$

To determine the value of the coordinate $a_{\frac{1}{2}}$ the steps to perform will be the following:

$$a_{\frac{1}{2}} = \frac{\sqrt{a_1^2+b_1^2+c_1^2}}{\sqrt{a_2^2+b_2^2+c_2^2}} \cdot \left(\sqrt{\frac{a_1^2+b_1^2}{a_1^2+b_1^2+c_1^2}} \cdot \sqrt{\frac{a_2^2+b_2^2}{a_2^2+b_2^2+c_2^2}} \right.$$

$$\left. + \sqrt{\frac{c_1^2}{a_1^2+b_1^2+c_1^2}} \cdot \sqrt{\frac{c_2^2}{a_2^2+b_2^2+c_2^2}} \right) \cdot \left(\sqrt{\frac{a_1^2}{a_1^2+b_1^2}} \cdot \sqrt{\frac{a_2^2}{a_2^2+b_2^2}} \right.$$

$$\left. + \sqrt{\frac{b_1^2}{a_1^2+b_1^2}} \cdot \sqrt{\frac{b_2^2}{a_2^2+b_2^2}} \right)$$

$$= \frac{1}{a_2^2+b_2^2+c_2^2} \cdot \left(\sqrt{a_1^2+b_1^2} \cdot \sqrt{a_2^2+b_2^2} + \sqrt{c_1^2} \cdot \sqrt{c_2^2} \right) \cdot$$

$$\cdot \left(\frac{\sqrt{a_1^2} \cdot \sqrt{a_2^2} + \sqrt{b_1^2} \cdot \sqrt{b_2^2}}{\sqrt{a_1^2+b_1^2} \cdot \sqrt{a_2^2+b_2^2}} \right)$$

$$= \frac{1}{a_2^2+b_2^2+c_2^2} \cdot \left[\left(\sqrt{a_1^2} \cdot \sqrt{a_2^2} + \sqrt{b_1^2} \cdot \sqrt{b_2^2} \right) \right.$$

$$\left. + \sqrt{c_1^2} \cdot \sqrt{c_2^2} \cdot \left(\frac{\sqrt{a_1^2} \cdot \sqrt{a_2^2} + \sqrt{b_1^2} \cdot \sqrt{b_2^2}}{\sqrt{a_1^2+b_1^2} \cdot \sqrt{a_2^2+b_2^2}} \right) \right]$$

$$= \frac{1}{a_2^2+b_2^2+c_2^2} \cdot \left(\sqrt{a_1^2} \cdot \sqrt{a_2^2} + \sqrt{b_1^2} \cdot \sqrt{b_2^2} \right) \cdot \left(1 + \frac{\sqrt{c_1^2} \cdot \sqrt{c_2^2}}{\sqrt{a_1^2+b_1^2} \cdot \sqrt{a_2^2+b_2^2}} \right)$$

To determine the value of the coordinate $b_{\frac{1}{2}}$ the steps to perform will be the following:

$$b_{\frac{1}{2}} = \frac{\sqrt{a_1^2+b_1^2+c_1^2}}{\sqrt{a_2^2+b_2^2+c_2^2}} \cdot \left(\sqrt{\frac{a_1^2+b_1^2}{a_1^2+b_1^2+c_1^2}} \cdot \sqrt{\frac{a_2^2+b_2^2}{a_2^2+b_2^2+c_2^2}} \right.$$

$$\left. + \sqrt{\frac{c_1^2}{a_1^2+b_1^2+c_1^2}} \cdot \sqrt{\frac{c_2^2}{a_2^2+b_2^2+c_2^2}} \right) \cdot \left(\sqrt{\frac{b_1^2}{a_1^2+b_1^2}} \cdot \sqrt{\frac{a_2^2}{a_2^2+b_2^2}} \right.$$

$$\left. - \sqrt{\frac{a_1^2}{a_1^2+b_1^2}} \cdot \sqrt{\frac{b_2^2}{a_2^2+b_2^2}} \right)$$

$$=\frac{1}{a_2^2+b_2^2+c_2^2}\cdot\left(\sqrt{a_1^2+b_1^2}\cdot\sqrt{a_2^2+b_2^2}+\sqrt{c_1^2}\cdot\sqrt{c_2^2}\right)\cdot$$

$$\cdot\left(\frac{\sqrt{b_1^2}\cdot\sqrt{a_2^2}-\sqrt{a_1^2}\cdot\sqrt{b_2^2}}{\sqrt{a_1^2+b_1^2}\cdot\sqrt{a_2^2+b_2^2}}\right)$$

$$=\frac{1}{a_2^2+b_2^2+c_2^2}\cdot\left[\left(\sqrt{b_1^2}\cdot\sqrt{a_2^2}-\sqrt{a_1^2}\cdot\sqrt{b_2^2}\right)\right.$$

$$\left.+\sqrt{c_1^2}\cdot\sqrt{c_2^2}\cdot\left(\frac{\sqrt{b_1^2}\cdot\sqrt{a_2^2}-\sqrt{a_1^2}\cdot\sqrt{b_2^2}}{\sqrt{a_1^2+b_1^2}\cdot\sqrt{a_2^2+b_2^2}}\right)\right]$$

$$=\frac{1}{a_2^2+b_2^2+c_2^2}\cdot\left(\sqrt{b_1^2}\cdot\sqrt{a_2^2}-\sqrt{a_1^2}\cdot\sqrt{b_2^2}\right)\cdot\left(1+\frac{\sqrt{c_1^2}\cdot\sqrt{c_2^2}}{\sqrt{a_1^2+b_1^2}\cdot\sqrt{a_2^2+b_2^2}}\right)$$

To determine the value of the coordinate $c_{\frac{1}{2}}$ the steps to perform will be the following:

$$c_{\frac{1}{2}}=\frac{\sqrt{a_1^2+b_1^2+c_1^2}}{\sqrt{a_2^2+b_2^2+c_2^2}}\cdot\left(\sqrt{\frac{c_1^2}{a_1^2+b_1^2+c_1^2}}\cdot\sqrt{\frac{a_2^2+b_2^2}{a_2^2+b_2^2+c_2^2}}+\right.$$

$$\left.-\sqrt{\frac{a_1^2+b_1^2}{a_1^2+b_1^2+c_1^2}}\cdot\sqrt{\frac{c_2^2}{a_2^2+b_2^2+c_2^2}}\right)=$$

$$=\frac{1}{a_2^2+b_2^2+c_2^2}\cdot\left[\sqrt{c_1^2}\cdot\sqrt{a_2^2+b_2^2}-\sqrt{c_2^2}\cdot\sqrt{a_1^2+b_1^2}\right]$$

These relations are valid in general, in the precise sense that they are also able to include cases where the coefficients a,b,c are zero (provided that we work with complete numbers not belonging in the line U). The only limitation in this regard is the need to avoid the following situation:

$$a_2^2+b_2^2+c_2^2=0$$

which confirms the impossibility to divide a complete number $o(t,\theta,\gamma)$ for zero (characterized by the values a_2, b_2, c_2 that make the above mentioned condition true).

Wanting to find relations that satisfy the division rule as a function of the effective coordinates of the complete numbers involved, we must assign to the roots the same sign of the coefficient located within them:

$$\sqrt{a^2} = a$$
$$\sqrt{b^2} = b$$
$$\sqrt{c^2} = c$$

The relations obtained will be the following:

$$(2.4)\qquad \begin{aligned} a_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (a_1 \cdot a_2 + b_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{\sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2}}\right) \\ b_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (b_1 \cdot a_2 - a_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{\sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2}}\right) \\ c_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left[c_1 \cdot \sqrt{a_2^2 + b_2^2} + c_2 \cdot \sqrt{a_1^2 + b_1^2}\right] \end{aligned}$$

Since the complete numbers involved are in standard representation, as determined by the theorem 2.11 we must consider the following relations:

$$\sqrt{a_1^2 + b_1^2} = \left|\sqrt{a_1^2 + b_1^2}\,\right|$$
$$\sqrt{a_2^2 + b_2^2} = \left|\sqrt{a_2^2 + b_2^2}\,\right|$$

that combined with those indicated by the formulas (2.4), proving the thesis. ∎

As an example of the theorem just proved, suppose you have to divide the complete numbers in standard representation provided with coordinates: $a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 1$.

Their modulus may be calculated in the following way:

$$t_1 = t_2 = \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

For their phases we should refer to the formulas related to the standard representation:

$$\begin{aligned} \gamma_1 = \gamma_2 &= \arctan\left(\frac{c_1}{\left|\sqrt{a_1^2 + b_1^2}\,\right|}\right) = \arctan\left(\frac{c_2}{\left|\sqrt{a_2^2 + b_2^2}\,\right|}\right) = \\ &= \arctan\left(\frac{1}{|\sqrt{2}|}\right) \simeq 35.26^\circ \\ \theta_1 = \theta_2 &= \arctan\left(\frac{b_1}{a_1}\right) = \arctan\left(\frac{b_2}{a_2}\right) = \arctan\left(\frac{1}{1}\right) = 45^\circ \end{aligned}$$

By applying the division rule we obtain as result the complete number provided with the following values of modulus and phases:

$$t_{\frac{1}{2}} = \frac{t_1}{t_2} = 1$$
$$\gamma_{\frac{1}{2}} = \gamma_1 - \gamma_2 = 0^\circ$$
$$\theta_{\frac{1}{2}} = \theta_1 - \theta_2 = 0^\circ$$

and the following coordinates:

$$\begin{aligned}
a_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \cos\left(\gamma_{\frac{1}{2}}\right) \cdot \cos\left(\theta_{\frac{1}{2}}\right) = 1 \cdot \cos\left(0°\right) \cdot \cos\left(0°\right) = 1\\
b_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \cos\left(\gamma_{\frac{1}{2}}\right) \cdot \sin\left(\theta_{\frac{1}{2}}\right) = 1 \cdot \cos\left(0°\right) \cdot \sin\left(0°\right) = 0\\
c_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \sin\left(\gamma_{\frac{1}{2}}\right) = 1 \cdot \sin\left(0°\right) = 0
\end{aligned}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$\begin{aligned}
a_{\frac{1}{2}} =&\frac{1}{a_2^2+b_2^2+c_2^2} \cdot (a_1 \cdot a_2 + b_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2+b_1^2}\right| \cdot \left|\sqrt{a_2^2+b_2^2}\right|}\right) =\\
=&\frac{1}{3} \cdot (1+1) \cdot \left(1+\frac{1}{2}\right) = 1\\
b_{\frac{1}{2}} =&\frac{1}{a_2^2+b_2^2+c_2^2} \cdot (b_1 \cdot a_2 - a_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2+b_1^2}\right| \cdot \left|\sqrt{a_2^2+b_2^2}\right|}\right) =\\
=&\frac{1}{3} \cdot (1-1) \cdot \left(1+\frac{1}{2}\right) = 0\\
c_{\frac{1}{2}} =&\frac{1}{a_2^2+b_2^2+c_2^2} \cdot \left[c_1 \cdot \left|\sqrt{a_2^2+b_2^2}\right| - c_2 \cdot \left|\sqrt{a_1^2+b_1^2}\right|\right] =\\
=&\frac{1}{3} \cdot [1 \cdot \sqrt{2} - 1 \cdot \sqrt{2}] = 0
\end{aligned}$$

Theorem 2.49. *With $o_1(t_1, \theta_1, \gamma_1)$ and $o_2(t_2, \theta_2, \gamma_2)$ in complementary representation, and both not belonging to the line U, their division may be expressed in the following way:*

$$o_{\frac{1}{2}}(a_{\frac{1}{2}}, b_{\frac{1}{2}}, c_{\frac{1}{2}})_{(t_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}})} = a_{\frac{1}{2}\left(\frac{t_1}{t_2}\right)} + i \cdot b_{\frac{1}{2}(\theta_1-\theta_2)} + u \cdot c_{\frac{1}{2}(\gamma_1-\gamma_2)}$$

where

$$\begin{aligned}
a_{\frac{1}{2}} &= \frac{1}{a_2^2+b_2^2+c_2^2} \cdot (a_1 \cdot a_2 + b_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2+b_1^2}\right| \cdot \left|\sqrt{a_2^2+b_2^2}\right|}\right)\\
b_{\frac{1}{2}} &= \frac{1}{a_2^2+b_2^2+c_2^2} \cdot (b_1 \cdot a_2 - a_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2+b_1^2}\right| \cdot \left|\sqrt{a_2^2+b_2^2}\right|}\right)\\
c_{\frac{1}{2}} &= \frac{1}{a_2^2+b_2^2+c_2^2} \cdot \left[c_2 \cdot \left|\sqrt{a_1^2+b_1^2}\right| - c_1 \cdot \left|\sqrt{a_2^2+b_2^2}\right|\right]
\end{aligned}$$

Proof. Since the complete numbers involved are in complementary representation, as determined by the theorem 2.15 we must consider the following relations:

$$\begin{aligned}
\sqrt{a_1^2+b_1^2} &= -\left|\sqrt{a_1^2+b_1^2}\right|\\
\sqrt{a_2^2+b_2^2} &= -\left|\sqrt{a_2^2+b_2^2}\right|
\end{aligned}$$

that combined with those indicated by the formulas (2.4), proving the thesis. ∎

As an example of the theorem just proved, suppose you have to divide the complete numbers in complementary representation provided with coordinates: $a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 1$.

Their modulus may be calculated in the following way:

$$t_1 = t_2 = \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

For their phases we should refer to the formulas related to the complementary representation:

$$\begin{aligned}\gamma_1 = \gamma_2 &= \arctan\left(\frac{c_1}{-\left|\sqrt{a_1^2+b_1^2}\,\right|}\right) = \arctan\left(\frac{c_2}{-\left|\sqrt{a_2^2+b_2^2}\,\right|}\right) = \\ &= \arctan\left(\frac{1}{-|\sqrt{2}|}\right) \simeq 144.73^\circ \\ \theta_1 = \theta_2 &= \arctan\left(\frac{-b_1}{-a_1}\right) = \arctan\left(\frac{-b_2}{-a_2}\right) = \arctan\left(\frac{-1}{-1}\right) = 225^\circ\end{aligned}$$

By applying the division rule we obtain as result the complete number provided with the following values of modulus and phases:

$$\begin{aligned}t_{\frac{1}{2}} &= \frac{t_1}{t_2} = 1 \\ \gamma_{\frac{1}{2}} &= \gamma_1 - \gamma_2 = 0^\circ \\ \theta_{\frac{1}{2}} &= \theta_1 - \theta_2 = 0^\circ\end{aligned}$$

and the following coordinates:

$$\begin{aligned}a_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \cos(\theta_{\frac{1}{2}}) = 1 \cdot \cos(0^\circ) \cdot \cos(0^\circ) = 1 \\ b_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \sin(\theta_{\frac{1}{2}}) = 1 \cdot \cos(0^\circ) \cdot \sin(0^\circ) = 0 \\ c_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \sin(\gamma_{\frac{1}{2}}) = 1 \cdot \sin(0^\circ) = 0\end{aligned}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$\begin{aligned}a_{\frac{1}{2}} &= \frac{1}{a_2^2+b_2^2+c_2^2} \cdot (a_1 \cdot a_2 + b_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2+b_1^2}\,\right| \cdot \left|\sqrt{a_2^2+b_2^2}\,\right|}\right) \\ &= \frac{1}{3} \cdot (1+1) \cdot \left(1 + \frac{1}{2}\right) = 1 \\ b_{\frac{1}{2}} &= \frac{1}{a_2^2+b_2^2+c_2^2} \cdot (b_1 \cdot a_2 - a_1 \cdot b_2) \cdot \left(1 + \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2+b_1^2}\,\right| \cdot \left|\sqrt{a_2^2+b_2^2}\,\right|}\right) \\ &= \frac{1}{3} \cdot (1-1) \cdot \left(1 + \frac{1}{2}\right) = 0 \\ c_{\frac{1}{2}} &= \frac{1}{a_2^2+b_2^2+c_2^2} \cdot \left[c_2 \cdot \left|\sqrt{a_1^2+b_1^2}\,\right| - c_1 \cdot \left|\sqrt{a_2^2+b_2^2}\,\right|\right] \\ &= \frac{1}{3} \cdot [1 \cdot \sqrt{2} - 1 \cdot \sqrt{2}] = 0\end{aligned}$$

Theorem 2.50. *With $o_1(t_1, \theta_1, \gamma_1)$ in standard representation and $o_2(t_2, \theta_2, \gamma_2)$ in complementary representation, and both not belonging to the line U, their division may be expressed in the following way:*

$$o_{\frac{1}{2}}(a_{\frac{1}{2}}, b_{\frac{1}{2}}, c_{\frac{1}{2}})_{(t_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}})} = a_{\frac{1}{2}(\frac{t_1}{t_2})} + i \cdot b_{\frac{1}{2}(\theta_1 - \theta_2)} + u \cdot c_{\frac{1}{2}(\gamma_1 - \gamma_2)}$$

where

$$a_{\frac{1}{2}} = \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (a_1 \cdot a_2 + b_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2 + b_1^2}\right| \cdot \left|\sqrt{a_2^2 + b_2^2}\right|}\right)$$

$$b_{\frac{1}{2}} = \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (b_1 \cdot a_2 - a_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2 + b_1^2}\right| \cdot \left|\sqrt{a_2^2 + b_2^2}\right|}\right)$$

$$c_{\frac{1}{2}} = \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left[-c_1 \cdot \left|\sqrt{a_2^2 + b_2^2}\right| - c_2 \cdot \left|\sqrt{a_1^2 + b_1^2}\right|\right]$$

Proof. Since the dividend is in standard representation, as determined by the theorem 2.11 we must consider the following relation:

$$\sqrt{a_1^2 + b_1^2} = \left|\sqrt{a_1^2 + b_1^2}\right|$$

while being the divisor in complementary representation, as determined by the theorem 2.15 we must consider the following relation:

$$\sqrt{a_2^2 + b_2^2} = -\left|\sqrt{a_2^2 + b_2^2}\right|$$

that combined with those indicated by the formulas (2.4), proving the thesis. ■

As an example of the theorem just proved, suppose you have to divide the complete number in standard representation provided with coordinates $a_1 = b_1 = c_1 = 1$ by that in complementary representation provided with the same coordinates coordinates: $a_2 = b_2 = c_2 = 1$.

Their modulus may be calculated in the following way:

$$t_1 = t_2 = \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

For their phases we should refer to the formulas related to the standard and complementary representations:

$$\gamma_1 = \arctan\left(\frac{c_1}{\left|\sqrt{a_1^2 + b_1^2}\right|}\right) = \arctan\left(\frac{1}{|\sqrt{2}|}\right) \simeq 35.26°$$

$$\gamma_2 = \arctan\left(\frac{c_2}{-\left|\sqrt{a_2^2 + b_2^2}\right|}\right) = \arctan\left(\frac{1}{-|\sqrt{2}|}\right) \simeq 144.73°$$

$$\theta_1 = \arctan\left(\frac{b_1}{a_1}\right) = \arctan\left(\frac{1}{1}\right) = 45°$$

$$\theta_2 = \arctan\left(\frac{-b_2}{-a_2}\right) = \arctan\left(\frac{-1}{-1}\right) = 225°$$

By applying the division rule we obtain as result the complete number provided with the following values of modulus and phases:

$$t_{\frac{1}{2}} = \frac{t_1}{t_2} = 1$$
$$\gamma_{\frac{1}{2}} = \gamma_1 - \gamma_2 \simeq -109.47°$$
$$\theta_{\frac{1}{2}} = \theta_1 - \theta_2 = -180°$$

and the following coordinates:

$$a_{\frac{1}{2}} = t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \cos(\theta_{\frac{1}{2}}) = 1 \cdot \cos(\simeq -109.47°) \cdot \cos(-180°) = \frac{1}{3}$$
$$b_{\frac{1}{2}} = t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \sin(\theta_{\frac{1}{2}}) = 1 \cdot \cos(\simeq -109.47°) \cdot \sin(-180°) = 0$$
$$c_{\frac{1}{2}} = t_{\frac{1}{2}} \cdot \sin(\gamma_{\frac{1}{2}}) = 1 \cdot \sin(\simeq -109.47°) = \frac{-2 \cdot \sqrt{2}}{3}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$a_{\frac{1}{2}} = \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (a_1 \cdot a_2 + b_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2 + b_1^2}\right| \cdot \left|\sqrt{a_2^2 + b_2^2}\right|}\right) =$$
$$= \frac{1}{3} \cdot (1+1) \cdot \left(1 - \frac{1}{2}\right) = \frac{1}{3}$$
$$b_{\frac{1}{2}} = \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (b_1 \cdot a_2 - a_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2 + b_1^2}\right| \cdot \left|\sqrt{a_2^2 + b_2^2}\right|}\right) =$$
$$= \frac{1}{3} \cdot (1-1) \cdot \left(1 - \frac{1}{2}\right) = 0$$
$$c_{\frac{1}{2}} = \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left[-c_1 \cdot \left|\sqrt{a_2^2 + b_2^2}\right| - c_2 \cdot \left|\sqrt{a_1^2 + b_1^2}\right|\right] =$$
$$= \frac{1}{3} \cdot [-1 \cdot \sqrt{2} - 1 \cdot \sqrt{2}] = \frac{-2 \cdot \sqrt{2}}{3}$$

Theorem 2.51. *With $o_1(t_1, \theta_1, \gamma_1)$ in complementary representation and $o_2(t_2, \theta_2, \gamma_2)$ in standard representation, and both not belonging to the line U, their division may be expressed in the following way:*

$$o_{\frac{1}{2}}(a_{\frac{1}{2}}, b_{\frac{1}{2}}, c_{\frac{1}{2}})_{(t_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}})} = a_{\frac{1}{2}(\frac{t_1}{t_2})} + i \cdot b_{\frac{1}{2}(\theta_1 - \theta_2)} + u \cdot c_{\frac{1}{2}(\gamma_1 - \gamma_2)}$$

where:

$$a_{\frac{1}{2}} = \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (a_1 \cdot a_2 + b_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2 + b_1^2}\right| \cdot \left|\sqrt{a_2^2 + b_2^2}\right|}\right)$$
$$b_{\frac{1}{2}} = \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (b_1 \cdot a_2 - a_1 \cdot b_2) \cdot \left(1 - \frac{c_1 \cdot c_2}{\left|\sqrt{a_1^2 + b_1^2}\right| \cdot \left|\sqrt{a_2^2 + b_2^2}\right|}\right)$$
$$c_{\frac{1}{2}} = \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot \left[c_1 \cdot \left|\sqrt{a_2^2 + b_2^2}\right| + c_2 \cdot \left|\sqrt{a_1^2 + b_1^2}\right|\right]$$

Proof. Since the dividend is in complementary representation, as determined by the theorem 2.15 we must consider the following relation:

$$\sqrt{a_1^2+b_1^2} = -\left|\sqrt{a_1^2+b_1^2}\,\right|$$

while being the divisor in standard representation, as determined by the theorem 2.11 we must consider the following relation:

$$\sqrt{a_2^2+b_2^2} = \left|\sqrt{a_2^2+b_2^2}\,\right|$$

that combined with those indicated by the formulas (2.4), proving the thesis. ∎

As an example of the theorem just proved, suppose you have to divide the complete number in complementary representation provided with coordinates $a_1 = b_1 = c_1 = 1$ by that in standard representation provided with the same coordinates coordinates: $a_2 = b_2 = c_2 = 1$.

Their modulus may be calculated in the following way:

$$t_1 = t_2 = \sqrt{a_1^2+b_1^2+c_1^2} = \sqrt{a_2^2+b_2^2+c_2^2} = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

For their phases we should refer to the formulas related to the complementary and standard representations:

$$\begin{aligned}
\gamma_1 &= \arctan\left(\frac{c_1}{-\left|\sqrt{a_1^2+b_1^2}\,\right|}\right) = \arctan\left(\frac{1}{-|\sqrt{2}|}\right) \simeq 144.73^\circ \\
\gamma_2 &= \arctan\left(\frac{c_2}{\left|\sqrt{a_2^2+b_2^2}\,\right|}\right) = \arctan\left(\frac{1}{|\sqrt{2}|}\right) \simeq 35.26^\circ \\
\theta_1 &= \arctan\left(\frac{-b_1}{-a_1}\right) = \arctan\left(\frac{-1}{-1}\right) = 225^\circ \\
\theta_2 &= \arctan\left(\frac{b_2}{a_2}\right) = \arctan\left(\frac{1}{1}\right) = 45^\circ
\end{aligned}$$

By applying the division rule we obtain as result the complete number provided with the following values of modulus and phases:

$$\begin{aligned}
t_{\frac{1}{2}} &= \frac{t_1}{t_2} = 1 \\
\gamma_{\frac{1}{2}} &= \gamma_1 - \gamma_2 \simeq 109.47^\circ \\
\theta_{\frac{1}{2}} &= \theta_1 - \theta_2 = 180^\circ
\end{aligned}$$

and the following coordinates:

$$\begin{aligned}
a_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \cos(\theta_{\frac{1}{2}}) = 1 \cdot \cos(\simeq 109.47^\circ) \cdot \cos(180^\circ) = \frac{1}{3} \\
b_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \sin(\theta_{\frac{1}{2}}) = 1 \cdot \cos(\simeq 109.47^\circ) \cdot \sin(180^\circ) = 0 \\
c_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \sin(\gamma_{\frac{1}{2}}) = 1 \cdot \sin(\simeq 109.47^\circ) = \frac{2 \cdot \sqrt{2}}{3}
\end{aligned}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$\begin{aligned}
a_{\frac{1}{2}} =& \frac{1}{a_2^2+b_2^2+c_2^2}\cdot(a_1\cdot a_2+b_1\cdot b_2)\cdot\left(1-\frac{c_1\cdot c_2}{\left|\sqrt{a_1^2+b_1^2}\right|\cdot\left|\sqrt{a_2^2+b_2^2}\right|}\right)\\
=&\frac{1}{3}\cdot(1+1)\cdot\left(1-\frac{1}{2}\right)=\frac{1}{3}\\
b_{\frac{1}{2}} =& \frac{1}{a_2^2+b_2^2+c_2^2}\cdot(b_1\cdot a_2-a_1\cdot b_2)\cdot\left(1-\frac{c_1\cdot c_2}{\left|\sqrt{a_1^2+b_1^2}\right|\cdot\left|\sqrt{a_2^2+b_2^2}\right|}\right)\\
=&\frac{1}{3}\cdot(1-1)\cdot\left(1-\frac{1}{2}\right)=0\\
c_{\frac{1}{2}} =& \frac{1}{a_2^2+b_2^2+c_2^2}\cdot\left[c_1\cdot\left|\sqrt{a_2^2+b_2^2}\right|+c_2\cdot\left|\sqrt{a_1^2+b_1^2}\right|\right]\\
=&\frac{1}{3}\cdot[1\cdot\sqrt{2}+1\cdot\sqrt{2}]=\frac{2\cdot\sqrt{2}}{3}
\end{aligned}$$

Theorem 2.52. *With only $o_1(t_1,\theta_1,\gamma_1)$ belonging to the line U and $o_2(t_2,\theta_2,\gamma_2)$ in standard representation, their division may be expressed in the following way:*

$$o_{\frac{1}{2}}(a_{\frac{1}{2}},b_{\frac{1}{2}},c_{\frac{1}{2}})_{(t_{\frac{1}{2}},\theta_{\frac{1}{2}},\gamma_{\frac{1}{2}})}=a_{\frac{1}{2}(\frac{t_1}{t_2})}+i\cdot b_{\frac{1}{2}(\theta_1-\theta_2)}+u\cdot c_{\frac{1}{2}(\gamma_1-\gamma_2)}$$

where:

$$\begin{aligned}
a_{\frac{1}{2}} &= \frac{1}{a_2^2+b_2^2+c_2^2}\cdot(c_1\cdot c_2)\cdot\frac{a_2\cdot\cos(\theta_1)+b_2\cdot\sin(\theta_1)}{\left|\sqrt{a_2^2+b_2^2}\right|}\\
b_{\frac{1}{2}} &= \frac{1}{a_2^2+b_2^2+c_2^2}\cdot(c_1\cdot c_2)\cdot\frac{a_2\cdot\sin(\theta_1)-b_2\cdot\cos(\theta_1)}{\left|\sqrt{a_2^2+b_2^2}\right|}\\
c_{\frac{1}{2}} &= \frac{1}{a_2^2+b_2^2+c_2^2}\cdot c_1\cdot\left|\sqrt{a_2^2+b_2^2}\right|
\end{aligned}$$

Proof. The division between two complete numbers, as we know, satisfies the following formula:

$$\begin{aligned}
o_{\frac{1}{2}}(t_{\frac{1}{2}},\theta_{\frac{1}{2}},\gamma_{\frac{1}{2}}) = \frac{t_1}{t_2}\cdot\{&[\cos(\gamma_1-\gamma_2)\cdot\cos(\theta_1-\theta_2)]\\
&+i\cdot[\cos(\gamma_1-\gamma_2)\cdot\sin(\theta_1-\theta_2)]+u\cdot[\sin(\gamma_1-\gamma_2)]\}
\end{aligned}$$

Since $o_1(t_1,\theta_1,\gamma_1)$ belongs to the line U will be provided with the following values of modulus and phases:

$$\begin{aligned}
t_1 &= \sqrt{c_1^2}\\
\gamma_1 &= \text{sign}(c_1)\cdot 90^\circ\\
\theta_1 &\text{ known } \neq \arctan\left(\frac{b_1}{a_1}\right)
\end{aligned}$$

unlike $o_2(t_2, \theta_2, \gamma_2)$ that will be provided with the following values:

$$t_2 = \sqrt{a_2^2 + b_2^2 + c_2^2}$$
$$\gamma_2 = \arctan\left(\frac{c_2}{\sqrt{a_2^2 + b_2^2}}\right)$$
$$\theta_2 = \arctan\left(\frac{b_2}{a_2}\right)$$

This means that we can write the coordinates sought in the following way:

$$a_{\frac{1}{2}} = \frac{\sqrt{c_1^2}}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \cdot \cos\left[\operatorname{sign}(c_1) \cdot 90^\circ - \arctan\left(\frac{c_2}{\sqrt{a_2^2 + b_2^2}}\right)\right] \cdot \cos\left[\theta_1 - \arctan\left(\frac{b_2}{a_2}\right)\right]$$
$$b_{\frac{1}{2}} = \frac{\sqrt{c_1^2}}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \cdot \cos\left[\operatorname{sign}(c_1) \cdot 90^\circ - \arctan\left(\frac{c_2}{\sqrt{a_2^2 + b_2^2}}\right)\right] \cdot \sin\left[\theta_1 - \arctan\left(\frac{b_2}{a_2}\right)\right]$$
$$c_{\frac{1}{2}} = \frac{\sqrt{c_1^2}}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \cdot \sin\left[\operatorname{sign}(c_1) \cdot 90^\circ - \arctan\left(\frac{c_2}{\sqrt{a_2^2 + b_2^2}}\right)\right]$$

To continue with the proof, we have to use the following trigonometric relations:

$$\cos(x - y) = \cos(x) \cdot \cos(y) + \sin(x) \cdot \sin(y)$$
$$\sin(x - y) = \sin(x) \cdot \cos(y) - \cos(x) \cdot \sin(y)$$
$$\cos\left[\arctan\left(\frac{c}{\sqrt{a^2 + b^2}}\right)\right] = \sqrt{\frac{a^2 + b^2}{a^2 + b^2 + c^2}}$$
$$\sin\left[\arctan\left(\frac{c}{\sqrt{a^2 + b^2}}\right)\right] = \sqrt{\frac{c^2}{a^2 + b^2 + c^2}}$$
$$\cos\left[\arctan\left(\frac{b}{a}\right)\right] = \sqrt{\frac{a^2}{a^2 + b^2}}$$
$$\sin\left[\arctan\left(\frac{b}{a}\right)\right] = \sqrt{\frac{b^2}{a^2 + b^2}}$$
$$\cos[\operatorname{sign}(x) \cdot 90^\circ - y] = \operatorname{sign}(x) \cdot sin(y)$$
$$\sin[\operatorname{sign}(x) \cdot 90^\circ - y] = \operatorname{sign}(x) \cdot cos(y)$$

To determine the value of the coordinate $a_{\frac{1}{2}}$ the steps to perform will be the following:

$$a_{\frac{1}{2}} = \operatorname{sign}(c_1) \cdot \frac{\sqrt{c_1^2}}{\sqrt{a_2^2+b_2^2+c_2^2}} \cdot \sqrt{\frac{c_2^2}{a_2^2+b_2^2+c_2^2}} \cdot$$
$$\cdot \left[\sqrt{\frac{a_2^2}{a_2^2+b_2^2}} \cdot \cos(\theta_1) + \sqrt{\frac{b_2^2}{a_2^2+b_2^2}} \cdot \sin(\theta_1) \right]$$
$$= \frac{1}{a_2^2+b_2^2+c_2^2} \cdot \operatorname{sign}(c_1) \cdot \sqrt{c_1^2} \cdot \sqrt{c_2^2} \cdot \frac{\sqrt{a_2^2} \cdot \cos(\theta_1) + \sqrt{b_2^2} \cdot \sin(\theta_1)}{\sqrt{a_2^2+b_2^2}}$$

To determine the value of the coordinate $b_{\frac{1}{2}}$ the steps to perform will be the following:

$$b_{\frac{1}{2}} = \operatorname{sign}(c_1) \cdot \frac{\sqrt{c_1^2}}{\sqrt{a_2^2+b_2^2+c_2^2}} \cdot \sqrt{\frac{c_2^2}{a_2^2+b_2^2+c_2^2}} \cdot$$
$$\cdot \left[\sqrt{\frac{a_2^2}{a_2^2+b_2^2}} \cdot \sin(\theta_1) - \sqrt{\frac{b_2^2}{a_2^2+b_2^2}} \cdot \cos(\theta_1) \right]$$
$$= \frac{1}{a_2^2+b_2^2+c_2^2} \cdot \operatorname{sign}(c_1) \cdot \sqrt{c_1^2} \cdot \sqrt{c_2^2} \cdot \frac{\sqrt{a_2^2} \cdot \sin(\theta_1) - \sqrt{b_2^2} \cdot \cos(\theta_1)}{\sqrt{a_2^2+b_2^2}}$$

To determine the value of the coordinate $c_{\frac{1}{2}}$ the steps to perform will be the following:

$$c_{\frac{1}{2}} = \operatorname{sign}(c_1) \cdot \frac{\sqrt{c_1^2}}{\sqrt{a_2^2+b_2^2+c_2^2}} \cdot \sqrt{\frac{a_2^2+b_2^2}{a_2^2+b_2^2+c_2^2}}$$
$$= \frac{1}{a_2^2+b_2^2+c_2^2} \cdot \operatorname{sign}(c_1) \cdot \sqrt{c_1^2} \cdot \sqrt{a_2^2+b_2^2}$$

These relations are valid in general, in the precise sense that they are also able to include cases where the coefficients a_2,b_2,c_2 are zero (provided that $o_2(a_2, b_2, c_2)$ remains in the context of the complete numbers not belonging in the line U).

The only limitation in this regard is the need to avoid the following situation:

$$a_2^2 + b_2^2 + c_2^2 = 0$$

which confirms the impossibility to divide a complete number $o(t, \theta, \gamma)$ for zero (characterized by the values a_2, b_2, c_2 that make the above mentioned condition true).

Wanting to find relations that satisfy the division rule as a function of the effective coordinates of the complete numbers involved, we must adopt for the coefficients a,b,c the convention $\sqrt{x^2} = x$, with the exception of c_1 for which we should adopt the convention $\sqrt{x^2} = |x|$. The reason is simple because if we adopt for c_1 the usual convention, we will have:

$$\operatorname{sign}(c_1) \cdot \sqrt{c_1^2} = |c_1|$$

and therefore a result of the division that depends on the modulus of the coordinate c1. While adopting $\sqrt{x^2} = |x|$ we will have:

$$\text{sign}\,(c_1) \cdot \sqrt{c_1^2} = c_1$$

and therefore a result of the division that depends on the effective value of this coordinate.

The relations obtained will be the following:

$$
\begin{aligned}
a_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (c_1 \cdot c_2) \cdot \frac{a_2 \cdot \cos(\theta_1) + b_2 \cdot \sin(\theta_1)}{\sqrt{a_2^2 + b_2^2}} \\
b_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (c_1 \cdot c_2) \cdot \frac{a_2 \cdot \sin(\theta_1) - b_2 \cdot \cos(\theta_1)}{\sqrt{a_2^2 + b_2^2}} \\
c_{\frac{1}{2}} &= \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot c_1 \cdot \sqrt{a_2^2 + b_2^2}
\end{aligned}
\tag{2.5}
$$

Since the number $o_2(a_2, b_2, c_2)$ is in standard representation, as determined by the theorem 2.11 we must consider the following relation:

$$\sqrt{a_2^2 + b_2^2} = \left|\sqrt{a_2^2 + b_2^2}\,\right|$$

that combined with those indicated by the formulas (2.5), proving the thesis. ∎

As an example of the theorem just proved, suppose you have to divide the outgoing numbers of coordinate: $c_1 = 1$ and phase $\theta_1 = 30°$ by a complete number in standard representation provided with coordinates: $a_2 = 1$, $b_2 = -1$, $c_2 = 1$.

Their modulus may be calculated in the following way:

$$
\begin{aligned}
t_1 &= \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{c_1^2} = \sqrt{1} = 1 \\
t_2 &= \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}
\end{aligned}
$$

For their phases in the case of the outgoing number we have:

$$
\begin{aligned}
\gamma_1 &= \text{sign}\,(c_1) \cdot 90° = 90° \\
\theta_1 &= 30°
\end{aligned}
$$

while in the case of the complete number we should refer to the formulas related to the standard representation:

$$
\begin{aligned}
\gamma_2 &= \arctan\left(\frac{c_2}{\left|\sqrt{a_2^2 + b_2^2}\,\right|}\right) = \arctan\left(\frac{1}{|\sqrt{2}|}\right) \simeq 35.26° \\
\theta_2 &= \arctan\left(\frac{b_2}{a_2}\right) = \arctan\left(\frac{-1}{1}\right) = -45°
\end{aligned}
$$

By applying the division rule we obtain as result the complete number provided with the following values of modulus and phases:

$$t_{\frac{1}{2}} = \frac{t_1}{t_2} = \frac{1}{\sqrt{3}}$$
$$\gamma_{\frac{1}{2}} = \gamma_1 - \gamma_2 \simeq 54.74°$$
$$\theta_{\frac{1}{2}} = \theta_1 - \theta_2 = 75°$$

and the following coordinates:

$$a_{\frac{1}{2}} = t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \cos(\theta_{\frac{1}{2}}) = \frac{1}{\sqrt{3}} \cdot \cos(\simeq 54.74°) \cdot \cos(75°) \simeq 0.09$$
$$b_{\frac{1}{2}} = t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \sin(\theta_{\frac{1}{2}}) = \frac{1}{\sqrt{3}} \cdot \cos(\simeq 54.74°) \cdot \sin(75°) \simeq 0.32$$
$$c_{\frac{1}{2}} = t_{\frac{1}{2}} \cdot \sin(\gamma_{\frac{1}{2}}) = \frac{1}{\sqrt{3}} \cdot \sin(\simeq 54.74°) = \frac{\sqrt{2}}{3}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$\begin{aligned}
a_{\frac{1}{2}} =& \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (c_1 \cdot c_2) \cdot \frac{a_2 \cdot \cos(\theta_1) + b_2 \cdot \sin(\theta_1)}{|\sqrt{a_2^2 + b_2^2}\,|} = \\
=& \frac{1}{3} \cdot 1 \cdot \frac{\cos(30°) - \sin(30°)}{|\sqrt{2}\,|} \simeq 0.09 \\
b_{\frac{1}{2}} =& \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (c_1 \cdot c_2) \cdot \frac{a_2 \cdot \sin(\theta_1) - b_2 \cdot \cos(\theta_1)}{|\sqrt{a_2^2 + b_2^2}\,|} = \\
=& \frac{1}{3} \cdot 1 \cdot \frac{\sin(30°) + \cos(30°)}{|\sqrt{2}\,|} \simeq 0.32 \\
c_{\frac{1}{2}} =& \frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot c_1 \cdot \left|\sqrt{a_2^2 + b_2^2}\right| = \frac{1}{3} \cdot 1 \cdot |\sqrt{2}\,| = \frac{\sqrt{2}}{3}
\end{aligned}$$

Theorem 2.53. *With only $o_1(t_1, \theta_1, \gamma_1)$ belonging to the line U and $o_2(t_2, \theta_2, \gamma_2)$ in complementary representation, their division may be expressed in the following way:*

$$o_{\frac{1}{2}}(a_{\frac{1}{2}}, b_{\frac{1}{2}}, c_{\frac{1}{2}})_{(t_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}})} = a_{\frac{1}{2}(\frac{t_1}{t_2})} + i \cdot b_{\frac{1}{2}(\theta_1 - \theta_2)} + u \cdot c_{\frac{1}{2}(\gamma_1 - \gamma_2)}$$

where:

$$a_{\frac{1}{2}} = -\frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (c_1 \cdot c_2) \cdot \frac{a_2 \cdot \cos(\theta_1) + b_2 \cdot \sin(\theta_1)}{|\sqrt{a_2^2 + b_2^2}\,|}$$
$$b_{\frac{1}{2}} = -\frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot (c_1 \cdot c_2) \cdot \frac{a_2 \cdot \sin(\theta_1) - b_2 \cdot \cos(\theta_1)}{|\sqrt{a_2^2 + b_2^2}\,|}$$
$$c_{\frac{1}{2}} = -\frac{1}{a_2^2 + b_2^2 + c_2^2} \cdot c_1 \cdot \left|\sqrt{a_2^2 + b_2^2}\right|$$

Proof. Since the number $o_2(a_2, b_2, c_2)$ is in complementary representation, as determined by the theorem 2.15 we must consider the following relation:

$$\sqrt{a_2^2 + b_2^2} = -\left|\sqrt{a_2^2 + b_2^2}\,\right|$$

that combined with those indicated by the formulas (2.5), proving the thesis. ■

As an example of the theorem just proved, suppose you have to divide the outgoing numbers of coordinate: $c_1 = 1$ and phase $\theta_1 = 30°$ by a complete number in complementary representation provided with coordinates: $a_2 = 1$, $b_2 = -1$, $c_2 = 1$.

Their modulus may be calculated in the following way:

$$t_1 = \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{c_1^2} = \sqrt{1} = 1$$
$$t_2 = \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$$

For their phases in the case of the outgoing number we have:

$$\gamma_1 = \text{ sign }(c_1) \cdot 90° = 90°$$
$$\theta_1 = 30°$$

while in the case of the complete number we should refer to the formulas related to the complementary representation:

$$\gamma_2 = \arctan\left(\frac{c_2}{-\left|\sqrt{a_2^2 + b_2^2}\,\right|}\right) = \arctan\left(\frac{1}{-|\sqrt{2}|}\right) \simeq 144.74°$$
$$\theta_2 = \arctan\left(\frac{-b_2}{-a_2}\right) = \arctan\left(\frac{1}{-1}\right) = 135°$$

By applying the division rule we obtain as result the complete number provided with the following values of modulus and phases:

$$t_{\frac{1}{2}} = \frac{t_1}{t_2} = \frac{1}{\sqrt{3}}$$
$$\gamma_{\frac{1}{2}} = \gamma_1 - \gamma_2 \simeq -54.74°$$
$$\theta_{\frac{1}{2}} = \theta_1 - \theta_2 = -105°$$

and the following coordinates:

$$a_{\frac{1}{2}} = t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \cos(\theta_{\frac{1}{2}}) = \frac{1}{\sqrt{3}} \cdot \cos(\simeq -54.74°) \cdot \cos(-105°) \simeq -0.09$$
$$b_{\frac{1}{2}} = t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \sin(\theta_{\frac{1}{2}}) = \frac{1}{\sqrt{3}} \cdot \cos(\simeq -54.74°) \cdot \sin(-105°) \simeq -0.32$$
$$c_{\frac{1}{2}} = t_{\frac{1}{2}} \cdot \sin(\gamma_{\frac{1}{2}}) = \frac{1}{\sqrt{3}} \cdot \sin(\simeq -54.74°) = -\frac{\sqrt{2}}{3}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$\begin{aligned}
a_{\frac{1}{2}} &= -\frac{1}{a_2^2+b_2^2+c_2^2}\cdot(c_1\cdot c_2)\cdot\frac{a_2\cdot\cos(\theta_1)+b_2\cdot\sin(\theta_1)}{|\sqrt{a_2^2+b_2^2}\,|} = \\
&= -\frac{1}{3}\cdot 1\cdot\frac{\cos(30^\circ)-\sin(30^\circ)}{|\sqrt{2}\,|} \simeq -0.09 \\
b_{\frac{1}{2}} &= -\frac{1}{a_2^2+b_2^2+c_2^2}\cdot(c_1\cdot c_2)\cdot\frac{a_2\cdot\sin(\theta_1)-b_2\cdot\cos(\theta_1)}{|\sqrt{a_2^2+b_2^2}\,|} = \\
&= -\frac{1}{3}\cdot 1\cdot\frac{\sin(30^\circ)+\cos(30^\circ)}{|\sqrt{2}\,|} \simeq -0.32 \\
c_{\frac{1}{2}} &= -\frac{1}{a_2^2+b_2^2+c_2^2}\cdot c_1\cdot\left|\sqrt{a_2^2+b_2^2}\right| = -\frac{1}{3}\cdot 1\cdot|\sqrt{2}\,| = -\frac{\sqrt{2}}{3}
\end{aligned}$$

Theorem 2.54. *With only $o_2(t_2,\theta_2,\gamma_2)$ belonging to the line U and $o_1(t_1,\theta_1,\gamma_1)$ in standard representation, their division may be expressed in the following way:*

$$o_{\frac{1}{2}}(a_{\frac{1}{2}},b_{\frac{1}{2}},c_{\frac{1}{2}})_{(t_{\frac{1}{2}},\theta_{\frac{1}{2}},\gamma_{\frac{1}{2}})} = a_{\frac{1}{2}(\frac{t_1}{t_2})} + i\cdot b_{\frac{1}{2}(\theta_1-\theta_2)} + u\cdot c_{\frac{1}{2}(\gamma_1-\gamma_2)}$$

where:

$$\begin{aligned}
a_{\frac{1}{2}} &= \frac{c_1}{c_2}\cdot\frac{a_1\cdot\cos(\theta_2)+b_1\cdot\sin(\theta_2)}{|\sqrt{a_1^2+b_1^2}\,|} \\
b_{\frac{1}{2}} &= \frac{c_1}{c_2}\cdot\frac{b_1\cdot\cos(\theta_2)-a_1\cdot\sin(\theta_2)}{|\sqrt{a_1^2+b_1^2}\,|} \\
c_{\frac{1}{2}} &= -\frac{1}{c_2}\cdot\left|\sqrt{a_1^2+b_1^2}\right|
\end{aligned}$$

Proof. The division between two complete numbers, as we know, satisfies the following formula:

$$\begin{aligned}
o_{\frac{1}{2}}(t_{\frac{1}{2}},\theta_{\frac{1}{2}},\gamma_{\frac{1}{2}}) = \frac{t_1}{t_2}\cdot\{&[\cos(\gamma_1-\gamma_2)\cdot\cos(\theta_1-\theta_2)] \\
&+ i\cdot[\cos(\gamma_1-\gamma_2)\cdot\sin(\theta_1-\theta_2)] + u\cdot[\sin(\gamma_1-\gamma_2)]\}
\end{aligned}$$

Since $o_2(t_2,\theta_2,\gamma_2)$ belongs to the line U will be provided with the following values of modulus and phases:

$$\begin{aligned}
&t_2 = \sqrt{c_2^2} \\
&\gamma_2 = \text{ sign }(c_2)\cdot 90^\circ \\
&\theta_2 \text{ known } \neq \arctan\left(\frac{b_2}{a_2}\right)
\end{aligned}$$

unlike $o_1(t_1, \theta_1, \gamma_1)$ that will be provided with the following values:

$$t_1 = \sqrt{a_1^2 + b_1^2 + c_1^2}$$
$$\gamma_1 = \arctan\left(\frac{c_1}{\sqrt{a_1^2 + b_1^2}}\right)$$
$$\theta_1 = \arctan\left(\frac{b_1}{a_1}\right)$$

This means that we can write the coordinates sought in the following way:

$$\begin{aligned}
a_{\frac{1}{2}} =& \frac{\sqrt{c_1^2 + b_1^2 + c_1^2}}{\sqrt{c_2^2}} \cdot \cos\left[\arctan\left(\frac{c_1}{\sqrt{a_1^2 + b_1^2}}\right) - \text{ sign }(c_2) \cdot 90°\right] \cdot \\
& \cdot \cos\left[\arctan\left(\frac{b_1}{a_1}\right) - \theta_2\right] \\
b_{\frac{1}{2}} =& \frac{\sqrt{c_1^2 + b_1^2 + c_1^2}}{\sqrt{c_2^2}} \cdot \cos\left[\arctan\left(\frac{c_1}{\sqrt{a_1^2 + b_1^2}}\right) - \text{ sign }(c_2) \cdot 90°\right] \cdot \\
& \cdot \sin\left[\arctan\left(\frac{b_1}{a_1}\right) - \theta_2\right] \\
c_{\frac{1}{2}} =& \frac{\sqrt{c_1^2 + b_1^2 + c_1^2}}{\sqrt{c_2^2}} \cdot \sin\left[\arctan\left(\frac{c_1}{\sqrt{a_1^2 + b_1^2}}\right) - \text{ sign }(c_2) \cdot 90°\right]
\end{aligned}$$

To continue with the proof, we have to use the following trigonometric relations:

$$\cos(x - y) = \cos(x) \cdot \cos(y) + \sin(x) \cdot \sin(y)$$
$$\sin(x - y) = \sin(x) \cdot \cos(y) - \cos(x) \cdot \sin(y)$$
$$\cos\left[\arctan\left(\frac{c}{\sqrt{a^2 + b^2}}\right)\right] = \sqrt{\frac{a^2 + b^2}{a^2 + b^2 + c^2}}$$
$$\sin\left[\arctan\left(\frac{c}{\sqrt{a^2 + b^2}}\right)\right] = \sqrt{\frac{c^2}{a^2 + b^2 + c^2}}$$
$$\cos\left[\arctan\left(\frac{b}{a}\right)\right] = \sqrt{\frac{a^2}{a^2 + b^2}}$$
$$\sin\left[\arctan\left(\frac{b}{a}\right)\right] = \sqrt{\frac{b^2}{a^2 + b^2}}$$
$$\cos[x - \text{ sign }(y) \cdot 90°] = \text{ sign }(y) \cdot sin(x)$$
$$\sin[x - \text{ sign }(y) \cdot 90°] = - \text{ sign }(y) \cdot cos(x)$$

To determine the value of the coordinate $a_{\frac{1}{2}}$ the steps to perform will be the following:

$$
\begin{aligned}
a_{\frac{1}{2}} &= \operatorname{sign}(c_2) \cdot \frac{\sqrt{c_1^2 + b_1^2 + c_1^2}}{\sqrt{c_2^2}} \cdot \sqrt{\frac{c_1^2}{a_1^2 + b_1^2 + c_1^2}} \cdot \\
&\quad \cdot \left[\sqrt{\frac{a_1^2}{a_1^2 + b_1^2}} \cdot \cos(\theta_2) + \sqrt{\frac{b_1^2}{a_1^2 + b_1^2}} \cdot \sin(\theta_2) \right] \\
&= \operatorname{sign}(c_2) \cdot \frac{\sqrt{c_1^2}}{\sqrt{c_2^2}} \cdot \frac{\sqrt{a_1^2} \cdot \cos(\theta_2) + \sqrt{b_1^2} \cdot \sin(\theta_2)}{\sqrt{a_1^2 + b_1^2}}
\end{aligned}
$$

To determine the value of the coordinate $b_{\frac{1}{2}}$ the steps to perform will be the following:

$$
\begin{aligned}
b_{\frac{1}{2}} &= \operatorname{sign}(c_2) \cdot \frac{\sqrt{c_1^2 + b_1^2 + c_1^2}}{\sqrt{c_2^2}} \cdot \sqrt{\frac{c_1^2}{a_1^2 + b_1^2 + c_1^2}} \cdot \\
&\quad \cdot \left[\sqrt{\frac{b_1^2}{a_1^2 + b_1^2}} \cdot \cos(\theta_2) - \sqrt{\frac{a_1^2}{a_1^2 + b_1^2}} \cdot \sin(\theta_2) \right] \\
&= \operatorname{sign}(c_2) \cdot \frac{\sqrt{c_1^2}}{\sqrt{c_2^2}} \cdot \frac{\sqrt{b_1^2} \cdot \cos(\theta_2) - \sqrt{a_1^2} \cdot \sin(\theta_2)}{\sqrt{a_1^2 + b_1^2}}
\end{aligned}
$$

To determine the value of the coordinate $c_{\frac{1}{2}}$ the steps to perform will be the following:

$$
c_{\frac{1}{2}} = -\operatorname{sign}(c_2) \cdot \frac{\sqrt{c_1^2 + b_1^2 + c_1^2}}{\sqrt{c_2^2}} \cdot \sqrt{\frac{a_1^2 + b_1^2}{a_1^2 + b_1^2 + c_1^2}} = -\operatorname{sign}(c_2) \cdot \frac{1}{\sqrt{c_2^2}} \cdot \sqrt{a_1^2 + b_1^2}
$$

These relations are valid in general, in the precise sense that they are also able to include cases where the coefficients a_1,b_1,c_1 are zero (provided that $o_1(a_1, b_1, c_1)$ remains in the context of the complete numbers not belonging in the line U).

The only limitation in this regard is the need to avoid the following situation:

$$
c_2^2 = 0
$$

which confirms the impossibility to divide a complete number $o(t, \theta, \gamma)$ for zero (characterized by the values c_2 that make the above mentioned condition true).

Wanting to find relations that satisfy the division rule as a function of the effective coordinates of the complete numbers involved, we must adopt for the coefficients a,b,c the convention $\sqrt{x^2} = x$, with the exception of c_2 for which we should adopt the convention $\sqrt{x^2} = |x|$. The reason is simple because if we adopt for c_2 the usual convention, we will have:

$$
\frac{\operatorname{sign}(c_2)}{\sqrt{c_2^2}} = \frac{1}{|c_2|}
$$

and therefore a result of the division that depends on the modulus of the coordinate c_2. While adopting $\sqrt{x^2} = |x|$ we will have:

$$\frac{\text{sign}\,(c_2)}{\sqrt{c_2^2}} = \frac{1}{c_2}$$

and therefore a result of the division that depends on the effective value of this coordinate.

The relations obtained will be the following:

$$\begin{aligned} a_{\frac{1}{2}} &= \frac{c_1}{c_2} \cdot \frac{a_1 \cdot \cos(\theta_2) + b_1 \cdot \sin(\theta_2)}{\sqrt{a_1^2 + b_1^2}} \\ b_{\frac{1}{2}} &= \frac{c_1}{c_2} \cdot \frac{b_1 \cdot \cos(\theta_2) - a_1 \cdot \sin(\theta_2)}{\sqrt{a_1^2 + b_1^2}} \\ c_{\frac{1}{2}} &= -\frac{1}{c_2} \cdot \sqrt{a_1^2 + b_1^2} \end{aligned} \tag{2.6}$$

Since the number $o_1(a_1, b_1, c_1)$ is in standard representation, as determined by the theorem 2.11 we must consider the following relation:

$$\sqrt{a_1^2 + b_1^2} = \left|\sqrt{a_1^2 + b_1^2}\,\right|$$

that combined with those indicated by the formulas (2.6), proving the thesis. ∎

As an example of the theorem just proved, suppose you have to divide the complete number in standard representation provided with coordinates: $a_1 = 1$, $b_1 = -1$, $c_1 = 1$ by an outgoing numbers of coordinate:$c_2 = 1$ and phase $\theta_2 = 30°$.

Their modulus may be calculated in the following way:

$$\begin{aligned} t_1 &= \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3} \\ t_2 &= \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{c_2^2} = \sqrt{1} = 1 \end{aligned}$$

For their phases in the case of the outgoing number we have:

$$\begin{aligned} \gamma_2 &= \text{ sign}\,(c_2) \cdot 90° = 90° \\ \theta_2 &= 30° \end{aligned}$$

while in the case of the complete number we should refer to the formulas related to the standard representation:

$$\begin{aligned} \gamma_1 &= \arctan\left(\frac{c_1}{\left|\sqrt{a_1^2 + b_1^2}\,\right|}\right) = \arctan\left(\frac{1}{|\sqrt{2}|}\right) \simeq 35.26° \\ \theta_1 &= \arctan\left(\frac{b_1}{a_1}\right) = \arctan\left(\frac{-1}{1}\right) = -45° \end{aligned}$$

By applying the division rule we obtain as result the complete number provided with the following values of modulus and phases:

$$\begin{aligned} t_{\frac{1}{2}} &= \frac{t_1}{t_2} = \sqrt{3} \\ \gamma_{\frac{1}{2}} &= \gamma_1 - \gamma_2 \simeq -54.74^\circ \\ \theta_{\frac{1}{2}} &= \theta_1 - \theta_2 = -75^\circ \end{aligned}$$

and the following coordinates:

$$\begin{aligned} a_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \cos(\theta_{\frac{1}{2}}) = \sqrt{3} \cdot \cos(\simeq -54.74^\circ) \cdot \cos(-75^\circ) \simeq 0.26 \\ b_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \sin(\theta_{\frac{1}{2}}) = \sqrt{3} \cdot \cos(\simeq -54.74^\circ) \cdot \sin(-75^\circ) \simeq -0.97 \\ c_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \sin(\gamma_{\frac{1}{2}}) = \sqrt{3} \cdot \sin(\simeq -54.74^\circ) = -\sqrt{2} \end{aligned}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$\begin{aligned} a_{\frac{1}{2}} &= \frac{c_1}{c_2} \cdot \frac{a_1 \cdot \cos(\theta_2) + b_1 \cdot \sin(\theta_2)}{|\sqrt{a_1^2 + b_1^2}\,|} = \frac{\cos(30^\circ) - \sin(30^\circ)}{|\sqrt{2}\,|} \simeq 0.26 \\ b_{\frac{1}{2}} &= \frac{c_1}{c_2} \cdot \frac{b_1 \cdot \cos(\theta_2) - a_1 \cdot \sin(\theta_2)}{|\sqrt{a_1^2 + b_1^2}\,|} = \frac{-\cos(30^\circ) - \sin(30^\circ)}{|\sqrt{2}\,|} \simeq -0.97 \\ c_{\frac{1}{2}} &= -\frac{1}{c_2} \cdot \left|\sqrt{a_1^2 + b_1^2}\right| = -\sqrt{2} \end{aligned}$$

Theorem 2.55. *With only $o_2(t_2, \theta_2, \gamma_2)$ belonging to the line U and $o_1(t_1, \theta_1, \gamma_1)$ in complementary representation, their division may be expressed in the following way:*

$$o_{\frac{1}{2}}(a_{\frac{1}{2}}, b_{\frac{1}{2}}, c_{\frac{1}{2}})_{(t_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}})} = a_{\frac{1}{2}(\frac{t_1}{t_2})} + i \cdot b_{\frac{1}{2}(\theta_1 - \theta_2)} + u \cdot c_{\frac{1}{2}(\gamma_1 - \gamma_2)}$$

where:

$$\begin{aligned} a_{\frac{1}{2}} &= -\frac{c_1}{c_2} \cdot \frac{a_1 \cdot \cos(\theta_2) + b_1 \cdot \sin(\theta_2)}{|\sqrt{a_1^2 + b_1^2}\,|} \\ b_{\frac{1}{2}} &= -\frac{c_1}{c_2} \cdot \frac{b_1 \cdot \cos(\theta_2) - a_1 \cdot \sin(\theta_2)}{|\sqrt{a_1^2 + b_1^2}\,|} \\ c_{\frac{1}{2}} &= \frac{1}{c_2} \cdot \left|\sqrt{a_1^2 + b_1^2}\right| \end{aligned}$$

Proof. Since the number $o_1(a_1, b_1, c_1)$ is in complementary representation, as determined by the theorem 2.15 we must consider the following relation:

$$\sqrt{a_1^2 + b_1^2} = -|\sqrt{a_1^2 + b_1^2}\,|$$

that combined with those indicated by the formulas (2.6), proving the thesis. ∎

As an example of the theorem just proved, suppose you have to divide a complete number in complementary representation provided with coordinates: $a_2 = 1$, $b_2 = -1$, $c_2 = 1$ by the outgoing numbers of coordinate: $c_1 = 1$ and phase $\theta_1 = 30°$.

Their modulus may be calculated in the following way:

$$t_1 = \sqrt{a_1^2 + b_1^2 + c_1^2} = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$$
$$t_2 = \sqrt{a_2^2 + b_2^2 + c_2^2} = \sqrt{c_2^2} = \sqrt{1} = 1$$

For their phases in the case of the outgoing number we have:

$$\gamma_2 = \text{ sign } (c_2) \cdot 90° = 90°$$
$$\theta_2 = 30°$$

while in the case of the complete number we should refer to the formulas related to the complementary representation:

$$\gamma_1 = \arctan\left(\frac{c_1}{-\left|\sqrt{a_1^2 + b_1^2}\,\right|}\right) = \arctan\left(\frac{1}{-|\sqrt{2}|}\right) \simeq 144.74°$$
$$\theta_1 = \arctan\left(\frac{-b_1}{-a_1}\right) = \arctan\left(\frac{1}{-1}\right) = 135°$$

By applying the division rule we obtain as result the complete number provided with the following values of modulus and phases:

$$t_{\frac{1}{2}} = \frac{t_1}{t_2} = \sqrt{3}$$
$$\gamma_{\frac{1}{2}} = \gamma_1 - \gamma_2 \simeq 54.74°$$
$$\theta_{\frac{1}{2}} = \theta_1 - \theta_2 = 105°$$

and the following coordinates:

$$a_{\frac{1}{2}} = t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \cos(\theta_{\frac{1}{2}}) = \sqrt{3} \cdot \cos(\simeq 54.74°) \cdot \cos(105°) \simeq -0.26$$
$$b_{\frac{1}{2}} = t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \sin(\theta_{\frac{1}{2}}) = \sqrt{3} \cdot \cos(\simeq 54.74°) \cdot \sin(105°) \simeq 0.97$$
$$c_{\frac{1}{2}} = t_{\frac{1}{2}} \cdot \sin(\gamma_{\frac{1}{2}}) = \sqrt{3} \cdot \sin(\simeq 54.74°) = \sqrt{2}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$a_{\frac{1}{2}} = -\frac{c_1}{c_2} \cdot \frac{a_1 \cdot \cos(\theta_2) + b_1 \cdot \sin(\theta_2)}{|\sqrt{a_1^2 + b_1^2}\,|} = -\frac{\cos(30°) - \sin(30°)}{|\sqrt{2}\,|} \simeq -0.26$$
$$b_{\frac{1}{2}} = -\frac{c_1}{c_2} \cdot \frac{b_1 \cdot \cos(\theta_2) - a_1 \cdot \sin(\theta_2)}{|\sqrt{a_1^2 + b_1^2}\,|} = -\frac{-\cos(30°) - \sin(30°)}{|\sqrt{2}\,|} \simeq 0.97$$
$$c_{\frac{1}{2}} = \frac{1}{c_2} \cdot \left|\sqrt{a_1^2 + b_1^2}\right| = \sqrt{2}$$

Theorem 2.56. *With $o_1(t_1, \theta_1, \gamma_1)$ and $o_2(t_2, \theta_2, \gamma_2)$ both belonging to the line U, their division may be expressed in the following way:*

$$o_{\frac{1}{2}}(a_{\frac{1}{2}}, b_{\frac{1}{2}}, c_{\frac{1}{2}})_{(t_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}})} = a_{\frac{1}{2}(\frac{t_1}{t_2})} + i \cdot b_{\frac{1}{2}(\theta_1 - \theta_2)} + u \cdot c_{\frac{1}{2}(\gamma_1 - \gamma_2)}$$

where:

$$\begin{aligned} a_{\frac{1}{2}} &= \frac{c_1}{c_2} \cdot \cos(\theta_1 - \theta_2) \\ b_{\frac{1}{2}} &= \frac{c_1}{c_2} \cdot \sin(\theta_1 - \theta_2) \\ c_{\frac{1}{2}} &= 0 \end{aligned}$$

Proof. The division between two complete numbers, as we know, satisfies the following formula:

$$\begin{aligned} o_{\frac{1}{2}}(t_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}}) = \frac{t_1}{t_2} \cdot \{[\cos(\gamma_1 - \gamma_2) \cdot \cos(\theta_1 - \theta_2)] + \\ + i \cdot [\cos(\gamma_1 - \gamma_2) \cdot \sin(\theta_1 - \theta_2)] + u \cdot [\sin(\gamma_1 - \gamma_2)]\} \end{aligned}$$

Since $o_1(t_1, \theta_1, \gamma_1)$ and $o_2(t_2, \theta_2, \gamma_2)$ belong to the line U will be provided with the following values of modulus and phases:

$$\begin{aligned} t_1 &= \sqrt{c_1^2} \\ t_2 &= \sqrt{c_2^2} \\ \gamma_1 &= \operatorname{sign}(c_1) \cdot 90^\circ \\ \gamma_2 &= \operatorname{sign}(c_2) \cdot 90^\circ \\ \theta_1 &\text{ known } \neq \arctan\left(\frac{b_1}{a_1}\right) \\ \theta_2 &\text{ known } \neq \arctan\left(\frac{b_2}{a_2}\right) \end{aligned}$$

This means that we can write the coordinates sought in the following way:

$$\begin{aligned} a_{\frac{1}{2}} &= \frac{\sqrt{c_1^2}}{\sqrt{c_2^2}} \cdot \cos[\operatorname{sign}(c_1) \cdot 90^\circ - \operatorname{sign}(c_2) \cdot 90^\circ] \cdot \cos(\theta_1 - \theta_2) \\ b_{\frac{1}{2}} &= \frac{\sqrt{c_1^2}}{\sqrt{c_2^2}} \cdot \cos[\operatorname{sign}(c_1) \cdot 90^\circ - \operatorname{sign}(c_2) \cdot 90^\circ] \cdot \sin(\theta_1 - \theta_2) \\ c_{\frac{1}{2}} &= \frac{\sqrt{c_1^2}}{\sqrt{c_2^2}} \cdot \sin[\operatorname{sign}(c_1) \cdot 90^\circ - \operatorname{sign}(c_2) \cdot 90^\circ] \end{aligned}$$

Considering that when c_1 and c_2 have the same sign we obtained:

$$\begin{aligned} &\cos[\operatorname{sign}(c_1) \cdot 90^\circ - \operatorname{sign}(c_2) \cdot 90^\circ] = \cos(\pm 0^\circ) = 1 = \operatorname{sign}(c_1) \cdot \operatorname{sign}(c_2) \\ &\sin[\operatorname{sign}(c_1) \cdot 90^\circ - \operatorname{sign}(c_2) \cdot 90^\circ] = \sin(\pm 0^\circ) = 0 \end{aligned}$$

and that when they have the opposite sign we obtained:

$$\cos\left[\,\text{sign}\,(c_1)\cdot 90^\circ - \,\text{sign}\,(c_2)\cdot 90^\circ\right] = \cos\left(\pm 180^\circ\right) = -1 = \,\text{sign}\,(c_1)\cdot\,\text{sign}\,(c_2)$$
$$\sin\left[\,\text{sign}\,(c_1)\cdot 90^\circ - \,\text{sign}\,(c_2)\cdot 90^\circ\right] = \sin\left(\pm 180^\circ\right) = 0$$

we can write:

$$\begin{aligned}
a_{\frac{1}{2}} &= \,\text{sign}\,(c_1)\cdot\,\text{sign}\,(c_2)\cdot\frac{\sqrt{c_1^2}}{\sqrt{c_2^2}}\cdot\cos\left(\theta_1-\theta_2\right)\\
b_{\frac{1}{2}} &= \,\text{sign}\,(c_1)\cdot\,\text{sign}\,(c_2)\cdot\frac{\sqrt{c_1^2}}{\sqrt{c_2^2}}\cdot\sin\left(\theta_1-\theta_2\right)\\
c_{\frac{1}{2}} &= 0
\end{aligned}$$

Wanting to find relations that satisfy the division rule as a function of the effective coordinates of the complete numbers involved, we must adopt for the coefficients c_1,c_2 the convention $\sqrt{x^2} = |x|$. In fact in this way we obtain:

$$\text{sign}\,(c_1)\cdot\sqrt{c_1^2} = c_1$$
$$\frac{\text{sign}\,(c_2)}{\sqrt{c_2^2}} = \frac{1}{c_2}$$

and therefore a result of the division that depends on the effective values of these coordinates. The relation that we obtain following these conventions proves the thesis. ∎

As an example of the theorem just proved, suppose you have to divide the outgoing numbers of coordinate: $c_1 = 1$ and phase $\theta_1 = 30^\circ$ by the outgoing number of coordinate: $c_2 = 1$ and phase $\theta_2 = 30^\circ$.

Their modulus may be calculated in the following way:

$$t_1 = \sqrt{a_1^2+b_1^2+c_1^2} = \sqrt{c_1^2} = \sqrt{1} = 1$$
$$t_2 = \sqrt{a_2^2+b_2^2+c_2^2} = \sqrt{c_2^2} = \sqrt{1} = 1$$

For their phases we have:

$$\begin{aligned}
\gamma_1 &= \,\text{sign}\,(c_1)\cdot 90^\circ = 90^\circ\\
\gamma_2 &= \,\text{sign}\,(c_2)\cdot 90^\circ = 90^\circ\\
\theta_1 &= 30^\circ\\
\theta_2 &= 30^\circ
\end{aligned}$$

By applying the division rule we obtain as result the complete number provided with the following values of modulus and phases:

$$\begin{aligned}
t_{\frac{1}{2}} &= \frac{t_1}{t_2} = 1\\
\gamma_{\frac{1}{2}} &= \gamma_1 - \gamma_2 = 0^\circ\\
\theta_{\frac{1}{2}} &= \theta_1 - \theta_2 = 0^\circ
\end{aligned}$$

and the following coordinates:

$$\begin{aligned}
a_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \cos(\theta_{\frac{1}{2}}) = 1 \cdot \cos(0^\circ) \cdot \cos(0^\circ) = 1 \\
b_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \cos(\gamma_{\frac{1}{2}}) \cdot \sin(\theta_{\frac{1}{2}}) = 1 \cdot \cos(0^\circ) \cdot \sin(0^\circ) = 0 \\
c_{\frac{1}{2}} &= t_{\frac{1}{2}} \cdot \sin(\gamma_{\frac{1}{2}}) = 1 \cdot \sin(0^\circ) = 0
\end{aligned}$$

At this point we can see how the formulas of the previous theorem make actually reach the same result:

$$\begin{aligned}
a_{\frac{1}{2}} &= \frac{\sqrt{c_1^2}}{\sqrt{c_2^2}} \cdot \cos(\theta_1 - \theta_2) = 1 \cdot \cos(0^\circ) = 1 \\
b_{\frac{1}{2}} &= \frac{\sqrt{c_1^2}}{\sqrt{c_2^2}} \cdot \sin(\theta_1 - \theta_2) = 1 \cdot \sin(0^\circ) = 0 \\
c_{\frac{1}{2}} &= 0
\end{aligned}$$

Theorem 2.57. *For the operation of division is defined indivisible the complete number 0, namely for:*

$$o_1(t_1, \theta_1, \gamma_1) = 0$$

we have:

$$\frac{o_1(t_1, \theta_1, \gamma_1)}{o_2(t_2, \theta_2, \gamma_2)} = 0$$

Proof. t_1,θ_1,γ_1,t_2,θ_2,γ_2 being real numbers, we can write:

$$\begin{aligned}
t_{\frac{1}{2}} &= \frac{t_1}{t_2} = \frac{0}{t_2} = 0 \\
\theta_{\frac{1}{2}} &= \theta_1 - \theta_2 = \theta_1 - \mathit{indeterminate} = \mathit{indeterminate} \\
\gamma_{\frac{1}{2}} &= \gamma_1 - \gamma_2 = \gamma_1 - \mathit{indeterminate} = \mathit{indeterminate}
\end{aligned}$$

proving the thesis. ∎

Theorem 2.58. *For the operation of division is defined neuter the complete number $1_{(S)}$, namely for:*

$$o_2(a_2, b_2, c_2)_{(S)} = 1_{(S)}$$

we have:

$$\frac{o_1(t_1, \theta_1, \gamma_1)}{o_2(t_2, \theta_2, \gamma_2)} = o_1(t_1, \theta_1, \gamma_1)$$

Proof. t_1,θ_1,γ_1,t_2,θ_2,γ_2 being real numbers, we can write:

$$\begin{aligned}
t_{\frac{1}{2}} &= \frac{t_1}{t_2} = \frac{t_1}{1} = t_1 \\
\theta_{\frac{1}{2}} &= \theta_1 - \theta_2 = \theta_1 - 0 = \theta_1 \\
\gamma_{\frac{1}{2}} &= \gamma_1 - \gamma_2 = \gamma_1 - 0 = \gamma_1
\end{aligned}$$

proving the thesis. ∎

Theorem 2.59. *For the operation of division is defined identical the same position with respect to the origin, namely for:*

$$o_2(a_2, b_2, c_2) = o_2(a_1, b_1, c_1)$$

we have:

$$\frac{o_1(t_1, \theta_1, \gamma_1)}{o_2(t_2, \theta_2, \gamma_2)} = 1_{(S)}$$

Proof. t_1,θ_1,γ_1,t_2,θ_2,γ_2 being real numbers, we can write:

$$t_{\frac{1}{2}} = \frac{t_1}{t_2} = \frac{t_1}{t_1} = 1$$
$$\theta_{\frac{1}{2}} = \theta_1 - \theta_2 = \theta_1 - \theta_1 = 0$$
$$\gamma_{\frac{1}{2}} = \gamma_1 - \gamma_2 = \gamma_1 - \gamma_1 = 0$$

proving the thesis. ∎

Theorem 2.60. *It is valid the invariantive property, namely:*

$$\frac{o_1(t_1, \theta_1, \gamma_1)}{o_2(t_2, \theta_2, \gamma_2)} = \frac{[o_1(t_1, \theta_1, \gamma_1) \cdot o_3(t_3, \theta_3, \gamma_3)]}{[o_2(t_2, \theta_2, \gamma_2) \cdot o_3(t_3, \theta_3, \gamma_3)]}$$
$$\frac{o_1(t_1, \theta_1, \gamma_1)}{o_2(t_2, \theta_2, \gamma_2)} = \frac{\frac{o_1(t_1,\theta_1,\gamma_1)}{o_3(t_3,\theta_3,\gamma_3)}}{\frac{o_2(t_2,\theta_2,\gamma_2)}{o_3(t_3,\theta_3,\gamma_3)}}$$

Proof. t_1,θ_1,γ_1,t_2,θ_2,γ_2,t_3,θ_3,γ_3 being real numbers, we can write:

$$t_{\frac{1}{2}} = \frac{t_1}{t_2}$$
$$\theta_{\frac{1}{2}} = (\theta_1 - \theta_2)$$
$$\gamma_{\frac{1}{2}} = (\gamma_1 - \gamma_2)$$

$$t_{\frac{(1\cdot 3)}{(2\cdot 3)}} = \frac{t_1 \cdot t_3}{t_2 \cdot t_3} = \frac{t_1}{t_2}$$
$$\theta_{\frac{(1\cdot 3)}{(2\cdot 3)}} = (\theta_1 + \theta_3) - (\theta_2 + \theta_3) = \theta_1 - \theta_2$$
$$\gamma_{\frac{(1\cdot 3)}{(2\cdot 3)}} = (\gamma_1 + \gamma_3) - (\gamma_2 + \gamma_3) = \gamma_1 - \gamma_2$$

$$t_{\frac{(\frac{1}{3})}{(\frac{2}{3})}} = \frac{\frac{t_1}{t_3}}{\frac{t_2}{t_3}} = \frac{t_1}{t_2}$$
$$\theta_{\frac{(\frac{1}{3})}{(\frac{2}{3})}} = (\theta_1 - \theta_3) - (\theta_2 - \theta_3) = \theta_1 - \theta_2$$
$$\gamma_{\frac{(\frac{1}{3})}{(\frac{2}{3})}} = (\gamma_1 - \gamma_3) - (\gamma_2 - \gamma_3) = \gamma_1 - \gamma_2$$

proving the thesis. ∎

Theorem 2.61. *It is not valid the distributive property of division over addition, namely for:*

$$o_1(t_1,\theta_1,\gamma_1) = o_3(t_3,\theta_3,\gamma_3) + o_4(t_4,\theta_4,\gamma_4)$$

we have:

$$\frac{o_1(t_1,\theta_1,\gamma_1)}{o_2(t_2,\theta_2,\gamma_2)} \neq \left[\frac{o_3(t_3,\theta_3,\gamma_3)}{o_2(t_2,\theta_2,\gamma_2)}\right] + \left[\frac{o_4(t_4,\theta_4,\gamma_4)}{o_2(t_2,\theta_2,\gamma_2)}\right]$$

Proof. Referring to the situation described by theorem 2.48 and considering that a_1,b_1,c_1,a_2,b_2,c_2, a_3,b_3,c_3,a_4,b_4,c_4 being real numbers, we can write:

$$c_{\frac{1}{2}} = \frac{1}{a_2^2+b_2^2+c_2^2} \cdot \left[c_1 \cdot \left|\sqrt{(a_2^2+b_2^2)}\right| - c_2 \cdot \left|\sqrt{(a_1^2+b_1^2)}\right|\right]$$

$$\begin{aligned}
c_{(\frac{3}{2})+(\frac{4}{2})} =& \frac{1}{a_2^2+b_2^2+c_2^2} \cdot \left[c_3 \cdot \left|\sqrt{(a_2^2+b_2^2)}\right| - c_2 \cdot \left|\sqrt{(a_3^2+b_3^2)}\right|\right] \\
&+ \frac{1}{a_2^2+b_2^2+c_2^2} \cdot \left[c_4 \cdot \left|\sqrt{(a_2^2+b_2^2)}\right| - c_2 \cdot \left|\sqrt{(a_4^2+b_4^2)}\right|\right] \\
=& \frac{1}{a_2^2+b_2^2+c_2^2} \cdot \Big\{(c_3+c_4) \cdot \left|\sqrt{(a_2^2+b_2^2)}\right| \\
&- c_2 \cdot \left[\left|\sqrt{(a_3^2+b_3^2)}\right| + \left|\sqrt{(a_4^2+b_4^2)}\right|\right]\Big\} \\
=& \frac{1}{a_2^2+b_2^2+c_2^2} \cdot \Big\{c_1 \cdot \left|\sqrt{(a_2^2+b_2^2)}\right| \\
&- c_2 \cdot \left[\left|\sqrt{(a_3^2+b_3^2)}\right| + \left|\sqrt{(a_4^2+b_4^2)}\right|\right]\Big\} \neq c_{\frac{1}{2}}
\end{aligned}$$

proving the thesis. ∎

Theorem 2.62. *It is not valid the distributive property of division over subtraction, namely for:*

$$o_1(t_1,\theta_1,\gamma_1) = o_3(t_3,\theta_3,\gamma_3) - o_4(t_4,\theta_4,\gamma_4)$$

we have:

$$\frac{o_1(t_1,\theta_1,\gamma_1)}{o_2(t_2,\theta_2,\gamma_2)} \neq \left[\frac{o_3(t_3,\theta_3,\gamma_3)}{o_2(t_2,\theta_2,\gamma_2)}\right] - \left[\frac{o_4(t_4,\theta_4,\gamma_4)}{o_2(t_2,\theta_2,\gamma_2)}\right]$$

Proof. Referring to the situation described by theorem 2.48 and considering that a_1,b_1,c_1,a_2,b_2,c_2, a_3,b_3,c_3,a_4,b_4,c_4 being real numbers, we can write:

$$c_{\frac{1}{2}} = \frac{1}{a_2^2+b_2^2+c_2^2} \cdot \left[c_1 \cdot \left|\sqrt{(a_2^2+b_2^2)}\right| - c_2 \cdot \left|\sqrt{(a_1^2+b_1^2)}\right|\right]$$

$$
\begin{aligned}
c_{(\frac{3}{2})-(\frac{4}{2})} =& \frac{1}{a_2^2+b_2^2+c_2^2} \cdot \left[c_3 \cdot \left| \sqrt{(a_2^2+b_2^2)} \right| - c_2 \cdot \left| \sqrt{(a_3^2+b_3^2)} \right| \right] \\
&- \frac{1}{a_2^2+b_2^2+c_2^2} \cdot \left[c_4 \cdot \left| \sqrt{(a_2^2+b_2^2)} \right| - c_2 \cdot \left| \sqrt{(a_4^2+b_4^2)} \right| \right] \\
=& \frac{1}{a_2^2+b_2^2+c_2^2} \cdot \left\{ (c_3 - c_4) \cdot \left| \sqrt{(a_2^2+b_2^2)} \right| + \right. \\
&\left. - c_2 \cdot \left[\left| \sqrt{(a_3^2+b_3^2)} \right| - \left| \sqrt{(a_4^2+b_4^2)} \right| \right] \right\} \\
=& \frac{1}{a_2^2+b_2^2+c_2^2} \cdot \left\{ c_1 \cdot \left| \sqrt{(a_2^2+b_2^2)} \right| \right. \\
&\left. - c_2 \cdot \left[\left| \sqrt{(a_3^2+b_3^2)} \right| - \left| \sqrt{(a_4^2+b_4^2)} \right| \right] \right\} \neq c_{\frac{1}{2}}
\end{aligned}
$$

proving the thesis. ∎

Theorem 2.63. *It is valid the equivalence between multiplication and division, namely:*

$$
\begin{aligned}
o_1(t_1,\theta_1,\gamma_1) \cdot o_2(t_2,\theta_2,\gamma_2) &= \frac{o_1(t_1,\theta_1,\gamma_1)}{\dfrac{1}{o_2(t_2,\theta_2,\gamma_2)}} \\
\frac{o_1(t_1,\theta_1,\gamma_1)}{o_2(t_2,\theta_2,\gamma_2)} &= o_1(t_1,\theta_1,\gamma_1) \cdot \frac{1}{o_2(t_2,\theta_2,\gamma_2)}
\end{aligned}
$$

Proof. t_1,θ_1,γ_1,t_2,θ_2,γ_2 being real numbers, we can write:

$$
\begin{aligned}
t_{1\cdot 2} &= t_1 \cdot t_2 \\
\theta_{1\cdot 2} &= \theta_1 + \theta_2 \\
\gamma_{1\cdot 2} &= \gamma_1 + \gamma_2
\end{aligned}
$$

$$
\begin{aligned}
t_{\frac{1}{\frac{1}{2}}} &= \frac{t_1}{\frac{1}{t_2}} = t_1 \cdot t_2 \\
\theta_{\frac{1}{\frac{1}{2}}} &= \theta_1 - (-\theta_2) = \theta_1 + \theta_2 \\
\gamma_{\frac{1}{\frac{1}{2}}} &= \gamma_1 - (-\gamma_2) = \gamma_1 + \gamma_2
\end{aligned}
$$

$$
\begin{aligned}
t_{\frac{1}{2}} &= \frac{t_1}{t_2} \\
\theta_{\frac{1}{2}} &= (\theta_1 - \theta_2) \\
\gamma_{\frac{1}{2}} &= (\gamma_1 - \gamma_2)
\end{aligned}
$$

$$
\begin{aligned}
t_{1\cdot\frac{1}{2}} &= t_1 \cdot \frac{1}{t_2} = \frac{t_1}{t_2} \\
\theta_{1\cdot\frac{1}{2}} &= \theta_1 + (-\theta_2) = \theta_1 - \theta_2 \\
\gamma_{1\cdot\frac{1}{2}} &= \gamma_1 + (-\gamma_2) = \gamma_1 - \gamma_2
\end{aligned}
$$

proving the thesis. ∎

2.6. N-th power

Definition 2.64. In the space RIU we can define n-th power of the complete number $o(t,\theta,\gamma)$, with n (natural number) known as exponent and $o(t,\theta,\gamma)$ known as base, as the number $o_{\uparrow n}(t_{\uparrow n},\theta_{\uparrow n},\gamma_{\uparrow n})$ also represented with the symbol $o(t,\theta,\gamma)^n$ that satisfies the following conditions:

1. $o(t,\theta,\gamma)^n = o(t,\theta,\gamma)\cdot \ldots \cdot o(t,\theta,\gamma) \quad$ for $n>0$
2. $o(t,\theta,\gamma)^n = \frac{o(t,\theta,\gamma)}{o(t,\theta,\gamma)} = 1 \quad$ for $n=0$
3. $o(t,\theta,\gamma)^n = \cfrac{1}{\cfrac{\cfrac{o(t,\theta,\gamma)}{\cdots}}{o(t,\theta,\gamma)}} \quad$ for $n<0$
4. $n>0 \quad$ for $o(t,\theta,\gamma)=0$

We note that the term $o(t,\theta,\gamma)$ in the first and third conditions is intended to appear $|n|$ times.

The first condition defines the repeated multiplication of the base by itself a positive number of times, the second a zero number of times, and finally the third a negative number of times. All these conditions correspond to require:

$$\begin{aligned} t_{\uparrow n} &= t^n \\ \theta_{\uparrow n} &= \theta\cdot n \\ \gamma_{\uparrow n} &= \gamma\cdot n \end{aligned}$$

The fourth condition gets its own justification by the impossibility of defining the n-th power module when to be multiplied by itself a zero number or a negative number of times is just the 0, because in this case would be present the following divisions for 0:

$$o(t,\theta,\gamma)^n = \frac{0}{0} = 1 \text{ for } n=0$$
$$o(t,\theta,\gamma)^n = \cfrac{1}{\cfrac{\cfrac{0}{\cdots}}{0}} \text{ for } n<0 \text{ with 0 that appears —n— times}$$

Theorem 2.65. *It is valid the product property of exponents, namely:*

$$(o^n)^m = o^{n\cdot m}$$

Proof. By applying to $(o^n)^m$ and $o^{n \cdot m}$ the definition of n-th power previously introduced, we really obtain the same result as we can observe by the following relations, when (m,n) are both greater than zero:

$$\begin{aligned}(o^n)^m &= (o \cdot o \cdot ... \cdot o) \cdot (o \cdot o \cdot ... \cdot o) \cdot ... \cdot (o \cdot o \cdot ... \cdot o) \\ o^{n \cdot m} &= (o \cdot o \cdot o \cdot o \cdot ... \cdot o)\end{aligned}$$

It is easy to verify how all pairs of obtainable relations show a total of $|n \cdot m|$ terms $o(t, \theta, \gamma)$ to the numerator or to the denominator. Since this result is not depending on the particular values assumed by $o(t, \theta, \gamma)$ we can consider the property examined here as generally valid. ∎

Theorem 2.66. *It is valid the sum property of exponents, namely:*

$$o^n \cdot o^m = o^{n+m}$$

Proof. By applying to $(o^n \cdot o^m)$ and o^{n+m} the definition of n-th power previously introduced, we really obtain the same result as we can observe by the following relations, when (m,n) are both greater than zero:

$$\begin{aligned}o^n \cdot o^m &= (o \cdot o \cdot o \cdot o \cdot ... \cdot o) \cdot (o \cdot o \cdot ... \cdot o) \\ o^{n+m} &= (o \cdot o \cdot o \cdot o \cdot o \cdot o \cdot o \cdot o \cdot ... \cdot o)\end{aligned}$$

It is easy to verify how all pairs of obtainable relations show a total of $|m + n|$ terms $o(t, \theta, \gamma)$ to the numerator or to the denominator. Since this result is not depending on the particular values assumed by $o(t, \theta, \gamma)$ we can consider the property examined here as generally valid. ∎

Theorem 2.67. *It is valid the difference property of exponents, namely:*

$$\frac{o^n}{o^m} = o^{n-m}$$

Proof. By applying to $(\frac{o^n}{o^m})$ and o^{n-m} the definition of n-th power previously introduced, we really obtain the same result as we can observe by the following relations, when (m,n) are both greater than zero:

$$\begin{aligned}\frac{o^n}{o^m} &= (o \cdot o \cdot o \cdot o \cdot ... \cdot o) && \text{if } n > m \\ o^{n-m} &= (o \cdot o \cdot o \cdot o \cdot o \cdot o \cdot o \cdot o \cdot ... \cdot o) && \text{if } n > m \\ \frac{o^n}{o^m} &= \frac{1}{(o \cdot o \cdot o \cdot o \cdot ... \cdot o)} && \text{if } n < m \\ o^{n-m} &= \frac{1}{(o \cdot o \cdot o \cdot o \cdot o \cdot o \cdot o \cdot o \cdot ... \cdot o)} && \text{if } n < m\end{aligned}$$

It is easy to verify how all pairs of obtainable relations show a total of $|m - n|$ terms $o(t, \theta, \gamma)$ to the numerator or to the denominator. Since this result is not depending on the particular values assumed by $o(t, \theta, \gamma)$ we can consider the property examined here as generally valid. ∎

Theorem 2.68. *It is valid the product property of bases, namely:*

$$o_1^n \cdot o_2^n = (o_1 \cdot o_2)^n$$

Proof. By applying to $(o_1^n \cdot o_2^n)$ and $(o_1 \cdot o_2)^n$ the definition of n-th power previously introduced, we really obtain the same result as we can observe by the following relations, when n is greater than zero:

$$\begin{aligned} o_1^n \cdot o_2^n &= (o_1 \cdot o_1 \cdot o_1 \cdot \ldots \cdot o_1) \cdot (o_2 \cdot o_2 \cdot o_2 \cdot \ldots \cdot o_2) \\ (o_1 \cdot o_2)^n &= (o_1 \cdot o_2) \cdot (o_1 \cdot o_2) \cdot \ldots \cdot (o_1 \cdot o_2) \cdot \end{aligned}$$

It is easy to verify how all pairs of obtainable relations show a total of $|n|$ terms $o_1(t_1, \theta_1, \gamma_1)$ and $|n|$ terms $o_2(t_2, \theta_2, \gamma_2)$ to the numerator or to the denominator. Since this result is not depending on the particular values assumed by $o_1(t_1, \theta_1, \gamma_1)$ and $o_2(t_2, \theta_2, \gamma_2)$ we can consider the property examined here as generally valid. ■

Theorem 2.69. *It is valid the quotient property of bases, namely:*

$$\frac{o_1^n}{o_2^n} = \left(\frac{o_1}{o_2}\right)^n$$

Proof. By applying to $\frac{o_1^n}{o_2^n}$ and $(\frac{o_1}{o_2})^n$ the definition of n-th power previously introduced, we really obtain the same result as we can observe by the following relations, when n is greater than zero:

$$\begin{aligned} \frac{o_1^n}{o_2^n} &= \frac{(o_1 \cdot o_1 \cdot o_1 \cdot \ldots \cdot o_1)}{(o_2 \cdot o_2 \cdot o_2 \cdot \ldots \cdot o_2)} \\ \left(\frac{o_1}{o_2}\right)^n &= \left(\frac{o_1}{o_2}\right) \cdot \left(\frac{o_1}{o_2}\right) \cdot \ldots \cdot \left(\frac{o_1}{o_2}\right) \end{aligned}$$

It is easy to verify how all pairs of obtainable relations show a total of $|n|$ terms $o_1(t_1, \theta_1, \gamma_1)$ to the numerator and $|n|$ terms $o_2(t_2, \theta_2, \gamma_2)$ to the denominator or vice versa. Since this result is not depending on the particular values assumed by $o_1(t_1, \theta_1, \gamma_1)$ and $o_2(t_2, \theta_2, \gamma_2)$ we can consider the property examined here as generally valid. ■

2.7. N-th root

Definition 2.70. In the space RIU we can define n-th root of the complete number $o(t, \theta, \gamma)$, with n (natural number) known as degree and $o(t, \theta, \gamma)$ known as radicand, as the number $o_{\downarrow n}(t_{\downarrow n}, \theta_{\downarrow n}, \gamma_{\downarrow n})$ also represented with the symbol $\sqrt[n]{o(t, \theta, \gamma)}$ that satisfies the following conditions:

1. $\sqrt[n]{o(t, \theta, \gamma)} \cdot \ldots \cdot \sqrt[n]{o(t, \theta, \gamma)} = o(t, \theta, \gamma) \quad$ for $n > 0$

2. $\dfrac{\dfrac{1}{\sqrt[n]{o(t, \theta, \gamma)}}}{\dfrac{\ldots}{\sqrt[n]{o(t, \theta, \gamma)}}} = o(t, \theta, \gamma) \quad$ for $n < 0$

3. $\theta_{\downarrow n} = \frac{\theta}{n}$, $\quad \gamma_{\downarrow n} = \frac{\gamma}{n}$

4. $n \neq 0 \quad$ for any $o(t, \theta, \gamma)$

5. $n \geq 0 \quad$ for $o(t, \theta, \gamma) = 0$

6. $\sqrt[n]{t} > 0, \quad t > 0$

We note that the term $\sqrt[n]{o(t, \theta, \gamma)}$ in the first and second conditions is intended to appear $|n|$ times.

The first condition defines the repeated multiplication of the root by itself a positive number of times, while the second a negative number of times. Both these conditions correspond to require:

$$t_{\downarrow n} = \sqrt[n]{t}$$
$$\theta_{\downarrow n} = \frac{\theta + k \cdot 360°}{n} \qquad \text{for } k = \pm 1, \pm 2, \pm 3, \pm 4, ...$$
$$\gamma_{\downarrow n} = \frac{\gamma + k \cdot 360°}{n} \qquad \text{for } k = \pm 1, \pm 2, \pm 3, \pm 4, ...$$

The third condition gets its own justification by the necessity of defining the n-th root in an univocal way. In fact, when that condition is not valid, there are n^2 different complete numbers able to satisfy such definition: one for each distinct pair of phases $\theta_{\downarrow n}$, $\gamma_{\downarrow n}$ given by the relations seen above.

Also the fourth condition gets its own justification by the necessity of defining the n-th root in an univocal way. In fact when that condition is not valid, the multiplication of the root by itself a number of times equal to 0 would require the use of the following expression:

$$\frac{\sqrt[n]{o(t, \theta, \gamma)}}{\sqrt[n]{o(t, \theta, \gamma)}} = 1$$

that would be satisfied by several values of $\sqrt[n]{o(t, \theta, \gamma)}$.

The fifth condition gets its own justification by the impossibility of defining values of n-th root that multiplied by itself a negative number of times are able to give as the result just 0 value. In fact the following expression:

$$\cfrac{1}{\cfrac{\sqrt[n]{o(t, \theta, \gamma)}}{\cfrac{...}{\sqrt[n]{o(t, \theta, \gamma)}}}} = 0 \text{ for } n < 0, \sqrt[n]{o(t, \theta, \gamma)} \text{ appears —n— times}$$

requires the existence of a divisor of 1 that can assign to it a quotient equal to 0: a thing that we know impossible.

The sixth condition gets its own justification by the need to make acceptable the n-th root in regard the modulus t of the complete number $o(t, \theta, \gamma)$.

Theorem 2.71. *It is valid the product property of degrees, namely:*

$$\sqrt[m]{\sqrt[n]{o}} = \sqrt[m\cdot n]{o}$$

Proof. By applying the principle according to which two numbers are equal if and only if they remain as such, also once we raise them to the same power, we can raise the two member of the previous equality to the number $(m \cdot n)$, obtaining:

$$\left(\sqrt[m]{\sqrt[n]{o}} \right)^{(m\cdot n)} = (\sqrt[m\cdot n]{o})^{(m\cdot n)}$$

At this point, we can verify the validity of the starting equality showing how the two members thus obtained are actually equal.

Considering the value $\sqrt[m]{\sqrt[n]{o}}$ of the first member as a complete number, it is possible to apply to it the theorem 2.65 concerning the product of exponents of the n-th power, obtaining:

$$\left(\sqrt[m]{\sqrt[n]{o}} \right)^{(m\cdot n)} = \left[\left(\sqrt[m]{\sqrt[n]{o}} \right)^{m} \right]^{n}$$

Then applying to this member the definition of n-th root, we obtain:

$$\left[\left(\sqrt[m]{\sqrt[n]{o}} \right)^{m} \right]^{n} = (\sqrt[n]{o})^{n} = o$$

By applying the same definition to the second member we obtain an equivalent final result:

$$(\sqrt[m\cdot n]{o})^{(m\cdot n)} = o$$

■

Theorem 2.72. *It is valid the product property of radicands, namely:*

$$\sqrt[n]{o_1} \cdot \sqrt[n]{o_2} = \sqrt[n]{o_1 \cdot o_2}$$

Proof. By applying the principle according to which two numbers are equal if and only if they remain as such, also once we raise them to the same power, we can raise the two member of the previous equality to the number n, obtaining:

$$(\sqrt[n]{o_1} \cdot \sqrt[n]{o_2})^{n} = (\sqrt[n]{o_1 \cdot o_2})^{n}$$

At this point, we can verify the validity of the starting equality showing how the two members thus obtained are actually equal.

Considering the values $\sqrt[n]{o_1}$ and $\sqrt[n]{o_2}$ of the first member as the complete numbers, it is possible to apply to them the theorem 2.68 concerning the product of bases of the n-th power, obtaining:

$$(\sqrt[n]{o_1} \cdot \sqrt[n]{o_2})^{n} = (\sqrt[n]{o_1})^{n} \cdot (\sqrt[n]{o_2})^{n}$$

Then applying to two factors of this member the definition of n-th root, we obtain:

$$(\sqrt[n]{o_1})^{n} \cdot (\sqrt[n]{o_2})^{n} = o_1 \cdot o_2$$

By applying the same definition to the second member we obtain an equivalent final result:

$$(\sqrt[n]{o_1 \cdot o_2})^{n} = o_1 \cdot o_2$$

■

Theorem 2.73. *It is valid the quotient property of radicands, namely:*

$$\frac{\sqrt[n]{o_1}}{\sqrt[n]{o_2}} = \sqrt[n]{\frac{o_1}{o_2}}$$

Proof. By applying the principle according to which two numbers are equal if and only if they remain as such, also once we raise them to the same power, we can raise the two member of the previous equality to the number n, obtaining:

$$\left(\frac{\sqrt[n]{o_1}}{\sqrt[n]{o_2}}\right)^n = \left(\sqrt[n]{\frac{o_1}{o_2}}\right)^n$$

At this point, we can verify the validity of the starting equality showing how the two members thus obtained are actually equal.

Considering the values $\sqrt[n]{o_1}$ and $\sqrt[n]{o_2}$ of the first member as the complete numbers, it is possible to apply to them the theorem 2.69 concerning the quotient of bases of the n-th power, obtaining:

$$\left(\frac{\sqrt[n]{o_1}}{\sqrt[n]{o_2}}\right)^n = \frac{(\sqrt[n]{o_1})^n}{(\sqrt[n]{o_2})^n}$$

Then applying to two factors of this member the definition of n-th root, we obtain:

$$\frac{(\sqrt[n]{o_1})^n}{(\sqrt[n]{o_2})^n} = \frac{o_1}{o_2}$$

By applying the same definition to the second member we obtain an equivalent final result:

$$\left(\sqrt[n]{\frac{o_1}{o_2}}\right)^n = \frac{o_1}{o_2}$$

■

2.8. Power with rational exponent

Definition 2.74. In the space RIU we can define power with rational exponent $\frac{m}{n}$ (n,m both natural numbers) of the complete number $o(t,\theta,\gamma)$, with $\frac{m}{n}$ known as rational exponent and $o(t,\theta,\gamma)$ known as base, as the number $o_{\uparrow m\downarrow n}(t_{\uparrow m\downarrow n},\theta_{\uparrow m\downarrow n},\gamma_{\uparrow m\downarrow n})$ also represented with the symbol $o(t,\theta,\gamma)^{\frac{m}{n}}$ or $\sqrt[n]{o(t,\theta,\gamma)^m}$ that satisfies the following conditions:

1. $\left[\sqrt[n]{o(t,\theta,\gamma)^m}\right]^n = o(t,\theta,\gamma)^m$
2. $m > 0$ for $o(t,\theta,\gamma) = 0$
3. $n \neq 0$ for any $o(t,\theta,\gamma)^m$ and therefore for any $o(t,\theta,\gamma)$

4. $n \geq 0$ for $o(t, \theta, \gamma)^m = 0$ and therefore for $o(t, \theta, \gamma) = 0$
5. $\theta_{\uparrow m \downarrow n} = \frac{\theta \cdot m}{n}$, $\gamma_{\uparrow m \downarrow n} = \frac{\gamma \cdot m}{n}$
6. $\sqrt[n]{t^m} > 0$, $t^m > 0$
7. $\sqrt[n]{t} > 0$, $t > 0$

The first condition defines the power with rational exponent as a n-th root of a m-th power.

The second condition is required for the correct definition of the m-th power.

The third, the fourth, the fifth and the sixth conditions are required for the correct definition of n-th root.

The seventh condition is required to make possible the reversal of the order between root and power, namely to write:

$$\left[\sqrt[n]{o(t, \theta, \gamma)}\,\right]^m$$

and therefore:

$$(\,\sqrt[n]{t}\,)^m$$

Theorem 2.75. *It is valid the inversion property between root and power, namely:*

$$o^{\frac{m}{n}} = (\,\sqrt[n]{o}\,)^m$$

Proof. For the proof we will make reference to the following formulation of the property just introduced:

$$\sqrt[n]{o^m} = (\,\sqrt[n]{o}\,)^m$$

By applying the principle according to which two numbers are equal if and only if they remain as such, also once we raise them to the same power, we can raise the two member of the previous equality to the number n obtaining:

$$\left(\sqrt[n]{o^m}\,\right)^n = [(\,\sqrt[n]{o}\,)^m]^n$$

At this point, we can verify the validity of the starting equality showing how the two members thus obtained are actually equal.

Considering the value $\sqrt[n]{o}$ of the second member as a complete number, it is possible to apply to it the theorem 2.65 concerning the product of exponents of the n-th power, obtaining:

$$[(\,\sqrt[n]{o}\,)^m]^n = (\,\sqrt[n]{o}\,)^{m \cdot n} = (\,\sqrt[n]{o}\,)^{n \cdot m} = [(\,\sqrt[n]{o}\,)^n]^m$$

Then applying to this member the definition of n-th root , we obtain:

$$[(\,\sqrt[n]{o}\,)^n]^m = o^m$$

By applying the same definition to the first member we obtain an equivalent final result:

$$\left(\sqrt[n]{o^m}\,\right)^n = o^m$$

∎

Theorem 2.76. *It is valid the equivalence property between exponent and degree, namely:*

$$o^{\frac{m}{n}} = o^{\frac{m\cdot p}{n\cdot p}}$$

Proof. For the proof we will make reference to the following formulation of the property just introduced:

$$\sqrt[n]{o^m} = \sqrt[n\cdot p]{o^{m\cdot p}}$$

By applying the principle according to which two numbers are equal if and only if they remain as such, also once we raise them to the same power, we can raise the two member of the previous equality to the number $(n \cdot p)$ obtaining:

$$\left(\sqrt[n]{o^m}\right)^{(n\cdot p)} = \left(\sqrt[n\cdot p]{o^{m\cdot p}}\right)^{(n\cdot p)}$$

At this point, we can verify the validity of the starting equality showing how the two members thus obtained are actually equal.

By applying to the first member the theorem 2.75 concerning the inversion between root and power of the power with rational exponent, we obtain:

$$\left(\sqrt[n]{o^m}\right)^{(n\cdot p)} = [(\sqrt[n]{o})^m]^{(n\cdot p)}$$

Considering the value $\sqrt[n]{o}$ of this member as a complete number, it is possible to apply to it the theorem 2.65 concerning the product of exponents of the n-th power, obtaining:

$$[(\sqrt[n]{o})^m]^{(n\cdot p)} = (\sqrt[n]{o})^{m\cdot n\cdot p} == (\sqrt[n]{o})^{n\cdot m\cdot p} = [(\sqrt[n]{o})^n]^{(m\cdot p)}$$

Then applying to this member the definition of n-th root , we obtain:

$$[(\sqrt[n]{o})^n]^{(m\cdot p)} = o^{m\cdot p}$$

By applying the same definition to the second member we obtain an equivalent final result:

$$\left(\sqrt[n\cdot p]{o^{m\cdot p}}\right)^{(n\cdot p)} = o^{m\cdot p}$$

∎

Theorem 2.77. *It is valid the product property of rational exponents, namely:*

$$(o^{\frac{m}{n}})^{\frac{p}{q}} = o^{\frac{m}{n}\cdot\frac{p}{q}} = o^{\frac{m\cdot p}{n\cdot p}}$$

Proof. For the proof we will make reference to the following formulation of the property just introduced:

$$\left(\sqrt[n]{o^m}\right)^{\frac{p}{q}} = \sqrt[n\cdot q]{o^{m\cdot p}}$$

At this point, we can verify the validity of the starting equality showing how the two members thus obtained are actually equal.

Let us start expressing the first member in the following way:

$$\left(\sqrt[n]{o^m}\right)^{\frac{p}{q}} = \sqrt[q]{\left(\sqrt[n]{o^m}\right)^p}$$

Considering the value o^m of this member as a complete number, it is possible to apply to it the theorem 2.75 concerning the inversion between root and power of the power with rational exponent, obtaining:

$$\sqrt[q]{(\sqrt[n]{o^m})^p} = \sqrt[q]{\sqrt[n]{(o^m)^p}}$$

Then applying to this member the theorem 2.71 concerning the product of degrees of the n-th root and the theorem 2.65 concerning the product of exponents of the n-th power, we obtain an expression coincident with the second member:

$$\sqrt[q]{\sqrt[n]{(o^m)^p}} = \sqrt[q\cdot n]{o^{m\cdot p}} \qquad \blacksquare$$

Theorem 2.78. *It is valid the sum property of rational exponents, namely:*

$$(o^{\frac{m}{n}}) \cdot (o^{\frac{p}{q}}) = o^{(\frac{m}{n}+\frac{p}{q})} = o^{\frac{(m\cdot q)+(p\cdot n)}{n\cdot q}}$$

Proof. For the proof we will make reference to the following formulation of the property just introduced:

$$\sqrt[n]{o^m} \cdot \sqrt[q]{o^p} = \sqrt[n\cdot q]{o^{(m\cdot q+p\cdot n)}}$$

At this point, we can verify the validity of the starting equality showing how the two members thus obtained are actually equal.

By applying to the first member the theorem 2.76 concerning the equivalence between exponent and degree of the power with rational exponent, we obtain:

$$\sqrt[n]{o^m} \cdot \sqrt[q]{o^p} = \sqrt[n\cdot q]{o^{m\cdot q}} \cdot \sqrt[q\cdot n]{o^{p\cdot n}}$$

Considering the values $o^{m\cdot q}$ and $o^{p\cdot n}$ of this member as the complete numbers, it is possible to apply to it the theorem 2.72 concerning the product of radicands of the n-th root, obtaining:

$$\sqrt[n\cdot q]{o^{m\cdot q}} \cdot \sqrt[q\cdot n]{o^{p\cdot n}} = \sqrt[n\cdot q]{o^{m\cdot q} \cdot o^{p\cdot n}}$$

Then applying to this member the theorem 2.66 concerning the sum of exponents of the n-th power, we obtain an expression coincident with the second member:

$$\sqrt[n\cdot q]{o^{m\cdot q} \cdot o^{p\cdot n}} = \sqrt[n\cdot q]{o^{(m\cdot q+p\cdot n)}} \qquad \blacksquare$$

Theorem 2.79. *It is valid the difference property of rational exponents, namely:*

$$\frac{(o^{\frac{m}{n}})}{(o^{\frac{p}{q}})} = o^{(\frac{m}{n}-\frac{p}{q})} = o^{\frac{(m\cdot q)-(p\cdot n)}{n\cdot q}}$$

Proof. For the proof we will make reference to the following formulation of the property just introduced:

$$\frac{\sqrt[n]{o^m}}{\sqrt[q]{o^p}} = \sqrt[n\cdot q]{o^{(m\cdot q - p\cdot n)}}$$

At this point, we can verify the validity of the starting equality showing how the two members thus obtained are actually equal.

By applying to the first member the theorem 2.76 concerning the equivalence between exponent and degree of the power with rational exponent, we obtain:

$$\frac{\sqrt[n]{o^m}}{\sqrt[q]{o^p}} = \frac{\sqrt[n\cdot q]{o^{m\cdot q}}}{\sqrt[q\cdot n]{o^{p\cdot n}}}$$

Considering the values $o^{m\cdot q}$ and $o^{p\cdot n}$ of this member as the complete numbers, it is possible to apply to it the theorem 2.73 concerning the quotient of radicands of the n-th root, obtaining:

$$\frac{\sqrt[n\cdot q]{o^{m\cdot q}}}{\sqrt[q\cdot n]{o^{p\cdot n}}} = \sqrt[n\cdot q]{\frac{o^{m\cdot q}}{o^{p\cdot n}}}$$

Then applying to this member the theorem 2.67 concerning the difference of exponents of the n-th power, we obtain an expression coincident with the second member:

$$\sqrt[n\cdot q]{\frac{o^{m\cdot q}}{o^{p\cdot n}}} = \sqrt[n\cdot q]{o^{(m\cdot q - p\cdot n)}} \qquad \blacksquare$$

Theorem 2.80. *It is valid the product property of bases, namely:*

$$(o_1^{\frac{m}{n}}) \cdot (o_2^{\frac{m}{n}}) = (o_1 \cdot o_2)^{\frac{m}{n}}$$

Proof. For the proof we will make reference to the following formulation of the property just introduced:

$$\sqrt[n]{o_1^m} \cdot \sqrt[n]{o_2^m} = \sqrt[n]{(o_1 \cdot o_2)^m}$$

At this point, we can verify the validity of the starting equality showing how the two members thus obtained are actually equal.

By applying to the second member the theorem 2.68 concerning the product of bases of the n-th power, we obtain:

$$\sqrt[n]{(o_1 \cdot o_2)^m} = \sqrt[n]{o_1^m \cdot o_2^m}$$

Considering the values o_1^m and o_2^m of this member as the complete numbers, it is possible to apply to it the theorem 2.72 concerning the product of radicands of the n-th root, obtaining an expression coincident with the first member:

$$\sqrt[n]{o_1^m \cdot o_2^m} = \sqrt[n]{o_1^m} \cdot \sqrt[n]{o_2^m} \qquad \blacksquare$$

Theorem 2.81. *It is valid the quotient property of bases, namely:*

$$\frac{o_1^{\frac{m}{n}}}{o_2^{\frac{m}{n}}} = \left(\frac{o_1}{o_2}\right)^{\frac{m}{n}}$$

Proof. For the proof we will make reference to the following formulation of the property just introduced:

$$\frac{\sqrt[n]{o_1^m}}{\sqrt[n]{o_2^m}} = \sqrt[n]{\left(\frac{o_1}{o_2}\right)^m}$$

At this point, we can verify the validity of the starting equality showing how the two members thus obtained are actually equal.

By applying to the second member the theorem 2.69 concerning the quotient of bases of the n-th power, we obtain:

$$\sqrt[n]{\left(\frac{o_1}{o_2}\right)^m} = \sqrt[n]{\frac{o_1^m}{o_2^m}}$$

Considering the values o_1^m and o_2^m of this member as the complete numbers, it is possible to apply to it the theorem 2.73 concerning the quotient of radicands of the n-th root, obtaining an expression coincident with the first member:

$$\sqrt[n]{\frac{o_1^m}{o_2^m}} = \frac{\sqrt[n]{o_1^m}}{\sqrt[n]{o_2^m}} \qquad \blacksquare$$

3. Numbers in the n dimensional space

3.1. N dimensional complete numbers

To identify the n dimensional complete numbers, we will use the following notations:

1. $o(a)$ or $o(t)$ for the real numbers
2. $o(a, b)$ or $o(t, \theta)$ for the complex numbers
3. $o(a, b, c)$ or $o(t, \theta, \gamma)$ for the complete number strictly speaking
4. $o(a, b, c, d)$ or $o(t, \theta, \gamma, \varphi)$ for the four dimensional complete numbers
5. $\cdots$
6. $o(a_1, a_2, .., a_n)$ or $o(t, \theta_2, \theta_3, .., \theta_n)$ for the n dimensional complete numbers

Definition 3.1. We can define n dimensional complete number $o(t, \theta_2, \theta_3, ..., \theta_n)$ as the position that can be reached starting from that unitary of the straight line V_1 first translating it of modulus t, then making the line R turn of the angle θ_2 in the plane V_1V_n, next making the plane V_1V_n turn of the angle θ_3 in the space $V_1V_{n-1}V_n$, after that making the space $V_1V_{n-1}V_n$ turn of the angle θ_4 in the hyperspace $V_1V_{n-2}V_{n-1}V_n$, and so on up to the rotation of angle θ_n of the n dimensional space $V_1V_2...V_{n-2}V_{n-1}V_n$.

In Figure 30 we can observe a complete number in the four dimensional space.

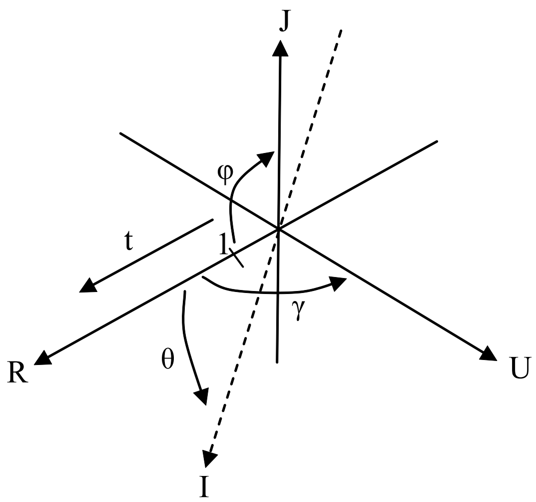

Figure 30: Cartesian representation of the four dimensional complete numbers

Theorem 3.2. *N dimensional complete numbers can be expressed in the following way:*

$$
\begin{aligned}
o(t, \theta_2, \theta_3, ..., \theta_n) = t \cdot \{ & v_1 \cdot [\cos(\theta_n) \cdot \cos(\theta_{n-1}) \cdot .. \cdot \cos(\theta_5) \cdot \cos(\theta_4) \cdot \cos(\theta_3) \cdot \cos(\theta_2)] \\
& + v_2 \cdot [\cos(\theta_n) \cdot \cos(\theta_{n-1}) \cdot .. \cdot \cos(\theta_5) \cdot \cos(\theta_4) \cdot \cos(\theta_3) \cdot \sin(\theta_2)] \\
& + v_3 \cdot [\cos(\theta_n) \cdot \cos(\theta_{n-1}) \cdot .. \cdot \cos(\theta_5) \cdot \cos(\theta_4) \cdot \sin(\theta_3)] \\
& + v_4 \cdot [\cos(\theta_n) \cdot \cos(\theta_{n-1}) \cdot .. \cdot \cos(\theta_5) \cdot \sin(\theta_4)] \\
& + ... + \\
& + v_{n-1} \cdot [\cos(\theta_n) \cdot \sin(\theta_{n-1})] \\
& + v_n \cdot [\sin(\theta_n)]\}
\end{aligned} \tag{3.1}
$$

with the symbols v_1,v_2,...,v_n that identify the versors concerning the orthogonal straight lines V_1,V_2,...,V_n that form the n dimensional space, the symbols θ_2, θ_3,...,θ_n the rotations used to introduce such lines (the line V_1 is introduced by the translating t), and the following symbols a_1,a_2,...,a_n constitute the coordinates of the complete number $o(t, \theta_2, \theta_3, ..., \theta_n)$ in the n dimensional space:

$$
\begin{aligned}
a_1 &= t \cdot [\cos(\theta_n) \cdot \cos(\theta_{n-1}) \cdot ... \cdot \cos(\theta_5) \cdot \cos(\theta_4) \cdot \cos(\theta_3) \cdot \cos(\theta_2)] \\
a_2 &= t \cdot [\cos(\theta_n) \cdot \cos(\theta_{n-1}) \cdot ... \cdot \cos(\theta_5) \cdot \cos(\theta_4) \cdot \cos(\theta_3) \cdot \sin(\theta_2)] \\
a_3 &= t \cdot [\cos(\theta_n) \cdot \cos(\theta_{n-1}) \cdot ... \cdot \cos(\theta_5) \cdot \cos(\theta_4) \cdot \sin(\theta_3)] \\
a_4 &= t \cdot [\cos(\theta_n) \cdot \cos(\theta_{n-1}) \cdot ... \cdot \cos(\theta_5) \cdot \sin(\theta_4)] \\
&... \\
a_n &= t \cdot [\sin(\theta_n)]
\end{aligned}
$$

Proof. By observing in Figure 31 how the addition of a new rotation allows us to express the coordinates of the complete numbers from one dimension to the next we obtain the previous relation. ∎

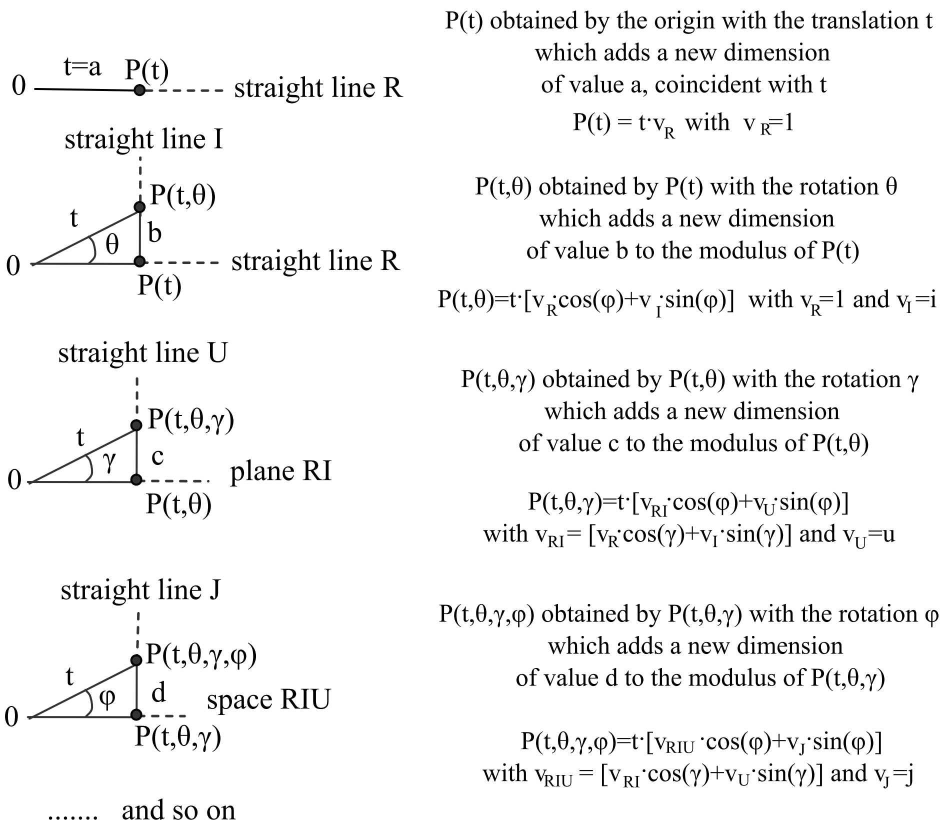

Figure 31: Construction of the n dimensional complete numbers

Theorem 3.3. *The modulus of the n dimensional complete numbers can be expressed in the following way:*

$$t = \sqrt{a_1^2 + a_2^2 + a_3^2 + ...a_n^2}$$

Proof. By applying Pythagoras' theorem to the steps leading us to the next dimensions, as shown by Figure 32 on the next page, we obtain the previous relation. ∎

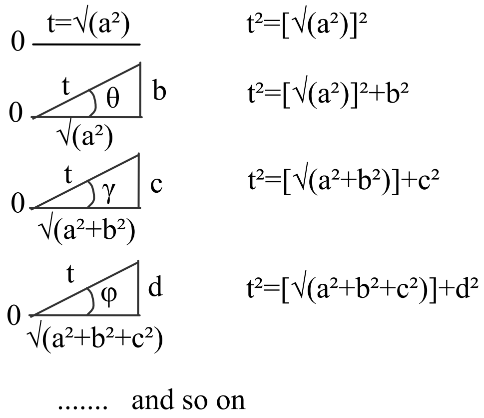

Figure 32: Representation of the modulus of the n dimensional complete numbers

Theorem 3.4. *The phases of the n dimensional complete numbers can be expressed in the following way:*

$$\theta_n = \arctan\left(\frac{a_n}{\sqrt{a_1^2 + a_2^2 + a_3^2 + ...a_{n-1}^2}}\right)$$

Proof. By applying the trigonometric relations of the function arctan() to the steps leading us to the next dimensions, as shown by Figure 33, we obtain the previous relation.

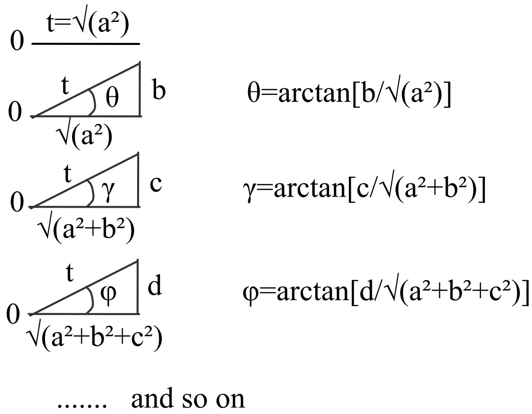

Figure 33: Representation of the phases of the n dimensional complete numbers

∎

Definition 3.5. An n dimensional complete numbers with coordinates (a_1,a_2,a_3,...,a_n) all non zero can be defined in standard representation if provided with phases (θ_2,θ_3,...,θ_n) that satisfy the conventions introduced hereunder.

For the positions $P(a_1, a_2, a_3, ..., a_n)$ in the region $V_1^+V_2^+V_3^+...V_n^+$, characterized by the values a_1,a_2,a_3,...,a_n all positives, the phases chosen will lie in the first quadrant, namely:

$$0^\circ < \theta_2, \theta_3, ..., \theta_n < 90^\circ$$

We can observe, with regard to this, Figure 34.

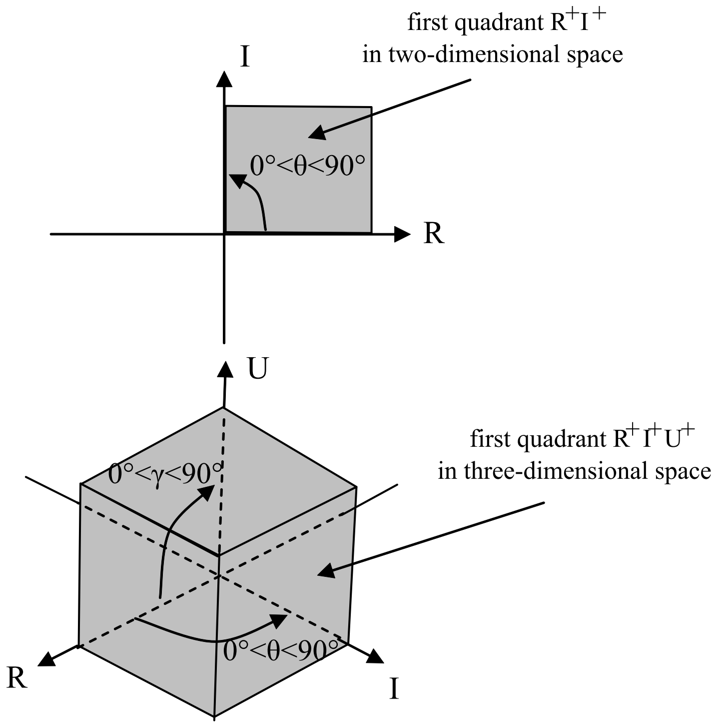

Figure 34: Standard representation of the phases θ,γ concerning the first quadrants

Since the following relations are valid:

$$\theta_2 = \arctan\left(\frac{a_2}{\sqrt{a_1^2}}\right)$$
$$\theta_3 = \arctan\left(\frac{a_3}{\sqrt{a_1^2 + a_2^2}}\right)$$
$$...$$
$$\theta_n = \arctan\left(\frac{a_n}{\sqrt{a_1^2 + a_2^2 + a_3^2 + ...a_{n-1}^2}}\right)$$

to allow the phases θ_2,θ_3,...,θ_n to have a value between $0°$ and $90°$ when the coefficients a_2,a_3,...,a_n are all positives, also the corresponding denominators should be positives. This means that the standard representation requires that we assign the positive solutions to the following roots:

$$\sqrt{a_1^2} = \left|\sqrt{a_1^2}\right|$$
$$\sqrt{a_1^2 + a_2^2} = \left|\sqrt{a_1^2 + a_2^2}\right|$$
$$...$$
$$\sqrt{a_1^2 + a_2^2 + a_3^2 + ...a_{n-1}^2} = \left|\sqrt{a_1^2 + a_2^2 + a_3^2 + ...a_{n-1}^2}\right|$$

For the positions $P(a_1, a_2, a_3, ..., a_n)$ in the region $V_1^- V_2^+ V_3^+ ...V_n^+$, characterized by the values a_2,a_3,a_4,...,a_n all positives and by the value a_1 negative, the phases chosen will be the following:

$$90° < \theta_2 < 180°$$
$$0° < \theta_3, \theta_4, ..., \theta_n < 90°$$

Since the following relations are valid:

$$\sin(180° - \theta_2) = \sin(\theta_2)$$
$$\cos(180° - \theta_2) = -\cos(\theta_2)$$

to impose the coefficient a_1 as the only negative value in the formula (3.1), will be enough to leave unchanged all phases θ_3,θ_4,...,θ_n at the value they have in the first quadrant, and change the value of $\theta_2 = \theta_2^*$ (that is the value that this phase assumes in the first quadrant) with $\theta_2 = (180° - \theta_2^*)$.

We can observe, with regard to this, Figure 35 on the facing page.

Since the following relations are valid:

$$\theta_2 = \arctan\left(\frac{a_2}{\sqrt{a_1^2}}\right)$$
$$\theta_3 = \arctan\left(\frac{a_3}{\sqrt{a_1^2 + a_2^2}}\right)$$
$$...$$
$$\theta_n = \arctan\left(\frac{a_n}{\sqrt{a_1^2 + a_2^2 + a_3^2 + ...a_{n-1}^2}}\right)$$

to allow the phases θ_3,θ_4,...,θ_n to have a value between 0° and 90° when the coefficients a_3,a_4,...,a_n are all positives, also the corresponding denominators should be positives. While to allow the phase θ_2 to have a value between 90° and 180° when the coefficient a_1 is negative and that a_2 is positive, we should consider the term which appears into its denominator as negative. This means that the standard representation requires that we assign the positive solutions to the following roots:

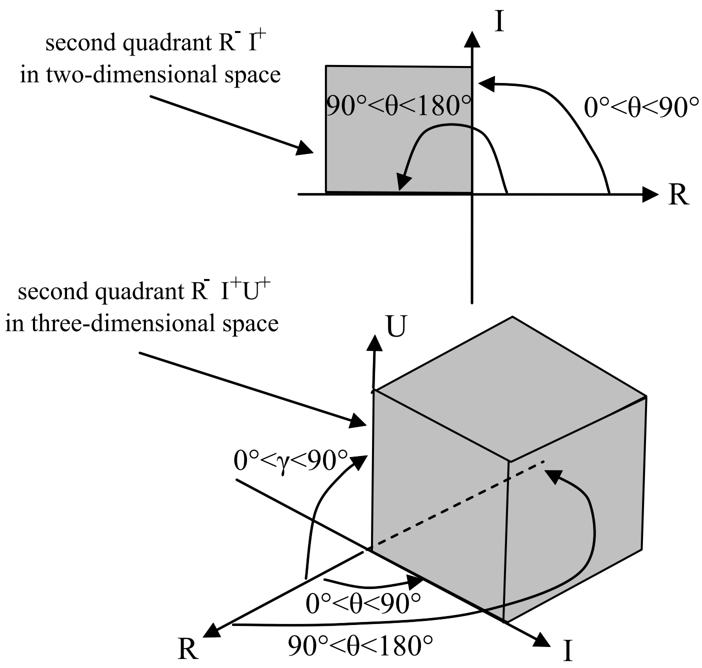

Figure 35: Standard representation of the phases θ,γ concerning the second quadrants

$$
\begin{aligned}
&\sqrt{a_1^2 + a_2^2} = \left|\sqrt{a_1^2 + a_2^2}\right| \\
&\sqrt{a_1^2 + a_2^2 + a_3^2} = \left|\sqrt{a_1^2 + a_2^2 + a_3^2}\right| \\
&... \\
&\sqrt{a_1^2 + a_2^2 + a_3^2 + ...a_{n-1}^2} = \left|\sqrt{a_1^2 + a_2^2 + a_3^2 + ...a_{n-1}^2}\right|
\end{aligned}
$$

and the negative solutions to:

$$\sqrt{a_1^2} = -\left|\sqrt{a_1^2}\right|$$

For the positions $P(a_1, a_2, a_3, ..., a_n)$ in the region $V_1^- V_2^- V_3^+ ... V_n^+$, characterized by the values a_3,a_4,a_5,...,a_n all positives and by the values a_1,a_2 negative, the phases chosen will be the following:

$$
\begin{aligned}
&180^\circ < \theta_2 < 270^\circ \\
&0^\circ < \theta_3, \theta_4, ..., \theta_n < 90^\circ
\end{aligned}
$$

Since the following relations are valid:

$$
\begin{aligned}
\sin(180^\circ + \theta_2) &= -\sin(\theta_2) \\
\cos(180^\circ + \theta_2) &= -\cos(\theta_2)
\end{aligned}
$$

to impose the coefficients a_1 and a_2 as the only negative values in the formula (3.1), will be enough to leave unchanged all phases θ_3,θ_4,...,θ_n at the value they have in the first quadrant, and change the value of $\theta_2 = \theta_2^*$ (that is the value that this phase assumes in the first quadrant) with $\theta_2 = (180° + \theta_2^*)$.

We can observe, with regard to this, Figure 36.

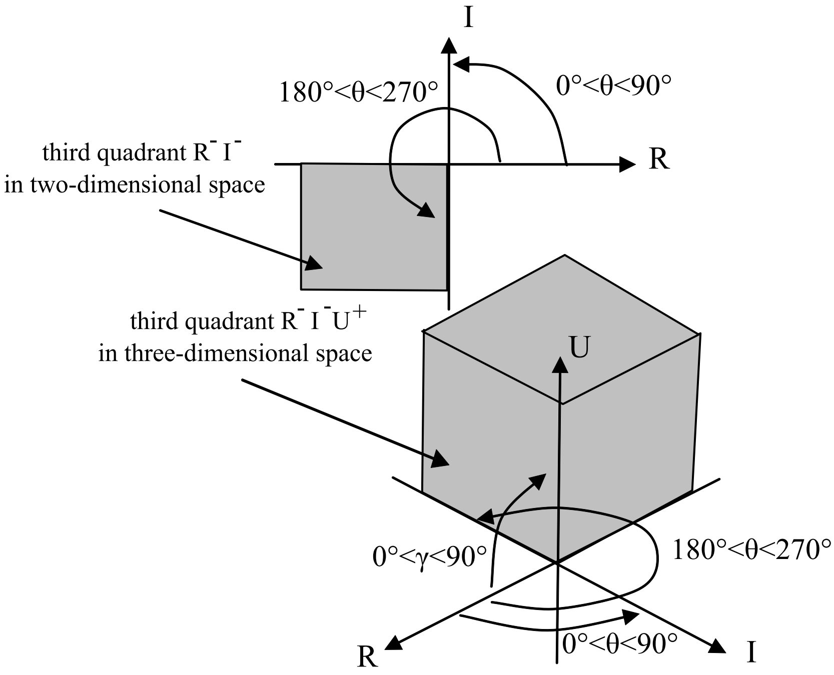

Figure 36: Standard representation of the phases θ,γ concerning the third quadrants

Since the following relations are valid:

$$\theta_2 = \arctan\left(\frac{a_2}{\sqrt{a_1^2}}\right)$$
$$\theta_3 = \arctan\left(\frac{a_3}{\sqrt{a_1^2 + a_2^2}}\right)$$
$$...$$
$$\theta_n = \arctan\left(\frac{a_n}{\sqrt{a_1^2 + a_2^2 + a_3^2 + ...a_{n-1}^2}}\right)$$

to allow the phases θ_3,θ_4,...,θ_n to have a value between 0° and 90° when the coefficients a_3,a_4,...,a_n are all positives, also the corresponding denominators should be positives. While to allow the phase θ_2 to have a value between 180° and 270° when the coefficient a_1 and a_2 are negative, we should consider the term which appears into its denominator as negative. This means that the standard representation requires that we assign the positive solutions to the following roots:

$$\sqrt{a_1^2+a_2^2} = \left|\sqrt{a_1^2+a_2^2}\,\right|$$
$$\sqrt{a_1^2+a_2^2+a_3^2} = \left|\sqrt{a_1^2+a_2^2+a_3^2}\,\right|$$
$$...$$
$$\sqrt{a_1^2+a_2^2+a_3^2+...a_{n-1}^2} = \left|\sqrt{a_1^2+a_2^2+a_3^2+...a_{n-1}^2}\,\right|$$

and the negative solutions to:

$$\sqrt{a_1^2} = -\left|\sqrt{a_1^2}\,\right|.$$

For the positions $P(a_1, a_2, a_3, ..., a_n)$ in the region $V_1^+V_2^-V_3^+...V_n^+$, characterized by the values a_1,a_3,a_4,...,a_n all positives and by the value a_2 negative, the phases chosen will be the following:

$$270° < \theta_2 < 360°$$
$$0° < \theta_3, \theta_4, ..., \theta_n < 90°$$

Since the following relations are valid:

$$\sin(360° - \theta_2) = -\sin(\theta_2)$$
$$\cos(360° - \theta_2) = \cos(\theta_2)$$

to impose the coefficient a_2 as the only negative value in the formula (3.1), will be enough to leave unchanged all phases θ_3,θ_4,...,θ_n at the value they have in the first quadrant, and change the value of $\theta_2 = \theta_2^*$ (that is the value that this phase assumes in the first quadrant) with $\theta_2 = (360° - \theta_2^*)$.

We can observe, with regard to this, Figure 37 on the following page.

Since the following relations are valid:

$$\theta_2 = \arctan\left(\frac{a_2}{\sqrt{a_1^2}}\right)$$
$$\theta_3 = \arctan\left(\frac{a_3}{\sqrt{a_1^2+a_2^2}}\right)$$
$$...$$
$$\theta_n = \arctan\left(\frac{a_n}{\sqrt{a_1^2+a_2^2+a_3^2+...a_{n-1}^2}}\right)$$

to allow the phases θ_3,θ_4,...,θ_n to have a value between 0° and 90° when the coefficients a_3,a_4,...,a_n are all positives, also the corresponding denominators should be positives. While to allow the phase θ_2 to have a value between 270° and 360° when the coefficient a_2 is negative and that a_1 is positive, we should consider the

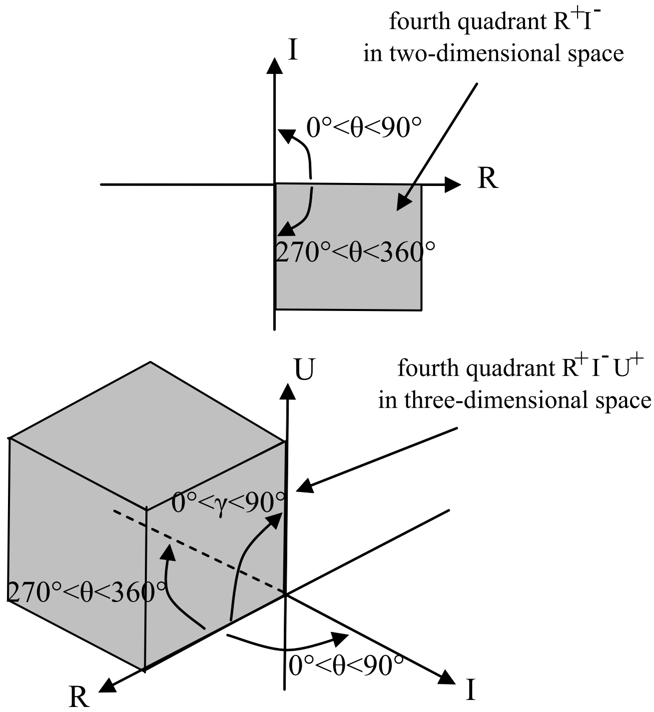

Figure 37: Standard representation of the phases θ,γ concerning the fourth quadrants

term which appears into its denominator as positive. This means that the standard representation requires that we assign the positive solutions to the following roots:

$$\sqrt{a_1^2} = \left|\sqrt{a_1^2}\right|$$
$$\sqrt{a_1^2 + a_2^2} = \left|\sqrt{a_1^2 + a_2^2}\right|$$
$$\sqrt{a_1^2 + a_2^2 + a_3^2 + ...a_{n-1}^2} = \left|\sqrt{a_1^2 + a_2^2 + a_3^2 + ...a_{n-1}^2}\right|$$

For the positions $P(a_1, a_2, a_3, ..., a_n)$ in the region $V_1V_2V_3...V_n$, characterized by the values a_3,a_4,a_5,...,a_n both positives and negative, the phases chosen will be the following:

$$0^\circ < \theta_i < 90^\circ \quad \text{for any } a_i > 0 \text{ with } i = 3, 4, 5, ..., n$$
$$270^\circ < \theta_i < 360^\circ \quad \text{for any } a_i < 0 \text{ with } i = 3, 4, 5, ..., n$$

Since the following relations are valid:

$$\sin(360^\circ - \theta_i) = -\sin(\theta_i)$$
$$\cos(360^\circ - \theta_i) = \cos(\theta_i)$$

to impose the negative sign to some of the coefficients a_3,a_4,...,a_n in the formula (3.1), will be enough to assign to the corresponding phases θ_3,θ_4,...,θ_n the opposite value with respect to that they have in the first quadrant (and therefore to assign them a value between 270° and 360°) and leave all the others unchanged.

We can observe, with regard to this, Figure 38.

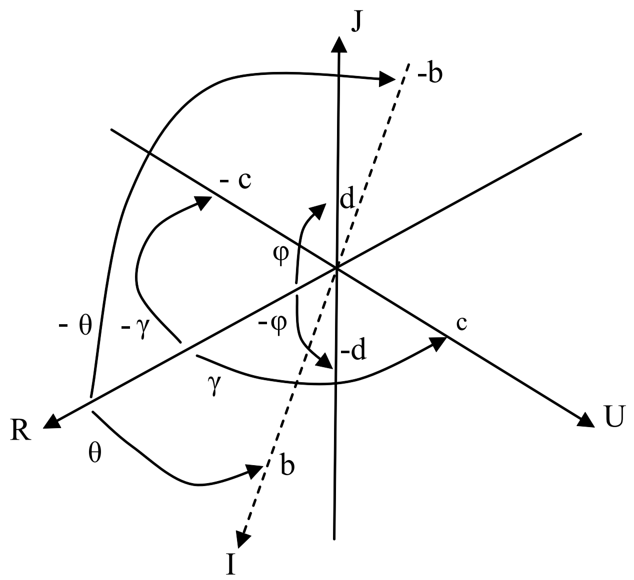

Figure 38: Variation of the sign of the phases due to the variation of sign of their corresponding coefficient

Since the following relations are valid:

$$\theta_3 = \arctan\left(\frac{a_3}{\sqrt{a_1^2 + a_2^2}}\right)$$
$$\theta_4 = \arctan\left(\frac{a_4}{\sqrt{a_1^2 + a_2^2 + a_3^2}}\right)$$
$$\ldots$$
$$\theta_n = \arctan\left(\frac{a_n}{\sqrt{a_1^2 + a_2^2 + a_3^2 + \ldots a_{n-1}^2}}\right)$$

to allow the phases θ_3,θ_4,...,θ_n to have a value between 0° and 90° when the corresponding coefficients a_3,a_4,...,a_n are positives, and a value between 270° and 360° when the corresponding coefficients are negative, the corresponding denominators should be all positives. This means that the standard representation requires that we assign the positive solutions to the following roots:

$$\sqrt{a_1^2 + a_2^2} = \left|\sqrt{a_1^2 + a_2^2}\right|$$
$$\sqrt{a_1^2 + a_2^2 + a_3^2} = \left|\sqrt{a_1^2 + a_2^2 + a_3^2}\right|$$
$$\sqrt{a_1^2 + a_2^2 + a_3^2 + \ldots a_{n-1}^2} = \left|\sqrt{a_1^2 + a_2^2 + a_3^2 + \ldots a_{n-1}^2}\right|$$

Since the management of sign of the coefficients a_3,a_4,...,a_n does not interfere with the angle θ_2, we can combine it with the management of signs of a_1 and a_2 according to the manner described above.

Theorem 3.6. *The standard representation of a n dimensional complete number of coordinates* $(a_1,a_2,a_3,...,a_n)$ *all non zero requires to give the following solutions to the following algebraic roots:*

$$\begin{aligned}
&\sqrt{a_1^2} = a_1\\
&\sqrt{a_1^2 + a_2^2} = \left|\sqrt{a_1^2 + a_2^2}\,\right|\\
&\sqrt{a_1^2 + a_2^2 + a_3^2} = \left|\sqrt{a_1^2 + a_2^2 + a_3^2}\,\right|\\
&...\\
&\sqrt{a_1^2 + a_2^2 + a_3^2 + ...a_{n-1}^2} = \left|\sqrt{a_1^2 + a_2^2 + a_3^2 + ...a_{n-1}^2}\,\right|
\end{aligned}$$

Proof. In the case of the representations previously examined the phases assume the values provided by the formulas:

$$\begin{aligned}
&\theta_2 = \arctan\left(\frac{a_2}{\sqrt{a_1^2}}\right)\\
&\theta_3 = \arctan\left(\frac{a_3}{\sqrt{a_1^2 + a_2^2}}\right)\\
&...\\
&\theta_n = \arctan\left(\frac{a_n}{\sqrt{a_1^2 + a_2^2 + a_3^2 + ...a_{n-1}^2}}\right)
\end{aligned}$$

when we give to the algebraic roots involved just the values considered here. And this immediately proves the thesis. ∎

For example in the four dimensional space the complete number with the expression:

$$\begin{aligned}
o(t,\theta,\gamma,\varphi) = &[\cos(\varphi)\cdot\cos(\gamma)\cdot\cos(\theta)] + i\cdot[\cos(\varphi)\cdot\cos(\gamma)\cdot\sin(\theta)]\\
&+ u\cdot[\cos(\varphi)\cdot\sin(\gamma)] + j\cdot[\sin(\varphi)]
\end{aligned}$$

associated to the position:

$$o(a,b,c,d) = o(-1,1,-1,1)$$

could be expressed in standard representation through the following phases:

$$\begin{aligned}
&\theta = \arctan\left(\frac{b}{a}\right) = \arctan\left(\frac{1}{-1}\right) = 135^\circ\\
&\gamma = \arctan\left(\frac{c}{|\sqrt{a^2+b^2}\,|}\right) = \arctan\left(\frac{-1}{\sqrt{2}}\right) \simeq -35.26^\circ\\
&\varphi = \arctan\left(\frac{d}{|\sqrt{a^2+b^2+c^2}\,|}\right) = \arctan\left(\frac{1}{\sqrt{3}}\right) = 30^\circ
\end{aligned}$$

and the following modulus:

$$t = \sqrt{a^2 + b^2 + c^2 + d^2} = \sqrt{4} = 2$$

To verify that the standard representation $o(\theta, \gamma, \varphi)$ thus obtained identifies just the position $o(-1, 0, 1, 0)$ it is sufficient to perform the following calculations:

$$\begin{aligned}
&a = t \cdot \cos(\varphi) \cdot \cos(\gamma) \cdot \cos(\theta) = 2 \cdot \cos(30°) \cdot \cos(\simeq -35.26°) \cdot \cos(135°) = -1 \\
&b = t \cdot \cos(\varphi) \cdot \cos(\gamma) \cdot \sin(\theta) = 2 \cdot \cos(30°) \cdot \cos(\simeq -35.26°) \cdot \sin(135°) = 1 \\
&c = t \cdot \cos(\varphi) \cdot \sin(\gamma) = 2 \cdot \cos(30°) \cdot \sin(\simeq -35.26°) = -1 \\
&d = t \cdot \sin(\varphi) = 2 \cdot \sin(30°) = 1
\end{aligned}$$

Definition 3.7. An n dimensional complete numbers with coordinates (a_1,a_2,a_3,...,a_n) all non zero can be defined in complementary representation if provided with phases obtained by the values: θ_2,θ_3,...,θ_n of the standard representation through those substitutions which allow us to identify the same positions.

Theorem 3.8. *If we call θ_2,θ_3,...,θ_n the phases that allow to an n dimensional complete number provided with coordinates (a_1,a_2,a_3,...,a_n) all non zero and in standard representation to identify any position of the space $V_1V_2V_3...V_n$, the alternative sets of phases able to individuate the same position can be obtained by the following values: θ, $(360° - \theta)$, $(180° - \theta)$, $(\theta + 180°)$.*

Proof. The ability to express through the formula (3.1) the same positions of the standard representation, assigning to the phases the following values: θ, $(360° - \theta)$, $(180° - \theta)$, $(\theta + 180°)$ comes from the fact that in this way we maintain the moduli unchanged and introduce signs which can neutralize each other, as shown by the following relations:

$$\begin{aligned}
&\cos(\theta) = \cos(\theta) \\
&\sin(\theta) = \sin(\theta) \\
&\cos(360° - \theta) = \cos(\theta) \\
&\sin(360° - \theta) = -\sin(\theta) \\
&\cos(180° - \theta) = -\cos(\theta) \\
&\sin(180° - \theta) = \sin(\theta) \\
&\cos(\theta + 180°) = -\cos(\theta) \\
&\sin(\theta + 180°) = -\sin(\theta)
\end{aligned}$$

Using this process it is possible to combine, for example, the standard representation concerning the fourth dimension:

$$\begin{aligned}
o(t, \theta, \gamma, \varphi) = &[\cos(\varphi) \cdot \cos(\gamma) \cdot \cos(\theta)] + i \cdot [\cos(\varphi) \cdot \cos(\gamma) \cdot \sin(\theta)] \\
&+ u \cdot [\cos(\varphi) \cdot \sin(\gamma)] + j \cdot [\sin(\varphi)]
\end{aligned}$$

to the following complementary representations:

$$\begin{aligned} o(t,\theta,\gamma+180°,180°-\varphi) = &[\cos(180°-\varphi)\cdot\cos(\gamma+180°)\cdot\cos(\theta)] \\ &+ i\cdot[\cos(180°-\varphi)\cdot\cos(\gamma+180°)\cdot\sin(\theta)] \\ &+ u\cdot[\cos(180°-\varphi)\cdot\sin(\gamma+180°)] \\ &+ j\cdot[\sin(180°-\varphi)] \end{aligned}$$

$$\begin{aligned} o(t,\theta+180°,360°-\gamma,180°-\varphi) = &[\cos(180°-\varphi)\cdot\cos(360°-\gamma)\cdot\cos(\theta+180°)] \\ &+ i\cdot[\cos(180°-\varphi)\cdot\cos(360°-\gamma)\cdot\sin(\theta+180°)] \\ &+ u\cdot[\cos(180°-\varphi)\cdot\sin(360°-\gamma)] \\ &+ j\cdot[\sin(180°-\varphi)] \end{aligned}$$

$$\begin{aligned} o(t,\theta+180°,180°-\gamma,\varphi) = &[\cos(\varphi)\cdot\cos(180°-\gamma)\cdot\cos(\theta+180°)] \\ &+ i\cdot[\cos(\varphi)\cdot\cos(180°-\gamma)\cdot\sin(\theta+180°)] \\ &+ u\cdot[\cos(\varphi)\cdot\sin(180°-\gamma)] \\ &+ j\cdot[\sin(\varphi)] \end{aligned}$$

■

For example if you want to identify a complementary representation of the following four dimensional complete number:

$$o(a,b,c,d) = o(-1,1,-1,1)$$

whose standard representation is provided with the following phases:

$$\begin{aligned} &\theta^* = 135° \\ &\gamma^* \simeq -35.26° \\ &\varphi^* = 30° \end{aligned}$$

and the following modulus:

$$t = 2$$

it is sufficient to perform the following calculations:

$$\begin{aligned} &\theta = \theta^* = 135° \\ &\gamma = \gamma^* + 180° \simeq (\simeq -35.26°) + 180° \simeq 144.74° \\ &\varphi = 180° - \varphi^* = 180° - 30° = 150° \end{aligned}$$

To verify that the complementary representation $o(\theta,\gamma,\varphi)$ thus obtained identifies just the position $o(-1,1,-1,1)$ it is sufficient to perform the following calculations:

$$\begin{aligned} &a = t\cdot\cos(\varphi)\cdot\cos(\gamma)\cdot\cos(\theta) = 2\cdot\cos(150°)\cdot\cos(\simeq 144.74°)\cdot\cos(135°) = -1 \\ &b = t\cdot\cos(\varphi)\cdot\cos(\gamma)\cdot\sin(\theta) = 2\cdot\cos(150°)\cdot\cos(\simeq 144.74°)\cdot\sin(135°) = 1 \\ &c = t\cdot\cos(\varphi)\cdot\sin(\gamma) = 2\cdot\cos(150°)\cdot\sin(\simeq 144.74°) = -1 \\ &d = t\cdot\sin(\varphi) = 2\cdot\sin(150°) = 1 \end{aligned}$$

Definition 3.9. The n dimensional complete numbers with coordinates (a_1,a_2,a_3,...,a_n) some of which are zero can be defined in standard representation if their phases besides to be consistent with those of the other standard representations (according to the definition 3.5) assume the zero value in case of indetermination.

The relations that give the values of the phases for the standard representation are the following:

$$\theta_2 = \arctan\left(\frac{a_2}{a_1}\right)$$
$$\theta_3 = \arctan\left(\frac{a_3}{|\sqrt{a_1^2+a_2^2}\,|}\right)$$
$$\theta_3 = \arctan\left(\frac{a_4}{|\sqrt{a_1^2+a_2^2+a_3^2}\,|}\right)$$
$$...$$
$$\theta_n = \arctan\left(\frac{a_n}{|\sqrt{a_1^2+a_2^2+a_3^2+...a_{n-1}^2}\,|}\right)$$

Due to coefficients with zero value, we can have the following notable cases:

$$\theta_2 = \arctan\left(\frac{0}{a_1}\right) = \begin{cases} 0^\circ & \text{for } a_1 > 0 \\ 180^\circ & \text{for } a_1 < 0 \end{cases}$$
$$\theta_i = \arctan\left(\frac{a_i}{0}\right) = \begin{cases} 90^\circ & \text{for } a_i > 0 \\ 270^\circ & \text{for } a_i < 0 \end{cases}$$
$$\theta_i = \arctan\left(\frac{0}{|x \neq 0|}\right) = 0^\circ$$
$$\theta_i = \arctan\left(\frac{0}{0}\right) = 0^\circ$$

For example in the four dimensional space the complete number with expression:

$$\begin{aligned} o(t,\theta,\gamma,\varphi) =& [\cos(\varphi)\cdot\cos(\gamma)\cdot\cos(\theta)] + i\cdot[\cos(\varphi)\cdot\cos(\gamma)\cdot\sin(\theta)] \\ &+ u\cdot[\cos(\varphi)\cdot\sin(\gamma)] + j\cdot[\sin(\varphi)] \end{aligned}$$

associated to the position:

$$o(a,b,c,d) = o(-1,0,1,0)$$

could be expressed in standard representation through the following phases:

$$\theta = \arctan\left(\frac{b}{a}\right) = \arctan\left(\frac{0}{-1}\right) = 180^\circ$$
$$\gamma = \arctan\left(\frac{c}{|\sqrt{a^2+b^2}\,|}\right) = \arctan\left(\frac{1}{1}\right) = 45^\circ$$
$$\varphi = \arctan\left(\frac{d}{|\sqrt{a^2+b^2+c^2}\,|}\right) = \arctan\left(\frac{0}{\sqrt{2}}\right) = 0^\circ$$

and the following modulus:

$$t = \sqrt{a^2 + b^2 + c^2 + d^2} = \sqrt{2}$$

To verify that the standard representation $o(\theta, \gamma, \varphi)$ thus obtained identifies just the position $o(-1, 0, 1, 0)$ it is sufficient to perform the following calculations:

$$\begin{aligned}
a &= t \cdot \cos(\varphi) \cdot \cos(\gamma) \cdot \cos(\theta) = \sqrt{2} \cdot \cos(0°) \cdot \cos(45°) \cdot \cos(180°) = -1 \\
b &= t \cdot \cos(\varphi) \cdot \cos(\gamma) \cdot \sin(\theta) = \sqrt{2} \cdot \cos(0°) \cdot \cos(45°) \cdot \sin(180°) = 0 \\
c &= t \cdot \cos(\varphi) \cdot \sin(\gamma) = \sqrt{2} \cdot \cos(0°) \cdot \sin(45°) = 1 \\
d &= t \cdot \sin(\varphi) = \sqrt{2} \cdot \sin(0°) = 0
\end{aligned}$$

Definition 3.10. The n dimensional complete numbers with coordinates $(a_1,a_2,a_3,...,a_n)$ some of which are zero can be defined in complementary representation if their phases besides to be consistent with those of the standard representations (according to the definition 3.5) show cases of indetermination in correspondence of which they do not assume the zero value.

For example in the four dimensional space the complete number with expression:

$$\begin{aligned}
o(t, \theta, \gamma, \varphi) =& [\cos(\varphi) \cdot \cos(\gamma) \cdot \cos(\theta)] + i \cdot [\cos(\varphi) \cdot \cos(\gamma) \cdot \sin(\theta)] + \\
&+ u \cdot [\cos(\varphi) \cdot \sin(\gamma)] + j \cdot [\sin(\varphi)]
\end{aligned}$$

associated to the position:

$$o(a, b, c, d) = o(0, 0, 1, 0)$$

could be expressed in complementary representation through the following phases:

$$\begin{aligned}
\theta &= \arctan\left(\frac{b}{a}\right) = \arctan\left(\frac{0}{0}\right) = 30° \neq 0° \\
\gamma &= \arctan\left(\frac{c}{|\sqrt{a^2 + b^2}\,|}\right) = \arctan\left(\frac{1}{0}\right) = 90° \\
\varphi &= \arctan\left(\frac{d}{|\sqrt{a^2 + b^2 + c^2}\,|}\right) = \arctan\left(\frac{0}{1}\right) = 0°
\end{aligned}$$

and the following modulus:

$$t = \sqrt{a^2 + b^2 + c^2 + d^2} = 1$$

To verify that the complementary representation $o(\theta, \gamma, \varphi)$ thus obtained identifies just the position $o(0, 0, 1, 0)$ it is sufficient to perform the following calculations:

$$\begin{aligned}
a &= t \cdot \cos(\varphi) \cdot \cos(\gamma) \cdot \cos(\theta) = 1 \cdot \cos(0°) \cdot \cos(90°) \cdot \cos(30°) = 0 \\
b &= t \cdot \cos(\varphi) \cdot \cos(\gamma) \cdot \sin(\theta) = 1 \cdot \cos(0°) \cdot \cos(90°) \cdot \sin(0°) = 0 \\
c &= t \cdot \cos(\varphi) \cdot \sin(\gamma) = 1 \cdot \cos(0°) \cdot \sin(90°) = 1 \\
d &= t \cdot \sin(\varphi) = 1 \cdot \sin(0°) = 0
\end{aligned}$$

Since the non zero values associated to the indeterminate phases are unlimited, unlimited will also be the complementary representation defined here.

Theorem 3.11. *N dimensional complete numbers (with $n > 2$) provided with coordinates $(a_1,a_2,a_3,...,a_n)$ cannot be expressed in the following way:*

$$o(a_1, a_2, ..., a_n) = v_1 \cdot a_1 + v_2 \cdot a_2 + ... + v_n \cdot a_n$$

namely:

$$o(t, \theta_2, ..., \theta_n) \neq o(a_1, a_2, ..., a_n) = v_1 \cdot a_1 + v_2 \cdot a_2 + ... + v_n \cdot a_n$$

Proof. The proof comes from the absence of bijection between translation and rotation operations of values $(t, \theta_2, ..., \theta_n)$ and the positions $(a_1, a_2, ..., a_n)$ of the n dimensional space, since always exists (for every dimension higher than the second) the complementary representation with the following phases:

$$o(t, \theta_2, ..., \theta_{(n-2)}, \theta_{(n-1)} + 180°, 180° - \theta_n)$$

In fact, if $(t, \theta_2, ..., \theta_n)$ are the values that make true the formula (3.1) of the n dimensional complete numbers, this same expression will also be satisfied by values:

$$(t, \theta_2, ..., \theta_{(n-2)}, \theta_{(n-1)} + 180°, 180° - \theta_n)$$

as shown by the following trigonometric relations:

$$\begin{aligned}
&\cos(180° - \theta_n) \cdot \cos(\theta_{(n-1)} + 180°) = \cos(\theta_n) \cdot \cos(\theta_{(n-1)}) \\
&\cos(180° - \theta_n) \cdot \sin(\theta_{(n-1)} + 180°) = \cos(\theta_n) \cdot \sin(\theta_{(n-1)}) \\
&\sin(180° - \theta_n) = \sin(\theta_n)
\end{aligned}$$

■

Since it is impossible to associate the complete numbers to the individual positions of the n dimensional space, we can always express them in terms of their coordinates $(a_1, a_2, ..., a_n)$, provided that we make explicit the phases involved as well.

In other words we should use the following notation:

$$o(a_1, a_2, ..., a_n)_{(t,\theta_2,...,\theta_n)} = v_1 \cdot a_{1(t)} + v_2 \cdot a_{2(\theta_2)} + v_3 \cdot a_{3(\theta_3)} + ... + v_n \cdot a_{n(\theta_n)}$$

where the values of $t,\theta_2,...,\theta_n$, if not yet given, should be reported to those which characterize the standard representation.

While any other notation of the following type:

$$o(a_1, a_2, ..., a_n) = v_1 \cdot a_1 + v_2 \cdot a_2 + ... + v_n \cdot a_n$$

that is devoid of sufficient information to trace the values of the phases $\theta_2,...,\theta_n$ will be able to represent the positions of the n dimensional space, but not the complete numbers.

3.2. N dimensional operations

Definition 3.12. In the space $V_1V_2V_3...V_n$ we can define addition between two positions $o(a_{11}, a_{21}, ..., a_{n1})$ and $o(a_{12}, a_{22}, ..., a_{n2})$ as the position $o(a_{1(1+2)}, a_{2(1+2)}, ..., a_{n(1+2)})$ represented also with the symbol $o(a_{11}, a_{21}, ..., a_{n1}) + o(a_{12}, a_{22}, ..., a_{n2})$ that satisfies the following condition:

$$o_{1+2}(a_{1(1+2)}, a_{2(1+2)}, ..., a_{n(1+2)}) = o_{1+2}(a_{11} + a_{12}, a_{11} + a_{12}, ..., a_{n1} + a_{n2})$$

This condition is equivalent to take the position of the space $V_1V_2V_3...V_n$ provided with the following coordinates:

$$\begin{aligned} a_{1(1+2)} &= a_{11} + a_{12} \\ a_{2(1+2)} &= a_{21} + a_{22} \\ &... \\ a_{n(1+2)} &= a_{n1} + a_{n2} \end{aligned}$$

For example in the fourth dimension we have:

$$o_{1+2}(a_{1+2}, b_{1+2}, c_{1+2}, d_{1+2}) = o_{1+2}(a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$$

with:

$$\begin{aligned} a_{1+2} &= a_1 + a_2 \\ b_{1+2} &= b_1 + b_2 \\ c_{1+2} &= c_1 + c_2 \\ d_{1+2} &= d_1 + d_2 \end{aligned}$$

It should be emphasized that the addition must be considered an operation that works on the positions and not on the complete numbers, at least for every dimension higher than the second, for which there is no bijection between the positions and the complete numbers.

To integrate the operation of addition, working on the positions, with the others, working on the complete numbers, will be enough making reference to the complete number that we can obtain assigning to the sum the phases of the standard representation.

Theorem 3.13. *The properties embodied by Theorems* 2.21, 2.22, 2.23, 2.24 *for the third dimension remain valid for the next dimensions as well.*

Proof. In practise, the proofs of these theorems can be merely extended to a number of dimensions at will, since each coordinate is treated independently of the others, and has the same properties. ∎

Definition 3.14. In the space $V_1V_2V_3...V_n$ we can define subtraction between two positions $o(a_{11}, a_{21}, ..., a_{n1})$ and $o(a_{12}, a_{22}, ..., a_{n2})$ as the position $o(a_{1(1-2)}, a_{2(1-2)}, ..., a_{n(1-2)})$ represented also with the symbol $o(a_{11}, a_{21}, ..., a_{n1}) - o(a_{12}, a_{22}, ..., a_{n2})$ that satisfies the following condition:

$$o_{1-2}(a_{1(1-2)}, a_{2(1-2)}, ..., a_{n(1-2)}) + o(a_{12}, a_{22}, ..., a_{n2}) = o(a_{11}, a_{21}, ..., a_{n1})$$

This condition defines the subtraction as the inverse operation of addition, and it is equivalent to require:

$$\begin{aligned}
&a_{1(1-2)} = a_{11} - a_{12}\\
&a_{2(1-2)} = a_{21} - a_{22}\\
&...\\
&a_{n(1-2)} = a_{n1} - a_{n2}
\end{aligned}$$

For example in the fourth dimension we have:

$$o_{1-2}(a_{1-2}, b_{1-2}, c_{1-2}, d_{1-2}) = o_{1-2}(a_1 - a_2, b_1 - b_2, c_1 - c_2, d_1 - d_2)$$

with:

$$\begin{aligned}
&a_{1-2} = a_1 - a_2\\
&b_{1-2} = b_1 - b_2\\
&c_{1-2} = c_1 - c_2\\
&d_{1-2} = d_1 - d_2
\end{aligned}$$

It should be emphasized that the subtraction must be considered an operation that works on the positions and not on the complete numbers, at least for every dimension higher than the second, for which there is no bijection between the positions and the complete numbers.

To integrate the operation of subtraction, working on the positions, with the others, working on the complete numbers, will be enough making reference to the complete number that we can obtain assigning to the difference the phases of the standard representation.

Theorem 3.15. *The properties embodied by Theorems* 2.26, 2.27, 2.28, 2.29 *for the third dimension remain valid for the next dimensions as well.*

Proof. In practice, the proofs of these theorems can be merely extended to a number of dimensions at will, since each coordinate is treated independently of the others, and has the same properties. ∎

Definition 3.16. In the space $V_1V_2...V_n$ we can define multiplication between two complete numbers $o_1(t_1, \theta_{21}, ..., \theta_{n1})$ and $o_2(t_2, \theta_{22}, ..., \theta_{n2})$ as the number $o_{1\cdot 2}(t_{1\cdot 2}, \theta_{2(1\cdot 2)}, ..., \theta_{n(1\cdot 2)})$ represented also with the symbol $o_1(t_1, \theta_{21}, ..., \theta_{n1}) \cdot o_2(t_2, \theta_{22}, ..., \theta_{n2})$ that satisfies the following condition:

$$o_{1\cdot 2}(t_{1\cdot 2}, \theta_{2(1\cdot 2)}, ..., \theta_{n(1\cdot 2)}) = o_{1\cdot 2}(t_1 \cdot t_2, \theta_{21} + \theta_{22}, ..., \theta_{n1} + \theta_{n2})$$

This condition defines the multiplication and it is equivalent to require:

$$\begin{aligned}
&t_{1\cdot 2} = t_1 \cdot t_2\\
&\theta_{2(1\cdot 2)} = \theta_{21} + \theta_{22}\\
&...\\
&\theta_{n(1\cdot 2)} = \theta_{n1} + \theta_{n2}
\end{aligned}$$

For example in the fourth dimension we have:

$$o_{1\cdot 2}(t_{1\cdot 2}, \theta_{1\cdot 2}, \gamma_{1\cdot 2}, \varphi_{1\cdot 2}) = o_{1\cdot 2}(t_1 \cdot t_2, \theta_1 + \theta_2, \gamma_1 + \gamma_2, \varphi_1 + \varphi_2)$$

with:

$$\begin{aligned} t_{1\cdot 2} &= t_1 \cdot t_2 \\ \theta_{1\cdot 2} &= \theta_1 + \theta_2 \\ \gamma_{1\cdot 2} &= \gamma_1 + \gamma_2 \\ \varphi_{1\cdot 2} &= \varphi_1 + \varphi_2 \end{aligned}$$

Theorem 3.17. *The properties embodied by Theorems* 2.40, 2.41, 2.42, 2.43, 2.44, 2.45, 2.46 *for the third dimension remain valid for the next dimensions as well.*

Proof. In practice, the proofs of these theorems can be merely extended to a number of dimensions at will, since each phases is treated independently of the others, and has the same properties.

Special reference also needs to be made to the distributive properties over addition and subtraction for which we must consider that the dimensions higher than the third, of fact extend them. This means that if these properties had been valid for the dimensions higher than the third, they had been such even in the third, as a sub-case, but we know that this does not happen. ∎

Definition 3.18. In the space $V_1V_2...V_n$ we can define division between two complete numbers $o_1(t_1, \theta_{21}, ..., \theta_{n1})$ and $o_2(t_2, \theta_{22}, ..., \theta_{n2})$ as the number $o_{\frac{1}{2}}(t_{\frac{1}{2}}, \theta_{2(\frac{1}{2})}, ..., \theta_{n(\frac{1}{2})})$ represented also with the symbol $\dfrac{o_1(t_1, \theta_{21}, ..., \theta_{n1})}{o_2(t_2, \theta_{22}, ..., \theta_{n2})}$ that satisfies the following conditions:

1. $o_{\frac{1}{2}}(t_{\frac{1}{2}}, \theta_{2(\frac{1}{2})}, ..., \theta_{n(\frac{1}{2})}) \cdot o_2(t_2, \theta_{22}, ..., \theta_{n2}) = o_1(t_1, \theta_{21}, ..., \theta_{n1})$
2. $o_2(t_2, \theta_{22}, ..., \theta_{n2}) \neq 0$

The first condition defines the division as the inverse operation of multiplication, and it is equivalent to require that:

$$\begin{aligned} t_{\frac{1}{2}} &= \frac{t_1}{t_2} \\ \theta_{2(\frac{1}{2})} &= \theta_{21} - \theta_{22} \\ &... \\ \theta_{n(\frac{1}{2})} &= \theta_{n1} - \theta_{n2} \end{aligned}$$

For example in the fourth dimension we have:

$$o_{1\cdot 2}(t_{\frac{1}{2}}, \theta_{\frac{1}{2}}, \gamma_{\frac{1}{2}}, \varphi_{\frac{1}{2}}) = o_{1\cdot 2}(t_1 \cdot t_2, \theta_1 - \theta_2, \gamma_1 - \gamma_2, \varphi_1 - \varphi_2)$$

with:

$$\begin{aligned} t_{\frac{1}{2}} &= \frac{t_1}{t_2} \\ \theta_{\frac{1}{2}} &= \theta_1 - \theta_2 \\ \gamma_{\frac{1}{2}} &= \gamma_1 - \gamma_2 \\ \varphi_{\frac{1}{2}} &= \varphi_1 - \varphi_2 \end{aligned}$$

The second condition gets its own justification by the necessity of defining the divisions in an univocal way. In fact when that condition is not valid, the expression:

$$o_{\frac{1}{2}}(t_{\frac{1}{2}}, \theta_{2(\frac{1}{2})}, ..., \theta_{n(\frac{1}{2})}) \cdot 0 = 0$$

besides to require a zero dividend $o_1(t_1, \theta_{21}, ..., \theta_{n1})$ as well, would be satisfied by more values of $o_{\frac{1}{2}}(t_{\frac{1}{2}}, \theta_{2(\frac{1}{2})}, ..., \theta_{n(\frac{1}{2})})$.

Theorem 3.19. *The properties embodied by Theorems* 2.57,2.58,2.59, 2.60, 2.61, 2.62, 2.63 *for the third dimension remain valid for the next dimensions as well.*

Proof. In practice, the proofs of these theorems can be merely extended to a number of dimensions at will, since each phases is treated independently of the others, and has the same properties.

Special reference also needs to be made to the distributive properties over addition and subtraction for which we must consider that the dimensions higher than the third, of fact extend them. This means that if these properties had been valid for the dimensions higher than the third, they had been such even in the third, as a sub-case, but we know that this does not happen. ■

Definition 3.20. In the space $V_1V_2...V_n$ we can define n-th power of the complete number $o(t, \theta_2, ..., \theta_i)$ with n (natural number) known as exponent and $o(t, \theta_2, ..., \theta_i)$ known as base, as the number $o_{\uparrow n}(t_{\uparrow n}, \theta_{2(\uparrow n)}, ..., \theta_{i(\uparrow n)})$ also represented with the symbol $o(t, \theta_2, ..., \theta_i)^n$ that satisfies the following conditions:

1. $o(t, \theta_2, ..., \theta_i)^n = o(t, \theta_2, ..., \theta_i) \cdot ... \cdot o(t, \theta_2, ..., \theta_i) \quad$ for $n > 0$
2. $o(t, \theta_2, ..., \theta_i)^n = \frac{o(t,\theta_2,...,\theta_i)}{o(t,\theta_2,...,\theta_i)} = 1 \quad$ for $n = 0$
3. $o(t, \theta_2, ..., \theta_i)^n = \cfrac{1}{\cfrac{o(t, \theta_2, ..., \theta_i)}{\cfrac{\cdots}{o(t, \theta_2, ..., \theta_i)}}} \quad$ for $n < 0$
4. $n > 0 \quad$ for $o(t, \theta_2, ..., \theta_i) = 0$

We note that the term $o(t, \theta_2, ..., \theta_i)$ in the first and third conditions is intended to appear $|n|$ times.

The first condition defines the repeated multiplication of the base by itself a positive number of times, the second a zero number of times, and finally the third a negative number of times. All these conditions correspond to require:

$$\begin{aligned}
&t_{\uparrow n} = t^n \\
&\theta_{2(\uparrow n)} = \theta_2 \cdot n \\
&... \\
&\theta_{i(\uparrow n)} = \theta_i \cdot n
\end{aligned}$$

For example in the fourth dimension we have:

$$o_{\uparrow n}(t, \theta, \gamma, \varphi)^n = o_{\uparrow n}(t_{\uparrow n}, \theta_{\uparrow n}, \gamma_{\uparrow n}, \varphi_{\uparrow n})$$

with:

$$\begin{aligned}
&t_{\uparrow n} = t^n \\
&\theta_{\uparrow n} = \theta \cdot n \\
&\gamma_{\uparrow n} = \gamma \cdot n \\
&\varphi_{\uparrow n} = \varphi \cdot n
\end{aligned}$$

The fourth condition gets its own justification by the impossibility of defining the n-th power module when to be multiplied by itself a zero number or a negative number of times is just the 0, because in this case would be present the following divisions for 0:

$$\begin{aligned}
o(t, \theta_2, ..., \theta_i)^n &= \frac{0}{0} = 1 \text{ for } n = 0 \\
o(t, \theta_2, ..., \theta_i)^n &= \frac{1}{\frac{\frac{0}{\cdots}}{0}} \text{ for } n < 0 \text{ with 0 that appears —n— times}
\end{aligned}$$

Theorem 3.21. *The properties embodied by Theorems* 2.65, 2.66, 2.67, 2.68, 2.69 *for the third dimension remain valid for the next dimensions as well.*

Proof. In practice, the proofs of these theorems can be repeated unchanged for dimensions higher than the third, since they do not depend on the number of dimensions considered but on the structure of the n-th power. ∎

Definition 3.22. In the space $V_1V_2...V_n$ we can define n-th root of the complete number $o(t, \theta_2, ..., \theta_i)$ with n (natural number) known as degree and $o(t, \theta_2, ..., \theta_i)$ known as radicand, as the number $o_{\downarrow n}(t_{\downarrow n}, \theta_{2(\downarrow n)}, ..., \theta_{i(\downarrow n)})$ also represented with the symbol $\sqrt[n]{o(t, \theta_2, ..., \theta_i)}$ that satisfies the following conditions:

1. $\sqrt[n]{o(t,\theta_2,...,\theta_i)} \cdot ... \cdot \sqrt[n]{o(t,\theta_2,...,\theta_i)} = o(t,\theta_2,...,\theta_i) \quad$ for $n > 0$

2. $\dfrac{1}{\dfrac{\sqrt[n]{o(t,\theta_2,...,\theta_i)}}{\dfrac{...}{\sqrt[n]{o(t,\theta_2,...,\theta_i)}}}} = o(t,\theta_2,...,\theta_i) \quad$ for $n < 0$

3. $\theta_{2(\downarrow n)} = \frac{\theta_2}{n}, \quad \theta_{3(\downarrow n)} = \frac{\theta_3}{n}, \quad ..., \quad \theta_{i(\downarrow n)} = \frac{\theta_i}{n}$

4. $n \neq 0 \quad$ for any $o(t,\theta_2,...,\theta_i)$

5. $n \geq 0 \quad$ for $o(t,\theta_2,...,\theta_i) = 0$

6. $\sqrt[n]{t} > 0, \quad t > 0$

We note that the term $\sqrt[n]{o(t,\theta_2,...,\theta_i)}$ in the first and second conditions is intended to appear $|n|$ times.

The first condition defines the repeated multiplication of the root by itself a positive number of times, while the second a negative number of times. Both these conditions correspond to require:

$$
\begin{aligned}
&t_{\downarrow n} = \sqrt[n]{t}\\
&\theta_{2(\downarrow n)} = \frac{\theta_2 + k \cdot 360°}{n} \qquad \text{for } k = \pm 1, \pm 2, \pm 3, \pm 4, ...\\
&...\\
&\theta_{i(\downarrow n)} = \frac{\theta_i + k \cdot 360°}{n} \qquad \text{for } k = \pm 1, \pm 2, \pm 3, \pm 4, ...
\end{aligned}
$$

The third condition gets its own justification by the necessity of defining the n-th root in an univocal way. In fact, when that condition is not valid, there are $n^{(i-1)}$ different complete numbers able to satisfy this definition: one for each distinct set of phases $\theta_{2(\downarrow n)}$,$\theta_{3(\downarrow n)}$,..., $\theta_{i(\downarrow n)}$ given by the relations seen above.

For example in the fourth dimension we have:

$$\sqrt[n]{o(t,\theta,\gamma,\varphi)} = o_{\downarrow n}(t_{\downarrow n}, \theta_{\downarrow n}, \gamma_{\downarrow n}, \varphi_{\downarrow n})$$

with:

$$
\begin{aligned}
t_{\downarrow n} &= \sqrt[n]{t}\\
\theta_{\downarrow n} &= \frac{\theta}{n}\\
\gamma_{\downarrow n} &= \frac{\gamma}{n}\\
\varphi_{\downarrow n} &= \frac{\varphi}{n}
\end{aligned}
$$

Also the fourth condition gets its own justification by the necessity of defining the n-th root in an univocal way. In fact when that condition is not valid, the multiplication of the root by itself a number of times equal to 0 would require the use of the following expression:

$$\frac{\sqrt[n]{o(t,\theta_2,...,\theta_i)}}{\sqrt[n]{o(t,\theta_2,...,\theta_i)}} = 1$$

that would be satisfied by several values of $\sqrt[n]{[o(t,\theta_2,...,\theta_i)]}$.

The fifth condition gets its own justification by the impossibility of defining values of n-th root that multiplied by itself a negative number of times are able to give as the result just 0 value. In fact the following expression:

$$\frac{1}{\frac{\sqrt[n]{o(t,\theta_2,...,\theta_i)}}{\frac{...}{\sqrt[n]{o(t,\theta_2,...,\theta_i)}}}} = 0 \text{ for } n < 0, \sqrt[n]{o(t,\theta_2,...,\theta_i)} \text{ appears —n— times}$$

requires the existence of a divisor of 1 that can assign to it a quotient equal to 0: a thing that we know impossible.

The sixth condition gets its own justification by the need to make acceptable the n-th root in regard the modulus t of the complete number $o(t,\theta_2,...,\theta_i)$.

Theorem 3.23. *The properties embodied by Theorems* 2.71, 2.72, 2.73 *for the third dimension remain valid for the next dimensions as well.*

Proof. In practice, the proofs of these theorems can be repeated unchanged for dimensions higher than the third, since they do not depend on the number of dimensions considered but on the structure of the n-th root. ■

Definition 3.24. In the space $V_1V_2...V_n$ we can define power with rational exponent $\frac{m}{n}$ (n,m both natural numbers) of the complete number $o(t,\theta_2,...,\theta_i)$ with $\frac{m}{n}$ known as the rational exponent and $o(t,\theta_2,...,\theta_i)$ known as base, as the number $o_{\uparrow m\downarrow n}(t_{\uparrow m\downarrow n},\theta_{2(\uparrow m\downarrow n)},..,\theta_{i(\uparrow m\downarrow n)})$ also represented with the symbol $o(t,\theta_2,..,\theta_i)^{\frac{m}{n}}$ or $\sqrt[n]{o(t,\theta_2,...,\theta_i)^m}$ that satisfies the following conditions:

1. $\left[\sqrt[n]{o(t,\theta_2,...,\theta_i)^m}\right]^n = o(t,\theta_2,...,\theta_i)^m$
2. $m > 0$ for $o(t,\theta_2,...,\theta_i) = 0$
3. $n \neq 0$ for any $o(t,\theta_2,...,\theta_i)^m$ and therefore for any $o(t,\theta_2,...,\theta_i)$
4. $n \geq 0$ for $o(t,\theta_2,...,\theta_i)^m = 0$ and therefore for $o(t,\theta_2,...,\theta_i) = 0$

5. $\theta_{2(\uparrow m\downarrow n)} = \dfrac{\theta_2 \cdot m}{n}, \quad \theta_{3(\uparrow m\downarrow n)} = \dfrac{\theta_3 \cdot m}{n}, \quad ..., \quad \theta_{i(\uparrow m\downarrow n)} = \dfrac{\theta_i \cdot m}{n}$

6. $\sqrt[n]{t^m} > 0, \quad t^m > 0$

7. $\sqrt[n]{t} > 0, \quad t > 0$

The first condition defines the power with rational exponent as a n-th root of a m-th power.

The second condition is required for the correct definition of the m-th power.

The third, the fourth, the fifth and the sixth conditions are required for the correct definition of n-th root.

For example in the fourth dimension we have:

$$\sqrt[n]{o(t,\theta,\gamma,\varphi)^m} = o_{\uparrow m\downarrow n}(t_{\uparrow m\downarrow n}, \theta_{\uparrow m\downarrow n}, \gamma_{\uparrow m\downarrow n}, \varphi_{\uparrow m\downarrow n})$$

with:

$$\begin{aligned} t_{\downarrow n} &= t^{\frac{m}{n}} \\ \theta_{\downarrow n} &= \frac{m}{n} \cdot \theta \\ \gamma_{\downarrow n} &= \frac{m}{n} \cdot \gamma \\ \varphi_{\downarrow n} &= \frac{m}{n} \cdot \varphi \end{aligned}$$

The seventh condition is required to make possible the reversal of the order between root and power, namely to write:

$$\left[\sqrt[n]{o(t,\theta_2,...,\theta_i)}\,\right]^m$$

and therefore:

$$(\sqrt[n]{t}\,)^m$$

Theorem 3.25. *The properties embodied by Theorems* 2.75, 2.76, 2.77, 2.78, 2.79, 2.80, 2.81 *for the third dimension remain valid for the next dimensions as well.*

Proof. In practice, the proofs of these theorems can be repeated unchanged for dimensions higher than the third, since they do not depend on the number of dimensions considered but on the structure of the power with rational exponent. ∎

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A RADICAL PROPERTY OF HYPERRINGS

A. Asokkumar
M. Velrajan

Department of Mathematics
Aditanar College of Arts and Science
Tiruchendur – 628216, Tamilnadu
India
e-mail: ashok_a58@yahoo.co.in

Abstract. In this paper we prove that Von Neumann regularity is a radical property on hyperrings.

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1. Introduction

The theory of hyperstructures was introduced in 1934 by Marty [11] at the 8^{th} Congress of Scandinavian Mathematicians. This theory has been subsequently developed by Corsini [5], [6], Mittas [12], [13], Stratigopoulos [17], Vougiouklis [20] and by various authors. Basic definitions and propositions about the hyperstructures are found in [5], [6] and [20]. Krasner [10] has studied the notion of hyperfields, hyperrings and then many researchers like Davvaz [7], Massouros [14] and others followed him.

There are different notions of hyperrings $(R, +, \cdot)$. If in a hyperring the addition $+$ is a hyperoperation and the multiplication $\cdot$ is a binary operation, then the hyperring is called a Krasner (additive) hyperring [10]. The monograph [8] of Davvaz and Leoreanu-Fotea contains many results about various hyperrings. Asokkumar and Velrajan [1], [4] have studied Von Neumann regularity in Krasner hyperrings. Rota [16] introduced multiplicative hyperrings, where the additions are binary operations and multiplications are hyperoperations. De Salvo [9] introduced hyperrings in which the additions and the multiplications are hyperoperations. These hyperrings are studied by Rahnamani Barghi [15] and also by Asokkumar and Velrajan [2], [3], [19].

In this paper we prove that regularity (Von Neumann) is a radical property on hyperrings, where the additions and the multiplications are hyperoperations. We also prove that if a hyperring R is regular, then for a hyperideal I of R both I and R/I are regular. Conversely, if R is a hyperring and if there exists a hyperideal I of R such that both I and R/I are regular, then R is regular.

2. Basic definitions and notations

This section explains some basic definitions that have been used in the sequel. A *hyperoperation* $\circ$ on a nonempty set H is a mapping of $H \times H$ into the family of nonempty subsets of H (i.e., $x \circ y \subseteq H$ for every $x, y \in H$). A *hypergroupoid* is a nonempty set H equipped with a hyperoperation $\circ$. For any two subsets A, B of a hypergroupoid H, the set $A \circ B$ means $\bigcup_{\substack{a\in A\\ b\in B}} (a \circ b)$. A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if $x\circ(y\circ z) = (x\circ y)\circ z$ for every $x, y, z \in H$(the associative axiom). A hypergroupoid $(H, \circ)$ is called a *quasihypergroup* if $x \circ H = H \circ x = H$ for every $x \in H$(the reproductive axiom). A reproductive semihypergroup is called a *hypergroup* (Marty). A comprehensive review of the theory of hypergroups appears in [5].

A nonempty set H with a hyperoperation $+$ is said to be a *canonical hypergroup* if the following conditions hold:

(i) for every $x, y, z \in H$, $x + (y + z) = (x + y) + z$,

(ii) for every $x, y \in H$, $x + y = y + x$,

(iii) there exists $0 \in H$ such that $0 + x = x$ for all $x \in H$,

(iv) for every $x \in H$ there exists an unique element denoted by $-x \in H$ such that $0 \in x + (-x)$,

(v) for every $x, y, z \in H$, $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$.

A nonempty subset N of a canonical hypergroup of H is called a *subcanonical hypergroup* of H if N itself is a canonical hypergroup under the same hyperoperation as that of H. Equivalently, for every $x, y \in N$, $x - y \subseteq N$. Moreover, for any subset A of H, $-A$ denotes the set $\{-a : a \in A\}$.

The following elementary facts in a canonical hypergroup easily follow from the axioms.

(i) $-(-a) = a$ for every $a \in R$;

(ii) 0 is the unique element such that for every $a \in R$, there is an element $-a \in R$ with the property $0 \in a + (-a)$;

(iii) $-0 = 0$;

(iv) $-(a + b) = -b - a$ for all $a, b \in R$.

Theorem 2.1 [19] *Let H be a canonical hypergroup and N be a subcanonical hypergroup of H. For any two elements $a, b \in H$, if we define a relation $a \sim b$ if and only if $a \in b + N$, then $\sim$ is an equivalence relation on H.*

Let $\overline{x}$ be the equivalence class determined by the element $x \in H$ and H/N be the collection of all equivalence classes.

Theorem 2.2 [19] *Let H be a canonical hypergroup and N be a subcanonical hypergroup of H. Then $\overline{x} = x + N$ for any $x \in H$.*

Theorem 2.3 [19] *Let H be a canonical hypergroup, N be a subcanonical hypergroup of H. If we define $\overline{x} \oplus \overline{y} = \{\overline{z} : z \in x + y\}$ for all $\overline{x}, \overline{y} \in H/N$, then H/N is a canonical hypergroup.*

A nonempty set R with two hyperoperations $+$ and $\cdot$ is said to be a *hyperring* if $(R, +)$ is a canonical hypergroup, $(R, \cdot)$ is a semihypergroup with $x \cdot 0 = 0 \cdot x = 0$ for all $x \in R$ (0 as a bilaterally absorbing element) and the hyperoperation $\cdot$ is *distributive* over $+$, i.e., for every $x, y, z \in R$, $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$. The hyperoperation $+$ is usually called *hyperaddition* and the hyperoperation $\cdot$ is called *hypermultiplication*.

Definition 2.4 Let R be a hyperring and I be a nonempty subset of R. Then I is called a *left* (resp. *right*) *hyperideal* of R if $(I, +)$ is a canonical subhypergroup of R and for every $a \in I$and $r \in R$, $ra \subseteq I$ (resp. $ar \subseteq I$). A *hyperideal* of R is one which is a left as well as a right hyperideal of R.

If I, J are left (resp. right) hyperideals of a hyperring R, then $I + J$ is a left (resp. right) hyperideal of R. If I, J are hyperideals of a hyperring R, then $I + J$ is a hyperideal of R. Let R be a hyperring, I a hyperideal of R and R/I be the set of all distinct equivalence classes of I in R obtained by considering I as a subcanonical hypergroup of R. Then R/I is a canonical hypergroup under the hyperaddition defined in the Theorem 2.3.

Theorem 2.5 [19] *If we define $\overline{x} \otimes \overline{y} = \{\overline{z} : z \in xy\}$ for all $\overline{x}, \overline{y} \in R/I$, then R/I is a hyperring.*

Definition 2.6 Let R_1 and R_2 be two hyperrings. A mapping ϕ from R_1 into R_2 is called a *homomorphism* if the following conditions hold for all $a, b \in R_1$:

(i) $\phi(a + b) \subseteq \phi(a) + \phi(b)$;

(ii) $\phi(ab) \subseteq \phi(a)\phi(b)$, and

(iii) $\phi(0) = 0$.

The mapping ϕ is called a *good homomorphism* or *a strong homomorphism* if

(i) $\phi(a+b) = \phi(a) + \phi(b)$;

(ii) $\phi(ab) = \phi(a)\phi(b)$, and

(iii) $\phi(0) = 0$ for all $a, b \in R_1$.

Definition 2.7 A homomorphism (resp. strong homomorphism)ϕ from a hyperring R_1 into a hyperring R_2 is said to be an *isomorphism* (*resp. strong isomorphism*) if ϕ is one to one and onto. In this case we say R_1 is *isomorphic* (resp. *strongly isomorphic*) to R_2 and is denoted by $R_1 \cong R_2$.

Definition 2.8 Let ϕ be a homomorphism from a hyperring R_1 into another hyperring R_2. Then the set $\{x \in R_1 : \phi(x) = 0\}$ is called the *kernel* of ϕ and is denoted by $Ker\phi$ and the set $\{\phi(x) : x \in R_1\}$ is called *Image* of ϕ and is denoted by $Im\phi$.

It is clear that $Ker\phi$ is a hyperideal of R_1 and $Im\phi$ is a subcanonical hypergroup of R_2 and $R_1/Ker\phi$ is a hyperring.

Theorem 2.9 [19] (First Isomorphism Theorem) *Let ϕ be a strong homomorphism from a hyperring R_1 onto a hyperring R_2 with kernel K. Then R_1/K is strongly isomorphic to R_2.*

Theorem 2.10 [19] (Second Isomorphism Theorem) *If I and J are hyperideals of a hyperring R then $J/(I \cap J) \cong (I+J)/I$.*

3. Regular hyperring

First, let us recall the definition of a regular ring. An element a in a ring R is said to be regular if $a \in aRa$. A ring R is called regular if every element of R is regular. We define a regular hyperring as follows.

Definition 3.1 [2] An element $a \in R$ is said to be regular if $a \in aRa$. That is, there exists an element $b \in R$ such that $a \in aba$. A hyperring R is said to be regular if every element of R is regular.

Proposition 3.2 [2] *Strong homomorphic image of a regular hyperring is a regular hyperring.*

Proposition 3.3 *If I is a hyperideal of a regular hyperring R, then I is regular.*

Proof. Consider a hyperideal I of R. Let $a \in I$. Since R is regular, there exists $x \in R$ such that $a \in axa$. Then $a \in a(xa) \subseteq (axa)(xa) = a(xax)a$ where $xax \subseteq I$. Thus I is regular. ∎

Theorem 3.4 *If I, J are regular hyperideals of a hyperring R, then $I \cap J$ is also a regular hyperideal of R.*

Proof. It is clear that $I \cap J$ is a hyperideal of R. Let $a \in I \cap J$. Then there exist $x \in I$ and $y \in J$ such that $a \in axa$ and $a \in aya$. Now,

$$a \in axa \subseteq (axa)x(aya) = a(xaxay)a.$$

Since I, J are hyperideals of R, $xaxay \subseteq I \cap J$. Thus $I \cap J$ is regular. ∎

4. Regularity is a radical property on hyperrings

In this section, we show that regularity is a radical property on hyperrings. We also prove that if a hyperring R is regular, then for a hyperideal I of R both I and R/I are regular. Conversely, if R is a hyperring and if there exists a hyperideal I of R such that both I and R/I are regular, then R is regular.

Definition 4.1 Let P be a property of hyperrings. A hyperring with the property P is called a *P-hyperring*. A hyperideal I of a hyperring R is called a *P-hyperideal* if the hyperideal I, as a hyperring, is a P-hyperring.

Definition 4.2 A P-hyperideal $P(R)$ of a hyperring R which contains every P-hyperideal of R is called the *P-hyperradical* of R.

Definition 4.3 A property P of a hyperring is called a *radical property* (in the sense of Amitsur and Kurosh [18]) if P satisfies the following conditions:

(i) Strong homomorphic image of a P-hyperring is a P-hyperring.

(ii) Every hyperring R has a P-hyperradical $P(R)$.

(iii) The hyperring $R/P(R)$ has no non-zero P-hyperideals.

Lemma 4.4 *Let R be a hyperring and $a \in R$. If there exists $x \in R$ and $c \in axa-a$ such that c is regular, then a is regular.*

Proof. Since $c \in axa - a$ is regular, there exists $d \in R$ such that $c \in cdc$. This means that

$$\begin{aligned} c &\in (axa-a)d(axa-a) \\ &= (axad-ad)(axa-a) \\ &\subseteq axadaxa - axada - adaxa + ada \\ &= a(xadaxa - xada - daxa + da) \\ &= a(xadax - xad - dax + d)a \end{aligned}$$

Hence $c \in aba$ for some $b \in xadax - xad - dax + d$. Since $c \in (axa-a)$, we get $a \in (axa - c) \subseteq axa - aba = a(x-b)a$. So $a \in aya$ for some $y \in x - b$. That is, a is regular. ∎

Theorem 4.5 *Let R be a regular hyperring and I be a hyperideal of R. Then I and R/I are regular. Conversely, if R is a hyperring and if there exists a hyperideal I of R such that both I and R/I are regular, then R is regular.*

Proof. Let R be a regular hyperring and I be a hyperideal of R. Then by the Proposition 3.3, I is a regular hyperideal. Let $x+I \in R/I$. Since R is regular, there exists $y \in R$ such that $x \in xyx$. Consider $\overline{y} = y + I$. Now, $\overline{x}\ \overline{y}\ \overline{x} = \{\overline{z} : z \in xyx\}$. Since $x \in xyx$ we have $\overline{x} \in \{\overline{z} : z \in xyx\}$. That is, $\overline{x} \in \overline{x}\ \overline{y}\ \overline{x}$. So $x + I$ is regular in R/I. Hence R/I is regular.

Conversely, suppose R is a hyperring and there exists a hyperideal I of R such that both I and R/I are regular. Let $a \in R$. Then $\overline{a} \in R/I$. Since R/I is regular, there exists an element $\overline{b} \in R/I$ such that $\overline{a} \in \overline{a}\ \overline{b}\ \overline{a} = \{\overline{z} : z \in aba\}$. This means that $\overline{a} = \overline{z}$ for some $z \in aba$. That is, $a + I = z + I$ for some $z \in aba$. Since $z \in a + I$, we get $z \in a + i$ for some $i \in I$. Therefore, $i \in -a + z = z - a \subseteq aba - a$. Thus $i \in aba - a$. Since I is regular, i is a regular element of I and therefore i is a regular element of R. Thus the set $aba - a$ contains a regular element i of R. Then by the Lemma 4.4, the element a is regular in R. Hence R is regular. ∎

Theorem 4.6 *Let R be a hyperring. If I and J are regular hyperideals of R, then $I + J$ is regular.*

Proof. Since $J/(I \cap J)$ is a homomorphic image of a regular hyperideal J, it is regular. By the Theorem 2.10, $J/(I \cap J)$ is isomorphic to $(I + J)/I$. Therefore, $(I + J)/I$ is regular. Since both I and $(I + J)/I$ are regular, by the Theorem 4.5, the hyperideal $I + J$ is regular. ∎

Theorem 4.7 *Any hyperring has a regular hyperradical.*

Proof. Let R be a hyperring. Consider the hyperideal (0) of R. Clearly, (0) is a regular hyperideal of R. If (0) is the only regular hyperideal of R, then this is the regular hyperradical.

Otherwise, let $\{I_i\}$ be the collection of all regular hyperideals in a hyperring R. Their sum is given by $M = \bigcup\{\sum_{finite} a_i : a_i \in I_i\}$. Clearly, M is a hyperideal of R. If $x \in M$, then $x \in a_i + a_j + a_k + \cdots + a_l$, where $a_i \in I_i$. By Theorem 4.6, $I_i + I_j + I_k + \cdots + I_l$ is a regular hyperideal. Therefore, x is regular. Hence, M is regular. Since M contains all regular hyperideals of R, we have M is the regular hyperradical of R. ∎

Theorem 4.8 *Let R be a hyperring and M be the regular hyperradical of R. Then the hyperring R/M has no non-zero regular hyperideals.*

Proof. Let J be a regular hyperideal of R/M. Then $J = I/M$ for some hyperideal I of R containing M. Since M and I/M are regular, by the Theorem 4.5, I is regular. By the definition of M, we have $I \subseteq M$. Hence $I = M$. Therefore, J is a zero hyperideal of R/M. ∎

Theorem 4.9 *The regularity is a radical property on hyperrings.*

Proof. The proof follows from the Proposition 3.2, and the Theorems 4.7, 4.8. ∎

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MULTI-OBJECTIVE DECISION MAKING BASED ON FUZZY EVENTS AND THEIR COHERENT (FUZZY) MEASURES

Antonio Maturo

University of Chieti-Pescara
Dep. of Social Sciences
Chieti
Italy
e-mail: amaturo@unich.it

Abstract. We propose a reformulation of the problem of making a decision with multiple objectives in terms of fuzzy scores and their consistent defuzzification, with respect to the logical point of view taken into account. The objectives are seen as subsets (or events) of a universal set U and the degree to which an alternative A_i satisfies the objective O_j is a conditional fuzzy event $A_i|O_j$, represented by a fuzzy set φ_{ij} defined on a partition π_{ij} of O_j. The elements of π_{ij} are the particular aspects of the objective O_j considered by A_i; the value assumed by an element $x \in \pi_{ij}$ is the extent to which A_i satisfies that particular aspect. Using an appropriate procedure of defuzzification fuzzy scores of alternatives with respect to the objectives are transformed into numerical scores belonging to the interval $[0, 1]$. We study the conditions of consistency of defuzzified scores taking into account the logical relations among the objectives and the alternatives. Finally, we develop criteria for the aggregation of scores of each alternative.

Keywords: multiobjective decision making, fuzzy events, coherent defuzzification, aggregation criteria.

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1. Introduction

A classical model of multi-objective decision making is based on a quadruple $(\mathbf{A}, \mathbf{O}, W, S)$, where $\mathbf{A}$ is the set of the *alternatives*, $\mathbf{O}$ is the set of the *objectives*, $W : \mathbf{O} \to [0, 1]$ is the *weight function*, which measures the weight, i.e. the importance, of the objectives; $S : \mathbf{A} \times \mathbf{O} \to [0, 1]$ is the *function score*, which, for each pair $(A_i, O_j) \in \mathbf{A} \times \mathbf{O}$, measures the score of A_i with respect to O_j, i.e. the extent to which the alternative A_i meets the objective O_j.

From now on we consider the case where $\mathbf{A}$ and $\mathbf{O}$ are finite. So we assume $\mathbf{A} = \{A_1, A_2, ..., A_m\}$, $\mathbf{O} = \{O_1, O_2, ..., O_n\}$, $W = \{w_1, w_2, ..., w_n\}$, where w_j is the weight of O_j.

Then the score function is represented by a matrix $S = (s_{ij})$, where s_{ij} measures the degree to which A_i meets the objective O_j. The rows of S are vectors associated with the alternatives and the columns are associated with the objectives.

Many authors, especially if they adopt the ranking procedure defined by the Analytic Hierarchy Process (AHP) [17], [3], [7], [11], [12], assume the conditions of normalization:

$$w_1 + w_2 + ... + w_n = 1, \tag{1}$$

$$\forall j \in 1, 2, ..., n, \ \ s_{1j} + s_{2j} + ... + s_{mj} = 1. \tag{2}$$

A classical formula to obtain the overall score $s(A_i)$ of the alternative A_i is as follows:

$$s(A_i) = w_1 s_{i1} + w_2 s_{i2} + ... + w_n s_{in}. \tag{3}$$

The preferred alternative is one that has the highest overall score.

We can observe that the role of objectives and alternatives is similar to that of events in subjective probability [5], [6], [4], [15]. Then let us extend the de Finetti's terminology to the decision problem. In particular, a family of objectives (resp. alternatives) two by two disjoint and exhaustive will be called *partition of the certain event.*

We note that, as in the subjective probability of de Finetti, each partition of the certain event is temporary, since each objective (resp. alternative) can be partitioned into sub-objectives (resp. sub-alternatives), i.e. for each partition of the certain event we can consider a finer. In this framework, an assignment of weights to a family of objectives (or alternatives) is similar to a probability assignment to a set of events, namely has the same formal properties, and then coherence conditions must be met. Then, from a formal point of view, the weight function $W : O_j \in \mathbf{O} \to w_j \in [0, 1]$ may be seen as a probability assignment on $\mathbf{O}$ and condition (1) follows from the assumption that $\mathbf{O}$ is a partition of the certain event Ω and W is a consistent assignment of probability [5], [4].

Similarly, the function $S : (A_i, O_j) \in \mathbf{A} \times \mathbf{O} \to s_{ij}$ plays the role of a conditional probability assignment [5], [4] where s_{ij} is the probability of the conditional event $A_i|O_j$. The conditions (2) follow from the hypothesis that $\mathbf{A}$ is a partition of the certain event and S is a coherent assessment of conditional probabilities. Then $s(A_i)$ can be interpreted (formally) as the probability of A_i and the formula (3) is a well-known formula of the theory of probability.

Then, in this order of ideas, if $\mathbf{O}$ (resp. $\mathbf{A}$) is not a partition of the certain event, consistency conditions are different from (1) (resp. (2)). They depend on the logical relationships between the events O_j (resp. A_i). These conditions reduce to the existence of nonnegative solutions of suitable linear systems [4].

In this paper we consider a more general point of view on the scores of the alternatives with respect to the objectives. The effect of an alternative A_i on an objective O_j is measured by a finite conditional fuzzy event $A_i|O_j$, represented by a fuzzy set φ_{ij} defined on a partition π_{ij} of O_j. The elements of π_{ij} are the particular aspects of the objective O_j considered by A_i; the value assumed by an element $x \in \pi_{ij}$ is the extent to which A_i satisfies that particular aspect.

The score s_{ij}, which provides an overall measure of the degree to which the alternative A_i meets the objective O_j, is interpreted as a defuzzification of φ_{ij}. In particular, in the notation of fuzzy events, s_{ij} is the probability of the conditional fuzzy event φ_{ij}.

The rest of the paper is organized as follows:

- In Section 2, we recall and introduce some concepts and results on fuzzy events and coherence conditions of their assignments of probability.
- In Section 3, we present a reformulation of the multi-objective decision making model in terms of fuzzy events and their coherent probabilities.
- In Section 4, we explore the problem of aggregation of the scores of each alternative with respect to the various objectives.
- In Section 5, we introduce fuzzy measures on fuzzy events, i.e., normalized monotonic measures, and we examine the conditions of consistency.
- Finally, in Section 6, we present some conclusions and research perspectives.

2. Fuzzy events and conditions of consistency of their assignments of probability

2.1. Basic concepts on fuzzy events

Fuzzy events and their assignments of probability were considered by Zadeh in [21] and [23]. The subjective probability of fuzzy events and conditions of coherence have been studied in [10]. In this subsection we introduce some basic concepts about fuzzy events, reworking and adapting the definitions given in [21] and [10], introducing some new concepts in view of the application to decision making problems.

Definition 2.1 A fuzzy event is a function $\varphi : \pi \to [0,1]$, where π is a partition of the certain event Ω. For all $x \in \pi$, $\varphi(x)$ is the *degree* to which the fuzzy event φ occurred if x occurs.

Let $Im(\varphi)$ be the image of φ. For each $y \in Im(\varphi)$ we indicate with $\varphi^{(-1)}(y)$ the union of the elements x of π such that $\varphi(x) = y$. The set $\pi^* = \{\varphi^{(-1)}(y) : y \in Im(\varphi)\}$ is a partition of Ω and the function $\varphi^* : \pi^* \to [0,1]$ such that $\varphi^*(x) = y$ if and only if $\varphi^{(-1)}(y) = x$ is a fuzzy event called the *reduced form* or *normal form* of φ. The elements $x \in \pi^*$ are called *atoms* or *constituents* of φ.

The fuzzy event φ is said to be *finite* if $Im(\varphi)$ is *finite*. In this case, if $\pi = \{x_1, x_2, ..., x_n\}, \varphi(x_i) = a_i$, using the notation of Zadeh [23] we write

$$\varphi = a_1/x_1 + a_2/x_2 + ... + a_n/x_n.$$

Remark 2.1 We note that, by identifying each event with its characteristic function, each event E can be seen as the fuzzy event φ_E with domain $\{E, E^c\}$ and such that $\varphi_E(E) = 1, \varphi_E(E^c) = 0$.

Definition 2.2 Let $\varphi_1 : \pi_1 \to [0,1]$ and $\varphi_2 : \pi_2 \to [0,1]$ two fuzzy events. We put:

$$\begin{gathered}\varphi_1 \le \varphi_2 \Leftrightarrow \forall x_1 \in \pi_1, x_2 \in \pi_2, x_1 \cap x_2 \neq \emptyset \Rightarrow \varphi_1(x_1) \le \varphi_2(x_2); \\ \varphi_1 = \varphi_2 \Leftrightarrow \varphi_1 \le \varphi_2, \varphi_2 \le \varphi_1.\end{gathered} \tag{4}$$

Remark 2.2 From the above definition it follows that two fuzzy events are equal if and only if they have the same reduced form. In particular, if two fuzzy events are equal then they have the same constituents.

If π_1 and π_2 are partitions of the certain event, let us denote with $\pi_1\pi_2$ their product, i.e. the partition $\{x_1 \cap x_2 : x_1 \in \pi_1, x_2 \in \pi_2, x_1 \cap x_2 \neq \emptyset\}$.

Definition 2.3 Let $\varphi_1 : \pi_1 \to [0,1]$ and $\varphi_2 : \pi_2 \to [0,1]$ be two fuzzy events and let $\star$ be an operation in $[0,1]$. We define $\varphi_1 \star \varphi_2 : \pi_1\pi_2 \to [0,1]$ as the fuzzy event with domain $\pi_1\pi_2$ and such that

$$\forall x_1 \in \pi_1, x_2 \in \pi_2 : x_1 \cap x_2 \neq \emptyset, (\varphi_1 \star \varphi_2)(x_1 \cap x_2) = \varphi_1(x_1) \star \varphi_2(x_2). \tag{5}$$

The most important are the following [8], [19]:

- $\star$ is a t-conorm, i.e. an operation in $[0,1]$ associative, commutative, with 0 as neutral element and increasing in each variable;
- $\star$ is a t-norm, i.e. an operation in $[0,1]$ associative, commutative, with 1 as neutral element and increasing in each variable.

Remark 2.3 We note that a constant $k \in [0,1]$ is a fuzzy event φ with domain $\pi = \{\Omega\}$ and $\varphi(\Omega) = k$, and the multiplication in $[0,1]$ is a t-norm. So the product of a fuzzy event by a scalar belonging to the interval $[0,1]$ is a special case of the formula (5).

Definition 2.4 Let F be a nonempty family of fuzzy events such that

$$k \in [0,1], \varphi \in F \Rightarrow k\varphi \in F,\ \varphi_1, \varphi_2 \in F, \varphi_1 + \varphi_2 \le 1 \Rightarrow \varphi_1 + \varphi_2 \in F. \tag{6}$$

A function $p : F \to [0,1]$ is said to be a *probability* on F if

P1 $\forall \varphi \in F, \inf(\varphi) \le p(\varphi) \le \sup(\varphi)$;

P2 $k \in [0,1], \varphi \in F \Rightarrow p(k\varphi) = kp(\varphi)$;

P3 $\varphi_1, \varphi_2 \in F, \varphi_1 + \varphi_2 \le 1 \Rightarrow p(\varphi_1 + \varphi_2) = p(\varphi_1) + p(\varphi_2)$.

As a consequence, we have the following corollary

Corollary 2.1 *Let $\varphi : \pi \to [0,1]$ be a finite fuzzy event, $\varphi = a_1/x_1 + a_2/x_2 + ... + a_n/x_n$. If $p : \pi \to [0,1]$ is a probability in π, then the probability of φ is the number:*

$$p(\varphi) = a_1 p(x_1) + a_2 p(x_2) + ... + a_n p(x_n). \tag{7}$$

2.2. Coherence of a probability assessment on fuzzy events

Definition 2.5 Let $\Phi = \{\varphi_1, \varphi_2, ..., \varphi_h\}$ be a finite family of finite fuzzy events $\varphi_i : \pi_i \to [0,1]$ in reduced form. The elements of the product $\pi = \pi_1 \pi_2 ... \pi_h$ are called *atoms* or *constituents* of Φ.

Referring to the notations of the definition 2.5, let $\pi = \{c_1, c_2, ..., c_s\}$ be the set of the atoms of Φ. If $c_r = x_1^r \cap x_2^r \cap ... \cap x_h^r$, $x_i^r \in \pi_i$, we can write:

$$\varphi_i = a_i^1/c_1 + a_i^2/c_2 + ... + a_i^s/c_s, \; a_i^r = \varphi_i(x_i^r) = \varphi_i(c_r). \tag{8}$$

If the probabilities of the atoms were assigned then by the formula (7) we obtain the probability of each fuzzy event belonging to Φ.

In practical applications, however, often occurs that, based on information, beliefs, rationales, expert opinions, we have an assessment of the probabilities $p_i = p(\varphi_i)$ of the fuzzy events belonging to Φ without knowing the probabilities of the atoms. In this case the question arises of whether these judgments are *consistent*, i.e. if there exists a probability distribution on the atoms that permits to get the $p(\varphi_i)$ by the formula (7). For this purpose we give the following definition.

Definition 2.6 An assignment of probabilities $p = (p_1, p_2, ..., p_h)$ to the family of finite fuzzy events $\Phi = \{\varphi_1, \varphi_2, ..., \varphi_h\}$, with $p_i = p(\varphi_i)$, is said to be coherent (or consistent) if there exists a probability distribution on the family of the atoms of Φ such that:

$$p(\varphi_i) = a_i^1 p(c_1) + a_i^2 p(c_2) + ... + a_i^s p(c_s). \tag{9}$$

with $a_i^r = \varphi_i(c_r)$.

Let $A = (a_{ir})$ be the matrix with $a_{ir} = \varphi_i(c_r)$ and let $P = [p_1, p_2, ..., p_h]^t$, $p_i = p(\varphi_i)$ be the column vector of probabilities assigned to the fuzzy events φ_i. Moreover let $Z = [z_1, z_2, ..., z_s]^t$ be the unknown vector of probabilities of the constituents. From the above definition the following theorem hold:

Theorem 2.1 *The assignment of probabilities $P = [p_1, p_2, ..., p_h]^t$ to the family of finite fuzzy events $\Phi = \{\varphi_1, \varphi_2, ..., \varphi_h\}$, with $p_i = p(\varphi_i)$, is consistent if and only if there exists a solution of the system:*

$$AZ = P, \; z_1 + z_2 + ... + z_s = 1, \; Z \geq 0. \tag{10}$$

2.3. Conditional fuzzy events and coherence of their probability assessments

An extension of the concept of fuzzy event is conditional fuzzy event.

Definition 2.7 Let H be a non impossible event. A *fuzzy event conditional* on H is a function $\varphi : \pi \to [0,1]$, where π is a partition of the event H. If $\pi = \{x_1, x_2, ..., x_n\}$, $\varphi(x_i) = a_i$, using a notation consistent with that of Zadeh [23] we write:

$$\varphi = a_1/(x_1|H) + a_2/(x_2|H) + ... + a_n/(x_n|H). \tag{11}$$

For each $y \in Im(\varphi)$ we indicate with $\varphi^{(-1)}(y)$ the union of the elements x of π such that $\varphi(x) = y$. The set $\pi^* = \{\varphi^{(-1)}(y) : y \in Im(\varphi)\}$ is a partition of H and the function $\varphi^* : \pi^* \to [0,1]$ such that $\varphi^*(x) = y$ if and only if $\varphi^{(-1)}(y) = x$ is a conditional fuzzy event called the *reduced form* or *normal form* of φ. The non impossible elements $x \in \pi^* \cup \{H^c\}$ are called *atoms* or *constituents* of φ. The conditional events $x|H, x \in \pi^*$ are the *conditional atoms* (or *conditional constituents*) of φ.

For $H = \Omega$ previous definitions are reduced to that of (unconditional) fuzzy events. The definitions 2.2, 2.3, and their consequences extend to fuzzy events conditional on H by simply replacing Ω with H. Definition 2.4 extends to the case where F is a family of conditional fuzzy events with the same conditioning H and formula (7) is replaced by:

$$p(\varphi) = a_1 p(x_1|H) + a_2 p(x_2|H) + ... + a_n p(x_n|H) \tag{12}$$

where $p(x_i|H)$ is the probability of the conditional event $x_i|H$.

Let $\Phi = \{\varphi_1, \varphi_2, ..., \varphi_h\}$, $\varphi_i : \pi_i \to [0,1]$, be a finite family of finite fuzzy events conditional on H in reduced form and let $\pi = \pi_1\pi_2...\pi_h$. The non impossible events belonging to $\pi \cup \{H^c\}$ are called *atoms* or *constituents* of Φ and the conditional events $x|H, x \in \pi$ are the *conditional atoms.* If $\pi = \{c_1, c_2, ..., c_s\}$, $c_r = x_1^r \cap x_2^r \cap ... \cap x_h^r$, $x_i^r \in \pi_i$, we can extend formula (8) replacing the atoms c_r with the conditional atoms $c_r|H$:

$$\varphi_i = a_i^1/(c_1|H) + a_i^2/(c_2|H) + ... + a_i^s/(c_s|H),\ a_i^r = \varphi_i(x_i^r) = \varphi_i(c_r). \tag{13}$$

Formula (9) is replaced by:

$$p(\varphi_i) = a_i^1 p(c_1|H) + a_i^2 p(c_2|H) + ... + a_i^s p(c_s|H),\ a_i^r = \varphi_i(c_r). \tag{14}$$

Similarly, we can can extend Theorem 2.1. In this case, however, the z_i have the meaning of the unknown probabilities of the atoms conditional on H.

In order to connect the consistency of conditional fuzzy events with the coherence of (unconditional) fuzzy events, let us introduce the following definition.

Definition 2.8 Let $\varphi : \pi \rightarrow [0,1]$ a fuzzy event conditional on $H \neq \Omega$, $\varphi = a_1/(x_1|H) + a_2/(x_2|H) + ... + a_n/(x_n|H)$. We call (*unconditional*) *fuzzy event associated* with φ the fuzzy event $\varphi^0 : \pi^0 = \pi \cup \{H^c\} \rightarrow [0,1]$ defined as $\varphi^0 = a_1/x_1 + a_2/x_2 + ... + a_n/x_n + 0/H^c$.

Remark 2.4 It is well known that, for every pair of events (E, H), $E \subseteq H$, $H \neq \emptyset$, $p(E) = p(E|H)p(H)$. Then from (7) and (12) it follows that, if φ is a fuzzy event conditional on H and φ^0 is the (unconditional) fuzzy event associated, $p(\varphi^0) = p(\varphi)p(H)$.

Then, we have the following theorem:

Theorem 2.2 *Let $p = (p_1, p_2, ..., p_h)$ an assessment of probabilities to the family of finite conditional fuzzy events $\Phi = \{\varphi_1, \varphi_2, ..., \varphi_h\}$, with φ_i conditional on H_i, $p_i = p(\varphi_i)$. If the assessment p implies that the probabilities of events H_i are all non-zero, then p is coherent if and only if the assessment $p^0 = (p_1 p(H_1), p_2 p(H_2), ..., p_h p(H_h))$ on the set of associated (unconditional) fuzzy events $\Phi^0 = (\varphi_1^0, \varphi_2^0, ..., \varphi_h^0)$ is coherent.*

Remark 2.5 It is worth noting that, from the theory on the consistency of conditional events [5], [4], [6] it follows that, if the assessment p does not imply that the probabilities of events H_i are all non-zero, then the consistency conditions on p are more complex than those of consistency of p^0.

3. A reformulation of the multi-objective decision model in terms of fuzzy events and their coherent probabilities

3.1. Weights and scores as coherent probabilities of fuzzy events

Assume, henceforth, that the objectives are events and that the effect of an alternative to an objective is represented by a finite conditional fuzzy event $A_i|O_j$ [10], [23], i.e. a fuzzy set $\varphi_{ij} : \pi_{ij} \rightarrow [0,1]$ with domain a finite partition π_{ij} of O_j. Each element $x \in \pi_{ij}$ is a particular aspect of the objective O_j and the value $\varphi_{ij}(x)$ is the extent to which the alternative A_i meets the facet x.

The score s_{ij}, which measures the degree to which, overall, the alternative A_i meets the objective O_j, is interpreted as a defuzzification of φ_{ij}. In particular, in this Sec., we assume that the scores s_{ij} meet, formally, the properties of a coherent assignment of probabilities on conditional fuzzy events φ_{ij}. In Sec. 5 we will consider other types of consistent defuzzification.

The weight w_j of the objective O_j is interpreted formally as its probability, so the product $w_j s_{ij}$ is the probability of the unconditional fuzzy event φ_{ij}^0 associated with φ_{ij}. If the weights w_j are positive, then by Theorem 2.2, the consistency of the assignment of probabilities s_{ij} on the conditional fuzzy events φ_{ij} is reduced to that of assigning coherent probabilities $w_j s_{ij}$ on the associated fuzzy events φ_{ij}^0.

Both logical reasons and to simplify the algorithms, you should first deal with the consistent assignment of weights $w_j > 0$ of the objectives O_j and then assigning consistent scores s_{ij} to the conditional fuzzy events φ_{ij}.

Let $\omega = \{o_1, o_2, ..., o_h\}$ be the set of atoms of the objectives O_j. Identifying the event O_j with its characteristic function we can write:

$$O_j = \delta_{j1}/o_1 + \delta_{j2}/o_2 + ... + \delta_{jh}/o_h,$$

with $\delta_{jr} = 1$ if $o_r \in O_j$, $\delta_{jr} = 0$ if $o_r \in O_j^c$.

Let $\Delta = (\delta_{jr})$ the matrix having as elements the numbers δ_{jr}. By Theorem 2.1, we have the following corollary:

Corollary 3.2 *The assignment of probabilities $W = [w_1, w_2, ..., w_n]^t$ to the events O_j is consistent if and only if there exist solutions $Z = [z_1, z_2, ..., z_h]^t$ of the system of equations and inequalities:*

$$\Delta Z = W, \ z_1 + z_2 + ... + z_h = 1, \ Z \geq 0. \tag{15}$$

Let us remark that if the objectives are incompatible and exhaustive events then the consistency of weights reduces to condition (1).

Once assigned positive weights w_j to the objectives O_j in a coherent way we pass to the second part of the algorithm: consistently assign probabilities $w_j s_{ij}$ to fuzzy events φ_{ij}^0.

Let $\pi = \{c_1, c_2, ..., c_s\}$ be the set of the atoms of the fuzzy events φ_{ij}^0 and let $\varphi_{ij}^0(c_r) = a_{ij}^r$. By Theorem 2.1, we have:

Corollary 3.3 *The assignment of probabilities $w_j s_{ij}$ to fuzzy events φ_{ij}^0 is consistent if and only if there exist solutions $Z = [z_1, z_2, ..., z_s]^t$ of the system of equations and inequalities:*

$$\forall (i,j), \ a_{ij}^1 z_1 + a_{ij}^2 z_2 + ... + a_{ij}^s z_s = w_j s_{ij} \tag{16}$$

$$z_1 + z_2 + ... + z_s = 1 \tag{17}$$

$$Z \geq 0 \tag{18}$$

If the scores are not consistent, then we must identify criteria and algorithms that allow us to gradually modify these scores and to get *closer to consistency* in every step.

There also seems useful to introduce the concept of *weak consistency*, which could replace the consistency in the case of complex decision problems with uncertain data.

3.2. Fuzzy coherence

Let $\Phi = \{\varphi_1, \varphi_2, ..., \varphi_h\}$ be a finite family of finite fuzzy events $\varphi_i : \pi_i \rightarrow [0,1]$, $\pi = \{c_1, c_2, ..., c_s\}$ the set of the atoms. Let $A = (a_{ir})$ be the matrix with $a_{ir} = \varphi_i(c_r)$.

From Theorem 2.1, an assessment of probabilities $p = (p_1, p_2, ..., p_h)$ on Φ is coherent if and only if p belongs to the set

$$S = \{X = [x_1, x_2, ..., x_h]^t : X = AZ,\ Z \in [0,1]^s,\ z_1 + z_2 + ... + z_s = 1\}.$$

Let us call S the *set of coherence* (or *consistence*) associated with Φ. S is bounded, closed, and contained in $[0,1]^h$, then the Euclidean distance between p and S is a nonnegative real number $d(p,S)$ less than or equal to $\sqrt{h}$. The finding that distance reduces to a quadratic programming problem. In this framework let us give the following definition.

Definition 3.9 We define *fuzzy coherence* the fuzzy set

$$\gamma : p \in [0,1]^h \to 1 - \frac{d^2(p,S)}{h}.$$

For every $p \in [0,1]^h$, $\gamma(p)$ is the *degree of coherence* of p.

In practical applications, where there is uncertainty about the values of fuzzy events, it seems appropriate to accept a slight inconsistency. So, given a decision problem, we propose to set a suitable positive number $\alpha < 1$, depending on the complexity of the problem (e.g. $\alpha = 0.9$). An assignments of probabilities p to a family of fuzzy events is said to be *weakly consistent* if $\gamma(p) \geq \alpha$.

We believe that in complex decision problems with uncertain data can be permitted to accept weakly consistent assignments of probabilities.

3.3. An algorithm to get closer to the consistency

Let Φ be a finite family of h finite fuzzy events and S its set of coherence. We can find, for each $i \in \{1, 2, ..., h\}$, two points $P_i = (a_1, a_2, ..., a_h)$ and $Q_i = (b_1, b_2, ..., b_h)$ such that P_i is a solution of the linear programming problem:

$$\text{(19)} \qquad \min x_i,\ (x_1, x_2, ..., x_h) \in S,$$

and Q_i is a solution of

$$\text{(20)} \qquad \max x_i,\ (x_1, x_2, ..., x_h) \in S.$$

The interval $[a_i, b_i]$ is the projection of S on the axis x_i. Let T be the convex set generated by the points $P_i,\ Q_i,\ i \in \{1, 2, ..., h\}$. Let G be the barycenter of T.

If p is not a coherent probability assessment on Φ, we propose the following algorithm to get closer to the consistency:

(step 1) We fix a small positive real number ε, indicating the extent to which we approach the consistency in each iteration.

(step 2) We urge decision makers to update the assignment p with a new assignment q with the condition that the dot product between the vectors pq and pG is not less than ε (and thus, for the Euclidean distances, $d(q,G) \leq d(p,G) - \varepsilon$).

(step 3) We assign p = q. If p is consistent, the algorithm ends, if p is not consistent we return to step 2.

4. Aggregation of scores

A usual choice is to aggregate the scores of the alternatives using the formula (3). This is acceptable if the decision maker is aware that, in this way, the score of each constituent is counted as many times as there are objectives in which the constituent is contained. If the decision maker believes that this assumption is correct for the decision problem under discussion, then it is right to use the formula (3).

We remark that from formulae (3), (9) and (16) the global score of the alternative A_i is the number:

$$s(A_i) = \sum_{j=1}^{n} s_{ij} w_j = \sum_{r=1}^{s} [\sum_{j=1}^{n} a_{ij}^r] z_r. \tag{21}$$

where $a_{ij}^r = \varphi_{ij}^0(c_r)$.

This means that the score assigned to the atom c_r is:

$$s(c_r) = \sum_{j=1}^{n} a_{ij}^r, \tag{22}$$

i.e., it is the sum of the scores of c_r with respect to the alternative A_i in all the objectives containing c_r and

$$s(A_i) = \sum_{r=1}^{s} s(c_r) z_r. \tag{23}$$

There are many other criteria to assess the scores of atoms. For instance, if the decision maker wants the score of each constituent contained in at least an objective is counted only once in the aggregation of the scores of each alternative, he can assume that in formula (23) the score of the atom c_r with respect to the alternative A_i is:

$$s(c_r) = \max_{j=1}^{n} a_{ij}^r. \tag{24}$$

Of course, there are many other possible formulae for $s(c_r)$. Precisely, we can assume:

$$s(c_r) = f(a_{i1}^r, a_{i2}^r, ..., a_{in}^r), \tag{25}$$

where f is a non negative real function, defined in $[0,1]^n$, null in $(0,0,...,0)$, continuous, symmetric respect to every pair of variables, and increasing respect to every argument. For instance, the operation of "sum" or of "max" can be replaced by a t-conorm.

We emphasize that, if the formula (22) holds, then the value $s(A_i)$ in formula (23) is independent on the solution $(z_1, z_2, ..., z_s)$ considered of the system (16)–(17) with the conditions (18). On the contrary, if a different formula is adopted for $s(c_r)$, then the value $s(A_i)$ depend on $(z_1, z_2, ..., z_s)$. From the continuity of the function f, the set of values $s(A_i)$ is a closed interval $[m_i, M_i]$ of the real line.

Of course m_i is obtained when $(z_1, z_2, ..., z_s)$ is a solution P_i of the mathematical programming problem:

$$\min s(A_i) = \sum_{r=1}^{s} f(a_{i1}^r, a_{i2}^r, ..., a_{in}^r)\ z_r, \tag{26}$$

with the constraints given by system (16)–(17) with the conditions (18).

Similarly M_i is obtained when $(z_1, z_2, ..., z_s)$ is a solution Q_i of the mathematical programming problem:

$$\max s(A_i) = \sum_{r=1}^{s} f(a_{i1}^r, a_{i2}^r, ..., a_{in}^r)\ z_r, \tag{27}$$

with the constraints given by system (16) - (17) with the conditions (18).

We propose, below, to assume that $s(A_i)$ is a suitable triangular fuzzy number. For definitions and results on fuzzy numbers, see, e.g., [21], [22], [23], [8], [20].

It seems natural to assume the support of $s(A_i)$ is the closed interval $[m_i, M_i]$. In order to define the core $c(A_i)$ of the fuzzy number $s(A_i)$, we propose to consider the convex set $H = [P_i, Q_i, i \in \{1, 2, ..., m\}]$ generated by the vertices P_i, Q_i, $i \in \{1, 2, ..., m\}$. H is contained in the set K of all the solutions of the system (16)–(17) with the conditions (18), that is also a convex set.

Let $G = (g_1, g_2, ..., g_s)$ be the barycenter of H. G belongs to H and it seems reasonable to assume that the core of $s(A_i)$ is the value

$$h_i = \sum_{r=1}^{s} f(a_{i1}^r, a_{i2}^r, ..., a_{in}^r)\ g_r, \tag{28}$$

So we propose $s(A_i)$ is the triangular fuzzy number (m_i, h_i, M_i).

5. The multi-objective decision model in terms of fuzzy events and their fuzzy measures

5.1. Fuzzy measures

Let us recall the concepts of *fuzzy measure*, Archimedean t-conorms and decomposable measure and some basic results (see, e.g., [18], [19], [8], [9]).

Definition 5.10 Let U be a set and $\mathcal{E}$ a σ-field of subsets of U. A real function, $m : \mathcal{E} \to R$, is said to be a *fuzzy measure* on $\mathcal{E}$ if:

FM1 $m(\emptyset) = 0$; $m(U) = 1$;

FM2 $\forall A, B \in \mathcal{E}, A \subseteq B \Rightarrow m(A) \leq m(B)$;

FM3 if $\{A_n\}_{n \in N}$ is a monotonic sequence of elements of $\mathcal{E}$ then:

$$\lim_{n \to +\infty} A_n = A \Rightarrow \lim_{n \to +\infty} m(A_n) = m(A).$$

If $\mathcal{E}$ is finite then conditions FM1 and FM2 imply FM3. If we want to generalize the concept of finitely additive probabilities considered by de Finetti [5] then we must define a *weak fuzzy measure*, satisfying only the first two conditions, regardless of whether the domain is finite or not. Then we introduce the following definition.

Definition 5.11 Let U be a set and F a family of subsets of U containing $\{\emptyset, U\}$. A real function, $m : F \to R$, is said to be a *weak fuzzy measure* on F if:

FM1 $m(\emptyset) = 0$; $m(U) = 1$;

FM2 $\forall A, B \in F, A \subseteq B \Rightarrow m(A) \leq m(B)$.

Remark 5.6 A coherent finitely additive probability satisfies condition FM1 and FM2, then the concept of *weak fuzzy measure* is a generalization of *coherent finitely additive probability.*

Definition 5.12 A t-conorm $\oplus$ is said to be *Archimedean* if it is continuous and $x \oplus x > x$, $\forall x \in (0, 1)$. An Archimedean t-conorm is called *strict* if it is strictly increasing in the open square $(0, 1)^2$.

The following representation theorem holds [9]:

Theorem 5.3 *A binary operation $\oplus$ on $[0, 1]$ is an Archimedean t-conorm if and only if there exists a strictly increasing and continuous function $g : [0, 1] \to [0, +\infty]$, with $g(0) = 0$, such that*

$$x \oplus y = g^{(-1)}(g(x) + g(y)).$$

Function $g^{(-1)}$ denotes the pseudo-inverse of g, i.e.:

$$g^{(-1)}(x) = g^{-1}(\min(x, g(1))).$$

The function g, called an additive generator *of $\oplus$, is unique up to a positive constant factor. Moreover $\oplus$ is strict if and only if $g(1) = +\infty$.*

A compromise between the very general concept of weak fuzzy measure and that of finitely additive probability, rather restrictive in some applications of decision theory, was considered by some authors, notably by Weber [19]. Here are the definitions and basic results that will be useful for the rest of this paper.

Definition 5.13 Let U be a set and F a field of subsets of U. A weak fuzzy measure m on F is said to be *a measure decomposable* w. r. to a t-conorm $\oplus$, or *a $\oplus$-decomposable measure*, if:

$$A \cap B = \emptyset \Rightarrow m(A \cup B) = m(A) \oplus m(B).$$

In [19], the following classification theorem is proved:

Theorem 5.4 *If the operation $\oplus$ in $[0, 1]$ is a strict Archimedean t-conorm, then $g \circ m : F \to [0, +\infty]$ is an infinite additive measure, whenever m is a $\oplus$-decomposable one.*

If $\oplus$ is a nonstrict Archimedean t-conorm, then $g \circ m$ is finite and one of the following cases occurs:

NSA *$g \circ m : F \to [0, +\infty]$ is a finite additive measure;*

NSP *$g \circ m$ is a finite set function which is only pseudo additive, i.e., if $\{A_n\}_{n \in \{1,2,...,s\}}$ is a family of pairwise disjoint elements of F, then:*

$$(g \circ m)\left(\bigcup_{n=1}^{s} A_n\right) < g(1) \Rightarrow (g \circ m)\left(\bigcup_{n=1}^{s} A_n\right) = \sum_{n=1}^{s}(g \circ m)(A_n);$$

$$(g \circ m)\left(\bigcup_{n=1}^{s} A_n\right) = g(1) \Rightarrow (g \circ m)\left(\bigcup_{n=1}^{s} A_n\right) \leq \sum_{n=1}^{s}(g \circ m)(A_n).$$

5.2. An extension of fuzzy measures to fuzzy events

Let $\oplus$ be a nonstrict Archimedean t-conorm and let g be an additive generator of $\oplus$ with $g(1) = 1$. Let $\star$ be a t-norm. We introduce the following definition:

Definition 5.14 Let $\varphi : \pi \to [0, 1]$ be a finite fuzzy event, $\varphi = a_1/x_1 + a_2/x_2 + ... + a_n/x_n$. If $m : \pi \to [0, 1]$ is a $\oplus$ - decomposable fuzzy measure in π, then the *measure of φ associated to the t-norm $\star$* is the number:

$$m(\varphi) = a_1 \star m(x_1) \oplus a_2 \star m(x_2) \oplus ... \oplus a_n \star m(x_n). \tag{29}$$

Example 5.1 Two notable t-norms are the usual multiplication $\cdot$ and the t-norm $\cdot^g$, associated to the pair $(\cdot, g)$, defined as follows (see [8], p. 75)

$$a \cdot^g b = g^{(-1)}(g(a) \cdot g(b)). \tag{30}$$

5.3. Coherence of a fuzzy measure assessment on fuzzy events

If we replace probabilities with $\oplus$ - decomposable fuzzy measures, then Definition 2.6 is replaced by the following definition.

Definition 5.15 An assignment of $\oplus$-decomposable fuzzy measures $m = (m_1, m_2, ..., m_h)$ to the family of finite fuzzy events $\Phi = (\varphi_1, \varphi_2, ..., \varphi_h)$, with $m_i = m(\varphi_i)$, is said to be coherent (or consistent) if there exists a $\oplus$ - decomposable fuzzy measure distribution on the family $\{c_1, c_2, ..., c_s\}$ of the atoms of Φ such that:

$$m(\varphi_i) = a_i^1 \star m(c_1) \oplus a_i^2 \star m(c_2) \oplus ... \oplus a_i^s \star m(c_s), \tag{31}$$

with $a_i^r = \varphi_i(c_r)$.

Definition 5.15 and Theorem 5.4 imply the following theorem.

Theorem 5.5 *The assignment of $\oplus$-decomposable fuzzy measures $m = [m_1, m_2, ..., m_h]^t$, $m_i < 1$, to the family of finite fuzzy events $\Phi = (\varphi_1, \varphi_2, ..., \varphi_h)$, with $m_i = m(\varphi_i)$, is consistent if and only if there exists a solution of the system:*

$$\forall i \in \{1, 2, ..., h\}, g(a_i^1 \star z_1) + g(a_i^2 \star z_2) + ... + g(a_i^s \star z_s) = g(m_i) \tag{32}$$

$$g(z_1) + g(z_2) + ... + g(z_s) \geq g(1) \tag{33}$$

$$Z \geq 0 \tag{34}$$

where z_r is the unknown measure of the atom c_r.

The previous system is not in general a linear system and is therefore difficult to solve. A substantial simplification is achieved, however, if the t-norm $\star$ is equal to $\cdot^g$. In fact, in this case it is reduced to the following system, linear with respect to the unknowns $g(z_r)$

$$\forall i, g(a_i^1)g(z_1) + g(a_i^2)g(z_2) + ... + g(a_i^s)g(z_s) = g(m_i) \tag{35}$$

$$g(z_1) + g(z_2) + ... + g(z_s) = 1 \geq g(1) \tag{36}$$

$$Z \geq 0. \tag{37}$$

5.4. Coherence of a fuzzy measure assessment as scores in a decision making problem

Let us refer to the notations used in Section 3.

Let $\pi = \{c_1, c_2, ..., c_s\}$ be the set of the atoms of the fuzzy events φ_{ij}^0 and let $\varphi_{ij}^0(c_r) = a_{ij}^r$.

From the results of the previous subsection, if the t-norm $\star$ is $\cdot^g$, then we have the following coherence theorem.

Theorem 5.6 *The assignment of $\oplus$-decomposable fuzzy measures $w_j s_{ij} < 1$, $w_j > 0$, to fuzzy events φ_{ij}^0 is consistent if and only if there exist solutions $Z = [z_1, z_2, ..., z_s]^t$ of the system of equations and inequalities:*

$$\forall (i, j),\ g(a_{ij}^1)g(z_1) + g(a_{ij}^2)g(z_2) + ... + g(a_{ij}^s)g(z_s) = g(w_j s_{ij}) \tag{38}$$

$$g(z_1) + g(z_2) + ... + g(z_s) \geq g(1) \tag{39}$$

$$Z \geq 0. \tag{40}$$

6. Conclusions and research perspectives

The aim of the paper is to stimulate a reflection on some key points in the decision process. In particular, we have explicated the hypotheses usually implicitly admitted in the decision-making processes and we have proposed criteria and procedures for assignment of weights and scores are consistent with the accepted assumptions and the logical relationships among the objectives and among the alternatives.

In the first 4 sections the reasoning and conclusions were bound by the idea of an additive aggregation of the weights or scores in a manner analogous to that which occurs in probability. In Sec. 5 were examined some implications arising from the idea of aggregations that follow logic other than additive.

The results can be helpful for the construction of consistent decision-making processes, i.e. taking into account the logical relations between objectives and alternatives, and the resulting numeric constraints in assigning weights and scores.

These constraints also depend on the ideas of measurement and aggregation of the measures that decision-makers see fit. We think it is important that these opinions and points of view are made explicit and that the assignments and criteria for aggregating measures adopted are consistent with these ideas.

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THE CATEGORY OF HYPER S-ACTS

Leila Shahbaz

Department of Mathematics
University of Maragheh
Maragheh 55181-83111
Iran
e-mail: leilashahbaz@yahoo.com

Abstract. The actions of a semigroup or a monoid S on sets have been studied and applied in many branches of mathematics. In this paper, we generalize this notion, and introduce the category of hyper S-acts with the homomorphisms between them. Then, using the usual notion of congruences defined for hyper S-acts, quotients are defined and isomorphism theorems are proved. Finally, limits and colimits in the category of hyper S-acts are studied.

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1. Introduction and preliminaries

The study of hyperstructures started in [7] by introducing hypergroups. Since then other classic hyperstructures have been studied in [2], [9], [10], [11], and the notion has been generalized to universal hyperalgebras and studied in [1], [3], [4], [8]. In this paper we introduce a special type of hyperstructure, namely, hyper S-acts, and study some notions such as congruences, quotients, isomorphism theorems, limits, and colimits in the category they form.

In the rest of this section we recall the definition of the category of S-acts. Let S be a semigroup. Recall that a (right) S*-act* or S*-system* is a set A together with a function $\lambda : A \times S \to A$, called the *action* of S (or the S-action) on A, such that for $a \in A$ and $s, t \in S$ (denoting $\lambda(a, s)$ by as) $a(st) = (as)t$. If S is a monoid with an identity e, we add the condition $ae = a$.

A morphism $f : A \to B$ between S-acts A, B is called an S*-map* if, for each $a \in A$, $s \in S$, $f(as) = f(a)s$.

Since id_A and the composite of two S-maps are S-maps, we have the category **Act-**S of all S-acts and S-maps between them (for more information about acts see [5] and [6]).

2. The category of hyper S-acts

In this section, first the notion of a hyper S-act over a monoid S is defined and then defining the proper homomorphisms between them, the category of hyper S-acts is introduced.

Definition 2.1 Let S be a monoid and A be a set. If we have a mapping

$$\begin{array}{rccc} \mu: & A \times S & \longrightarrow & \mathcal{P}(A) \\ & (a,s) & \mapsto & \mu(a,s) = a \circ s \in \mathcal{P}(A) \end{array}$$

called the *hyper action* of S (or the hyper S-action) on A, such that for $a \in A$ and $s, t \in S$

(i) $a \in a \circ e$,

(ii) $a \circ (st) = (a \circ s) \circ t$, where

$$B \circ s = \bigcup_{b \in B} b \circ s, \forall B \subseteq A.$$

Then we call A a *right hyper S-act* or a *right hyper act* over S and write $A_{\mathcal{H}}$.

Analogously, we define a *left hyper S-act* A and write ${}_{\mathcal{H}}A$.

Remark 2.2 Every S-act A_S is naturally a hyper S-act, by defining

$$\begin{array}{rccc} \mu: & A \times S & \longrightarrow & \mathcal{P}(A) \\ & (a,s) & \longmapsto & \mu(a,s) = \{as\}. \end{array}$$

But there are hyper S-acts which are not ordinary as above. Take $S = \{1, s\}$ where $s^2 = s$ and $A = \{a, b\}$ with the action $a \circ 1 = \{a\}$, $a \circ s = \{a, b\}$, $b \circ 1 = \{b\}$, $b \circ s = \{a, b\}$. Then $A_{\mathcal{H}}$ is a right hyper S-act which is not a right S-act.

Definition 2.3 A function $f : A_{\mathcal{H}} \to B_{\mathcal{H}}$, where A and B are hyper S-acts, is called a homomorphism if $f(a \circ s) \subseteq f(a) \circ s$ for all $a \in A, s \in S$.
f is called a strong homomorphism if $f(a \circ s) = f(a) \circ s$ for all $a \in A, s \in S$.

Definition 2.4 A homomorphism $f : A_{\mathcal{H}} \to B_{\mathcal{H}}$, where A and B are hyper S-acts, is called an isomorphism if it is bijective.

One can easily see that the hyper S-acts with their homomorphism form a category, denoted by $\mathcal{H}Act - S$.

3. Congruences and quotients

This section is devoted to the study of congruences and quotients of hyper S-acts.

Definition 3.5 The equivalence relation θ on a hyper S-act $A_{\mathcal{H}}$ is called a *congruence* if for every $a, b \in A$ and $s \in S$,

$$a\theta b \Longrightarrow \frac{a \circ s}{\theta} = \frac{b \circ s}{\theta}$$

where, for $X \subseteq A$, $\frac{X}{\theta} = \{\frac{x}{\theta} = [x]_\theta : x \in X\}$, and $[x]_\theta$ is the equivalence class of x with respect to θ.

Notice that for every $X, Y \subseteq A$, $\frac{X}{\theta} = \frac{Y}{\theta}$ if and only if $X\overline{\theta}Y$, where $X\overline{\theta}Y$ means that for every $x \in X$ there exists $y \in Y$ such that $x\theta y$ and for every $y \in Y$ there exists $x \in X$ such that $x\theta y$.

The set of all equivalence relations on a hyper S-act $A_{\mathcal{H}}$ is denoted by $Eq(A_{\mathcal{H}})$, and the set of all congruences on $A_{\mathcal{H}}$ is denoted by $Con(A_{\mathcal{H}})$.

Remark 3.6 If A_S is an S-act then an equivalence relation θ on A_S is a congruence on A_S if and only if it is a congruence on A_S as a hyper S-act.

Definition 3.7 Let $A_{\mathcal{H}}$ be a hyper S-act and $\theta \in Eq(A_{\mathcal{H}})$. We define a hyper operation $\circ_{\frac{A_{\mathcal{H}}}{\theta}}$ on $\dfrac{A_{\mathcal{H}}}{\theta}$ as follows:

$$\begin{array}{rccc} \circ_{\frac{A_{\mathcal{H}}}{\theta}} : & \dfrac{A_{\mathcal{H}}}{\theta} \times S & \longrightarrow & \mathcal{P}\left(\dfrac{A_{\mathcal{H}}}{\theta}\right) \\ & \left(\dfrac{a}{\theta}, s\right) & \longmapsto & \displaystyle\bigcup_{x \in \frac{a}{\theta}} \frac{x \circ s}{\theta} \end{array}$$

for all $a \in A$ and $s \in S$.

We call $\dfrac{A_{\mathcal{H}}}{\theta}$ with this hyper operation, the *quotient hyper S-act* of $A_{\mathcal{H}}$ with respect to a congruence θ. Notice that if θ is a congruence on $A_{\mathcal{H}}$ then

$$\frac{a}{\theta} \circ s = \frac{a \circ s}{\theta}.$$

Theorem 3.8 *Let $A_{\mathcal{H}}$ be a hyper S-act and $\theta \in Eq(A_{\mathcal{H}})$. Then we have the following:*

(i) *The natural map $\pi : A_{\mathcal{H}} \to \dfrac{A_{\mathcal{H}}}{\theta}$ given by $\pi(a) = \dfrac{a}{\theta}$ is a homomorphism.*

(ii) *The natural map $\pi : A_{\mathcal{H}} \to \dfrac{A_{\mathcal{H}}}{\theta}$ is a strong homomorphism if and only if θ is a congruence and it is called a canonical epimorphism.*

Proof. (i) Let $a \in A$ and $s \in S$. Then we have $\pi(a \circ s) = \dfrac{a \circ s}{\theta} \subseteq \displaystyle\bigcup_{x \in \frac{a}{\theta}} \frac{x \circ s}{\theta} =$ $\pi(a) \circ s$. Hence π is a homomorphism.

(ii) Let $\theta \in Con(A_{\mathcal{H}})$. Then for $a \in A, x \in \dfrac{a}{\theta}$, and $s \in S$, $\dfrac{a \circ s}{\theta} = \dfrac{x \circ s}{\theta}$ and so $\pi(a \circ s) = \dfrac{a \circ s}{\theta} = \displaystyle\bigcup_{x \in \frac{a}{\theta}} \frac{x \circ s}{\theta} = \pi(a) \circ s$. Thus π is a strong homomorphism. Conversely, let π be a strong homomorphism, $a, b \in A, a\theta b$, and $s \in S$. Then

$$\frac{a \circ s}{\theta} \subseteq \bigcup_{x \in \frac{b}{\theta}} \frac{x \circ s}{\theta} = \frac{b}{\theta} \circ s = \pi(b) \circ s = \pi(b \circ s) = \frac{b \circ s}{\theta}.$$

Similarly, $\dfrac{b \circ s}{\theta} \subseteq \dfrac{a \circ s}{\theta}$, and hence θ is a congruence on $A_{\mathcal{H}}$. ▪

Theorem 3.9 *Let $f : A_{\mathcal{H}} \to B_{\mathcal{H}}$ be a strong homomorphism of hyper S-acts and $\theta \in Con(B_{\mathcal{H}})$. Then $(f \times f)^{-1}(\theta) = \{(x, y) : f(x)\theta f(y)\}$ is a congruence on $A_{\mathcal{H}}$. In particular, $Kerf = \{(x, y) : f(x) = f(y)\}$ is a congruence on $A_{\mathcal{H}}$.*

Proof. Let $a, b \in A, s \in S$ and $(a, b) \in (f \times f)^{-1}(\theta)$. Then $f(a)\theta f(b)$. So $f(a \circ s) = (f(a) \circ s)\overline{\theta}(f(b) \circ s) = f(b \circ s)$. Thus for every $x \in a \circ s$ there exists $y \in b \circ s$ such that $f(x)\theta f(y)$, or equivalently $(x, y) \in (f \times f)^{-1}(\theta)$. Similarly, for every $y \in b \circ s$ there exists $x \in a \circ s$ such that $(x, y) \in (f \times f)^{-1}(\theta)$. Hence $(f \times f)^{-1}(\theta)$ is a congruence on $A_{\mathcal{H}}$. The second part follows from the first part using the fact that $Kerf = (f \times f)^{-1}(\Delta_B)$ where $\Delta_B = \{(b, b) : b \in B\}$ is the identity congruence on $B_{\mathcal{H}}$. ▪

Corollary 3.10 *For a hyper S-act $A_{\mathcal{H}}$, the following are equivalent:*

(i) $\theta \in Con(A_{\mathcal{H}})$,

(ii) *There exists a strong homomorphism $f : A_{\mathcal{H}} \to B_{\mathcal{H}}$ such that $\theta = Kerf$.*

Proof. (i) $\Rightarrow$ (ii) Take f to be the canonical epimorphism $\pi : A_{\mathcal{H}} \to \dfrac{A_{\mathcal{H}}}{\theta}$.

(ii) $\Rightarrow$ (i) It is clear by the above Theorem. ▪

Remark 3.11 The ordered set $(Eq(A_{\mathcal{H}}), \subseteq)$ is a complete lattice with $\cap$ as infimum and supremum, $\bigvee$, as follows

$$\bigvee_{i \in I} \theta_i = \{(a, b) : \exists x_0, ..., x_n \in A, \exists i_1, ..., i_n \in I \ s.t. \ a = x_0 \ \theta_{i_1} \ x_1 ... \theta_{i_n} \ x_n = b\}$$

for $\theta_i \in Eq(A_{\mathcal{H}})$.

Theorem 3.12 *For every hyper S-act $A_{\mathcal{H}}$, $(Con(A_{\mathcal{H}}), \subseteq)$ is a complete lattice.*

Proof. We show that for $\{\theta_i\}_{i \in I} \in Con(A_{\mathcal{H}}), \bigvee_{i \in I} \theta_i = \theta$ is a congruence. Let $a, b \in A, s \in S$ and $a\theta b$. Then, by the definition of $\bigvee$ which is given in the above Remark, there exist $x_0, ..., x_n \in A, i_1, \ldots, i_n \in I$ such that $a = x_0\theta_{i_1}x_1...\theta_{i_n}x_n = b$. Since each θ_i is a congruence, we have $(x_{t-1} \circ s) \ \overline{\theta_{i_t}} \ (x_t \circ s)$ for $t = 1, ..., n$. Thus $(a \circ s)\overline{\theta}(b \circ s)$ and so θ is a congruence. Therefore, arbitrary supremums exist in $Con(A_{\mathcal{H}})$ and hence it is a complete lattice. ▪

4. The isomorphism theorems

In this section, using the usual notion of a congruence defined for hyper S-acts, we prove the decomposition theorem and the generalized version of the second isomorphism theorem from S-act to hyper S-acts.

Theorem 4.13 (Decomposition Theorem) *Let $f : A_{\mathcal{H}} \to B_{\mathcal{H}}$ and $g : A_{\mathcal{H}} \to C_{\mathcal{H}}$ be strong homomorphisms, g be onto and $Kerg \subseteq Kerf$. Then there exists a unique strong homomorphism $h : C_{\mathcal{H}} \to B_{\mathcal{H}}$ such that $hg = f$.*

Proof. Define h by $h(c) = f(a)$ where $c = g(a)$. Then, h is well-defined since $Kerg \subseteq Kerf$. It is enough to show that h is a strong homomorphism. Let $c \in C, g(a) = c$ for some $a \in A$. Then $h(c \circ s) = h(g(a) \circ s) = h(g(a \circ s)) = f(a \circ s) = f(a) \circ s = hg(a) \circ s = h(c) \circ s$ and hence h is a strong homomorphism. The uniqueness of h follows from its definition. ∎

Corollary 4.14 (First Isomorphism Theorem) *Let $f : A_{\mathcal{H}} \to B_{\mathcal{H}}$ be an onto strong homomorphism. Then $A_{\mathcal{H}}/Kerf \cong B_{\mathcal{H}}$.*

Proof. Apply the above Theorem to $\pi : A_{\mathcal{H}} \to A_{\mathcal{H}}/Kerf$ instead of g. Since π is onto and $Ker\pi = Kerf$, there exists $h : A_{\mathcal{H}}/Kerf \to B_{\mathcal{H}}$ such that $h\pi = f$. Since f, π are onto, so is h. Also, if $h(c) = h(c')$ then $f(a) = f(a')$, where $\pi(a) = c, \pi(a') = c'$. But, since $Ker\pi = Kerf$, we get $\pi(a) = \pi(a')$, that is $c = c'$. Hence h is one-one and thus an isomorphism. ∎

Notation 4.15 *Let A be a set, and $\theta, \psi \in Eq(A)$ and $\theta \subseteq \psi$. We denote the set $\{(x/\theta, y/\theta) \in (A/\theta)^2 : (x, y) \in \psi\}$ by ψ/θ.*

Theorem 4.16 (Second Isomorphism Theorem) *Let $A_{\mathcal{H}}$ be a hyper S-act and $\theta \in Eq(A_{\mathcal{H}})$. Then*

(i) *For $\psi \in Con(A_{\mathcal{H}})$ with $\theta \subseteq \psi, \psi/\theta$ is a congruence on $A_{\mathcal{H}}/\theta$ and $(A_{\mathcal{H}}/\theta)/(\psi/\theta) \cong A_{\mathcal{H}}/\psi$.*

(ii) *If θ is a congruence on $A_{\mathcal{H}}$, then all congruences on $A_{\mathcal{H}}/\theta$ are of the form ψ/θ for some $\psi \in Con(A_{\mathcal{H}})$ with $\theta \subseteq \psi$.*

Proof. (i) First we show that the map $f : A_{\mathcal{H}}/\theta \to A_{\mathcal{H}}/\psi$ given by $f(a/\theta) = a/\psi$ is a strong homomorphism. So, let $a \in A$ and $s \in S$. Then

$$\begin{aligned} f\left(\frac{a}{\theta} \circ s\right) &= f\left(\bigcup_{x \in \frac{a}{\theta}} \frac{x \circ s}{\theta}\right) = \bigcup_{x \in \frac{a}{\theta}} f\left(\frac{x \circ s}{\theta}\right) = \bigcup_{x \in \frac{a}{\theta}} \frac{x \circ s}{\psi} = \bigcup_{x \in \frac{a}{\theta}} \frac{a \circ s}{\psi} \\ &= \frac{a \circ s}{\psi} = \frac{a}{\psi} \circ s = f\left(\frac{a}{\theta}\right) \circ s. \end{aligned}$$

Thus f is a strong homomorphism and $\psi/\theta = Kerf \in Con(A_{\mathcal{H}}/\theta)$ and so by Corollary 4.14, $(A_{\mathcal{H}}/\theta)/(\psi/\theta) \cong A_{\mathcal{H}}/\psi$.

(ii) Let φ be a congruence on $A_{\mathcal{H}}/\theta$. Take $\psi = \{(a, b) : (a/\theta, b/\theta) \in \varphi\}$. Then, $\theta \subseteq \psi$ and $\varphi = \psi/\theta$. Further ψ is a congruence on $A_{\mathcal{H}}$. Since θ and ψ/θ are congruences on $A_{\mathcal{H}}$, $\gamma_\theta : A_{\mathcal{H}} \to A_{\mathcal{H}}/\theta$ and $\gamma : A_{\mathcal{H}}/\theta \to \dfrac{A_{\mathcal{H}}/\theta}{\psi/\theta}$ are strong homomorphisms. Thus, $f = \gamma\gamma_\theta$ is also a strong homomorphism. But

$$\begin{aligned} Kerf &= \{(x, y) : f(x) = f(y)\} = \{(x, y) : (x/\theta)/\psi = (y/\theta)/\psi\} \\ &= \{(x, y) : (x/\theta)(\psi/\theta)(y/\theta)\} = \psi \end{aligned}$$

Thus ψ is a congruence by Corollary 3.10. ∎

5. Limits and colimits in the category $\mathcal{H}ACT - S$

In this section the limits and colimits of hyper S-acts are studied.

Remark 5.17 For a semigroup S, the set of all hyper S-actions on a fixed set X is denoted by $H = H(X)$. Let $\circ_i, \circ_j$ be two elements of $H(X)$. Define $\circ_i \leq \circ_j$ if for every $x \in X$ and $s \in S$, $x \circ_i s \subseteq x \circ_j s$. Then $H(X)$ with the relation $\leq$ is a complete Boolean algebra, with $\bigwedge H, \bigvee H$ given by

$$x\left(\bigwedge H\right)s = \bigcap_{\circ \in H}(x \circ s),$$
$$x\left(\bigvee H\right)s = \bigcup_{\circ \in H}(x \circ s),$$

for $x \in X$ and $s \in S$. Specially, $\mathbf{0}, \mathbf{1}$ in $H(X)$ are given by $x\mathbf{0}s = \emptyset, x\mathbf{1}s = X$. Also, the complement $\circ'$ of an element $\circ \in H(X)$ is defined as $x \circ' s = X - (x \circ s)$.

Lemma 5.18 *Let S be a semigroup and X be a set, $F = \{f_i : X \to A_i \mid i \in I\}$, $G = \{g_i : B_i \to X \mid i \in I\}$ be families of functions, where A_i, B_i are hyper S-acts, for all $i \in I$. Then, the greatest (smallest) hyper S-action on the set X, for which $f_i \circ g_i$ are homomorphisms, exists. This hyper S-action on X is called the hyper S-action induced by $F(G)$ and is denoted by $\circ^{\to}(F)(\circ^{\leftarrow}(G))$, or simply by $\circ^{\to}(\circ^{\leftarrow})$.*

Proof. Let H be the set of all hyper S-actions on a set X which makes each f_i a homomorphism. Take $\circ^{\to} = \bigvee H$. It is enough to show that $\circ^{\to} \in H$. Let $i \in I$ be fixed. We prove that each f_i is a homomorphism from hyper S-act $(X, \circ^{\to})$ to $(A_i, \circ_i)$. Let $x \in X, s \in S$. For every $\circ \in H, f_i(x \circ s) \subseteq f_i(x) \circ_i s$ where $\circ_i$ is the hyper S-action on A_i. Then

$$f_i(x \circ^{\to} s) = f_i\left[\bigcup_{\circ \in H}(x \circ s)\right] = \bigcup_{\circ \in H} f_i(x \circ s) \subseteq f_i(x) \circ_i s.$$

Thus f_i is a homomorphism. Dually, taking K to be the set of all hyper S-actions on X which makes each g_i a homomorphism, and $\circ^{\leftarrow} = \bigwedge K$ it can be shown that $\circ^{\leftarrow} \in K$. ∎

Theorem 5.19 *Let $D : I \to \mathcal{H}Act - S$ be a diagram and $U : \mathcal{H}Act - S \to$ **Set** be the forgetful functor. If $\{f_i : A \to UA_i\}_{i \in I}$ is a limit of $U \circ D : I \to$ **Set**, then $\{f_i : A \to A_i\}_{i \in I}$ is a limit of D, where the hyper S-action on A is induced by $\{f_i : A \to A_i\}_{i \in I}$.*

Proof. Let $\{h_i : C \to A_i\}_{i \in I}$ be a source of D in $\mathcal{H}Act - S$. Consider $\{Uh_i : UC \to UA_i\}_{i \in I}$ in **Set**. Since $\{f_i : A \to UA_i\}_{i \in I}$ is a limit of $U \circ D$, there exists $h : UC \to A$ such that $h_i = f_i h$, for all $i \in I$. Now, it is enough to show that h is a homomorphism, where the hyper S-action on A, say $\circ^{\to}$, is induced by $\{f_i : A \to A_i\}_{i \in I}$. Define another hyper S-action $\circ_A$ on A as follows:

$$h(x) \circ_A s = \bigcup_{h(y) = h(x)} h(y \circ_C s)$$

for $x \in C, s \in S$, and for other elements of A, $a \circ_A s = a \circ^{\rightarrow} s$. Then $\circ_A$ is a hyper S-action on A which makes each f_i a homomorphism. Indeed, for $i \in I, x \in C, s \in S$,

$$\begin{aligned} f_i[h(x) \circ_A s] &= f_i[\textstyle\bigcup_{h(y)=h(x)} h(y \circ_C s)] \\ &= \bigcup_{h(y)=h(x)} f_i h(y \circ_C s) \\ &\subseteq \bigcup_{h(y)=h(x)} h_i(y \circ_C s) \\ &\subseteq \bigcup_{h(y)=h(x)} h_i(y) \circ_i s \\ &= \textstyle\bigcup_{h(y)=h(x)} f_i h(y) \circ_i s \\ &= f_i h(x) \circ_i s. \end{aligned}$$

The result for the other elements of A follows from the same property of $\circ^{\rightarrow}$. So, $\circ_A \leq \circ^{\rightarrow}$, and then for every $x \in C, s \in S$,

$$h(x \circ_C s) \subseteq h(x) \circ_A s \subseteq h(x) \circ^{\rightarrow} s.$$

Thus, h is a homomorphism, as required. ∎

Theorem 5.20 *Let $D : I \to \mathcal{H}Act - S$ be a diagram and $U : \mathcal{H}Act - S \to$ **Set** be the forgetful functor. If $\{g_i : UA_i \to A\}_{i \in I}$ is a colimit of $U \circ D : I \to$ **Set**, then $\{g_i : A_i \to A\}_{i \in I}$ is a colimit of D, where the hyper S-action on A is induced by $\{g_i : A_i \to A\}_{i \in I}$.*

Proof. Similar to the proof of the above theorem, let $\{k_i : A_i \to C\}_{i \in I}$ be a sink of D in $\mathcal{H}Act - S$. Consider $\{Uk_i : UA_i \to UC\}_{i \in I}$ in **Set**. Since $\{g_i : UA_i \to A\}_{i \in I}$ is a colimit of $U \circ D$, there exists $k : A \to UC$ such that $kg_i = k_i$, for all $i \in I$. Now, we show that k is a homomorphism, where the hyper S-action on A, say $\circ^{\leftarrow}$, is induced by $\{g_i : A_i \to A\}_{i \in I}$. Define another hyper S-action $\circ_A$ on A as follows:

$$a \circ_A s = k^{-1}(k(a) \circ_C s)$$

for $a \in A, s \in S$. Then $\circ_A$ is a hyper S-action on A which makes each g_i a homomorphism. So, $\circ^{\leftarrow} \leq \circ_A$, and then for every $a \in A, s \in S$,

$$a \circ^{\leftarrow} s \subseteq a \circ_A s = k^{-1}(k(a) \circ_C s)$$

and hence $k(a \circ^{\leftarrow} s) \subseteq k(a) \circ_C s$. Thus, k is a homomorphism. ∎

As a corollary of the two preceding theorems we have the following.

Corollary 5.21 *The category $\mathcal{H}Act - S$ has all limits and colimits and $U : \mathcal{H}Act - S \to$ **Set** preserves limits and colimits.*

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MAXIMAL PARTIAL LINE SPREADS OF $PG(3, q)$, q EVEN

Maria Scafati Tallini

Abstract. Applying the representation of $PG(3, q)$ over $AG(2, q)$, [3], we construct a maximal partial line spread of $PG(3, q)$, $q = 2^{2n}$, n an integer, $n \geq 1$, of size $q^2 = q + 2$. This size is the greatest known till now, except a sporadic case, found by O. Heden [2], for $q = 7$.

1. Introduction

Using the representation of $PG(3, q)$ over $AG(2, q)$ explained in [3], we construct a maximal partial line spread of $PG(3, q)$, $q = 2^{2n}$, n an integer, of size $q^2 = q + 2$. A spread of this cardinality has been constructed by J.W. Freeman [1]. This cardinality is the greatest known till now, except a sporadic case for $q = 7$, found by O. Heden [2].

For the notations and theorems about the representation of $PG(3, 2^{2n})$ over $AG(2, 2^n)$, we refer to the paper [3] cited in the bibliography, which the reader must know before reading this text.

Let $GF(q)$ be the Galois field of order q, with $q = 2^{2n}$, n an integer, $n \geq 1$. An element $x \in GF(q)$ is called *cube*, if there is $y \in GF(q)$ such that $x = y^3$. Let $\mathcal{C}$ be the set of cubes of $GF(q)$. The multiplicative group $\mathcal{G}$ of $GF(q)$ is cyclic and then it admits a generator g. It follows that $\mathcal{G} = \{g, g^2, ..., g^{q-1} = 1\}$ and that $|\mathcal{G}| \geq 3$.

Theorem 1. *If g is a generator of $GF(2^{2n})$, then $g \notin \mathcal{C}$.*

Proof. Assume $g \in \mathcal{C}$. There is then $b \in GF(2^{2n})$, such that $g = b^3$. Moreover, $b = g^m$, m and integer and $1 \leq m \leq q - 1$. Therefore, $g = g^{3m}$, whence $3m \equiv 1 \bmod (q - 1)$. By this and by $1 \leq m \leq q - 1$ (which implies $3 \leq 3m \leq 3q - 3$), it follows

(i) $3m = q$,

(ii) $3m = 2q - 1$,

(iii) $3m = 3q - 2$.

The condition (i) is not true, since q is not a multiple of 3, (iii) is also not true, since 2 is not a multiple of 3. Therefore, m must satisfy (ii). We get:

$$q = 2^{2n} = (3-1)^{2n} = (3+(-1))^{2n} = \sum_{j=0}^{2n} \binom{2n}{j} 3^j (-1)^{2n-j}.$$

It follows that q is of the form:

$$q = 3^M + 1,$$

with M an integer. By this and by (ii), it follows:

$$3m = 2(3M+1) - 1 = 6M + 1,$$

a contradiction, since 1 is not a multiple of 3. This contradiction proves that g is not a cube. ∎

From this theorem, we get that $GF(2^{2n}) - \mathcal{C} \neq \emptyset$. Since 1 is a cube, it follows that $\mathcal{C} - \{0\} \neq \emptyset$. Therefore, *in* $GF(2^{2n})$ *there are cubes and not cubes.*

Now, let $m \in \mathcal{C} - \{0\}$, $\overline{m} \in GF(2^{2n}) - \mathcal{C}$. Let $PG(2, 2^{2n})$ be the projective space of dimension 3 over $GF(2^{2n})$ and let $AG(2, 2^{2n})$ be the affine plane over $GF(2^{2n})$. Fix a coordinate system (X, Y) in $AG(2, 2^{2n})$. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the parabolas of $AG(2, 2^{2n})$ with the equations:

$$\mathcal{P}_1 : y = mx^2,$$
$$\mathcal{P}_2 : \overline{m}x^2.$$

Let $P_1(X_1, Y_1)$ and $P_2(X_2, Y_2)$ be two points of $AG(2, 2^{2n})$, with $P_1 \in \mathcal{P}_1$, $P_2 \in \mathcal{P}_2$, $P_1 \neq P_2$. The line through P_1 parallel to the x axis and the line through P_2 parallel to the y axis meet at the point $A(X_2, Y_1)$. The line through P_1 parallel to the y axis and the line through P_2 parallel to the x axis meet at the point $B(X_1, Y_2)$. Obviously, $A \neq B$. We call the ordered pair (A, B) the *pair associated* with the pair (P_1, P_2). Let (A', B') be the pair associated with (P_1', P_2'), with $(A', B') \neq (A, B)$. We remark that $A \neq A'$, $B \neq B'$. For, if $A = A'$, then $P_1 = P_1'$, $P_2 = P_2'$ and then $B = B'$, whence $(A, B) = (A', B')$, a contradiction. This contradiction proves that $A \neq A'$. Similarly, we prove that $B \neq B'$.

Theorem 2. *The lines AA' and BB' are not parallel.*

Proof. Let us distinguish the following three cases:

(a) AA' is parallel to the y axis,

(b) BB' is parallel to the y axis,

(c) neither AA', or BB' are parallel to the y axis.

Let us prove (a).

If the line AA' is parallel to the y axis, the lines AP_1 and $A'P_1'$ coincide and then necessarily $P_1 = P_1'$. It follows that the line BB' is parallel to the x axis and then AA' and BB' are not parallel.

Let us prove (b). If the line BB' is parallel to the y axis, then the lines BP_2 and $B'P_2'$ coincide and then necessarily $P_2 = P_2'$. It follows that the line AA' is parallel to the x axis and then AA' and BB' are not parallel.

Let us prove (c). Now, let AA' and BB' be not parallel to the y axis. Let $m(A,A')$ be the slope of the line AA' and $m(B,B')$ the slope of the line BB'.

We get:

$$\begin{aligned} m(A,A') &= \frac{Y_2 - Y_2'}{X_1 - X_1'}, \\ m(B,B') &= \frac{Y_1 - Y_1'}{X_2 - X_2'}, \end{aligned}$$

with $X_1 \neq X_1'$, $X_2 \neq X_2'$.

Then

$$\begin{aligned} &AA' \text{ parallel to } BB' \iff m(A,A') = m(B,B') \\ &\iff (Y_2 - Y_2')(X_2 - X_2') = (Y_1 - Y_1')(X_1 - X_1') \\ &\iff Y_1X_1 - Y_1X_1' - Y_1'X_1 + Y_1'X_1' = Y_2X_2 - Y_2X_2' - Y_2'X_2 + Y_2'X_2'. \end{aligned}$$

Since the characteristic of $GF(2^{2n})$ is two, since $Y_1 = mX_1^2$, $Y_1' = mX_1'^2$ and $Y_2 = \overline{m}X_2^2$, $Y_2' = \overline{m}X_2'^2$, we get:

$$\overline{m} = \frac{m(X_1 + X_1')^3}{(X_2 - X_2')^3}\,.$$

Therefore AA' and BB' are parallel if and only if

$$\overline{m} = \frac{m(X_1 + X_1')^3}{(X_2 - X_2')^3}\,.$$

Then AA' and BB' are not parallel, otherwise $\overline{m}$ is a cube ($m \in \mathcal{C}$), but $\overline{m} \in GF(2^{2n}) - \mathcal{C}$. Therefore the theorem is completely proved. ∎

Remark that the line AB is distinct from the y axis. For, if this line coincides with the y axis, then P_1 and P_2 belonged both to the y axis, a contradiction, otherwise they should coincide with the origin O. The contradiction proves the remark.

Remark also that $A \neq O$, $B \neq O$. For, if $A = O$, then $P_2 = O$, $P_1 = O$, a contradiction, since $P_1 \neq P_2$. Then $A \neq O$ and similarly $B \neq O$.

Theorem 3. *The line AB does not pass through the origin O.*

Proof. If $O \in AB$, since the line AB is distinct from the y axis, it has the equation $y = \alpha x$, $\alpha \in GF(2^{2n})$. Moreover $A \neq O$, $B \neq O$, and then $X_1 \neq O$, $X_2 \neq O$. Then we get:

$$\alpha = \frac{Y_2}{X_1} = \frac{Y_1}{X_2},$$

that is $X_2Y_D = X_1Y_1$.

From this and by $Y = \overline{m}X_2^2$, $Y_1 = mX_1^2$, we get

$$\overline{m}X_2^3 = mX_1^3,$$

whence $\overline{m} \in \mathcal{C}$, a contradiction, since $\overline{m} \notin \mathcal{C}$. The contradiction proves that the line AB does not pass through O, that is Theorem 3. ∎

2. Construction of a maximal partial line spread of $PG(3, 2^{2n})$, n integer, $n \geq 1$

Denote by r_0 the line of $PG(3, 2^{2n})$ belonging to the class b) of [3] represented in $AG_2(2, 2^{2n})$ (see Sections 2 and 3 of [3]) by the proper line pencil with centre O. Let

$$\begin{aligned} \mathcal{S} &= \{(P_1, P_2) : P_1 \in \mathcal{P}_1,\ P_2 \in \mathcal{P}_2,\ P_1 \neq P_2\}, \\ S &= \{(A, B) : (A, B) \text{ is the pair associated with the pair } (P_1, P_2), \\ &\qquad \text{with } (P_1, P_2) \in \mathcal{S}\}. \end{aligned}$$

Denote by $\ell(U_1, U_2)$ the line of $PG(3, 2^{2n})$ belonging to the class a) of [3], represented by the ordered pair of distinct points (U_1, U_2) of $AG(2, 2^{2n})$ and let

$$\mathcal{F} = \{v, r_0\} \cup \{\ell(A, B)\}_{(A,B) \in S}.$$

Let us prove the

Theorem 4. *The set of lines $\mathcal{F}$ of $PG(3, 2^{2n})$ is a total spread.*

Proof. We get:

(α) $v \cap r_0 = \emptyset$, since r_0, which is a line of the plane π (see [3]), is represented by a proper pencil of lines of $AG(2, 2^{2n})$ and then does not contain $Y = v \cap \pi$.

(β) $v \cap \ell(A, B) = \emptyset$, $\forall (A, B) \in S$, since the ordered pairs of distinct points of $AG(2, 2^{2n})$ represent the lines of the class a) of [3] of $PG(3, 2^{2n})$ not meeting v and not in π.

(γ) $r_0 \cap \ell(A, B) = \phi$, $\forall (A, B) \in S$, since in Theorem 3 we have proved that the line AB, with $(A, B) \in S$ does not pass through the origin O.

(δ) Two distinct lines $\ell(A, B)$ and $\ell'(A', B')$ with $(A, B) \in S$, $(A', B') \in S$ are not incident, since we proved in Theorem 2 that the lines AA' and BB' are not parallel.

Since the pairs of S, associated with distinct pairs of $\mathcal{S}$ are distinct, it follows

$$|S| = |\mathcal{S}| = q^2 - 1,$$

because the number of pairs of points $(P_1, P_2) \in \mathcal{S}$ except the pair $(0,0)$ is $q^2 - 1$.

By that and since the lines of $PG(2, 2^{2n})$ represented by distinct pairs of S are distinct, it follows that

$$|\{\ell(A,B)\}_{(A,B)\in S}| = q^2 - 1.$$

By the previous arguments and by the definition of $\mathcal{F}$, it follows that

$$|\mathcal{F}| = q^2 + 1.$$

Then $\mathcal{F}$ is a total spread, since it is a covering of $PG(3, 2^{2n})$. ∎

Now, let us call *regulus* of $PG(3, 2^{2n})$ a regulus of a hyperbolic quadric of $PG(3, 2^{2n})$. A total spread $\mathcal{F}'$ of $PG(3, 2^{2n})$ is called *regular*, if for any three distinct lines of $\mathcal{F}'$ the regulus containing such lines consists of lines of $\mathcal{F}'$.

Now, let us prove the following

Theorem 5. *Let t_1 and t_2 be two distinct and not parallel lines of $AG(2, 2^{2n})$ and let O be their common point. Let A be a point of $t_1 - \{O\}$ and B a point of $t_2 - \{O\}$. Let r_0 be the line of the plane π (see* [3], Theorem 4 of Section 2, for $r = 3$) *represented in $AG(2, 2^{2n})$ by the pencil with centre O. Let ℓ be the line of $PG(3, 2^{2n})$ represented by the ordered pair of distinct points (A, B) (see* [3], Theorem 3, for $r = 3$)*, the lines v (see* [3]*), r_0 and ℓ being mutually skew. Denote by $\mathcal{I}$ the hyperbolic quadric of $PG(3, 2^{2n})$ determined by v, r_0, ℓ and let $\mathcal{R}$ be the regulus containing such three lines. We prove that the remaining lines of $\mathcal{R}$ are represented in $AG(2, 2^{2n})$ by the ordered pairs of distinct points (A', B'), with $A' \neq O$, $A' \neq A$, $B' \neq O$, $B' \neq B$ and $A'B'$ parallel to AB.*

Proof. (see Figure 1). The line u_1 of $PG(3, 2^{2n})$ represented in $AG(2, 2^{2n})$ in the following way (see [3], Theorem 3, Section 2, for $r = 3$):

$$u_1 : \{(t_1, t), \text{ with } t \text{ a line of } AG(2, 2^{2n}) \text{ parallel to } t_1 \ (t \neq t_1)\}$$

contains U_1, meets r_0 at the point T_1, represented by the line t_1 and meets ℓ at the point of $PG(3, 2^{2n})$, represented in $AG(2, 2^{2n})$ by the ordered pair of distinct lines (t_1, t_1'), where t_1' is the line parallel to t_1 through B. The line u_2 of $PG(3, 2^{2n})$ represented in $AG(2, 2^{2n})$ as follows:

$$u_2 : \{(t, t_2), \text{ with } t \text{ a line of } AG(2, 2^{2n}) \text{ parallel to } t_2 \ (t_1 \neq t_2)\}$$

contains U_2, meets r_0 at the point T_2, represented in $AG(2, 2^{2n})$ by the line t_2 and meets ℓ at the point represented in $PG(2, 2^{2n})$ by the ordered pair (t_2', t_2), where t_2' is the line through A parallel to t_2. The line s of the plane π, represented in $AG(2, 2^{2n})$ by the improper pencil of lines parallel to AB, meets v at Y, r_0 at T, distinct from T_1 and T_2, represented in $AG(2, 2^{2n})$ by the line through O parallel to AB and meets ℓ at the point L, belonging to the plane π, represented by the line AB. Therefore, the lines u_1, u_2 and s belong to the regulus $\mathcal{R}'$ of $\mathcal{I}$, opposite to $\mathcal{R}$. Now, let $A' \in t' - \{O, A\}$ and $B' \in t_2 - \{O, B\}$, such that A', B' is parallel to AB. The line ℓ' of $PG(3, 2^{2n})$, represented by the pair (A', B') meets u_1 at the

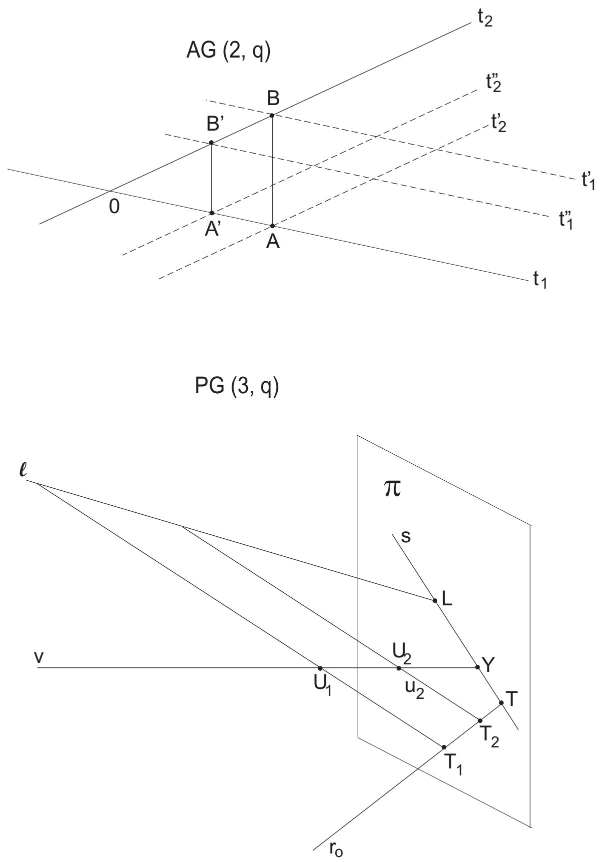

Figure 1

point represented by the ordered pair (t_1, t_1''), where t_1'' is the line through B' parallel to t_1. The line ℓ' meets u_2 at the point represented by the ordered pair (t_2'', t_2), where t_2'' is the line through A' parallel to t_2'. The line ℓ' meets s at the point represented by the line $A'B'$. Therefore the line ℓ' ($\ell' \neq \ell, v, r_0$) meets u_1, u_2, s. It follows that $\ell' \in \mathcal{R}$. By varying of A' in $t_1 - \{O\}$, we obtain $q-2$ pairs (A', B'), representing the lines of the regulus $\mathcal{R}$, distinct from v, r_0, ℓ. Therefore, we get in the whole $q+1$ lines of $\mathcal{R}$, that is the whole regulus $\mathcal{R}$ which is so represented by the lines v, r_0, ℓ and by the pairs (A', B'), with $A' \in t_1 - \{O, A\}$. ■

Now, let us prove the following

Theorem 6. *The spread $\mathcal{F}$ is not regular.*

Proof. Let $\overline{P}_2$ be a point of $\mathcal{P}_2 - \{O\}$ and let $(\overline{A}, \overline{B})$ the pair of S associated with the pair $(O, \overline{P}_2)$ of $\mathcal{S}$. Obviously, $\overline{A} \in y$ axis $-\{O\}$, $\overline{B} \in x$ axis $- \{O\}$. Let

$\mathcal{I}$ be the hyperbolic quadric of $PG(3,2^{2n})$ containing the lines v, r_0 and $\ell(\overline{A},\overline{B})$ of $\mathcal{F}$. Let $\mathcal{R}$ be the regulus of $\mathcal{I}$ containing such lines. Let ℓ' be a line of $\mathcal{R}$ distinct from v, r_0 and $\ell(\overline{A},\overline{B})$. Since the line ℓ' does not meet v (since it belongs to the same regulus of v) and does not belong to the plane π, since it does not meet r_0, it belongs to the class a) of [3] and therefore is represented by an ordered pair (A', B') of distinct points of $AG(2,2^{2n})$.

By Theorem 5, we get:

$$\begin{aligned} &A' \in y \text{ axis} - \{O, \overline{A}\},\\ &B' \in x \text{ axis} - \{O, \overline{B}\},\\ &A'B' \text{ is parallel to } AB. \end{aligned}$$

We remark that $\ell' \notin \mathcal{F}'$. For, ℓ is obviously distinct from v and r_0. Moreover, it is easy to prove that $\ell' \neq \ell(AB)$, for any pair (A,B) associated with a pair (P_1,P_2) of $\mathcal{S}$, with $P_1 \neq O$. It is now to prove that ℓ' is distinct from each of the lines $\ell(AB)$, with (A,B) associated with (O,P_2), $P_2 \in \mathcal{P}_2 - \{O\}$. To do this, let T be the point common to the line through A' parallel to the x axis and to the line through B' parallel to the y axis. The distinct points O, T_2, T are collinear over a line b, as it is easy to prove. If the pair (A',B') is associated with a pair (O,P_2), $P_2 \in \mathcal{P}_2 - \{O\}$, necessarily $T_2 = P_2$ and therefore $T \in \mathcal{P}_2$, a contradiction, since the line b cannot meet $\mathcal{P}_2$ at three distinct points. The contradiction proves that (A',B') is not associated with any pair (O,P_2), $P_2 \in \mathcal{P}_2 - \{O\}$ and then $\ell' \in \mathcal{F}'$. The previous remark is therefore proved. It follows that $\mathcal{R}$ is not entirely consisting of lines of $\mathcal{I}$ and hence it is not regular. ∎

By the above arguments, we get that in $PG(3,2^{2n})$ there is a total non-regular line spread. As such a spread gives rise to an affine non-desarguesian translation plane of order 2^{4n}, we get the following

Theorem 7. *For any $q = 2^{2n}$, n an integer, $n \geq 1$, there exists a non-desarguesian affine plane of order 2^{4n}.*

Let $\mathcal{T}$ be the following set of lines of $PG(3,2^{2n})$:

$$\mathcal{T} = \{\ell(A,B) : (A,B) \text{ is associated with } \{(O,P_2),\ P_2 \in \mathcal{P}_2 - \{O\} \cup \{v, r_0\}.$$

The set $\mathcal{T}$ is a subset of $\mathcal{F}$ and has size $q+1$, but $\mathcal{T}$ is not a regulus of $PG(3,2^{2n})$, since $\mathcal{T}$ contains v, r_0 and $\ell(\overline{A},\overline{B})$ and does not contain $\ell'(A',B')$ (see Theorem 6) which is a line of the regulus $\mathcal{R}$ containing v, r_0 and $\ell(\overline{A},\overline{B})$. Let T_1 and T_2 be the points of $\pi - \{Y\}$ represented by the x axis and the y axis, respectively. The line U_2T_1 of $PG(3,2^{2n})$ meets v at U_2, r_0 at T_1 and $\ell(A,B) \in \mathcal{T} - \{v, r_0\}$ at the point represented by the ordered pair $(t_A, x \text{ axis})$, where t_A is the line of $AG(2,2^{2n})$ through A and parallel to the x axis. It follows that the line U_2T_1 meets all the lines of $\mathcal{T}$. The line U_1T_2 of $PG(3,2^{2n})$ meets v at U_1, r_0 at T_2 and $\ell(A,B) \in \mathcal{T} - \{v, r_0\}$ at the point represented by the ordered pair $(y \text{ axis}, t_B)$, where t_B is the line of $AG(2,2^{2n})$ through B parallel to the y axis. It follows that

U_1T_2 meets all the lines of $\mathcal{T}$. The line U_1T_2 of $PG(3,2^{2n})$ meets v at U_1, r_0 at T_2 and $\ell(A,B) \in \mathcal{T} - \{v, r_0\}$ at the point represented by the ordered pair (y axis, t_B), where t_B is the line of $AG(2,2^{2n})$ through B parallel to the y axis. It follows that U_1T_2 meets all the lines of $\mathcal{T}$. The lines U_1T_2 and U_2T_1 are mutually skew (as it is easy to prove by using the representation [3] of $U_1T_2 - \{U_1\}$ and of $U_2T_1 - \{U_2\}$ in $AG(2,2^{2n})$, or equivalently considering that the lines of $\mathcal{T}$ are mutually skew). Now let

$$\widetilde{\mathcal{F}} = (\mathcal{F} - \mathcal{T}) \cup \{U_1T_2, U_2T_1\}.$$

Obviously, $\widetilde{\mathcal{F}}$ is a line spread of $PG(3,2^{2n})$. Moreover, $\widetilde{\mathcal{F}}$ is also maximal. For, let ℓ be a line of $PG(3,2^{2n})$ not meeting any line of $\mathcal{F}$.

Then the points of ℓ range over the $q+1$ lines of $\mathcal{T}$ and it is $\ell \cap U_1T_2 = \emptyset$, $\ell \cap U_2T_1 = \emptyset$. Then the hyperbolic quadric of $PG(3,2^{2n})$ containing the three lines U_1T_2, U_2T_1 and ℓ admits $\mathcal{T}$ as one of its reguli. A contradiction, since $\mathcal{T}$ is not a regulus of $PG(3,2^{2n})$. The contradiction proves that every line of $PG(3,2^{2n})$ meets some line of $\mathcal{F}$ and then $\mathcal{F}$ is maximal. Moreover

$$|\mathcal{F}| = q^2 - q + 2.$$

Therefore, the following theorem holds:

Theorem 8. *In $PG(3,2^{2n})$, n integer $n \geq 1$, there is a maximal non-total line spread of size $q^2 - q + 2$.*

This result was obtained by Freeman [1] in 1980, who constructed an example which was the only before this research. Here, we construct a maximal non-total line spread for q even of $PG(3,2^{2n})$, using only the geometry of the affine plane $AG(2,2^{2n})$. The cardinality q^2-q+2 is the maximum known till now, except the sporadic case, for $q=7$, found by Heden [2].

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H_v-STRUCTURES AND THE BAR IN QUESTIONNAIRES

Pipina Nikolaidou
Thomas Vougiouklis

Democritus University of Thrace
School of Education
681 00 Alexandroupolis
Greece
e-mail: *pnikolai@eled.duth.gr*
tvougiou@eled.duth.gr

Abstract. The class of hyperstructures called H_v-structures has been studied from several aspects as well as in connection with many other topics of mathematics. Here we present applications obtained from social sciences mainly the ones used questionnaires. Moreover we improve the procedure of the filling the questionnaires, using the bar instead of Likert scale, on computers where we write down automatically the results so they are ready for research.

Key Words and Phrases: hyperstructures, H_v-structures, hopes.

AMS Subject Classification: 20N20, 16Y99.

1. Basic definitions

We deal with the theory of hyperstructures introduced by Marty in 1934 [12]. For basic definitions and applications on the related theory one can see the books [3],[4],[7],[16] and related survey papers as the [6]. More specifically we focus on the large class of hyperstructures called H_v*-structures* introduced in 1990 [15], which satisfy the *weak axioms* where the non-empty intersection replaces the equality. Basic definitions on the topic are the following:

In a set H equipped with a hyperoperation (abbreviation *hyperoperation* = **hope**) $\cdot : H \times H \to P(H) - \{\varnothing\}$, we abbreviate by

WASS the *weak associativity*: $(xy)z \cap x(yz) \neq \varnothing,\ \forall x, y, z \in H$ and by

COW the *weak commutativity*: $xy \cap yx \neq \varnothing,\ \forall x, y \in H$

The hyperstructure $(H, \cdot)$ is called H_v*-semigroup* if it is *WASS*, it is called H_v*-group* if it is reproductive H_v-semigroup, i.e., $xH = Hx = H$, $\forall x \in H$. The hyperstructure $(R, +, \cdot)$ is called H_v*-ring* if both $(+)$ and $(\cdot)$ are *WASS*, the reproduction axiom is valid of $(+)$ and $(\cdot)$ is weak distributive with respect to $(+)$:

$$x(y + z) \cap (xy + xz) \neq \varnothing,\ (x + y)z \cap (xz + yz) \neq \varnothing,\ \forall x, y, z \in R.$$

Motivations. The motivation for H_v-structures is the following [16]: We know that the quotient of a group with respect to an invariant subgroup is a group. F. Marty from 1934, states that, the quotient of a group with respect to any subgroup is a hypergroup. Finally, the quotient of a group with respect to any partition (or equivalently to any equivalence relation) is an H_v-group.

Specifying this motivation we remark: Let $(G,\cdot)$ be a group and R be an equivalence relation (or a partition) in G, then $(G/R,\cdot)$ is an H_v-group, therefore we have the quotient $(G/R,\cdot)/\beta^*$ which is a group, the *fundamental* one. Remark that the classes of the fundamental group $(G/R,\cdot)/\beta^*$ are a union of some of the R-classes. Otherwise, the $(G/R,\cdot)/\beta^*$ has elements classes of G where they form a partition which classes are larger than the classes of the original partition R.

In an H_v-semigroup the powers of an element $h \in H$ are defined as follows: $h^1 = \{h\}, h^2 = h \cdot h, ..., h^n = h \circ h \circ ... \circ h$, where $(\circ)$ denotes the *n-ary circle hope*, i.e. take the union of hyperproducts, n times, with all possible patterns of parentheses put on them. An H_v-semigroup $(H,\cdot)$ is called *cyclic of period s*, if there exists an element g, called *generator*, and a natural number s, the minimum one, such that $H = h^1 \cup h^2 ... \cup h^s$. Analogously the cyclicity for the infinite period is defined [16]. If there is an element h and a natural number s, the minimum one, such that $H = h^s$, then $(H,\cdot)$ is called *single-power cyclic of period s.*

The main tool to study hyperstructures are the fundamental relations β^*, γ^* and ϵ^*, which are defined, in H_v-groups, H_v-rings and H_v-vector spaces, resp., as the smallest equivalences so that the quotient would be group, ring and vector space, resp. The relation β^* was introduced by M. Koskas in 1970 [11] and was mainly studied intensively and in depth by Corsini [3]. The relations γ^* and ϵ^*, were introduced by T. Vougiouklis [15],[16],[17] and he named them Fundamental. A way to find the fundamental classes is given by theorems as the following [16]:

Theorem 1.1 *Let $(H,\cdot)$ be an H_v-group and denote by $\boldsymbol{U}$ the set of all finite products of elements of H. We define the relation β in H by setting $x\beta y$ iff $\{x,y\} \subset \boldsymbol{u}$ where $u \in U$. Then β^* is the transitive closure of β.*

An element is called single if its fundamental class is singleton [16].

Fundamental relations are used for general definitions. Thus, an H_v-ring $(R,+,\cdot)$ is called H_v-field if R/γ^* is a field.

Let $(H,\cdot),(H,*)$ be H_v-semigroups defined on the same set H. $(\cdot)$ is called smaller than $(*)$, and $(*)$ greater than $(\cdot)$, iff there exists an

$$f \in Aut(H,*) \text{ such that } xy \subset f(x*y),\ \forall x,y \in H.$$

Then we write $\cdot \le *$ and we say that $(H,*)$ contains $(H,\cdot)$. If $(H,\cdot)$ is a structure then it is called *basic structure* and $(H,*)$ is called $H_b - structure$.

Theorem 1.2 (The Little Theorem). *Greater hopes than the ones which are WASS or COW, are also WASS or COW, respectively.*

This Theorem leads to a partial order on H_v-structures and mainly to a correspondence between hyperstructures and posets. The determination of all H_v-groups and H_v-rings is very interesting but hard.

To compare classes we can see the small sets. The problem of enumeration and classification of H_v-structures, or of classes of them, was started very early but recently we have interesting results by using computers. The problem is complicate in H_v-structures because we have great numbers. The partial order introduced in H_v-structures restrict the problem in finding the minimal, up to *isomorphisms*, H_v-structures. In this direction we have recently results by Bayon and Lygeros as the following [1]:

In a set with two elements, then there are 20 H_v-groups, up to isomorphism.

In sets with three elements: There are 6.494 minimal isomorphisms-groups. The 137 are abelians and 6.357 are not; the 6.152 are cyclic and 342 are not. The number of H_v-groups with three elements is 1.026.462. The 7.926 are abelians, 1.018.536 are not; 1.013.598 are cyclic and 12.864 are not. 16 are very thin.

The number of H_v-groups with 4 elements with scalar unit is 631.609. There are 8.028.299.905 abelian H_v-groups, the 7.995.884.377 are cyclic and the 32.415.528 are not. There are 10.614.362 abelian hypergroups: the 10.607.666 are cyclic and the 6.696 are not. Notice that there are only 97 canonical hypergroups.

Definition 1.3 [18],[19]. Let $(H, \cdot)$ be hypergroupoid. We *remove* $h \in H$, if we consider the restriction of $(\cdot)$ in the set $H - \{h\}$. $\underline{h} \in H$ *absorbs* $h \in H$ if we replace h by $\underline{h}$ and h does not appear in the structure. $\underline{h} \in H$ merges with $h \in H$, if we take as product of any $x \in H$ by $\underline{h}$, the union of the results of x with both h, $\underline{h}$, and consider h and $\underline{h}$ as one class with representative $\underline{h}$, therefore, h does not appear in the hyperstructure.

In 1989 Corsini and Vougiouklis introduced a method to obtain stricter algebraic structures from given ones through hyperstructure theory. This method was introduced before of the H_v-structures, but in fact the H_v-structures appeared in the procedure.

Definition 1.4 The *uniting elements method* is the following: Let G be a structure and d be a property, which is not valid, and it is described by a set of equations. Consider the partition in G for which it is put together, in the same class, every pair of elements that causes the non-validity of d. The quotient G/d is an H_v-structure. Then quotient of G/d by the fundamental relation β^*, is a stricter structure $(G/d)\beta^*$ for which d is valid.

An application of the uniting elements is if more than one property desired. The reason for this is some of the properties lead straighter to the classes: commutativity and the reproductivity are easily applicable. One can do this because there is a related theorem [16].

The Lie-Santilli *isotopies* born to solve Hadronic Mechanics problems. Santilli proposed [13] a 'lifting' of the trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The *isofields* needed correspond to H_v-structures called *e-hyperfields* which are used in physics or biology. *Definition*: Let $(H_o,+,\cdot)$ be the attached H_v-field of the H_v-semigroup $(H,\cdot)$. If $(H,\cdot)$ has a left and right scalar unit e, then $(H_o,+,\cdot)$ is an e-hyperfield, the *attached* H_v-field of $(H,\cdot)$.

Most of H_v-structures are used in Representation (abbreviate by rep) Theory. Reps of H_v-groups can be considered either by generalized permutations or by H_v-matrices [14],[16],[19]. Reps by generalized permutations can be achieved by using translations. In the rep theory the singles are playing a crucial role.

The rep problem by H_v-matrices is the following:

H_v-*matrix* is called a matrix if has entries from an H_v-ring. The hyperproduct of H_v-matrices $A=(a_{ij})$ and $B=(b_{ij})$, of type $m\times n$ and $n\times r$, respectively, is a set of $m\times r$ H_v-matrices, defined in a usual manner:

$$A\cdot B=(a_{ij})\cdot(b_{ij})=\{C=(c_{ij})|c_{ij}\in\oplus\sum a_{ik}\cdot b_{kj}\},$$

where $(\oplus)$ denotes the n-ary circle hope on the hyperaddition.

Definition 1.5 Let $(H,\cdot)$ be an H_v-group, $(R,+,\cdot)$ an H_v-ring, $M_R=\{(a_{ij})|a_{ij}\in R\}$, then any map

$$\mathbf{T}:H\to\mathbf{M}_R:h\to T(h)\ with\ T(h_1h_2)\cap T(h_1)T(h_2)\neq\varnothing,\forall h_1,h_2\in H,$$

is called H_v-*matrix rep*. If $T(h_1h_2)\subset T(h_1)T(h_2)$, then $\boldsymbol{T}$ is an *inclusion rep*, if $T(h_1h_2)=T(h_1)T(h_2)$, then $\boldsymbol{T}$ is a *good rep*.

Hopes on any type of matrices can be defined, these are called helix hopes [8], [25].

2. The ∂-hopes

In [20],[21],[22] we defined a hope, in a groupoid with a map f on it called *theta* ∂.

Definition 2.1 Let $(G,\cdot)$ be groupoid (resp., hypergroupoid) and $f:G\to G$ be a map. We define a hope (∂), on G as follows

$x\partial y=\{f(x)\cdot y,x\cdot f(y)\}$, $\forall x,y\in G$. (resp. $x\partial y=(f(x)\cdot y)\cup(x\cdot f(y))$, $\forall x,y\in G$

If $(\cdot)$ is commutative then (∂) is commutative. If $(\cdot)$ is *COW* then (∂) is *COW*.

Let $(G,\cdot)$ be groupoid (resp., hypergroupoid) and $f:G\to P(G)-\{\varnothing\}$ be any multivaued map. We define the (∂), on G as follows $x\partial y=(f(x)\cdot y)\cup(x\cdot f(y))$, $\forall x,y\in G$.

Let $(G,\cdot)$ be groupoid $f_i:G\to G, i\in I$, be a set of maps on G. The $f_\cup:G\to P(G):f_\cup(x)=\{f_i(x)|i\in I\}$, is the *union* of $f_i(x)$. We have the *union theta-hope* (∂), on G if we take $f_\cup(x)$. If we take $\underline{f}\equiv f\cup(id)$, then we have the *b-theta-hope*.

Motivation for the definition of the theta-hope is the map *derivative* where only the multiplication of functions can be used.

Properties 2.2 *If $(G, \cdot)$ is a semigroup, then:*

1. *For every f, the (∂) is WASS*
2. *If f is homomorphism and projection , i.e $f^2 = f$, then (∂) is associative.*
3. *If $(G, \cdot)$ is a semigroup then, for every f, the b-theta-hope (∂) is WASS.*
4. *Reproductivity. If $(\cdot)$ is reproductive then (∂) is also reproductive.*
5. *Commutativity. If $(\cdot)$ is commutative then (∂) is commutative. If f is into the centre of G, then (∂) is a commutative. If $(\cdot)$ is a COW then, (∂) is a COW.*
6. *Unit elements. u is right unit if $x\partial u = \{f(x)\cdot u, x\cdot f(u)\} \ni x$. So$f(u) = e$, if e is a unit in $(G, \cdot)$.The elements of the kernel of f, are the units of (G, ∂). In hypergroups does not necessarily exist any unit element and if there exists a unit this is not necessarily unique. Moreover the ∂-hopes do not have always the unit element of the group as unit for the corresponding ∂-hope. This is so because $e\partial e = \{f(e)e, ef(e)\} = \{f(e)\}$*
7. *Inverse elements. Let $(G, \cdot)$ be a monoid with unit e and u be a unit in (G, ∂), then $f(u) = e$.. For given x, the x' is an inverse with respect to u, if $x\partial x' = \{f(x) \cdot x', x \cdot f(x')\} \ni u$ and $x'\partial x = \{f(x') \cdot x, x' \cdot f(x)\} \ni u$. So, $x' = (f(x))^{-1}u$ and $x' = u(f(x))^{-1}$ are the right and left inverses, respectively.We have two-sided inverses iff $f(x)u = uf(x)$*

Proposition 2.3 *Let $(G, \cdot)$ be, then for all maps $f : G \to G$, the (G, ∂) is an H_v-group.*

Motivation. For the definition of the theta-hope is the map *derivative* where only the multiplication of functions can be used. Therefore, in these terms, for two functions $s(x)$, $t(x)$, we have $s\partial t = \{s't, st'\}$, where $(')$ denotes the derivative.

Proposition 2.4 *Let $g \in G$ is a generator of the group $(G, \cdot)$. Then,*

(a) *for every f, g is a generator in $(G\partial)$, with period at most n.*

(b) *suppose that there exists an element w such that $f(w) = g$, then the element w is a generator in (G, ∂), with period at most n.*

There is connection of ∂-hopes with other hyperstructures:

Example. P-hopes [16]. Let $(G, \cdot)$ be a commutative semigroup and $P \subset G$. Consider the multivalued map f such that $f(x) = P \cdot x$, $\forall x, y \in G$.

Then we have $x\partial y = x \cdot y \cdot P$, $\forall x, y \in G$.

So, the ∂-hope coincides with the well known class of P-hopes [22].

One can define ∂-hopes on rings and other more complicate structures, where more than one ∂-hopes can be defined. Moreover, one can replace structures by hyper ones or by H_v-structures, as well.

3. The bar in questionnaires

During last decades hyperstructures seem to have a variety of applications not only in other branches of mathematics but also in many other sciences including the social ones. These applications range from biomathematics and hadronic physics to automata theory, to mention but a few. This theory is closely related to fuzzy theory; consequently, hyperstructures can now be widely applicable in industry and production, too.

In several papers, such as [2], [5], [13], [24], one can find numerous applications; similarly, in the books [4], [7] a wide variety of applications is also presented.

An important new application, which combines hyperstructure theory and fuzzy theory, is to replace in questionnaires the scale of Likert by the bar of Vougiouklis & Vougiouklis. The suggestion is the following [10]:

Definition 3.1 "In every question substitute the Likert scale with 'the bar' whose poles are defined with '0' on the left end, and '1' on the right end:

0 ________________ 1

The subjects/participants are asked instead of deciding and checking a specific grade on the scale, to cut the bar at any point s/he feels expresses her/his answer to the specific question".

The use of the bar of Vougiouklis & Vougiouklis instead of a scale of Likert has several advantages during both the filling-in and the research processing. The final suggested length of the bar, according to the Golden Ratio, is 6.2cm, see [24]. Several advantages on the use of the bar instead of scale one can find in [10].

4. A computerizing filling questionnaires

We present now a program of filling a questionnaire on a computer such that the results automatically can be transferred for research elaboration.

There are several advantages of the bar one of them is the time of filling the questionnaire. The only disadvantage of the bar is to transfer the data collection to a computer for elaboration. At this point, we present an implemented application to overcome the problems raised during the transferring the data. This application overcomes the problem of inputting data from questionnaires to processing and eliminates time of data collection, transferring data directly for any kind of elaboration.

The application has been implemented using Visual Basic and the data is being saved on a Microsoft Access Database. The application is based on "events" and an OleDbConnection is used to connect the program with the database.

Filling-in such questionnaire can be easily achieved by using this application, as it is based on a very simple user interface. The participants have to "click" on the bar, in order to indicate the point that satisfies their answer on the question made. The user has the opportunity to change his answer by "clicking" on another point anytime before submit.

The results are being saved on a simple database (Microsoft Access Database) indicating the exact point each participant has "cut" the bar

5. Applications

One problem in research is to describe mathematical models using theta-hopes. Such a problem is the following [23]:

Problem 5.1 *In the research processing suppose that we want to use Likert scale dividing the continuum* [01] *both by, first, into equal steps (segments) and, second, into equal-area spaces according to Gauss distribution* [9], [24]. *If we consider both types of divisions into n segments, then the continuum* [01] *is divided into* $2n-1$ *segments, if n is odd number and into* $2(n-1)$ *segments, if n is even number. We can number the segments and we can consider as an organized devise the group* $(Zk, \oplus)$ *where* $k = 2n-1$ *or* $2(n-1)$. *Then we can obtain several hyperstructures using* ∂*-hopes as the following way: We can have two partitions of the final segments, into n classes either using the division into equal steps or the Gauss distribution by putting in the same class all segments that belong* (a) *to the equal step or* (b) *to equal-area spaces according to Gauss distribution. Then we can consider two kinds of maps* (i) *a multi-map where every element corresponds to the hole class or* (ii) *a map where every element corresponds to one special fixed element of the same class. Using these maps we define the* ∂*-hopes and we obtain the corresponding* H_v*-structure.*

An application on this direction is the following construction [23]:

Construction 5.2 *Consider a group* $(G, \cdot)$ *and suppose take a partition* G_i, $i \in I$, *of the* ***G***. *Select and fix an element* g_i *of each partition class* G_i, *and consider the map*

$$f : G \to G \text{ such that } f(x) = g_i,\ \forall x \in G_i,$$

then (G, ∂) *is an* H_v*-group. Moreover, the fundamental group* $(G/R, \cdot)/\beta^*$ *is* (*up to isomorphism*) *a subgroup of the corresponding fundamental group* $(G, \partial)/\beta^*$.

Remark. In the above construction, if one of the selected elements is the unit element e of the group $(G, \cdot)$, otherwise, if there exist an element $z \in G$ such that $f(z) = e$, then we have $(G/R, \cdot)/\beta^* = (G, \partial)/\beta^*$.

Proposition 5.3 *Suppose* $(G, \cdot)$ *be a group and* G_i, $i \in I$ *be a partition of* G. *For any class we fix a* $g_i \in G_i$, *and take the map* $f : G \to G : f(x) = g_i, \forall x \in G_i$. *If for the unit element e, in* $(G, \cdot)$, *we have* $f(e) = e$, *i.e. e is any fixed element, then e is also a unit element of the* H_v*-group* (G, ∂). *Moreover* $(f(x))^{-1}$ *is an inverse element in the* ∂*-*H_v*-group* (G, ∂), *of x.*

Now, we conclude with an example of the above Construction:

Example 5.4 Suppose that we take the case of the Likert scale with 5 equal steps: $[0 - 1.24 - 2.48 - 3.72 - 4.96 - 6.2]$ and the Gauss 5 equal areas: $[0 - 2.4 - 2.9 - 3.3 - 3.8 - 6.2]$ we have 9 segments as follows

$$[0 - 1.24 - 2.4 - 2.48 - 2.9 - 3.3 - 3.72 - 3.8 - 4.96 - 6.2]$$

Therefore, if we consider the set $(Z_9,+)$ and if we name the above segments by 0, 1, 2, ..., 8 then if we consider the Gauss partition: $\{0,1\},\{2,3\},\{4\},\{5,6\},\{7,8\}$ we take, according to the above Construction, the map f such that $f(0) = 0$, $f(1) = 0$, $f(2) = 2$, $f(3) = 2$, $f(4) = 4$, $f(5) = 5$, $f(6) = 5$, $f(7) = 7$, $f(8) = 7$, then we obtain the following table:

∂	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
0	0	0,1	2	2,3	4	5	5,6	7	7,8
1	0,1	1	2,3	3	4,5	5,6	6	7,8	8
2	2	2,3	4	4,5	6	7	7,8	0	0,1
3	2,3	3	4,5	5	6,7	7,8	8	0,1	1
4	4	4,5	6	6,7	8	0	0,1	2	2,3
5	5	5,6	7	7,8	0	1	1,2	3	4,3
6	5,6	6	7,8	8	0,1	1,2	2	3,4	4
7	7	7,8	0	0,1	2	3	3,4	5	5,6
8	7,8	8	0,1	1	2,3	4,3	4	5,6	6

Remark that, for the H_v-group (Z_9, ∂), the elements 0 and 1 are unit elements. (Z_9, ∂) is cyclic where the elements 2, 3, 4, 5, 6, 7 and 8 are generators with period 6, 7, 6, 9, 6, 7 and 7 respectively.

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RELATED FIXED POINT THEOREM FOR SIX MAPPINGS ON THREE MODIFIED INTUITIONISTIC FUZZY METRIC SPACES

Sushil Sharma

Department of Mathematics
Madhav Science College
Vikram University
Ujjain-456010
India
e-mail: sksharma2005@yahoo.com

Prashant Tilwankar

Department of Mathematics
Shri Vaishnav Institute of Management
Indore-452009
India
e-mail:prashant_tilwankar@yahoo.com

Abstract. Related fixed point theorems on two or three metric spaces have been proved in different ways. Sharma, Deshpande and Thakur were the first who have established related fixed point theorem for four mappings on two complete fuzzy metric spaces. Their work was maiden in this line. In this paper we obtain a related fixed point theorem for six mappings on three complete modified intuitionistic fuzzy metric spaces. Of course this is a new result on this line.

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Keywords: modified intuitionistic fuzzy metric space, common fixed point, Cauchy sequence.

1. Introduction

Motivated by the potential applicability of fuzzy topology to quantum particle physics particularly in connection with both string and $e^{(\infty)}$ theory developed by El Naschie [10], [11], Park introduced and discussed in [24] a notion of intuitionistic fuzzy metric spaces which is based on the idea of intuitionistic fuzzy sets due to Atanassov [3] and the concept of fuzzy metric space given by George and Veeramani [18]. Actually, Park's notion is useful in modelling some phenomena where it is necessary to study the relationship between two probability functions. It has direct physics motivation in the context of the two-slit experiment as the foundation of E-infinity of high energy physics, recently studied by El Naschie [12], [13].

Alaca et al. [2] using the idea of intuitionistic fuzzy sets, they defined the notion of intuitionistic fuzzy metric space as Park [24] with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [22]. Further, they introduced the notion of Cauchy sequences in intuitionistic fuzzy metric spaces and proved the well known fixed point theorems of Banach [4] and Edelstein [9] extended to intuitionistic fuzzy metric spaces with the help of Grabiec [13]. Turkoglu et al. [30] introduced the concept of compatible maps and compatible maps of types (α) and (β) in intuitionistic fuzzy metric spaces and gave some relations between the concepts of compatible maps and compatible maps of types (α) and (β).

Since the intuitionistic fuzzy metric space has extra conditions, Saadati, Sedghi and Shobe [28] modified the idea of intuitionistic fuzzy metric spaces and gave the new notion of intuitionistic fuzzy metric spaces with the help of the notion of continuous t-representable.

Related fixed point theorems on two or three metric spaces were proved by Fisher[14],[15], Nung[23], Popa [24], Jain, Sahu and Fisher [19], Jain, Shrivastava and Fisher [20], Cho, Kang and Kim [5], Fisher and Murthy [16] and many others. Sharma, Deshpande and Thakur [29] established a related fixed point theorem for four mappings on two complete fuzzy metric spaces. Deshpande and Pathak [8] intuitionistically fuzzified the results of Sharma, Deshpande and Thakur [29] and proved a related fixed point theorem for two pairs of mappings on two intuitionistic fuzzy metric spaces. In this paper, we extend the results of Deshpande and Pathak [8] and prove a related fixed point theorem for six mappings on three complete modified intuitionistic fuzzy metric spaces.

2. Preliminaries

Definition 2.1. ([26]) A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if $*$ is satisfying the following conditions:

(i) $*$ is commutative and associative,

(ii) $*$ is continuous,

(iii) $a * 1 = a$ for all $a \in [0,1]$,

(iv) $a * b \le c * d$ whenever $a \le c$ and $b \le d$, $a, b, c, d \in [0,1]$.

Definition 2.2. ([26]) A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is continuous t-conorm if $\diamond$ is satisfying the following conditions:

(i) $\diamond$ is commutative and associative,

(ii) $\diamond$ is continuous,

(iii) $a \diamond 0 = a$ for all $a \in [0,1]$,

(iv) $a \diamond b = c \diamond d$ whenever $a \le c$ and $b \le d$, $a, b, c, d \in [0,1]$.

Lemma 2.1. ([7]) *Consider the set L^* and operation $\leq_{L^*}$ defined by*

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1\}$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \geq y_2, \text{ for every } (x_1, x_2), (y_1, y_2) \in L^*.$$

Then $(L^*, \ \leq_{L^*})$ is a complete lattice.

Definition 2.3. ([3]) An intuitionistic fuzzy set $A_{\zeta,\eta}$ in a universe U is an object $A_{\zeta,\eta} = \{(\zeta_A(u), \eta_A(u))|\ u \in U\}$, where, for all $u \in U$, $\zeta_A(u) \in [0, 1]$ and $\eta_A(u) \in [0, 1]$ are called the membership degree and the non-membership degree, respectively, of u in $A_{\zeta,\eta}$, and, furthermore, they satisfy $\zeta_A(u) + \eta_A(u) \leq 1$.

For every $z_i = (x_i, y_i) \in L^*$, if $c_i \in [0, 1]$ such that $\sum\limits_{j=1}^{n} c_j = 1$, then it is easy that

$$c_1(x_1, y_1) + \cdots + c_n(x_n, y_n) = \sum_{j=1}^{n} c_j \ (x_j, \ y_j) = \left(\sum_{j=1}^{n} c_j x_j, \sum_{j=1}^{n} c_j y_j\right) \in L^*. \tag{2.1}$$

We denote its units by $0_{L^*} = (0, 1)$ and $1_{L^*} = (1, 0)$. Classically, a triangular norm $* = T$ on $[0, 1]$ is defined as an increasing, commutative, associative mapping $T : [0, 1]^2 \to [0, 1]$ satisfying $T(1, \ x) = 1 * x = x$, for all $x \in [0, 1]$. A triangular conorm $\mathrm{S} = \diamond$ is defined as an increasing, commutative, associative mapping $S : [0, 1]^2 \to [0, 1]$ satisfying $S(0, \ x) = 0 \diamond x = x$, for all $x \in [0, 1]$. Using the lattice $(L^*, \ \leq_{L^*})$ these definitions can be straightforwardly extended.

Definition 2.4. ([6]) A triangular norm (t-norm) on L^* is a mapping $\tau : (L^*)^2 \to L^*$ satisfying the following conditions:

$(\forall\ x \in L^*)(\tau(x, 1_{L^*}) = x)$ (boundary condition),
$(\forall(x, y) \in (L^*)^2)(\tau(x, y) = \tau(y, x))$ (commutativity),
$(\forall(x, y, z) \in (L^*)^3)(\tau(x, \tau(y, z)) = \tau(\tau(x, y), z))$ (associativity),
$(\forall(x, x', y, y') \in (L^*)^4)(x \leq_{L^*} x')$ and $(y \leq_{L^*} y' \to \tau(x, y) \leq_{L^*} \tau(x', y'))$ (monotonicity).

Definition 2.5. ([6], [7]) A continuous t-norm τ on L^* is called continuous t-representable if and only if there exist a continuous t-norm $*$ and a continuous t-conorm $\diamond$ on [0, 1] such that, for all $x = (x_1, x_2), \ y = (y_1, y_2) \in L^*$,

$$\tau(x, y) = (x_1 * y_1, x_2 \diamond y_2).$$

Now, define a sequence τ^n recursively by $\tau^1 = \tau$ and

$$\tau^n(x^{(1)}, \ ..., x^{(n+1)}) \ = \tau(\tau^{n-1}(x^{(1)}, \ ..., \ x^{(n)}, \ x^{(n+1)}) \text{ for } n \geq 2 \text{ and } x^{(i)} \in L^*.$$

Definition 2.6. ([28]) Let M, N are fuzzy sets from $X^2 \times (0, +\infty)$ to $[0, 1]$ such that $M(x, y, t) + N(x, y, t) \leq 1$ for all $x, y \in X$ and $t > 0$. The 3-tuple $(X, \mathcal{M}_{M,N}, \tau)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary (non-empty) set, τ is a continuous t-representable and $\mathcal{M}_{M,N}$ is a mapping $X^2 \times (0, +\infty) \to L^*$ (an intuitionistic fuzzy set, see Definition 2.3) satisfying the following conditions for every $x, \ y \in X$ and $t, s > 0$:

(a) $\mathcal{M}_{M,N}(x,y,t) >_{L^*} 0_{L^*}$;

(b) $\mathcal{M}_{M,N}(x,y,t) = 1_{L^*}$ if and only if $x = y$;

(c) $\mathcal{M}_{M,N}(x,y,t) = \mathcal{M}_{M,N}(y,x,t)$;

(d) $\mathcal{M}_{M,N}(x,y,t+s) \geq_{L^*} \tau(\mathcal{M}_{M,N}(x,z,t), \mathcal{M}_{M,N}(z,y,s))$;

(e) $\mathcal{M}_{M,N}(x,y,\cdot) : (0,\infty) \to L^*$ is continuous.

In this case, $\mathcal{M}_{M,N}$ is called an *intuitionistic fuzzy metric.*

Here,

$$\mathcal{M}_{M,N}(x,y,t) = (M(x,y,t), N(x,y,t)).$$

Example 2.1. ([28]) Let (X,d) be a metric space. Denote

$$\tau(a,b) = (a_1 b_1, \min(a_2 + b_2, 1))$$

for all $a = (a_1, a_2)$ and $b = (b_1, b_2) \in L^*$ and let M and N be fuzzy sets on $X^2 \times (0,\infty)$ defined as follows:

$$\mathcal{M}_{M,\ N}(x,y,t)=(M(x,y,t), N(x,y,t))=\left(\frac{ht^n}{ht^n\ +\ md(x,\ y)}, \frac{md(x,\ y)}{ht^n\ +\ md(x,\ y)}\right)$$

for all $t, h, m, n \in R^+$.

Then, $(X, \mathcal{M}_{M,N}, \tau)$ is an intuitionistic fuzzy metric space.

Example 2.2. ([28]) Let $X = N$. Define

$$\tau(a,b) = (\max(0, a_1 + b_1 - 1), a_2 + b_2 - a_2 b_2)$$

for all $a = (a_1, a_2)$ and $b = (b_1, b_2) \in L^*$ and let M and N be fuzzy sets on $X^2 \times (0,\infty)$ defined as follows:

$$\mathcal{M}_{M,N}(x,y,t) = (M(x,y,t), N(x,y,t)) = \left\{ \begin{array}{ll} \left(\dfrac{x}{y}, \dfrac{y-x}{y}\right) & \text{if } x \leq y, \\ \left(\dfrac{y}{x}, \dfrac{x-y}{x}\right) & \text{if } y \leq x, \end{array} \right\}$$

for all $x, y \in X$ and $t > 0$. Then $(X, \mathcal{M}_{M,\ N}, \tau)$ is an intuitionistic fuzzy metric space.

Definition 2.7. ([28]) A sequence $\{x_n\}$ in an intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,\ N}, \tau)$ is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in N$ such that

$$\mathcal{M}_{M,N}(x_n, y_m, t) >_{L^*} (N_s(\varepsilon), \varepsilon)$$

and for each $n, m \geq n_0$, here N_s is the standard negator. The sequence $\{x_n\}$ is said to be convergent to $x \in X$ in the intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,N}, \tau)$ and denoted by $x_n \overset{\mathcal{M}_{M,N}}{\longrightarrow} x$ if $\mathcal{M}_{M,N}(x_n, x, t) \to 1_{L^*}$ whenever $n \to \infty$ for every $t > 0$. An intuitionistic fuzzy metric space is said to be complete if and only if every Cauchy sequence is convergent.

Lemma 2.2. ([27]) *Let $\mathcal{M}_{M,\ N}$ be an intuitionistic fuzzy metric space. Then, for any $t > 0$, $\mathcal{M}_{M,N}(x, y, t)$ is non-decreasing with respect to t, in $(L^*, \leq_{L^*})$, for all x, y in X.*

Lemma 2.3. ([1]) *Let $(X, \mathcal{M}_{M,N}, \tau)$ be a modified intuitionistic fuzzy metric space. For each $\lambda \in (0, 1)$, define the map $E_\lambda : X^2 \to R^+ \cup \{0\}$ by*

$$E_\lambda(x, y) = \inf\{t > 0 : \mathcal{M}_{M,N}(x, y, t) >_{L^*} (1 - \lambda, \lambda)\},$$

then

(a) *For each $\lambda \in (0, 1)$, we have a $\mu \in (0, 1)$ such that*

$$E_\lambda(x_1, x_n) \leq E_\mu(x_1, x_2) + E_\mu(x_2, x_3) + \cdots + E_\mu(x_{n-1}, x_n),$$

for any $x_1, x_2, x_3, ..., x_n \in X$.

(b) *The sequence $\{x_n\}_{n\in N}$ in X is convergent to x if and only if $E_\lambda(x_n, x) \to 0$.*

Also, the sequence $\{x_n\}_{n\in N}$ is a Cauchy sequence in X if and only if it is a Cauchy sequence with respect to E_λ.

Lemma 2.4. ([21]) *Let $(X, \mathcal{M}_{M,\ N}, \tau)$ be an intuitionistic fuzzy metric space. If for a sequence $\{x_n\}$ in X, there exists $k \in (0, 1)$ such that*

$$\mathcal{M}_{M,N}(x_n, x_{n+1}, kt) \geq_{L^*} \mathcal{M}_{M,N}(x_{n-1}, x_n, t), \text{ for all } n \text{ and for all } t,$$

then $\{x_n\}$ is a Cauchy sequence in X.

Proof. Let $(X, \mathcal{M}_{M,\ N}, \tau)$ be an intuitionistic fuzzy metric space. Let for a sequence $\{x_n\}$ in X, there exists $k \in (0, 1)$ such that

$$\mathcal{M}_{M,\ N}(x_n, x_{n+1}, kt) \geq_{L^*} \mathcal{M}_{M,\ N}(x_{n-1}, x_n, t), \text{ for all } n \text{ and } t,$$

then

$$\mathcal{M}_{M,N}(x_n, x_{n+1}, t) \geq_{L^*} \mathcal{M}_{M,N}\left(x_{n-1}, x_n, \frac{t}{k}\right) \geq_{L^*} \mathcal{M}_{M,N}\left(x_{n-2}, x_{n-1}, \frac{t}{k^2}\right)$$
$$\ldots \geq_{L^*} \mathcal{M}_{M,N}(x_0, x_1, \frac{t}{k^n}), \text{ for all } n.$$

Now

$$\begin{aligned}
E_\lambda(x_{n+1}, x_n) &= \inf\{t > 0 : \mathcal{M}_{M,N}(x_{n+1}, x_n, t) \geq_{L^*} (1-\lambda, \lambda)\} \\
&\leq \inf\left\{t > 0 : \mathcal{M}_{M,N}\left(x_1, x_0, \frac{t}{k^n}\right) \geq_{L^*} (1-\lambda, \lambda)\right\} \\
&= \inf\{k^n t > 0 : \mathcal{M}_{M,N}(x_1, x_0, t) \geq_{L^*} (1-\lambda, \lambda)\} \\
&= k^n \inf\{t > 0 : \mathcal{M}_{M,N}(x_1, x_0, t) \geq_{L^*} (1-\lambda, \lambda)\} \\
&= k^n\ E_\lambda(x_0, x_1). \\
E_\lambda(x_{n+1}, x_n) &\leq k^n E_\lambda(x_0, x_1) \ldots (A)
\end{aligned}$$

Again from Lemma 2.3, for $\lambda \in (0,1)$, there exists $\mu \in (0,1)$ such that

$$\begin{aligned}
E_\lambda(x_n, x_{n+p}) &\leq E_\mu(x_n, x_{n+1}) + E_\mu(x_{n+1}, x_{n+2}) + \cdots + E_\mu(x_{n+p-1}, x_{n+p}) \\
&\leq k^n\ E_\mu(x_0, x_1) + k^{n+1}\ E_\mu(x_0, x_1) + \cdots + k^{n+p-1}\ E_\mu(x_0, x_1), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{using (A)} \\
&= (k^n + k^{n+1} + \cdots + k^{n+p-1}) E_\mu(x_0, x_1), \\
&= \frac{k^n}{1-k}\ E_\mu(x_0, x_1), \text{ as } 0 < k < 1,
\end{aligned}$$

which tends to 0, as $n \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence in X. ∎

Lemma 2.5. ([21]) *In an intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,\ N}, \tau)$, if for some x, y in X there exists $k \in (0,1)$ such that*

$$\mathcal{M}_{M,\ N}(x, y, kt) \geq_{L^*} \mathcal{M}_{M,\ N}(x, y, t), \textit{for all } t,$$

then $x = y$.

Proof. Let for $\lambda \in (0,1)$

$$\begin{aligned}
E_\lambda(x, y) &= \inf\{t > 0 : \mathcal{M}_{M,N}(x, y, t) \geq_{L^*} (1-\lambda, \lambda)\} \\
&\leq \inf\{t > 0 : \mathcal{M}_{M,N}(x, y, t/k) \geq_{L^*} (1-\lambda, \lambda)\} \\
&= \inf\{kt > 0 : \mathcal{M}_{M,N}(x, y, t) \geq_{L^*} (1-\lambda, \lambda)\} \\
&= k \inf\{t > 0 : \mathcal{M}_{M,N}(x, y, t) \geq_{L^*} (1-\lambda, \lambda)\} \\
&= k\, E_\lambda(x, y).
\end{aligned}$$

Therefore, $E_\lambda(x, y) = 0$. Hence $x = y$. ∎

Sharma, Deshpande and Thakur [29] established the following related fixed point theorem for four mappings on two complete fuzzy metric spaces.

Theorem A. *Let $(X, M_1, *)$ and $(Y, M_2, *)$ be two complete fuzzy metric spaces. Let A, B be mappings from X into Y and S, T be mappings from Y into X satisfying the inequalities:*

(i) $M_1(SAx, TBx', kt) \geq M_1(x, x', t) * M_1(x, SAx, t) * M_1(x', TBx', t) * M_1(SAx, TBx', t)$

(ii) $M_2(BSy, ATy', kt) \geq M_2(y, y', t) * M_2(y, BSy, t) * M_2(y', ATy', t) * M_2(BSy, ATy', kt)$

for all x, x' in X and y, y' in Y. If one of the mappings A, B, S, T is continuous, then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y. Further, $Az = Bz = w$ and $Sw = Tw = z$.

Deshpande and Pathak [8] intuitionistically fuzzify the results of Sharma, Deshpande and Thakur [29] and proved the following:

Theorem B. *$(X, M_1, N_1, *, \diamond)$ and $(Y, M_2, N_2, *, \diamond)$ be two complete intuitionistic fuzzy metric spaces. Let A, B be mappings from X into Y and let S, T be mappings from Y into X satisfying the inequalities:*

(i) $M_1(SAx, TBx\prime, kt) \geq M_1(x, x\prime, t) * M_1(x, SAx, t) * M_1(x\prime, TBx\prime, t) * M_1(SAx, TBx\prime, t)$

and

$N_1(SAx, TBx\prime, kt) \leq N_1(x, x\prime, t) \diamond N_1(x, SAx, t) \diamond N_1(x\prime, TBx\prime, t) \diamond N_1(SAx, TBx\prime, t)$

(ii) $M_2(BSy, ATy\prime, kt) \geq M_2(y, y\prime, t) * M_2(y, BSy, t) * M_2(y\prime, ATy\prime, t) * M_2(BSy, ATy\prime, t)$

and

$N_2(BSy, ATy\prime, kt) \leq N_2(y, y\prime, t) \diamond N_2(y, BSy, t) \diamond N_2(y\prime, ATy\prime, t) \diamond N_2(BSy, ATy\prime, t)$

for all $x, x\prime$ in X and $y, y\prime$ in Y. If one of the mappings A, B, S, T is continuous, then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y. Further, $Az = Bz = w$ and $Sw = Tw = z$.

We extend the results of Deshpande and Pathak [8] and prove a related fixed point theorem for six mappings on three complete modified intuitionistic fuzzy metric spaces.

3. Main result

Theorem 3.1. *Let $(X, \mathcal{M}_{M_1,\ N_1}, \tau)$, $(Y, \mathcal{M}_{M_2,\ N_2}, \tau)$ and $(Z, \mathcal{M}_{M_3,\ N_3}, \tau)$ be three complete intuitionistic fuzzy metric spaces. Let A, B be continuous mappings from X into Y, let S, T be continuous mappings from Y into Z and let P, Q be continuous mappings from Z into X satisfying the inequalities:*

$$\begin{aligned}(3.1)\quad \mathcal{M}_{M_1,N_1}(PSAx, QTBx', kt) \geq_{L^*} & \mathcal{M}_{M_1,\ N_1}(x, x', t) * \mathcal{M}_{M_1,N_1}(x, PSAx, t)\\ & * \mathcal{M}_{M_1,N_1}(x', QTBx', t) * \mathcal{M}_{M_1,N_1}(PSAx, QTBx', t)\end{aligned}$$

$$\begin{aligned}(3.2)\quad \mathcal{M}_{M_2,N_2}(APSy, BQTy', kt) \geq_{L^*} & \mathcal{M}_{M_2,N_2}(y, y', t) * \mathcal{M}_{M_2,N_2}(y, APSy, t)\\ & * \mathcal{M}_{M_2,N_2}(y', BQTy', t) * \mathcal{M}_{M_2,N_2}(APSy, BQTy', t)\end{aligned}$$

$$\begin{aligned}(3.3)\quad \mathcal{M}_{M_3,N_3}(SAPz, TBQz', kt) \geq_{L^*} & \mathcal{M}_{M_3,N_3}(z, z', t) * \mathcal{M}_{M_3,N_3}(z, SAPz, t)\\ & * \mathcal{M}_{M_3,N_3}(z', TBQz', t) * \mathcal{M}_{M_3,N_3}(SAPz, TBQz', t)\end{aligned}$$

for all x, x' in X, y, y' in Y and z, z' in Z, $t > 0$ and $k \in (0, 1)$, then PSA and QTB have a unique common fixed point u in X, APS and BQT have a unique common fixed point v in Y and SAP and TBQ have a unique common fixed point w in Z. Further, $Au = Bu = v, Sv = Tv = w$ and $Pw = Qw = u$.

Proof. Let $x = x_0$ be an arbitrary point in X and define sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in X, Y and Z respectively as follows:

Choose a point $z_1 = Sy_1$, a point $y_1 = Ax_0$, a point $x_1 = Pz_1$, a point $z_2 = Ty_2$, a point $y_2 = Bx_1$ and a point $x_2 = Qz_2$. In general, having chosen x_{2n-2} in X, choose a point $y_{2n-1} = Ax_{2n-2}$, a point $y_{2n} = Bx_{2n-1}$, a point $z_{2n-1} = Sy_{2n-1}$, a point $z_{2n} = Ty_{2n}$, a point $x_{2n-1} = Pz_{2n-1}$ and a point $x_{2n} = Qz_{2n}$ for all $n = 1, 2, ...$

Applying inequality (3.1), we have

$$\begin{aligned}\mathcal{M}_{M_1,N_1}(x_{2n+1}, x_{2n}, kt) &= \mathcal{M}_{M_1,N_1}(PSAx_{2n}, QTBx_{2n-1}, kt)\\ &\geq_{L^*} \mathcal{M}_{M_1,N_1}(x_{2n}, x_{2n-1}, t) * \mathcal{M}_{M_1,N_1}(x_{2n}, PSAx_{2n}, t)\\ &* \mathcal{M}_{M_1,\ N_1}(x_{2n-1}, QTBx_{2n-1}, t) * \mathcal{M}_{M_1,N_1}(PSAx_{2n}, QTBx_{2n-1}, t)\\ &= \mathcal{M}_{M_1,N_1}(x_{2n}, x_{2n-1}, t) * \mathcal{M}_{M_1,N_1}(x_{2n}, x_{2n+1}, t)\\ &* \mathcal{M}_{M_1,N_1}(x_{2n-1}, x_{2n}, t) * \mathcal{M}_{M_1,N_1}(x_{2n+1}, x_{2n}, t)\\ &\geq_{L^*} \mathcal{M}_{M_1,N_1}(x_{2n}, x_{2n-1}, t) * \mathcal{M}_{M_1,N_1}(x_{2n}, x_{2n+1}, t)\end{aligned} \tag{3.4}$$

Similarly, we have

$$\begin{aligned}\mathcal{M}_{M_1,N_1}(x_{2n+2}, x_{2n+1}, kt) \geq_{L^*} & \mathcal{M}_{M_1,N_1}(x_{2n+1}, x_{2n}, t)\\ & * \mathcal{M}_{M_1,N_1}(x_{2n+1}, x_{2n+2}, t).\end{aligned} \tag{3.5}$$

Thus, from (3.4) and (3.5), it follows that

$$\mathcal{M}_{M_1,N_1}(x_{n+1}, x_{n+2}, kt) \geq_{L^*} \mathcal{M}_{M_1,N_1}(x_n, x_{n+1}, t) * \mathcal{M}_{M_1,N_1}(x_{n+1}, x_{n+2}, t),$$

for $n = 1, 2, ...$

Consequently, for positive integers n, p we have

$$\mathcal{M}_{M_1,N_1}(x_{n+1}, x_{n+2}, kt) \geq_{L^*} \mathcal{M}_{M_1,N_1}(x_n, x_{n+1}, t) * \mathcal{M}_{M_1,N_1}(x_{n+1}, x_{n+2}, t/k^p).$$

Thus, since $\mathcal{M}_{M_1,\ N_1}(x_{n+1}, x_{n+2}, kt) \to 1_{L^*}$ as $p \to \infty$, we have

$$\mathcal{M}_{M_1,N_1}(x_{n+1}, x_{n+2}, kt) \geq_{L^*} \mathcal{M}_{M_1,N_1}(x_n, x_{n+1}, t) \tag{3.6}$$

Similarly, applying inequality (3.2) and (3.3), we have

$$\mathcal{M}_{M_2,N_2}(y_{n+1}, y_{n+2}, kt) \geq_{L^*} \mathcal{M}_{M_2,N_2}(y_n, y_{n+1}, t) \tag{3.7}$$

$$\mathcal{M}_{M_3,N_3}(z_{n+1}, z_{n+2}, kt) \geq_{L^*} \mathcal{M}_{M_3,N_3}(z_n, z_{n+1}, t) \tag{3.8}$$

By Lemma 2.4, $\{x_n\}$ is a Cauchy sequence in a complete intuitionistic fuzzy metric space X and so has a limit u in X. It follows similarly that the sequences $\{y_n\}$ and $\{z_n\}$ are also Cauchy sequences in complete intuitionistic fuzzy metric space Y and Z and so have limits v in Y and w in Z.

Using (3.1), we have

$$\begin{aligned}
\mathcal{M}_{M_1,N_1}(PSAx_{2n}, u, kt) &\geq_{L^*} \mathcal{M}_{M_1,N_1}(PSAx_{2n}, x_{2n}, \tfrac{kt}{2}) * \mathcal{M}_{M_1,N_1}(x_{2n}, u, \tfrac{kt}{2}) \\
&= \mathcal{M}_{M_1,N_1}(PSAx_{2n}, QTBx_{2n-1}, \tfrac{kt}{2}) * \mathcal{M}_{M_1,N_1}(x_{2n}, u, \tfrac{kt}{2}) \\
&\geq_{L^*} \mathcal{M}_{M_1,N_1}(x_{2n}, x_{2n-1}, \tfrac{t}{2}) * \mathcal{M}_{M_1,N_1}(x_{2n}, PSAx_{2n}, \tfrac{t}{2}) \\
&\qquad * \mathcal{M}_{M_1,N_1}(x_{2n-1}, QTBx_{2n-1}, \tfrac{t}{2}) \\
&* \mathcal{M}_{M_1,N_1}(PSAx_{2n}, QTBx_{2n-1}, \tfrac{t}{2}) * \mathcal{M}_{M_1,N_1}(x_{2n}, u, \tfrac{kt}{2}) \\
&\geq_{L^*} \mathcal{M}_{M_1,N_1}(x_{2n}, x_{2n-1}, \tfrac{t}{2}) * \mathcal{M}_{M_1,N_1}(x_{2n}, x_{2n+1}, \tfrac{t}{2}) \\
&* \mathcal{M}_{M_1,N_1}(x_{2n-1}, x_{2n}, \tfrac{t}{2}) * \mathcal{M}_{M_1,N_1}(x_{2n+1}, x_{2n}, \tfrac{t}{2}) * \mathcal{M}_{M_1,N_1}(x_{2n}, u, \tfrac{kt}{2})
\end{aligned}$$

Taking limit $n \to \infty$, we have

$$\lim_{n\to\infty} \mathcal{M}_{M_1,\ N_1}(PSAx_{2n}, u, kt) \to 1_{L^*}.$$

Thus, we have

$$\lim_{n\to\infty} PSAx_{2n} = u = \lim_{n\to\infty} PSy_{2n+1} \tag{3.9}$$

Similarly, we can prove that

$$\lim_{n\to\infty} QTBx_{2n-1} = u = \lim_{n\to\infty} QTy_{2n} \tag{3.10}$$

$$\lim_{n\to\infty} APSy_{2n-1} = v = \lim_{n\to\infty} APz_{2n-1} \tag{3.11}$$

$$\lim_{n\to\infty} BQTy_{2n} = v = \lim_{n\to\infty} BQz_{2n} \tag{3.12}$$

$$\lim_{n\to\infty} SAPz_{2n} = w = \lim_{n\to\infty} SAx_{2n} \tag{3.13}$$

$$\lim_{n\to\infty} TBQz_{2n-1} = w = \lim_{n\to\infty} TBx_{2n-1} \tag{3.14}$$

Since A and B are continuous, we have

$$\lim_{n\to\infty} Ax_{2n} = Au = v, \quad \lim_{n\to\infty} Bx_{2n-1} = Bu = v. \tag{3.15}$$

Using inequality (3.1), we have

$$\begin{aligned}\mathcal{M}_{M_1,N_1}(PSAu, QTBx_{2n-1}, kt) \geq_{L^*} \mathcal{M}_{M_1,N_1}(u, x_{2n-1}, t) * \mathcal{M}_{M_1,N_1}(u, PSAu, t)\\ *\mathcal{M}_{M_1,N_1}(x_{2n-1}, QTBx_{2n-1}, t) * \mathcal{M}_{M_1,N_1}(PSAu, QTBx_{2n-1}, t).\end{aligned}$$

Letting $n \to \infty$ and using (3.10), we have

$$\mathcal{M}_{M_1,\ N_1}(PSAu, u, kt) \geq_{L^*} \mathcal{M}_{M_1,\ N_1}(u, PSAu, t).$$

Therefore, by Lemma 2.5 and using (3.15), we have $PSAu = u = PSv$.

Using inequality (3.1), we have

$$\begin{aligned}\mathcal{M}_{M_1,N_1}(PSAx_{2n}, QTBu, kt) \geq_{L^*} \mathcal{M}_{M_1,N_1}(x_{2n}, u, t) * \mathcal{M}_{M_1,N_1}(x_{2n}, PSAx_{2n}, t)\\ *\mathcal{M}_{M_1,N_1}(u, QTBu, t) * \mathcal{M}_{M_1,N_1}(PSAx_{2n}, QTBu, t).\end{aligned}$$

Letting $n \to \infty$ and using (3.9), we have

$$\mathcal{M}_{M_1,N_1}(u, QTBu, kt) \geq_{L^*} \mathcal{M}_{M_1,N_1}(u, QTBu, t).$$

Therefore, by Lemma 2.5 and using (3.15), we have $QTBu = u = QTv$.

Since S and T are continuous, we have

$$\lim_{n\to\infty} Sy_{2n-1} = Sv = w, \quad \lim_{n\to\infty} Ty_{2n} = Tv = w. \tag{3.16}$$

Using inequality (3.2), we have

$$\begin{aligned}\mathcal{M}_{M_2,\ N_2}(APSv, BQTy_{2n}, kt) \geq_{L^*} \mathcal{M}_{M_2,\ N_2}(v, y_{2n}, t) * \mathcal{M}_{M_2,\ N_2}(v, APSv, t)\\ *\mathcal{M}_{M_2,\ N_2}(y_{2n}, BQTy_{2n}, t) * \mathcal{M}_{M_2,\ N_2}(APSv, BQTy_{2n}, t).\end{aligned}$$

Letting $n \to \infty$ and using (3.12), we have

$$\mathcal{M}_{M_2,\ N_2}(APSv, v, kt) \geq_{L^*} \mathcal{M}_{M_2,\ N_2}(v, APSv, t).$$

Therefore, by Lemma 2.5 and using (3.16), we have $APSv = v = APw$.

Using inequality (3.2), we have

$$\begin{aligned}&\mathcal{M}_{M_2,N_2}(APSy_{2n-1}, BQTv, kt)\\ &\quad\geq_{L^*} \mathcal{M}_{M_2,N_2}(y_{2n-1}, v, t) * \mathcal{M}_{M_2,N_2}(y_{2n-1}, APSy_{2n-1}, t)\\ &\quad *\mathcal{M}_{M_2,N_2}(v, BQTv, t) * \mathcal{M}_{M_2,N_2}(APSy_{2n-1}, BQTv, t).\end{aligned}$$

Letting $n \to \infty$ and using (3.11), we have

$$\mathcal{M}_{M_2,\ N_2}(v, BQTv, kt) \geq_{L^*} \mathcal{M}_{M_2,\ N_2}(v, BQTv, t).$$

Therefore, by Lemma 2.5 and using (3.16), we have $BQTv = v = BQw$.

Since P and S are continuous, we have

$$\lim_{n\to\infty} Pz_{2n} = Pw = u, \quad \lim_{n\to\infty} Qz_{2n-1} = Qw = u. \tag{3.17}$$

Using inequality (3.3), we have

$$\mathcal{M}_{M_3,N_3}(SAPw, TBQz_{2n-1}, kt) \geq_{L^*} \mathcal{M}_{M_3,N_3}(w, z_{2n-1}, t) * \mathcal{M}_{M_3,N_3}(w, SAPw, t) \\ *\mathcal{M}_{M_3,N_3}(z_{2n-1}, TBQz_{2n-1}, t) * \mathcal{M}_{M_3,N_3}(SAPw, TBQz_{2n-1}, t).$$

Letting $n \to \infty$ and using (3.14), we have

$$\mathcal{M}_{M_3,\ N_3}(SAPw, w, kt) \geq_{L^*} \mathcal{M}_{M_3,\ N_3}(w, SAPw, t).$$

Therefore, by Lemma 2.5 and using (3.17), we have $SAPw = w = SAu$.

Using inequality (3.3), we have

$$\mathcal{M}_{M_3,\ N_3}(SAPz_{2n}, TBQw, kt) \geq_{L^*} \mathcal{M}_{M_3,\ N_3}(z_{2n}, w, t) * \mathcal{M}_{M_3,\ N_3}(z_{2n}, SAPz_{2n}, t) \\ * \mathcal{M}_{M_3,\ N_3}(w, TBQw, t) * \mathcal{M}_{M_3,\ N_3}(SAPz_{2n}, TBQw, t).$$

Letting $n \to \infty$ and using (3.13), we have

$$\mathcal{M}_{M_3,\ N_3}(w, TBQw, kt) \geq_{L^*} \mathcal{M}_{M_3,\ N_3}(w, TBQw, t).$$

Therefore, by Lemma 2.5 and using (3.17), we have $TBQw = w = TBu$.

Thus, we have

$$\text{(3.18)} \qquad \left\{ \begin{array}{l} PSAu = QTBu = PSv = QTv = Pw = Qw = u, \\ APSv = BQTv = APw = BQw = Au = Bu = v, \\ SAPw = TBQw = SAu = TBu = Sv = Tv = w. \end{array} \right\}$$

To prove the uniqueness of the fixed point, suppose that PSA and QTB have a common fixed point u' also.

Using inequality (3.1), we have

$$\mathcal{M}_{M_1,N_1}(PSAu, QTBu', kt) \geq_{L^*} \mathcal{M}_{M_1,N_1}(u, u', t) * \mathcal{M}_{M_1,N_1}(u, PSAu, t) \\ *\mathcal{M}_{M_1,N_1}(u', QTBu', t) * \mathcal{M}_{M_1,N_1}(PSAu, QTBu', t).$$

Therefore, we have

$$\mathcal{M}_{M_1,\ N_1}(u, u', kt) \geq_{L^*} \mathcal{M}_{M_1,\ N_1}(u, u', t).$$

By Lemma 2.5, we have $u = u'$. Similarly we can prove that v and w are unique common fixed point of APS and BQT and of SAP and TBQ. This completes the proof. ∎

If we put $M_1 = M_2 = M_3 = M$ and $N_1 = N_2 = N_3 = N$ in Theorem 3.1, we get the following:

Corollary 1. *Let $(X, \mathcal{M}_{M,\ N}, \tau)$, $(Y, \mathcal{M}_{M,\ N}, \tau)$ and $(Z, \mathcal{M}_{M,\ N}, \tau)$ be three complete intuitionistic fuzzy metric spaces. Let A, B be continuous mappings from X into Y, let S, T be continuous mappings from Y into Z and let P, Q be continuous mappings from Z into X satisfying the inequalities:*

$$\mathcal{M}_{M,N}(PSAx, QTBx', kt) \geq_{L^*} \mathcal{M}_{M,N}(x, x', t) * \mathcal{M}_{M,N}(x, PSAx, t) * \mathcal{M}_{M,N}(x', QTBx', t) * \mathcal{M}_{M,N}(PSAx, QTBx', t) \tag{3.1}$$

$$\mathcal{M}_{M,N}(APSy, BQTy', kt) \geq_{L^*} \mathcal{M}_{M,N}(y, y', t) * \mathcal{M}_{M,N}(y, APSy, t) * \mathcal{M}_{M,N}(y', BQTy', t) * \mathcal{M}_{M,N}(APSy, BQTy', t) \tag{3.2}$$

$$\mathcal{M}_{M,N}(SAPz, TBQz', kt) \geq_{L^*} \mathcal{M}_{M,\ N}(z, z', t) * \mathcal{M}_{M,\ N}(z, SAPz, t) * \mathcal{M}_{M,\ N}(z', TBQz', t) * \mathcal{M}_{M,\ N}(SAPz, TBQz', t) \tag{3.3}$$

for all x, x' in X, y, y' in Y and z, z' in $Z, t > 0$ and $k \in (0,1)$, then PSA and QTB have a unique common fixed point u in X, APS and BQT have a unique common fixed point v in Y and SAP and TBQ have a unique common fixed point w in Z. Further, $Au = Bu = v, Sv = Tv = w$ and $Pw = Qw = u$. ∎

If we put $A = B, S = T$ and $P = Q$ in Theorem 3.1, we get the following:

Corollary 2. *Let $(X, \mathcal{M}_{M_1,N_1}, \tau)$, $(Y, \mathcal{M}_{M_2,N_2}, \tau)$ and $(Z, \mathcal{M}_{M_3,N_3}, \tau)$ be three complete intuitionistic fuzzy metric spaces. Let A be continuous mapping from X into Y, let S be continuous mapping from Y into Z and let P be continuous mapping from Z into X satisfying the inequalities:*

$$\mathcal{M}_{M_1,N_1}(PSAx, PSAx', kt) \geq_{L^*} \mathcal{M}_{M_1,N_1}(x, x', t) * \mathcal{M}_{M_1,N_1}(x, PSAx, t) * \mathcal{M}_{M_1,N_1}(x', PSAx', t) * \mathcal{M}_{M_1,N_1}(PSAx, PSAx', t) \tag{3.4}$$

$$\mathcal{M}_{M_2,N_2}(APSy, APSy', kt) \geq_{L^*} \mathcal{M}_{M_2,N_2}(y, y', t) * \mathcal{M}_{M_2,N_2}(y, APSy, t) * \mathcal{M}_{M_2,N_2}(y', APSy', t) * \mathcal{M}_{M_2,N_2}(APSy, APSy', t) \tag{3.5}$$

$$\mathcal{M}_{M_3,N_3}(SAPz, SAPz', kt) \geq_{L^*} \mathcal{M}_{M_3,N_3}(z, z', t) * \mathcal{M}_{M_3,N_3}(z, SAPz, t) * \mathcal{M}_{M_3,N_3}(z', SAPz', t) * \mathcal{M}_{M_3,N_3}(SAPz, SAPz', t) \tag{3.6}$$

for all x, x' in X, y, y' in Y and z, z' in Z, $t > 0$ and $k \in (0,1)$, then PSA has a unique common fixed point u in X, APS has a unique common fixed point v in Y and SAP has a unique common fixed point w in Z. Further, $Au = v, Sv = w$ and $Pw = u$. ∎

Remark 3.1. From Theorem 3.1, with $P = Q = Ix$ (the identity mapping on X), we obtain modified intuitionistic version of the results of Sharma, Deshpande and Thakur [29] and Deshpande and Pathak [8].

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GEOMETRIC EQUIVALENCE BETWEEN THE VEBLEN AND DESARGUES THEOREMS AND BETWEEN THE PAPPUS–PASCAL AND THE "THREE STARS THEOREMS"

Maria Scafati Tallini

Summary. Let $P(r,k)$ and $A(2,k)$ be the projective r-dimensional space over the field k and the projective plane over the same field k, respectively. Let $PG(3,q)$ be the three-dimensional projective space over the Galois field $GF(q)$ and $AG(2,q)$ be the affine plane over $GF(q)$. Referring to the representation of $P(r,k)$ over $A(2,k)$ called also "Crashing" (see [1]), we prove the equivalence, from the geometric point of view, between the Veblen axiom in $PG(3,q)$ and the Desargues theorem in $AG(2,q)$. Moreover, we get a representation in $PG(3,q)$ of the Pappus-Pascal theorem in $AG(2,q)$, consisting of a suitable configuration of planes, called the "Three stars theorem", which turns out to be a geometric equivalence between those two theorems. For the notations and theorems about the representation of $P(r,k)$ over $A(2,k)$ (and therefore in particular of $PG(3,q)$ over $AG(2,q)$), we refer to the paper [1], cited in the bibliography, which the reader must know before reading this text.

1. Geometric equivalence between the Veblen and Desargues configurations

It is known that in $PG(3,q)$, the projective three dimensional space over the field k, the following *Veblen axiom* holds:

For any two lines z and t of $PG(3,q)$ meeting at O, if r_1 and r_2 are two lines each meeting both z and t at two distinct points, distinct from O, also r_1 and r_2 are incident.

Let z and t be two lines of $PG(3,q)$ meeting at Z (see Fig. 1). Let Z_1 and Z_2 be two distinct points of z, both different from Z. Let T_1 and T_2 be two distinct points of t, both distinct from Z. Let r_1 be the line Z_1T_1 and r_2 the line Z_2T_2. Since in $PG(3,q)$ the Veblen axiom holds, the lines r_1 and r_2 meet at a point X. The lines z, t, r_1, r_2 belong to the same plane α. Now, let π be a plane of $PG(3,q)$ through z and distinct from α. Let Y be a point of $z - \{Z, Z_1, Z_2\}$. Finally, let v be a line through Y and not belonging either to π, or to α (see Fig. 1).

Let us represent (using the crashing [1]) the Veblen configuration of $PG(3,q)$ in the affine plane $AG(2,q)$. The points Z, Z_1 and Z_2 are represented by three distinct lines z, z_1, z_2 of $AG(2,q)$. The line t belongs to the class a) of [1] and therefore is represented in $AG(2,q)$ by an ordered pair of distinct points A and B of z'. The point T_1 of t is transformed in an ordered pair of parallel and distinct lines through A and B, let they be a and b respectively. The point T_2 is transformed in the ordered pair of parallel and distinct lines a' and b' through A and B, respectively.

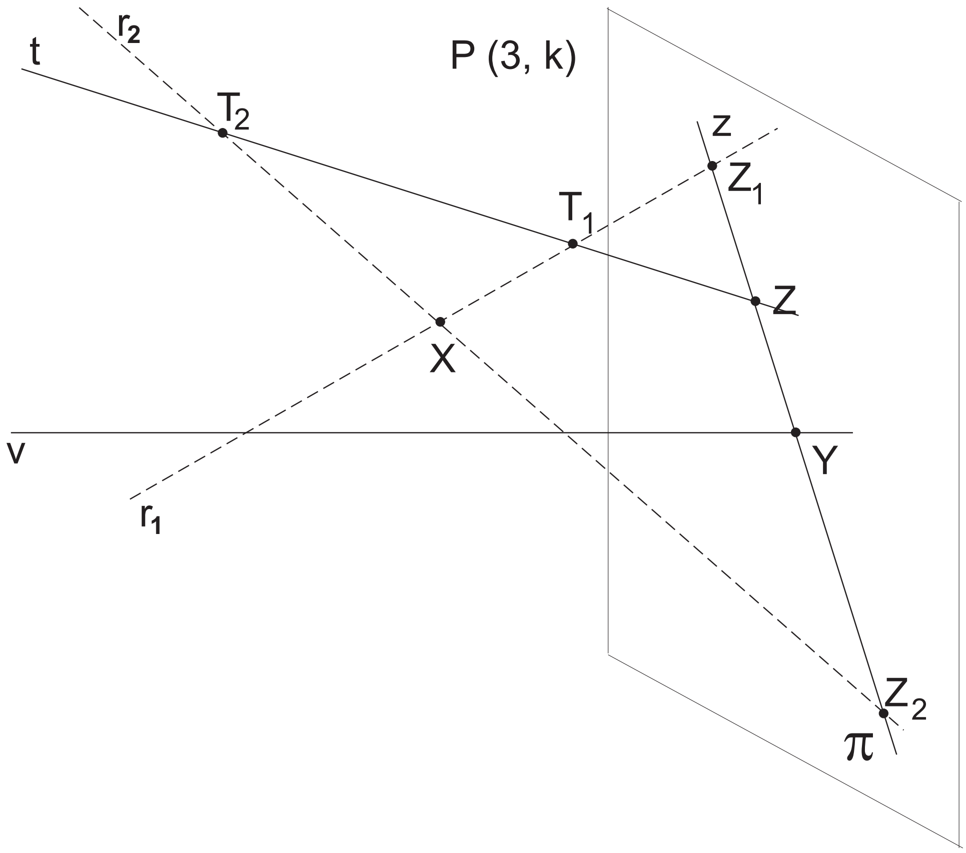

Figure 1. The Veblen configuration.

The line Z_1T_1 is represented in $AG(2,q)$ by the ordered pair of distinct points (A', B'), with $A' = \alpha \cap z$, $B' = b \cap z_1$. The line T_2Z_2 is represented in $AG(2,q)$ by the ordered pair of distinct points (A'', B''), with $A'' = a' \cap z_2$, $B'' = b' \cap z_2$. By the Veblen axiom in $PG(3,q)$, the lines Z_1T_1 and Z_2T_2 meet at a point X which does not belong either to v, or to π, therefore such a point is represented by an ordered pair of parallel and distinct lines of $AG(2,q)$. It follows that the line $A'A''$ and the line $B'B''$ of $AG(2,q)$ are parallel (see Fig. 2), since the line represented by the ordered pair (A', B') and the line represented by the ordered pair (A'', B'') must have a point in common, which is necessarily represented by an ordered pair of parallel and distinct lines.

The configuration obtained in this way in $AG(2, q)$ is the affine plane Desargues configuration. Conversely, let us consider the affine Desargues configuration in $AG(2, q)$, as in Fig. 2. The line t of $PG(3, q)$ represented by the ordered pair of distinct points A and B of z' and the line z of π containing the points represented by the lines z, z', z_2 meet at the point Z, represented by the line z'. The line r_1 of $PG(3, q)$ (see Fig. 1) represented in $AG(2, q)$ by the ordered pair (A', B') and the line r_2 of $PG(3, q)$, represented by the ordered pair (A'', B'') both meet z and t. By the Desargues theorem the lines $B'B''$ and $A'A''$ of $AG(2, q)$ are parallel. It follows that the lines r_1 and r_2 meet at X, represented by the ordered pair of parallel and distinct lines $A'A''$ and $B'B''$. Therefore, the Desargues configuration (see Fig. 2) changes to the Veblen configuration of $PG(3, q)$ (see Fig. 1). So, the geometric equivalence of Veblen and Desargues configurations is proved. ∎

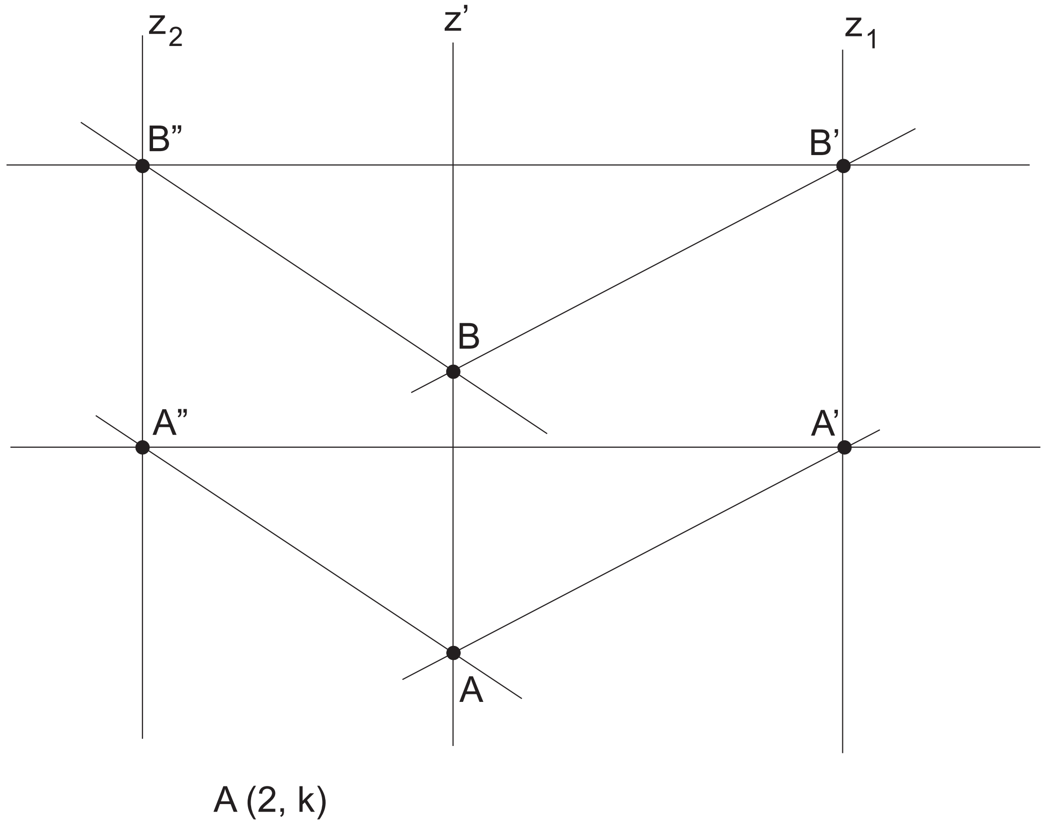

Figure 2. The Desargues affine configuration.

2. The three stars theorem in $PG(3, q)$ and its equivalence with the Pappus–Pascal theorem in the affine plane $AG(2, q)$

Theorem 1. The "three stars theorem". *Let $PG(3, q)$ be the projective three dimensional space over the field k. Let v be a line of $PG(3, q)$ and let π be a plane not through v meeting v at a point Y. Let U_1 and U_2 be two points of v, distinct between them and from Y. Let A_1 and A_2 be two distinct points of π such that the line $A_1A_2 = d$ does not contain Y. Let s_1, s_2, s_3 be three lines of π through A_1 distinct between them and from d and not through Y. Let s'_1, s'_2, s'_3 be three lines of π not through A_2, distinct between them and from d and not through Y. Denoting by $\gamma(U_i, s_j)$ the plane through U_i and the line s_j $(i = 1, 2,\ j = 1, 2, 3)$, assume that the following two conditions are satisfied:*

1) *The planes $\gamma(U_1, s_1)$, $\gamma(U_2, s_2)$, $\gamma(U_1, s'_2)$ and $\gamma(U_2, s'_1)$ belong to a star.*
2) *The planes $\gamma(U_1, s_2)$, $\gamma(U_2, s_3)$, $\gamma(U_1, s'_3)$ and $\gamma(U_2, s'_2)$ belong to a star* (see Fig. 3).

Then, the four planes $\gamma(U_1, s_3)$, $\gamma(U_2, s_1)$, $\gamma(U_1, s'_1)$ and $\gamma(U_2, s'_3)$ belong to a star.

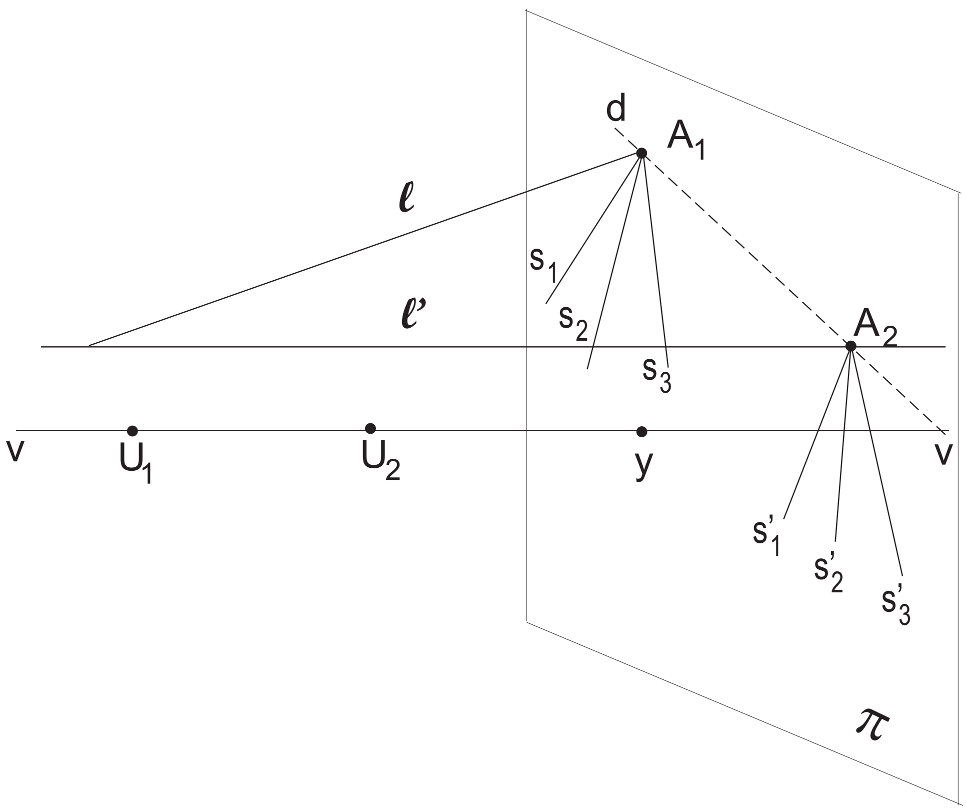

Figure 3.

Proof. Let r_1 and r_2 be the distinct lines of $AG(2,q)$ representing the points A_1 and A_2, respectively (see [1]). Such lines are not parallel, since the line d does not pass through Y and so they meet at a point D. The lines s_1, s_2, s_3 are represented in $AG(2,q)$, by three pencils of lines with centres at D_1, D_2, D_3, respectively, which are distinct between them and from D and all contained in r_1.

Similarly, the lines s'_1, s'_2, s'_3 are represented in $AG(2,q)$ by three pencils of lines with centres D'_1, D'_2, D'_3, respectively, all contained in r_2. We remark that condition 1) is equivalent to the incidence of the lines $\ell = \gamma(U_1, s_1) \cap \gamma(U_2, s_2)$ and $\ell' = \gamma(U_1, s'_2) \cap \gamma(U_2, s'_1)$. Now, let us prove that the line ℓ, which belongs to the class a) of [1], in the crashing is represented by the ordered pair of distinct points (D_1, D_2). For, let S_1 be a point of s_1, distinct from $s_1 \cap s_2$ and let t be the line U_1S_1. Such a line t meets the line f of the class a), represented by the pair (D_1, D_2), since the dotted line t (which belongs to the class e) meets f at the point represented by the ordered pair (h_1, h'_1), where h_1 is the line representing S_1 and h'_1 is the line through D_2 and parallel to h_1.

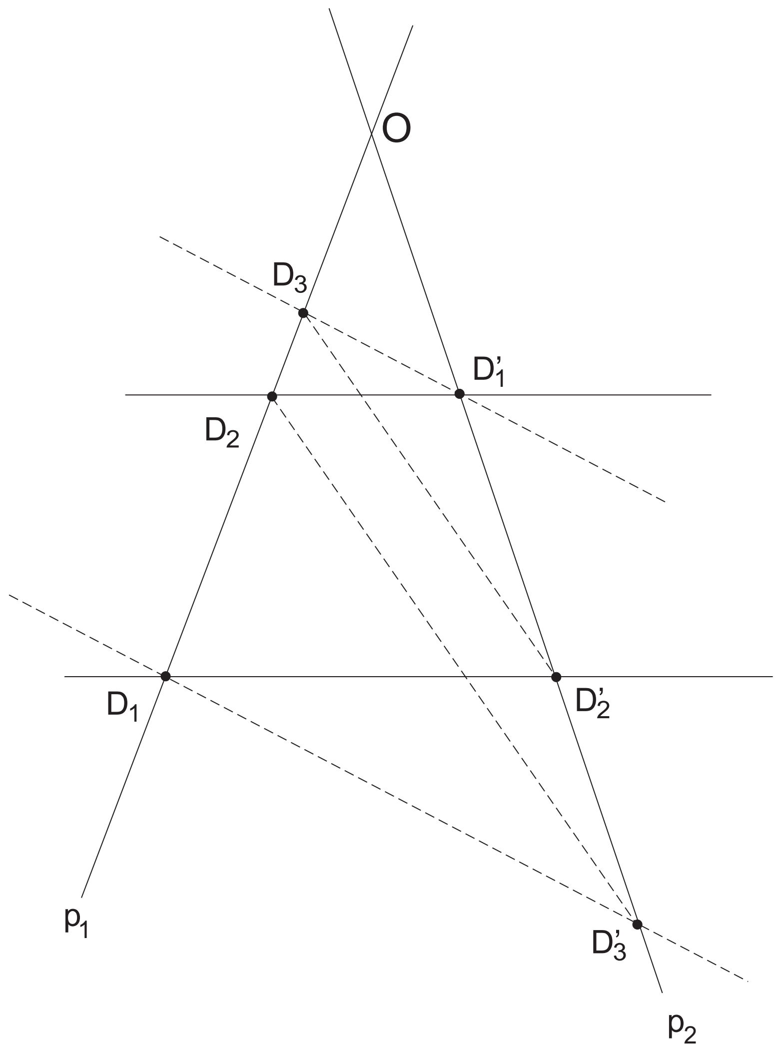

Figure 4.

By varying S_1 in $s_1 - (s_1 \cap s_2)$, the line t meets the line f. It follows that f is contained in the plane $\gamma(U_1, s_1)$. Similarly, we prove that f is contained in the plane $\gamma(U_2, s_2)$ and then $f = \gamma(U_1, s_1) \cap \gamma(U_2, s_2)$. Therefore, f coincides with ℓ. Similarly, ℓ' is represented by the ordered pair (D'_2, D'_1). Since ℓ and ℓ' meet and since $A_1 = \ell \cap \pi$, $A_2 = \ell' \cap \pi$ are two distinct points, by the representation of the lines of the class c) to which ℓ and ℓ' belong, it follows that $D_1D'_2$ and $D_2D'_1$ are parallel, since the ordered pair $(D_1D'_2, D_2D'_1)$ represents the point $\ell \cap \ell'$. In a similar way we prove that the condition 2) implies the parallelism between the lines D'_3D_2 and $D_2D'_3$ (see Fig. 4). The conditions 1) and 2) of Theorem 1 in $AG(2, q)$ by using the crashing become the conditions of the Pappus–Pascal configuration, which are the following (see Fig. 4): given two lines p_1, p_2 of $AG(2, q)$ meeting at O, let D_1, D_2, D_3 be three distinct points different from O of p_1. Let D'_1, D'_2, D'_3 be three distinct points not coincident with O of p_2. Let the lines $D_2D'_1$, $D_1D'_2$ be parallel like the lines $D_3D'_2$, $D_2D'_3$. Then, $D_3D'_1$ and $D_1D'_3$ are parallel. From such a parallelism it follows that the line h_1 of $PG(3, q)$ represented by the ordered pair (D_3, D_1) meets the line h_2, represented by the ordered pair (D'_1, D'_3).

Moreover, $h_1 = \gamma(U_1, s_3) \cap \gamma(U_2, s_1)$ and $h_2 = \gamma(U_1, s'_1) \cap \gamma(U_2, s'_3)$ similarly to the case of the lines s_1 and s_2. By the incidence of the lines h_1 and h_2 the thesis follows, that is, the fact that the planes $\gamma(U'_1, s_3)$, $\gamma(U_2, s_1)$, $\gamma(U_1, s'_1)$, $\gamma(U_2, s''_3)$ belong to the same star.

References

[1] Scafati Tallini, M., *Representation of the projective space $P(r, k)$ in the affine plane $A(2, k)$*, Proc. Conference on Error-Correcting Codes, Cryptography and Finite Geometries, Amer. Math. Soc. (Eds. A. Bruen and D. Wehlan) (2010), 109-122.

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THREE REPRESENTATIONS OF A HYPERBOLIC QUADRIC OF $PG(3,q)$ IN $AG(2,q)$

Maria Scafati Tallini

Summary. We construct three different representations of a hyperbolic quadric of a projective Galois space PG(3,q) in the affine Galois plane $AG(2,q)$. To do this, we use the representation R, or $R(U_1,U_2,\pi,3)$ of the projective space $P(r,k)$, over the field k, in the affine plane $A(2,k)$, over the same field k, called also "Crashing", cited in the bibliography [1]. Further applications of this representation are the construction of maximal partial line spreads in PG, q even, a geometric proof of the equivalence between the Desargues and the Veblen theorems and a geometric proof of the equivalence between the Pappus-Pascal theorem and the "Three stars theorem". Those results will soon appear.

1. First representation

Theorem 1. Theorem of the hyperbola and the hyperbolic quadric. *Let $PG(3,q)$ be the projective space of dimension three over the Galois field $GF(q)$, let $AG(2,q)$ be the affine plane over the same field and let R be an $R(U_1,U_2,\pi,3)$-representation of $PG(3,q)$, as in* [1]. *Let $\mathcal{I}$ be a hyperbola of $AG(2,q)$ and let t_1 and t_2 be the asymptots of $\mathcal{I}$. Let T_1 and T_2 be the points of π represented through R by the lines t_1 and t_2 of $AG(2,q)$, respectively. For any point X of $\mathcal{I}$, let X_1 be the point common to t_1 and to the line through X, parallel to t_2, let X_2 be the point common to the line t_2 and to the line through X parallel to t_1, let ℓ_X be the line of $PG(3,q)$ represented through R, by the ordered pair (X_1,X_2) and, finally, let $\overline{\ell}_X$ be the line of $PG(3,q)$ represented through R by the ordered pair (X_2,X_1). Then, the following sets of $PG(3,q)$:*

$$\begin{aligned}\mathcal{R} &= \{\ell_X\}_{X\in\mathcal{I}} \cup \{U_1T_2, U_2T_1\},\\ \overline{\mathcal{R}} &= \{\overline{\ell}_X\}_{X\in\mathcal{I}} \cup \{U_1T_1, U_2T_2\},\end{aligned}$$

where U_iT_j, $i,j=1,2$, denotes the line of $PG(3,q)$ through the points U_i and T_j, are the two reguli of a hyperbolic quadric of $PG(3,q)$ meeting π at a non-degenerate conic, admitting v as a secant line.

Proof. Let $PG(3,q)$ be the three dimensional projective space over $GF(q)$, let $AG(2,q)$ be the affine plane over $GF(q)$ and let R be an $R(U_1,U_2,\pi,3)$-representation of $PG(3,q)$. Let $\mathcal{I}$ be a hyperbola of $AG(2,q)$, let t_1 and t_2 be the asymptots of $\mathcal{I}$ and let O be the common point of t_1 and t_2. For any $i = 1, 2$, the line t_i represents, through R, a point T_i of $\pi - \{Y\}$ (see Fig. 1), where Y is the point $U_1U_2 \cap \pi$. The line of π joining T_1 and T_2 does not contain Y, since in $AG(2, q)$ the lines t_1 and t_2 meet at O. It follows that the lines U_1T_1 and U_2T_2 of $PG(3, q)$ are skew, like the lines U_1T_2 and U_2T_1. Obviously, the four lines U_iT_j, $i, j = 1, 2$, form a skew quadrangle, denoted by Q.

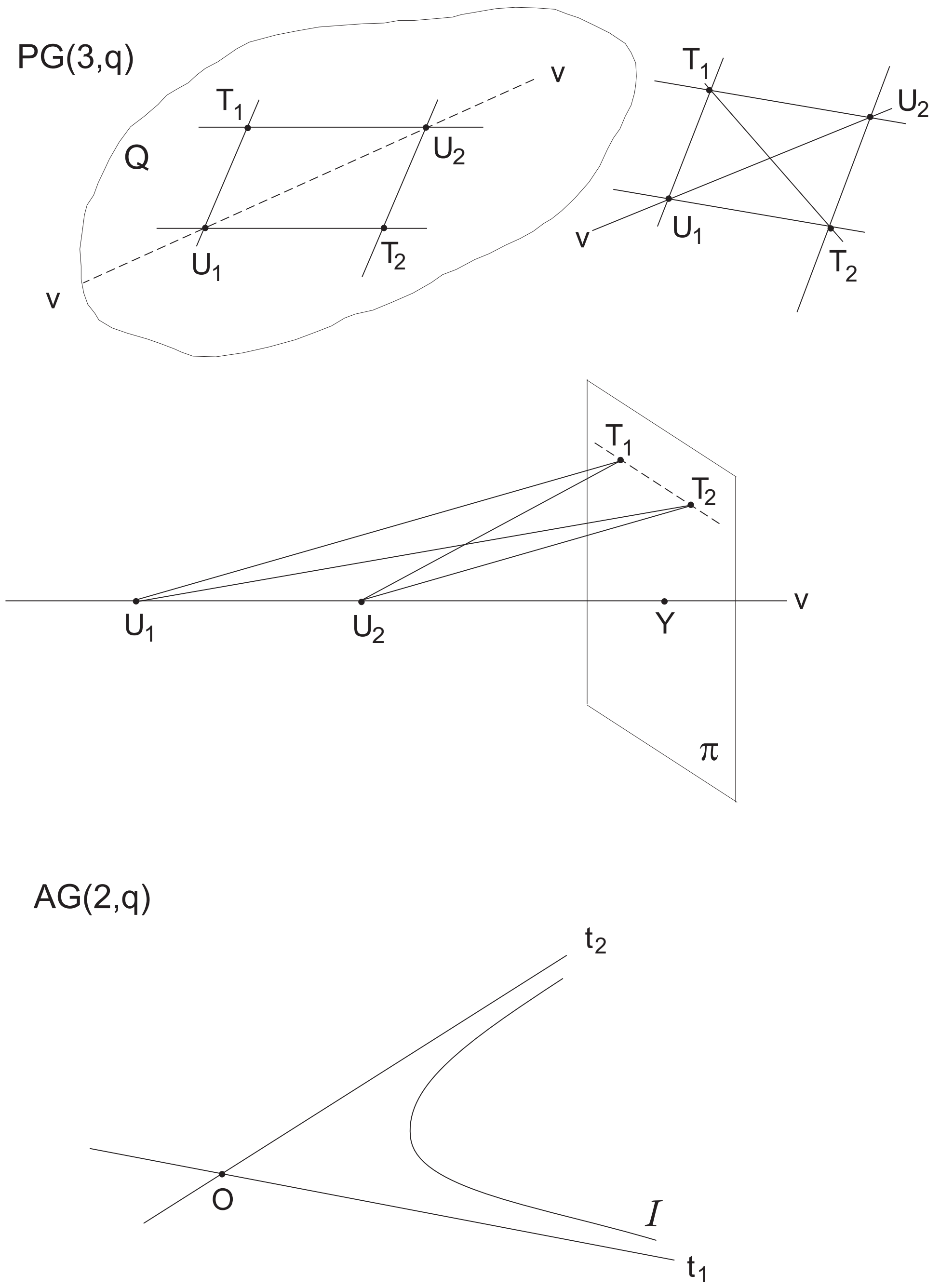

Figure 1.

The lines U_iT_j, $i,j=1,2$, of $PG(3,q)$ are represented by R in $AG(2,q)$ in the following way:

$$\begin{aligned} U_1T_1 &: \{(t_1\} \cup \{(t_1,t)\}_{t\in\mathcal{T}_1}, \\ U_1T_2 &: \{(t_2\} \cup \{(t_2,t)\}_{t\in\mathcal{T}_2}, \\ U_2T_1 &: \{(t_1\} \cup \{(t,t_1)\}_{t\in\mathcal{T}_1}, \\ U_2T_2 &: \{(t_2\} \cup \{(t,t_2)\}_{t\in\mathcal{T}_2}, \end{aligned}$$

where $\mathcal{T}_i, i=1,2$, is the set of the lines of $AG(2,q)$ parallel to t_i and distinct from t_i.

Now, let A be a point of $\mathcal{I}$ (see Fig. 2).

Let t_1' be the line of $AG(2,q)$ through A and parallel to t, and t_2' the line through A parallel to t_2. Moreover, let $A_1 = t_1 \cap t_2'$, $A_2 = t_2 \cap t_1'$. It is $A_1 \neq A_2$, since $A_i \in t_i - \{O\}$, for any $i=1,2$. The ordered pair of distinct points (A_1, A_2) of $AG(2,q)$ represents, by R, a line ℓ of $PG(3,q)$ not meeting v and not in π (see Fig. 2). By the representations of ℓ and U_iT_j, $i,j=1,2$, in $AG(2,q)$, we get:

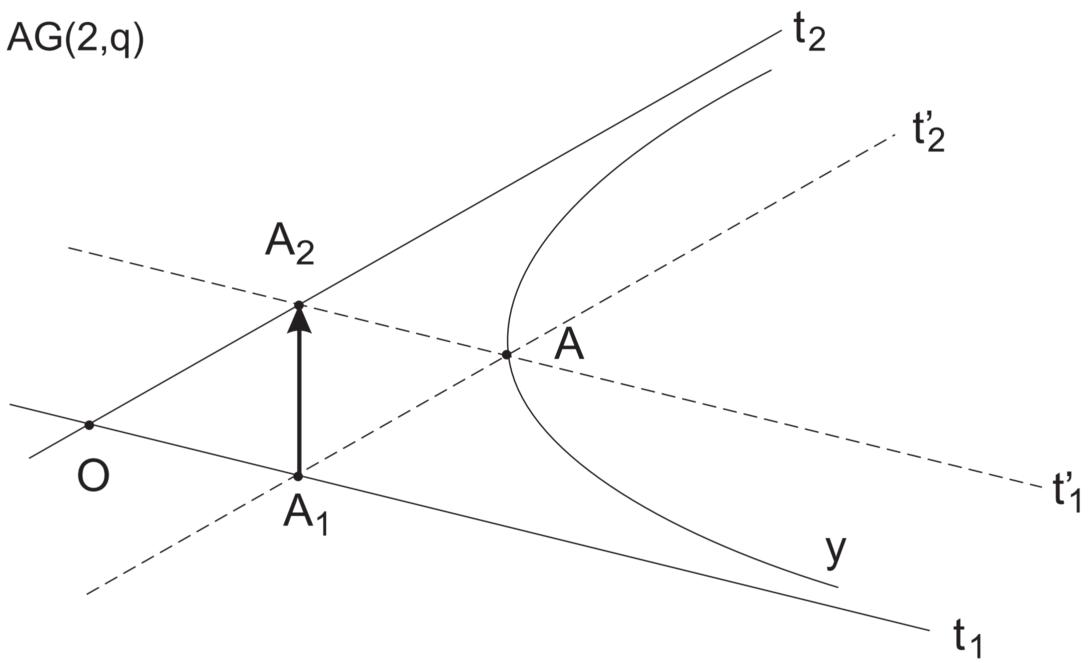

Figure 2.

1) The line ℓ meets the line U_1T_1 at L' represented by the ordered pair (t_1, t'_1). Such a point L' is distinct from U_1 and T_1, because the ordered pairs of distinct lines of $AG(2, q)$ represent the points of $PG(3, q)$ not in π (see Fig. 3).

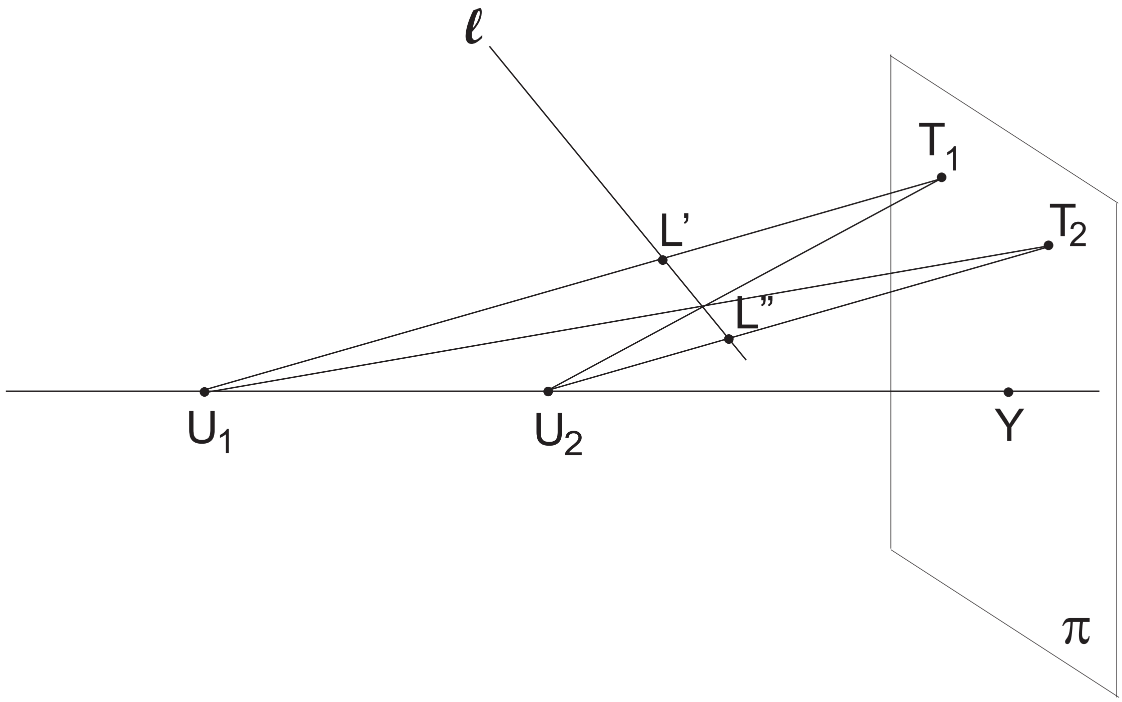

Figure 3.

2) The line ℓ meets the line U_2T_2 at the point L'' represented by the ordered pair (t'_2, t_2) and such a point is distinct from U_2 and T_2.

3) The line ℓ does not meet either U_1T_2, or U_2T_1.

By 3) and since the lines U_2T_1 and U_1T_2 of $PG(3, q)$ are mutually skew, it follows that the lines ℓ, U_1T_2 and U_2T_1 are two by two skew. Let us denote by $\mathcal{H}$ the hyperbolic quadric of $PG(3, q)$ containing ℓ, U_1T_2 and U_2T_1. Then, call $\mathcal{R}$ the regulus of $\mathcal{H}$ containing ℓ, U_1T_2 and U_2T_1. By 1) and 2), it follows that U_1T_1 and U_2T_2 belong to the regulus $\overline{\mathcal{R}}$ of $\mathcal{H}$ opposite to $\mathcal{R}$. The ordered pair (A_2, A_1) represents a line $\overline{\ell}$ of $PG(3, q)$ not meeting v and not in π.

By the representations of $\overline{\ell}$ and U_iT_j, $i, j = 1, 2$, in $AG(2, q)$, we get:

4) The line $\overline{\ell}$ meets U_2T_1, at the point $\overline{L}'$, represented by the ordered pair (t'_1, t_1); such a point $\overline{L}'$ is distinct from U_2 and T_1.

5) The line $\overline{\ell}$ meets U_1T_2, at the point $\overline{L}''$, represented by the ordered pair (t_2, t'_2); such a point is distinct from U_1 and T_2.

6) The line $\overline{\ell}$ does not meet either U_1T_1, or U_2T_2.

7) The line $\overline{\ell}$ meets ℓ at the point P of π represented by the line A_1A_2 of $AG(2, q)$.

By 4), 5), 6) and 7) it follows that the line $\overline{\ell}$ is a line of $\overline{\mathcal{R}}$ distinct from U_1T_1 and U_2T_2 (see Fig. 4).

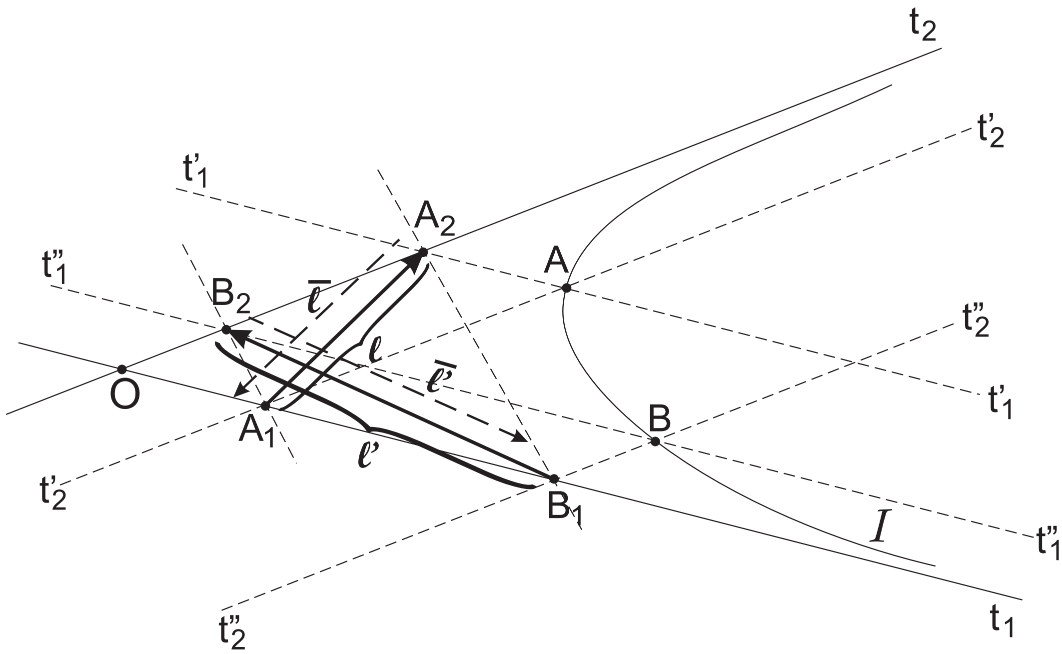

Figure 4.

Now, let B be a point of $\mathcal{I} - \{A\}$. Let t_1'' be the line of $AG(2, q)$ through B and parallel to t_1 and let t_2'' be the line of $AG(2, q)$ through B parallel to t_2. Let $B_1 = t_1 \cap t_2''$ and $B_2 = t_2 \cap t_1''$. The ordered pair of distinct points (B_1, B_2) represents a line ℓ' of $PG(3, q)$ not meeting v and not in π. Such a line meets U_1T_1 at the point of $PG(3, q)$ represented by the ordered pair (t_1, t_1''). The line ℓ' meets U_2T_1 at the point $PG(3, q)$ represented by the ordered pair (t_2'', t_2). Let us prove that ℓ' meets $\overline{\ell}$. To do this, choose a coordinate system in $AG(2, q)$ such that $T = T(0, 0)$, $A_1 = A_1(1, 0)$, $A_2 = A_2(0, 1)$. In such a system, the coordinates of the point A are $(1, 1)$ and the hyperbola $\mathcal{I}$ has the equation $xy = 1$. It follows that

$$B = B\left(x_0, \frac{1}{x_0}\right),$$

$$B_1 = B_1(x_0, 0),$$

$$B_2 = B_2\left(0, \frac{1}{x_0}\right),$$

with $x_0 \neq 0$. The slopes of the lines A_1B_2 and A_2B_1 are both equal to $a - \dfrac{\ell}{x_0}$, therefore such two lines are parallel. It follows that ℓ' meets $\overline{\ell}$. Since ℓ' meets U_1T_1, U_2T_2 and $\overline{\ell}$ which belong to $\overline{\mathcal{R}}$, it follows that $\ell' \in \mathcal{R}$. Similarly, we prove that the line ℓ' of $PG(3,q)$ represented by the ordered pair (B_2, B_1) belongs to $\overline{\mathcal{R}}$.

For any $X \in \mathcal{I}$, let X_1 be the point common to t_1 and to the line through X parallel to t_2 and let X_2 be the point common to t_2 and to the line through X parallel to t_1 (see Fig. 5).

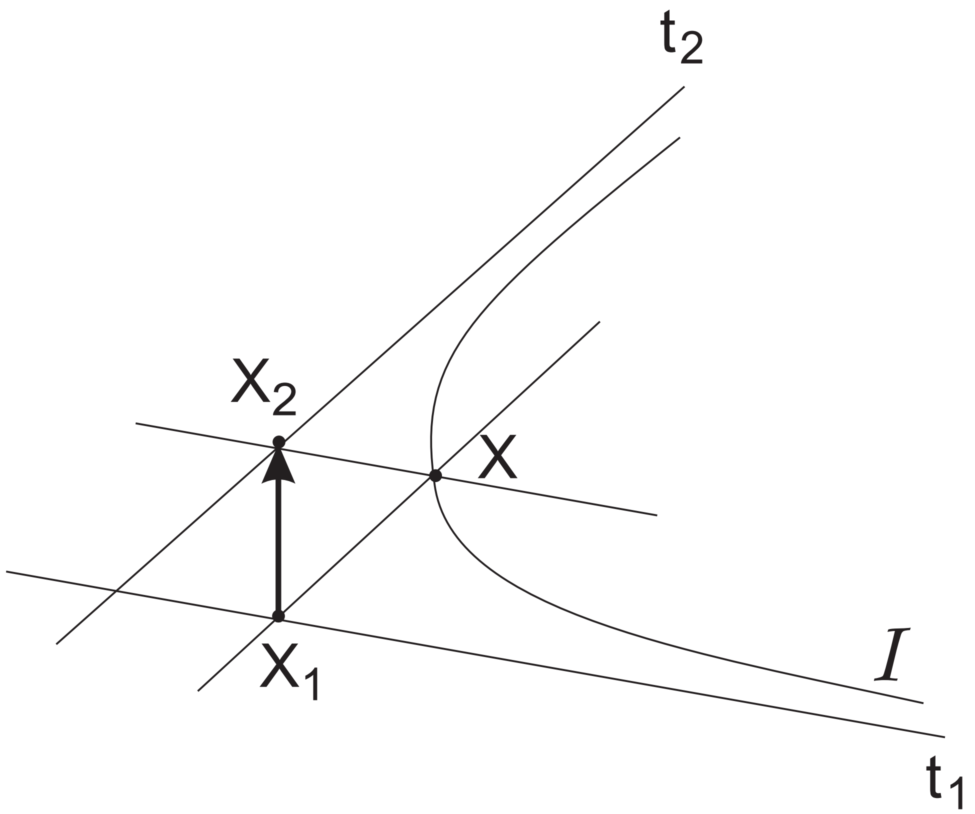

Figure 5.

Let ℓ_X be the line of $PG(3,q)$ represented by the ordered pair (X_1, X_2) and let $\overline{\ell}_X$ the line represented by the ordered pair (X_2, X_1). By the previous results, we get:

$$\begin{aligned} \mathcal{F}_1 &= \{\ell_X\}_{x\in\mathcal{I}} \subset \mathcal{R}, \\ \overline{\mathcal{F}}_1 &= \{\overline{\ell}_X\}_{x\in\mathcal{I}} \subset \overline{\mathcal{R}}. \end{aligned}$$

The above inclusions are proper, since there are lines of $\mathcal{R}$ not in $\mathcal{F}_1$ (U_1U_2 and U_2T_1) and lines of $\overline{\mathcal{R}}$ not in $\overline{\mathcal{F}}_1$ (U_1T_1 and U_2T_2).

We remark that there is no line of $\mathcal{H}$ contained in π. For, let b a line of π and in $\mathcal{H}$. then, either $b \in \mathcal{R}$, or $b \in \overline{\mathcal{R}}$.

Let $b \in \mathcal{R}$. The line b meets $\overline{\ell}$ (because b and $\overline{\ell}$ belong to opposite reguli of $\mathcal{H}$). Since that and since $b \subset \pi$, it follows that b meets $\overline{\ell}$ at the point P common to $\overline{\ell}$

and π, represented by R in $AG(2,q)$ by the line A_1A_2. But such a point P belongs also to the line b. Therefore, ℓ and b have P in common. Since ℓ and b are lines of the same regulus $\mathcal{R}$ of $\mathcal{H}$, it follows $b = \ell$: a contradiction, since ℓ is not a line of π, while $b \subset \pi$. The contradiction proves that $b \notin \mathcal{R}$. Similarly, we prove that $b \notin \overline{\mathcal{R}}$. So, we get a contradiction, because from $b \subset \mathcal{H}$, it follows that $b \in \mathcal{R} \cup \overline{\mathcal{R}}$. The contradiction proves that there is no line of $\mathcal{H}$ contained in π. So, the remark is proved.

From this remark it follows that $\mathcal{H}$ meets π at a non-degenerate conic. Obviously, every line of $\mathcal{H}$ is a line of $\mathcal{R}$ not meeting v, while every line of $\overline{\mathcal{F}}$ is a line of $\overline{\mathcal{R}}$ not meeting v.

Now, let us prove that every line of $\mathcal{R}$ not meeting v is a line of $\mathcal{F}$.

Let $\widetilde{\ell}$ be a line of $\mathcal{R}$ not meeting v. Since $\widetilde{\ell}$ does not meet v and, since $\widetilde{\ell}$ is not a line of π (we already proved that no line of $\mathcal{H}$ is contained in π), it follows that $\widetilde{\ell}$ is represented by an ordered pair (L_1, L_2) of distinct points of $AG(2,q)$. The line $\widetilde{\ell}$, which belongs to $\mathcal{R}$, meets U_1T_1, U_2T_2 and $\overline{\ell}$, which belong to $\overline{\mathcal{R}}$. By the representations of $\widetilde{\ell}, U_1T_1$ and U_2T_2 and since $\widetilde{\ell}$ meets U_1T_1 and U_2T_2, we get

$$L_1 \in t_1,\ L_2 \in t_2.$$

We remark that $L_1 \neq O$. In fact, if $L_1 = O$, the distinct points L_1 and L_2 belong both to t_2 and $\widetilde{\ell}$ contains T_2. It follows that $\widetilde{\ell} = U_1T_2$, since $\widetilde{\ell} \in \mathcal{R}$, $U_1T_2 \in \mathcal{R}$, $T_2 \in \widetilde{\ell}$, $T_2 \in U_1T_2$: a contradiction, since $\widetilde{\ell}$ does not meet v, while U_1T_2 meets v and U_1. The contradiction proves the remark. Similarly, we prove that $L_2 \neq O$.

By the above remark and since $L_1 \in t_1$, $L_2 \in t_2$, it follows

$$L_1 \in t_1 - \{O\},\ L_2 \in t_2 - \{O\}.$$

By the previous results, it follows immediately that $L_1 \neq A_2$, $L_2 \neq A_1$.

As $\widetilde{\ell}$ meets $\overline{\ell}$, it follows that in $AG(2,q)$ the line L_1A_2 is parallel to L_2A_1 (maybe coinciding with it). Let L be the point of $AG(2,q)$ common to the line through L_2 parallel to t_1 and to the line through L_1 parallel to t_2. Let us prove that $L \in \mathcal{I}$. In the coordinate system that we chose before, let m be the slope of the lines parallel to A_1L_2 and A_2L_1. Such a slope does exist, since $L_1 \in t_1 - \{O\}$ and it is different from zero because $L_2 \in t_2 - \{O\}$. The points L_1 and L_2 have coordinates $L_1\left(-\dfrac{1}{m}, 0\right)$, $L_2(0,-m)$. It follows that L has coordinates $\left(-\dfrac{1}{m}, -m\right)$ and then $L \in \mathcal{I}$ (remember that in our coordinate system the hyperbola $\mathcal{I}$ has the equation $xy = 1$). By the above results and by the definition of $\mathcal{F}$, it follows that $\widetilde{\ell} \in \mathcal{F}$. So, every line of $\mathcal{R}$ not meeting v is a line of $\mathcal{F}$. So, the result is proved. Similarly, we prove that every line of $\overline{\mathcal{R}}$ not meeting v is a line of $\overline{\mathcal{F}}$. It follows that all the lines of $\mathcal{F}$ coincide with the lines of $\mathcal{R}$ not meeting v, while the lines of $\overline{\mathcal{F}}$ coincide with the lines of $\overline{\mathcal{R}}$ not meeting v. We remark that the lines U_2T_2 and U_2T_1 coincide with the lines of $\mathcal{R}$ meeting v. For, U_1T_2 and U_2T_1 are lines of $\mathcal{R}$ meeting v. Conversely, every line of $\mathcal{R}$ meeting v coincides either with U_1T_2, or with U_2T_1. For, let $\ell_{\mathcal{R}}$ be a line of $\mathcal{R}$ meeting v distinct from U_1T_2 and U_2T_1. Then, the point $L = \ell_{\mathcal{R}} \cap v$ is distinct from either U_1, or U_2. Then the line v, having three distinct points in common with $\mathcal{H}$, is a line of $\mathcal{H}$. It follows $v \in \overline{\mathcal{R}}$, since v meets $\ell_{\mathcal{R}}$, U_1T_2 and U_2T_1, belonging to $\mathcal{R}$. The line v meets also $U_2T_2 \in \overline{\mathcal{R}}$. It follows that $v = U_2T_2$, a contradiction, since $T_2 \neq Y$. The contradiction proves that no line of $\mathcal{R}$ meeting v and distinct from U_1T_2 and U_2T_1 exists, whence every line of $\mathcal{R}$ meeting v coincides either with U_1T_2, or with U_2T_1. So, the remark is proved. Similarly, we prove that U_1T_1 and U_2T_2 coincide with the lines of $\overline{\mathcal{R}}$ meeting v. By the above arguments, it follows that

$$\begin{aligned} \mathcal{R} &= \mathcal{F} \cup \{U_1T_2, U_2T_1\}, \\ \overline{\mathcal{R}} &= \overline{\mathcal{F}} \cup \{U_1T_1, U_2T_2\}. \end{aligned}$$

As all the hyperbolic quadrics of $PG(3,q)$ are equivalent, it follows that for every hyperbolic quadric $\mathcal{H}$ of $PG(3,q)$ there is a representation $R(U_1, U_2, \pi, 3)$ of $PG(3,q)$ which represents $\mathcal{H}$ by a hyperbola of $AG(2,q)$.

2. Second representation

Let $AG(2,q)$ be the affine plane over the Galois field $GF(q)$. In $AG(2,q)$, let t_1 and t_2 be two distinct lines meeting at a point O. Let A be a point of $t_1 - \{O\}$ (see Fig. 6), let B be a point of $t_2 - \{O\}$ and, finally, let t_3 be the line through A and B.

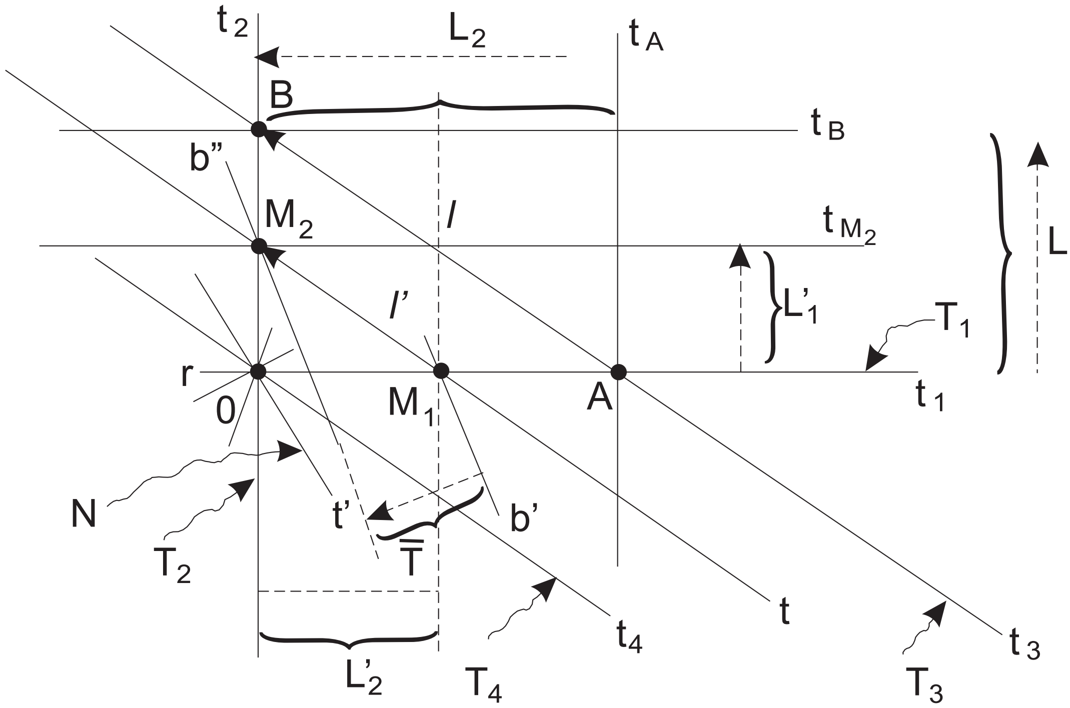

Figure 6.

From now on, the symbol d_{MN} denotes the direction of the line of $AG(2,q)$ through the distinct points M and N. Let t_4 be the line through O with direction d_{AB}. Let R be an $R(U_1, U_2, \pi, 3)$-representation of $PG(3,q)$ (see [1]). Let ℓ be the line of $PG(3,q)$ represented by the ordered pair of distinct points (A, B). Let r be the line of π (not through Y) represented by the proper pencil of lines with centre O. The lines v, r and ℓ are two by two skew. Let $\mathcal{H}$ be the hyperbolic quadric of $PG(3,q)$ containing v, r and ℓ and let $\mathcal{R}$ be the regulus of $\mathcal{H}$ determined by v, r, ℓ and $\overline{\mathcal{R}}$ the opposite regulus. Let T_1 be the point of $\pi - \{Y\}$ represented by the line t_1 and let T_2 be the point of $\pi - \{Y\}$ represented by the line t_2. Let F be the improper pencil (that is the pencil of parallel lines) consisting of the lines of $AG(2,q)$ with direction d_{AB} and let z be the line of π (through Y) represented by F. The line z meets v at Y, meets r at T_4 represented by the line t_4 and meets *ell* at the point T_3, represented by the line t_3. It follows that z is a line of $\overline{\mathcal{R}}$. The line U_1T_1 of $PG(3,q)$ meets v at the point U_1, meets r at T_1 and ℓ at L_2 represented by the ordered pair (t_1, t_B), where t_B is the line of $AG(2,q)$ through B and parallel to t_1. It follows that $U_1T_1 \in \overline{\mathcal{R}}$ (see Fig. 7).

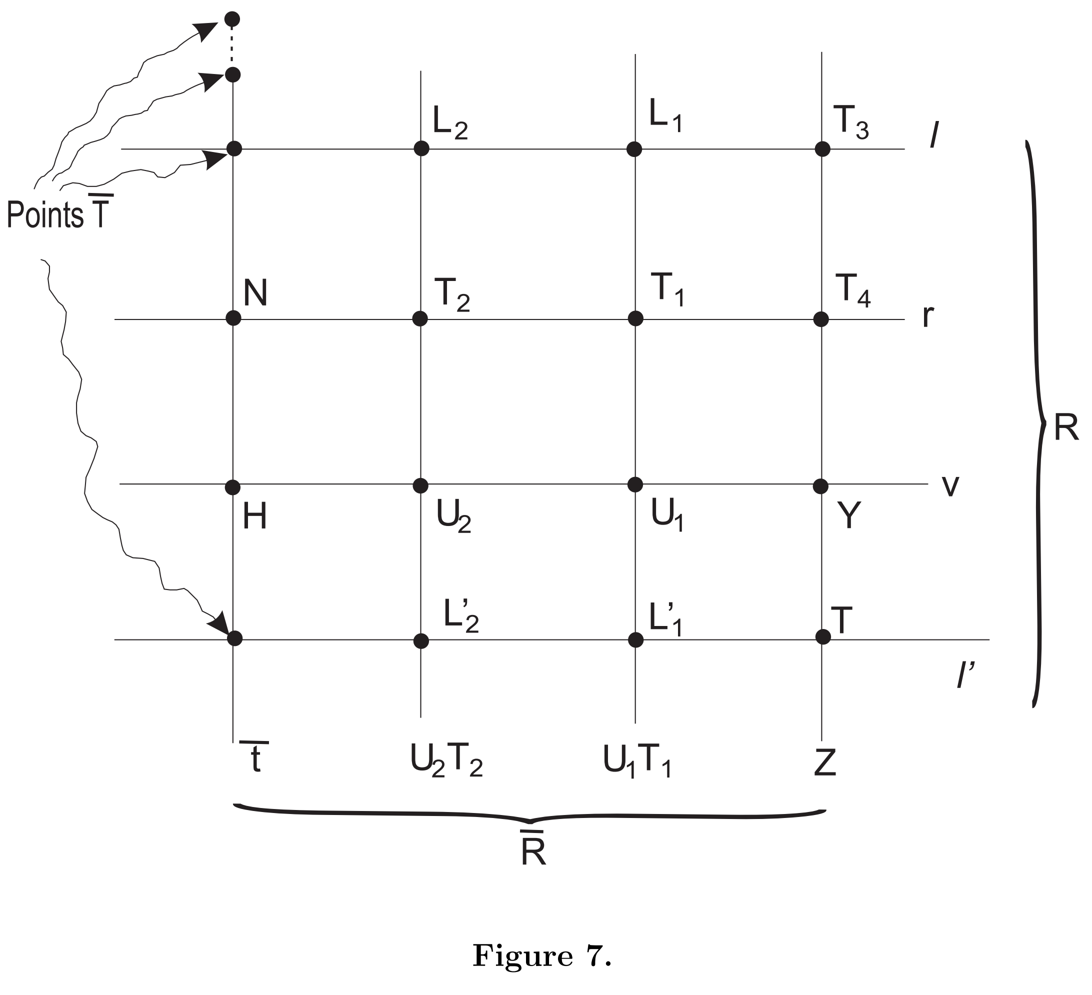

Figure 7.

Now, let M_1 and M_2 be two points of $AG(2,q)$ mutually distinct and such that $M_1 \in t_1 - \{O\}$, $M_2 \in t_2 - \{O\}$, $d_{M_1M_2} = d_{AB}$. Let t be the line of $AG(2,q)$ through M_1 and M_2. Let ℓ' be the line of $PG(3,q)$ represented by the ordered pair (M_1, M_2). The line ℓ' meets z at T, represented by the line t, meets U_1T_1 at L_1', represented by the ordered pair (t_1, t_{M_2}), where t_{M_2} is the line of $AG(2,q)$ through M_2 and parallel to t_1, meeting U_2T_2 at the point L_2', represented by the ordered pair (t_{M_1}, t_2), where t_{M_1} is the line of $AG(2,q)$ through M_1 and parallel to t_2. It follows that $\ell' \in \mathcal{R}$. By the above arguments it follows that every ordered pair of distinct points (M_1, M_2), with $M_1 \in t_1 - \{O\}$, $M_2 \in t_2 - \{O\}$, $d_{M_1M_2} = d_{AB}$, represents a line of $\mathcal{R}$, distinct from v and r. Conversely, let $m \in \mathcal{R} - \{v, r\}$. The line m is not a line of π, since m and r are skew and r is a line of π. It follows that m is represented by an ordered pair of distinct points (M_1, M_2). The line m, as a line of $\mathcal{R}$, meets U_1T_1, U_2T_2 and z, which are lines of $\overline{\mathcal{R}}$. Since m meets z (at point distinct from Y), it follows that $d_{M_1M_2} = d_{AB}$. Since m meets U_1T_1, it follows that $M_1 \in T_1$, while, since m meets U_2T_2, it follows that $M_2 \in t_2$. Moreover, by $M_1 \neq M_2$, $d_{M_1M_2} = d_{AB}$ and since neither t_1, nor t_2 have the direction d_{AB}, it follows that $M_1 \in t_1 - \{O\}$ and $M_2 \in t_2 - \{O\}$. By the previous arguments, it follows that every line of $\mathcal{R} - \{v, r\}$ is represented by an ordered pair of distinct points (M_1, M_2), with $M_1 \in t_1 - \{O\}$, $M_2 \in t_2 - \{O\}$, $d_{M_1M_2} = d_{AB}$. So, we prove that the ordered pairs of distinct points (M_1, M_2), with $M_1 \in t_1 - \{O\}$, $M_2 \in t_2 - \{O\}$, $d_{M_1M_2} = d_{AB}$, represent exactly all the lines of $\mathcal{R} - \{v, r\}$. If we denote by $\ell_{M_1M_2}$ the line of $PG(3,q)$ represented in $AG(2,q)$ by the ordered pair of distinct points (M_1, M_2), we get

$$\begin{aligned}\mathcal{R} = \{v, r\} \cup \{\ell_{M_1M_2} : M_1 \in t_1 - \{O\},\\ M_2 \in t_2 - \{O\},\ d_{M_1M_2} = d_{AB}\}.\end{aligned}$$

Now, let α be a plane of $PG(3,q)$ containing v but not through U_1T_1, U_2T_2 and z. Such a plane α is tangent to $\mathcal{H}$ at a point $H \in v - \{Y, U_1, U_2\}$ and contains therefore the line $\bar{t}$ of $\overline{\mathcal{R}}$ through H. Moreover, α is spanned in $AG(2,q)$ by a direction d' distinct from d_{AB} and distinct either from the direction of t_1, or that of t_2. Consider the points of $\bar{t} - \{H\}$. They are the intersections of the lines of $\mathcal{R}$ distinct from v with the plane α. Such points are therefore represented as follows:

1) The point $N = \bar{t} \cap r = \alpha \cap r$ is represented by the line t' of $AG(2,q)$ through O and of direction d'.

2) Each point $\overline{T}$ of $\bar{t} - \{H, N\}$ is represented by an ordered pair (b', b''), where b' and b'' are the lines of $AG(2,q)$ with direction d' and through the points M_1, M_2 respectively, with

$$M_1 \in t_1 - \{O\},\ M_2 \in t_2 - \{O\},$$
$$d_{M_1M_2} = d_{AB}.$$

By varying of the direction d' in $\mathcal{D} - \{d_{AB}, d_1, d_2\}$, where $\mathcal{D}$ is the set of the directions of $AG(2, q)$ and d_1, d_2 are the directions of t_1 and t_2 respectively, we get the representations of all the lines of $\overline{\mathcal{R}} - \{z, U_1T_1, U_2T_2\}$, each of them being deprived of their point in common with v (see Fig. 8).

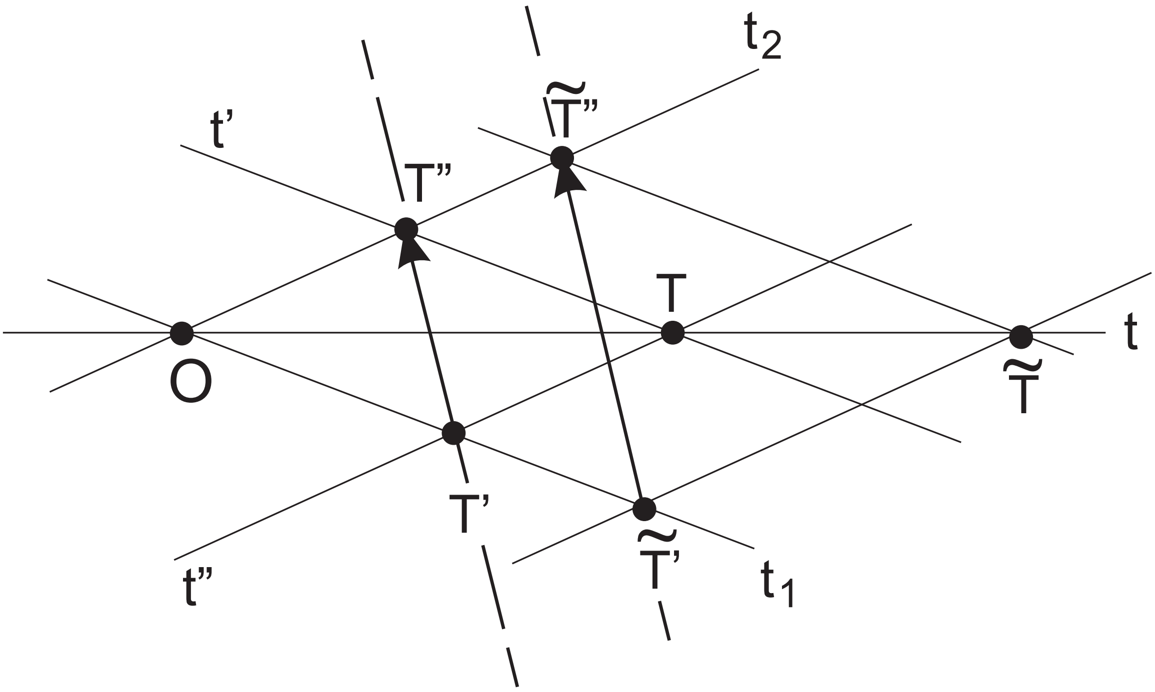

Figure 8.

Now, let t_1, t_2 and t be three distinct lines of $AG(2, q)$ through the same point O. Let T be a point of t distinct from O. Let t' be the line through R and parallel to t_1 and let $T'' = t_2 \cap t'$. Let t' be the lines through T parallel to t_2 and let $T' = t_1 \cap t''$. By the Desargues theorem it follows that the direction of $T'T''$ does not depend on T.

By that and by the previous arguments, it follows:

Theorem 2. *Let $PG(3, q)$ be the three-dimensional projective space over the field q and let $AG(2, q)$ be the affine plane over q. Let R be an $R(U_1, U_2, \pi, 3)$-representation of $PG(3, q)$. Let t_1, t_2 and t be three distinct lines of $AG(2, q)$ through the same point O and let d_1 and d_2 be the directions of t_1 and t_2, respectively. Let T_1 be the point of π represented by the line t_1 and let T_2 be the point of π represented by t_2. For any $T \in t - \{O\}$, let T'_T be the point common to the line t_1 and to the line through T parallel to t_2 and let T''_T be the point common to the line t_2 and to the line through T parallel to t_1. Then the direction d of the line joining T'_T and T''_T does not depend on the choice of T in $t - \{O\}$. Let $\ell(T'_T, T''_T)$*

be the line of $PG(3, q)$ *represented by the ordered pair of distinct points* (T'_T, T''_T) *and let* r *be the line of* π *represented by the proper pencil* (*that is the pencil of lines through the same point* O) *of lines with centre* O. *Then, the following set of lines*

$$\mathcal{R} = \{v, r\} \cup \{\ell(T'_T, T''_T)\}_{T \in t - \{O\}}$$

is the regulus of a hyperbolic quadric $\mathcal{H}$ *in* $PG(3, q)$.

Let z *be the line of* π *represented by the improper pencil* (*that is the pencil of parallel lines*) *of lines of* $AG(2, q)$ *with direction* d. *Finally, let* d' *be a direction of* $AG(2, q)$ *distinct from* d_1, d_2 *and* d *and let* n *be the line of* $AG(2, q)$ *through* O *and having direction* d'. *For any* $T \in t - \{O\}$, *let* $b'(T)$ *and* $b''(T)$ *be the lines of* $AG(2, q)$ *having direction* d' *and through* T'_T *and* T''_T, *respectively. The following set*

$$\{b'(T), b''(T)\}_{T \in t - \{O\}} \cup \{n\}$$

represents a line of the regulus $\overline{\mathcal{R}}$ *of* $\mathcal{H}$ *opposite to* $\mathcal{R}$, *deprived of its point in common with* v. *Such a line will be denoted by* $\ell(d')$. *Then the regulus* $\overline{\mathcal{R}}$ *is*

$$\overline{\mathcal{R}} = \{z, U_1T_1, U_2T_2\} \cup \{\ell(d')\}_{d' \in \mathcal{D} - \{d_1, d_2, d\}},$$

where $\mathcal{D}$ *is the set of the directions of* $AG(2, q)$.

Since the hyperbolic quadrics of $PG(3, q)$ are all equivalent, it follows that for any hyperbolic quadric $\mathcal{H}$ of $PG(3, q)$ there is an $R(U_1, U_2, \pi, 3)$-representation of $PG(3, q)$ which represents $\mathcal{H}$ by means of an ordered triple (t_1, t_2, t) of distinct lines of $AG(2, q)$ of the same pencil.

3. Third representation

Let $PG(3, q)$ be the three dimensional projective space over the field k and let $AG(2, q)$ be the affine plane over k. Let $\mathcal{S}$ be the set of points of $AG(2, q)$ and R an $R(U_1, U_2, \pi, 3)$-representation of $PG(3, q)$. In $AG(2, q)$ let t_1 and t_2 be two distinct lines meeting at O (see Fig. 9) and let r_0 be the line of π represented (see [1]), through R, by the proper pencil of lines of $AG(2, q)$ through O.

Let T_1 and T_2 be the points of π represented, through R, by the lines t_1 and t_2 of $AG(2, q)$. Let X be a point of $AG(2, q) - t_1 \cup t_2$ and let x be the line of π represented by the proper pencil of lines with centre X. Let I be the so defined set

$$I = \{(X_1, X_2) \in \mathcal{S} \times \mathcal{S} : X_1 \in t_1 - \{O\},\ X_2 \in t_2 - \{O\} \text{ and } X_1, X_2, X \text{ collinear}\}.$$

Denote by $\ell(X', X'')$ the line of $PG(3, q)$ represented by the ordered pair (X', X'') of distinct points of $AG(2, q)$, let

$$\mathcal{L} = \{\ell(X_1, X_2)\}_{(X_1, X_2) \in I}.$$

The lines U_1T_1 and U_2T_2 of $PG(3, q)$ are mutually skew, since the line of π through T_1 and T_2 does not contain $Y = \pi \cap v$ (t_1 and t_2 are not parallel). The line x and U_1T_1 are skew, like the line x and U_2T_2, since $x \in \pi$, $T_1 \notin x$, $T_2 \notin x$ ($X \notin t_1,\ X \notin t_2$). It follows that x, U_1T_1 and U_2T_2 are mutually skew.

Let $\mathcal{H}$ be the hyperbolic quadric of $PG(3,q)$ containing the lines x, U_1T_1 and U_2T_2 and let $\overline{\mathcal{R}}$ be the regulus of $\mathcal{H}$ to which such lines belong. The line r_0 meets x at the point of π represented by the line OX of $AG(2,q)$ and meets U_1T_1 and U_2T_2 at T_1 and T_2, respectively. It follows that r_0 belongs to the regulus $\mathcal{R}$ of $\mathcal{H}$ opposite to $\overline{\mathcal{R}}$.

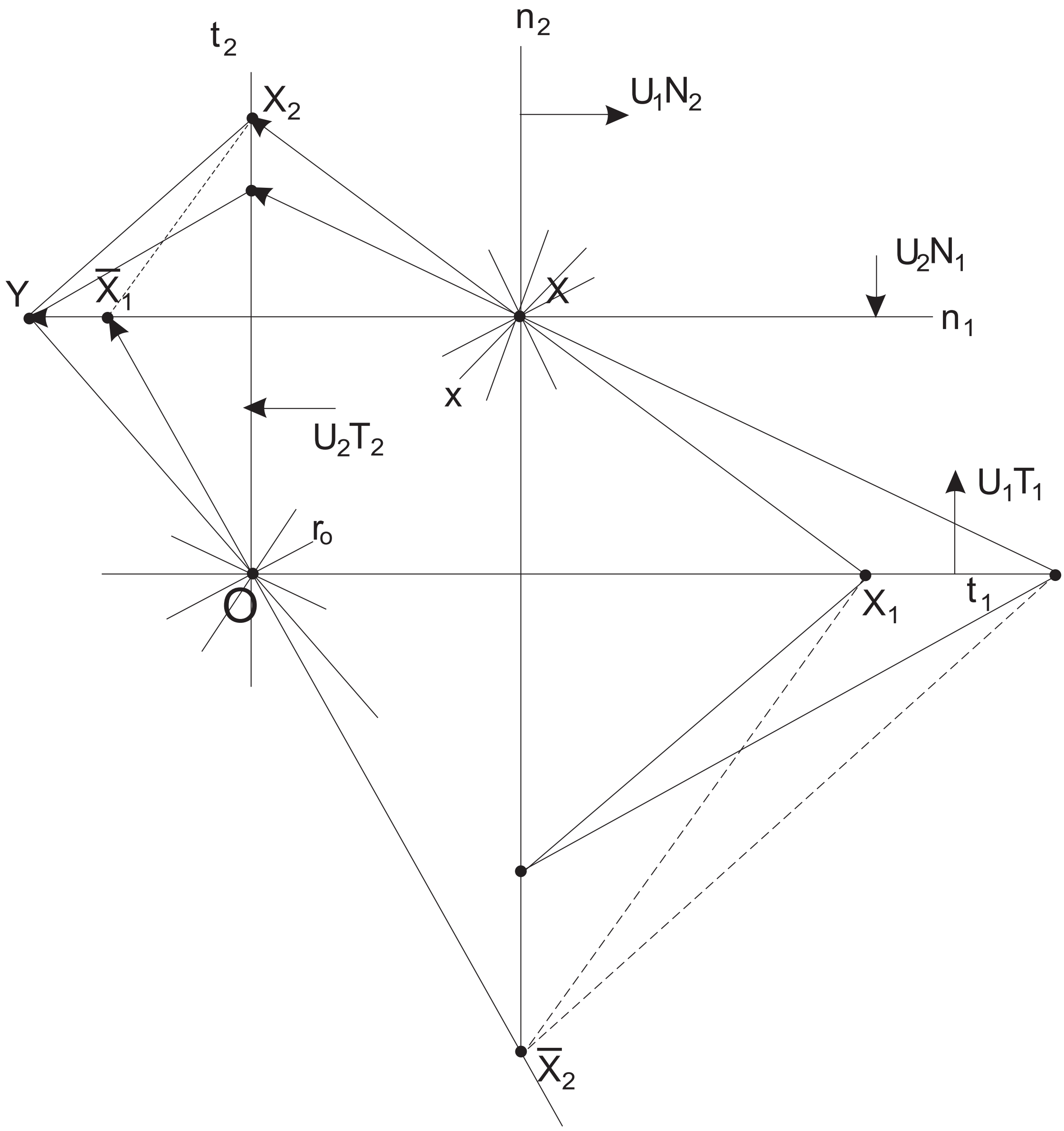

Figure 9.

Moreover, it is easy to check that every line of $\mathcal{L}$ meets x, U_1T_1 and U_2T_2. It follows that every line of $\mathcal{L}$ belongs to $\mathcal{R}$. Now, let n_1 and n_2 be the lines of $AG(2,q)$ through X and parallel to t_1 and t_2, respectively. Then, let N_1 and N_2 be the points of π represented by the lines n_1 and n_2, respectively. The line U_2N_1 of $PG(3,q)$ meets U_2T_2 at U_2, U_1T_1 at the point represented by the ordered pair (t_1,n_1) and x and N_1. It follows that $U_2N_1 \in \mathcal{R}$. The line U_1N_2 meets U_1T_1 at U_1, U_2T_2 at the point represented by the ordered pair (n_2,t_2) and x at N_2. Then $U_1N_2 \in \mathcal{R}$. By the above arguments, it follows

$$E = \{r_0, U_1N_2, U_2N_1\} \cup \mathcal{L} \subseteq \mathcal{R}.$$

Now, let us prove that *every line of $\mathcal{R}$ is a line of E.*

Proof. Assume that r' is a line of $\mathcal{R}$ not in E. Then r' is not in π, otherwise $r_0 \cap r' \neq \emptyset$, while r' and r_0 are skew, since r and r_0 are two distinct lines of $\mathcal{R}$ $(r_0 \in E,\ r' \in E)$. The line r' does not contain U_1 and U_2, since r' is distinct from U_1N_2 and U_2N_1 $(U_1N_2 \in E,\ U_2N_1 \in E,\ r' \notin E)$. Remark that r' does not meet v. For, $M = r' \cap v$. By $U_1 \notin r'$, $U_2 \notin r'$, it follows that $M \neq U_1$, $M \neq U_2$. Then, v contains the three distinct points M, U_1 and U_2 of $\mathcal{H}$ and then v is a line of $\mathcal{H}$. Since v meets U_1T_1 and U_2T_2, which belong to $\overline{\mathcal{R}}$, it follows that $v \in \mathcal{R}$ and then v meets $x \in \overline{\mathcal{R}}$: a contradiction, because x does not pass through $Y = \pi \cap v$ (x is represented in $AG(2,q)$ by the proper pencil of lines with centre X). The contradiction proves the remark. Therefore, r' is a line of $PG(3,q)$ not in π and not meeting v. It follows that r' is represented by an ordered pair of distinct points (X_1,X_2) of $AG(2,q)$. By $r' \in \mathcal{R}$ and $x \in \overline{\mathcal{R}}$, it follows that $r' \cap x \neq \emptyset$ and then the line of $AG(2,q)$ joining X_1 and X_2 contains X. By $r' \in \mathcal{R}$ and $U_1T_1 \in \overline{\mathcal{R}}$, it follows that $r' \cap U_1T_1 \neq \emptyset$ and then $X_1 \in t_1$. By $r' \in \mathcal{R}$ and $U_2T_2 \in \overline{\mathcal{R}}$ it follows $r' \cap U_2T_2 \neq \emptyset$ and then $X_2 \in r_2$. Moreover, we get $X_1 \neq O$, $x_2 \neq O$, that is $X_1 \in t_1 - \{O\}$, $X_2 \in t_2 - \{O\}$, since the points X_1, X_2 and X are collinear and $X_1 \neq X_2$. Then $X_1 \in t_1 - \{O\}$, $X_2 \in t_2 - \{O\}$ and X_1, X_2, X are collinear. By that, it follows $(X_1,X_2) \in I$ and then $r' = \ell(X_1,X_2) \in \mathcal{L} \subset E$, so $r' \in E$ is a contradiction, since $r' \notin E$. The contradiction proves that every line of $\mathcal{R}$ belongs to E and then $\mathcal{R} \subseteq E$. By that and by $E \subseteq \mathcal{R}$ it follows $\mathcal{R} = E$, that is

$$\mathcal{R} = \{r_0, U_1N_2, U_2N_1\} \cup \mathcal{L}.$$

Now, let $\overline{I}$ be the following set

$$\overline{I} = \{(\overline{X}_2,\overline{X}_1) \in \mathcal{S}\times\mathcal{S} : \overline{X}_1 \in n_1 - \{X\},\ X_2 \in n_2 - \{X\},\ \overline{X}_1, \overline{X}_2 \text{ and } O \text{ collinear}\}.$$

Let

$$\overline{\mathcal{L}} = \{\ell(\overline{X}_2,\overline{X}_1)\}_{(\overline{X}_2,\overline{X}_1)\in\overline{I}}.$$

In a similar way as before, we get

$$\overline{\mathcal{R}} = \{x, U_1T_1, U_2T_2\} \cup \overline{\mathcal{L}}.$$

So, the following theorem is proved.

Theorem 3. *Let $PG(3,q)$ be the projective three-dimensional space over the field q, let $AG(2,q)$ be the affine plane over q and let $\mathcal{S}$ be the set of points of $AG(2,q)$ and R an $R(U_1,U_2,\pi,3)$-representation of $PG(3,q)$ (see [1]). In $AG(2,q)$ let t_1 and t_2 be two distinct lines meeting at a point O and let r_0 be the line of π represented, through R, by the proper pencil of lines of $AG(2,q)$ with centre O. Let T_1 and T_2 be the points of π represented through R, by the lines t_1 and t_2 of $AG(2,q)$ respectively. Let X be a point of $AG(2,q)-(t_1\cup t_2)$ and let x be the line of π represented by the proper pencil of lines of $AG(2,q)$ with centre X.*

Let n_1 and n_2 be the lines of $AG(2,q)$ through X and parallel to t_1 and t_2, respectively. Then, let N_1 and N_2 be the points of π represented through R, by the lines n_1 and n_2, respectively. Let I and $\overline{I}$ be the following sets:

$$\begin{aligned} I &= \{(X_1,X_2)\in\mathcal{S}\times\mathcal{S} : X_1\in t_1-\{O\},\ X_2\in t_2-\{O\} \\ &\qquad\qquad\qquad\qquad\textit{and } X_1,X_2,X \textit{ collinear}\}, \\ \overline{I} &= \{(\overline{X}_2,\overline{X}_1)\in\mathcal{S}\times\mathcal{S} : \overline{X}_1\in n_1-\{X\},\ \overline{X}_2\in n_2-\{X\} \\ &\qquad\qquad\qquad\qquad\textit{and } \overline{X}_1,\overline{X}_2,O \textit{ collinear}\}. \end{aligned}$$

Denote by $\ell(X',X'')$ the line of $PG(3,q)$ represented through R by the ordered pair of distinct points (X',X'') of $AG(2,q)$.

Let

$$\begin{aligned} \mathcal{L} &= \{\ell(X_1,X_2)\}_{(X_1,X_2)\in I}, \\ \overline{\mathcal{L}} &= \{\ell(\overline{X}_2,\overline{X}_1)\}_{(\overline{X}_2,\overline{X}_1)\in\overline{I}}. \end{aligned}$$

Then the following sets of lines or $PG(3,q)$

$$\begin{aligned} \mathcal{R} &= \{r_0,U_1N_2,U_2N_1\}\cup\mathcal{L}, \\ \overline{\mathcal{R}} &= \{x,U_1T_1,U_2T_2\}\cup\overline{\mathcal{L}}, \end{aligned}$$

are the two reguli of a hyperbolic quadric of $PG(3,q)$ admitting the plane π as tangent plane, the contact point being the point of π represented by the line OX of $AG(2,q)$ and the line v being a secant line.

The above theorem allows us to represent in the plane $AG(2,q)$ the reguli and then the points of a hyperbolic quadric $\mathcal{H}$, deprived of two points which are not represented. For, the line r_0 is represented by the pencil of lines of $AG(2,q)$ with centre O, the line x is represented by the pencil of lines of $AG(2,q)$ with centre X, the lines U_1N_2, U_2N_1, U_1T_1 U_2T_2 are represented as follows:

$U_1N_2-\{U_1\}:\{n_2\}\cup\{n_2,n\}:n$ parallel to n_2 and distinct from n_2,

$U_2N_1-\{U_2\}:\{n_1\}\cup\{n,n_1\}:n$ parallel to n_1 and distinct from n_1,

$U_1T_1-\{U_1\}:\{t_1\}\cup\{t_1,t\}:t$ parallel to t_1 and distinct from t_1,

$U_2T_2-\{U_2\}:\{t_2\}\cup\{t,t_2\}:t$ parallel to t_2 and distinct from t_2,

Since the hyperbolic quadrics are all equivalent in $PG(3,q)$, it follows that for any hyperbolic quadric $\mathcal{H}$ of $PG(3,q)$ there is an $R(U_1,U_2,\pi,3)$-representation which allows us to represent $\mathcal{H}$ as in Theorem 3.

Those three different representations of a hyperbolic quadric of $PG(3, q)$ in $AG(2, q)$ show a further application of the representation called also "crashing" cited in the bibliography.

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RECOGNITION OF A_{10} AND $L_4(4)$ BY TWO SPECIAL CONJUGACY CLASS SIZES[1]

Yanheng Chen

School of Mathematics and Statistics
Southwest University
Chongqing 400715
P.R. China
and
School of Mathematics and Statistics
Chongqing Three Gorges University
Chongqing
404100
P.R. China

Guiyun Chen[2]

School of Mathematics and Statistics
Southwest University
Chongqing 400715
P.R. China
e-mail: gychen@swu.edu.cn
math_yan@126.com

Abstract. It is well-known that A_{10} is the smallest (by order) nonabelian simple group with connected prime graph and $L_4(4)$ is the smallest nonabelian simple group of Lie type with connected prime graph. In 2009, A.V. Vasil'ev first dealt with the groups with connected prime graph and proved that Thompson's conjecture holds for A_{10} and $L_4(4)$ (see [1]). In this work, the authors characterize finite simple groups A_{10} and $L_4(4)$ by their orders and largest and smallest conjugacy class sizes greater than 1, and partially generalize A.V. Vasil'ev's work.

Keywords: finite simple groups, conjugacy class size, prime graph.

AMS Mathematics Subject Classification(2010): 20D08, 20D60.

1. Introduction

Throughout this paper, groups under consideration are finite. For any group G, $\pi(G)$ denotes the set of prime divisors of $|G|$. We associate to $\pi(G)$ a simple graph called prime graph of G, denoted by $\Gamma(G)$. Prime graph $\Gamma(G)$ is defined as

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[2]Corresponding author.

follows: the vertex of $\Gamma(G)$ is the set of all prime divisors of the order of G, two distinct vertexes p and q are adjacent by edge if and only if there is an element of order pq in G(see [10]). Denote the connected components of the prime graph by $T(G) = \{\pi_i(G)|1 \leqslant i \leqslant t(G)\}$, where $t(G)$ is the number of the prime graph components of G. If the order of G is even, we always assume that $2 \in \pi_1(G)$. In addition, for $x \in G$, $cl_G(x)$ denotes the conjugacy class in G containing x and $C_G(x)$ denotes the centralizer of x in G. Let $cs(G) = \{n \in \mathbb{N}|G$ has a conjugacy class C such that $|C| = n\}$. For $p \in \pi(G)$, we denote G_p and $Syl_p(G)$ a Sylow $p-$subgroup of G and the set of all of its Sylow $p-$subgroups, respectively. We also denote $Soc(G)$ the socle of G which is the subgroup generated by the set of a minimal normal subgroups of G. The other notation and terminologies in this paper are standard and the reader is referred to [8] if necessary. The second author G.Y. Chen once worked on J.G. Thompson's conjecture posed by J.G. Thompson in 1980s, which is about characterizing finite simple groups by the set of lengths of its conjugacy classes as following (ref. to [[9], Problem 12.38]):

Thompson's conjecture. *Let G be a finite group with $Z(G) = 1$ and L is a finite non-abelian simple group satisfying that $cs(G) = cs(L)$, then $G \simeq L$.*

In 1994, G.Y. Chen proved in his Ph.D. dissertation [3] that if G is a group with $Z(G) = 1$, and L a non-abelian simple group with non-connected prime graph such that $cs(G) = cs(L)$, then $G \simeq L$ (also ref. to [4], [5], [6]). In 2009, A.V. Vasil'ev first dealt with the groups with connected prime graph and proved that Thompson's conjecture holds for A_{10} and $L_4(4)$ (see [1]). In 2011, N. Ahanjideh in [2] proved that Thompson's conjecture is true for $L_n(q)$. Recently, G.Y. Chen and J.B. Li contributed their interests on special class sizes of finite simple groups, and characterize successfully sporadic simple groups (see J.B. Li's Ph.D. dissertation [15])and simple K_3-groups (to prepared) by their orders and few special class sizes greater than 1. In their papers, they provided two new ways to characterize finite simple group by group order and largest class size, or smallest class size greater than 1. More importantly, one of two methods doesn't consider about connection of prime graph of group. Thus it is may be effective to deal with simple groups which have connected prime graph. In this paper, we focus our attention on simple groups A_{10} and $L_4(4)$ which have connected prime graphs, and characterize A_{10} and $L_4(4)$ by their orders, and largest and smallest conjugacy class sizes greater than 1, respectively. In addition, we partially generalize A.V. Vasil'ev's work (see [1]) and prove that Thompson's conjecture holds for A_{10} and $L_4(4)$ at the same time. That is the following theorem. For convenience, $lcs(G)$ and $scs(G)$ denote largest and smallest conjugacy class size greater than 1 of group G, respectively.

Main Theorem. *Let G be a group and L one of A_{10} and $L_4(4)$. Then $G \simeq L$ if and only if $|G| = |L|$ and $lcs(G) = lcs(L)$ and $scs(G) = scs(L)$.*

If Main Theorem is proved, then the following corollary holds, which proves Thompson's conjecture for A_{10} and $L_4(4)$.

Corollary. *Thompson's conjecture holds for finite simple group A_{10} and $L_4(4)$.*

Proof. Let G be a group and L one of A_{10} and $L_4(4)$. Under the hypothesis of Thompson's conjecture, it is proved in [1] that $|G| = |L|$. Hence the corollary follows from Main Theorem.

2. Preliminaries

First, we generalize a simple fact which is used many times in G.Y. Chen and J.B. Li's works. It is important to prove our Main Theorem.

Lemma 2.1. *Let G be a group, $\overline{G} = G/Z(G)$. $\overline{N}$ is a minimal normal subgroup of $\overline{G}$, and N is the pre-image of $\overline{N}$ in G. If $p \in \pi(\overline{N})$ for some $p \in \pi(G)$ and $N_p \in Syl_p(N)$ satisfying $|N_p| < scs(G)$, then $\overline{N}$ is not solvable.*

Proof. Assume that $\overline{N}$ is solvable. Then $\overline{N}$ is an elementary abelian $p-$group with $|\overline{N}| = p^t$, $t \geq 1$, and N is a nilpotent normal subgroup of G by the hypothesis. Hence N_p is a normal subgroup of G, and N_p is not a subgroup of $Z(G)$. So there exists an element x of $N_p - Z(G)$ satisfying that

$$1 < |cl_G(x)| = |G : C_G(x)| \leq |N_p| < scs(G),$$

violating the hypothesis.

By Lemma 2.1, the fact can easily be obtained as a corollary following.

Corollary 2.2. *Let G be a group, $\overline{G} = G/Z(G)$. If $|G_p| < scs(G)$ for any $p \in \pi(G)$, then $Soc(\overline{G}) \trianglelefteq \overline{G} \lesssim Aut(Soc(\overline{G}))$.*

Proof. Suppose that $\overline{N}$ is any minimal normal subgroup of $\overline{G}$, and N is the pre-image of $\overline{N}$ in G. By the hypothesis, N satisfies that $|N_p| \leq |G_p| < scs(G)$, so every minimal normal subgroup of $\overline{G}$ is not solvable by Lemma 2.1. Let $S_1, S_2, \ldots, S_k (k \geq 1)$ be all minimal normal subgroup of $\overline{G}$. Let $M = Soc(\overline{G})$, hence $M = Soc(\overline{G}) = S_1 \times S_2 \times \cdots \times S_k$ and S_i is a direct product of some isomorphic non-abelian simple groups for $i = 1, 2, \ldots, k$. Now, we assert that $C_{\overline{G}}(M) = 1$. If not, there exists a minimal normal subgroup $\overline{S}$ of $\overline{G}$ such that $\overline{S} \leq C_{\overline{G}}(M) \bigcap M$. Thus $\overline{S}$ is an abelian group, a contradiction. By N/C theorem, we have $M \trianglelefteq \overline{G} = \overline{G}/C_{\overline{G}}(M) \lesssim \mathrm{Aut}(M)$, as desired.

Lemma 2.3. *Let K be a normal subgroup of a group G, and $\overline{G} = G/K$. If $\overline{x}$ is the image of an element x of G in $\overline{G}$, and $(|x|, |K|) = 1$, then $C_{\overline{G}}(\overline{x}) = C_G(x)K/K$. In particular, if $K = Z(G)$, then $C_{\overline{G}}(\overline{x}) = C_G(x)/Z(G)$.*

Proof. This is an immediate consequence of Theorem 1.6.2 in [14] or Lemma 5 in [1]. For $\pi(A_{10}), \pi(L_4(4)) \subseteq \{2,\ 3,\ 5,\ 7,\ 17\}$, we need to list all the non-abelian simple groups L satisfying with $\pi(L) \subseteq \{2, 3, 5, 7, 17\}$.

Lemma 2.4. *Let L be a non-abelian simple group. If $\pi(L) \subseteq \{2, 3, 5, 7, 17\}$, then L is isomorphic to one of simple groups of Table 1. Especially, $\{2, 3\} \subseteq \pi(L)$, and if $L \neq S_6(2), S_8(2)$, then $\pi(Out(L)) \subseteq \{2, 3\}$.*

Proof. This is Lemma 2.5 in [7].

Table 1. Non-abelian simple groups L with $\pi(L) \subseteq \{2,3,5,7,17\}$

L	Order of L	$\lvert\mathrm{Out}(L)\rvert$	L	Order of L	$\lvert\mathrm{Out}(L)\rvert$
A_5	$2^2 \cdot 3 \cdot 5$	2	A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	2
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2
A_6	$2^3 \cdot 3^2 \cdot 5$	2^2	$S_4(4)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 17$	4
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3	$S_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1
$L_2(17)$	$2^4 \cdot 3^2 \cdot 17$	2	$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	$\lvert D_8\rvert$
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2
$L_2(16)$	$2^4 \cdot 3 \cdot 5 \cdot 17$	4	A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	2
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2	$O_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	$\lvert S_3\rvert$
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	$O_8^-(2)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$	2^2
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	$\lvert D_{12}\rvert$	$L_4(4)$	$2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17$	2
$U_4(2)$	$2^6 \cdot 3^4 \cdot 5$	2	He	$2^{10} \cdot 3^3 \cdot 7^3 \cdot 17$	2
$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	2^2	$S_8(2)$	$2^{16} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$	1
$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	$\lvert S_3\rvert$			

A group G is said to be an almost simple group related to L if and only if $L \trianglelefteq G \leq \mathrm{Aut}(L)$ for some non-abelian simple group L. Almost simple groups related to L with $\pi(L) \subseteq \{2,3,5,7,17\}$ are listed in the following lemma.

Lemma 2.5. *Let L be a non-abelian simple group such that $\pi(L) \subseteq \{2,3,5,7,17\}$. If $L \trianglelefteq G \leq Aut(L)$, then G is isomorphic to one of the groups listed in Table 2.*

Table 2. Almost simple groups $L \trianglelefteq G \leq Aut(L)$ with $\pi(L) \subseteq \{2,3,5,7,17\}$

L	G	L	G	L	G	L	G
A_5	L	$L_3(4)$	L	$S_6(2)$	L	$L_4(4)$	L
	$L \cdot 2$		$L \cdot 2_1$	A_{10}	L		$L \cdot 2_1$
$L_2(7)$	L		$L \cdot 3$		$L \cdot 2$		$L \cdot 2_2$
	$L \cdot 2$		$L \cdot 6$	$S_4(7)$	L		$L \cdot 2_3$
A_6	L		$L \cdot 2_2$		$L \cdot 2$		$L \cdot 2^2$
	$L \cdot 2_1$		$L \cdot 3 \cdot 2_2$	$O_8^+(2)$	L	$L_2(16)$	L
	$L \cdot 2_2$		$L \cdot 2_3$		$L \cdot 2$		$L \cdot 2$
	$L \cdot 2_3$		$L \cdot 3 \cdot 2_3$		$L \cdot 3$		$L \cdot 4$
	$L \cdot 2^2$		$L \cdot 2^2$		$L \cdot S_3$	$O_8^-(2)$	L
$L_2(8)$	L		$L \cdot D_{12}$	$U_4(3)$	L		$L \cdot 2$
	$L \cdot 3$	$L_2(49)$	L		$L \cdot 2_1$	$S_4(4)$	L
A_7	L		$L \cdot 2_1$		$L \cdot 4$		$L \cdot 2$
	$L \cdot 2$		$L \cdot 2_2$		$L \cdot 2_2$		$L \cdot 4$
$U_3(3)$	L		$L \cdot 2_3$		$L \cdot (2^2)_{122}$		
	$L \cdot 2$		$L \cdot 2^2$		$L \cdot (2^2)_{133}$		
A_8	L	$U_3(5)$	L		$L \cdot D_8$		
	$L \cdot 2$		$L \cdot 2$	$L_2(17)$	L		
$U_4(2)$	L		$L \cdot 3$		$L \cdot 2$		
	$L \cdot 2$		$L \cdot S_3$	He	L		
A_9	L	J_2	L		$L \cdot 2$		
	$L \cdot 2$		$L \cdot 2$	$S_8(2)$	L		

Proof. All almost simple groups not related to $L_2(17)$, $L_2(16)$, $S_4(4)$, $S_8(2)$, $O_8^-(2)$, $L_4(4)$, and He listed in Table 2 were given in Proposition 1 in [11]. Those related to one of $L_2(17)$, $L_2(16)$, $S_4(4)$, $S_8(2)$, $O_8^-(2)$, $L_4(4)$, and He are easily obtained by an algorithm from [12].

Lemma 2.6. *Let $R = R_1 \times \cdots \times R_k$, where R_i is a direct product of n_i isomorphic copies of a non-abelian simple group H_i, where H_i and H_j are not isomorphic if $i \neq j$. Then* $\mathrm{Aut}(R) \simeq \mathrm{Aut}(R_1) \times \cdots \times \mathrm{Aut}(R_k)$ *and* $\mathrm{Aut}(R_i) \simeq (\mathrm{Aut}(H_i) \wr S_{n_i}$, *where in this wreath product* $\mathrm{Aut}(H_i)$ *appears in its right regular representation and the symmetric group S_{n_i} in its natural permutation representation. Moreover, these isomorphisms induce outer automorphisms* $\mathrm{Out}(R) \simeq \mathrm{Out}(R_1) \times \cdots \times \mathrm{Out}(R_k)$ *and* $\mathrm{Out}(R_i) \simeq (\mathrm{Out}(H_i) \wr S_{n_i}$.

Proof. This is Theorem 3.3.20 in [13].

3. Proof of the Main Theorem

We divide the sufficient proof of Main Theorem into two lemmas.

Lemma 3.1. *Let G be a group with $|G| = 2^7 \cdot 3^4 \cdot 5^2 \cdot 7$. If $lcs(G) = 2^4 \cdot 3^4 \cdot 5^2 \cdot 7$ and $scs(G) = 2^4 \cdot 3 \cdot 5$. Then $G \simeq A_{10}$.*

Proof. It is clear that one has that $Z(G) \leqslant C_G(x)$ for any $x \in G$. Set x, and $y \in G$ such that $lcs(G) = |cl_G(x)| = 2^4 \cdot 3^4 \cdot 5^2 \cdot 7$ and $scs(G) = |cl_G(y)| = 2^4 \cdot 3 \cdot 5$. Since $|G| = 2^7 \cdot 3^4 \cdot 5^2 \cdot 7$ and $lcs(G) = 2^4 \cdot 3^4 \cdot 5^2 \cdot 7$, $Z(G)$ is a proper subgroup of G with $|Z(G)| \mid 2^3$ by the hypothesis. Thus 3, 5, and $7 \notin \pi(Z(G))$. Considering $\overline{G} = G/Z(G)$. For any prime $p \in \pi(G)$, the order of Sylow $p-$subgroup of G is less than $scs(G)$. By Corollary 2.2, we know that every minimal normal subgroup of $\overline{G} = G/Z(G)$ is non-solvable and $Soc(\overline{G}) \trianglelefteq \overline{G} \leq \mathrm{Aut}(Soc(\overline{G}))$. Let $M = Soc(\overline{G})$ and $S_1, S_2, \cdots, S_k (k \geq 1)$ be all minimal normal subgroups of $\overline{G}$, hence $M = Soc(\overline{G}) = S_1 \times S_2 \times \cdots \times S_k$ and S_i is a direct product of some isomorphic non-abelian simple groups for $i = 1, 2, \ldots, k$.

We assert that $3 \in \pi(M)$. Otherwise, M is a simple K_3-group with $\pi(M) = \{2, 5, 7\}$. This is impossible by Table 1.

We assert that $5 \in \pi(M)$. Otherwise, M is a simple K_3-group with $\pi(M) = \{2, 3, 7\}$ and $5 \in \pi(\mathrm{Out}(G))$. By Table 1, we find that M is isomorphic to one of the following simple groups: $L_2(7), L_2(8)$, and $U_3(3)$. By Lemma 2.4, we see that $|\mathrm{Out}(L_2(7))| = |\mathrm{Out}(U_3(3))| = 2$, and $|\mathrm{Out}(L_2(8))| = 3$, a contradiction to $5 \in \pi(\mathrm{Out}(G))$.

We also assert that $7 \in \pi(M)$. Otherwise, $\pi(M) = \{2, 3, 5\}$ and $7 \in \pi(\mathrm{Out}(G))$. By Table 1, M may be isomorphic to one of the following groups:

$$A_5, A_6, U_4(2), A_5 \times A_5, A_5 \times A_6, \text{ and } A_6 \times A_6.$$

By Lemma 2.4 and Lemma 2.6, we see that outer automorphism groups of these groups are $2-$groups, contradicting to $7 \in \pi(\mathrm{Out}(G))$. Hence $\{3, 5, 7\} \subseteq \pi(M)$. By Table 1 again, M may be isomorphic to one of the following groups:

$$A_7, A_8, L_3(4), A_9, J_2, A_{10}, A_5 \times L_2(7), A_5 \times L_2(8), A_5 \times U_3(3), A_5 \times A_7,$$
$$L_2(7) \times A_6, \text{ and } L_2(8) \times A_6.$$

Now, let us recall that $M \trianglelefteq \overline{G} \leq \mathrm{Aut}(M)$. If M is isomorphic to one of $A_7, A_8, L_3(4), A_9, A_5 \times L_2(7), A_5 \times L_2(8), A_5 \times U_3(3), L_2(7) \times A_6$, and $L_2(8) \times A_6$, then we have that $5 \parallel |\overline{G}|$ by Table 1 and Lemma 2.6. Hence $5 \in \pi(Z(G))$, a contradiction. If M is isomorphic to one of J_2, and $A_5 \times A_7$, then by the same reasoning $3 \in \pi(Z(G))$, a contradiction.

Hence $M \simeq A_{10}$ and G must be isomorphic to A_{10} by comparing the orders of M and G, as claimed.

Lemma 3.2. *Let G be a group with $|G| = 2^{12}\cdot 3^4\cdot 5^2\cdot 7\cdot 17$. If $lcs(G) = 2^{10}\cdot 3^2\cdot 5^2\cdot 7\cdot 17$ and $scs(G) = 3^2 \cdot 5 \cdot 7 \cdot 17$. Then $G \simeq L_4(4)$.*

Proof. First, for any $x \in G$, $Z(G)$ is contained in $C_G(x)$. By the hypothesis, there exist y, and $z \in G$ such that $scs(G)= |cl_G(y)| =3^2 \cdot 5 \cdot 7 \cdot 17$ and $lcs(G)= |cl_G(z)| =2^{10}\cdot 3^2\cdot 5^2\cdot 7\cdot 17$. Since $|G| = 2^{12}\cdot 3^4\cdot 5^2\cdot 7\cdot 17$, and $lcs(G) = 2^{10}\cdot 3^2\cdot 5^2\cdot 7\cdot 17$, we have that $Z(G)$ is a proper subgroup of G of order dividing 36. Similar to Lemma 3.1, Considering $\overline{G} = G/Z(G)$. For any prime $p \in \pi(G)$, the order of Sylow $p-$subgroup of G is less than $scs(G)$. Hence by Corollary 2.2, we know that every minimal normal subgroup of $\overline{G}$ is non-solvable and $Soc(\overline{G}) \trianglelefteq \overline{G} \leq \mathrm{Aut}(Soc(\overline{G}))$. Let $M = Soc(\overline{G})$ and $S_1,\ S_2,\ \ldots,\ S_k (k \geq 1)$ be all minimal normal subgroups of $\overline{G}$. Hence $M = Soc(\overline{G}) = S_1 \times S_2 \times \cdots \times S_k$ and S_i is a direct product of some isomorphic non-abelian simple groups for $i = 1, 2, \ldots, k$. Similar to the proof Lemma 3.1, we prove 5, 7, and $17 \in \pi(M)$.

If $5 \notin \pi(M)$, then $5 \in \pi(\mathrm{Out}(M))$. Applying Table 1 and possible order of M, M may be isomorphic to one of the following groups:

$$L_2(7),\ L_2(8),\ L_2(17),\ U_3(3),\ L_2(7) \times L_2(17),\ \text{and } L_2(8) \times L_2(17).$$

By Lemmas 2.4 and 2.6, we see that outer automorphism groups of groups above are $\{2,\ 3\}-$groups, a contradiction to $5 \in \pi(\mathrm{Out}(M))$. Therefore, $5 \in \pi(M)$.

If $7 \notin \pi(M)$, then $7 \in \pi(\mathrm{Out}(M))$. By Table 1 and $5 \in \pi(M)$, M may be isomorphic to one of the following groups:

$$A_5,\ A_6,\ L_2(16),\ U_4(2),\ S_4(4),\ A_5 \times A_5,\ A_5 \times A_6,\ A_6 \times A_6,\ A_5 \times L_2(17),$$
$$A_5 \times L_2(16), A_6 \times L_2(17),\ \text{and } A_6 \times L_2(16).$$

By Table 1 and Lemma 2.6, we see that outer automorphism groups of these groups are $2-$ groups, a contradiction. Hence $7 \in \pi(M)$.

If $17 \notin \pi(M)$, then $17 \in \pi(\mathrm{Out}(G))$. By Table 1 and $\{5, 7\} \subseteq \pi(M)$, M may be isomorphic to one of the following groups:

$$A_7,\ A_8,\ L_3(4),\ A_9,\ J_2,\ S_6(2),\ A_{10},\ A_5 \times L_2(7),\ A_5 \times L_2(8),\ A_5 \times A_7,\ A_5 \times A_8,$$
$$A_5 \times L_3(4),\ A_6 \times L_2(7),\ A_6 \times L_2(8),\ A_6 \times A_7,\ A_6 \times A_8,\ A_6 \times L_3(4),$$
$$\text{and } A_5 \times L_2(7) \times A_6.$$

By Table 1 and Lemma 2.6, we know that 17 is not a prime divisor of outer automorphism of those groups above, a contradiction. Hence $17 \in \pi(M)$. For convenience, we assume that $7 \in \pi(S_i)$, and $17 \in \pi(S_j)$ for $i, j \in \{1, 2, \cdots, k\}$.

If $i \neq j$, then S_i and S_j are two non-isomorphic simple groups. By Table 1 and possible order of M again, we see that M may be isomorphic to one of the following groups:

$L_2(7) \times L_2(16),\ L_2(7) \times L_2(17),\ L_2(7) \times S_4(4),\ L_2(8) \times L_2(16),$
$L_2(8) \times L_2(17),\ L_2(8) \times S_4(4), A_7 \times L_2(16),\ A_7 \times L_2(17),\ U_3(3) \times L_2(16),$
$A_8 \times L_2(16), A_8 \times L_2(17),\ L_3(4) \times L_2(16),\ L_3(4) \times L_2(17),$
$A_5 \times L_2(7) \times L_2(16),\ A_5 \times L_2(7) \times L_2(17),\ A_5 \times L_2(8) \times L_2(16),$
and $A_6 \times L_2(7) \times L_2(16)$.

If M is isomorphic to one of the following groups:

$L_2(7) \times L_2(16),\ L_2(7) \times L_2(17), Ł_2(8) \times L_2(16),\ U_3(3) \times L_2(16),\ A_8 \times L_2(17),$
$L_3(4) \times L_2(17),\ L_2(8) \times L_2(17),\ A_7 \times L_2(17),$ and $A_5 \times L_2(7) \times L_2(17),$

then, by Table 1 and Lemma 2.6, we come to $5 \in \pi(Z(G))$ by comparing the orders of M, $\overline{G}$, and $\mathrm{Aut}(M)$, a contradiction.

If $M \simeq L_2(7) \times S_4(4)$, then

$$\mathrm{Aut}(M) = \mathrm{Aut}(L_2(7)) \times \mathrm{Aut} S_4(4)) = L_2(7) \cdot 2 \times S_4(4) \cdot 4$$

by Lemma 2.6 and Table 2.

Recall that $M \trianglelefteq \overline{G} \leq \mathrm{Aut}(M)$, then $|Z(G)| = 3$ or 6. So there exists an element w of order 7 in G such that $C_G(w)/Z(G) = C_{\overline{G}}(\overline{w}) \geq \langle \overline{w} \rangle \times S_4(4)$ by Lemma 2.3, where $\overline{w}$ is the image of w in $\overline{G}$. Hence $|C_G(w)| \geq 2^8 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 17$ such that $1 < |cl_G(w)| < scs(G)$, a contradiction to minimality of $scs(G)$.

If $M \simeq L_2(8) \times S_4(4)$, then by same way above we come to $\mathrm{Aut}(M) = L_2(8) \cdot 3 \times S_4(4) \cdot 4$ such that $|Z(G)| \leq 2$. So there exists an element w of order 7 in G such that such that $1 < |cl_G(w)| < scs(G)$, a contradiction.

In a similar way used above, we can deal with the remaining cases of M, and can always find an element of G such that its conjugacy class length is less than $scs(G)$, leading to a contradiction.

Hence $i = j$. Without loss of generality, assume that 7, and $17 \in \pi(S_1)$. Then S_1 is a non-abelian simple group and isomorphic to $O_8^-(2)$ or $L_4(4)$ by Table 1. Therefore $k = 1$, and M may be isomorphic to one of following groups: $O_8^-(2)$, and $L_4(4)$.

If $M \simeq O_8^-(2)$, then, by Table 1 and Table 2, $\overline{G} \simeq O_8^-(2)$ or $O_8^-(2) \cdot 2$ and $\mathrm{Aut}(M) = O_8^-(2) \cdot 2$. Comparing the orders of M, $\overline{G}$, and $\mathrm{Aut}(M)$, we see that $|Z(G)| = 5$ and $\overline{G} \simeq O_8^-(2)$. If G is a split extension $O_8^-(2)$ by $Z(G)$, then $G = O_8^-(2) \times Z(G)$. Therefore, by [8], there exists a non-central element w of order 2 in G such that $1 < |cl_G(w)| < scs(G)$, leading to a contradiction. Hence G is not a split extension $O_8^-(2)$ by $Z(G)$, which implies that 5 divides $|\mathrm{Mult}(O_8^-(2))|$, a contradiction to $|\mathrm{Mult}(O_8^-(2))| = 2$ by [8].

Hence $M \simeq L_4(4)$ and so G must be isomorphic to $L_4(4)$ by $|G| = |M|$, as desired.

Proof of the Main Theorem. The necessity is obvious by [8] and the sufficiency follows from Lemmas 3.1 and 3.2.

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THE GROUPS OF TWO CLASSES OF CERTAIN CYCLICALLY PRESENTED GROUPS ARE ESSENTIALLY 3-GENERATED

Dedicated to Dr. D.L. Johnson

Devon Roy Stoddart

Caribbean Institute of Technology
1 Pimento Way, Free Port, Montego Bay
P.O. Box 349
Jamaica W.I.
e-mail: devon.roy@gmail.com

Abstract. Two classes of cyclically presented groups were introduced in [3] and proven infinite for $n \succeq 3$ in [2]. I show that the groups of these classes of certain cyclically presented groups are essentially 3-generated. The groups G_n and H_n for $n = 3$ and 4 were shown to be 2-generated in [9] and [1], while the abelianized groups G_n^{ab} of G_n were dealt with in [8]. Naturally, the groups G_n and H_n for $n = 1$ and 2 are trivial. Showing that the groups of these two classes are essentially 3–generated has been the most difficult to solve thus far.

Keywords and phrases: Cyclically presented groups; 2-generated, 3-generated.

1. Introduction

The cyclically presented groups

$$H_n = \left\langle x_i | x_{i+1}^{-1} x_{i+2} x_{i+1}^{-1} x_{i+2} x_{i+1}^{-2} x_i x_{i+1}^{-1} x_i \right\rangle_n,$$

where subscripts are reduced mod n to lie in the set $\{1, 2, ..., n\}$ belong to a second class of groups introduced in [3]. They have the Alexander polynomial $f(t) = 2t^2 - 5t + 2$, which is equal to the polynomial associated with the cyclically presented groups, of knot with 6 crossings denoted by 6_1 and is equivalent to the 2-bridge knot $b(9, 4)$ in [7].

The other class of groups

$$G_n = \left\langle x_i | x_{i+1}^{-1} x_{i+2} x_{i+1}^{-1} x_{i+2} x_i x_{i+1}^{-1} x_i \right\rangle_n$$

has the Alexander polynomial $f(t) = 2t^2 - 3t + 2$, which is equal to the polynomial associated with the cyclically presented groups, of knot with 5 crossings denoted by 5_2 in [7] were previously dealt with in [8] and [9].

A detailed study of the connections between these group presentations and closed 3-dimensional manifolds can be found in [4] and [5].

2. A general relation of each class of groups

Before we can approach the matter of showing that the groups of these classes of groups are essentially 3-generated, we will need to compute an essential formula for each class.

Lemma 2.1. *For any H_n,*

$$x_1x_2...x_{n-2}x_{n-1}x_n = 1,$$

and for any G_n,

$$x_nx_{n-1}x_{n-2}...x_2x_1 = 1,$$

for $n \in N$.

Proof. From the relations of

$$H_n = \left\langle x_i | x_{i+1}^{-1}x_{i+2}x_{i+1}^{-1}x_{i+2}x_{i+1}^{-2}x_ix_{i+1}^{-1}x_i \right\rangle_n,$$

starting with the first relation derived from $i = n-1$, and post-multiplying the next one derived from $i = n-2$, successively, until you have multiplied the last relation derived from $i = n$, gives:

$$\begin{aligned} x_n^{-1}x_1x_n^{-1}x_1x_n^{-2}x_{n-1}x_n^{-1}x_{n-1}x_{n-1}^{-1}x_nx_{n-1}^{-1}x_nx_{n-1}^{-2}x_{n-2}x_{n-1}^{-1}x_{n-2}... \\ x_2^{-1}x_3x_2^{-1}x_3x_2^{-2}x_1x_2^{-1}x_1x_1^{-1}x_2x_1^{-1}x_2x_1^{-2}x_nx_1^{-1}x_n = 1, \end{aligned} \tag{1}$$

which means,

$$x_n^{-1}x_{n-1}^{-1}x_{n-2}^{-1}...x_2^{-1}x_1^{-1} = 1,$$

or

$$x_1x_2...x_{n-2}x_{n-1}x_n = 1, \tag{2}$$

for $n \in N$.

Similarly, for any G_n,

$$x_nx_{n-1}x_{n-2}...x_2x_1 = 1, \tag{3}$$

for $n \in N$. So all the $x_i's$ within each relation above are related.

3. The groups $H_n's$ are essentially 3-generated

Looking at patterns in the relations of the groups $H_n's$, we are choosing new generators and then reducing the number of generators to the least possible. We then simplify the presentations of the groups $H_n's$.

Theorem 3.1. *The groups $H_n's$ are essentially* 3*-generated.*

The relations of H_n are shown below:

$$x_2^{-1}x_3x_2^{-1}x_3x_2^{-2}x_1x_2^{-1}x_1 = 1, \tag{4}$$
$$x_3^{-1}x_4x_3^{-1}x_4x_3^{-2}x_2x_3^{-1}x_2 = 1, \tag{5}$$
$$x_4^{-1}x_5x_4^{-1}x_5x_4^{-2}x_3x_4^{-1}x_3 = 1, \tag{6}$$
$$|, \tag{7}$$
$$|, \tag{8}$$
$$|, \tag{9}$$
$$x_{n-2}^{-1}x_{n-1}x_{n-2}^{-1}x_{n-1}x_{n-2}^{-2}x_{n-3}x_{n-2}^{-1}x_{n-3} = 1, \tag{10}$$
$$x_{n-1}^{-1}x_nx_{n-1}^{-1}x_nx_{n-1}^{-2}x_{n-2}x_{n-1}^{-1}x_{n-2} = 1, \tag{11}$$
$$x_n^{-1}x_1x_n^{-1}x_1x_n^{-2}x_{n-1}x_n^{-1}x_{n-1} = 1, \tag{12}$$
$$x_1^{-1}x_2x_1^{-1}x_2x_1^{-2}x_nx_1^{-1}x_n = 1. \tag{13}$$

Now, pre-multiplying these relations, starting with the first one to the 3^{th} to last one, we get

$$x_{n-1}^{-1}x_nx_{n-1}^{-1}x_nx_{n-1}^{-1}x_{n-2}^{-1}x_{n-3}^{-1}...x_4^{-1}x_3^{-1}x_2^{-2}x_1x_2^{-1}x_1 = 1, \tag{14}$$
$$x_n^{-1}x_1x_n^{-1}x_1x_n^{-2}x_{n-1}x_n^{-1}x_{n-1} = 1, \tag{15}$$
$$x_1^{-1}x_2x_1^{-1}x_2x_1^{-2}x_nx_1^{-1}x_n = 1. \tag{16}$$

However, from equation (2)

$$x_{n-1}^{-1}x_{n-2}^{-1}x_{n-3}^{-1}...x_4^{-1}x_3^{-1}x_2^{-1} = x_nx_1,$$

and therefore H_n can be re-written as

$$(x_{n-1}^{-1}x_n)^2x_nx_1(x_2^{-1}x_1)^2 = 1, \tag{17}$$
$$(x_n^{-1}x_1)^2x_n^{-1}(x_n^{-1}x_{n-1})^2 = 1, \tag{18}$$
$$(x_1^{-1}x_2)^2x_1^{-1}(x_1^{-1}x_n)^2 = 1. \tag{19}$$

Having looked at patterns in the relators, we set

$$\begin{aligned}
z &= x_1^{-1}x_n^{-1}(x_n^{-1}x_{n-1})^2,\\
u &= (x_{n-1}^{-1}x_n)^2x_nx_1x_2^{-1}x_1\\
t &= (x_n^{-1}x_1)^2x_n^{-2}x_{n-1},\\
s &= x_1^{-2}x_nx_1^{-1}x_n,\\
r &= x_n^{-1}x_1.
\end{aligned}$$

Therefore,

$$x_1 = r^{-2}s^{-1}, \tag{20}$$
$$x_n = r^{-2}s^{-1}r^{-1}, \tag{21}$$
$$x_n^{-2}x_{n-1} = r^{-2}t, \tag{22}$$
$$x_2^{-1}x_1 = zu, \tag{23}$$
$$x_2 = r^{-2}s^{-1}u^{-1}z^{-1}, \tag{24}$$
$$x_{n-1} = (r^{-2}s^{-1}r^{-1})^2r^{-2}t. \tag{25}$$

We now simplify the presentation in terms of u, t and r, but we start by using the above equations to re-write the above relations in terms of r, z, u, s and t. Using relator (17), we get:

$$uzu = 1 \Rightarrow z = u^{-2}, \tag{26}$$

while from relator (18), we get:

$$s^{-1}z = 1 \Rightarrow z = s, \tag{27}$$

and from relator (19), we get:

$$(u^{-1}z^{-1})^2s = 1 \Rightarrow s = (zu)^2. \tag{28}$$

Now the relations (26) and (27) imply that $s = z = u^{-2}$. So z and s can be eliminated from the set of generators, as they can be expressed in terms of u. Hence the groups $H_n's$ are generated by t, u and r, thus the groups $H_n's$ are 3-generated. We know, however from [1], that the groups H_1 and H_2 are trivial, while H_3 and H_4 are 2-generated.

Theorem 3.2. *The groups $H_n's$ can be re-written as:*

$$\left\langle r, t, u | r^{-2}u^2r^{-3}t^2 \right\rangle.$$

Clearly,

$$z = x_1^{-1}x_n^{-1}(x_n^{-1}x_{n-1})^2, \tag{29}$$
$$z = sr^3sr^2(rsr^2(r^{-2}s^{-1}r^{-1})^2r^{-2}t)^2, \tag{30}$$
$$z = str^{-2}s^{-1}r^{-3}t, \tag{31}$$
$$tr^{-2}s^{-1}r^{-3}t = 1, \tag{32}$$

since $z = s$. Now, replacing s from the above relation (32), we get:

$$tr^{-2}u^2r^{-3}t = 1,$$

which gives

$$r^{-2}u^2r^{-3}t^2 = 1. \tag{33}$$

All the other manipulations of relations give this same relation, so

$$\langle r, t, u | r^{-2} u^2 r^{-3} t^2 \rangle .$$

4. The groups $G_n^{\prime} s$ are essentially 3-generated

Looking at patterns in the relations of the groups $G_n^{\prime} s$, we are choosing new generators and then reducing the number of generators to the least possible. We then simplify the presentations of the groups $G_n^{\prime} s$.

Theorem 4.1. *The groups $G_n^{\prime} s$ are essentially 3-generated.*

The relations of G_n are shown below:

$$x_2^{-1} x_3 x_2^{-1} x_3 x_1 x_2^{-1} x_1 = 1, \tag{34}$$
$$x_3^{-1} x_4 x_3^{-1} x_4 x_2 x_3^{-1} x_2 = 1, \tag{35}$$
$$x_4^{-1} x_5 x_4^{-1} x_5 x_3 x_4^{-1} x_3 = 1, \tag{36}$$
$$|, \tag{37}$$
$$|, \tag{38}$$
$$|, \tag{39}$$
$$x_{n-2}^{-1} x_{n-1} x_{n-2}^{-1} x_{n-1} x_{n-3} x_{n-2}^{-1} x_{n-3} = 1, \tag{40}$$
$$x_{n-1}^{-1} x_n x_{n-1}^{-1} x_n x_{n-2} x_{n-1}^{-1} x_{n-2} = 1, \tag{41}$$
$$x_n^{-1} x_1 x_n^{-1} x_1 x_{n-1} x_n^{-1} x_{n-1} = 1, \tag{42}$$
$$x_1^{-1} x_2 x_1^{-1} x_2 x_n x_1^{-1} x_n = 1. \tag{43}$$

Now, pre-multiplying these relations, starting with the first one to the 3^{th} to the last one, we get

$$x_{n-1}^{-1} x_n x_{n-1}^{-1} x_n x_{n-1} x_{n-2} ... x_4 x_3 x_1 x_2^{-1} x_1 = 1, \tag{44}$$
$$x_n^{-1} x_1 x_n^{-1} x_1 x_{n-1} x_n^{-1} x_{n-1} = 1, \tag{45}$$
$$x_1^{-1} x_2 x_1^{-1} x_2 x_n x_1^{-1} x_n = 1. \tag{46}$$

However, from equation (3)

$$x_n x_{n-1} x_{n-2} ... x_4 x_3 = x_1^{-1} x_2^{-1},$$

and therefore G_n can be re-written as

$$x_{n-1}^{-1} x_n x_{n-1}^{-1} x_1^{-1} x_2^{-1} x_1 x_2^{-1} x_1 = 1, \tag{47}$$
$$x_n^{-1} x_1 x_n^{-1} x_1 x_{n-1} x_n^{-1} x_{n-1} = 1, \tag{48}$$
$$x_1^{-1} x_2 x_1^{-1} x_2 x_n x_1^{-1} x_n = 1. \tag{49}$$

Having looked at patterns in the relators, we set

$$\begin{aligned}
u &= x_{n-1}^{-1}x_n x_{n-1}^{-1}x_1^{-1}x_2^{-1}x_1,\\
t &= (x_n^{-1}x_1)^2 x_{n-1},\\
z &= x_1 x_{n-1} x_n^{-1} x_{n-1},\\
s &= x_n x_1^{-1} x_n\\
r &= x_1^{-1}x_n.
\end{aligned}$$

Therefore,

$$x_2^{-1}x_1 = zu, \tag{50}$$

$$x_n = sr^{-1}, \tag{51}$$

$$x_2 = sr^{-2}u^{-1}z^{-1}, \tag{52}$$

$$x_{n-1} = r^2 t, \tag{53}$$

$$x_1 = sr^{-2}. \tag{54}$$

We now simplify the presentation in terms of u, r and t, but we start by using the above equations to re-write the above relations in terms of r, s, u, z and t. Using relator (47), we get:

$$uzu = 1 \Rightarrow z = u^{-2}, \tag{55}$$

while from relator (48), we get:

$$s^{-1}z = 1 \Rightarrow z = s, \tag{56}$$

and from relator (49), we get:

$$(u^{-1}z^{-1})^2 s = 1 \Rightarrow s = (zu)^2, \tag{57}$$

and so from equations (55) and (56), $s = z = u^{-2}$. This means z and s can be eliminated from the set of generators. Hence the groups G_n's are generated by t, u and r. Thus the groups G_n's are 3-generated. We know, however, that the groups G_1 and G_2 are trivial, while G_3 and G_4 are 2-generated as proven in the paper [9] derived from my 2000 thesis and also in [1]. It was also proven that G_5 is 3-generated in the latter.

Theorem 4.2. *The groups G_n's can be re-written as:*

$$\left\langle t, r, u | ru^2r^2t^2 \right\rangle.$$

Clearly,

$$z = x_1 x_{n-1} x_n^{-1} x_{n-1}, \tag{58}$$

$$z = sr^{-2}r^2trs^{-1}r^2t, \tag{59}$$

$$z = strs^{-1}r^2t, \tag{60}$$

$$trs^{-1}r^2t = 1, \tag{61}$$

since $z = s$. Now, replacing s from relation (61), we get:

$$tru^2r^2t = 1, \tag{62}$$

which gives

$$ru^2r^2t^2 = 1. \tag{63}$$

All other manipulations of relations give this same relation, so

$$\langle t, r, u | ru^2r^2t^2 \rangle .$$

5. Remark

These groups, actually, have 'small' generating sets as purported by Dr. D.L. Johnson – Remark 4.4 in [3], which was questioned by Professor M.F. Newman in [6].

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n-FOLD POSITIVE IMPLICATIVE HYPER K-IDEALS

P. Babari

M. Pirasghari

Department of Mathematics
Faculty of Mathemetics
Tarbiat Modares University
Tehran
Iran
e-mail: p.babari@modares.ac.ir
m.pirasghari@modraes.ac.ir

M.M. Zahedi

Department of Mathematics
Shahid Bahonar University of Kerman, Kerman
Iran
e-mail: zahedi_mm@modares.ac.ir

Abstract. In this paper we are supposed to introduce the definitions of n-fold positive implicative hyper K-ideals. These definitions are the generalizations of the definitions of positive implicative hyper K-ideals, which have been defined in [13]. Then we obtain some related results. In particular we determine the relationships between those n-fold positive implicative hyper K-ideals which satisfy the simple condition.

Keywords: hyper K-algebra; weak hyper K-ideal; hyper K-ideal; n-fold positive implicative hyper K-ideals; simple condition.

Introduction

The theory of hyper compositional structure has been introduced by F. Marty in 1934 during the 8th congress of Scandinavian Mathematicians, where he presented his work [10]. Today the research in the hyper compositional structures field is very vivid. In particular Y.B. Jun, M.M. Zahedi, X.L. Xin and R.A. Borzooei introduced the notions of hyper BCK-algebra and hyper K-algebra in 2000 [4], [8]. The concepts of an n-fold positive implicative hyper K-ideals are the generalizations of the concepts of positive implicative hyper K-ideals, which are related to the concepts of positive implicative ideals of a BCK-algebra [15]. The relationships between positive implicative hyper K-ideals have been studied by M.M. Zahedi and T. Roodbari [12]. They defined 27 types of positive implicative hyper K-ideals, and proved some propositions and theorems in this field. Now in this manuscript we define 27 types of n-fold positive implicative hyper K-ideals, and we concentrate on their relationships. Then we study the relationships between those n-fold positive implicative hyper K-ideals which satisfy the simple condition.

1. Preliminaries

In this paper we use the definitions of hyper K-algebra and hyper K-ideal as the most important definitions.

Definition 1.1. [4] Let H be a nonempty set, and " $\circ$ " be a hyperoperation on H, that " $\circ$ " is a function from $H \times H$ to $P^*(H)$=$P(H)$-$\varnothing$. Then H is called a hyper K-algebra if it contains "0" and satisfies the following axioms:

$HK-1$ $\quad (x \circ z) \circ (y \circ z) < x \circ y$;

$HK-2$ $\quad (x \circ y) \circ z = (x \circ z) \circ y$;

$HK-3$ $\quad x < x$;

$HK-4$ $\quad x < y,\ y < x \Rightarrow x = y$;

$HK-5$ $\quad 0 < x$;

for all $x, y, z \in H$, where $x < y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A < B$ is defined by $\exists a \in A$, $\exists b \in B$ such that $a < b$.

Note that if $A, B \subseteq H$, then by $A \circ B$ we mean that the subset $\bigcup a \circ b$ of H for all $a \in A$ and $b \in B$.

Theorem 1.2.[2] *Let $(H, \circ, 0)$ be a hyper K-algebra. Then for all $x, y, z \in H$ and for all non-empty subsets A, B and C of H the following relations hold:*

(1) $\quad (x \circ y) < z \Leftrightarrow (x \circ z) < y$;

(2) $\quad (x \circ z) \circ (x \circ y) < (y \circ z)$;

(3) $\quad x \circ (x \circ y) < y$;

(4) $\quad x \circ y < x$;

(5) $\quad A \circ B < A$;

(6) $\quad A \subseteq B \Rightarrow A < B$;

(7) $\quad x \in x \circ 0$;

(8) $\quad (\mathrm{A} \circ \mathrm{C}) \circ (\mathrm{A} \circ \mathrm{B}) < (\mathrm{B} \circ \mathrm{C})$;

(9) $\quad (\mathrm{A} \circ \mathrm{C}) \circ (\mathrm{B} \circ \mathrm{C}) < (\mathrm{A} \circ \mathrm{B})$;

(10) $\quad (\mathrm{A} \circ \mathrm{B}) < C \Leftrightarrow (\mathrm{A} \circ \mathrm{C}) < B$;

(11) $\quad A \circ B < A$;

(12) $\quad (\mathrm{A} \circ \mathrm{C}) \circ B = (\mathrm{A} \circ \mathrm{B}) \circ C$;

Theorem 1.3. [6] *Let x,y,z be some elements in hyper K-algebra H. Then the following hold:*

(1) *$x < y$ implies that $z \circ y < z \circ x$,*

(2) *$x < y$ implies that $x \circ z < y \circ z$.*

Definition 1.4. [2] Let I be a nonempty subset of a hyper K-algebra H and $0 \in I$. Then

(1) I is called a weak hyper K-ideal of H if $x \circ y \subseteq I$ and $y \in I$ imply that $x \in I$ for all $x, y \in H$.

(2) I is called a hyper K-ideal of H if $x \circ y < I$ and $y \in I$ imply that $y \in I$ for all $x, y \in H$.

Note that in any hyper K-algebra H, $\{0\} \subseteq H$ is a hyper K-ideal.

Theorem 1.5. [2] *Any hyper K-ideal of a hyper K-algebra H is a weak hyper K-ideal.*

Definition 1.6. [3] Let I be a nonempty subset of a hyper K-algebra H. Then we say that I is closed, whenever $x<y$, $y \in I$ imply that $x \in I$ for all $x,y \in H$.

Definition 1.7. [2] Let H be a hyper K-algebra. An element $a \in H$ is called a left (resp. right) scalar if $|a \circ x| = 1$ (resp. $|x \circ a| = 1$) for all $x \in H$.

Theorem 1.8. [12] *Let I be a hyper K-ideal of a hyper K-algebra H. Then the following statements are equivalent:*

(1) $(x \circ y) < I$,

(2) $(x \circ y) \cap I \neq \varnothing$.

Definition 1.9. [12] Let $H = \{0,1,2\}$ be a hyper K-algebra. We say that H satisfies the simple condition if the conditions $1 \not< 2$ and $2 \not< 1$ hold.

Definition 1.10. [12] A hyper K-algebra H is called simple if for all distinct elements $a, b \in H - 0$, $a \not< b$ and $b \not< a$.

Theorem 1.11. [12] *Let H satisfies the simple condition. Then,*

(i) $a \circ 0 = \{a\}$, *for all* $a \in H - \{0\}$,

(ii) $a \in a \circ b$, *for all distinct elements* $a, b \in H$,

(iii) $H - \{a\} \subseteq H \circ a$, *for all* $a \in H$,

(iv) $a \in b \circ c \Longleftrightarrow c \in b \circ a$, *for all distinct elements* $a, c \in H$, *and* $b \in H - \{0\}$,

(v) $x < x \circ a \Longleftrightarrow x \in x \circ a$, *for all* $a, x \in H$,

(vi) $A < A \circ b \Longleftrightarrow A \cap (A \circ b) \neq \varnothing$, *for all* $b \in H$ *and* $\varnothing \neq A \subseteq H$,

(vii) $(x \circ y) \circ z < x \circ (y \circ z)$, *for all* $x, y, z \in H$,

(viii) *If* $0 \in I \subseteq H$, *then* $A \circ B < I \Longleftrightarrow (A \circ B) \cap I \neq \varnothing$, *for all non-empty subsets A and B of H.*

In the rest of this paper, by H we denote a hyper K-algebra.

2. n-fold positive implicative hyper K-ideals

In this section we define the notions of n-fold positive implicative hyper K-ideals of types $1'$,$2'$,$3'$,and $4'$. Then we define 27 other types, and we give many examples to show that these notions are different from each other. Finally we prove some theorems and obtain some related result.

Definition 2.1. Let I be a nonempty subset of a hyper K-algebra H such that $o\in I$. If n is a natural number, then I is called an n-fold positive implicative hyper K-ideal of

(i) type $1'$, if for all $x,y\in H$, $x\circ y^{n+1}\subseteq I$ implies that $x\circ y^n\subseteq I$,
(ii) type $2'$, if for all $x,y\in H$, $x\circ y^{n+1}\subseteq I$ implies that $x\circ y^n<I$,
(iii) type $3'$, if for all $x,y\in H$, $x\circ y^{n+1}<I$ implies that $x\circ y^n\subseteq I$,
(v) type $4'$, if for all $x,y\in H$, $x\circ y^{n+1}<I$ implies that $x\circ y^n<I$.

Theorem 2.2. *Let A be a weak hyper K-ideal and I be hyper K-ideal of hyper K-algebra H such that $I\subseteq A$. If I is an n-fold positive implicative hyper K-ideal of type $1'$ or $3'$, so is A.*

Proof. Assume I is an n-fold positive implicative hyper K-ideal of type $1'$, and $x\circ y^{n+1}\subseteq A$. Then by Theorem 1.2 $x\circ y^{n+1}<A$. Since $0\in(x\circ y^{n+1})\circ(x\circ y^{n+1})$, and $0\in I$, we obtain $0\in((x\circ y^{n+1})\circ(x\circ y^{n+1}))\circ I$. Therefore we have

$$(x\circ(x\circ y^{n+1}))\circ y^{n+1}=(x\circ y^{n+1})\circ(x\circ y^{n+1})<I$$

On the other hand, I is an n-fold positive implicative hyper K-ideal of type $1'$. So, $(x\circ(x\circ y^{n+1}))\circ y^n\subseteq I$. Hence, $(x\circ(x\circ y^{n+1}))\circ y^n\subseteq A$, thus $(x\circ y^n)\circ(x\circ y^{n+1})\subseteq A$. Moreover, A is a weak hyper K-ideal and $x\circ y^{n+1}\subseteq A$. So, $x\circ y^n\subseteq A$. It means A is an n-fold positive implicative hyper K-ideal of type $1'$.

Similarly, we can prove for type $3'$.

Theorem 2.3. *Let A and I be hyper K-ideals of hyper K-algebra H such that $I\subseteq A$. If I is an n-fold positive implicative hyper K-ideal of type $2'$ or $4'$, so is A.*

Proof. Assume I is an n-fold positive implicative hyper K-ideal of type $2'$, and $x\circ y^{n+1}\subseteq A$. Then by Theorem 1.2 $x\circ y^{n+1}<A$.

Since $(x\circ(x\circ y^{n+1}))\circ y^{n+1}=(x\circ y^{n+1})\circ(x\circ y^{n+1})<I$, so $(x\circ(x\circ y^{n+1}))\circ y^{n+1}<I$. By hypothesis I is an n-fold positive implicative hyper K-ideal of type $2'$, we have $(x\circ(x\circ y^{n+1}))\circ y^n<I$. Therefore, $(x\circ(x\circ y^{n+1}))\circ y^n<A$, thus, $(x\circ y^n)\circ(x\circ y^{n+1})<A$. Moreover, A is a hyper K-ideal and $x\circ y^{n+1}\subseteq A$, therefore $x\circ y^n<A$. It means A is an n-fold positive implicative hyper K-ideal of type $2'$.

Similarly, we can prove for type $4'$.

Definition 2.4. Let I be a nonempty subset of a hyper K-algebra H, such that $0\in I$. If n is a natural number, then I is called an n-fold positive implicative hyper K-ideal of:

(i) type 1, if for all $x,y,z\in H$, $(x\circ y)\circ z^n\subseteq I$ and $(y\circ z^n)\subseteq I$ imply that $(x\circ z^n)\subseteq I$,
(ii) type 2, if for all $x,y,z\in H$, $(x\circ y)\circ z^n\subseteq I$ and $(y\circ z^n)\subseteq I$ imply that $(x\circ z^n)\cap I\neq\varnothing$,
(iii) type 3, if for all $x,y,z\in H$, $(x\circ y)\circ z^n\subseteq I$ and $(y\circ z^n)\subseteq I$ imply that $(x\circ z^n)<I$,
(iv) type 4, if for all $x,y,z\in H$, $(x\circ y)\circ z^n\subseteq I$ and $(y\circ z^n)\cap I\neq\varnothing$ imply that $(x\circ z^n)\subseteq I$,

(v) type 5, if for all $x,y,z \in H$, $(x \circ y) \circ z^n \subseteq I$ and $(y \circ z^n) \cap I \neq \varnothing$ imply that $(x \circ z^n) \cap I \neq \varnothing$,

(vi) type 6, if for all $x,y,z \in H$, $(x \circ y) \circ z^n \subseteq I$ and $(y \circ z^n) \cap I \neq \varnothing$ imply that $(x \circ z^n) < I$,

(vii) type 7, if for all $x,y,z \in H$, $(x \circ y) \circ z^n \subseteq I$ and $(y \circ z^n) < I$ imply that $(x \circ z^n) < I$,

(viii) type 8, if for all $x,y,z \in H$, $(x \circ y) \circ z^n \subseteq I$ and $(y \circ z^n) < I$ imply that $(x \circ z^n) \cap I \neq \varnothing$,

(ix) type 9, if for all $x,y,z \in H$, $(x \circ y) \circ z^n \subseteq I$ and $(y \circ z^n) < I$ imply that $(x \circ z^n) \subseteq I$,

(x) type 10, if for all $x,y,z \in H$, $((x \circ y) \circ z^n) \cap I \neq \varnothing$ and $(y \circ z^n) \subseteq I$ imply that $(x \circ z^n) \cap I \neq \varnothing$,

(xi) type 11, if for all $x,y,z \in H$, $((x \circ y) \circ z^n) \cap I \neq \varnothing$ and $(y \circ z^n) \subseteq I$ imply that $(x \circ z^n) \subseteq I$,

(xii) type 12, if for all $x,y,z \in H$, $((x \circ y) \circ z^n) \cap I \neq \varnothing$ and $(y \circ z^n) \subseteq I$ imply that $(x \circ z^n) < I$,

(xiii) type 13, if for all $x,y,z \in H$, $((x \circ y) \circ z^n) \cap I \neq \varnothing$ and $(y \circ z^n) \cap I \neq \varnothing$ imply that $(x \circ z^n) \subseteq I$,

(xiv) type 14, if for all $x,y,z \in H$, $((x \circ y) \circ z^n) \cap I \neq \varnothing$ and $(y \circ z^n) \cap I \neq \varnothing$ imply that $(x \circ z^n) \cap I \neq \varnothing$,

(xv) type 15, if for all $x,y,z \in H$, $((x \circ y) \circ z^n) \cap I \neq \varnothing$ and $(y \circ z^n) \cap I \neq \varnothing$ imply that $(x \circ z^n) < I$,

(xvi) type 16, if for all $x,y,z \in H$, $((x \circ y) \circ z^n) \cap I \neq \varnothing$ and $(y \circ z^n) < I$ imply that $(x \circ z^n) < I$,

(xvii) type 17, if for all $x,y,z \in H$, $((x \circ y) \circ z^n) \cap I \neq \varnothing$ and $(y \circ z^n) < I$ imply that $(x \circ z^n) \cap I \neq \varnothing$,

(xviii) type 18, if for all $x,y,z \in H$, $((x \circ y) \circ z^n) \cap I \neq \varnothing$ and $(y \circ z^n) < I$ imply that $(x \circ z^n) \subseteq I$,

(xix) type 19, if for all $x,y,z \in H$, $(x \circ y) \circ z^n < I$ and $(y \circ z^n) \cap I \neq \varnothing$ imply that $(x \circ z^n) < I$,

(xx) type 20, if for all $x,y,z \in H$, $(x \circ y) \circ z^n < I$ and $(y \circ z^n) \cap I \neq \varnothing$ imply that $(x \circ z^n) \subseteq I$,

(xxi) type 21, if for all $x,y,z \in H$, $(x \circ y) \circ z^n < I$ and $(y \circ z^n) \cap I \neq \varnothing$ imply that $(x \circ z^n) \cap I \neq \varnothing$,

(xxii) type 22, if for all $x,y,z \in H$, $(x \circ y) \circ z^n < I$ and $(y \circ z^n) \subseteq I$ imply that $(x \circ z^n) \subseteq I$,

(xxiii) type 23, if for all $x,y,z \in H$, $(x \circ y) \circ z^n < I$ and $(y \circ z^n) \subseteq I$ imply that $(x \circ z^n) < I$,

(xxiv) type 24, if for all $x,y,z \in H$, $(x \circ y) \circ z^n < I$ and $(y \circ z^n) \subseteq I$ imply that $(x \circ z^n) \cap I \neq \varnothing$,

(xxv) type 25, if for all $x,y,z \in H$, $(x \circ y) \circ z^n < I$ and $(y \circ z^n) < I$ imply that $(x \circ z^n) < I$,

(xxvi) type 26, if for all $x,y,z\in H$, $(x\circ y)\circ z^n < I$ and $(y\circ z^n) < I$ imply that $(x\circ z^n)\cap I\neq\varnothing$,

(xxvii) type 27, if for all $x,y,z\in H$, $(x\circ y)\circ z^n < I$ and $(y\circ z^n) < I$ imply that $(x\circ z^n)\subseteq I$.

For simplicity of notation we use n-fold PIHKI instead of n-fold Positive Implicative Hyper K-ideal .

Remark. From this definition, we conclude that the notions of 1-fold PIHKI of type j and PIHKI of type j of H coincide, for any $j = 1, 2, ..., 27$.

Theorem 2.5. *Let I be a hyper K-ideal of hyper K-algebra H. If I is an n-fold PIHK of type* 2,3,5,6,7,8,10,12,15,16,19,21,23,24,25, *or* 26. *Then it is also, $n+1$-fold PIHKI of type* 2,3,5,6,7,8,10,12,15,16,19,21,23,24,25, *or* 26, *respectively.*

Proof. Let I be an n-fold PIHKI of type 2, $(x\circ y)\circ z^{n+1}\subseteq I$, and $y\circ z^{n+1}\subseteq I$. By Theorem 1.2 we have $x\circ z^{n+1} < x\circ z^n$. Since I is of type 2, we have $x\circ z^n < I$. On the other hand, I is a hyper K-ideal. So $x\circ z^{n+1} < I$. It means I is an n+1-fold PIHKI of type 2.

For other types the proof is similar.

Open problem. *If I is an n-fold PIHKI of type* 1,2,..., *or* 27, *then is it also n+1-fold PIHKI of type* 1,2,..., *or* 27, *respectively?*

Example 2.6. (1) The following table shows a hyper K-algebra structure on $H = \{0, 1, 2\}$.

$\circ$	0	1	2
0	{0,1}	{0}	{0,1}
1	{1,2}	{0,1}	{0,2}
2	{2}	{1,2}	{0,1,2}

It is easy to check that $I = \{0, 2\}$ is a 2-fold PIHKI of types 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 ,12 , 14, 15 ,16, 17, 19,21, 22, 23, 24, 25 and 26, while $I = \{0, 2\}$ is not a 2-fold PIHKI of type 11, because $((0\circ 2)\circ 0^2)\cap I\neq\varnothing$, $2\circ 0^2\subseteq I$, but $0\circ 0^2\nsubseteq I$. Also $I = \{0, 2\}$ is not a 2-fold PIHKI of type 13, because $((0\circ 2)\circ 0^2\cap I\neq\varnothing$, $2\circ 0^2\subseteq I$, but $0\circ 0^2\subseteq I$. Similarly, by considering $x = 0$, $y = 2$, $z = 0$, we have $I = \{0, 2\}$ is not a 2-fold PIHKI of types 18, 20, 27.

(2) Consider the following hyper K-algebra

$\circ$	0	1	2
0	{0,1}	{0}	{0,1}
1	{1,2}	{0,1}	{0,2}
2	{2}	{1,2}	{0,1,2}

we have $I = \{0 , 2\}$ is a 2-fold PIHKI of types 11, 13.

(3) Consider the following hyper K-algebra

$\circ$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0}	{1}
2	{2}	{0,2}	{0,2}

It can be checked that $I = \{0\ ,\ 2\}$ is a 2-fold PIHKI of types 27.

(4) Let $(X\ ,\ *\ ;\ 0)$ be a BCK-algebra and define a hyper opration "$\circ$" on X by $x{\circ}y{=}\{x{*}y\}$ for all $x,\ y\ {\in}X$. If I is an n-fold positive implicative ideal of the BCK-algebra X, then it is easy to see that $(I\ ,\ *\ ;\ 0)$ is an n-fold PIHKI of types 1, 2, 3,..., or 27.

Theorem 2.7. *Let I be a non-empty subset of H. Then the following statements hold:*

(1) *If I is n-fold PIHKI of type* 4*, then I is n-fold PIHKI of types* 1, 6,

(2) *If I is n-fold PIHKI of type* 5*, then I is n-fold PIHKI of types* 2, 6,

(3) *If I is n-fold PIHKI of type* 6*, then I is n-fold PIHKI of type* 3*,*

(4) *If I is n-fold PIHKI of type* 8*, then I is n-fold PIHKI of type* 7*,*

(5) *If I is n-fold PIHKI of type* 9*, then I is n-fold PIHKI of types* 7, 8*,*

(6) *If I is n-fold PIHKI of type* 11*, then I is n-fold PIHKI of types* 10, 12,

(7) *If I is n-fold PIHKI of type* 10*, then I is n-fold PIHKI of type* 12,

(8) *If I is n-fold PIHKI of type* 13*, then I is n-fold PIHKI of types* 14, 15,

(9) *If I is n-fold PIHKI of type* 14*, then I is n-fold PIHKI of* 15,

(10) *If I is n-fold PIHKI of type* 18*, then I is n-fold PIHKI of* 16, 17,

(11) *If I is n-fold PIHKI of type* 17*, then I is n-fold PIHKI of type* 16,

(12) *If I is n-fold PIHKI of type* 20*, then I is n-fold PIHKI of type* 3*,*

(13) *If I is n-fold PIHKI of type* 21*, then I is n-fold PIHKI of type* 19*,*

(14) *If I is n-fold PIHKI of type* 24*, then I is n-fold PIHKI of type* 23*,*

(15) *If I is n-fold PIHKI of type* 22*, then I n-fold PIHKI of type* 24*,*

(16) *If I is n-fold PIHKI of type* 27*, then I is n-fold PIHKI of type* 26.

Proof. The proof is straightforward.

The following examples show that the converse of the statements of Theorem 2.7 are not true in general.

Example 2.8. The following tables show some hyper K-algebra structures on H $= \{0, 1, 2\}$.

(1):

$\circ$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0}	{1}
2	{2}	{0,1}	{0,1,2}

We can see that $I = \{0, 1\}$ is a 2-fold PIHKI of type 6, while I is not a 2-fold PIHKI of type 4, because $((2\circ 1)\circ 0^2)\subseteq I$ and $(1\circ 0^2)\cap I\neq\varnothing$, but $(2\circ 0^2)\nsubseteq I$.

(2):

$\circ$	0	1	2
0	{0}	{0,1,2}	{0,1,2}
1	{1}	{0,2}	{1,2}
2	{2}	{0,1}	{0,1,2}

We can see that $I = \{0, 1\}$ is a 2-fold PIHKI of type 7, while I is not a 2-fold PIHKI of type 8, because $((2\circ 1)\circ 0^2)\subseteq I$ and $(1\circ 0^2)<I$, but $(2\circ 0^2)\cap I=\varnothing$. Also $I = \{0, 1\}$ is not a 2-fold PIHKI of type 9, since $((2\circ 1)\circ 0^2)\subseteq I$ and $(1\circ 0^2)<I$, but $(2\circ 0^2)\nsubseteq I$.

(3):

$\circ$	0	1	2
0	{0,1}	{0}	{0,1}
1	{1,2}	{0,1}	{0,2}
2	{2}	{1,2}	{0,1,2}

$I = \{0, 2\}$ is a 2-fold PIHKI of type 10, while I is not a 2-fold PIHKI of type 11, because $((0\circ 1)\circ 2^2)\cap I\neq\varnothing$ and $(1\circ 2^2)\subseteq I$, but $(0\circ 2^2)\subseteq I$.

(4):

$\circ$	0	1	2
0	{0}	{0,1,2}	{0,1,2}
1	{1}	{0,2}	{1,2}
2	{2}	{0,1}	{0,1,2}

$I = \{0, 1\}$ is a 2-fold PIHKI of type 12, while I is not a 2-fold PIHKI of type 11, because $((2\circ 1)\circ 0^2)\cap I\neq\varnothing$ and $(1\circ 0^2)\subseteq I$, but $(2\circ 0^2)\nsubseteq I$.

(5):

$\circ$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0}	{1}
2	{2}	{0,1}	{0,1,2}

$I = \{0, 1\}$ is a 2-fold PIHKI of type 15, while I is not a 2-fold PIHKI of type 13, because $((2\circ 1)\circ 2^2)\cap I\neq\varnothing$ and $(1\circ 2^2)\cap I\neq\varnothing$, but $(2\circ 2^2)\nsubseteq I$.

(6):

$\circ$	0	1	2
0	{0,1}	{0}	{0,1}
1	{1,2}	{0,1}	{0,2}
2	{2}	{1,2}	{0,1,2}

$I = \{0, 2\}$ is a 2-fold PIHKI of type 14, while I is not a 2-fold PIHKI of type 13, because $((0\circ 1)\circ 2^2)\cap I\neq\varnothing$ and $(1\circ 2^2)\cap I\neq\varnothing$, but $(0\circ 2^2)\nsubseteq I$.

(7):

$\circ$	0	1	2
0	{0}	{0,1,2}	{0,1,2}
1	{1}	{0,2}	{1,2}
2	{2}	{0,1}	{0,1,2}

$I = \{0, 2\}$ is a 2-fold PIHKI of type 15, while I is not a 2-fold PIHKI of type 14, because

$((1\circ 2)\circ 0^2)\cap I\neq\varnothing$ and $(2\circ 0^2)\cap I\neq\varnothing$, but $(1\circ 0^2)\cap I=\varnothing$.

(8):

$\circ$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0}	{1}
2	{2}	{0,2}	{0,2}

$I = \{0, 1\}$ is a 2-fold PIHKI of type 16, while I is not a 2-fold PIHKI of type 18, because $((2\circ 1)\circ 2^2)\cap I\neq\varnothing$ and $(1\circ 2^2)\cap I\neq\varnothing$, but $(2\circ 2^2)\nsubseteq I$.

(9):

$\circ$	0	1	2
0	{0,1}	{0}	{0,1}
1	{1,2}	{0,1}	{0,2}
2	{2}	{1,2}	{0,1,2}

We see that $I = \{0, 2\}$ is a 2-fold PIHKI of type 17, while I is not a 2-fold PIHKI of type 18, because $((0\circ 1)\circ 2^2)\cap I\neq\varnothing$ and $(1\circ 2^2)<I$, but $(0\circ 2^2)\nsubseteq I$.

(10):

$\circ$	0	1	2
0	{0}	{0,1,2}	{0,1,2}
1	{1}	{0,2}	{1,2}
2	{2}	{0,1}	{0,1,2}

$I = \{0, 1\}$ is a 2-fold PIHKI of type 16, while I is not a 2-fold PIHKI of type 17, because $((2\circ 1)\circ 0^2)\cap I\neq\varnothing$ and $(1\circ 0^2)<I$, but $(2\circ 0^2)\cap I=\varnothing$.

Also we see that $I = \{0, 1\}$ is a 2-fold PIHKI of type 19, while I is not a 2-fold PIHKI of type 20, becaus $((2\circ 1)\circ 2^2)\cap I\neq\varnothing$ and $(1\circ 2^2)\cap I=\varnothing$, but $(2\circ 2^2)\nsubseteq I$.

(11):

$\circ$	0	1	2
0	{0,1}	{0}	{0,1}
1	{1,2}	{0,1}	{0,2}
2	{2}	{1,2}	{0,1,2}

$I = \{0, 2\}$ is a 2-fold PIHKI of type 24, while I is not a 2-fold PIHKI of type 22, because $((1\circ 2)\circ 0^2)<I$ and $(2\circ 0^2)\subseteq I$, but $(1\circ 0^2)\nsubseteq I$.

(12):

$\circ$	0	1	2
0	{0}	{0,1,2}	{0,1,2}
1	{1}	{0,2}	{1,2}
2	{2}	{0,1}	{0,1,2}

We see that $I = \{0, 1\}$ is a 2-fold PIHKI of type 25, but I is not a 2-fold PIHKI of type 26, because $((2\circ 1)\circ 0^2)<I$ and $(1\circ 0^2)<I$, $(2\circ 0^2)\cap I=\varnothing$.

(13):

$\circ$	0	1	2
0	{0,1}	{0}	{0,1}
1	{1,2}	{0,1}	{0,2}
2	{2}	{1,2}	{0,1,2}

$I = \{0, 2\}$ is a 2-fold PIHKI of type 26, while I is not a 2-fold PIHKI of type 27, because $((0\circ 1)\circ 2^2) < I$ and $(1\circ 2^2) < I$, but $(0\circ 2^2) \nsubseteq I$.

Also we see that $I = \{0, 1\}$ is a 2-fold PIHKI of types 2 and 3, while I is not a 2-fold PIHKI of type 1, because $((2\circ 1)\circ 0^2) \subseteq I$ and $(1\circ 0^2) \subseteq I$, but $(2\circ 0^2) \nsubseteq I$.

Theorem 2.9. *Let I be a hyper K-algebra of H. Then the following statement are equivalent:*

(1) $x\circ y^n < I$,

(2) $(x\circ y^n) \cap I \neq \varnothing$.

Proof. (1)$\Rightarrow$(2) Assume $x\circ y^n < I$, then there exist $a \in I$, and $t \in x\circ y^n$ such that $t < a$. Thus $0 \in t\circ a$. Now, since $0 \in I$ and $0 \in 0\circ a$, then $t\circ a < I$. So $t \in I$. Hence, $(x\circ y^n) \cap I \neq \varnothing$.

(2)$\Rightarrow$(1) It is obvious.

Theorem 2.10. *Let I be a hyper K-ideal of a hyper K-algebra H. Then the following statements are equivalent:*

(1) *I is an n-fold PIHKI of type* 14*,*

(2) *I is an n-fold PIHKI of type* 15*,*

(3) *I is an n-fold PIHKI of type* 16*,*

(4) *I is an n-fold PIHKI of type* 17*,*

(5) *I is an n-fold PIHKI of type* 19*,*

(6) *I is an n-fold PIHKI of type* 21*,*

(7) *I is an n-fold PIHKI of type* 25*,*

(8) *I is an n-fold PIHKI of type* 26*.*

Proof. (1)$\Rightarrow$(2) Let I be an n-fold PIHKI of type 14. So for all $x,y,z \in H$, if $((x\circ y)\circ z^n) \cap I \neq \emptyset$, and $(y\circ z^n) \cap I \neq \emptyset$, then $(x\circ z^n) \cap I \neq \emptyset$. On the other hand, by Theorem 2.9 we have $(x\circ z^n) < I$. Thus I is of type 15.

(8)$\Rightarrow$(1) Let I be an n-fold PIHKI of type 26. So for all $x,y,z \in H$, if $((x\circ y)\circ z^n) < I$, and $y\circ z^n < I$, then $(x\circ z^n) \cap I \neq \emptyset$. Now, by Theorem 2.9 we have $((x\circ y)\circ z^n) \cap I \neq \emptyset$, and so, $(y\circ z^n) \cap I \neq \emptyset$ implies that $(x\circ z^n) \cap I \neq \emptyset$. Thus I is of type 14.

The proof of other statements can be obtained by the same way.

Theorem 2.11. *Let I be a hyper K-ideal of a hyper K-algebra H. Then the following statements are equivalent:*

(1) *I is an n-fold PIHKI of type* 13*,*

(2) *I is an n-fold PIHKI of type* 18,

(3) *I is an n-fold PIHKI of type* 20,

(4) *I is an n-fold PIHKI of type* 27.

Proof. By considering Theorem 2.9 the proof is easy.

Theorem 2.12. *Let I be a hyper K-ideal of a hyper K-algebra H. Then the following statements are equivalent:*

(1) *I is an n-fold PIHKI of type* 10,

(2) *I is an n-fold PIHKI of type* 23,

(3) *I is an n-fold PIHKI of type* 12,

(4) *I is an n-fold PIHKI of type* 24.

Proof. By considering Theorem 2.9 the proof is easy.

Theorem 2.13. *Let I be a hyper K-ideal of a hyper K-algebra H. Then the following statements hold:*

(1) *If I is of type* $3'$ *then it is of type* 3,7,8,9,13,14,15,,16,17,18,20,26, *and* 27.

(2) *If I is of type* $4'$ *then it is of type* 3,7,10,12,14,15,16,17,19,21,23,24,25, *and* 26.

Proof. Assume that $(x \circ y) \circ z^n \subseteq I$ and $y \circ z^n < I$. Since:

$$((x \circ z^n) \circ z^n) \circ ((x \circ y) \circ z^n) < (x \circ z^n) \circ (x \circ y) < (y \circ z^n) < I$$

then

$$((x \circ z^n) \circ z^n) \circ ((x \circ y) \circ z^n) < I.$$

Since I is a hyper K-ideal, $((x \circ z^n) \circ z^n) < I$. Then by our hypothesis $x \circ z^n \subseteq I$, i.e. I is of type 9. Thus, by Theorem 2.7 it is of type 7,8.

By the same way, it can be proved it is of type 27. Thus, by Theorem 2.7 it is of type 26. Other types can be obtained by Theorems 2.11, and 2.7, similarly.

By the same way, and by considering Theorems 2.7, 2.10, and 2.12, it can be proved 2 is true.

Theorem 2.14. *Let* $0 \in H$ *be a right scalar element of a hyper K-algebra H and I be an n-fold PIHKI of type* 11,13,14,21,22 *or* 24. *Then I is a hyper K-ideal.*

Proof. Let $x,y \in H$, I be an n-fold PIHKI of type 11, $(x \circ y) \cap I \neq \varnothing$, and $y \in I$. Since $0 \in H$ is a right scalar element, we have $((x \circ y) \circ 0^n) \cap I \neq \varnothing$ and $\{y\} = y \circ 0 = y \circ 0^n \subseteq I$. Thus $\{x\} = x \circ 0 = x \circ 0^n \subseteq I$, then $x \in I$. Therefore I is a hyper K-ideal. The proof of each of the n-fold PIHKI of types 13,14,20,21,22 or 24 is the same.

Example 2.15. (1) The following table shows a hyper K-algebra structure on $H = \{0, 1, 2\}$.

$\circ$	0	1	2
0	{0,1}	{0,1,2}	{0,1,2}
1	{1}	{0,1}	{1,2}
2	{1,2}	{0,1,2}	{0,1,2}

Then $I = \{0\ ,\ 1\}$ is an n-fold PIHKI of type 21, for any $n \in N$, while I is not a hyper K-ideal, because $(2 \circ 1) \cap I \neq \varnothing$, and $1 \in I$, but $2 \notin I$. Also we see that $0 \in H$ is not a right scalar element.

(2) Consider the following hyper K-algebra

$\circ$	0	1	2
0	{0,1}	{0}	{0,1}
1	{1,2}	{0,1}	{0,2}
2	{2}	{1,2}	{0,1,2}

We see that $0 \in H$ is not a right scalar element and $I = \{0\ ,\ 2\}$ is an n-fold PIHKI of types 11, 14, 22, and 24, for any $n \in N$, while I is not a hyper K-ideal, because $(1 \circ 2) \cap I \neq \varnothing$, and $2 \in I$, but $1 \notin I$. $I = \{0\ ,\ 1\}$ is an n-fold PIHKI of type 13, for any $n \in N$, while I is not a hyper K-ideal, because $(2 \circ 1) \cap I \neq \varnothing$, and $1 \in I$, but $2 \notin I$.

Note that Example 2.15 shows the condition $0 \in H$ is a right scalar is necessary in Theorem 2.14.

Theorem 2.16. *Let $0 \in H$ be a right scalar element of a hyper K-algebra H and I be closed. If I is an n-fold PIHKI of type* 12, 15, 16, 19 *or* 23*, then I is a weak hyper K-ideal.*

Proof. Let I be an n-fold PIHKI of type 12, $x,y \in H$, $(x \circ y) \subseteq I$ and $y \in I$. Since $0 \in H$ is a right scalar element, $((x \circ y) \circ 0^n) \cap I \neq \varnothing$ and $(y \circ 0) \subseteq I$ imply that $(x \circ 0^n) < I$. So there exists $i \in I$ such that $x \circ 0^n < i$. Therefore $(x \circ 0^{n-1}) \circ i < 0$. Thus, there exists $k \in (x \circ 0^{n-1}) \circ i$ such that $k < 0$. Hence, by $0 < k$ we have $k = 0$, i.e. $0 \in (x \circ 0^{n-1}) \circ i$. It means $x \circ 0^{n-1} < i$. Repeatedly using this way it follows $x < i$. Now since I is closed, we obtain that $x \in I$. Therefore I is a weak hyper K-ideal. The proof of each of the n-fold PIHKI of types 15, 16, 19, or 23 is the same.

Theorem 2.17. *Let $0 \in H$ be a right scalar element of a hyper K-algebra H and I be an n-fold PIHKI of type* 18, 20, 26 *or* 27*. Then I is a weak hyper K-ideal.*

Proof. The proof is similar to the proof of Theorem 2.16.

Example 2.18. Consider the following hyper K-algebra

$\circ$	0	1	2
0	{0,1}	{0}	{0,1}
1	{1,2}	{0,1}	{0,2}
2	{2}	{1,2}	{0,1,2}

In this example $I = \{0\ ,\ 2\}$ is an n-fold PIHKI of types 12, 15, 16, 19, and 23, for any $n \in N$, while I is not a hyper K-ideal, because we see that I is not closed and $0 \in H$ is not a right scalar.

Definition 2.19. Let H be a hyper K-algebra and $I \subseteq H$ and $a \in I$. We define $I_{a^n} = \{\ x \in H \mid (x \circ a^n) \cap I \neq \varnothing\ \}$.

Theorem 2.20. *Let H be a hyper K-algebra. Then I is an n-fold PIHKI of type* 14 *if and only if for all $a \in H$, I_{a^n} is a hyper K-ideal.*

Proof. Let for all x,y,$a \in H$, $((x \circ y) \cap I_{a^n}) \neq \varnothing$, $y \in I_{a^n}$. Then $((x \circ y) \circ a^n) \cap I \neq \varnothing$, $(y \circ a^n) \cap I \neq \varnothing$. Since I is an n-fold PIHKI of type 14, $(x \circ a^n) \cap I \neq \varnothing$. Therefore $x \in I_{a^n}$, i.e. I is a hyper K-ideal.

Conversely, let for all x,y,$a \in H$, $((x \circ y) \circ a^n) \cap I \neq \varnothing$ and $(y \circ a^n) \cap I \neq \varnothing$. Then, $x \circ y \subseteq I_{a^n}$. So,by Theorem 2.9 $x \circ y < I_{a^n}$. Now, since $y \in I_{a^n}$ and I_{a^n} is a hyper K-ideal, we obtain $x \in I_{a^n}$. Thus, $(x \circ a^n) \cap I \neq \varnothing$, i.e. I is an n-fold PIHKI of type 14.

3. n-fold positive implicative hyper K-ideals in simple hyper K-algebras

In this part $(H, \circ, 0)$ is a simple hyper K-algebra, unless otherwise is stated.

Theorem 3.1. *Let $0 \in I \subseteq H$. Then*

(i) *I is an n-fold PIHKI of type* 2 *if and only if I is an n-fold PIHKI of type* 3*,*

(ii) *I is an n-fold PIHKI of type* 4 *if and only if I is an n-fold PIHKI of type* 9*,*

(iii) *I is an n-fold PIHKI of type* 5 *if and only if I is an n-fold PIHKI of type* 6(7,8)*,*

(iv) *I is an n-fold PIHKI of type* 11 *if and only if I is an n-fold PIHKI of type* 22*,*

(v) *I is an n-fold PIHKI of type* 10 *if and only if I is an n-fold PIHKI of type* 12(23,24)*,*

(vi) *I is an n-fold PIHKI of type* 13 *if and only if I is an n-fold PIHKI of type* 18(20,27)*,*

(vii) *I is an n-fold PIHKI of type* 14 *if and only if I is an n-fold PIHKI of type* 15(16,17,19,21,25,26).

Proof. The proof follows from Definition 2.4 and Theorem 1.11.

Theorem 3.2. *Let $a \in H - \{0\}$ and $I = H - \{a\}$ be a hyper K-ideal. Then I is an n-fold PIHKI of type* 25(14,15,16,17,19,21,26) *if and only if $|a \circ b^n| = 1$, for all $b \in I$.*

Proof. Let I be an n-fold PIHKI of type 25. Then we prove that $|a \circ b^n| = 1$, for all $b \in I$. On the contrary, let $|a \circ b^n| > 1$, for some $b \in I$. By Theorem 1.11(ii) we have $a \in a \circ b^n$. So there exists $c \in H - \{a\}$ such that $c \in a \circ b^n$. Thus $(a \circ 0) \circ b^n = (a \circ b^n) \circ 0 < I$ and $0 \circ b^n < I$ imply that $a \circ b^n < I$. It means $(a \circ b) \circ b^{n-1} < I$. So $(a \circ b) \circ b^{n-1} < I$ and $b \circ b^{n-1} < I$ imply that $a \circ b^{n-} < I$. Repeatedly using this way it follows $a \circ b < I$. Since I is a hyper K-ideal and $b \in I$ we have $a \in I$, which is a contradiction. Therefore $|a \circ b^n| = 1$, for all $b \in I$.

Conversely, let $|a \circ b^n| = 1$, for all $b \in I$. We show that I is an n-fold PIHKI of type 25. On the contrary, let $(x \circ y) \circ z^n < I$ and $y \circ z^n < I$, but $x \circ z^n \not< I$, for some $x, y, z \in H$. $x \circ z^n \not< I$ implies that $x \neq z$. By Theorem 1.11(ii) $x \in x \circ z$. Thus by hypothesis we obtain $x = a$. If $x = y$, then $y \circ z^n = a \circ z^n = \{a\} \not< I$, which is a contradiction. If $x \neq y$, then $(x \circ y) \circ z^n = a \circ z^n = \{a\} \not< I$, which is a contradiction. Therefore I is an n-fold PIHKI of type 25.

Theorem 3.3. *Let Let $a \in H - \{0\}$ and $I = H - \{a\}$. If I is an n-fold PIHKI of type* 27(13,18,20), *then*

(i) $|a \circ b^n| = 1$, *for all $b \in I$,*

(ii) $b \circ c^n \neq H$, *for all $b, c \in H$.*

Proof. (i) On the contrary, let $|a \circ b^n| > 1$, for some $b \in I$. Then there exists $t \in H - \{a\}$ such that $t \in a \circ b^n$. So $(a \circ t) \circ b^n < I$. Thus $(a \circ t) \circ b^n < I$ and $t \circ b^n < I$ imply that $a \circ b^n \subseteq I$, which is a contradiction. Because By Theorem 1.11(ii) $a \in a \circ b^n \subseteq I$ and so $a \in I$. Therefore $|a \circ b^n| = 1$, for all $b \in I$.

(ii) If there exist $b, c \in H$ such that $b \circ c^n \neq H$, then $(b \circ 0) \circ c^n < I$ and $o \circ c^n < I$ imply that $H = b \circ c^n \subseteq I$, which is impossible. Therefore $b \circ c^n \neq H$, for all $b, c \in H$.

The following example shows that the converse of the above theorem is not true in general.

Example 3.4. The following table shows a simple hyper K-algebra structure on $H = \{0, 1, 2, 3\}$.

$\circ$	0	1	2	3
0	{0}	{0}	{0,2}	{0}
1	{1}	{0}	{1,2}	{1}
2	{2}	{2}	{0}	{2}
3	{3}	{3}	{2,3}	{0}

We can see that $2 \circ b^2 = \{2\}$, for all $b \in H - \{2\}$, and $b \circ c^2 \neq H$, for all $b, c \in H$, but $I = H - \{2\}$ is not a 2-fold PIHKI of type 27. Because $(2 \circ 0) \circ 2^2 < I$ and $0 \circ 2^2 < I$, while $2 \circ 2^2 = \{0, 2\} \nsubseteq I$.

Theorem 3.5. *Let $a \in H - \{0\}$ and $I = H - \{a\}$. Then I is an n-fold PIHKI of type* $10(12, 23, 24)$ *if and only if $|a \circ b^n| = 1$, for all $b \in I$.*

Proof. The proof is similar to the proof of Theorem 3.2, by imposing some modifications.

Theorem 3.6. *Let $a \in H - \{0\}$ and $I = H - \{a\}$. If $|a \circ b^n| = 1$, for all $b \in I$, then I is an n-fold PIHKI of type* $6(5, 7, 8)$.

Proof. Let $(x \circ y) \circ z^n \subseteq I$, and $(y \circ z^n) < I$. We show that $x \circ z^n < I$. If $x = z$, it is clear that $x \circ z^n < I$. Now let $x \neq z$. Consider two cases: case(1): $x \neq a$, and case(2): $x = a$. Case(1): By Theorem 1.11(ii) we obtain $x \in x \circ z^n$ and so $x \circ z^n < I$. case(2): We consider the following two sub-cases and show that $(x \circ y) \circ z^n \nsubseteq I$ or $y \circ z \nless I$.

case(i'): $y = x$ implies that $\{a\} = y \circ z^n = x \circ z^n = a \circ z^n \nless I$.

case(ii'): $y \neq x$ implies that $\{a\} = (x \circ y) \circ z^n \nsubseteq I$. Therefore I is an n-fold PIHKI of type 7.

The following example shows that the converse of the above theorem is not true in general.

Example 3.7. The following table shows a simple hyper K-algebra structure on $H = \{0, 1, 2, 3\}$.

$\circ$	0	1	2	3
0	{0}	{0}	{0,2}	{0}
1	{1}	{0}	{1,2}	{1}
2	{2}	{2}	{0}	{2}
3	{3}	{3}	{2,3}	{0}

We can see that $I = H - \{1\}$ is an n-fold PIHKI of type 5,6,7 and 8, but $|1 \circ 2^n| \neq 1$.

Theorem 3.8. *Let $a \in H - \{0\}$ and $I = H - \{a\}$. Then I is an n-fold PIHKI of type* 10(12,14,15,16,17,19,21,23,24,25,26) *if and only if I is a hyper K-ideal of H.*

Proof. The proof follows from Theorems 3.2 and 3.5.

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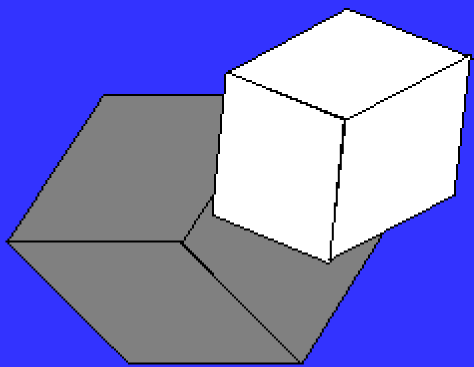

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