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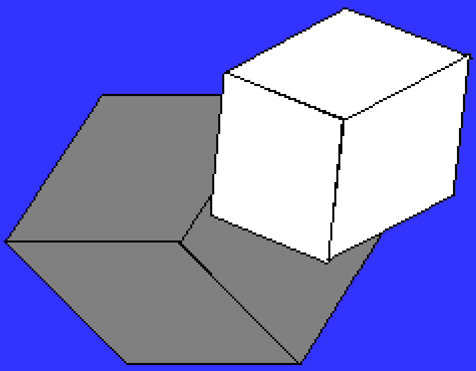

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Professor Emeritus Ioannis Mittas

In memoriam

In April 2012, Professor Emeritus Ioannis Mittas passed away. He was born in 1921 in Veria, Greece and he received his B.S. in Mathematics from Aristotle University of Thessaloniki and Ph.D in Mathematics from the National Technical University of Athens in 1970. In 1973, he was elected Professor, Chair of Advanced Mathematics, Faculty of Engineering Aristotle University of Thessaloniki, where he served till his retirement in 1985, occupying important posts including that of the Dean of the Faculty; straight after his retirement he was awarded the title of Professor Emeritus. He was a member of a numerous societies and member of the editorial committee of several journals. He was awarded *The National Resistance 1941-1944 Medal* for his participation in the resistance during the German Occupation of Greece.

Professors Mittas's contribution to hyperstructure theory was pioneering as he published numerous papers in several journals and proceedings among which his most famous paper *Hypergroupes canoniques*, Math. Balkanica, V.2, (1972). He continued working and publishing results and research papers and his ex PhD students and co-researchers go on publishing results on the topics he introduced and founded.

Professor Emeritus Ioannis Mittas was a great mathematician and a great personality and everyone was eager to work and collaborate with him.

Thomas Vougiouklis

To the first-rate presentation by Prof. Vougiouklis, the Chief-Editor can add only few words:

Prof. Mittas gave first-rate contributions to the Science which assure him a high-ranking place in the History of Mathematics.
Not only his ex PhD students and co-researchers could value his results. For instance I myself also, began to work in Hyperstructures (and I continue still), after having read, by chance, about 40 years ago the paper of his, mentioned above. At that time I did not know him, I met him in the following years, and I could appreciate also his excellent human qualities.

Piergiulio Corsini

The Editorial Board is close to Ioannis Mittas's family

The “Italian Journal of Pure and Applied Mathematics” cannot more take advantage of the precious collaboration of Prof. Ioannis Mittas, who has suddenly passed away.

The members of the Editorial Board and Managing Board express their deep sorrow for this loss.

The Chief-Editor wants to express his most heartfelt sympathy to Prof. Mittas's family for the demise of a first-class scientist and a very dear friend.

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CONVEXITY IN NORMED HYPERVECTOR SPACES

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Abstract. In this paper we obtain some results on convexity in a normed hypervector space. We also investigate the concept of absorbing and balanced set and generalize the corresponding results of vector space.

Keywords: norm; convex set; strictly convex set; hypervector space.

AMS Subject Classification: 46A55; 46B99.

1. Introduction

In 1934 Marty [5] introduced a new mathematical structure as a generalization of groups and called it hypergroup. Subsequently, many authors worked on this new field and constructed some other generalizations such as hyperrings, hypermodules, and hyperfields. In 1988 the notion of hypervector space was given by Tallini [12]. She studied some algebraic properties of this new structure in [8], [9], [10], and [11]. A wealth of applications of these new constructions in: geometry, hypergraphs, binary relations, combinatorics, codes, cryptography, probability, and etc. can be found in [2]. Recently, we studied hypervector spaces in the viewpoint of analysis and generalized some definitions and proved many interesting theorems about them in [6] and [7]. In this paper we focus on some other properties of these spaces. We define convex, strictly convex, balanced, and absorbing subsets of a normed hypervector space and prove some theorems about them.

Let $P(X)$ be the power set of a set X, $P^\star(X) = P(X)\backslash\{\varnothing\}$, and K a field. A *hypervector space* over K that is defined in [8], is a quadruplet $(X, +, \circ, K)$ such that $(X, +)$ is an abelian group and

$$\circ : K \times X \longrightarrow P^\star(X)$$

is a mapping that for all a, $b \in K$ and x, $y \in X$ the following properties holds:

(i) $(a + b) \circ x \subseteq (a \circ x) + (b \circ x)$,

(ii) $a \circ (x + y) \subseteq (a \circ x) + (a \circ y)$,

(iii) $a \circ (b \circ x) = (ab) \circ x$, where $a \circ (b \circ x) = \{\, a \circ y \ : \ y \in b \circ x \,\}$,

(iv) $(-a) \circ x = a \circ (-x)$

(v) $x \in 1 \circ x$.

Note that every vector space is a hypervector space and specially, every field is a hypervector space over itself.

A non-empty subset of a hypervector space X over a field K is called a *subspace* of X if the following holds:

(i) $H - H \subseteq H$,

(ii) $a \circ H \subseteq H$, for every $a \in K$.

Let $(X, +, \circ, K)$ be a hypervector space. Suppose that for every $a \in K$, $|a|$ denoted the valuation of a in K. A *pseudonorm* on X that is defined in [9], is a mapping

$$\|\cdot\| : X \longrightarrow \mathbb{R}$$

that for all $a \in K$ and x, $y \in X$ has the following properties:

(i) $||0|| = 0$,

(ii) $||x + y|| \leqslant ||x|| + ||y||$,

(iii) $\sup ||a \circ x|| = |a|\, ||x||$.

A pseudonorm on X is called a *norm*, if:

$$||x|| = 0 \Longleftrightarrow x = 0.$$

2. Main results

The norm that is defined on hypervector space $(X, +, \circ, \|\cdot\|, K)$ induced a topology on X as following:

Let $(X, +, \circ, ||\ .\ ||, K)$ be a normed hypervector space. For $x \in X$ and $\epsilon > 0$ the open ball $B_\epsilon(x)$ is defined as

$$B_\epsilon(x) = \{\, y \in X \,:\, ||x - y|| < \epsilon \,\}.$$

The $\{\, B_\epsilon(x) \,:\, x \in X, \epsilon > 0 \,\}$ is a basis for a topology on X which is the topology induced by this norm.

Clearly, the intersection of any collection of convex sets is convex.

Definition 2.1. A non-empty subset E of a hypervector space $X = (X, +, \circ, K)$ is called convex if for any x, $y \in E$ and $0 \leqslant \lambda \leqslant 1$,

$$\lambda \circ x + (1 - \lambda) \circ y \subseteq E.$$

Also X is called strictly convex if $\sup \left\| \frac{1}{2} \circ x + \frac{1}{2} \circ y \right\| < 1$, whenever x, $y \in X$, $x \neq y$, and $||x|| = ||y|| = 1$.

Lemma 2.2 *Let $X = (X, +, \circ, ||\cdot||, K)$ be a normed hypervector space, $x \in X$, and $\epsilon > 0$. If*

$$B_\epsilon(x) = \{\, y \in X : ||x - y|| < \epsilon\},$$

then $B_\epsilon(x)$ is a convex set.

Proof. First, suppose that $x = 0$. Let z, $w \in B_\epsilon(x)$ and $0 \leqslant \lambda \leqslant 1$. Since for all a, $b \in X$, $||a + b|| \leqslant ||a|| + ||b||$ and by the definition of a norm, we have

$$\begin{aligned} \sup ||\lambda \circ z + (1 - \lambda) \circ w|| &\leqslant \sup ||\lambda \circ z|| + \sup ||(1 - \lambda) \circ w|| \\ &= \lambda\, ||z|| + (1 - \lambda)\, ||w|| \leqslant \lambda\epsilon + (1 - \lambda)\epsilon = \epsilon. \end{aligned}$$

On the other hand, it is easy to check that $B_\epsilon(x) = B_\epsilon(0) + x$. So the proof is complete. ■

Theorem 2.3. *Let $X = (X, +, \circ, \|\cdot\|, K)$ be a normed hypervector space, $x, y \in X$, $x \neq y$, and $||x|| = ||y|| = 1$. If X is strictly convex, then*

$$\sup ||t \circ x + (1 - t) \circ y|| < 1,$$

for $0 < t < 1$. Also X is strictly convex if there is some t, $0 < t < 1$, such that

$$\sup ||t \circ x + (1 - t) \circ y|| < 1,$$

for x, $y \in X$ with $x \neq y$, and $||x|| = ||y|| = 1$.

Proof. Suppose $0 < t_0 < 1$ and

$$\sup ||t_0 \circ x + (1 - t_0) \circ y|| < 1.$$

If $0 < t < t_0$, then let $s = \dfrac{t}{t_0}$. So $0 < s < 1$ and

$$\begin{aligned} t \circ x + (1-t) \circ y &= (st_0) \circ x + [s(1-t_0) + 1 - s] \circ y \\ &\subseteq (st_0) \circ x + (s(1-t_0)) \circ y + (1-s) \circ y. \end{aligned}$$

Therefore

$$\begin{aligned} \sup ||t \circ x + (1-t) \circ y|| &\leqslant \sup ||(st_0) \circ x + (s(1-t_0)) \circ y + (1-s) \circ y|| \\ &\leqslant \sup ||(st_0) \circ x + (s(1-t_0)) \circ y|| + \sup ||(1-s) \circ y|| \\ &\leqslant s \sup ||\frac{1}{s}((st_0) \circ x + (s(1-t_0)) \circ y|| + (1-s)||y|| \\ &\leqslant s \sup ||t_0 \circ x + (1-t_0) \circ y|| + (1-s)||y|| \\ &< 1 - s + s = 1, \end{aligned}$$

because $a \sup ||b \circ x|| = ab||x|| = \sup ||(ab) \circ x||$, for every a, $b > 0$.

If $t_0 < t < 1$, let $s = \dfrac{1-t}{1-t_0}$. Then $0 < s < 1$ and

$$\begin{aligned} t \circ x + (1-t) \circ y &= (st_0 + 1 - s) \circ x + s(1-t_0) \circ y \\ &\subseteq (1-s) \circ x + (st_0) \circ x + s(1-t_0) \circ y. \end{aligned}$$

Hence

$$\begin{aligned} \sup ||t \circ x + (1-t) \circ y|| &\leqslant \sup ||(1-s) \circ x + (st_0) \circ x + s(1-t_0) \circ y|| \\ &\leqslant \sup ||(1-s) \circ x|| + \sup ||(st_0) \circ x + s(1-t_0) \circ y|| \\ &\leqslant (1-s)||x|| + s \sup ||t_0 \circ x + (1-t_0) \circ y|| \\ &< 1 - s + s = 1, \end{aligned}$$

as before. Thus

$$\sup ||t \circ x + (1-t) \circ y|| < 1,$$

for $0 < t < 1$, if $\sup ||t_0 \circ x + (1-t_0) \circ y|| < 1$, for some $0 < t_0 < 1$. On the other hand if X is strictly convex, then

$$\sup ||\frac{1}{2} \circ x + \frac{1}{2} \circ y|| < 1.$$

This proves both parts of the theorem. ■

Before proving the next theorem, it is necessary to note the following useful remark.

Remark 2.4. If $X = (X, +, \circ, ||\cdot||, K)$ is a normed hypervector space such that for every x, $y \in X$, $||(x+y)|| = \sup ||1 \circ x + 1 \circ y||$, then for every $0 \neq a \in K$, we have $\sup ||a \circ (x+y)|| = \sup ||a \circ x + a \circ y||$. Because

$$\begin{aligned} \sup ||a \circ (x+y)|| &= |a| \sup ||1 \circ (x+y)|| \\ &= |a| \sup ||1 \circ x + 1 \circ y|| \\ &= \sup ||a \circ (1 \circ x + 1 \circ y)||, \end{aligned}$$

and therefore

$$\begin{aligned}\sup ||a \circ x + a \circ y|| &= |a| \sup ||\tfrac{1}{a} \circ (a \circ x + a \circ y)|| \\ &\leqslant |a| \sup ||1 \circ x + 1 \circ y|| = \sup ||a \circ (x+y)|| \\ &\leqslant \sup ||a \circ x + a \circ y||.\end{aligned}$$

It shows that

$$\sup ||a \circ (x+y)|| = \sup ||a \circ x + a \circ y||.$$

Theorem 2.5. *Let $X = (X, +, \circ, || \cdot ||, K)$ be a normed hypervector space such that for every x, $y \in X$ and $c \in K$,*

(i) $x \in c \circ y$ *implies* $y \in c^{-1} \circ x$,

(ii) $x \in 1 \circ y$ *and* $y \in 1 \circ x$ *implies* $x = y$,

(iii) $||(x+y)|| = \sup ||1 \circ x + 1 \circ y||$,

(iv) $||c \circ x||$ *is a closed set.*

Then X is strictly convex if and only if for all non-zero elements x, $y \in X$, $||x+y|| = ||x|| + ||y||$ implies $x \in c \circ y$, for some $c > 0$.

Proof. First, suppose the condition holds. Let x, $y \in X$ be such that $||x|| = ||y|| = 1$ and $\sup ||\frac{1}{2} \circ x + \frac{1}{2} \circ y|| = 1$. So

$$1 = ||x|| = ||y|| = \sup ||\frac{1}{2} \circ x + \frac{1}{2} \circ y|| = \frac{1}{2}||x+y||$$

and, therefore, $||x+y|| = 2 = ||x|| + ||y||$. Hence, there is $c > 0$ such that $x \in c \circ y$. Since $y \in c^{-1} \circ x$, we have

$$1 = ||x|| \leqslant \sup ||c \circ y|| = c||y|| \leqslant c \sup ||c^{-1} \circ x|| = cc^{-1}||x|| = 1,$$

and, therefore, $c = 1$. It means that $x \in 1 \circ y$ and $y \in 1 \circ x$. So, $x = y$ and X is strictly convex.

To complete the proof, suppose X is strictly convex and let non-zero elements $x, y \in X$ be such that $||x+y|| = ||x|| + ||y||$. Also let $x_1 \in \frac{1}{||x||} \circ x$ and $y_1 \in \frac{1}{||y||} \circ y$ be such that $||x_1|| = ||y_1|| = 1$. Since $x \in ||x|| \circ x_1$ and $y \in ||y|| \circ y_1$, then

$$||x+y|| \leqslant \sup ||\frac{1}{||x||} \circ x_1 + \frac{1}{||y||} \circ y_1|| \leqslant \sup ||1 \circ x + 1 \circ y|| = ||x+y||.$$

So $\sup ||||x|| \circ x_1 + ||y|| \circ y_1|| = ||x|| + ||y||$. It means that

$$\sup ||\frac{||x||}{||x|| + ||y||} \circ x_1 + \frac{||y||}{||x|| + ||y||} \circ y_1|| = 1,$$

where $0 < \dfrac{||x||}{||x|| + ||y||} < 1$. By Theorem 2.3, we have $x_1 = y_1$. Hence

$$x \in ||x|| \circ x_1 = ||x|| \circ y_1 \subseteq \frac{||x||}{||y||} \circ y.$$

This completes the proof. ■

Example 2.6. Let $(\mathbb{R}^n, +)$ be the classical additive group over $\mathbb{R}^n$ and for every $a \in \mathbb{R}$,

$$a \circ x = \{\, tax \,:\, 0 \leqslant t \leqslant 1 \,\},$$

where tax is the classical multiplication of $\mathbb{R}$ over $\mathbb{R}^n$. Now, let $||x||$ be the distance of x from the origin in $\mathbb{R}^n$. Then it is easily seen that $(\mathbb{R}^n, +, \circ, ||\cdot||, \mathbb{R})$ is a normed hypervector space which is satisfied in the hypothesis of Theorem 2.5.

Definition 2.7. A non-empty subset E of a hypervector space $X = (X, +, \circ, K)$ is called balanced if $k \circ x \subseteq E$, whenever $x \in E$ and $k \in K$ with $|k| \leqslant 1$, and it is called absorbing if for every $x \in X$ there exists $r > 0$ such that

$$\frac{1}{r} \circ x \subseteq E.$$

Theorem 2.8. *Let $X = (X, +, \circ, ||\cdot||, K)$ be a normed hypervector space such that*

(i) $0 \circ x = k \circ 0 = \{0\}$,

(ii) $x \in k \circ y$ *implies that* $y \in k^{-1} \circ x$,

(iii) $\{\, k \circ x \,:\, k \in K \,\}$ *is a subspace of* X,

for every $x, y \in X$, $k \in K$. Also suppose that E is a convex, balanced and absorbing subset of X such that no non-zero subspace of X is contained in E. If

$$||x|| = \inf\{\, r > 0 \,:\, \frac{1}{r} \circ x \subseteq E \,\}, \tag{1}$$

then $||\cdot||$ is a norm on X, and that

$$\{\, x \in X : ||x|| < 1 \,\} \subseteq E \subseteq \{\, x \in X : ||x|| \leqslant 1\}.$$

Furthermore, for every norm in a hypervector space X, there is a convex, balanced and absorbing subset E of X such that it stabilizes (1).

Proof. Let $x \in X$ and suppose that $||x|| = \inf S_x$, where

$$S_x = \{\, r > 0 : \frac{1}{r} \circ x \subseteq E \,\}.$$

Then $S_x \subseteq (0,\, \infty)$, and hence $||x|| \geqslant 0$, for every $x \in X$. If $x \in E$ then $-x \in (-1) \circ x \subseteq E$, because E is balanced. So

$$\{0\} = \frac{1}{2} \circ 0 = \frac{1}{2} \circ (x - x) \subseteq \frac{1}{2} \circ x + \frac{1}{2} \circ (-x) \subseteq E,$$

because E is convex. Thus $\{0\} = \frac{1}{r} \circ 0 \subseteq E$, for every $r > 0$, and therefore $S_0 = (0,\, \infty)$ and $||0|| = \inf S_0 = 0$.

Now, suppose that $x \neq 0$. Since $0 \neq x \in 1 \circ x \subseteq Y$, then $Y = \{\, k \circ x \,:\, k \in K \,\}$ is a non-zero subspace of X. So, Y is not contained in E. Hence there is $k_1 \in K$ such that $k_1 \circ x \nsubseteq E$. Clearly, $k_1 \neq 0$. Suppose that $0 < r < \dfrac{1}{|k_1|}$ and $r \in S_x$. Then, $\dfrac{1}{r} \circ x \subseteq E$, and since $|K_1 r| < 1$ and E is balanced, we have

$$k_1 \circ x = (k_1 r) \circ \left(\frac{1}{r} \circ x \right) \subseteq E.$$

This contradiction shows that if $r \in S_x$, then $\dfrac{1}{|k_1|} \leqslant r$. Thus, we have proved that $||x|| = 0$ if and only if $x = 0$.

Next, assume that $k \neq 0$ and $r \in S_{k \circ x} = \bigcup\limits_{y \in k \circ x} S_y$. Then there is $y \in k \circ x$ such that $r \in S_y$. Therefore, $\dfrac{1}{r} \circ y \subseteq E$, $x \in k^{-1} \circ y$ and since E is balanced, we have

$$\frac{|k|}{r} \circ x = \frac{|k|}{k} \circ \left(\frac{k}{r} \circ x \right) \subseteq \frac{|k|}{k} \circ \left(\frac{k}{r} \circ (k^{-1} \circ y) \right) = \frac{|k|}{k} \circ \left(\frac{1}{r} \circ y \right) \subseteq E.$$

It means that $\dfrac{r}{|k|} \in S_x$ and then $||x|| \leqslant \dfrac{r}{|k|}$. So, for every $r \in S_y$ that $y \in k \circ x$, we have $|k| ||x|| \leqslant r$ and hence

$$|k| ||x|| \leqslant ||y||.$$

So, for every $k \neq 0$,

$$|k| ||x|| \leqslant \sup ||k \circ x||.$$

By changing k to $\frac{1}{k}$, we have

$$\left| \frac{1}{k} \right| ||y|| \leqslant \sup \left\| \frac{1}{k} \circ y \right\|,$$

for every $y \in k \circ x$. Therefore, there is $z \in \dfrac{1}{k} \circ y$ such that $\left| \dfrac{1}{k} \right| ||y|| \leqslant ||z||$. So,

$$|k| ||z|| \leqslant ||y|| \leqslant |k| ||z||.$$

It means that $\frac{1}{|k|}||y|| = ||z||$. Since $z \in \frac{1}{k} \circ y \subseteq \frac{1}{k} \circ (k \circ x) = 1 \circ x$, we conclude that

$$\sup ||k \circ x|| \leqslant |k| \sup ||1 \circ x||.$$

Now, we show that $\sup ||1 \circ x|| = ||x||$. Let $z \in 1 \circ x$. Then, $x \in 1 \circ z$ and $\frac{1}{r} \circ z \subseteq E$, for every $r \in S_z$, and also

$$\frac{1}{r} \circ x \subseteq \frac{1}{r} \circ (1 \circ z) = \frac{1}{r} \circ z \subseteq E.$$

It means that $r \in S_x$. Similarly, one can show that if $r \in S_x$, then $r \in S_z$. Hence $||z|| = ||x||$. So by the last inequality, we have

$$\sup ||k \circ x|| \leqslant |k|||x||,$$

and, therefore, we have shown that

$$\sup ||k \circ x|| = |k|||x||,$$

for every $k \neq 0$. Clearly, it is true for $k = 0$.

To prove the triangular inequality, let x, $y \in X$. Given $\epsilon > 0$, we can find $r \in S_x$ and $s \in S_y$ such that $r < ||x|| + \epsilon$ and $s < ||y|| + \epsilon$. Then $\frac{1}{r} \circ x \subseteq E$, $\frac{1}{r} \circ y \subseteq E$, and since E is convex, we have

$$\frac{1}{r+s} \circ (x+y) \subseteq \frac{1}{r+s} \circ x + \frac{1}{r+s} \circ y = \frac{r}{r+s} \circ \left(\frac{1}{r} \circ x\right) + \frac{s}{r+s} \circ \left(\frac{1}{s} \circ y\right) \subseteq E.$$

Hence $r + s \in S_{x+y}$, and

$$||x+y|| \leqslant r + s < ||x|| + ||y|| + 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have

$$||x+y|| \leqslant ||x|| + ||y||.$$

Thus $|| \cdot ||$ is a norm on X.

Next, let $||x|| < 1$, then there is $r \in S_x$ with $||x|| < r < 1$. Now, since E is balanced, $\frac{1}{r} \circ x \subseteq E$ and $x \in 1 \circ x = r \circ \left(\frac{1}{r} \circ x\right) \subseteq E$, and therefore

$$\{x \in X : ||x|| < 1\} \subseteq E.$$

If $x \in E$, then $1 \circ x \subseteq E$, because E is balanced and hence $||x|| \leqslant 1$. Thus

$$E \subseteq \{x \in X : ||x|| \leqslant 1\}.$$

For the last part of the theorem, let $X = (X, +, \circ, || \cdot ||, K)$ be a normed hypervector space. Put $E = \{\, x \in X : ||x|| < 1\}$. Then by Lemma 2.2, E is convex. If $x \in E$ and $r \in K$ such that $|r| \leqslant 1$, then $\sup ||r \circ x|| = |r|||x|| < 1$ and therefore $r \circ x \subseteq E$ and E is balanced. For $0 \neq x \in X$, let $r = 2||x||$, then $\sup \left\| \frac{1}{2||x||} \circ x \right\| = \frac{1}{2}$ and hence E is absorbing. Let $x \in X$ and

$$S_x = \left\{ r > 0 : \frac{1}{r} \circ x \subseteq E \right\}.$$

If $r \in S_x$, then $\frac{1}{r} \circ x \subseteq E$ and so $\frac{1}{r}||x|| = \sup \left\| \frac{1}{r} \circ x \right\| < 1$. Thus $||x|| < r$, for every $r \in S_x$. This implies that $||x|| \leqslant \inf S_x$. If $||x|| < \inf S_x$, then choose r with $||x|| < r < \inf S_x$. Thus, $\sup \left\| \frac{1}{r} \circ x \right\| = \frac{1}{r}||x|| < 1$, and so $\frac{1}{r} \circ x \subseteq E$, that is $r \in S_x$. This is not possible since $r < \inf S_x$. So, $||x|| = \inf S_x$ and the proof is complete. ■

Example 2.9. One can easily check that Example 2.6 satisfies in the hypothesis of Theorem 2.8, too.

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DOMINATION IN THE INTERSECTION GRAPHS OF RINGS AND MODULES

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Abstract. In this paper we obtain domination number in the intersection graphs of ideals of rings, and intersection graphs of submodules of modules.

Keywords: intersection graph, ideal, ring, module, domination.

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1. Introduction

The graph theory and algebraic notation and terminology follow from [9] and [7], respectively. Let $F = \{S_i : i \in I\}$ be an arbitrary family of sets. The *intersection graph* $G(F)$ of F is the graph whose vertices are S_i, $i \in I$ and in which the vertices S_i and S_j $(i, j \in I)$ are adjacent if and only if $S_i \neq S_j$ and $S_i \cap S_j \neq \emptyset$. It is known that every simple graph is an intersection graph, [8].

Let $G = (V, E)$ be a graph. The *(open) neighborhood* $N(v)$ of a vertex $v \in V$ is the set of vertices which are adjacent to v. For a subset S of vertices, $N(S) = \bigcup\limits_{v \in S} N(v)$ and $N[S] = N(S) \sup S$. A set of vertices S in G is a *dominating set*, (or just DS), if $N[S] = V(G)$. The *domination number*, $\gamma(G)$, of G is the minimum cardinality of a dominating set of G. For references on Domination Theory, see [6].

It is interesting to study the intersection graphs $G(F)$ when the members of F have an algebraic structure. Bosak [1], in 1964, studied graphs of semigroups. Then Csákány and Pollák [3], in 1969, studied the graphs of subgroups of a finite group. Zelinka [10], in 1975, continued the work on intersection graphs of nontrivial subgroups of finite abelian groups. Recently, Chakrabarty et al. [2] studied intersection graphs of ideals of rings. Jafari Rad et al. [4] considered intersection graph of subspaces of a vector space. They also studied the intersection graphs of submodules of a module [5].

In this paper, we study domination in the intersection graphs of ideals of rings and domination in the intersection graphs of submodules of modules. In section 2, we determine domination number in the intersection graphs of ideals of rings. In section 3, we determine the domination number in the intersection graphs of submodules of modules.

Throughout this paper, R is a commutative ring R with 1. For a ring R the intersection graph of ideals of R, denoted by $\Gamma(R)$, is the graph whose vertices are in a one-to-one correspondence with proper nontrivial ideals of R and two distinct vertices are adjacent if and only if the corresponding ideals of R have a nontrivial (nonzero) intersection. For an R-module M, the intersection graph of submodules of M, denoted by $\Gamma(M)$, is the graph whose vertices are in a one-to-one correspondence with proper nontrivial submodules of M and two distinct vertices are adjacent if and only if the corresponding submodules have a nontrivial (nonzero) intersection.

For a ring R, we define $\gamma(\Gamma(R)) = 0$ if R is a field, and for an R-module M, we define $\gamma(\Gamma(M)) = 0$ if M is simple.

2. Ring

In this section we determine the domination number in the intersection graphs of ideals of rings. We begin with the following obvious lemma.

Lemma 2.1 *Let* R_1, R_2 *be two rings with* 1. *Then* $I \trianglelefteq R_1 \times R_2$ *if and only if* $I = I_1 \times I_2$, *where* $I_i \trianglelefteq R_i$ *for* $i = 1, 2$.

Lemma 2.2 *Let* R_1, R_2 *be two rings with* 1. *Then* $\gamma(\Gamma(R_1 \times R_2)) \leq 2$.

Proof. Notice that $\{R_1 \times 0, 0 \times R_2\}$ is a DS of $\Gamma(R_1 \times R_2)$. ■

Theorem 2.3 *Let* R_1, R_2 *be two rings with* 1. *Then* $\gamma(\Gamma(R_1 \times R_2)) = 1$ *if and only if* $\gamma(\Gamma(R_1)) = 1$ *or* $\gamma(\Gamma(R_2)) = 1$.

Proof. ($\Longrightarrow$) Let $\{I \times J\}$ be a DS for $\Gamma(R_1 \times R_2)$. Since $R_1 \times 0$ and $0 \times R_2$ are dominated by $\{I \times J\}$, we obtain that I or J is a nontrivial proper ideal. Without loss of generality, assume that I is a nontrivial proper ideal. Now each ideal I_1 of R_1, $(I_1 \times 0) \cap (I \times J) \neq 0$. So $I \cap I_1 \neq 0$. This implies that $\{I\}$ is a DS for $\Gamma(R_1)$, and so $\gamma(\Gamma(R_1)) = 1$.

($\Longleftarrow$) Let $\gamma(\Gamma(R_1)) = 1$, and $\{I\}$ be a DS for $\Gamma(R_1)$. It follows that $\{I \times R_2\}$ is a DS for $\Gamma(R_1 \times R_2)$. This completes the proof. ■

A ring R is indecomposable if, for any pair of nontrivial rings R_1, R_2,

$$R \ncong R_1 \times R_2.$$

Lemma 2.4 *If* R *is an indecomposable ring with* 1 *such that* $\Gamma(R) \neq \emptyset$, *then* $\gamma(\Gamma(R)) = 1$.

Proof. Let M be a maximal ideal of R. Let I be an arbitrary proper nontrivial ideal of R. If $I \cap M = 0$ then $I + M = R$ and so $R \simeq I \times M$, a contradiction. So $I \cap M \neq 0$. We conclude that $\{M\}$ is a DS for $\Gamma(R)$. ■

Corollary 2.5 *Let R be a ring with* 1. *Then* $\gamma(\Gamma(R)) \leq 2$.

Proof. The result follows from Lemmas 2.2 and 2.4. ■

Theorem 2.6 *Let R be an Artinian commutative ring with* 1. *Then* $\gamma(\Gamma(R)) = 2$ *if and only if* $R = R_1 \times R_2 \times ... \times R_t$, *where* $t \geq 2$ *and R_i is a field for* $i = 1, 2, ..., t$.

Proof. ($\Longrightarrow$) Since R is Artinian, we have $R = R_1 \times R_2 \times ... \times R_t$, where R_i is a local ring for $i = 1, 2, ..., t$. By Theorem 2.3, $\gamma(\Gamma(R_i)) \neq 1$. This implies that $\Gamma(R_i)$ is the null graph, and so R_i is a field. But $\gamma(\Gamma(R)) = 2$. So $\Gamma(R) \neq \emptyset$, and then $t \geq 2$.

($\Longleftarrow$) Follows from Theorem 2.3. ■

3. Module

In this section we determine the domination number in the intersection graphs of submodules of modules. An R-module M is *semisimple* if $M \cong M_1 \times M_2 \times ... \times M_k$, where M_i is a simple R-module for $i = 1, 2, ..., k$.

Lemma 3.7 *Let M be an Artinian R-module and* $N =< \{K : K$ *is a minimal submodule of* $M\} >$. *Then* $N \cong K_1 \times K_2 \times ... \times K_t$, *where K_i is minimal* (*simple*).

Proof. Assume to the contrary that $N \ncong K_1 \times K_2 \times ... \times K_t$, for any t and minimal submodules K_i. Let K_1 be a minimal submodule of M. Since $M \neq K_1$, there exists a minimal submodule K_2 such that $K_1 \cap K_2 = 0$. Since $N \neq K_1 \oplus K_2$, then there exists a minimal submodule K_3 such that $(K_1 + K_2) \cap K_3 = 0$. By continuing this method, we obtain minimal submodules $K_1, K_2, ...$ such that $\sum_{i \in \mathbb{N}} K_i = \bigoplus_{i \in \mathbb{N}} K_i$. Since $\bigoplus_{i \in \mathbb{N}} K_i$ is not Artinian R-module, we obtain a contradiction. Notice that K_i is simple, since it is minimal, for each i. ■

Lemma 3.8 *If M is a semisimple module, then* $\gamma(\Gamma(M)) \neq 1$.

Proof. If M is simple, then $\gamma(\Gamma(M)) = 0$. So we assume that $M = M_1 \oplus M_2 \oplus ... \oplus M_k$, where M_i is simple and $t \geq 2$. Assume that $\{N\}$ is a DS for $\Gamma(M)$. For any i, $N \cap M_i \neq 0$, and so $M_i \subseteq N$. Then $M \subseteq N$, a contradiction. ■

Corollary 3.9 *Let M be an Artinian R-module. Then* $\gamma(\Gamma(M)) = 1$ *if and only if M is not semisimple.*

Proof. The result follows from Lemmas 3.7 and 3.8. ■

So in the rest of this section we consider semisimple modules.

Theorem 3.10 ([4]) *If V is a vector space of dimension $d \geq 2$ over a filed F of order q, then* $\gamma(G(V)) = q + 1$.

Lemma 3.11 *Let M be finite semisimple and* $M = M_1 \oplus M_2 \oplus ... \oplus M_k$, *where* $k \geq 1$ *and M_i is simple for* $i = 1, 2, ..., k$. *If for any i, j,* $Ann(M_i) = Ann(M_j)$, *then* $\gamma(\Gamma(M)) = |M_1| + 1$.

Proof. Let $m = Ann(M_1)$. Since M_1 is simple , m is maximal. By assumption, M is an $\frac{R}{m}$-module, and so is a vector space over $\frac{R}{m}$. By Theorem 3.10, $\gamma(\Gamma(M)) = \left|\frac{R}{m}\right| + 1 = |M_1| + 1$. ∎

Lemma 3.12 *Let M be a semisimple R-module and $M = M_1 \oplus M_2 \oplus ... \oplus M_k$, where $k \geq 1$ and M_i is simple for $i = 1, 2, ..., k$. If $Ann(M_1) = Ann(M_2) = ... = Ann(M_t)$ and $Ann(M_1) \neq Ann(M_i)$ for $t + 1 \leq i \leq k$, where $t < k$, then any submodule N of M is in the form $N_1 + N_2$, where $N_1 \leq M_1 + M_2 + ...M_t$ and $N_2 \leq M_{t+1} + ... + M_k$.*

Proof. Let $x \in N$ and $x = x_1 + x_2$, where $x_1 \in M_1 + M_2 + ...M_t$ and $x_2 \in M_{t+1} + ... + M_k$. Let $Ann(M_i) = m_i$ for each i. Then $m_1 + (m_{t+1} \cap m_{t+2} \cap ... \cap m_k) = R$. This implies that there are $a \in m_1$ and $b \in m_{t+1} \cap m_{t+2} \cap ... \cap m_k$ such that $a + b = 1$. Now $bx = (1 - a)x_1 + bx_2 = x_1 - ax_1 + 0 = x_1$ and then $x_1 \in N$. This implies that $x_2 \in N$. This completes the proof. ∎

Corollary 3.13 *Let M be a semisimple R-module and $M = M_1 \oplus M_2 \oplus ... \oplus M_k$, where $k \geq 1$ and M_i is simple for $i = 1, 2, ..., k$. Assume that $Ann(M_1) = Ann(M_2) = ... = Ann(M_t)$ and $Ann(M_1) \neq Ann(M_i)$ for $t + 1 \leq i \leq k$, where $t < k$. Then $\gamma(\Gamma(M)) = 2$.*

Proof. By Corollary 3.9, $\gamma(\Gamma(M)) \geq 2$, and by Lemma 3.12, $\{M_1 + M_2 + ... + M_k, M_{k+1} + ... + M_t\}$ is a DS for $\Gamma(M)$. We conclude that $\gamma(\Gamma(M)) = 2$. ∎

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NOTES FOR HARTLEY TRANSFORMS OF GENERALIZED FUNCTIONS

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Abstract. The classical Hartley transform, originally introduced by Hartley as a real transform with a number of properties being similar to the properties of Fourier transform. In this work, we extend the Hartley transform to certain space of distributions of compact support. Further, we establish that the Hartley transform and its inverse are one to one and onto mappings in the space of Boehmians. Moreover, continuity with respect to δ and Δ convergence is discussed in some detail. Certain theorems are also proved.

Keywords: Hartly transform; Boehmian space; convolution; smooth function; distribution of compact support.

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1. Introduction

The Hartley transform is a spectral transform closely related to the Fourier transforms. It contain the same information that the Fourier transform does, and no advantage accrues in its use for complex signals. However, for real signal, the Hartley transform is real and this can offer computational advantages in signal processing applications that traditionally make use of Fourier transforms [15]. Moreover, the Hartley transform can be analytically continued into the complex plane, and for real functions it is Hermitian symmetry on reflection in the real axis. The Hartley transform in the complex plane is an entire function of exponential type with zeros close to the real axis, a property shared with Fourier transform [15, p.p. 414]. The Hartley transform $H(\upsilon)$, of $f(x)$, is defined by [13]-[17]

$$H(\upsilon) = \int_{\mathbb{R}} f(x)\, cas\,(2\pi x\upsilon)\, dx, \tag{1}$$

where

$$cas\,(2\pi\upsilon x) = \cos(2\pi\upsilon x) + \sin(2\pi\upsilon x).$$

Therefore, it follows that the inverse Hartley transform is given by

$$f(x) = \int_{\mathbb{R}} H(v)\, cas\,(2\pi xv)\, dv. \tag{2}$$

The scalling and linearity conditions of the Hartley transform have been described in [16] . Theorems for the Hartley transform, analogous to those for the Fourier transform, can be easily derived from definitions. The more complicated, compared to the Fourier transform, is the convolution theorem which can sometimes amount to a disadvantage of the Hartley transform. For functions f and g, L^1 functions, the convolution theorem of the Hartley is defined by

$$H(f \bullet g)(v) = \frac{1}{2}\boldsymbol{G}(Hf \times Hg)(v), \tag{3}$$

where

$$\boldsymbol{G}(f \times g)(v) = f(v)\, g(v) + f(v)\, g(-v) + f(-v)\, g(v) - f(-v)\, g(-v). \tag{4}$$

and $\bullet$ is the usual convolution product.

2. Boehmian spaces

The general construction of Boehmians is algebraic in nature. Boehmians were first constructed as a generalization of regular Mikusinski operators [5]. The minimal structure necessary for the construction of Boehmians consists of the following elements:

(i) A nonempty set $\mathbb{X}$ and a commutative semigroup $(\mathbb{Y}, *)$;

(ii) An operation $\circ : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{X}$ such that for each $x \in \mathbb{X}$ and $s_1, s_2, \in \mathbb{Y}$, $x \circ (s_1 * s_2) = (x \circ s_1) \circ s_2$;

(iii) A collection $\Delta \subset \mathbb{Y}^N$ such that:

  a) If $x, y \in \mathbb{X}, (s_n) \in \Delta$, $x \circ s_n = y \circ s_n$ for all n, then $x = y$;
  b) If $(s_n), (t_n) \in \Delta$, then $(s_n * t_n) \in \Delta$.

Elements of Δ are called delta sequences.

Let $\mathcal{F} = \{(x_n, s_n) : x_n \in \mathbb{X}, s_n \in \Delta, x_n \circ s_m = x_m \circ s_n, \forall m, n \in \mathbf{N}\}$. Consider $(x_n, s_n), (y_n, t_n) \in \mathcal{F}, x_n \circ t_m = y_m \circ s_n, \forall m, n \in \mathbf{N}$, then we say $(x_n, s_n) \sim (y_n, t_n)$. The relation $\sim$ is an equivalence relation in $\mathcal{F}$. The space of equivalence clases in $\mathcal{F}$ is denoted by β. Elements of β are called Boehmians. A typical element of β is written as $\left[\frac{x_n}{s_n}\right]$. Between $\mathbb{X}$ and β there is a canonical embedding expressed as $x \longrightarrow \frac{x \circ s_n}{s_n}$. The operation $\circ$ can be extended to $\beta \times \mathbb{X}$ by $\frac{x_n}{s_n} \circ t = \frac{x_n \circ t}{s_n}$. In β, two types of convergence:

Type 1: *A sequence* (h_n) *in* β *is said to be* δ **convergent** *to* h *in* β, *denoted by* $h_n \xrightarrow{\delta} h$, *if there exists a delta sequence* (s_n) *such that* $(h_n \circ s_n), (h \circ s_n) \in \mathbb{X}$, $\forall k, n \in \mathbf{N}$, *and* $(h_n \circ s_k) \to (h \circ s_k)$ *as* $n \to \infty$,*in* $\mathbb{X}$, *for every* $k \in \mathbf{N}$.

Type 2: *A sequence* (h_n) *in* β *is said to be* Δ **convergent** *to* h *in* β, *denoted by* $h_n \xrightarrow{\Delta} h$, *if there exists a* $(s_n) \in \Delta$ *such that* $(h_n - h) \circ s_n \in \mathbb{X}, \forall n \in \mathbf{N}$, *and* $(h_n - h) \circ s_n \to 0$ *as* $n \to \infty$ *in* $\mathbb{X}$. For more details, see [1]-[5], [7]-[9], [11], [12].

3. The Boehmian Space $\mathbb{M}(\mathbb{E}, \mathbb{D}, \Delta, \bullet)$

By $\mathbb{D}(\mathbb{R})$, we denote the space of test functions of compact support and $\mathbb{E}(\mathbb{R})$ the space of all infinitely smooth functions on $\mathbb{R}$ equipped with the sequence of multinorms $\xi_k(f) = \sup_{x \in K} \left| f^{(k)}(x) \right|$ for every $f \in \mathbb{E}(\mathbb{R})$, where K run through compact subsets of $\mathbb{R}$. The kernel function of the Hartley transform, $cas2\pi x\upsilon$, is certainly a member of $\mathbb{E}(\mathbb{R})$ for arbitrary but fixed υ. This justifies the extension of the Hartley transform to the context of distributions through the formula

$$Hf(\upsilon) = \langle f(x), cas2\pi\upsilon x \rangle$$

for each $f \in \mathbb{E}'(\mathbb{R})$, the strong dual of $\mathbb{E}(\mathbb{R})$ of distriutions of compact support [10], [19]. Let Δ be the family of sequences (γ_n) from $\mathbb{D}(\mathbb{R})$ such that

$$(5) \qquad \int_{\mathbb{R}} \gamma_n(x)\,dx = 1, \qquad \text{for every } n \in \mathbf{N}.$$

$$(6) \qquad \int_{\mathbb{R}} |\gamma_n(x)|\,dx \le M, \qquad \text{for some positive } M.$$

$$(7) \qquad \operatorname{supp} \gamma_n(x) \subset (-\varepsilon_n, \varepsilon_n), \qquad \varepsilon_n \to 0 \text{ as } n \to \infty.$$

Members of Δ are named as delta sequences .

Lemma 3.1. *Let* $f \in \mathbb{E}(\mathbb{R})$ *and* $\phi \in \mathbb{D}(\mathbb{R})$ *then* $f \bullet \phi \in \mathbb{E}(\mathbb{R})$.

Lemma 3.2. *Given* $f_1, f_2 \in \mathbb{E}(\mathbb{R})$ *and* $\phi \in \mathbb{D}(\mathbb{R})$ *then for every* $\alpha \in \mathbb{C}$,

$$(f_1 + f_2) \bullet \phi = f_1 \bullet \phi + f_2 \bullet \phi \text{ and } \alpha(f_1 \bullet \phi) = (\alpha f_1) \bullet \phi = f_1 \bullet (\alpha\phi).$$

Lemma 3.3. *Let* $f_n \to f$ *in* $\mathbb{E}(\mathbb{R})$ *and* $\phi \in \mathbb{D}(\mathbb{R})$ *then* $f_n \bullet \phi \to f \bullet \phi$.

Proofs of Lemmas 3.1-3.3 are straightforward from the properties of the integral operator.

Lemma 3.4. *Let* $f_n \to f$ *in* $\mathbb{E}(\mathbb{R})$ *as* $n \to \infty$ *and* $\phi \in \mathbb{D}(\mathbb{R})$ *then* $f_n \bullet \phi \to f$ *as* $n \to \infty$.

Proof. If $f_n \to f$ in $\mathbb{E}(\mathbb{R})$ then from Eq. (5) we get $\left| D^k (f_n \bullet \phi - f)(x) \right| \le M\xi_k (f_n(x - t) - f(x)) \to 0$ as $n \to \infty$. Hence $\xi_k(f_n \bullet \phi - f) \to 0$ as $n \to \infty$. The lemma is completely proved.

The Boehmian Space $\mathbb{M}\left(\mathbb{E},\mathbb{D},\Delta,\bullet\right)$ is constructed. The sum of two Boehmians and multiplication by a scalar in $\mathbb{M}\left(\mathbb{E},\mathbb{D},\Delta,\bullet\right)$ can be defined in a natural way $\left[\frac{f_n}{\phi_n}\right]+\left[\frac{g_n}{\psi_n}\right]=\left[\frac{f_n\bullet\psi_n+g_n\bullet\phi_n}{\phi_n\bullet\psi_n}\right]$ and $\alpha\left[\frac{f_n}{\phi_n}\right]=\left[\alpha\frac{f_n}{\phi_n}\right], \alpha\in\mathbb{C}$. The operation $\bullet$ and the differentiation are defined by $\left[\frac{f_n}{\phi_n}\right]\bullet\left[\frac{g_n}{\psi_n}\right]=\left[\frac{f_n\bullet g_n}{\phi_n\bullet\psi_n}\right]$ and $D^{\alpha}\left[\frac{f_n}{\phi_n}\right]=\left[\frac{D^{\alpha}f_n}{\phi_n}\right]$. The relationship between the notion of convergence and the product $\bullet$ is given by:

1$-$If $f_n\to f$ as $n\to\infty$ in $\mathbb{E}\left(\mathbb{R}\right)$ and, $\phi\in\mathbb{D}\left(\mathbb{R}\right)$ is fixed, then $f_n\bullet\phi\to f\bullet\phi$ in $\mathbb{E}\left(\mathbb{R}\right)$, as $n\to\infty$; 2$-$ If $f_n\to f$ as $n\to\infty$ in $\mathbb{E}\left(\mathbb{R}\right)$ and $\left(\delta_n\right)\in\Delta$, then $f_n\bullet\delta_n\to f$ in $\mathbb{E}\left(\mathbb{R}\right)$, as $n\to\infty$. The operation $\bullet$ can be extended to $\mathbb{M}\left(\mathbb{E},\mathbb{D},\Delta,\bullet\right)\times\mathbb{D}$ in the sense that *If* $[f_n/\delta_n]\in\mathbb{M}\left(\mathbb{E},\mathbb{D},\Delta,\bullet\right)$ *and* $\phi\in\mathbb{D}\left(\mathbb{R}\right)$*,then* $[\frac{f_n}{\phi_n}]\bullet\phi=[\frac{f_n\bullet\phi}{\delta_n}]$.

Convergence in $\mathbb{M}\left(\mathbb{E},\mathbb{D},\Delta,\bullet\right)$, is defined as follows: *A sequence of* $\left(\beta_n\right)$ *in* $\mathbb{M}\left(\mathbb{E},\mathbb{D},\Delta,\bullet\right)$ *is said to be* δ *convergent to a Boehmian* β *in* $\mathbb{M}\left(\mathbb{E},\mathbb{D},\Delta,\bullet\right)$, *denoted by* $\beta_n\xrightarrow{\delta}\beta$, *if there exists a delta sequence* $\left(\delta_n\right)$ *such that* $\beta_n\bullet\delta_n,\beta\bullet\delta_n\in\mathbb{E}\left(\mathbb{R}\right),\forall k,n\in\mathbf{N}$, *and* $\beta_n\bullet\delta_k\to\beta\bullet\delta_k$ *as* $n\to\infty$, *in* $\mathbb{E}\left(\mathbb{R}\right)$, *for every* $k\in\mathbf{N}$. This can be interpreted to mean: $\beta_n\xrightarrow{\delta}\beta\ (n\to\infty)$ *in* $\mathbb{M}\left(\mathbb{E},\mathbb{D},\Delta,\bullet\right)$ *if and only if there is* $f_{n,k},f_k\in\mathbb{E}\left(\mathbb{R}\right)$ *and* $\left(\delta_k\right)\in\Delta$ *such that* $\beta_n=\left[\frac{f_{n,k}}{\delta_k}\right],\beta=\left[\frac{f_k}{\delta_k}\right]$ *and for each* $k\in\mathbf{N},f_{n,k}\to f_k$ as $n\to\infty$ in $\mathbb{E}\left(\mathbb{R}\right)$. It is more often convenient to use another kind of convergence**:**

A sequence $\left(\beta_n\right)$ *in* $\mathbb{M}\left(\mathbb{E},\mathbb{D},\Delta,\bullet\right)$ *is said to be* Δ *convergent to a* β *in* $\mathbb{M}\left(\mathbb{E},\mathbb{D},\Delta,\bullet\right)$, *denoted by* $\beta_n\xrightarrow{\Delta}\beta$, *if there exists a* $\left(\delta_n\right)\in\Delta$ *such that* $\left(\beta_n-\beta\right)\bullet\delta_n\in\mathbb{E}\left(\mathbb{R}\right),\forall n\in\mathbf{N}$, *and* $\left(\beta_n-\beta\right)\bullet\delta_n\to0$ *as* $n\to\infty$ *in* $\mathbb{E}\left(\mathbb{R}\right)$.

4. The Space $\mathbb{M}_H\left(\mathbb{E},\mathbb{D}^H,\Delta^H,\Upsilon\right)$

To extend the Hartley transform to Boehmians we describe another space of Boehmians as follows. First, it will be necessary to know that, if $(\gamma_n)\in\Delta$ then

$$H\gamma_n\left(\upsilon\right)\to1 \text{ as } n\to\infty. \tag{8}$$

and that

$$H\gamma_n\left(-\upsilon\right)\to1 \text{ as } n\to\infty. \tag{9}$$

uniformly on compact subsets. Let a mapping Υ between f and $H\phi$ be defined by

$$\left(f\ \Upsilon\ H\phi\right)\left(\upsilon\right)=\frac{1}{2}\boldsymbol{G}\left(f\times H\phi\right)\left(\upsilon\right). \tag{10}$$

$\boldsymbol{G}$ has the usual meaning in (4). Denote by $\mathbb{D}^H\left(\mathbb{R}\right)$, the set of all Hartley transform of functions from $\mathbb{D}\left(\mathbb{R}\right)$. Define $\Delta^H=\left\{H\gamma_n:\gamma_n\in\Delta,\ \forall n\in\mathbf{N}\ \right\}$. We prove the following

Lemma 4.1. *Let* $f\in\mathbb{E}\left(\mathbb{R}\right),H\phi\in\mathbb{D}^H\left(\mathbb{R}\right)$ *then* $\left(f\ \Upsilon\ H\phi\right)\left(\upsilon\right)\in\mathbb{E}\left(\mathbb{R}\right)$.

Proof. Let $k \in \mathbf{N}$ then for each $f \in \mathbb{E}(\mathbb{R})$ and $H\phi \in \mathbb{D}^H \subset \mathbb{E}(\mathbb{R})$ we have $H\phi(\mp \upsilon) f(\pm \upsilon) \in \mathbb{E}(\mathbb{R})$. If K is a compact subset of $\mathbb{R}$ containing supp ϕ then by using Eq. (10) and Eq. (4) we get

$$\sup_{\upsilon \in K} \left| D_\upsilon^k (f \curlyvee H\phi)(\upsilon) \right| < \infty. \tag{11}$$

Allowing K traverses the set of real numbers yields $f \curlyvee H\phi \in \mathbb{E}(\mathbb{R})$. This completes the proof of the lemma.

Lemma 4.2. *A mapping* $\mathbb{E} \times \mathbb{D}^H \rightarrow \mathbb{E}$ *defined by*

$$(f, H\phi) \rightarrow f \curlyvee H\phi \tag{12}$$

satisfies the following

(i) *If* $H\phi, H\psi \in \mathbb{D}^H(\mathbb{R})$ then $(H\phi \curlyvee H\psi)(\upsilon) \in \mathbb{D}^H(\mathbb{R})$.

(ii) *If* $f, g \in \mathbb{E}(\mathbb{R}), H\phi \in \mathbb{D}^H(\mathbb{R})$ *then* $((f+g) \curlyvee H\phi)(\upsilon) = (f \curlyvee H\phi)(\upsilon) + (g \curlyvee H\psi)(\upsilon)$.

(iii) $(H\phi \curlyvee H\psi)(\upsilon) = (H\psi \curlyvee H\phi)(\upsilon), \forall H\phi, H\psi \in \mathbb{D}^H(\mathbb{R})$.

(iv) *If* $f \in \mathbb{E}(\mathbb{R}), H\phi, H\psi \in \mathbb{D}^H(\mathbb{R})$ *then* $(f \curlyvee H\phi) \curlyvee H\psi = f \curlyvee (H\phi \curlyvee H\psi)$.

Proof. (i) Let $\upsilon \in \mathbb{R}$ then it is clear that $(H\phi \curlyvee H\psi)(\upsilon) = H(\phi \bullet \psi)(\upsilon)$. But Lemma 3.1 implies $\phi \bullet \psi \in \mathbb{D}(\mathbb{R})$. Hence $H\phi \curlyvee H\psi \in \mathbb{D}^H(\mathbb{R})$.

(ii) is obvious.

(iii) Since $\phi \bullet \psi \in \mathbb{D}(\mathbb{R}), \phi \bullet \psi = \psi \bullet \phi$. Applying the Hartley transform yields $H(\psi \bullet \phi)(\upsilon) = H(\phi \bullet \psi)(\upsilon)$. This implies $H\phi \curlyvee H\psi = H\psi \curlyvee H\phi$.

(iv) can be easily established by routine calculation from Eq. (4) and Eq. (10). The lemma is completely proved.

Lemma 4.3. *Let* $f_1, f_2 \in \mathbb{E}(\mathbb{R}), H\gamma_n \in \Delta^H, \forall n, f_1 \curlyvee H\gamma_n = f_2 \curlyvee H\gamma_n, \forall n,$ *then* $f_1 = f_2$ *in* $\mathbb{E}(\mathbb{R})$.

Proof. From hypothesis $f_1 \curlyvee H\gamma_n = f_2 \curlyvee H\gamma_n, \forall n$. Invoking Eq. (8) and Eq. (9) in Eq. (10), yields $f_1(\upsilon) = f_2(\upsilon), \forall \upsilon \in \mathbb{R}$. Thus $f = g$. This completes the proof of the lemma.

Lemma 4.4.

(1) *If* $f_n \rightarrow f$ *in* $\mathbb{E}(\mathbb{R})$ *as* $n \rightarrow \infty$ *and* $H\phi \in \mathbb{D}^H(\mathbb{R})$ *then* $f_n \curlyvee H\phi \rightarrow f \curlyvee H\phi$ *as* $n \rightarrow \infty$.

(2) *If* $f_n \rightarrow f$ *in* $\mathbb{E}(\mathbb{R})$ *as* $n \rightarrow \infty$ *and* $H\gamma_n \in \Delta^H$ *then* $f_n \curlyvee H\gamma_n \rightarrow f$ *as* $n \rightarrow \infty$ *in* $E(\mathbf{R})$.

The proof of the above lemma is a result of Eq. (8) and Eq. (9).

Lemma 4.5. *Let* $(H\gamma_n)_1^\infty, (H\xi_n)_1^\infty \in \Delta^H$ *then* $(H\gamma_n \curlyvee H\xi_n)_1^\infty \in \Delta^H$.

Proof. Let $(\gamma_n), (\xi_n) \in \Delta$ then $(\gamma_n \bullet \xi_n) \in \Delta$. Thus, $(\gamma_n \bullet \xi_n) \in \Delta$ is the sequence such that $(H\gamma_n \curlyvee H\xi_n)(\upsilon) = H(\gamma_n \bullet \xi_n)(\upsilon) \in \Delta^H$. Hence the lemma is completely proved. Further, it can be observed that

$$H\gamma_n \curlyvee H\xi_n \to 1 \text{ as } n \to \infty,$$

for every $(\gamma_n), (\xi_n) \in \Delta$. Hence, Δ^H satisfies the necessary conditions for delta sequences. The Boehmian space $\mathbb{M}_H(\mathbb{E}, \mathbb{D}^H, \Delta^H, \curlyvee)$, or $\mathbb{M}_H$, is constructed. Addition, scalar multiplication, differentiation, convolution and convergence can be defined in a natural way. For some detail, the sum and multiplication by a scalar on $\mathbb{M}_H(\mathbb{E}, \mathbb{D}^H, \Delta^H, \curlyvee)$ is defined as $\left[\frac{Hf_n}{H\phi_n}\right] + \left[\frac{Hg_n}{H\psi_n}\right] = \left[\frac{Hf_n \curlyvee H\psi_n + Hg_n \curlyvee H\phi_n}{H\phi_n \curlyvee H\psi_n}\right]$ and $\alpha\left[\frac{Hf_n}{H\phi_n}\right] = \left[\alpha\frac{Hf_n}{H\phi_n}\right], \alpha \in \mathbb{C}$. The operation $\curlyvee$ and the differentiation are defined by $\left[\frac{Hf_n}{H\phi_n}\right] \curlyvee \left[\frac{Hg_n}{H\psi_n}\right] = \left[\frac{Hf_n \curlyvee Hg_n}{H\phi_n \curlyvee H\psi_n}\right]$ and $D^\alpha\left[\frac{Hf_n}{H\phi_n}\right] = \left[\frac{D^\alpha Hf_n}{H\phi_n}\right]$. Convergence on $\mathbb{M}_H(\mathbb{E}, \mathbb{D}^H, \Delta^H, \curlyvee)$ is defined by:

A sequence of $(H\beta_n)$ *in* $\mathbb{M}_H(\mathbb{E}, \mathbb{D}^H, \Delta^H, \curlyvee)$ *is said to be* δ **convergent** *to a Boehmian* $H\beta$, *denoted by* $H\beta_n \xrightarrow{\delta} H\beta$, *if there exists a delta sequence* $(H\delta_n)$ *such that* $H\beta_n \curlyvee H\delta_n, H\beta \curlyvee H\delta_n \in \mathbb{E}(\mathbb{R}), \forall k, n \in \mathbf{N}$, *and* $H\beta_n \curlyvee H\delta_k \to H\beta \curlyvee H\delta_k$ *as* $n \to \infty$,*in* $\mathbb{E}(\mathbb{R})$, *for every* $k \in \mathbf{N}$. This can be interpreted to mean: $H\beta_n \xrightarrow{\delta} H\beta$ $(n \to \infty)$ *if and only if there is* $Hf_{n,k}, Hf_k \in \mathbb{E}(\mathbb{R})$ *and* $(H\delta_k) \in \Delta^H$ *such that* $H\beta_n = \left[\frac{Hf_{n,k}}{H\delta_k}\right], H\beta = \left[\frac{Hf_k}{H\delta_k}\right]$ *and for each* $k \in \mathbf{N}, Hf_{n,k} \to Hf_k$ as $n \to \infty$ in $\mathbb{E}(\mathbb{R})$.

It is more often convenient to use another kind of convergence: *A sequence* $(H\beta_n) \in \mathbb{M}_H(\mathbb{E}, \mathbb{D}^H, \Delta^H, \curlyvee)$ *is said to be* Δ **convergent** *to a* $H\beta \in \mathbb{M}_H(\mathbb{E}, \mathbb{D}^H, \Delta^H, \curlyvee)$, *denoted by* $H\beta_n \xrightarrow{\Delta} H\beta$, *if there exists a* $(H\delta_n) \in \Delta^H$ *such that* $(H\beta_n - H\beta) \curlyvee H\delta_n \in \mathbb{E}(\mathbb{R})$, $\forall n \in \mathbf{N}$, *and* $(H\beta_n - H\beta) \curlyvee H\delta_n \to 0$ *as* $n \to \infty$ *in* $\mathbb{E}(\mathbb{R})$.

Theorem 4.6. *The mapping*

$$\begin{array}{rcl} \mathbb{E} & \to & \mathbb{M}_H(\mathbb{E}, \mathbb{D}^H, \Delta^H, \curlyvee) \\ f & \to & \left[\frac{f \curlyvee H\gamma_n}{H\gamma_n}\right] \end{array} \tag{13}$$

is a continuous imbedding of $\mathbb{E}(\mathbb{R})$ *into* $\mathbb{M}_H(\mathbb{E}, \mathbb{D}^H, \Delta^H, \curlyvee)$ *with respect to* δ *convergence.*

Proof. To show the mapping is one to one let $\left[\frac{f \curlyvee H\gamma_n}{H\gamma_n}\right] = \left[\frac{g \curlyvee Ht_n}{Ht_n}\right]$. Then

$$(f \curlyvee H\gamma_n) \curlyvee Ht_m = (g \curlyvee Ht_m) \curlyvee H\gamma_n.$$

For large values of m and n, $Ht_m, H\gamma_n \to 1$. The above equation is therefore reduced to $f = g$. To establish continuity of Eq. (13) with respect to $\delta-$convergence, let $f_n \to 0$ as $n \to \infty$ then $f_n \curlyvee H\gamma_n \to 0$ as $n \to \infty$. Hence $\left[\frac{f_n \curlyvee H\gamma_n}{H\gamma_n}\right] \to 0$ as $n \to \infty$. The theorem is completely proved.

5. Hartley transform of Boehmians

Let $\left[\frac{f_n}{\gamma_n}\right] \in \mathbb{M}\left(\mathbb{E}, \mathbb{D}, \Delta, \bullet\right)$ then, in view of analysis established in Section 4, we define the extended Hartley transform by

$$\boldsymbol{\kappa}\left[\frac{f_n}{\gamma_n}\right] = \left[\frac{Hf_n}{H\gamma_n}\right]. \tag{14}$$

in $\mathbb{M}\left(\mathbb{E}, \mathbb{D}^H, \Delta^H, \curlyvee\right)$.

Theorem 5.1. $\boldsymbol{\kappa} : \mathbb{M}\left(\mathbb{E}, \mathbb{D}, \Delta, \bullet\right) \to \mathbb{M}_H$ *is well-defined.*

Proof. Let $\left[\frac{f_n}{\gamma_n}\right] = \left[\frac{g_n}{t_n}\right]$ then $f_n \bullet t_m = g_m \bullet \gamma_n$. Employing the Hartley transform and using Eq. (10) we get $Hf_n \curlyvee Ht_m = Hg_m \curlyvee H\gamma_n$. Therefore

$$\left[\frac{Hf_n}{H\gamma_n}\right] = \left[\frac{Hg_n}{Ht_n}\right].$$

The theorem is completely proved.

Theorem 5.2. $\boldsymbol{\kappa}$:$\mathbb{M}\left(\mathbb{E}, \mathbb{D}, \Delta, \bullet\right) \to \mathbb{M}_H$ *is one to one.*

Proof. Let $\boldsymbol{\kappa}\left[\frac{f_n}{\gamma_n}\right] = \boldsymbol{\kappa}\left[\frac{g_n}{t_n}\right]$ then $Hf_n \curlyvee Ht_m = Hg_m \curlyvee H\gamma_n$. Using the fact that the classical Hartley transform is one to one and upon employing Eq. (10) we get $f_n \bullet t_m = g_m \bullet \gamma_n$. Therefore $\left[\frac{f_n}{\gamma_n}\right] = \left[\frac{g_n}{t_n}\right]$. Hence the theorem.

Theorem 5.3. $\boldsymbol{\kappa}$:$\mathbb{M}\left(\mathbb{E}, \mathbb{D}, \Delta, \bullet\right) \to \mathbb{M}_H$ *is continuous with respect to δ convergence.*

Proof. Let $x_n \to 0$ in $\mathbb{M}\left(\mathbb{E}, \mathbb{D}, \Delta, \bullet\right)$ as $n \to \infty$, then using [7], $x_n = \left[\frac{f_{n,i}}{\gamma_i}\right]$ for some $f_{n,i}$ where $f_{n,i} \to 0$ as $n \to \infty$. Applying the Hartley transform yields $Hf_{n,i} \to 0$ as $n \to \infty$. Thus $\boldsymbol{\kappa}x_n \to 0$ as $n \to \infty$. This completes the proof.

Theorem 5.4. $\boldsymbol{\kappa}$:$\mathbb{M}\left(\mathbb{E}, \mathbb{D}, \Delta, \bullet\right) \to \mathbb{M}_H$ *is linear* .

Proof of this theorem is straightforward.

Definition 5.5. *Let* $\left[\frac{Hf_n}{H\gamma_n}\right] \in \mathbb{M}_H$ *then we define the inverse generalized Hartley transform to be the mapping*

$$\boldsymbol{\kappa}^{-1}\left[\frac{Hf_n}{H\gamma_n}\right] = \left[\frac{f_n}{\gamma_n}\right] \tag{15}$$

in $\mathbb{M}\left(\mathbb{E}, \mathbb{D}, \Delta, \bullet\right)$.

Theorem 5.6. $\boldsymbol{\kappa}^{-1}:\mathbb{M}_H\rightarrow\mathbb{M}\left(\mathbb{E},\mathbb{D},\Delta,\bullet\right)$ *is a well-defined and linear mapping.*

The proof is analoguous to that of Theorems 5.1 and 5.4, and thus avoided.

Theorem 5.7. $\boldsymbol{\kappa}^{-1}:\mathbb{M}_H\rightarrow\mathbb{M}\left(\mathbb{E},\mathbb{D},\Delta,\bullet\right)$ *is one to one.*

Theorem 5.8. $\boldsymbol{\kappa}^{-1}:\mathbb{M}_H\rightarrow\mathbb{M}\left(\mathbb{E},\mathbb{D},\Delta,\bullet\right)$ *is continuous with respect to $\delta-$convergence.*

The proof of Theorems 5.7 and 5.8 are analoguous to that of Theorem 5.2 and Theorem 5.3, respectively. Details are avoided.

Theorem 5.9. *The mapping* $\boldsymbol{\kappa}$:$\mathbb{M}\left(\mathbb{E},\mathbb{D},\Delta,\bullet\right)\rightarrow\mathbb{M}_H$ *is surjective.*

Proof. Let $\left[\frac{Hf_n}{H\gamma_n}\right]\in\mathbb{M}_H$ be arbitrary, then $Hf_n\curlyvee H\gamma_m=Hf_m\curlyvee H\gamma_n$ for every $m,n\in\mathbf{N}$. Using Eq. (10), $H\left(f_n\bullet\gamma_m\right)=H\left(f_m\bullet\gamma_n\right),$ for every $m,n\in\mathbf{N}$. Hence the Boehmian $\left[\frac{f_n}{\gamma_n}\right]\in\mathbb{M}\left(\mathbb{E},\mathbb{D},\Delta,\bullet\right)$ satisfies the equation $\boldsymbol{\kappa}\left[\frac{f_n}{\gamma_n}\right]=\left[\frac{Hf_n}{H\gamma_n}\right]$. This complete the proof of the lemma.

Theorem 5.10. $\boldsymbol{\kappa}$:$\mathbb{M}\left(\mathbb{E},\mathbb{D},\Delta,\bullet\right)\rightarrow\mathbb{M}_H$, $\boldsymbol{\kappa}^{-1}:\mathbb{M}_H\rightarrow\mathbb{M}\left(\mathbb{E},\mathbb{D},\Delta,\bullet\right)$ *are continuous with respect to Δ convergence.*

Proof. Let $x_n\overset{\Delta}{\rightarrow}x$ in $\mathbb{M}\left(\mathbb{E},\mathbb{D},\Delta,\bullet\right)$ as $n\rightarrow\infty$. Then, there is $f_n\in\mathbb{E}\left(\mathbb{R}\right)$ and $(\gamma_n)\in\Delta$ such that $\left(x_n-x\right)\bullet\gamma_n=\left[\frac{f_n\bullet\gamma_n}{\gamma_n}\right]$ and $f_n\rightarrow0$ as $n\rightarrow\infty$. Employing the Hartley transform implies $\boldsymbol{\kappa}\left(\left(x_n-x\right)\bullet\gamma_n\right)=\left[\frac{H(f_n\bullet\gamma_n)}{H\gamma_n}\right]$. Hence $\boldsymbol{\kappa}\left(\left(x_n-x\right)\bullet\gamma_n\right)=\left[\frac{Hf_n\curlyvee H\gamma_n}{H\gamma_n}\right]\sim Hf_n\rightarrow0$ as $n\rightarrow\infty$ in $\mathbb{M}_H$. Therefore $\boldsymbol{\kappa}x_n\rightarrow\boldsymbol{\kappa}x$ as $n\rightarrow\infty$. Next, let $y_n\overset{\Delta}{\rightarrow}y$ in $\mathbb{M}_H$ as $n\rightarrow\infty$, then we find $F_k\in\mathbb{E}\left(\mathbb{R}\right)$ such that $\left(y_n-y\right)\curlyvee\gamma_n=\left[\frac{F_n\curlyvee H\gamma_n}{H\gamma_n}\right]$ and $F_n\rightarrow0$ as $n\rightarrow\infty$ for some $(\gamma_n)\in\Delta$ and $F_n=Hf_n$.

Next, applying Eq. (10),

$$\boldsymbol{\kappa}^{-1}\left(\left(y_n-y\right)\curlyvee H\gamma_n\right)=\left[\frac{H^{-1}\left(F_n\curlyvee H\gamma_n\right)}{\gamma_n}\right].$$

Thus $\boldsymbol{\kappa}^{-1}\left(\left(y_n-y\right)\curlyvee H\gamma_n\right)=\left[\frac{f_n\bullet\gamma_n}{\gamma_n}\right]\sim f_n\rightarrow0$ as $n\rightarrow\infty$ in $\mathbb{E}\left(\mathbb{R}\right)$. Thus $\boldsymbol{\kappa}^{-1}\left(\left(y_n-y\right)\curlyvee H\gamma_n\right)=\left(\boldsymbol{\kappa}^{-1}y_n-\boldsymbol{\kappa}^{-1}y\right)\bullet\gamma_n\rightarrow0$ as $n\rightarrow\infty$. Hence, $\boldsymbol{\kappa}^{-1}y_n\overset{\Delta}{\rightarrow}\boldsymbol{\kappa}^{-1}y$ as $n\rightarrow\infty$ in $\mathbb{M}\left(\mathbb{E},\mathbb{D},\Delta,\bullet\right)$. This completes the proof of the theorem.

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NONPARAMETRIC ESTIMATION OF A MULTIVARIATE PROBABILITY DENSITY FOR MIXING SEQUENCES BY THE METHOD OF WAVELETS

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Abstract. The mathematical theory of wavelet and their applications in statistics have become a well-known technique for non-parametric curve estimation: see e.g. Meyer (1990), Daubachies (1992), Chui (1992), Donoho and Johnstone (1995) and Vidakovic (1999). We Consider the problem of estimation of the partial derivatives of a multivariate probability density f of mixing sequences, using wavelet-based method. Many stochastic processes and time series are known to be mixing. Under certain weak assumptions autoregressive and more generally bilinear time series models are strongly mixing with exponential mixing coefficients. The problem of density estimation from dependent samples is often considered. For instance quadratic losses were considered by Ango Nze and Doukhan (1993). Bosq (1995) and Doukhan and Loen (1990). We investigate the variance and the rate of the almost convergence of wavelet-based estimators. Rate of convergence of estimators when f belongs to the Besov space is also established.

Keywords and phrases: nonparametric estimation of partial derivatives, multivariate density, wavelet, mixing process.

1. Introduction

Methods of nonparametric estimation of a multivariate probability density function and regression function are discussed in Prakasa Rao (1983,1999). The problem of estimation of partial derivatives of multivariate probability density is of interest of Singh (1981), Prakasa Rao (1983), especially to detect concavity or convexity properties of the regression function.

Kernel-type estimation the functional $I_2(f)$ has been investigated by Hall and Marron (1987). and Bickel and Ritov (1988) among others. Prakasa Rao (1996) studied nonparametric estimation of the derivative of a density by wavelets and obtained a precise asymptotic expression for the mean integrated squared error following techniques of Masry (1994). Estimation of the integrated squared density derivatives was discussed in Prakasa Rao (1999) by the method of wavelets and a precise asymptotic expression for the mean squared error had been obtained.

Prakasa Rao (2000) also obtained the almost sure convergence of estimation of the partial derivatives of a probability density. We now extend the result to the case of strongly mixing process . We show that the L_p error of the proposed estimator attains the same rate when the observations are independent. Certain week dependence conditions are imposed to the $\{x_i\}$ defined in $\{\Omega, N, P\}$.

Let N_k^m denote the σ-algebra generated by events $X_k \in A_k, ..., X_m \in A_m$. We consider the following classical mixing conditions:

1. Strong mixing (s.m) also called α-mixing:

$$\sup \sup |p(AB) - p(A)p(B)| = \alpha(s) \rightarrow 0 \qquad as \quad s \rightarrow \infty$$

2. Complete regularity (c.r.), also called β-mixing:

$$\sup E\{var|p(B|N_1^m) - p(B)|\} = \beta(s) \rightarrow 0 \qquad as \quad s \rightarrow \infty$$

3. Uniformely strong mixing (u.s.m.), also called $\phi - mixing$:

$$\sup \sup \frac{|p(AB) - p(A)p(B)|}{p(A)} = \phi(s) \rightarrow 0 \qquad as \quad s \rightarrow \infty$$

4. ρ-mixing:

$$\sup \sup |corr(X, Y)| = \rho(s) \rightarrow 0 \qquad as \quad s \rightarrow \infty$$

Among various mixing conditions used in the literature, α-mixing is reasonably weak, and has many practical applications. Many stochastic processes and time series are known to be mixing. Under certain weak assumptions autoregressive and more generally bilinear time series models are strongly mixing with exponential mixing coefficients. The problem of density estimation from dependent samples is often considered. For instance quadratic losses were considered by

Ango Nze and Doukhan (1993), Bosq (1995) and Doukhan and Loen (1990). Linear wavelet estimators were also used in context: Doukhan (1988) and Doukhan and Loen (1990). Leblance (1996) also established that the $L_{\acute{p}}$-loss ($2 \leq \acute{p} < \infty$) of the linear wavelet density estimators for a stochastic process converges at the rate $N^{\frac{-s}{(2s+1)}}$ ($s = 1/p + 1/\acute{p}$), when the density of f belongs to the Besov space $B^s_{p,\acute{p}}$. Dooti, Niroumand and Afshari (2006) extended the above result for derivative of a density.

2. Discussion of Theorem's Assumptions

Consider the following conditions:

C_1: The process is ρ-mixing and $\sum_{t=1}^{\infty} \rho(t) \leq R < \infty$.

C_2: The process is ϕ-mixing and $\sum_{t=1}^{\infty} \phi^{1/2}(t) \leq \phi < \infty$.

Since the inequality $\rho(t) \leq 2\varphi^{1/2}(t)$ holds (see Doukhan 1994), C_2 implies C_1. Also note that if X and Y are random variables , then the following covariance inequalities hold.(see Doukhan, 1994, section 1.2.2)

$$cov(X,Y) \leq 2\rho(j-i)\|X\|_2.\|Y\|_2 \tag{2.1}$$

$$cov(X,Y) \leq 2\phi^{1/p}(j-i)\|X\|_p.\|Y\|_q \tag{2.2}$$

for any $p, q \geq 1$ and $1/p + 1/q = 1$.

3. Preliminaries

A multiresolution in R^d is a decomposition of the space $L^2(R^d)$ in to an interesting sequence of closed subspaces V_j, $-\infty < j < \infty$ such that

(i) $\bigcap_{j=-\infty}^{\infty} V_j = 0$,

(ii) $\overline{\bigcup_{j=-\infty}^{\infty} V_j} = L^2(R^d)$,

(iii) there exists a scaling function $\varphi \in V_0$ such that

$$\int_{R^d} \varphi(x)dx = 1$$

and $\{\varphi(x-k), k \in Z^d\}$ is an orthogonal basis for V_0 and for all $h \in L^2(R^d)$,

(iv) for all $k \in z^d$, $h(X) \in V_0 \to h(x-k) \in V_0$ and

(v) $h(x) \in V_j \to h(2x) \in V_{j+1}$.

In fact, the family $\{\varphi_{j,k} = 2^{\frac{jd}{2}}(2^j x - k), k \in Z^d\}$ is an orthonormal basis for V_j.

Definition 3.1. The multiresolution analysis is said to be *r-regular* if $\varphi \in C^{(r)}$ and all its partial derivatives up to total order r are rapidly decreasing, that is for any integer $m \geq 1$ there exists a constant c_m such that

$$|(D^\beta \varphi)(x)| \leq \frac{c_m}{(1+\|x\|)^m} \tag{3.1}$$

for all $|\beta| \leq r$, where

$$(D^\beta \varphi)(x) = \frac{\partial \varphi(x)}{\partial x_1^{\beta_1} \ldots \partial x_n^{\beta_n}} \tag{3.2}$$

and

$$\beta = (\beta_1, \ldots, \beta_d), |\beta| = \sum_{i=1}^{d} \beta_i. \tag{3.3}$$

Define $\{X_n, n \geq 1\}$ be i.i.d d-dimensional random vectors with density $f(x)$. Suppose f is partially differentiable up to total order r. The problem is to estimate

$$(D^\beta \varphi)(x) = \frac{\partial \varphi(x)}{\partial x_1^{\beta_1} \ldots \partial x_n^{\beta_n}}$$

We assume that $D^\beta f = L^2(R^d)$. Let $(D^\beta f)_l$ be the orthogonal projection of $D^\beta f$ on V_l. Then

$$(D^\beta f)(x) = \sum_{k \in z^d} a_{lk} \varphi_{l,k}(x) \tag{3.4}$$

where

$$a_{lk} = \int_{R^d} (D^\beta f)(u) \varphi_{l,k}(u) du \tag{3.5}$$

$$= (-1)^{\sum_{i=1}^{d} \beta_i} \int_{R^d} (D^\beta \varphi_{l,k}) f(u) du \tag{3.6}$$

Hence

$$\hat{a}_{lk} = \frac{(-1)^{\sum_{i=1}^{d} \beta_i}}{n} \sum_{i=1}^{n} (D^\beta \varphi_{l,k})(x_i) \tag{3.7}$$

Let us estimate $D^\beta f$ by

$$(D^\beta f)_n(x) = \sum_{k \in z^d} \hat{a}_{lk} \varphi_{l,k}(x) \tag{3.8}$$

Assume that the support of φ is compact. then the series above has a finite number of nonzero terms for any fixed x. Define the kernel $K(u,v)$ by

$$K(u,v) = \sum_{k \in z^d} \varphi(u-k)(D^\beta \varphi)(v-k)$$

Let

$$E(u,v) = \sum_{k \in z^d} \varphi(u-k)\varphi(v-k) \tag{3.9}$$

Note that

$$\partial_v^\beta E(u,v) = \sum_{k \in z^d} \varphi(u-k)(D^\beta \varphi)(v-k) = K(u,v) \tag{3.10}$$

Then there exists constants $C_m > 0$, for any $m \geq 1$, such that

$$|\partial_u^v \partial_v^x K(u,v)| \leq C_m(1 + \|u-v\|)^{-m}, m \geq 1 \tag{3.11}$$

for $|\alpha| \leq r$ and $|\alpha + \beta| \leq r$ (cf. Meyer (1992)). Choosing $\alpha = 0 = \gamma$, we obtain that in particular

$$|K(u,v)| \leq C_m(1 + \|u-v\|)^{-m} \tag{3.12}$$

In particular

$$|K(u,v)| \leq C_{d+1}(1 + \|u-v\|)^{-(d+1)} \tag{3.13}$$

and

$$\int_{R^d} |K(u,v)|^j dv \leq G_j(d) \tag{3.14}$$

where

$$G_j(d) = 2\pi^{d/2} \frac{\Gamma(d)\Gamma(j + d(j-1))}{\Gamma(d/2)\Gamma((d+1)j))} C_{d+1}^j \tag{3.15}$$

Note that

$$(D^\beta f)_n(x) = \sum_{k \in z^d} \varphi_{l,k}(x) \frac{(-1)^{\sum_{i=1}^d \beta_i}}{n} \sum_{i=1}^n (D^\beta \varphi_{l,k})(x_i)$$

$$= \frac{(-1)^{|\beta_i|}}{n} \sum_{i=1}^n \sum_{k \in z^d} \varphi_{l,k}(x)(D^\beta \varphi_{l,k})(x_i) = \frac{(-1)^{|\beta|}}{n} 2^{ld + l|\beta|} \sum_{i=1}^n K(2^l x, 2^l X_i) \tag{3.16}$$

Hence

$$\begin{aligned}&Var[(D^\beta f)_n(x)]\\&=\frac{2^{2ld+2l|\beta|}}{n^2}\left[\sum_{i=1}^n Var(K(2^l x,2^l X_i))+2\sum_{i>j}Cov(K(2^l x,2^l X_i),K(2^l x,2^l X_j))\right].\end{aligned}$$

Following along the lines of Rao (1999), one may easily get

$$n2^{-l(d+2|\beta|)}Var[K(2^l x,2^l X_i)]\le M_1G_2(d)(1+o(1)). \tag{3.17}$$

Applying (2.2), it is straightforward to have

$$\begin{aligned}&Cov(K(2^l x,2^l X_i),K(2^l x,2^l X_j))\\&\quad=2\phi^{1/2}(j-i)\|K(2^l x,2^l x_i)\|_2\|K(2^l x,2^l x_j)\|_2\\&\quad d=2\phi^{1/2}(j-i)\left(\int 2^{-2ld}K^2(2^l x,u)du\right)^{1/2}\left(\int 2^{-2ld}K^2(2^l x,v)dv\right)^{1/2}\\&\quad=2^{-2ld+1}\phi^{1/2}(j-i)G_2(d).\end{aligned}\tag{3.18}$$

Having C_2 in mind with (3.17) and (3.18), it follows that

$$\sup_{x\in R^d}Var[(D^\beta f)_n(x)]\le M_1\frac{2^{l(d+2\beta)}}{n}+M_2\frac{2^{l(d+2\beta)}}{n}. \tag{3.19}$$

Let D be a compact set in R^d and $L(n); n\ge 1$ be a non-negative sequence to be chosen later. Suppose that the set D can be covered by a finite number $L(n)$ of cubes $I_j=I_{nj}$ with centers $x_j=x_{nj}$ with sides of length m_n for $j=1,...,L(n)$, $m_n=const/L^{1/d}(n)$. This is possible since D is compact. Then

$$\begin{aligned}&\sup_{x\in D}|(D^\beta f_n)(x)-E((D^\beta f_n)(X))|\\&\qquad=\max_{1\le j\le L(n)}\sup_{x\in D\cap I_j}|(D^\beta f_n)(x)-E((D^\beta f_n)(X))|\\&\qquad\le\max_{1\le j\le L(n)}\sup_{x\in D\cap I_j}|(D^\beta f_n)(x)-(D^\beta f_n)(X_j)|\\&\qquad+\max_{1\le j\le L(n)}|(D^\beta f_n)(x_j)-E((D^\beta f_n)(X))|\\&\qquad+\max_{1\le j\le L(n)}\sup_{x\in D\cap I_j}|E((D^\beta f_n)(X))-E((D^\beta f_n)(X_j))|\\&\qquad=T_1+T_2+T_3.\end{aligned}\tag{3.20}$$

Applying (3.23),(3.24) in Rao (1999), one may have

$$T_1=T_2=O\left(\frac{2^{l(d+|\beta|+1)}}{L^{1/d}(n)}\right). \tag{3.21}$$

Now, for any x,

$$\begin{aligned} Z_n(x) &= (D^\beta f_n)(x) - E[(D^\beta f_n)(X)] \\ &= = \frac{(-1)^{|\beta| 2^{ld+l|\beta|}}}{n} \sum_{i=1}^{n} K(2^l x, 2^l X_i) - E(K(2^l x, 2^l X_i)) = 1/n \sum_{i=1}^{n} Y_{ni}, \end{aligned}$$

where

$$Y_{ni} = (-1)^{|\beta| 2^{ld+l|\beta|}} \{K(2^l x, 2^l X_i) - E(K(2^l x, 2^l X_i))\}.$$

Note that relation (3.12) implies that

$$|K((u,v)| \le C_2 \tag{3.22}$$

Lemma 3.1. (Doukhan (1994)) *Assume the sequence to satisfy the* ϕ*-mixing condition and* $n\phi(n)$ *is bounded and*

(i) $\forall t \in N, EX_t = 0$

(ii) $\exists \sigma^2 \in R^+, \forall n, m \in N : 1/mE(X_n, ..., X_m)^2 \le \sigma^2$

(iii) $\forall t \in N, |x_t| \le M.$

Then, there exists constants a *and* $b > 0$ *such that*

$$P\left(\sum_{i=1}^{n} |X_t| \ge x\sqrt{n}\right) \le a \exp(-bx^2).$$

It is easy to check (i), (ii) and (iii) in the above Lemma applying (3.19),(3.23) as well as the definition of Y_{ni}. Using (3.19) and (3.22), one may easily apply the above result to conclude that for any $\eta_n > 0$ and $t_n > 0$,

$$P(|Z_n(x)| \ge \eta_n) \le P\left(|\sum_{i=1}^{n} Y_{ni}| \ge n\eta_n\right) \le a \exp(-nb\eta_n^2) \tag{3.23}$$

therefore

$$P\left(\max_{1 \le j \le L(n)} |Z_n(x)| \ge \eta_n\right) \le L(n) \exp(-nb\eta_n^2). \tag{3.24}$$

Let

$$\eta_n = \frac{2^{l(d+1+|\beta|)}}{L^{1/d}(n)}.$$

Then

$$P\left(\max_{1 \le j \le L(n)} |Z_n(x)| \ge \eta_n\right) \le L(n) \exp\left(-nb\frac{2^{l(d+1+\beta)}}{L^{2/d}(n)}\right). \tag{3.25}$$

Let

$$L(n) = (2^{l(d+1+\beta)} n / \log n)^{d/2}.$$

Then

$$T_1 + T_3 = O\left(2^{l(d/2)+\beta}\frac{(\log)^{1/2}}{n^{1/2}}\right). \tag{3.26}$$

Suppose that $l_n \to \infty$ and

$$\frac{2^{l(d+1+\beta)}\log n}{n} \to 0$$

as $n \to \infty$. Following the relation (3.25), it can be checked that

$$\sum_{n=1}^{\infty} P(|Z_n(x)| \geq \eta_n) \leq \sum_{n=1}^{\infty} \frac{L(n)}{n^a} < \infty$$

for a suitable constant $a > 1$. Hence, by Borel-Cantelli Lemma, it follows that

$$T_2 = \max_{1\leq j\leq L(n)} |Z_n(x_j)| \leq \eta_n \tag{3.27}$$

Combining (3.26) and (3.27), it follows that

$$T_1 + T_2 + T_3 = O\left(2^{l(d/2)+\beta}\frac{(\log)^{1/2}}{n^{1/2}}\right) = O\left(\left(\frac{2^{l(d)+2\beta}\log}{n}\right)^{1/2}\right). \tag{3.28}$$

Note that

$$E[(D^\beta f_n)(x)] = \sum_{k\in z^d} a_{lk}\varphi_{l,k}(x) = P_{V_i}(D^\beta f_n)(x).$$

Hence

$$(D^\beta f_n)(x) - E[(D^\beta f_n)(x)] = (D^\beta f_n)(x) - P_{V_i}(D^\beta f_n)(x) = \sum_{j\geq 1} P_{W_j}(D^\beta f_n)(x).$$

If $(D^\beta f_n)(x) \in B_{spq}$, where B_{spq} is a Besov space, for some $0 < s < r, 1 \leq p, q < \infty$ with $s > d/p$ and if the multiresolution analysis is r-regular, then it follows by arguments in Masry and Kerkyacharian and Picard (1992) that

$$\begin{aligned} &\sup_{x\in R^d} (D^\beta f_n)(x) - E[(D^\beta f_n)(x)] \\ &\sup_{x\in R^d} (D^\beta f_n)(x) - P_{V_i}(D^\beta f_n)(x) \\ &\leq (constant) 2^{-(s-d/p)l} J_{spq}(D^\beta f), \end{aligned} \tag{3.29}$$

where

$$J_{spq}(D^\beta f) = \|P_{v_0 g}\|_{L_P} + \left(\sum_{j\geq 0}(2^{js}\|P_{w_j g}\|_{L_P})^q\right)^{1/q}$$

for $g \in B_{spq}$. Combining (3.28) and (3.29), we have

$$\begin{aligned} &\sup_{x\in R^d} (D^\beta f_n)(x) - Et[(D^\beta f_n)(x)] \\ &= O\left(\left(\frac{2^{l(d)+2\beta}\log n}{n}\right)^{1/2}\right) + O(2^{-(s-d/p)l}) \ a.s. \end{aligned} \tag{3.30}$$

Now we are ready to have the main result of the paper.

5. Main Results

Let the sequence $\{X_n\}_{n\geq 1}$ be the ϕ-mixing sequence of random variables with the property C_2 such that $n\phi(n)$ is bounded.

Theorem 5.1. *Suppose the multivariate probability density $f(x)$ is bounded with partial derivative up to order r. Further, suppose that $D^\beta f \in L^2(R^d)$ and $l = l_n \to \infty$ such that*

$$\frac{2^{l(d+1+\beta)}\log n}{n} \to 0$$

Then, there exists constants $M_1, M_2 > 0$ such that

$$\sup_{x\in R^d} Var[(D^\beta f)_n(x)] \leq M_1 \frac{2^{l(d+2\beta)}}{n} + M_2 \frac{2^{l(d+2\beta)}}{n}$$

Furthermore, for any compact set $D \in R^d$,

$$\sup_{x\in D} |(D^\beta f_n)(x) - E((D^\beta f_n)(X))| = O\left(\left(\frac{2^{l(d)+2\beta}\log}{n}\right)^{1/2}\right) \quad a.s.$$

In addition if $D \in B_{spq}$ for some $0 < s < r$, $1 \leq p$, $q < \infty$ with $s > d/p$, then

$$\sup_{x\in R^d} (D^\beta f_n)(x) - E[(D^\beta f_n)(x)] = O\left(\left(\frac{2^{l(d)+2\beta}\log n}{n}\right)^{1/2}\right) + O\left(2^{-(s-d/p)l}\right) \quad a.s.$$

and if $D \in B_{s\infty\infty}$, then

$$\sup_{x\in R^d} (D^\beta f_n)(x) - E[(D^\beta f_n)(x)] = O\left(\left(\frac{2^{l(d)+2\beta}\log n}{n}\right)^{1/2}\right) + O(2^{-sl}) \quad a.s.$$

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LOWER AND UPPER BOUNDS OF THE ČEBYŠEV FUNCTIONAL FOR THE RIEMANN-STIELTJES INTEGRAL

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Abstract. Lower and upper bounds of the Čebyšev functional for the Riemann-Stieltjes integral, in the monotonicity case of one function, are given. Applications in relation with the Steffensen generalisation of the Čebyšev inequality are provided.

Keywords and phrases: Riemann-Stieltjes integral, Čebyšev inequality, Steffensen inequality, Integral inequalities.

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1. Introduction

In [3], S.S. Dragomir introduced the following *Čebyšev functional for the Riemann-Stieltjes integral*:

$$\begin{aligned}(1.1)\quad T(f,g;u) := {}& \frac{1}{u(b)-u(a)}\int_a^b f(t)\,g(t)\,du(t)\\ &-\frac{1}{u(b)-u(a)}\int_a^b f(t)\,du(t)\cdot\frac{1}{u(b)-u(a)}\int_a^b g(t)\,du(t),\end{aligned}$$

provided $u(b)\neq u(a)$ and the involved Riemann-Stieltjes integrals exist.

In order to bound the error in approximating the Riemann-Stieltjes integral of the product in terms of the product of the integrals, as described in the definition of the Čebyšev functional (1.1), the first author obtained the inequality:

$$\begin{aligned}(1.2)\quad & |T(f,g;u)|\\ &\leq \frac{1}{2}(M-m)\cdot\frac{1}{|u(b)-u(a)|}\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty \bigvee_a^b(u),\end{aligned}$$

provided u is of bounded variation, f, g are continuous on $[a, b]$ and $m \leq f(t) \leq M$ for any $t \in [a, b]$. The constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

Moreover, if f, g are as above and u is monotonic nondecreasing on $[a, b]$, then

$$
\begin{aligned}
(1.3)\quad & |T(f, g; u)| \\
& \leq \frac{1}{2}(M-m) \frac{1}{u(b)-u(a)} \int_a^b \left| g(t) - \frac{1}{u(b)-u(a)} \int_a^b g(s)\, du(s) \right| du(t),
\end{aligned}
$$

and the constant $\frac{1}{2}$ here is also sharp.

Finally, if f and g are Riemann integrable and u is Lipschitzian with the constant $L > 0$, then also

$$
\begin{aligned}
(1.4)\quad & |T(f, g; u)| \\
& \leq \frac{1}{2} L (M-m) \frac{1}{|u(b)-u(a)|} \int_a^b \left| g(t) - \frac{1}{u(b)-u(a)} \int_a^b g(s)\, du(s) \right| dt,
\end{aligned}
$$

provided $m \leq f(t) \leq M$, $t \in [a, b]$. The multiplicative constant $\frac{1}{2}$ is best possible in (1.4).

For results concerning bounds for the Čebyšev functional $T(f, g; u)$ see [4] and [5]. For other recent results on inequalities for the Riemann-Stieltjes integral, see [1], [2] and [6].

The main aim of this paper is to provide an upper and a lower bound for the functional $T(f, g; u)$ under the monotonicity assumption on the function f. An application for the Čebyšev inequality for Riemann-Stieltjes integrals that is related to Steffensen's result from [8] is given as well.

2. The results

The following result providing upper and lower bounds for the quantity

$$[h(b) - h(a)]\, T(f, g, h; a, b)$$

can be stated:

Theorem 2.1 *Let* $f, g, h : [a, b] \to \mathbb{R}$ *be such that* $h(a) \neq h(b)$ *and the Riemann-Stieltjes integrals* $\int_a^b f(t)\, dh(t)$, $\int_a^b g(t)\, dh(t)$ *and* $\int_a^b f(t)\, g(t)\, dh(t)$ *exist. If* f

is monotonic nondecreasing, then

$$\begin{aligned}(2.1)\quad &[f(b)-f(a)]\inf_{t\in[a,b]}\left\{\int_t^b g(s)\,dh(s)-\frac{h(b)-h(t)}{h(b)-h(a)}\cdot\int_a^b g(\tau)\,dh(\tau)\right\}\\ &\le\int_a^b f(t)\,g(t)\,dh(t)-\frac{1}{h(b)-h(a)}\cdot\int_a^b f(t)\,dh(t)\cdot\int_a^b g(t)\,dh(t)\\ &\le[f(b)-f(a)]\sup_{t\in[a,b]}\left\{\int_t^b g(s)\,dh(s)-\frac{h(b)-h(t)}{h(b)-h(a)}\cdot\int_a^b g(\tau)\,dh(\tau)\right\}.\end{aligned}$$

If f is monotonic nonincreasing, then:

$$\begin{aligned}(2.2)\quad &[f(a)-f(b)]\inf_{t\in[a,b]}\left\{\int_a^t g(s)\,dh(s)-\frac{h(t)-h(a)}{h(b)-h(a)}\cdot\int_a^b g(\tau)\,dh(\tau)\right\}\\ &\le\int_a^b f(t)\,g(t)\,dh(t)-\frac{1}{h(b)-h(a)}\cdot\int_a^b f(t)\,dh(t)\cdot\int_a^b g(t)\,dh(t)\\ &\le[f(a)-f(b)]\sup_{t\in[a,b]}\left\{\int_a^t g(s)\,dh(s)-\frac{h(t)-h(a)}{h(b)-h(a)}\cdot\int_a^b g(\tau)\,dh(\tau)\right\}.\end{aligned}$$

Inequalities (2.1) *and* (2.2) *are sharp.*

Proof. We use the following Abel type inequality obtained by Mitrinović et al. in [7, p. 336]:

Let u be a nonnegative and monotonic nondecreasing function on $[a,b]$ and $v,w:[a,b]\to\mathbb{R}$ such that the Riemann-Stieltjes integrals $\int_a^b v(t)\,dw(t)$ and $\int_a^b u(t)\,v(t)\,dw(t)$ exist. Then

$$\begin{aligned}(2.3)\qquad u(b)\inf_{t\in[a,b]}\left\{\int_t^b v(t)\,dw(t)\right\}&\le\int_a^b u(t)\,v(t)\,dw(t)\\ &\le u(b)\sup_{t\in[a,b]}\left\{\int_t^b v(t)\,dw(t)\right\}.\end{aligned}$$

We also use the representation (see [3])

$$\begin{aligned}(2.4)\quad &T(f,g,h;a,b)\\ &=\frac{1}{h(b)-h(a)}\int_a^b[f(t)-\gamma]\left[g(t)-\frac{1}{h(b)-h(a)}\int_a^b g(s)\,dh(s)\right]dh(t),\end{aligned}$$

which holds for any $\gamma\in\mathbb{R}$.

Now, if we choose $\gamma=f(a)$, then we observe that the function $u(t)=f(t)-f(a)$ is nonnegative and monotonic nondecreasing on $[a,b]$ and applying

(2.3) for $w(t)=h(t)$ and $v(t)=g(t)-\frac{1}{h(b)-h(a)}\int_a^b g(s)\,dh(s)$ we deduce:

$$\begin{aligned}(2.5)\qquad & [f(b)-f(a)]\inf_{t\in[a,b]}\left\{\int_t^b\left[g(s)-\frac{1}{h(b)-h(a)}\cdot\int_a^b g(\tau)\,dh(\tau)\right]dh(s)\right\}\\ &\le [h(b)-h(a)]\,T(f,g,h;a,b)\\ &\le [f(b)-f(a)]\sup_{t\in[a,b]}\left\{\int_t^b\left[g(s)-\frac{1}{h(b)-h(a)}\cdot\int_a^b g(\tau)\,dh(\tau)\right]dh(s)\right\},\end{aligned}$$

which is equivalent with the desired inequality (2.1).

For the second inequality, we use (2.4) with $\gamma=f(b)$ and the following Abel type result for functions u which are monotonic nonincreasing and nonnegative (see [7, p. 336]):

$$\begin{aligned}(2.6)\qquad u(a)\inf_{t\in[a,b]}\left\{\int_a^t v(t)\,dw(t)\right\} &\le \int_a^b u(t)\,v(t)\,dw(t)\\ &\le u(a)\sup_{t\in[a,b]}\left\{\int_a^t v(t)\,dw(t)\right\}.\end{aligned}$$

The details are omitted.

Let us prove for instance the sharpness of the second inequality in (2.1).

If we choose $h(t)=t$ and $g(t)=\operatorname{sgn}\left(t-\frac{a+b}{2}\right)$, $t\in[a,b]$ then we have to show that the inequality:

$$(2.7)\qquad \int_a^b f(t)\operatorname{sgn}\left(t-\frac{a+b}{2}\right)dt\le[f(b)-f(a)]\sup_{t\in[a,b]}\left\{\int_t^b\operatorname{sgn}\left(s-\frac{a+b}{2}\right)ds\right\}$$

is sharp provided f is monotonic nondecreasing on $[a,b]$.

Notice that

$$\lambda(t):=\int_t^b\operatorname{sgn}\left(s-\frac{a+b}{2}\right)ds=\begin{cases} t-a & \text{if } t\in\left[a,\frac{a+b}{2}\right]\\ b-t & \text{if } t\in\left(\frac{a+b}{2},b\right]\end{cases}$$

and then $\sup_{t\in[a,b]}\lambda(t)=\frac{b-a}{2}$.

Therefore (2.7) becomes

$$(2.8)\qquad \int_a^b f(t)\operatorname{sgn}\left(t-\frac{a+b}{2}\right)dt\le[f(b)-f(a)]\cdot\frac{b-a}{2}.$$

Now, if in this inequality we choose $f(t)=\operatorname{sgn}\left(t-\frac{a+b}{2}\right)$, which is monotonic nondecreasing on $[a,b]$, we get in both sides of (2.8) the same quantity $b-a$.

The sharpness of the other inequalities can be shown in a similar way. The details are omitted. ■

Remark 2.1 We observe that

$$\begin{aligned}
&\int_t^b g(s)\,dh(s)-\frac{h(b)-h(t)}{h(b)-h(a)}\cdot\int_a^b g(\tau)\,dh(\tau)\\
&=\int_t^b g(s)\,dh(s)-\frac{h(b)-h(t)}{h(b)-h(a)}\cdot\left[\int_a^t g(s)\,dh(s)+\int_t^b g(s)\,dh(s)\right]\\
&=\frac{h(t)-h(a)}{h(b)-h(a)}\int_t^b g(s)\,dh(s)-\frac{h(b)-h(t)}{h(b)-h(a)}\int_a^t g(s)\,dh(s)\\
&=\frac{[h(t)-h(a)][h(b)-h(t)]}{h(b)-h(a)}\\
&\qquad\times\left[\frac{1}{h(b)-h(t)}\int_t^b g(s)\,dh(s)-\frac{1}{h(t)-h(a)}\int_a^t g(s)\,dh(s)\right].
\end{aligned}$$

Therefore, if we denote by $\Delta(g,h;t,a,b)$ the difference

$$\frac{1}{h(b)-h(t)}\int_t^b g(s)\,dh(s)-\frac{1}{h(t)-h(a)}\int_a^t g(s)\,dh(s),$$

provided $h(t)\neq h(a),h(b)$ for $t\in[a,b]$, then from (2.1) we get

$$\begin{aligned}
&[f(b)-f(a)]\inf_{t\in[a,b]}\left\{\frac{[h(t)-h(a)][h(b)-h(t)]}{h(b)-h(a)}\Delta(g,h;t,a,b)\right\}\\
&\le\int_a^b f(t)g(t)\,dh(t)-\frac{1}{h(b)-h(a)}\int_a^b f(t)\,dh(t)\cdot\int_a^b g(t)\,dh(t)\\
&\le[f(b)-f(a)]\sup_{t\in[a,b]}\left\{\frac{[h(t)-h(a)][h(b)-h(t)]}{h(b)-h(a)}\Delta(g,h;t,a,b)\right\},
\end{aligned}\tag{2.9}$$

provided f is monotonic nondecreasing on $[a,b]$.

A similar result can be stated from (2.2) on noticing that

$$\begin{aligned}
&\int_a^t g(s)\,dh(s)-\frac{h(t)-h(a)}{h(b)-h(a)}\cdot\int_a^b g(\tau)\,dh(\tau)\\
&\qquad\qquad=-\frac{[h(t)-h(a)][h(b)-h(t)]}{h(b)-h(a)}\Delta(g,h;t,a,b).
\end{aligned}$$

Indeed, since

$$\begin{aligned}
&\inf_{t\in[a,b]}\left(\sup_{t\in[a,b]}\right)\left\{\int_a^t g(s)\,dh(s)-\frac{h(t)-h(a)}{h(b)-h(a)}\cdot\int_a^b g(s)\,dh(s)\right\}\\
&=\inf_{t\in[a,b]}\left(\sup_{t\in[a,b]}\right)\left\{-\frac{[h(t)-h(a)][h(b)-h(t)]}{h(b)-h(a)}\Delta(g,h;t,a,b)\right\}\\
&=-\sup_{t\in[a,b]}\left(\inf_{t\in[a,b]}\right)\left\{\frac{[h(t)-h(a)][h(b)-h(t)]}{h(b)-h(a)}\Delta(g,h;t,a,b)\right\},
\end{aligned}$$

then from (2.2) we get

$$\begin{aligned}
&[f(a)-f(b)]\inf_{t\in[a,b]}\left\{\frac{[h(t)-h(a)][h(b)-h(t)]}{h(b)-h(a)}\Delta(g,h;t,a,b)\right\}\\
&\le\frac{1}{h(b)-h(a)}\int_a^b f(t)\,dh(t)\int_a^b g(t)\,dh(t)-\int_a^b f(t)\,g(t)\,dh(t)\\
&\le[f(a)-f(b)]\sup_{t\in[a,b]}\left\{\frac{[h(t)-h(a)][h(b)-h(t)]}{h(b)-h(a)}\Delta(g,h;t,a,b)\right\}
\end{aligned}$$

provided that f is monotonic nonincreasing on $[a,b]$.

The following corollary gives a particular result of interest for Riemann weighted integrals.

Corollary 2.1 *Let* $f,g,w:[a,b]\to\mathbb{R}$ *be such that the Riemann integrals* $\int_a^b f(t)\,w(t)\,dt$, $\int_a^b g(t)\,w(t)\,dt$, $\int_a^b f(t)\,g(t)\,w(t)\,dt$ *and* $\int_a^b w(t)\,dt$ *exist, and* $\int_a^b w(t)\,dt\neq 0$.

If f *is monotonic nondecreasing, then*

$$\begin{aligned}
(2.10)\quad &[f(b)-f(a)]\inf_{t\in[a,b]}\left\{\int_t^b g(s)\,w(s)\,ds-\frac{\int_t^b w(s)\,ds}{\int_a^b w(s)\,ds}\cdot\int_a^b g(\tau)\,w(\tau)\,d\tau\right\}\\
&\le\int_a^b f(t)\,g(t)\,w(t)\,dt-\frac{1}{\int_a^b w(s)\,ds}\cdot\int_a^b f(t)\,w(t)\,dt\cdot\int_a^b g(t)\,w(t)\,dt\\
&\le[f(b)-f(a)]\sup_{t\in[a,b]}\left\{\int_t^b g(s)\,w(s)\,ds-\frac{\int_t^b w(s)\,ds}{\int_a^b w(s)\,ds}\cdot\int_a^b g(\tau)\,w(\tau)\,d\tau\right\}.
\end{aligned}$$

If f *is monotonic nonincreasing, then*

$$\begin{aligned}
(2.11)\quad &[f(a)-f(b)]\inf_{t\in[a,b]}\left\{\int_a^t g(s)\,w(s)\,ds-\frac{\int_a^t w(s)\,ds}{\int_a^b w(s)\,ds}\cdot\int_a^b g(\tau)\,w(\tau)\,d\tau\right\}\\
&\le\int_a^b f(t)\,g(t)\,w(t)\,dt-\frac{1}{\int_a^b w(s)\,ds}\cdot\int_a^b f(t)\,w(t)\,dt\cdot\int_a^b g(t)\,w(t)\,dt\\
&\le[f(a)-f(b)]\sup_{t\in[a,b]}\left\{\int_a^t g(s)\,w(s)\,ds-\frac{\int_a^t w(s)\,ds}{\int_a^b w(s)\,ds}\cdot\int_a^b g(\tau)\,w(\tau)\,d\tau\right\}.
\end{aligned}$$

Remark 2.2 If we define

$$\tilde{\Delta}(g,w;t,a,b):=\frac{1}{\int_t^b w(s)\,ds}\int_t^b g(s)\,w(s)\,ds-\frac{1}{\int_a^t w(s)\,ds}\int_a^t g(s)\,w(s)\,ds,$$

provided $\int_a^t w(s)\,ds,\ \int_t^b w(s)\,ds \neq 0$, then, under the assumptions of Corollary 2.1, we have:

$$\begin{aligned}(2.12)\quad & [f(b)-f(a)]\inf_{t\in[a,b]}\left\{\frac{\int_a^t w(s)\,ds\int_a^b w(s)\,ds}{\int_a^b w(s)\,ds}\cdot\tilde{\Delta}(g,w;t,a,b)\right\}\\ &\le \int_a^b f(t)\,g(t)\,w(t)\,dt-\frac{1}{\int_a^b w(s)\,ds}\cdot\int_a^b f(t)\,w(t)\,dt\cdot\int_a^b g(t)\,w(t)\,dt\\ &\le [f(b)-f(a)]\sup_{t\in[a,b]}\left\{\frac{\int_a^t w(s)\,ds\int_a^b w(s)\,ds}{\int_a^b w(s)\,ds}\cdot\tilde{\Delta}(g,w;t,a,b)\right\},\end{aligned}$$

provided f is monotonic nondecreasing on $[a,b]$, and

$$\begin{aligned}(2.13)\quad & [f(a)-f(b)]\inf_{t\in[a,b]}\left\{\frac{\int_a^t w(s)\,ds\int_a^b w(s)\,ds}{\int_a^b w(s)\,ds}\cdot\tilde{\Delta}(g,w;t,a,b)\right\}\\ &\le \frac{1}{\int_a^b w(s)\,ds}\int_a^b f(t)\,w(t)\,dt\cdot\int_a^b g(t)\,w(t)\,dt-\int_a^b f(t)\,g(t)\,w(t)\,dt\\ &\le [f(a)-f(b)]\sup_{t\in[a,b]}\left\{\frac{\int_a^t w(s)\,ds\int_a^b w(s)\,ds}{\int_a^b w(s)\,ds}\cdot\tilde{\Delta}(g,w;t,a,b)\right\}\end{aligned}$$

if f is monotonic nonincreasing on $[a,b]$.

Remark 2.3 In the particular case where $w(t)=1$, $t\in[a,b]$, we get the simpler inequalities:

$$\begin{aligned}(2.14)\quad & [f(b)-f(a)]\inf_{t\in[a,b]}\left\{\int_t^b g(s)\,ds-\frac{b-t}{b-a}\int_a^b g(\tau)\,d\tau\right\}\\ &\le \int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\int_a^b g(t)\,dt\\ &\le [f(b)-f(a)]\sup_{t\in[a,b]}\left\{\int_t^b g(s)\,ds-\frac{b-t}{b-a}\int_a^b g(\tau)\,d\tau\right\}\end{aligned}$$

in the case where f is monotonic nondecreasing on $[a,b]$.

If f is monotonic nonincreasing on $[a,b]$, then

$$\begin{aligned}(2.15)\quad & [f(a)-f(b)]\inf_{t\in[a,b]}\left\{\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(\tau)\,d\tau\right\}\\ &\le \int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\int_a^b g(t)\,dt\\ &\le [f(a)-f(b)]\sup_{t\in[a,b]}\left\{\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(\tau)\,d\tau\right\}.\end{aligned}$$

If we denote

$$\bar{\Delta}\left(g;t,a,b\right):=\frac{1}{b-t}\int_{t}^{b}g\left(s\right)ds-\frac{1}{t-a}\int_{a}^{t}g\left(s\right)ds,$$

then we get from (2.14)

$$\begin{aligned}&\frac{\left[f\left(b\right)-f\left(a\right)\right]}{b-a}\inf_{t\in[a,b]}\left\{\left(t-a\right)\left(b-t\right)\bar{\Delta}\left(g;t,a,b\right)\right\}\\ &\leq\int_{a}^{b}f\left(t\right)g\left(t\right)dt-\frac{1}{b-a}\int_{a}^{b}f\left(t\right)dt\cdot\int_{a}^{b}g\left(t\right)dt\\ &\leq\frac{\left[f\left(b\right)-f\left(a\right)\right]}{b-a}\sup_{t\in[a,b]}\left\{\left(t-a\right)\left(b-t\right)\bar{\Delta}\left(g;t,a,b\right)\right\},\end{aligned}$$

provided f is monotonic nondecreasing and from (2.15)

$$\begin{aligned}&\frac{\left[f\left(a\right)-f\left(b\right)\right]}{b-a}\inf_{t\in[a,b]}\left\{\left(t-a\right)\left(b-t\right)\bar{\Delta}\left(g;t,a,b\right)\right\}\\ &\leq\frac{1}{b-a}\int_{a}^{b}f\left(t\right)dt\cdot\int_{a}^{b}g\left(t\right)dt-\int_{a}^{b}f\left(t\right)g\left(t\right)dt\\ &\leq\frac{\left[f\left(a\right)-f\left(b\right)\right]}{b-a}\sup_{t\in[a,b]}\left\{\left(t-a\right)\left(b-t\right)\bar{\Delta}\left(g;t,a,b\right)\right\}\end{aligned}$$

if f is monotonic nonincreasing on $[a,b]$.

3. Applications for the Čebyšev inequality

Let $f,g:[a,b]\rightarrow\mathbb{R}$ be integrable functions, both increasing or both decreasing. Furthermore, let $p:[a,b]\rightarrow[0,\infty)$ be an integrable function, then [7, p. 239]:

$$\int_{a}^{b}p\left(x\right)dx\int_{a}^{b}p\left(x\right)f\left(x\right)g\left(x\right)dx\geq\int_{a}^{b}p\left(x\right)f\left(x\right)dx\int_{a}^{b}p\left(x\right)g\left(x\right)dx. \tag{3.1}$$

This inequality in known in the literature as Čebyšev's inequality.

For various other results related to this classical fact, see Chapter IX of the book [7] .

Proposition 3.1 *Let $f,g,h:[a,b]\rightarrow\mathbb{R}$ be such that the Riemann-Stieltjes integrals $\int_{a}^{b}f\left(t\right)dh\left(t\right)$, $\int_{a}^{b}g\left(t\right)dh\left(t\right)$ and $\int_{a}^{b}f\left(t\right)g\left(t\right)dh\left(t\right)$ exist. If $h\left(b\right)>h\left(a\right)$, f is monotonic nondecreasing (nonincreasing) and*

$$\left[h\left(b\right)-h\left(a\right)\right]\int_{a}^{t}g\left(s\right)dh\left(s\right)\geq\left[h\left(b\right)-h\left(t\right)\right]\int_{a}^{b}g\left(s\right)dh\left(s\right) \tag{3.2}$$

for any $t\in[a,b]$, then

$$\left[h\left(b\right)-h\left(a\right)\right]\int_{a}^{b}f\left(t\right)g\left(t\right)dh\left(t\right)\geq(\leq)\int_{a}^{b}f\left(t\right)dh\left(t\right)\cdot\int_{a}^{b}g\left(t\right)dh\left(t\right). \tag{3.3}$$

The proof follows by Theorem 2.1 by using

$$\int_t^b g(s)\,dh(s) - \frac{h(b)-h(t)}{h(b)-h(a)}\int_a^b g(s)\,dh(s)$$
$$= -\left[\int_a^t g(s)\,dh(s) - \frac{h(t)-h(a)}{h(b)-h(a)}\int_a^b g(s)\,dh(s)\right].$$

Remark 3.4 The above proposition implies the following Čebyšev type inequality for weighted integrals (with not necessarily positive weights).

Let $f, g, w : [a,b] \to \mathbb{R}$ be such that the Riemann integrals, $\int_a^b w(t)\,dt$, $\int_a^b f(t)\,w(t)\,dt$, $\int_a^b g(t)\,w(t)\,dt$ and $\int_a^b f(t)\,g(t)\,w(t)\,dt$ exist.

If $\int_a^b w(t)\,dt > 0$, f is monotonic nondecreasing (nonincreasing) and

$$\int_a^b w(s)\,ds \int_t^b g(s)\,w(s)\,ds \geq \int_t^b w(s)\,ds \int_a^b g(s)\,w(s)\,ds \tag{3.4}$$

for any $t \in [a,b]$, then

$$\int_a^b w(t)\,dt \int_a^b f(t)\,g(t)\,w(t)\,dt \geq (\leq) \int_a^b f(t)\,w(t)\,dt \int_a^b g(t)\,w(t)\,dt. \tag{3.5}$$

In particular (i.e., if $w(s) = 1$), if f is monotonic nondecreasing (nonincreasing) and if

$$(b-a)\int_a^t g(s)\,ds \geq (b-t)\int_a^b g(s)\,ds \tag{3.6}$$

for any $t \in [a,b]$, then

$$(b-a)\int_a^b f(t)\,g(t)\,dt \geq (\leq) \int_a^b f(t)\,dt \int_a^b g(t)\,dt. \tag{3.7}$$

Remark 3.5 Notice that, the weighted inequality (3.5), as pointed out in [7, p. 246], can be also obtained from the Steffensen result [8] which states that: if F, G, H are integrable functions on $[a,b]$ such that for all $x \in [a,b]$

$$\frac{\int_a^x G(t)\,dt}{\int_a^b G(t)\,dt} \leq \frac{\int_a^x H(t)\,dt}{\int_a^b H(t)\,dt},$$

then

$$\frac{\int_a^b F(t)\,G(t)\,dt}{\int_a^b G(t)\,dt} \geq \frac{\int_a^b F(t)\,H(t)\,dt}{\int_a^b H(t)\,dt}, \tag{3.8}$$

provided F is monotonic nondecreasing on $[a,b]$.

The choice $F(t) \equiv f(t)$, $H(t) = w(t)$, and $G(t) = g(t)w(t)$ in (3.8) produces (3.5) under the condition that (3.4) holds and f is monotonic nondecreasing.

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ENUMERATION OF HYPERCOMPOSITIONAL STRUCTURES DEFINED BY BINARY RELATIONS

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Abstract. This paper deals with hyperoperations that derive from binary relations and it studies the hypercompositional structures that are created by them. It is proved that if ρ is a binary relation on a non-void set H, then the hypercomposition $xy = \{z \in H : (x, z) \in \rho \text{ and } (z, y) \in \rho\}$ satisfies the associativity or the reproductivity only when it is total. There also appear routines that calculate (with the use of small computing power) the number of non isomorphic hypergroupoids, when the cardinality of H is finite.

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1. Introduction

The theory of hypercompositional structures was born in 1934, when F. Marty introduced the notion of the hypergroup [5]. A hypergroup is a pair $(H, \cdot)$ where H is a non empty set and ”·” a hypercomposition, i.e., a mapping from $H \times H$ to the power set $P(H)$ of H, which satisfies the axioms:

(i) $a(bc) = (ab)c$ for every $a, b, c \in H$ (associativity)

(ii) $aH = Ha = H$ for every $a \in H$ (reproductivity)

In a hypergroup, the result of the hypercomposition is always a nonempty set. Indeed, suppose that for two elements $a, b \in H$ it holds that $ab = \varnothing$. Then $H = aH = a(bH) = (ab)H = \varnothing H = \varnothing$, which is absurd. Thus, if a non empty set H is endowed with a hypercomposition which does not satisfy the associative and the reproductive low, then the void set can possibly be among its results. A pair $(H, \cdot)$ where H is a non empty set and ” · ” a hypercomposition, is called partial hypegroupoid, while it is called hypegroupoid if $ab \neq \varnothing$, for all $a, b \in H$. A hypergroupoid in which the associativity is valid, is called semi–hypergroup, while it is called quasi–hypergroup if only the reproductivity holds.

Several papers dealing with the construction of hypergroupoids and hypergroups appear in the relevant bibliography, since hypergroups are much more varied than groups, e.g. for each prime number p, there exists only one group, up to isomorphism, with cardinality p, while the number of pairwise non isomorphic hypergroups is very large. For example there exist 3999 non isomorphic hypergroups with 3 elements [12]. Nieminen [8] studied hypergroups associated with graphs and G. G. Massouros studied hypergroups associated with automata [7]. Also Chvalina [1], Rosenberg [9], Corsini [2], De Salvo and Lo Faro [3] studied hypergroupoids and hypergroups defined in terms of binary relations. This paper deals with the hypergroupoids defined by Corsini, it proves that this family of hypergroupoids contains only one semihypergroup and only one quasihypergroup, the total hypergroup and enumerates the hypergroupoids with $2, 3, 4$ and 5 elements. The order n of a finite hypergroupoid H is defined to be the number of elements in the set H.

Let H be a non empty set and ρ a binary relation on H. Corsini introduced in H the hypercomposition.

$$x \cdot y = \{z \in H : (x, z) \in \rho \text{ and } (z, y) \in \rho\}. \tag{1.1}$$

With the above hypercomposition, $(H, \cdot)$ becomes a partial hypergroupoid, while it becomes a hypergroupoid if for each pair of elements $x, y \in H$, there exists $z \in H$ such that $(x, z) \in \rho$ and $(z, y) \in \rho$. Since $\rho^2 = \rho \circ \rho = \{(x, y) \in H^2 : (x, z), (z, y) \in \rho \text{ for some } z \in H\}$, it derives that $(H, \cdot)$ is a hypegroupoid if $\rho^2 = H^2$.

2. The hypercompositional structures defined by ρ

Let H_ρ denote the hypercompositional structure defined by (1.1) through the binary relation ρ. One can observe that the reproductivity is valid in H_ρ if and only if $(x, y) \in \rho$, for all $x, y \in H_\rho$. Indeed let x be an arbitrary element of H_ρ. For the reproductivity to be valid, it must hold: $y \in xH_\rho$, for all $y \in H_\rho$. Hence, for all $x, y \in H_\rho$, the pair (x, y) must belong to ρ. Thus:

Proposition 2.1. *H_ρ is a quasihypergroup, if and only if $(x, y) \in \rho$ for all $x, y \in H_\rho$.*

Next, suppose that H_ρ is a hypergroupoid. Then:

Lemma 2.1. *If H_ρ is a semihypergroup and $(z, z) \notin \rho$ for some $z \in H_\rho$, then $(s, z) \in \rho$ implies that $(z, s) \notin \rho$.*

Proof. Suppose that $(s, z) \in \rho$ and $(z, s) \in \rho$. Then for zz and ss we have

$$zz = \{x \in H_\rho : (z, x) \in \rho \text{ and } (x, z) \in \rho\},$$

thus $s \in zz$ and,

$$ss = \{x \in H_\rho : (s, x) \in \rho \text{ and } (x, s) \in \rho\}$$

thus $z \in ss$. Now $z \in (zz)s$ since $ss \subseteq (zz)s$. But $z \notin z(zs)$, because:

$$z(zs){=}z\{x \in H_\rho : (z,x) \in \rho \text{ and } (x,s) \in \rho\}{=}\{y \in H_\rho : (z,y) \in \rho \text{ and } (y,x) \in \rho\}$$

and $(z,z) \notin \rho$. Hence the associativity is not valid, which contradicts the assumption that H_ρ is a semihypergroup.

Corollary 2.1. *If H_ρ is a semihypergroup and ρ is not reflexive, then ρ is not symmetric.*

Lemma 2.2. *If H_ρ is a semihypergroup, then ρ is reflexive.*

Proof. Suppose that $(x,x) \notin \rho$, for some $x \in H_\rho$. Then, according to Lemma 2.1, for every element t in H_ρ such that $(x,t) \in \rho$, it derives that $(t,x) \notin \rho$. But $xx = \{y \in H_\rho : (x,y) \in \rho \text{ and } (y,x) \in \rho\}$. Therefore $xx = \emptyset$, which is absurd, since H_ρ is a semihypergroup.

Lemma 2.3. *If any pair of elements of H_ρ does not belong to ρ, then H_ρ is not a semihypergroup.*

Proof. According to Lemma 2.2, if $(x,x) \notin \rho$ for some $x \in H_\rho$, then H_ρ is not a semihypergroup. So, let t, z be two elements of H_ρ such that $t \neq z$ and $(t,z) \notin \rho$. Then:

$$t(tz) = t\{s \in H : (t,s) \in \rho \text{ and } (s,z) \in \rho\} = \{y \in H : (t,y) \in \rho \text{ and } (y,s) \in \rho\}$$

According to Lemma 2.2, it holds $(t,t) \in \rho$. Also $(t,s) \in \rho$. Therefore $t \in t(tz)$. On the other hand:

$$(tt)z = \{r \in H : (t,r) \in \text{ and } (r,t) \in \rho\}z = \{w \in H : (r,w) \in \text{ and } (w,z) \in \rho\}$$

Thus $(tt)z \subseteq \{w \in H : (w,z) \in \rho\}$, therefore $t \notin (tt)z$. Hence the associativity is not valid.

From the above series of lemmas, it derives that:

Proposition 2.2. *H_ρ is a semihypergroup if and only if $(x,y){\in}\rho$, for all $x,y{\in}H_\rho$.*

Now, if $(x,y) \in \rho$ for all $x,y \in H_\rho$, then the hypercomposition which is defined through ρ is total, i.e. $xy = H_\rho$, for all $x,y \in H_\rho$. But if a hypercompositional structure is endowed with the total hypercomposition, then it is a hypergroup. Therefore, from Propositions 2.1 and 2.2, it derives that:

Theorem 2.1. *The only semihypergroup and the only quasihypergroup defined by the binary relation ρ is the total hypergroup.*

3. Enumeration of the finite hypergroupoids

Every relation ρ in a finite set H with card$H = n$, is represented by a Boolean matrix M_ρ and conversely every $n \times n$ Boolean matrix defines in H a binary relation. Indeed, let H be the set $\{a_1, \cdots, a_n\}$. Then a $n \times n$ Boolean matrix is constructed as follows: the element (i, j) of the matrix is 1, if $(a_i, a_j) \in \rho$ and it is 0 if $(a_i, a_j) \notin \rho$ and vice versa. Hence, in every set with n elements, 2^{n^2} partial hypergroupoids can be defined.

Recall that in Boolean algebra it holds: $0 + 1 = 1 + 0 = 1 + 1 = 1$, while $0 + 0 = 0$. Also $0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$ and $1 \cdot 1 = 1$. Let H_ρ be the above mentioned partial hypergroupoid, which is defined by a binary relation ρ. Then H_ρ is a hypergroupoid if and only if $M_\rho^2 = T$, where $T = (t_{ij})$ with $t_{ij} = 1$ for all i, j [2]. The matrix M_ρ is called good, if H_ρ is a hypergroupoid. Since the element c_{ij} of M_ρ^2 is equal to $\sum_{s=1}^{n} x_{is}y_{sj}$, it derives that matrices having a column or a row consisting only of 0 elements are not good.

Now, from Proposition 2.1 it derives:

Proposition 3.1. *H_ρ is quasihypergroup if and only if $M_\rho = T$.*

Also, Proposition 2.2 gives:

Proposition 3.2. *H_ρ is semihypergroup if and only if $M_\rho = T$.*

Hence, the theorem holds:

Theorem 3.1. *The only relation ρ that gives a semihypergroup or a quasihypergroup is the one which has $M_\rho = T$, and so H_ρ is the total hypergroup.*

Spartalis and Mamaloukas [11] wrote, in Visual Basic code, a 190-lines long program that enumerates the hypergroupoids associated with binary relations of orders 2, 3 and 4. Though, the following few lines of a Mathematica [13] program produces these results through a considerably shorter process. It simply collects in variable `c` all the Boolean matrices of size n and computes their squares. Boolean minimum entry of these squares is recorded in table `z`. In return, we count the nonzero elements of `z`.

```
Good[n_] :=
Module[{c, i1, z},
        c = Tuples[Tuples[{0, 1}, n], n];
        z = Table[Min[Flatten[c[[i1]].c[[i1]]]]
                  ,{i1, 1, 2^(n*n)}];
        Return[Count[z, _?Positive]]
      ]
```

Which gives:

```
In[1]:= Good[2]
Out[1]= 3
In[2]:= Good[3]
Out[2]= 73
In[3]:= Good[4]
Out[3]= 6003
In[4]:= Good[5]
Out[4]= 2318521
```

Thus, it is confirmed that there exist $3, 73$ and 6003 binary relations that form a hypergroupoid of orders $2, 3$ and 4 respectively. It took the above program only a few minutes to count 2318521 hypergroupoids of order 5. For $n = 6$, the function `Good` fails due to memory restrictions of a small computer. One can proceed with a more slow but reliable package and form one by one the various Boolean matrices and their squares.

Remark. Notice that the above enumeration coincides with the enumeration of square roots of the total Boolean matrix, i.e. the Boolean matrix with all entries equal 1.

3.1. Isomorphisms

Naturally, the question arises: When two hypergroupoids, are isomorphic?

Proposition 3.3. *If in the Boolean matrix M_ρ the i and j rows are interchanged and, at the same time, the corresponding i and j columns are interchanged as well, then the deriving new matrix and the initial one, give isomorphic hypergroupoids.*

Proof. Suppose that $H = \{a_1, ..., a_n\}$ is a finite set and let $(H_{\rho_1}, \bullet_{\rho_1})$ be the hypergroupoid defined by a binary relation ρ_1. Let M_{ρ_1} be the Boolean matrix defined by ρ_1. Now suppose that the i and j rows and columns are interchanged and let M_{ρ_2} be the new Boolean matrix. Then a new binary relation ρ_2 is defined on H. Obviously for ρ_1 and ρ_2 it holds:

$$\begin{array}{lll}
(a_k, a_i) \in \rho_1 \Longleftrightarrow (a_k, a_j) \in \rho_2 & (a_i, a_k) \in \rho_1 \Longleftrightarrow (a_j, a_k) \in \rho_2 & \text{if } k \neq i, j \\
(a_i, a_j) \in \rho_1 \Longleftrightarrow (a_j, a_i) \in \rho_2 & (a_j, a_i) \in \rho_1 \Longleftrightarrow (a_i, a_j) \in \rho_2 & \\
(a_i, a_i) \in \rho_1 \Longleftrightarrow (a_j, a_j) \in \rho_2 & (a_j, a_j) \in \rho_1 \Longleftrightarrow (a_i, a_i) \in \rho_2 &
\end{array}$$

If $(H_{\rho_2}, \bullet_{\rho_2})$ is the hypercompositional structure defined by M_{ρ_2}, then the mapping $\phi : H_{\rho_1} \longrightarrow H_{\rho_2}$ with:

$$\phi(x) = \begin{cases} x & \text{if } x \neq a_i, a_j \\ a_i & \text{if } x = a_j \\ a_j & \text{if } x = a_i \end{cases}$$

is an isomorphism. Obviously ϕ is $1 - 1$ and onto. Next we distinguish the cases:

1. $\phi(a_i \bullet_{\rho_1} a_j) = \phi\{x \in H : (a_i, x) \in \rho_1 \text{ and } (x, a_j) \in \rho_1\}$
   (a) If $a_i \bullet_{\rho_1} a_j \cap \{a_i, a_j\} = \emptyset$. Then
   $\phi\{x \in H : (a_i, x) \in \rho_1 \text{ and } (x, a_j) \in \rho_1\}$
   $= \{\phi(x) \in H : (a_i, x) \in \rho_1 \text{ and } (x, a_j) \in \rho_1\}$
   $= \{x \in H : (a_i, x) \in \rho_1 \text{ and } (x, a_j) \in \rho_1\}$
   $= \{x \in H : (a_j, x) \in \rho_2 \text{ and } (x, a_i) \in \rho_2\} = a_j \bullet_{\rho_2} a_i = \phi(a_i) \bullet_{\rho_2} \phi(a_j)$
   (b) If $a_i \bullet_{\rho_1} a_j \cap \{a_i, a_j\} \neq \emptyset$. Assume e.g. that a_i belongs to $a_i \bullet_{\rho_1} a_j$, then
   $\phi\{x \in H : (a_i, x) \in \rho_1 \text{ and } (x, a_j) \in \rho_1\}$
   $= \{\phi(x) \in H : (a_i, x) \in \rho_1 \text{ and } (x, a_j) \in \rho_1\}$
   $= \{x \in H, x \neq a_i : (a_i, x) \in \rho_1 \text{ and } (x, a_j) \in \rho_1\} \bigcup \{a_j\}$
   $= \{x \in H : (a_j, x) \in \rho_2 \text{ and } (x, a_i) \in \rho_2\} = a_j \bullet_{\rho_2} a_i = \phi(a_i) \bullet_{\rho_2} \phi(a_j)$

Similar is the proof of the rest cases, i.e. $a_j$to be in $a_i \bullet_{\rho_1} a_j$ or both a_i, a_j to be in $a_i \bullet_{\rho_1} a_j$. Also, since the principle of duality is valid [4], the dual statement holds, i.e. $\phi(a_j \bullet_{\rho_1} a_i)$=$\phi(a_j) \bullet_{\rho_2} \phi(a_i)$.

2. If $a_k, a_\lambda \notin \{a_i, a_j\}$ then
$\phi(a_k) \bullet_{\rho_1} \phi(a_\lambda) = \phi\{x \in H : (a_k, x) \in \rho_1 \text{ and } (x, a_\lambda) \in \rho_1\} =$
   (a) if neither a_i nor a_j belongs to $a_k \bullet_{\rho_1} a_\lambda$ then
   $\phi\{x \in H : (a_k, x) \in \rho_1 \text{ and } (x, a_\lambda) \in \rho_1\}$
   $= \{\phi(x) \in H : (a_k, x) \in \rho_1 \text{ and } (x, a_\lambda) \in \rho_1\}$
   $= \{x \in H : (a_k, x) \in \rho_1 \text{ and } (x, a_\lambda) \in \rho_1\}$
   $= \{x \in H : (a_k, x) \in \rho_2 \text{ and } (x, a_\lambda) \in \rho_2\} = a_k \bullet_{\rho_2} a_\lambda$
   $= \phi(a_k) \bullet_{\rho_2} \phi(a_\lambda)$
   (b) if $a_k \bullet_{\rho_1} a_\lambda \cap \{a_i, a_j\} \neq \emptyset$. Assume e.g. that both a_i, a_j belong to $a_k \bullet_{\rho_1} a_\lambda$ then
   $\phi\{x \in H : (a_k, x) \in \rho_1 \text{ and } (x, a_\lambda) \in \rho_1\}$
   $= \{\phi(x) \in H : (a_k, x) \in \rho_1 \text{ and } (x, a_\lambda) \in \rho_1\}$
   and since $\phi(a_i) = a_j$, $\phi(a_j) = a_i$ this is equal to
   $\{x \in H : (a_k, x) \in \rho_1 \text{ and } (x, a_\lambda) \in \rho_1\}$
   or $\{x \in H : (a_k, x) \in \rho_2 \text{ and } (x, a_\lambda) \in \rho_2\}$ which is
   $a_k \bullet_{\rho_2} a_\lambda$ or $\phi(a_k) \bullet_{\rho_2} \phi(a_\lambda)$

Similar is the proof for the cases $\phi(a_k) \bullet_{\rho_1} \phi(a_i)$, $\phi(a_k) \bullet_{\rho_2} \phi(a_j)$ and their duals.

From the above proposition, the following theorem derives.

Theorem 3.3. *If the Boolean matrix M_σ derives from M_ρ by interchanging rows and the corresponding columns, then the hypergroupoids H_σ and H_ρ are isomorphic.*

The isomorphic classes of these hypergroupoids are not computed in [11]. These can be counted with a proper modification of the function `Good[]`, which will then return all the binary matrices that form a hypergroupoid. Thus, the above function changes in one of its lines and can be found in the appendix as a module of the package.

Check, for example, the three binary relations with matrices of size 2

```
In[4]:= h2 = Good1[2]
Out[4]= {{{0, 1}, {1, 1}}, {{1, 1}, {1, 0}}, {{1, 1}, {1, 1}}}
```

We are able now to give a function that forms all $n!$ isomorphisms of a given binary relation.

```
IsomorphTest1[a_List] :=
Module[{p, a1},
        p = Permutations[Range[1, Length[a]]];
        Return[Table[a1 = a;
                a1 = ReplaceAll[a1, a1[[All, Table[j2,
```

```
                                   {j2, 1, Length[a1]}]]] ->
                                      a1[[All, p[[j1]]]]];
                      ReplaceAll[a1, a1[[Table[j2,
                                   {j2, 1, Length[a]}]]] ->
                                      a1[[p[[j1]]]]],
                  {j1, 1, Length[p]}]
              ]]
```

Let us see the six permutations of the matrix

$$M_\rho = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

which are defined by corresponding binary relations, that give isomorphic hypergroupoids:

```
In[5]:= IsomorphTest1[{{1, 0, 1}, {1, 1, 0}, {0, 1, 1}}]
Out[5]:= {{{1,0,1}, {1,1,0}, {0,1,1}}, {{1,1,0}, {0,1,1}, {1,0,1}},
          {{1,1,0}, {0,1,1}, {1,0,1}}, {{1,0,1}, {1,1,0}, {0,1,1}},
          {{1,0,1}, {1,1,0}, {0,1,1}}, {{1,1,0}, {0,1,1}, {1,0,1}}}
```

In order to count the number of the different nonisomorphic classes of hypergroupoids of order n, a n–digit array, called `cardinalities`, is used by the program. Each time the routine encounters an isomorphic class, it drops it from variable `h2`.

```
Cardin[d_] :=
Module[{h2, cardinalities, len, temp1, temp},
  h2 = Good1[d];
  cardinalities = Table[0, {j1, 1, Factorial[d]}];
  While[Length[h2] > 0,
        temp = Union[IsomorphTest1[h2[[1]]]];
        len = Length[Union[temp]];
        cardinalities[[len]] = cardinalities[[len]] + 1;
        h2 = Complement[h2, temp]
        ];
  Return[cardinalities]]
```

Then we get

```
In[6]:= Cardin[2]
Out[6]:= {1, 1}
In[7]:= Total[%]
Out[7]:= 2
In[8]:= Cardin[3]
Out[8]:= {2, 1, 5, 0, 0, 9}
```

```
In[9]:= Total[%]
Out[9]:= 17
In[10]:= Cardin[4]
Out[10]:= {2, 0, 1, 5, 0, 7, 0, 4, 0, 0, 0, 78,
          0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 207}
In[11]:= Total[%]
Out[11]:= 304
In[12]:= Cardin[5]
Out[12]= {2, 0, 0, 0, 5, 0, 0, 0, 0, 13, 0, 1, 0, 0, 8,
          0, 0, 0, 0, 78, 0, 0, 0, 3, 0, 0, 0, 0, 0, 152,
          0, 0, 0, 0, 0, 0, 0, 0, 0, 42, 0, 0, 0, 0, 0, 0,
          0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2206, 0,
          0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
          0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
          0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
          0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 18150}
In[13]:= Total[%]
Out[13]= 20660
```

So, there are $2, 17, 304$ and 20660 isomorphic classes (I.C.) of orders $2, 3, 4$ and 5 respectively. For example, for order 4 there are 2 I.C. of cardinality 1, 1 I.C. of cardinality 3, 5 I.C. of cardinality 4, 7 I.C. of cardinality 6, 4 I.C. of cardinality 8, 78 I.C. of cardinality 12 and 207 I.C. of cardinality 24. These $2 + 1 + 5 + 7 + 4 + 78 + 207 = 304$ I.C. form the

$$2 \cdot 1 + 1 \cdot 3 + 5 \cdot 4 + 7 \cdot 6 + 4 \cdot 8 + 78 \cdot 12 + 207 \cdot 24 = 6003$$

non–isomorphic hypergroupoids of order 4.

We also mention that there are $2^{n\times n}$ binary matrices of size n. We may count the non–isomorphic ones by simply changing the line

```
h2 = Good[n];
```

in the routine `Cardin[]` by the line

```
h2=Tuples[Tuples[0,1,n],n].
```

Then we get

```
In[14]:= Cardin[1]
Out[14]= {2}
In[15]:= Cardin[2]
Out[15]= {4, 6}
In[16]:= Total[%]
Out[16]= 10
In[17]:= Cardin[3]
Out[17]= {4, 2, 28, 0, 0, 70}
```

```
In[18]:= Total[%]
Out[18]= 104
In[19]:= Cardin[4]
Out[19]= {4, 0, 4, 28, 0, 32, 0, 16, 0, 0, 0, 496,
          0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2464}
In[20]:= Total[%]
Out[20]= 3044
```

The integer sequence $2, 10, 104, 3044$ etc. coincides with the integer sequence A000595, appeared in [10] and represents the number of non–isomorphic unlabeled binary relations on n nodes.

3.2. Weak Associativity

As proved in Section 2 above, the total hypergroup is the only hypergroupoid that fulfills the property of associativity. Thus, we checked a weaker property, which is called Weak Associativity:

$$a(bc)\bigcap(ab)c \neq \emptyset \text{ for all } a, b, c \in H. \tag{3.1}$$

Having, up to this point, constructed all the hypergroupoids of order $2, 3, 4$ and 5, we check the validity of this property to all of them and we count the ones that verify it. The package is given and explained in the appendix.

Its results are:

```
In[21]:= BinaryTest[2]
Out[21]= 3
In[22]:= BinaryTest[3]
Out[22]= 43
In[23]:= BinaryTest[4]
Out[23]= 2619
In[24]:= BinaryTest[5]
Out[24]= 602431
```

The counting of the hypergoupoids of orders $n \geq 6$ is time consuming, so we discontinued at $n = 5$.

4. Conclusions

This paper shows that the total hypergroup is the only hypergroup which can be produced by hypercomposition (1.1). Since it is a hypergoup it is also a semihypergroup and a quasihypergroup. No other semi- or quasi- hypergroups can be produced by (1.1). On the other hand there exist lots of hypergoupoids that can be produced by (1), the number of which is calculated with the use of Mathematica packages that are constructed for this purpose and consist part of the contents of this paper. The results of these calculations are given in the cumulative Table 1 below for the orders $2, 3, 4$ and 5:

Table 1: Cumulative results

order →	2	3	4	5
Boolean matrices (BM)	16	512	65536	33554432
BM forming Hypergroupoids	3	73	6003	2318521
BM forming Weak-Associative Hypergroupoids	3	43	2619	602431
Nonisomorphic BM	10	104	3044	291968
Nonisomorphic BM forming Hypergroupoids	2	17	304	20660

5. The Mathematica package

The Mathematica package referred to in Section 3, is given bellow.

```
BeginPackage["BinaryTest‘"];
Clear["BinaryTest‘*"];

BinaryTest::usage = "BinaryTest[n] counts the binary
relations of dimension n that form a hypergroupoid.
It also counts the Weak-associative binary hypergroupoids"

Begin["‘Private‘"];
Clear["BinaryTest‘Private‘*"];

BinaryTest[n_] :=
Module[{c, ch},
   c = Good1[n];
   ch = Table[AssociativityWeakTest[HyperGroupoid[c[[j1]]]],
             {j1, 1, Length[c]}];
   Return[Count[ch, True]]];

Good1[n_] :=
Module[{c, i1, z},
        c = Tuples[Tuples[{0, 1}, n], n];
        z = Table[Min[Flatten[c[[i1]].c[[i1]]]]
                  ,{i1, 1, 2^(n*n)}];
        Return[Select[Transpose[{c, z}], #[[2]] > 0 &][[All, 1]]]
      ]

AssociativityWeakTest[a_List] :=
Module[{i, j, k, test},
  i = 1; j = 1; k = 1; test = True;
  While[test && i <= Length[a],
    test = Intersection[
```

```
    Union[Flatten[Union[Extract[a,
                  Distribute[{a[[i, j]], {k}}, List]]]]],
    Union[Flatten[Union[Extract[a,
                  Distribute[{{i}, a[[j, k]]}, List]]]]]
                       ] != {};
    k = k + 1;
    If[k > Length[a], k = 1; j = j + 1;
                      If[j > Length[a], i = i + 1; j = 1];
      ];
        ];
   Return[test]
       ];

HyperGroupoid[a_List] :=
Table[Table[Intersection[
            Floor[(a[[j1, 1 ;; Length[a]]]
                 + a[[1 ;; Length[a], j2]])/2
                 ]*
            Table[j3, {j3, 1, Length[a]}],
            Table[j3, {j3, 1, Length[a]}]
                        ],
      {j1, 1, Length[a]}],
{j2, 1, Length[a]}];

End[];
EndPackage[];
```

The package consists of four functions. The three internal ones are:

`Good1:` that returns all the hypergroupoids associated to a binary relation of order n.

`HyperGroupoid:` that constructs the hypergroupoid associated to a given Boolean matrix of a binary relation.

`AssociativityWeakTest:` that tests property (3.1) by forming all n^3 possible products of all triplets of elements of H.

and the main one, which is:

`BinaryTest:` After calling `Good`, it constructs the table of the deriving hypergroupoids, using the function `HyperGroupoid`. Finally, it counts the number of those, which satisfy the property of the weak associativity.

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DEGENERACY OF SOME CLUSTER SETS

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Abstract. Using θ-closure [39], δ-closure [39] of a subset and β-open [1] sets, we initiate two new kinds of cluster sets for functions and multifunctions. An explicit expression of each kind of cluster sets are given in terms of filters and grills [38] and also several of their properties are investigated. In the process, the degeneracy of such cluster sets are used as tools to obtain new characterizations of various separation axioms. As application, such investigations ultimately provide new techniques for studying the covering property β-closedness [7].subjclass[2000]Primary 54D20, 54D30, 54C60, 54C99.

Keywords: β-open, β-closure, θ-closure, δ-closure, β-closed space, β-θ-cluster set, β-δ-cluster set, weakly Hausdorff, Urysohn.

1. Introduction

Cluster sets for functions and multifunctions in general topology have been investigated recently by good many researchers although such notion was developed long ago and studied extensively within the framework of real and analytic function theory (see [42]). In this regard, the classical book of Collingwood and Lohwater [12] and the research papers of Weston [40] and Hunter [20] are worth to be mentioned where a comprehensive collections of works are found. Subsequently, Hamlett [18], [19], Joseph [22] and many others (see [26], [27]) have extended this notion for investigation of various characterizations of different covering properties like compactness, Lindeloffness, H-closedness [39], minimal Hausdorffness, P-closedness [3], S-closedness [37].

Since the introduction of β-open sets [1] (= semipreopen [4]), many topological concepts are studied in terms of β-open sets. Only very few we mention which are found in the papers [1, 2, 4, 5, 6, 7, 8, 11, 16, 17, 21, 30, 31, 32, 33]. In this paper, using β-open sets and θ-closure [39] (resp. δ-closure [39]) due to Veličko [39], we introduce two new kinds of cluster sets called β-θ-cluster (resp. β-δ-cluster) sets which provide new techniques for the study of the covering property β-closedness [7]. (A space X is called β-closed if for every covering of X by β-open sets has finite subfamily whose β-closures cover X.) In the course of study, the degeneracy of such cluster sets are used as tools to obtain new characterizations of various separation axioms like Hausdorffness, weakly Hausdorffness, Urysohnness and other relevant properties.

Throughout the paper, X and Y denote topological spaces without any separation axioms, and $\psi : X \rightarrow Y$ denotes a single valued function from X into Y. By a multifunction $F : X \rightarrow Y$, we mean a function mapping points of X into the nonempty subsets of Y. For a subset S of X, clS and $intS$ represent the closure of S and interior of S in X respectively. We recall the following well known definitions: A subset S of a space (X, τ) or X is said to be α-open [28] (resp. semi-open [24], pre-open [25], β-open [1] or semi-preopen [4]) if $S \subset intclintS$ (resp. $S \subset clintS$, $S \subset intclS$ or $S \subset clintclS$). We denote the classes of all α-open (resp. semi-open, pre-open, β-open or semi-preopen) sets in a space X by τ_α (resp. $SO(X), PO(X), \beta O(X) = SPO(X)$). It is well known that $\tau \subset \tau_\alpha = PO(X) \cap SO(X) \subset PO(X) \cup SO(X) \subset \beta O(X)$. The family of all open (resp. β-open) sets containing a subset S is denoted by $\tau(S)$ (resp. $\beta O(X, S)$). If in particular $S = \{x\}$ then they are respectively denoted by $\tau(x)$ and $\beta O(X, x)$. The complement of a β-open set is called a β-closed set. Pre-closed and semi-closed sets are defined similarly. The β-closure=sp-closure [4] (resp. pre-closure, semi-closure) of S denoted by βclS (resp. $pclS$, $sclS$) is the intersection of all β-closed (resp. pre-closed and semi-closed) subsets of X containing S. A space X is quasi-H-closed [10] (resp. S-closed [37], P-closed [3]) if every open (resp. semi-open, pre-open) cover of X has finite subfamily whose closures (resp. closures, pre-closures) cover X. For a subset S of X, the θ-closure [39] (resp. δ-closure [39]) of S, denoted by θ-clS (resp. δ-clS) is the set $\{x \in X : clU \cap S \neq \emptyset$ for each $U \in \tau(x)\}$ (resp. $\{x \in X : intclU \cap S \neq \emptyset)$ for each $U \in \tau(x)\}$). S is called

θ-closed (resp. δ-closed) if $S = \theta\text{-}clS$ (resp. $S = \delta\text{-}clS$). A space X is called weakly Hausdorff [36], if each point x of X is the intersection of all regular closed sets containing x.

A filter base $\mathcal{F}$ on a space X is said to be β-θ-adhere [7] at x if $F \cap \beta clU \neq \emptyset$, for each $F \in \mathcal{F}$ and $U \in \beta O(X, x)$. Thron [38] has defined a non-empty family $\mathcal{G}$ of non-empty subsets of X to be a grill if (i) $A \in \mathcal{G}$ and $A \subseteq B \Rightarrow B \in \mathcal{G}$ and (ii) $A \cup B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$. A grill $\mathcal{G}$ on X is said to β-θ-converges to a point x of (X, τ), if for each $U \in \beta O(X, x)$ there is a $G \in \mathcal{G}$ with $G \subseteq \beta clU$. A subset A of a space X is said to be NC-set [35] if for every cover of A by means of regular open or δ-open sets of X has a finite subcover.

2. Prerequisites

The following definitions and results will be frequently used in the subsequent sections.

Definition 2.1 A subset S of a space (X, τ) is said to be β-regular (= semipre-regular [30]) if it is both β-open as well as β-closed.

The family of all β-regular sets of a space X and that containing a point x of X are respectively denoted by $\beta R(X)$ and $\beta R(X, x)$.

Lemma 2.2 [30] *For a subset S of a space X, $S \in \beta O(X)$ if and only if $\beta clS \in \beta R(X)$.*

Definition 2.3 A point $x \in X$ is said to be in the β-θ-closure (= sp-θ-closure [30]) of S, denoted by $\beta\text{-}\theta\text{-}cl(S)$, if $S \cap \beta clV \neq \emptyset$ for every $V \in \beta O(X, x)$. If $\beta\text{-}\theta\text{-}clS = S$, then S is said to be β-θ-closed (=sp-θ-closed [30]). The complement of a β-θ-closed set is said to be β-θ-open (=sp-θ-open [30]).

Lemma 2.4 [30] *For a subset A of a space X, $\beta\text{-}\theta\text{-}cl(A) = \cap\{R : A \subset R$ and $R \in \beta R(X)\}$.*

Lemma 2.5 [30] *Let A and B be any subsets of a space X. Then the following properties hold:*

(i) *$x \in \beta\text{-}\theta\text{-}cl(A)$ if and only if $A \cap V \neq \emptyset$ for each $V \in \beta R(X, x)$,*

(ii) *if $A \subset B$ then $\beta\text{-}\theta\text{-}clA \subset \beta\text{-}\theta\text{-}clB$,*

(iii) *$\beta\text{-}\theta\text{-}cl(\beta\text{-}\theta\text{-}clA) = \beta\text{-}\theta\text{-}clA$,*

(iv) *intersection of an arbitrary family of β-θ-closed sets in X is β-θ-closed in X,*

(v) *A is β-θ-open if and only if for each $x \in A$, there exists $V \in \beta R(X, x)$ such that $x \in V \subset A$,*

(vi) *If* $A \in \beta O(X)$ *then* $\beta clA = \beta\text{-}\theta\text{-}clA$,

(vii) *If* $A \in \beta R(X)$ *then* A *is* β*-*θ*-closed.*

Remark 2.6 [30] Noiri has shown that β-regular $\Rightarrow$ β-θ-open $\Rightarrow$ β-open. But the converses are not necessarily true.

A space X is called β-closed [7] if every β-open cover has a finite subfamily whose β-closures cover X. Equivalently, X is β-closed [7] if every β-regular (resp. β-θ-open) cover has a finite subcover.

3. β-θ-cluster sets and β-δ-cluster sets

Definition 3.1 Let $\psi : (X, \tau) \rightarrow (Y, \tau')$ be a function. Then β-θ-cluster (resp. β-δ-cluster) set of ψ at $x \in X$, denoted by $\beta_\theta^\tau(\psi; x)$ (resp. $\beta_\delta^\tau(\psi; x)$), is defined to be the set $\bigcap\{\theta\text{-}cl\psi(\beta clU) : U \in \beta O(X, x)\}$ (resp. $\bigcap\{\delta\text{-}cl\psi(\beta clU) : U \in \beta O(X, x)\}$).

When no topology is mentioned on X then we write $\beta_\theta(\psi; x)$ (resp. $\beta_\delta(\psi; x)$) instead of $\beta_\theta^\tau(\psi; x)$ (resp. $\beta_\delta^\tau(\psi; x)$).

Theorem 3.2 *The following are equivalent for a function* $\psi : X \rightarrow Y$*:*

(a) $y \in \beta_\delta(\psi; x)$,

(b) *there is a grill* $\bigwedge$ *on* X *such that* $\bigwedge$ β*-*θ*-converges to* x *and* $y \in \cap\{\delta\text{-}cl\psi(A) : A \in \bigwedge\}$,

(c) *the filter base* $\{\psi^{-1}(intclV) : V \in \tau(y)\}$ β*-*θ*-adheres at* x.

Proof. (a)$\Rightarrow$(c). Let $y \in \beta_\delta(\psi; x)$. Then $intclV \cap \psi(\beta clU) \neq \emptyset$ i.e. $\beta clU \cap \psi^{-1}(intclV) \neq \emptyset$ for each $V \in \tau(y)$ and for each $U \in \beta O(X, x)$. Therefore the filter base $\{\psi^{-1}(intclV) : V \in \tau(y)\}$ β-θ-adheres at x.
(c)$\Rightarrow$(b). Let $\bigwedge = \{A \subset X : A \cap F \neq \emptyset, \forall F \in \mathcal{F} = \{\psi^{-1}(intclW) : W \in \tau(y)\}\}$. We shall first show that $\bigwedge$ is a grill on X. It is clear that $\bigwedge \neq \emptyset$ with $\emptyset \notin \bigwedge$ and $A_2 \in \bigwedge$ whenever $A_1 \in \bigwedge$ with $A_1 \subset A_2$. If $A_1 \cup A_2 \in \bigwedge$ i.e. if $(A_1 \cup A_2) \cap F \neq \emptyset$, for all $F \in \mathcal{F}$ then either $A_1 \in \bigwedge$ or $A_2 \in \bigwedge$. Suppose $A_1 \notin \bigwedge$ and $A_2 \notin \bigwedge$. Then there exist an $F_1 \in \mathcal{F}$ such that $A_1 \cap F_1 = \phi$ and an $F_2 \in \mathcal{F}$ such that $A_2 \cap F_2 = \phi$. Since $\mathcal{F}$ is a filter base on X, there exists an $F_3 \in \mathcal{F}$ such that $F_3 \subset F_1 \cap F_2$ and hence $(A_1 \cup A_2) \cap F_3 = \emptyset$ — a contradiction. So, $\bigwedge$ is a grill on X. Let $W \in \tau(y)$ and $V \in \beta O(X, x)$. Since $\mathcal{F}$ β-θ-adheres at x (by hypothesis (c)), $\beta clV \cap \psi^{-1}(intclW) \neq \emptyset$. Hence $\beta clV \cap F \neq \emptyset$ for each $F \in \mathcal{F}$. The definition of $\bigwedge$ shows that $\beta clV \in \bigwedge$ for all $V \in \beta O(X, x)$. Hence the grill $\bigwedge$ β-θ-converges to x. From the definition of $\bigwedge$, it is clear that $intclW \cap \psi(A) \neq \emptyset$ for all $A \in \bigwedge$. So $y \in \cap\{\delta\text{-}cl\psi(A) : A \in \bigwedge\}$.
(b)$\Rightarrow$(a). Let $\bigwedge$ be a grill on X that β-θ-converges to x and for that grill $y \in \cap\{\delta\text{-}cl\psi(A) : A \in \bigwedge\}$. Hence $\beta clU \in \bigwedge$, for each $U \in \beta O(X, x)$ (as $\bigwedge$ β-θ-converges to x) and thus $y \in \cap\{\delta\text{-}cl\psi(\beta clU) : U \in \beta O(X, x)\} = \beta_\delta(\psi; x)$. ■

Theorem 3.3 *The following are equivalent for a function* $\psi : X \rightarrow Y$*:*

(a) $y \in \beta_\theta(\psi; x)$,

(b) *there is a grill* $\bigwedge$ *on* X *such that* $\bigwedge$ β-θ*-converges to* x *and* $y \in \bigcap\{\theta\text{-}cl\psi(A) : A \in \bigwedge\}$,

(c) *the filter base* $\{\psi^{-1}(clV) : V \in \tau(y)\}$ β-θ*-adheres at* x.

Proof. The proof is similar to the proof of Theorem 3.2. ■

It is clear from the definitions that for a function $\psi : X \rightarrow Y$, $\beta_\delta(\psi; x) \subseteq \beta_\theta(\psi; x)$, for each point $x \in X$. However, we give an example of a function where the later inclusion is proper.

Example 3.4 Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Clearly, $\beta R(X, \tau) = \{\phi, X, \{a\}, \{b\}, \{b, c\}, \{a, c\}\}$. Consider the identity function $\psi : (X, \tau) \rightarrow (X, \tau)$. Then $\beta_\theta(\psi; c) = X$ whereas $\beta_\delta(\psi; c) = \{c\}$. So, $\beta_\delta(\psi; c) \underset{\neq}{\subset} \beta_\theta(\psi; c)$.

The following theorem shows when such cluster sets are equal.

Theorem 3.5 *Let* $\psi : X \rightarrow Y$ *be a function. Then* $\beta_\theta(\psi; x) = \beta_\delta(\psi; x)$*, when* Y *is regular.*

Proof. Since for a regular space Y, $\delta\text{-}cl(A) = \theta\text{-}cl(A)$ for each $A \subset Y$, the proof therefore quite obvious. ■

Definition 3.6 A function $\psi : X \rightarrow Y$ is strongly β-irresolute [30] if for each $x \in X$ and each $V \in \beta O(Y, \psi(x))$, there exists an $U \in \beta O(X, x)$ such that $\psi(\beta clU) \subset V$.

Theorem 3.7 *A space* Y *is weakly Hausdorff if and only if for any space* X*, and any strongly* β*-irresolute surjective function* $\psi : X \rightarrow Y$*,* $\beta_\delta(\psi; x)$ *is degenerate for each* $x \in X$.

Proof. *Sufficiency part.* Let for a space X, $\psi : X \rightarrow Y$ be a strongly β-irresolute surjective function. So for points y, z in Y with $y \neq z$ $\exists x$ and $x_0 \in X$ such that $\psi(x) = y$ and $\psi(x_0) = z$. Because of degeneracy of $\beta_\delta(\psi; x)$ and $\beta_\delta(\psi; x_0)$, we have $\beta_\delta(\psi; x) = \{\psi(x)\} = \{y\}$ and $\beta_\delta(\psi; x_0) = \{\psi(x_0)\} = \{z\}$. So there exist $V \in \tau(y)$ and $W \in \beta O(X, x_0)$ such that $\psi(\beta clW) \cap intclV = \emptyset$ and hence $\psi(\beta clW) \subset Y - intclV = U$ (say), which is of course a regular closed set of Y. Clearly $z = \psi(x_0) \in U$ but $y \notin U$. Therefore Y is weakly Hausdorff.

Necessity part. Let for any space X, $\psi : X \rightarrow Y$ be a surjective strongly β-irresolute function. Hence for each $x \in X$ and each $V \in \beta O(Y, \psi(x))$, there exists an $W \in \beta O(X, x)$ such that $\psi(\beta clW) \subset V$. From the definition of β-δ-cluster set of ψ at x, it is clear that $\beta_\delta(\psi; x) = \bigcap\{\delta\text{-}cl\psi(\beta clU) : U \in \beta O(X, x)\} \subseteq \bigcap\{\delta\text{-}clV) : V \in \beta O(Y, \psi(x))\}$. Since Y is weakly Hausdorff for each $y \neq \psi(x)$, there exists a regular closed set U containing $\psi(x)$ such that $y \notin U$. As ψ is strongly β-irresolute and every regular closed set is β-open, there exists a $W \in \beta O(X,x)$ such that $\psi(\beta clW) \subset U$. Since $Y - U = intcl(Y - U) \cap U = \emptyset$, $intcl(Y - U) \cap \psi(\beta clW) = \emptyset$. So, $y \notin \delta\text{-}cl\psi(\beta clW)$ and hence $y \notin \beta_\delta(\psi;x)$. Thus $\beta_\delta(\psi;x)=\{\psi(x)\}$ for each $x \in X$. ■

A function $\psi : X \to Y$ is called strongly θ-β-continuous [30] if each point $x \in X$ and each open set U containing $\psi(x)$, there exists a $V \in \beta O(X,x)$ such that $\psi(\beta clV) \subset U$.

Theorem 3.8 *A space Y is Hausdorff if and only if for any space X and for any strongly θ-β-continuous surjective function $\psi : X \to Y$, $\beta_\theta(\psi;x)$ is degenerate for each $x \in X$.*

Proof. Let Y be a Hausdorff space and for any space X, $\psi : X \to Y$ is strongly θ-β-continuous surjection. Let $y \in Y$ with $y \neq \psi(x)$. Since $\psi : X \to Y$ is strongly θ-β-continuous, for each open set U containing $\psi(x)$, there is a $V \in \beta O(X,x)$ such that $\psi(\beta clV) \subset U$. Now $\beta_\theta(\psi;x)=\cap\{\theta\text{-}cl\psi(\beta clV) : V \in \beta O(X,x)\} \subseteq \cap\{\theta\text{-}clU : U \in \tau(\psi(x))\} = \cap\{clU : U \in \tau(\psi(x))\}$. Since Y is Hausdorff, there exist disjoint open sets W_1 , W_2 with $y \in W_1$ and $\psi(x) \in W_2$. Hence $W_1 \cap clW_2=\emptyset$. As $y \notin clW_2$, then $y \notin \beta_\theta(\psi;x)$. Thus $\beta_\theta(\psi;x)=\{\psi(x)\}$.

Conversely, let for a space X and for any strongly θ-β-continuous surjective function $\psi : X \to Y$, $\beta_\theta(\psi;x)$ is degenerate for each $x \in X$. So, for any two points y, z in Y with $y \neq z$ $\exists x$ and $x_0 \in X$ such that $y = \psi(x)$ and $z = \psi(x_0) \notin \beta_\theta(\psi;x)$. Hence there exist an $U \in \tau(z)$ and a $W \in \beta O(X,x)$ such that $clU \cap \psi(\beta clW)=\phi$ i.e. $\psi(\beta clW) \subset Y - clU$. Since $Y - clU \in \tau(y)$ and $U \in \tau(z)$ then Y is Hausdorff.■

Theorem 3.9 *Let Y be an Urysohn space. Then for some space X and for some function $\psi : X \to Y$, $\beta_\theta(\psi;x)$ is degenerate for each $x \in X$.*

Proof. Suppose Y is Urysohn. Suppose also that for each space X and every function $\psi : X \to Y$, $\beta_\theta(\psi;x)$ is not degenerate at some point $x \in X$. So for the identity function $f : Y \to Y$, there exist point, say x of Y and a point $y \in Y$ with $y \neq x$ $(= f(x)) \in \beta_\theta(f;x)$. Then for each $W \in \beta O(X,x)$ and each $V \in \tau(y)$, $\beta clW \cap clV \neq \phi$. Hence, in particular, $\beta clW \cap clV \neq \phi$ for each $W \in \tau(x)$ and each $V \in \tau(y)$. But for an open set W, one can cheek that $\beta clW = intclintW = intclW$. So, $intclW \cap clV \neq \emptyset$ and hence $clW \cap clV \neq \emptyset$ for each $W \in \tau(x)$ and each $V \in \tau(y)$. This contradicts that Y is Urysohn. ■

Theorem 3.10 *A regular topological space Y is Urysohn if for some space X and some surjective function $\psi : X \to Y$, $\beta_\theta(\psi;x)$ is degenerate for each $x \in X$.*

Proof. Let for some space X and for some surjection $\psi : X \to Y$, where Y is regular, $\beta_\theta(\psi;x)$ is degenerate for each $x \in X$. So, for $y_1, y_2 \in Y$ with $y_1 \neq y_2$, $\exists$ $x_1, x_2 \in X$ such that $y_1 = \psi(x_1)$ and $y_2 = \psi(x_2)$. The degeneracy of $\beta_\theta(\psi;x)$ at each $x \in X$ ensures that $y_1 = \psi(x_1) \notin \beta_\theta(\psi;x_2)$. So, there exists $V \in \tau(y_1)$ and

$W \in \beta O(X,x)$ such that $clV \cap \psi(\beta clW)=\emptyset$. Clearly $Y - clV \in \tau(y_2)$. Since Y is regular, there exists an open set $U \in \tau(y_2)$ such that $y_2 \in U \subset clU \subset Y - clV$. So we have $V \in \tau(y_1)$ and $U \in \tau(y_2)$ with $clV \cap clU = \emptyset$. Thus Y is Urysohn. ■

Corollary 3.11 *A regular topological space Y is Urysohn if and only if for some space X and some function $\psi : X \to Y$, $\beta_\theta(\psi; x)$ is degenerate for each $x \in X$.*

In the above discussion, we have observed that the degeneracy of $\beta_\delta(\psi; x)$ and $\beta_\theta(\psi; x)$ at each point x of the domain space X characterize certain separation axioms of the codomain space Y. We now investigate some other situations where separation axioms like almost regularity, Hausdorffness on the domain space ensure the degeneracy of such cluster sets of certain functions.

Definition 3.12 A function $\psi : X \to Y$ is called a θ-closed [13] (resp. δ-closed [29]) function if image of each θ-closed (resp. δ-closed) set in X is θ-closed (resp. δ-closed) in Y.

A δ-closed function $\psi : X \to Y$ is called δ-perfect [29] if each fiber is an NC-set in X.

Theorem 3.13 *Let X be a Hausdorff space. Then for any δ-perfect function $\psi : X \to Y$ (where Y is any space), $\beta_\delta(\psi; x)$ is degenerate for each $x \in X$.*

Proof. Since for any subset S of X, $\beta clS \subseteq \delta\text{-}clS$, then we have $\beta_\delta(\psi; x) = \cap\{\delta - cl\psi(\beta clW) : W \in \beta O(X,x)\} \subseteq \cap\{\delta - cl\psi(\delta - clW) : W \in \beta O(X,x)\}$. Since ψ is δ-perfect and hence is a δ-closed function, then

$$\beta_\delta(\psi; x) \subseteq \cap\{\psi(\delta - clW) : W \in \beta O(X,x)\}.$$

Again since ψ is δ-perfect, $\psi^{-1}(y)$ is an NC-set for each $y \in Y$. It is obvious that every NC-set in a Hausdorff space is a θ-closed set. Let $y \neq \psi(x)$. As $\psi^{-1}(y)$ is θ-closed, there exists an open set $V \in \tau(x)$ such that $\psi^{-1}(y) \cap clV = \emptyset$. Hence $y \notin \psi(clV) = \psi(\delta\text{-}clV)$ (as V is open). Since $\tau(x) \subset \beta O(X,x)$, then from above deduction, $y \notin \beta_\delta(\psi; x)$. Therefore $\beta_\delta(\psi; x) = \{\psi(x)\}$ for each $x \in X$. ■

Definition 3.14 A subset S of space (X,τ) is called α-paracompact [23] (resp. α-nearly paracompact [23]) if every cover of S by open (resp. regular open) sets of (X,τ) has an open X-locally finite refinement.

Theorem 3.15 *Let X is a Hausdorff space. Then for any δ-closed function $\psi : X \to Y$ (where Y is any space), if $\psi^{-1}(y)$ is α-nearly paracompact for each $y \in Y$, then $\beta_\delta(\psi; x)$ is degenerate for each $x \in X$.*

Proof. As every α-nearly paracompact set in a Hausdorff space is θ-closed [15], the proof is quite similar to the proof of the Theorem 3.13. ■

Theorem 3.16 *Let X is a Hausdorff paracompact space. Then for any δ-closed function $\psi : X \to Y$ (where Y is any space), if for each $y \in Y$ $\psi^{-1}(y)$ is any of the following:*

(a) *α-paracompact,*
(b) *α-nearly paracompact,*
(c) *δ-closed,*
(d) *θ-closed,*
(e) *closed,*
(f) *g-closed.*

Then, $\beta_\delta(\psi; x)$ is degenerate for each $x \in X$.

Proof. Dontchev and Noiri (Corollary 3.7 [15]), have proved that for a subset A of Hausdorff paracompact space the following are equivalent: (a) A is α-paracompact, (b) A is α-nearly paracompact, (c) A is δ-closed, (d) A is θ-closed, (e) A is closed, (f) A is g-closed. So the theorem follows from Theorem 3.15. ■

Definition 3.17 A space (X, τ) is called almost regular [34] if for every regular closed set S in X and for each $x \in S$, there exist disjoint open sets U and V such that $x \in U$ and $S \subset V$.

Theorem 3.18 *Let $\psi : X \to Y$ be θ-closed map from a almost regular space X into a space Y. If $\psi^{-1}(y)$ is θ-closed in X for each $y \in Y$, then $\beta_\theta(\psi; x)$ is degenerate for each $x \in X$.*

Proof. Since for each subset A of X, $\beta clA \subset \theta\text{-}clA$ and since in a almost regular space X, $\theta\text{-}clA$ is always θ-closed the proof is thus quite similar to the proof of Theorem 3.13 and is therefore omitted. ■

Theorem 3.19 *Let X be an almost regular Hausdorff space. If $\psi : X \to Y$ (where Y is any space) is an injective θ-closed function then $\beta_\theta(\psi; x)$ is degenerate for each $x \in X$.*

Proof. Since X be an almost regular then from the argument given in Theorem 3.18, we get $\beta_\theta(\psi; x) \subseteq \{\psi(\theta\text{-}clV) : V \in \beta O(X, x)\}$. Now as X is being Hausdorff, for any point x_1 in X with $x \neq x_1$, there is an open set $W \in \tau(x) \subseteq \beta O(X, x)$ such that $x_1 \notin \theta\text{-}clW$. Since ψ is injective, so $\psi(x_1) \notin \psi(\theta\text{-}clW)$. Therefore $\beta_\theta(\psi; x) = \{\psi(x)\}$. ■

For a σ-ideal $\mathcal{I}$ on X, the two topologies τ_1 and τ_2 on X are called equivalent modulo $\mathcal{I}$, denoted by $\tau_1 \equiv \tau_2(mod\mathcal{I})$ if for any subset S of X, the difference set $(\tau_1)\beta\text{-}\theta\text{-}clS - (\tau_2)\beta\text{-}\theta\text{-}clS \in \mathcal{I}$, where $(\tau_i)\beta\text{-}\theta\text{-}clS$ is the β-θ-closure of S in the space (X, τ_i) for $i = 1, 2$.

Theorem 3.20 *Let τ_1 and τ_2 be two topologies on X such that $\tau_1 \equiv \tau_2(mod\mathcal{I})$, where $\mathcal{I}$ is a σ-ideal of subsets of X and also let Y be a second countable space. Then for any function $\psi : X \to Y$, there is an $I \in \mathcal{I}$ such that $\beta_\delta^{\tau_1}(\psi; x) = \beta_\delta^{\tau_2}(\psi; x)$, for each $x \in X - I$.*

Proof. Consider the sets

$$L = \{x \in X : \beta_\delta^{\tau_1}(\psi; x) \nsubseteq \beta_\delta^{\tau_2}(\psi; x)\} \text{ and } R = \{x \in X : \beta_\delta^{\tau_2}(\psi; x) \nsubseteq \beta_\delta^{\tau_1}(\psi; x)\}.$$

Since Y is second countable it has a countable base, (say) $\mathcal{B} = \{V_n \, n \geq 1\}$. If we take $L_n = \{x \in X : x \in (\tau_1)\beta - \theta - cl\psi^{-1}(\beta clV_n)$ but $x \notin (\tau_2)\beta - \theta - cl\psi^{-1}(\beta clV_n)\}$, for each $n = 1, 2, ...$, then each $L_n \in \mathcal{I}$ and hence $\cup\{L_n : n \geq 1\} \in \mathcal{I}$ (as $\tau_1 \equiv \tau_2(mod\mathcal{I})$ and $\mathcal{I}$ is a σ-ideal). We claim that $L \subseteq \cup\{L_n : n \geq 1\}$. Indeed, let $x \in L$ then their exists an $y_0 \in \beta_\delta^{\tau_1}(\psi; x)$ but $y_0 \notin \beta_\delta^{\tau_2}(\psi; x)$. Theorem 3.2 asserts that the filter base $\bigwedge = \{\psi^{-1}(\beta clV) : V \in \tau(y_0)\}$ β-θ-adheres at x in (X, τ_1) but not in (X, τ_2). So there exists an open set $U \in \tau(y_0)$ such that $x \notin (\tau_2)\beta$-θ-$cl\psi^{-1}(\beta clU)$ and hence $x \notin (\tau_2)\beta$-θ-$cl\psi^{-1}(\beta clV_n)$ for some $V_n \in \mathcal{B}$ for which $y_0 \in V_n \subset U$. Obviously, $x \in (\tau_1)\beta$-θ-$cl\psi^{-1}(\beta clV_n)$. Hence $x \in L_n$. So $L \subseteq \cup\{L_n : n \geq 1\}$ and since $\mathcal{I}$ is a σ-ideal, $L \in \mathcal{I}$. Similarly $R \in \mathcal{I}$. Thus the set $\{x \in X : \beta_\delta^{\tau_1}(\psi; x) \neq \beta_\delta^{\tau_2}(\psi; x)\} = L \cup R \in \mathcal{I}$. Hence the proof is complete. ■

Theorem 3.21 *Let τ_1 and τ_2 be two topologies on X with $\tau_1 \equiv \tau_2(mod\mathcal{I})$, where $\mathcal{I}$ is a σ-ideal of subsets of X and also let Y be a second countable space. Then for any function $\psi : X \to Y$, there is an $I \in \mathcal{I}$ such that $\beta_\theta^{\tau_1}(\psi; x) = \beta_\theta^{\tau_2}(\psi; x)$, for each $x \in X - I$.*

Proof. The proof is similar to the above theorem and is thus omitted. ■

We now establish a theorem which asserts that under certain conditions the equivalence of $\beta_\delta(\psi; x)$, $\beta_\theta(\psi; x)$ and some other cluster sets which are already found in literature like P-cluster set $P(\psi; x)$ [27], S-cluster set $S(\psi; x)$ [26], cluster set $C(\psi; x)$ [40]. For this, we need the following definition.

Definition 3.22 A space (X, τ) is said to be submaximal [10] if every dense subset of X is open. It is called extremally disconnected [28] if closure of each open set is open in X.

Theorem 3.23 *Let X be a submaximal extremally disconnected space and Y be a regular space then for a function $\psi : X \to Y$ the following sets are all equal:*

(a) $\beta_\delta(\psi; x)$,
(b) $\beta_\theta(\psi; x)$,
(c) $C(\psi; x)$,
(d) $S(\psi; x)$,
(e) $P(\psi; x)$.

Proof. Since the space (X, τ) is a submaximal extremally disconnected then $\tau = \tau_\alpha = SO(X) = PO(X) = \beta O(X)$ [8]. Hence the proof follows. ■

4. β-θ-cluster set and β-δ-cluster set for multifunctions and characterizations of β-closed spaces

Definition 4.1 Let $F : X \to Y$ be a multifunction and $x \in X$. Then β-θ-cluster (resp. β-δ-cluster) set of X at $x \in X$, is denoted by $\beta_\theta(F; x)$ (resp. $\beta_\delta(F; x)$), is defined as the set $\bigcap\{\theta\text{-}clF(\beta clU) : U \in \beta O(X, x)\}$ (resp. $\bigcap\{\delta\text{-}clF(\beta clU) : U \in \beta O(X, x)\}$).

For a multifunction $F : X \to Y$ and a subset S of X, the notation $\beta_\theta(F; S)$ (resp. $\beta_\delta(F; S)$) stands for the set $\bigcup_{x \in S} \beta_\theta(F; x)$ (resp. $\bigcup_{x \in S} \beta_\delta(F; x)$).

Definition 4.2 A multifunction $F : X \to Y$ is said to have a β-θ-closed (resp. β-δ-closed) graph if for each $(x, y) \notin G(F)$, there exist $U \in \beta O(X, x)$ and $V \in \tau(y)$ such that $(\beta clU \times clV) \cap G(F) = \emptyset$ (resp. $(\beta clU \times intclV) \cap G(F) = \emptyset$).

Definition 4.3 For a subset $S \subset X \times Y$, the (2)β-θ-closure of S, denoted by (2)β-θ-$cl(S)$ is defined as the set $\{(x, y) \in X \times Y$: for all $V \in \beta O(X, x)$ and for all $W \in \tau(y)$, $(\beta clV \times clW) \cap S \neq \emptyset\}$. If $S =$ (2)β-θ-$cl(S)$, then S is called (2)β-θ-closed.

Theorem 4.4 *Let $F : X \to Y$ be a multifunction. Then*

(a) $\beta_\theta(F; x) = \Pi_y((\{x\} \times Y) \cap (2)\beta\text{-}\theta\text{-}clG(F))$ *for each* $x \in X$,

(b) $\beta_\delta(F; x) = \Pi_y((\{x\} \times Y) \cap (2)\beta\text{-}\delta\text{-}clG(F))$ *for each* $x \in X$.

Proof. (a) Let $x \in X$. Then $y \in \beta_\delta(F; x)$ if and only if for each $U \in \beta O(X, x)$ and each $V \in \tau(y)$, $clV \cap F(\beta clU) \neq \emptyset$ if and only if $(\beta clU \times clV) \cap G(F) \neq \emptyset$ if and only if $(x, y) \in (2)\beta\text{-}\theta\text{-}clG(F)$ if and only if $y \in \Pi_y((\{x\} \times Y) \cap (2)\beta\text{-}\theta\text{-}clG(F))$.
(b) The proof is quite similar to (a) and is thus omitted. ■

Theorem 4.5 *Let $F : X \to Y$ be a multifunction. Then following statements are equivalent:*

(a) $\beta_\theta(F; x) = F(x)$ *for each* $x \in X$,

(b) *F has a β-θ-closed graph,*

(c) $F(x) = \Pi_y((\{x\} \times Y) \cap (2)\beta\text{-}\theta\text{-}clG(F))$ *for each* $x \in X$.

Proof. (a)⇒(b). Let $(x, y) \notin G(F)$. Then $y \notin F(x)$. Hence by (a), $y \notin \beta_\theta(F; x)$. So there exist $U \in \beta O(X, x)$ and $V \in \tau(y)$ such that $clV \cap F(\beta clU) = \emptyset$. Thus $(\beta clU \times clV) \cap G(F) = \emptyset$. So F has a β-θ-closed graph.
(b)⇒(c). Let $x \in X$ and $y \notin F(x)$. Then $(x, y) \notin G(F)$. Since F has a β-θ-closed graph, there exist $U \in \beta O(X, x)$ and $V \in \tau(y)$ such that $(\beta clU \times clV) \cap G(F) = \emptyset$ and hence $(x, y) \notin (2)\beta\text{-}\theta\text{-}clG(F)$. Thus $y \notin \Pi_y((\{x\} \times Y) \cap (2)\beta\text{-}\theta\text{-}clG(F))$. The reverse inclusion is also similar.
(c)⇒(a). Follows from Theorem 4.4 (a). ■

Theorem 4.6 *For a multifunction $F : X \to Y$ the following statements are equivalent:*

(a) $\beta_\delta(F; x) = F(x)$ *for each* $x \in X$,

(b) *F has a β-δ-closed graph,*

(c) $F(x) = \Pi_y((\{x\} \times Y) \cap (2)\beta\text{-}\delta\text{-}clG(F))$ *for each* $x \in X$.

Proof. The proof is quite similar to Theorem 4.5. ■

Definition 4.7 A multifunction $F : X \to Y$ is said to be upper strongly θ-β-continuous if for each $x \in X$ and each $V \in \tau(F(x))$, there is an $U \in \beta O(X, x)$ such that $F(\beta clU) \subset V$.

Theorem 4.8 *If $F : X \to Y$ is upper strongly θ-β-continuous multifunction, then for each $x \in X$, $\beta_\theta(F; x) = \theta\text{-}clF(x)$.*

Proof. From the definition of $\beta_\theta(F; x)$, it is clear that $\theta\text{-}clF(x) \subset\beta_\theta(F; x)$. Since F is a upper strongly θ-β-continuous, for each $V \in \tau(F(x))$, there is an $U \in \beta O(X, x)$ such that $F(\beta clU) \subset V$. Hence the filter subbase $F(\beta R(X, x))$ is stronger than the filter subbase $\tau(F(x))$ in Y. So $\beta_\theta(F; x) \subset ad_\theta\tau(F(x)) = ad\tau(F(x)) = \theta\text{-}clF(x)$ (since for a subset K of a space X, $\theta\text{-}clK = ad\tau(K)$ [22]). Therefore $\beta_\theta(F; x) = \theta\text{-}clF(x)$. ■

Theorem 4.9 *An upper strongly θ-β-continuous multifunction has a β-θ-closed graph if and only if it has θ-closed point images.*

Proof. The proof follows from Theorem 4.5 and Theorem 4.8. ■

Corollary 4.10 *A strongly θ-β-continuous [30] surjection $\psi : X \to Y$ has a β-θ-closed graph if and only if Y is Hausdorff.*

Proof. Since a space Y is Hausdorff if and only if its points are θ-closed, the proof follows from Theorem 4.9. ■

We say a multifunction $F : X \to Y$ has a θ-closed graph (resp.δ-closed graph) if $G(F)$ is a θ-closed (resp.δ-closed) subset of $X \times Y$.

Theorem 4.11 *If a multifunction $F : X \to Y$ has*

(i) *θ-closed graph then $\beta_\theta(F; x) = F(x)$ for each $x \in X$,*

(ii) *δ-closed graph then $\beta_\delta(F; x) = F(x)$, for each $x \in X$.*

Proof. We proof this theorem for the multifunction having θ-closed graph only. For the case of multifunction having δ-closed graph, the proof is similar. Let $x \in X$ and $y \in \beta_\theta(F; x)$. Then for each $V \in \beta O(X, x)$ and each $U \in \tau(y)$, $F(\beta clV) \cap clU \neq \emptyset$ i.e $F^-(clU) \cap \beta clV \neq \emptyset$. Thus for any basic open set $W_1 \times W_2$ in $X \times Y$ containing (x, y), $F^-(clW_1) \cap F^-(clW_2) \neq \emptyset$ (as $\tau(x) \subset \beta O(X, x)$) and hence $cl(W_1 \times W_2) = (clW_1 \times clW_2) \cap G(F) \neq \emptyset$ (as $\beta clW_1 \subset clW_1$). So $(x, y) \in \theta\text{-}clG(F) = G(F)$. Hence $y \in F(x)$. Thus $\beta_\theta(F; x) \subseteq F(x)$. Also it is obvious that $F(x) \subseteq \beta_\theta(F; x)$. So $\beta_\delta(F; x) = F(x)$, for each $x \in X$. ■

Lemma 4.12 *Let X and Y be topological spaces, $M \subseteq X$, $N \subseteq Y$. Then*

(i) *$\beta cl(M \times N) \subseteq \beta clM \times \beta clN$,*

(ii) *If $M \in \beta O(X, x)$ and $N \in \beta O(Y, y)$ then $M \times N \in \beta O(X \times Y, (x, y))$.*

Proof. (i) $\beta cl(M\times N)=(M\times N)\cup intclint(M\times N)=(M\times N)\cup(intclintM\times intclintN)\subseteq(M\cup intclintM)\times(N\cup intclintN)=\beta clM\times\beta clN$.
(ii) $M\times N\subseteq(clintclM)\times(clintclN)=clintcl(M\times N)$. ∎

Theorem 4.13 *If a multifunction $F:X\to Y$ satisfies any one of the following:*

(a) $\beta_\theta(F;x)=F(x)$*, for each* $x\in X$*,*

(b) $\beta_\delta(F;x)=F(x)$*, for each* $x\in X$*.*

Then, the graph $G(F)$ is sp-θ-closed [30].

Proof. (a) Let $F:X\to Y$ be the multifunction satisfies (a) and let $(x,y)\notin G(F)$. Then $y\notin F(x)=\beta_\theta(F;x)$ and so there exist $U\in\beta O(X,x)$ and $V\in\tau(y)\subseteq\beta O(Y,y)$ such that $clV\cap F(\beta clU)=\emptyset$ and hence $(\beta clU\times\beta clV)\cap G(F)=\emptyset$ (as $\beta clU\subset clV$). But from the lemma 4.12, we have $\beta cl(U\times V)\cap G(F)=\emptyset$, where $U\times V\in\beta O(X\times Y,(x,y))$. So $G(F)$ is *sp*-θ-closed.
(b) Since for a subset S of Y, $\beta clintS=intclintS$ [6], then for any open set V of Y $\beta clV=intclV$. Because of this fact, the proof is now quite similar to the proof as for the case (a). ∎

Theorem 4.14 *A subset S of a topological space X is β-closed relative to X (i.e. β-set) if and only if for every filter base Ω on X with $F\cap K\neq\emptyset$ for each $F\in\Omega$ and each $K\in\beta O(X,S)$, $S\cap\beta$-θ-$ad\Omega\neq\emptyset$.*

Proof. Let S be not β-closed relative to X. Then there exists a cover $\mathcal{V}=\{V_\alpha:\alpha\in I\}$ by β-open sets of X such that $S\not\subset\cup\{\beta clV_\alpha:\alpha\in I_0\}$ for any finite subset I_0 of I. Clearly the family $\Omega=\{S-\cup_{\alpha\in I_0}\beta clV_\alpha : I_0$ is a finite subset of $I\}$ is a filterbase on X. As $\emptyset\neq F\subseteq S$ and $S\subseteq K$ for each $F\in\Omega$ and $K\in\beta O(X,S)$ then $F\cap K\neq\emptyset$ for each $F\in\Omega$ and $K\in\beta O(X,S)$. Clearly $S\cap\beta$-θ-$ad\Omega=\emptyset$. In fact if $x\in S$ then there exists $V_\alpha\in\mathcal{V}$ such that $x\in V_\alpha$ and $X-\beta clV_\alpha\in\Omega$ satisfying $\beta clV_\alpha\cap(X-\beta clV_\alpha)=\emptyset$, a contradiction. So S is β-closed relative to X.

Conversely, if possible let $S\cap\beta$-θ-$ad\Omega=\emptyset$, when S is β-closed relative to X and Ω satisfies condition of the hypothesis. Then for each $x\in S$, there exists a $V_x\in\beta O(X,x)$ and an $F_x\in\Omega$ such that $\beta clV_x\cap F_x=\emptyset$. Since S is β-closed relative to X, there exist $x_1,x_2,\ldots\ldots,x_n\in S$ such that $S\subset\cup_{i=1}^n\beta clV_{x_i}$. Since Ω is a filterbase, there exists an $F\in\Omega$ such that $F\subseteq\cup_{i=1}^n\beta clF_{x_i}$. Clearly $F\cap(\cup_{i=1}^n\beta clV_{x_i})=\emptyset$ and $\cup_{i=1}^n\beta clV_{x_i}\in\beta O(X,S)$. This contradicts the hypothesis which Ω satisfies. ∎

Lemma 4.15 *If S is a β-θ-closed subset of a β-closed space X then S is β-closed relative to X.*

Proof. The proof is quite clear. ∎

Theorem 4.16 *The following statements are equivalent for a space X:*

(a) *X is β-closed,*

(b) *For a multifunction* $F : X \to Y$*, where* Y *is any topological space,* $\cap\{\delta\text{-}clF(V) : V \in \beta O(X, S)\} \subseteq \beta_\delta(F, S)$*, for each* β*-*δ*-closed set* S *of* X*,*

(c) *For each multifunction* $F : X \to Y$*, where* Y *is any topological space,* $\cap\{\beta\text{-}\theta\text{-}clF(V) : V \in \beta O(X, S)\} \subseteq \beta_\theta(F, S)$*, for each* β*-*θ*-closed set* S *of* X*.*

Proof. (a)⇒(b). Since X is β-closed and S is β-θ-closed, then by Lemma 4.15, S is β-closed and X. If $y \in \cap\{\delta\text{-}clF(V) : V \in \beta O(X, S)\}$ then for each $U \in \tau(y)$, $F(V) \cap intclU \neq \emptyset$ i.e. $V \cap F^-(intclV) \neq \emptyset$. Since S is β-closed relative to X and the filter base $\Omega = \{F^-(intclU) : U \in \tau(y)\}$ satisfies the condition of Theorem 4.14, then $S \cap \beta\text{-}\theta\text{-}ad\Omega \neq \emptyset$. Let $x \in S \cap \beta\text{-}\theta\text{-}ad\Omega$. Then for each $W \in \beta O(X, x)$ and each $U \in \tau(y)$, $F^-(intclU) \cap \beta clW \neq \emptyset$ i.e. $intclU \cap F(\beta clW) \neq \emptyset$; which shows that $y \in \delta\text{-}clF(\beta clW)$, for each $W \in \beta O(X, x)$. Thus $y \in \beta_\delta(F, x) \subseteq \beta_\delta(F, S)$.
(b)⇒(c). Since we know that for a subset A of X, $\beta clintA = intclintA$ [6], then for an open U, $\beta clU = intclU$. Hence from the definition of β-θ-closure of a subset A, we have $\beta\text{-}\theta\text{-}clA \subseteq \delta\text{-}clA$. So (b)⇒(c) follows immediately.
(c)⇒(a). Let Ω be a filter base on X. Let $z \notin X$. Now consider $Y = X \cup \{z\}$ and $\tau_Y = \{U \subseteq Y$: either $z \notin U$ or there exists $F \in \Omega$ such that $F \subseteq U$ when $z \in U\}$. Then τ_Y is a topology on Y [22]. So for the identity map $\psi : X \to Y$, we have from hypothesis (c) that $\beta_\delta(\psi, X) \supseteq \cap\{\beta\text{-}\theta\text{-}cl\psi(V) : V \in \beta O(Y, X)\} = \cap\{\beta\text{-}\theta\text{-}clV : V \in \beta O(Y, X)\} = \beta\text{-}\theta\text{-}clX$ (in (Y, τ_Y)). To prove the last equality, it is enough to prove that $z \in \beta\text{-}\theta\text{-}clX$ (in (Y, τ_Y)). Clearly $\{z\}$ is not open in (Y, τ_Y) and also $\{z\}$ is not β-open in (Y, τ_Y). In fact $\{z\} \not\subseteq clintcl\{z\}$ (as $cl\{z\} = \{z\}$ and hence $intcl\{z\} = \emptyset$). So $\beta clU \cap X \neq \emptyset$, for all $U \in \beta O(Y, z)$. Thus $z \in \beta\text{-}\theta\text{-}clX$. Hence $z \in \beta_\delta(\psi, x)$ for some $x \in X$. Let $F \in \Omega$. Then from the construction of τ_Y, it is clear that $Y - (F \cup \{z\})$ as well as $F \cup \{z\}$ are both open in (Y, τ_Y). Also $\beta cl(F \cup \{z\}) = intcl(F \cup \{z\})$ (as for an open set W, $\beta clW = intclW$). So for each $V \in \beta O(X, x)$, and each $F \in \Omega$, we have $\beta clV \cap F = \psi(\beta clV) \cap (F \cup \{z\})$ (as $z \notin \psi(\beta clV)) = \psi(\beta clV) \cap \beta cl(F \cup \{z\})$ (since $\{x_0\}$ is open and hence β-open in (Y, τ_Y) whenever a point x_0 in X satisfies $x_0 \notin F) = \psi(\beta clV) \cap intcl(F \cup \{z\}) \neq \emptyset$ (as $z \in \beta_\delta(\psi, x)$ and $F \cup \{z\} \in \tau_Y$). So $x \in \beta\text{-}\theta\text{-}ad\Omega$ and hence X is β-closed as we know that a space X is β-closed if and only if every filter base on X, β-θ-adheres at a some point in X [7]. ∎

Theorem 4.17 *The following statements are equivalent for a space* X*:*

(a) X *is* β*-closed,*

(b) *For a multifunction* $F : X \to Y$*, where* Y *is any topological space,* $\cap\{\theta\text{-}clF(V) : V \in \beta O(X, S)\} \subseteq \beta_\theta(F, S)$*, for each* β*-*θ*-closed set* S *of* X*,*

(c) *For each multifunction* $F : X \to Y$*, where* Y *is any topological space,* $\cap\{\beta\text{-}\theta\text{-}clF(V) : V \in \beta O(X, S)\} \subseteq \beta_\theta(F, S)$*, for each* β*-*θ*-closed set* S *of* X*.*

Proof. The proof is quite similar to the proof of the Theorem 4.16 and is thus omitted. ∎

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INÉGALITÉS DE TYPE FAIBLE POUR L'OPÉRATEUR MAXIMAL FRACTIONNAIRE DANS LES ESPACES DE MORREY PAR RAPPORT À LA CAPACITÉ DE HAUSDORFF

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Abstract. We prove a boundedness property for the fractional maximal operator in the Morrey type spaces with respect to the Hausdorff content. As an application of this result, we obtain a Fefferman-Stein inequality.

1. Introduction

Soit n un entier positif non nul.

Sauf mention spécifique, α, λ, δ et p sont des nombres réels vérifiant:

$$0 \leq \alpha < n \ ; \quad 0 < \delta \leq n \ ; \quad 0 \leq \lambda \leq \delta \ ; \quad 1 \leq p < +\infty.$$

On appellera cube, tout cube Q de $\mathbb{R}^n$ dont les côtés sont parallèles aux axes de coordonnées.

Si Q est un cube, alors $\ell(Q)$ et $\overset{\circ}{Q}$ désignent respectivement la longueur de ses côtés et son intérieur.

Si E est un sous-ensemble de $\mathbb{R}^n$ alors $|E|$ est sa mesure de Lebesgue et χ_E sa fonction caractéristique.

Soit E un sous-ensemble de $\mathbb{R}^n$.

La δ-capacité de Hausdorff $H^\delta(E)$ de E est

$$H^\delta(E) = \inf_{(Q_i)_i} \left\{ \sum_i \ell(Q)^\delta, \ E \subset \bigcup_i Q_i \right\} \tag{1}$$

où l'infinimum est pris sur tous les recouvrements dénombrables $(Q_i)_i$ de E, Q_i étant un cube pour tout i.

Si dans l'expression (1) on prend l'infinimum sur tous les recouvrements dénombrables de E par des cubes dyadiques, on obtient la δ-capacité dyadique de Hausdorff $H^{\delta}_{\Delta}(E)$ de E.

Soit f une fonction localement intégrable de $\mathbb{R}^n$.

- La fonction maximale fractionnaire d'ordre α de f est

$$M_\alpha f(x) = \sup_{\substack{Q \\ Q \ni x}} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f(y)|dy$$

où le supremum est pris sur tous les cubes Q contenant x.

Bien entendu, lorsque $\alpha = 0$, $M_0 = M$ est l'opérateur maximal de Hardy-Littlewood.

- On rappelle que f appartient à l'espace de Morrey classique $L^{p,\lambda}(dx)$ si et seulement si la quantité

$$\|f\|_{L^{p,\lambda}(dx)} = \sup_Q \left[\frac{1}{\ell(Q)^\lambda} \int_Q |f(y)|^p dy\right]^{\frac{1}{p}} \text{ est finie};$$

et f appartient à l'espace de Morrey de type faible $L^{p,\lambda}_*(dx)$ si et seulement si la quantité

$$\|f\|_{L^{p,\lambda}_*(dx)} = \sup_{\substack{t>0 \\ Q}} t \left[|\{x \in \mathbb{R}^n \,/\, |f(x)| > t\} \cap Q|\right]^{\frac{1}{p}} \ell(Q)^{\frac{-\lambda}{p}} \text{ est finie.}$$

Par analogie, on dit que f appartient à l'espace faible de Morrey $L^{p,\lambda}(H^\delta)$ par rapport à la δ-capacité de Hausdorff, si et seulement si la quantité

$$\|f\|_{L^{p,\lambda}_*(H^\delta)} = \sup_{\substack{t>0 \\ Q}} t \left[H^\delta\left(\{x \in \mathbb{R}^n \,/\, |f(x)| > t\} \cap Q\}\right]^{\frac{1}{p}} \ell(Q)^{\frac{-\lambda}{p}} \text{ est finie.}$$

On remarquera que $L^{p,0}(H^\delta)$ et $L^{p,0}_*(H^\delta)$ sont respectivement l'espace (fort) de Lebesgue $L^p(H^\delta)$ et l'espace faible de Lebesgue $L^p_*(H^\delta)$ par rapport à la δ-capacité de Hausdorff étudiés par de nombreux auteurs comme D.R. Adams ([1] et [2] par exemple), J. Orobitg et J. Verdera ([6] par exemple).

Enfin C désigne une constante dont la valeur peut changer d'une proposition à une autre.

Dans [5], Kuznetsov a montré l'inégalité suivante:

$$\left\| \left(\sum_{i=1}^{\infty} (M_\alpha f_i)^p \right)^{\frac{1}{p}} \right\|_{L^1_*(H^{n-\alpha})} \leq C \frac{p}{p-1} \sum_{i=1}^{\infty} \|f_i\|_{L^1(dx)}, \tag{2}$$

où $\{f_i\}_i$ est une famille de fonctons et $1 < p < +\infty$.

Dans la preuve de (2), l'inégalité suivante de type Fefferman-Stein:

$$\left\|\left(\sum_{i=1}^{\infty} f_i^p\right)^{\frac{1}{p}}\right\|_{L^1_*(H^\delta)} \leq C\frac{p}{p-1}\sum_{i=1}^{\infty}\|f_i\|_{L^1_*(H^\delta)} \tag{3}$$

a joué un role central. En effet dans [2], D.R. Adams a montré que pour toute fonction f et $\delta \geq \frac{d}{n}(n-\alpha)$ on a l'inégalité

$$\|M_\alpha f\|_{L_*^{\frac{\delta}{n-\alpha}}(H^\delta)} \leq C\|f\|_{L^{\frac{d}{n}}(H^d)}. \tag{4}$$

Il est clair que (4) équivaut à

$$\|M_\alpha f\|_{L_*^{\frac{\delta}{n-\alpha}}(H^\delta)} \leq C\|f\|_{L^1(dx)} \text{ avec } \delta \geq n-\alpha. \tag{5}$$

Il suffit de prendre dans (4) $d = n$ pour avoir (5).

Réciproquement, comme pour toute fonction positive f et tout réel d tel que $0 < d \leq n$ on a

$$\int_{\mathbb{R}^n} f(x)dx \leq \frac{n}{d}\left(\int_{\mathbb{R}^n} f^{\frac{d}{n}}dH^d\right)^{\frac{n}{d}}$$

(voir [6], Lemme 3), de (5) on obtient (4) car $\frac{d}{n} \leq 1$.

En prenant $\delta = n-\alpha$ et en écrivant (3) pour $M_\alpha f_i$ au lieu de f_i puis en utilisant (5), on obtient (2).

L'inégalité (5) exprime le fait que: *si une fonction f appartient à l'espace de Lebesgue $L^1(dx)$ alors sa fonction maximale fractionnaire $M_\alpha f$ appartient à l'espace $L_*^{1,0}(H^\delta)$.*

Celà nous amène à la question suivante: *à quelle classe de fonctions appartient la fonction maximale fractionnaire $M_\alpha f$ lorsque f appartient à l'espace de Morrey $L^{1,\lambda}(dx)$?*

Remarquons qu'il existe bien des fonctions appartenant à $L^{1,\lambda}(dx)$ mais qui n'appartiennent pas à $L^1(dx)$. Il suffit, par exemple, de considérer la fonction

$$f(x) = \prod_{i=1}^{n}|x_i|^{\frac{\lambda}{n}-1}.$$

Le théorème 1 suivant donne une réponse à cette question.

Théorème 1. *Supposons que $0 \leq \alpha < n$, $\delta \geq \lambda \geq 0$, $\delta \geq n-\alpha$ et posons $\mu = \frac{\lambda}{\delta}(n-\alpha)$. Alors il existe une constante $C > 0$ telle que pour toute fonction $f \in L^{1,\mu}(dx)$ on ait*

$$\|M_\alpha f\|_{L_*^{\frac{\delta}{n-\alpha},\lambda}(H^\delta)} \leq C\|f\|_{L^{1,\mu}(dx)}.$$

Le théorème suivant généralise l'inégalité (2).

Théorème 2. *Soient* $0 \leq \alpha < n$, $\lambda \leq \delta = n - \alpha$ *et* $1 < p < +\infty$. *Alors il existe une constante* $C > 0$ *telle que pour toutes fonctions* $\{f_i\}_{i=1,\ldots,\infty}$, *on ait*

$$\left\| \left(\sum_{i=1}^{\infty} (M_\alpha f_i)^p \right)^{\frac{1}{p}} \right\|_{L_*^{1,\lambda}(H^\delta)} \leq C \frac{p}{p-1} \sum_{i=1}^{\infty} \|f_i\|_{L^{1,\lambda}(dx)}.$$

Il s'obtient par une argumentation similaire à la preuve de l'inégalité (2) figurant dans [5], moyennant l'utilisation de la généralisation suivante de l'inégalité (3).

Théorème 3. *Soient* $1 < p < +\infty$, $0 < \delta \leq n$, $0 \leq \lambda \leq n$ *et soit* $\{f_i\}_{i=1,\ldots,\infty}$ *des fonctions positives telles que* $\sum_{i=1}^{\infty} \|f_i\|_{L_*^{1,\lambda}(H^\delta)} < \infty$. *Alors il existe une constante* $C > 0$ *indépendante des* f_i *telle que:*

$$\left\| \left(\sum_{i=1}^{\infty} f_i^p \right)^{\frac{1}{p}} \right\|_{L_*^{1,\lambda}(H^\delta)} \leq C \frac{p}{p-1} \sum_{i=1}^{\infty} \|f_i\|_{L_*^{1,\lambda}(H^\delta)}.$$

2. Preuve du Théorème 1

Afin de prouver ce théorème, on montre d'abord le lemme suivant:

Lemme 1. *Soient* Q *et* P *deux cubes vérifiant:* $\overset{\circ}{Q} \cap \overset{\circ}{\widehat{3P}} = \emptyset$ *et* $(3P) \cap Q \neq \emptyset$. *Alors* $Q \subset 9P$ *ou* $3P \subset \overset{\circ}{\widehat{3Q}}$.

Preuve. Supposons que $\ell(3P) \geq \ell(Q)$. Alors $Q \subset 3(3P) = 9P$.

Supposons maintenant que $\ell(3P) < \ell(Q)$. Soit u un élément de l'intersection des frontières des cubes $3P$ et Q. $\forall k \in \{1, 2, \ldots, n\}$, on a

$$|x_{P,k} - u_k| \leq \tfrac{\ell(3P)}{2} \text{ et } |u_k - x_{Q,k}| \leq \tfrac{\ell(Q)}{2}$$

où $x_{P,k}$ et $x_{Q,k}$ sont les k-ièmes coordonnées respectives des centres x_P et x_Q des cubes P et Q. Donc

$$|x_{P,k} - x_{Q,k}| \leq \frac{\ell(3P)}{2} + \frac{1}{2}\ell(Q) < \ell(Q).$$

Soit $x \in 3P$. On a:

$$\begin{array}{rcl} |x_k - x_{Q,k}| & \leq & |x_k - x_{P,k}| + |x_{P,k} - x_{Q,k}| \\ & < & \frac{\ell(3P)}{2} + \ell(Q) < \frac{\ell(3Q)}{2} \end{array}$$

c'est à dire $x \in 3Q$. D'où $3P \subset \overset{\circ}{\widehat{3Q}}$. ∎

Rappelons également quelques propriétés de la capacité de Hausdorff essentielles dans la preuve du Théorème 1.

Proposition 1. (voir [1] page 117) *Soit* $0 < \delta \leq n$.

P1: H^δ_Δ *et* H^δ *sont comparables c'est-à-dire qu'il existe des constantes* A *et* B *telles que*

$$H^\delta_\Delta(E) \leq AH^\delta(E) \leq BH^\delta_\Delta(E)$$

pour tout $E \subset \mathbb{R}^n$.

P2: H^δ_Δ *est continue pour les suites croissantes c'est-à-dire si* $(E_j)_j$ *est une suite croissante de sous-ensembles de* $\mathbb{R}^n$ *telle que* $E = \cup E_j$ *alors*

$$\lim_{j\to\infty} H^\delta_\Delta(E_j) = H^\delta_\Delta(E).$$

Preuve du Théorème 1. Soit Q un cube de $\mathbb{R}^n$. Fixons $t > 0$ et posons

$$\Omega_t = \{x \in Q / M_\alpha f(x) > t\} = Q \cap \{x \in \mathbb{R}^n / M_\alpha f(x) > t\}$$

Étape 1: Soit k l'unique entier de $\mathbb{Z}$ tel que $2^{-k-1} < l(Q) \leq 2^{-k}$. Alors Q rencontre au moins un cube et au plus 2^n cubes dyadiques d'ordre k. Plus précisément, on a: $Q \subset \underset{j\in J}{\cup} Q_j$ avec $card(J) \leq 2^n$, où les Q_j sont des cubes dyadiques d'ordre k rencontrant Q et $Q \subset 3Q_j$.

Posons maintenant, pour tout $j \in J$, $\Omega^j_t = \{x \in Q_j / M_\alpha f(x) > t\}$. On a:

$$\begin{aligned} x \in \Omega_t \;&\Leftrightarrow\; x \in Q \text{ et } M_\alpha f(x) > t \\ &\Rightarrow\; \exists j \in J,\ x \in Q_j,\ M_\alpha f(x) > t \\ &\Rightarrow\; \exists j \in J,\ x \in \Omega^j_t. \end{aligned}$$

D'où $\Omega_t \subset \underset{j\in J}{\cup} \Omega^j_t$ et donc

$$H^\delta(\Omega_t) \leq \sum_{j\in J} H^\delta(\Omega^j_t).$$

Soit j_0 un élément de J pour lequel $H^\delta(\Omega^{j_0}_t)$ réalise le maximum des $H^\delta(\Omega^j_t)$. Ainsi donc $H^\delta(\Omega_t) \leq 2^n H^\delta(\Omega^{j_0}_t)$ avec $\ell(Q) < \ell(Q_{j_0}) \leq 2\ell(Q)$. D'où

$$\begin{aligned} & t\left[H^\delta\left(Q \cap \{x \in \mathbb{R}^n / M_\alpha f(x) > t\}\right)\right]^{\frac{n-\alpha}{\delta}} \ell(Q)^{-\lambda\frac{n-\alpha}{\delta}} \\ &\quad \leq 2^{(n+\lambda)(\frac{n-\alpha}{\delta})} t\left[H^\delta\left(Q_{j_0} \cap \{x \in \mathbb{R}^n / M_\alpha f(x) > t\}\right)\right]^{\frac{n-\alpha}{\delta}} \ell(Q_{j_0})^{-\lambda\frac{n-\alpha}{\delta}}. \end{aligned}$$

Pour cette raison on peut supposer dans la suite que le cube Q est dyadique.

Étape 2: Supposons que f est bornée et à support compact.

Il existe alors une famille $\{Q_i\}_{i\in I}$ de cubes dyadiques maximaux (pour l'inclusion) vérifiant:

pour tout $i \in I$

$$\frac{1}{|Q_i|^{1-\frac{\alpha}{n}}} \int_{Q_i} |f(y)|dy > C_{n,\alpha} t \quad \text{et} \quad \{x \in \mathbb{R}^n / M_\alpha f(x) > t\} \subset \bigcup_{i\in I} 3Q_i$$

(voir [4], page 137 et [3] Lemme 2-1) où $C_{n,\alpha} = 2^{\alpha-2n}$ est une constante qui ne dépend pas de f.
Alors tout cube Q_i de la famille $\{Q_i\}_{i\in I}$ est tel que:

$$\ell(Q) \leq \left(\frac{1}{C_{n,\alpha}t}\int_{Q_i}|f(y)|dy\right)^{\frac{1}{n-\alpha}}. \tag{6}$$

On a:

$$\begin{aligned}
\Omega_t &= Q\cap\{x\in\mathbb{R}^n/M_\alpha f(x)>t\}\subset \underset{i\in I}{\cup}(3Q_i)\cap Q\\
&\subset \bigcup_{Q_i\subsetneq Q}(3Q_i\cap Q)\cup\bigcup_{Q\subset Q_i}(3Q_i\cap Q)\cup\bigcup_{\overset{\circ}{Q}\cap\overset{\circ}{Q}_i=\emptyset}(3Q_i\cap Q)\\
&\subset \bigcup_{Q_i\subsetneq Q}(3Q_i\cap Q)\cup\bigcup_{Q\subset Q_i}(3Q_i\cap Q)\\
&\cup \bigcup_{\substack{\overset{\circ}{Q}\cap 3\overset{\circ}{\overset{\frown}{Q}}_i=\emptyset\\ \overset{\circ}{Q}\cap\overset{\circ}{Q}_i=\emptyset}}(3Q_i\cap Q)\cup\bigcup_{\substack{\overset{\circ}{Q}\cap 3\overset{\circ}{\overset{\frown}{Q}}_i\neq\emptyset\\ \overset{\circ}{Q}\cap\overset{\circ}{Q}_i=\emptyset}}(3Q_i\cap Q).
\end{aligned}$$

Par suite

$$\begin{aligned}
\Omega_t &\subset \bigcup_{Q_i\subsetneq Q}(3Q_i\cap Q)\cup\bigcup_{Q\subset Q_i}(3Q_i\cap Q)\cup\bigcup_{\substack{\overset{\circ}{Q}\cap 3\overset{\circ}{\overset{\frown}{Q}}_i=\emptyset\\ \overset{\circ}{Q}\cap\overset{\circ}{Q}_i=\emptyset}}(3Q_i\cap Q)\\
&\cup \bigcup_{\substack{\overset{\circ}{Q}\cap 3\overset{\circ}{\overset{\frown}{Q}}_i\neq\emptyset,\ \ell(3Q_i)<\ell(Q)\\ \overset{\circ}{Q}\cap\overset{\circ}{Q}_i=\emptyset}}(3Q_i\cap Q)\cup\bigcup_{\substack{\overset{\circ}{Q}\cap 3\overset{\circ}{\overset{\frown}{Q}}_i\neq\emptyset,\ \ell(3Q_i)\geq\ell(Q)\\ \overset{\circ}{Q}\cap\overset{\circ}{Q}_i=\emptyset}}(3Q_i\cap Q)\\
&\subset \bigcup_{Q_i\subsetneq Q}(3Q_i\cap Q)\cup\bigcup_{Q\subset Q_i}(3Q_i\cap Q)\cup\bigcup_{\substack{\overset{\circ}{Q}\cap 3\overset{\circ}{\overset{\frown}{Q}}_i=\emptyset\\ \overset{\circ}{Q}\cap\overset{\circ}{Q}_i=\emptyset}}(3Q_i\cap Q)\\
&\cup \bigcup_{\substack{\overset{\circ}{Q}\cap 3\overset{\circ}{\overset{\frown}{Q}}_i\neq\emptyset,\ 3Q_i\subset 3Q\\ \overset{\circ}{Q}\cap\overset{\circ}{Q}_i=\emptyset}}(3Q_i\cap Q)\cup\bigcup_{\substack{\overset{\circ}{Q}\cap 3\overset{\circ}{\overset{\frown}{Q}}_i\neq\emptyset,\ Q\subset 9Q_i\\ \overset{\circ}{Q}\cap\overset{\circ}{Q}_i=\emptyset}}(3Q_i\cap Q).
\end{aligned}$$

D'où

$$\Omega_t\subset\bigcup_{3Q_i\subset 3Q}3Q_i\cup\bigcup_{Q\subset 9Q_i}3Q_i\cup\bigcup_{\overset{\circ}{Q}\cap 3\overset{\circ}{\overset{\frown}{Q}}_i=\emptyset}(3Q_i\cap Q).$$

Par le Lemme 1, on obtient

$$\Omega_t \subset \bigcup_{3Q_i \subset 3Q} 3Q_i \cup \bigcup_{Q \subset 9Q_i} 3Q_i.$$

Supposons qu'il existe i_0 tel que $Q \subset 9Q_{i_0}$. Comme $\Omega_t \subset Q$, on a:

$$\begin{aligned} H^\delta(\Omega_t) &\leq \ell(Q)^\delta \\ H^\delta(\Omega_t)\ell(Q)^{-\lambda} &\leq \ell(Q)^{\delta-\lambda} \leq 9^{\delta-\lambda}\ell(Q_{i_0})^{\delta-\lambda}, \end{aligned}$$

et par suite, d'après l'inégalité (6),

$$H^\delta(\Omega_t)\ell(Q)^{-\lambda} \leq 9^{\delta-\lambda}\ell(Q_{i_0})^{-\lambda}\,(C_{n,\alpha}t)^{-\frac{\delta}{n-\alpha}}\left(\int_{Q_{i_0}}|f(y)|dy\right)^{\frac{\delta}{n-\alpha}},$$

c'est-à-dire:

$$t\left[H^\delta(\Omega_t)\right]^{\frac{n-\alpha}{\delta}}\ell(Q)^{-\lambda\frac{n-\alpha}{\delta}} \leq C\left[\frac{1}{\ell(Q_{i_0})^\mu}\int_{Q_{i_0}}|f(y)|dy\right].$$

Supposons que $\{i/\ \ Q \subset 9Q_i\} = \emptyset$. Alors $\Omega_t \subset \displaystyle\bigcup_{3Q_i \subset 3Q} 3Q_i$ et donc, en utilisant l'inégalité (6) et ensuite $\dfrac{\delta}{n-\alpha} \geq 1$, on obtient:

$$\begin{aligned} H^\delta(\Omega_t) &\leq 3^\delta \sum_{3Q_i \subset 3Q} \ell(Q_i)^\delta \\ &\leq 3^\delta\,(C_{n,\alpha}t)^{-\frac{\delta}{n-\alpha}} \sum_{3Q_i \subset 3Q}\left(\int_{Q_i}|f(y)|dy\right)^{\frac{\delta}{n-\alpha}} \\ &\leq 3^\delta\,(C_{n,\alpha}t)^{-\frac{\delta}{n-\alpha}}\left(\sum_{3Q_i \subset 3Q}\int_{Q_i}|f(y)|dy\right)^{\frac{\delta}{n-\alpha}} \\ &\leq 3^\delta\,(C_{n,\alpha}t)^{-\frac{\delta}{n-\alpha}}\left(\int_{\bigcup_{3Q_i \subset 3Q} Q_i}|f(y)|dy\right)^{\frac{\delta}{n-\alpha}} \\ &\leq 3^\delta\,(C_{n,\alpha}t)^{-\frac{\delta}{n-\alpha}}\left(\int_{3Q}|f(y)|dy\right)^{\frac{\delta}{n-\alpha}}. \end{aligned}$$

$$t\left[H^\delta(\Omega_t)\right]^{\frac{n-\alpha}{\delta}}\ell(Q)^{-\lambda\frac{n-\alpha}{\delta}} \leq C\left[\frac{1}{\ell(3Q)^\mu}\int_{3Q}|f(y)|dy\right].$$

Dans tous les cas on a

$$t\left[H^\delta(\Omega_t)\right]^{\frac{n-\alpha}{\delta}}\ell(Q)^{-\lambda\frac{n-\alpha}{\delta}} \leq C\|f\|_{L^{1,\mu}(dx)}. \tag{7}$$

Étape 3: Revenons au cas général où f est une fonction localement intégrable. Posons pour tout entier positif non nul k,

$$f_k = \min\{k, \ \ |f|\chi_{B(0,k)}\} \quad \text{et} \quad \Omega_{k,t} = \{x \in Q, \ \ M_\alpha f_k(x) > t\},$$

où $B(0,k)$ est la boule de centre 0 et de rayon k. On a:

$$f_k \uparrow |f| \quad \text{et} \quad \Omega_{k,t} \uparrow \cup_k \Omega_{k,t} = \Omega_t.$$

De plus, (7) est vraie pour $f = f_k$ et $\Omega_t = \Omega_{k,t}$ et on a

$$t\left[H^\delta(\Omega_{k,t})\right]^{\frac{n-\alpha}{\delta}} \ell(Q)^{-\lambda\frac{n-\alpha}{\delta}} \ \ \leq \ \ C\|f_k\|_{L^{1,\mu}(dx)} \leq C\|f\|_{L^{1,\mu}(dx)}.$$

En vertu de la Proposition 1, H_Δ^δ et H^δ sont comparables et

$$\lim_{k\to+\infty} H_\Delta^\delta(\Omega_{k,t}) = H_\Delta^\delta(\Omega_t).$$

Par conséquent,

$$\begin{aligned} t\left[H^\delta(\Omega_t)\right]^{\frac{n-\alpha}{\delta}} \ell(Q)^{-\lambda\frac{n-\alpha}{\delta}} &\leq A^{-1}Bt\left[H_\Delta^\delta(\Omega_t)\right]^{\frac{n-\alpha}{\delta}} \ell(Q)^{-\lambda\frac{n-\alpha}{\delta}} \\ &= A^{-1}Bt\lim_{k\to\infty}\left[H_\Delta^\delta(\Omega_{k,t})\right]^{\frac{n-\alpha}{\delta}} \ell(Q)^{-\lambda\frac{n-\alpha}{\delta}} \\ &\leq A^{-1}BAt\lim_{k\to\infty}\left[H^\delta(\Omega_{k,t})\right]^{\frac{n-\alpha}{\delta}} \ell(Q)^{-\lambda\frac{n-\alpha}{\delta}} \\ &\leq A^{-1}BAC\|f\|_{L^{1,\mu}(dx)}. \end{aligned}$$

D'où

$$\|M_\alpha f\|_{L_*^{\frac{\delta}{n-\alpha},\lambda}(H^\delta)} \leq C\|f\|_{L^{1,\mu}(dx)}. \qquad \blacksquare$$

3. Preuve des Théorèmes 3 et 2

La démonstration reprend dans les grandes lignes la preuve du Corollaire 5.2.3 de [5] avec quelques aménagements.

Preuve du Théorème 3. Nous montrons ce résultat pour H_Δ^δ au lieu de H^δ puisque H_Δ^δ et H^δ sont comparables.

- Soit N un entier positif, on suppose que

$$\|f_i\|_{L_*^{1,\lambda}(H_\Delta^\delta)} := \sup_{\substack{t>0\\ Q}} H_\Delta^\delta\left(\{x \in \mathbb{R}^n \, / |f_i(x)| > t\} \cap Q\right) \ell(Q)^{-\lambda} = 1$$

pour tout $i = 1, \ldots, N$ et $\{\alpha_i\}_{i=1,\ldots,N}$ des nombres réels positifs tels que

$$\sum_{i=1}^N \alpha_i = 1.$$

On montre qu'il existe une constante C indépendante des f_i telle que

$$\left\| \left(\sum_{i=1}^{N} (\alpha_i f_i)^p \right)^{\frac{1}{p}} \right\|_{L_*^{1,\lambda}(H_\Delta^\delta)} \leq C \frac{p}{p-1}. \tag{8}$$

En effet, (8) est trivial pour $N = 1$.

Si $N \geq 2$, on pose

$$F = \left(\sum_{i=1}^{N} (\alpha_i f_i)^p) \right)^{\frac{1}{p}}.$$

Pour tout cube Q et pour tout $t > 0$, on considère les ensembles

$$\begin{aligned}
\Omega_{t,i} &= \{x \in Q;\ \alpha_i f_i(x) > t\} \,\forall\, i, \\
\widetilde{\Omega}_t &= \left\{ x \in Q \backslash \bigcup_{i=1}^{N} \Omega_{t,i} \,/\, F(x) > t \right\} \quad \text{et} \\
\widetilde{\Omega}_{t,i} &= \left\{ x \in Q;\ t \geq \alpha_i f_i(x) > \frac{t}{N^{\frac{1}{p}}} \right\} \forall\, i.
\end{aligned}$$

On a

$$\begin{aligned}
\{x \in Q \,/\, F(x) > t\} &\subset \left(\bigcup_{i=1}^{N} \Omega_{t,i} \right) \cup \widetilde{\Omega}_t \quad \text{et} \\
\widetilde{\Omega}_t &\subset \bigcup_{i=1}^{N} \widetilde{\Omega}_{t,i}.
\end{aligned}$$

Et

$$\begin{aligned}
& t H_\Delta^\delta \left(\{x \in Q \backslash\, F(x) > t\} \right) \ell(Q)^{-\lambda} \\
& \qquad \leq t H_\Delta^\delta \left(\widetilde{\Omega}_t \right) \ell(Q)^{-\lambda} + \sum_{i=1}^{N} \alpha_i \|f_i\|_{L_*^{1,\lambda}(H_\Delta^\delta)} \\
& \qquad \leq t H_\Delta^\delta \left(\widetilde{\Omega}_t \right) \ell(Q)^{-\lambda} + 1.
\end{aligned} \tag{9}$$

$$\begin{aligned}
H_\Delta^\delta \left(\widetilde{\Omega}_t \right) &\leq H_\Delta^\delta \left(\bigcup_{i=1}^{N} \widetilde{\Omega}_{t,i} \right) \\
&\leq \sum_{i=1}^{N} H_\Delta^\delta \left(\left\{ x \in Q;\ \alpha_i f_i(x) > \frac{t}{N^{\frac{1}{p}}} \right\} \right) \\
&\leq \frac{N^{\frac{1}{p}}}{t} \ell(Q)^{\lambda}.
\end{aligned} \tag{10}$$

Puisque $F(x) > t$ pour tout $x \in \widetilde{\Omega}_t$, on a

$$\begin{aligned} t^p H_\Delta^\delta\left(\widetilde{\Omega}_t\right) &\leq \int_0^{+\infty} H_\Delta^\delta\left(\left\{x \in Q \,/\, F(x)^p \chi_{\widetilde{\Omega}_t}(x) > s\right\}\right) ds \\ &:= \int_Q (F(x))^p \chi_{\widetilde{\Omega}_t}(x) dH_\Delta^\delta \\ &\leq C \sum_{i=1}^N \int_Q (\alpha_i f_i(x))^p \chi_{\widetilde{\Omega}_t}(x) dH_\Delta^\delta. \qquad (11) \end{aligned}$$

Comme pour tout $x \in \widetilde{\Omega}_t$ $\alpha_i f_i(x) \leq t$, on a:

$$\begin{aligned} \int_Q (\alpha_i f_i(x))^p \chi_{\widetilde{\Omega}_t}(x) dH_\Delta^\delta &= \int_0^{t^p} H_\Delta^\delta\left(\left\{x \in Q \,/\, (\alpha_i f_i(x))^p \chi_{\widetilde{\Omega}_t}(x) > s\right\}\right) ds \\ &\leq \int_0^{\alpha_i \frac{t^p}{N^{\frac{1}{p}}}} H_\Delta^\delta\left(\widetilde{\Omega}_t\right) \\ &\quad + \int_{\alpha_i \frac{t^p}{N^{\frac{1}{p}}}}^{t^p} H_\Delta^\delta\left(\left\{x \in Q \,/\, (\alpha_i f_i(x))^p > s\right\}\right) ds \\ &\leq C\, \alpha_i t^{p-1} \frac{p}{p-1} \ell(Q)^\lambda \quad \text{d'après (10).} \end{aligned}$$

En utilisant (9) et l'inégalité ci-dessus on obtient:

$$t H_\Delta^\delta\left(\{x \in Q \,/\, F(x) > t\}\right) \leq C \frac{p}{p-1}.$$

D'où

$$\left\| \left(\sum_{i=1}^N (\alpha_i f_i)^p \right)^{\frac{1}{p}} \right\|_{L_*^{1,\lambda}(H_\Delta^\delta)} \leq C \frac{p}{p-1}.$$

Supposons maintenant que

$$\sum_{i=1}^N \|f_i\|_{L_*^{1,\lambda}(H_\Delta^\delta)} < \infty.$$

En considérant les fonctions

$$\frac{f_i}{\|f_i\|_{L_*^{1,\lambda}(H_\Delta^\delta)}}$$

et les réels

$$\frac{\|f_i\|_{L_*^{1,\lambda}(H_\Delta^\delta)}}{\sum_{i=1}^N \|f_i\|_{L_*^{1,\lambda}(H_\Delta^\delta)}},$$

on a d'après ce qui précède que

$$\left\| \left(\sum_{i=1}^N f_i^p \right)^{\frac{1}{p}} \right\|_{L_*^{1,\lambda}(H_\Delta^\delta)} \leq C \frac{p}{p-1} \sum_{i=1}^N \|f_i\|_{L_*^{1,\lambda}(H_\Delta^\delta)}.$$

- Pour le cas général où $\sum_{i=1}^{\infty} \|f_i\|_{L_*^{1,\lambda}(H^\delta)} < \infty$, on a

$$g_N = \left(\sum_{i=1}^{N}(f_i)^p\right)^{\frac{1}{p}} \uparrow \left(\sum_{i=1}^{+\infty}(f_i)^p\right)^{\frac{1}{p}} = g.$$

La propriété P2 de la Proposition 1 assure que

$$\begin{aligned} tH_\Delta^\delta\left(\{x \in Q \,/\, g(x) > t\}\right)\ell(Q)^{-\lambda} &= \lim_{N\to+\infty} tH_\Delta^\delta\left(\{x \in Q \,/\, g_N(x) > t\}\right)\ell(Q)^{-\lambda} \\ &\leq \lim_{N\to+\infty} \|g_N\|_{L_*^{1,\lambda}(H_\Delta^\delta)} \\ &\leq \lim_{N\to+\infty} C\frac{p}{p-1}\sum_{i=1}^{N}\|f_i\|_{L_*^{1,\lambda}(H_\Delta^\delta)}. \end{aligned}$$

Et donc

$$\left\|\left(\sum_{i=1}^{\infty} f_i^p\right)^{\frac{1}{p}}\right\|_{L_*^{1,\lambda}(H_\Delta^\delta)} \leq C\frac{p}{p-1}\sum_{i=1}^{\infty}\|f_i\|_{L_*^{1,\lambda}(H_\Delta^\delta)}. \qquad \blacksquare$$

Preuve du théorème 2. Sous les hypothèses du Théorème 2, le Théorème 3 donne pour tout i,

$$\|M_\alpha f_i\|_{L_*^{1,\lambda}(H^\delta)} \leq C\|f_i\|_{L^{1,\lambda}(dx)}.$$

Et,

$$\begin{aligned} \left\|\left(\sum_{i=1}^{\infty}(M_\alpha f_i)^p\right)^{\frac{1}{p}}\right\|_{L_*^{1,\lambda}(H^\delta)} &\leq C\frac{p}{p-1}\sum_{i=1}^{\infty}\|M_\alpha f_i\|_{L^{1,\lambda}(dx)} \\ &\leq C\frac{p}{p-1}\sum_{i=1}^{\infty}\|f_i\|_{L^{1,\lambda}(dx)}. \end{aligned}$$

Il est clair que, pour $\lambda = 0$, on obtient l'inégalité (2).

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INEQUALITIES FOR MARKS IN MULTIDIGRAPHS

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Abstract. An r-digraph (multidigraph) D is an orientation of a multigraph that is without loops and contains at most two edges between any pair of distinct vertices. So 1-digraph is an oriented graph, and complete 1-digraph is a tournament. Define $p_v = r(n-1) + d_v^+ - d_v^-$, the mark ($r$-score) of a vertex v in an r-digraph D, where d_v^+ and d_v^-, respectively denote the outdegree and indegree of v and n is the number of vertices in D. In this paper, we obtain some stronger inequalities for marks in r-digraphs.

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1. Introduction

A tournament is an orientation of a complete simple graph. The score s_{v_i} (or simply, s_i) of a vertex v_i in a tournament is the outdegree of v_i. The score sequence of a tournament is formed by listing the vertex scores in non-decreasing order. The following result of Landau [3] gives necessary and sufficient conditions for a sequence of non-negative integers to be the score sequence of some tournament.

Theorem 1.1. *A sequence $[s_i]_1^n$ of nonnegative integers in non-decreasing order is the score sequence of some tournament if and only if*

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2}, \textit{ for } 1 \leq k \leq n,$$

with equality when $k = n$.

With the marking system, the mark p_v of a vertex v in a tournament is given by $p_v = 2s_v + n - 1$, and Landaus conditions become

$$\sum_{i=1}^{k} p_i \geq k(n + k - 2), \text{ for } 1 \leq k \leq n,$$

with equality for $k = n$.

An oriented graph is a digraph with no symmetric pairs of directed arcs and without loops. Avery [1] defined a_{v_i} (or, simply, a_i) $= n - 1 + d_v^+ - d_v^-$, the score of a vertex v_i in an oriented graph, where d_v^+ and d_v^-, respectively denote the outdegree and indegree of v_i and n is the number of vertices. The score sequence of an oriented graph is formed by listing the vertex scores in non-decreasing order. The following result is due to Avery [1].

Theorem 1.2. *A sequence $[a_i]_1^n$ of non-negative integers in non-decreasing order is the score sequence of some oriented graph if and only if*

$$\sum_{i=1}^{k} a_i \geq k(k-1), \textit{ for } 1 \leq k \leq n,$$

with equality when $k = n$.

Once again, with the marking system, the mark p_v of a vertex v in an oriented graph is given by $p_v = a_v + n - 1$, and Averys conditions become

$$\sum_{i=1}^{k} p_i \geq k(n + k - 2), \text{ for } 1 \leq k \leq n,$$

with equality for $k = n$.

A digraph D is semicomplete if for any pair of vertices $u \neq v$ in D, there is an arc from u to v or an arc v to u (or both). The following necessary and sufficient conditions for a non-decreasing sequence of integers to be the score sequence for a semicomplete digraph is given by Reid and Zhang [8].

Theorem 1.3. *A sequence $[s_i]_1^n$ of integers in non-decreasing order is the score sequence of some semicomplete digraph if and only if*

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2} \quad \textit{and} \quad s_k \leq n - 1, \quad \textit{for all } k,\ 1 \leq k \leq n.$$

An r-digraph (multidigraph) D is an orientation of a multigraph that is without loops and contains at most r edges between any pair of distinct vertices. Let D be an r-digraph with vertex set $V = \{v_1, v_2, ..., v_n\}$, and let d_v^+ and d_v^-, respectively denote the outdegree and indegree of a vertex v. Define $p_v = r(n-1) + d_v^+ - d_v^-$, the mark ($r$-score) of v, so that $0 \leq p_v < 2r(n-1)$. The sequence $P = [p_i]_1^n$, where $p_i = p_{v_i}$, in nondecreasing order is the mark sequence of D. An r-digraph can be interpreted as the result of a competition in which the participants play each other at most r times, with an arc from u to v if and only u defeats v. A player receives two points for each win, and one point for each tie (draw). With this marking system, player v obtains a total of p_v points. A sequence P of non-negative integers in nondecreasing order is said to be realizable if there exists an r-digraph with mark sequence P. The following existence criteria for realizability is due to Pirzada and Samee [5]. Various results on marks in digraphs can be found in [6], [7].

Theorem 1.4. *A sequence $[p_i]_1^n$ of non-negative integers in non-decreasing order is the mark sequence of some r-digraph if and only if*

$$\sum_{i=1}^{k} p_i \geq 2k(k-1), \quad \textit{for } 1 \leq k \leq n,$$

with equality when $k = n$.

Some stronger inequalities for scores in tournaments are given by Brualdi and Shen [2], and for scores in oriented graphs are given by Pirzada and Samee [4].

2. Stronger Inequalities

A regular r-digraph on n vertices is one whose all vertices have marks $2r(n-1)$. The converse D' of an r-digraph D is obtained by reversing each arc of D. If u and v are vertices in an r-digraph, we denote by $u(x-y)v$ to mean that there are x arcs directed from u to v and y arcs directed from v to u. Clearly, $0 \leq x, y \leq 2$ and $0 \leq x + y \leq r$. A triple in an r-digraph is an induced r-digraph with three vertices and is of the form $u(x_1 - x_2)v(y_1 - y_2)w(z_1 - z_2)u$, where for $1 \leq i \leq r$, $0 \leq x_i, y_i, z_i \leq r$, and $0 \leq \sum_{i=1}^{2} x_i, \sum_{i=1}^{2} y_i, \sum_{i=1}^{2} z_i \leq r$. In an r-digraph, a 1-triple is an induced 1-subdigraph with three vertices. An 1-triple is said to be transitive if it is of the form $u(1-0)v(1-0)w(0-1)u$, or $u(1-0)v(0-1)w(0-0)u$, or $u(1-0)v(0-0)w(0-1)u$, or $u(1-0)v(0-0)w(0-0)u$, or $u(0-0)v(0-0)w(0-0)u$, otherwise it is intransitive. An r-digraph is said to be transitive if every of its 1-triple is transitive. The inequalities given below in Theorems 2.1, 2.2, 2.3, 2.4 are the generalizations of the inequalities on scores in tournaments due to Brualdi and Shen [2].

The following result gives a lower bound for $\sum_{i \in I} p_i$.

Theorem 2.1. *A sequence $P = [p_i]_1^n$ of non-negative integers in non-decreasing order is a mark sequence of an r-digraph if and only if for every subset $I \subseteq [n] = \{1, 2, ..., n\}$,*

$$\sum_{i\in I} p_i \geq r \sum_{i\in I}(i-1) + r\binom{|I|}{2} \tag{2.1.1}$$

with equality when $|I| = n$.

Proof. *Sufficiency.* Let the sequence $P = [p_i]_1^n$ of non-negative integers in non-decreasing order satisfy equation (2.1.1). Now, for any $I \subseteq [n]$, we have

$$\sum_{i\in I}(i-1) \geq \sum_{i=1}^{|I|}(i-1) = \binom{|I|}{2}.$$

Therefore, from equation (2.1.1), we have

$$\sum_{i\in I} p_i \geq r\sum_{i\in I}(i-1) + r\binom{|I|}{2} \geq r\binom{|I|}{2} + r\binom{|I|}{2} = 2r\binom{|I|}{2}.$$

Hence, by Theorem 1.4, P is a mark sequence.

Necessity. Assume that $P = [p_i]_1^n$ is a mark sequence of some r-digraph. For any subset $I \subseteq [n]$, define

$$f(I) = \sum_{i\in I} p_i - r\sum_{i\in I}(i-1) - r\binom{|I|}{2}.$$

Claim $I = \{i : 1 \leq i \leq |I|\}$. If not, then there exists $i \notin I$ and $j \in I$ such that $j = i + 1$. So, $p_i \leq p_j$.

For $j \in I$, we have

$$\begin{aligned} f(I) &= \sum_{t\in I} p_t - r\sum_{t\in I}(t-1) - r\tbinom{|I|}{2} \\ &= \sum_{\substack{t\in I\\ j\notin I}} p_t + p_j - r\left(\sum_{\substack{t\in I\\ j\notin I}}(t-1) + (j-1)\right) - r\tbinom{|I|}{2}. \end{aligned}$$

Therefore

$$\begin{aligned} f(I) - f(I - \{j\}) &= p_j - r(j-1) - r\tbinom{|I|}{2} + r\tbinom{|I|-1}{2} \\ &= p_j - r(j-1) - r(|I|-1) = p_j - r(j + |I| - 2). \end{aligned}$$

Since $f(I) - f(I - \{j\}) < 0$, therefore $p_j - r(j + |I| - 2) < 0$.

Again $f(I \cup \{i\}) = \sum_{i\in I} p_t + p_i - r\left(\sum_{t\in I}(t-1) + (i-1)\right) - r\tbinom{|I|+1}{2}$.

So $f(I\cup\{i\})-f(I)=p_i-r(i-1)-r\binom{|I|+1}{2}+r\binom{|I|}{2}=p_i-r(i+|I|-1)$.
As $f(I\cup\{i\})-f(I)\geq 0$, therefore $p_i-r(i+|I|)-1)\geq 0$.
Thus $p_j<r(j+|I|-2)$ and $p_i\geq r(i+|I|-1)$.
Therefore $r(i+|I|-1)\leq p_i\leq p_j<r(j+|I|-2)$.
Since $j=i+1$, therefore $r(i+|I|-1)<r(i+1+|I|-2)$.
That is, $2(i+|I|-1)<2(i+|I|-1)$, which is a contradiction.
Hence

$$\begin{aligned} f(I) &= \sum_{i=1}^{|I|}p_i-r\sum_{i=1}^{|I|}(i-1)-r\tbinom{|I|}{2}=\sum_{i=1}^{|I|}p_i-r\tbinom{|I|}{2}-r\tbinom{|I|}{2} \\ &\geq r|I|(|I|-1)-2r\tbinom{|I|}{2}=0. \qquad \text{(by Theorem 1.4)} \end{aligned}$$

Thus

$$\sum_{i\in I}p_i-r\sum_{i\in I}(i-1)-r\binom{|I|}{2}\geq 0,$$

that is ,

$$\sum_{i\in I}p_i\geq r\sum_{i\in I}(i-1)+r\binom{|I|}{2}.$$

This proves the necessity. ■

We note that equality can occur often in equation (2.1.1). For example, in the transitive r-digraph of order n with mark sequence $[0,2r,4r,...,2r(n-1)]$, and in the regular r-digraph of order n with mark sequence $[r(n-1),r(n-1),...,r(n-1)]$. We further observe that Theorem 2.1 is best possible, since for any real $\varepsilon>0$, the inequality

$$\sum_{i\in I}p_i\geq(1+\varepsilon)r\sum_{i\in I}(i-1)+(1-\varepsilon)r\binom{|I|}{2}$$

fails for some I, and some r-digraphs. This can been seen, for example, in the transitive r-digraph of order n with mark sequence $[0,2r,4r,...,2r(n-1)]$, and in the regular r-digraph of order n with mark sequence $[r(n-1),r(n-1),...,r(n-1)]$.

The next result gives a set of upper bounds for $\sum_{i\in I}p_i$ and is equivalent to the set of lower bounds for $\sum_{i\in I}p_i$ in Theorem 2.1.

Theorem 2.2. *A sequence $P=[p_i]_1^n$ of non-negative integers in non-decreasing order is a mark sequence of an r-digraph if and only if for every subset $I\subseteq[n]=\{1,2,...,n\}$,*

$$\sum_{i\in I}p_i\leq r\sum_{i\in I}(i-1)+\frac{1}{2}\,r|I|(2n-|I|-1),$$

with equality when $|I|=n$.

Proof. We have $[n] = \{1, 2, ..., n\}$. Let $J = [n] - I$, so that $I + J = [n]$ and $|J| + |I| = n$. Therefore, by Theorem 2.1, P is a mark sequence if and only if

$$\sum_{i \in [n]} p_i = rn(n-1) \quad \text{and} \quad \sum_{i \in J} p_i \geq r \sum_{i \in J} (i-1) + r\binom{|J|}{2}$$

if and only if

$$\sum_{i \in I} p_i + \sum_{i \in J} p_i = rn(n-1) \;\; \text{and} \;\; \sum_{i \in J} p_i \geq r \sum_{i \in J} (i-1) + r\binom{|J|}{2}$$

if and only if

$$\begin{aligned}
\sum_{i \in I} p_i &= rn(n-1) - \sum_{i \in J} p_i \leq rn(n-1) - \left(r \sum_{i \in J} (i-1) \right) + r\binom{|J|}{2} \\
&= rn(n-1) - \left(r\, \frac{n(n-1)}{2} - r \sum_{i \in I} (i-1) \right) + r\tbinom{n-|I|}{2} \\
&\qquad \left(\text{because } r \sum_{i \in I} (i-1) + r \sum_{i \in J} (i-1) = r\tbinom{n}{2} \text{ and } |I| + |J| = n \right) \\
&= rn(n-1) - r\, \frac{n(n-1)}{2} + r \sum_{i \in I} (i-1) + \frac{r}{2}\, (n - |I|)(n - |I| - 1) \\
&= r \sum_{i \in I} (i-1) + \frac{r}{2}\, |I|(2n - |I| - 1),
\end{aligned}$$

which proves the result. ∎

We now have the following results.

Theorem 2.3. *If $P = [p_i]_1^n$ is a mark sequence of an r-digraph, then for each i,*

$$r(i-1) \leq p_i \leq r(n+i-2).$$

Proof. Let $I = \{i\}$ in Theorem 2.1 and Theorem 2.2. Then

$$\sum_{i \in I} p_i \geq r \sum_{i \in I} (i-1) + r\binom{|I|}{2}$$

implies that

$$p_i \geq r(i-1),$$

and

$$\sum_{i \in I} p_i \leq r \sum_{i \in I} (i-1) + \frac{r}{2}\, |I|(2n - |I| - 1)$$

implies that

$$p_i \leq r(n+i-2).$$

Therefore

$$r(i-1) \leq p_i = r(n+i-2).$$

Second proof. We first show that $r(i-1) \leq p_i$. Suppose on the contrary that $p_i < r(i-1)$. Then, for every $k < i$, we have $p_k \leq p_i < r(i-1)$. That is, $p_1 < r(i-1)$, $p_2 < r(i-1)$,...,$p_i < r(i-1)$. Adding these inequalities, we have $\sum_{k=1}^{i} p_k < ri(i-1)$, which is a contradiction to Theorem 1.4. Therefore $r(i-1) \leq p_i$.

The second inequality is dual to the first. In the converse r-digraph with mark sequence $P' = [p'_i]_1^n$, we have

$$p'_{n-i+1} \geq r((n-i+1)-1) = r(n-i). \qquad \text{(by the first inequality)}.$$

But $p_i = 2r(n-1) - p'_{n-i+1}$. So $p_i \leq 2r(n-1) - r(n-i) = r(n+i-2)$. Therefore $p_i \leq r(n+i-2)$. Hence the result. ∎

For any integers r and s with $r \leq s$, let $[r,s]$ denotes the set of all integers between r and s.

Theorem 2.4. *Let $P = [p_i]_1^n$ be a mark sequence of an r-digraph. If*

$$\sum_{i \in I} p_i = r \sum_{i \in I} (i-1) + r \binom{|I|}{2}, \tag{2.4.1}$$

for some $I \subseteq [n]$, then one of the following holds.

(a) *$I = [1, |I|]$ and $\sum_{i=1}^{|I|} p_i = r|I|(|I|-1)$.*

(b) *$I = [t, t+|I|-1]$ for some t, $2 \leq t \leq n-|I|+1$,*

$$\sum_{i=1}^{t+|I|-1} p_i = r(t+|I|-1)(t+|I|-2)$$

and $p_i = r(t+|I|-2)$ for all $i \leq t+|I|-1$.

(c) *$I = [1,m] \cup [m+t, t+|I|-1]$ for some r and t such that $1 \leq m \leq |I|-1$ and $2 \leq t \leq n-|I|+1$,*

$$\sum_{i=1}^{m} p_i = rm(m-1), \quad \sum_{i=1}^{t+|I|-1} p_i = r(t+|I|-1)(t+|I|-2)$$

and $p_i = 2(m+t+|I|-2)$ for all i, $m+1 \leq i \leq t+|I|-1$.

Theorem 2.5. *If $P = [p_i]_1^n$ is a mark sequence of an r-digraph, then*

(a) $\sum_{i=1}^{t} p_i^2 \geq \sum_{i=1}^{t} (2rt - 2r - p_i)^2$, *for* $1 \leq t \leq n$, *with equality when* $t = n$.

(b) *For* $1 < g < \infty$, $\frac{1}{g} + \frac{1}{h} = 1$, $\sum_{i=1}^{t} p_i^g \geq t(rt - r)^g$, *where* $1 \leq t \leq n$, *with equality when* $t = n$ *and* $p_1 = p_2 = \cdots = p_t$.

Proof. (a). By Theorem 1.4, we have

$$rt(t-1) \leq \sum_{i=1}^{t} p_i, \text{ for } 1 \leq t \leq n \text{ with equality when } t = n$$

or

$$\sum_{i=1}^{t} p_i^2 + 2(rt - 2r)rt(t-1) \leq \sum_{i=1}^{t} p_i^2 + 2(2rt - 2r)\sum_{i=1}^{t} p_i,$$
$$\text{for } 1 \leq t \leq n \text{ with equality when } t = n,$$

or

$$\sum_{i=1}^{t} p_i^2 + t(2rt - 2r)^2 - 2(2rt - 2r)\sum_{i=1}^{t} p_i \leq \sum_{i=1}^{t} p_i^2,$$
$$\text{for } 1 \leq t \leq n \text{ with equality when } t = n,$$

or

$$\begin{aligned} p_1^2 + \cdots + p_t^2 &+ \underbrace{(2rt - 2r)^2 + \cdots + (2rt - 2r)^2}_{k-\text{times}} \\ &- 2(2rt - 2r)p_1 - \cdots - 2(2rt - 2r)p_t \leq \sum_{i=1}^{t} p_i^2, \end{aligned}$$
$$\text{for } 1 \leq t \leq n \text{ with equality when } t = n,$$

or

$$(2rt - 2r - p_1)^2 + \cdots + (2rt - 2r - p_t)^2 \leq \sum_{i=1}^{t} p_i^2, \text{ for } 1 \leq t \leq n \text{ with equality when } t = n,$$

or

$$\sum_{i=1}^{t} (2rt - 2r - p_i)^2 \leq \sum_{i=1}^{t} p_i^2, \text{ for } 1 \leq t \leq n \text{ with equality when } t = n.$$

(b) Again, by Theorem 1.4, we have

$$\begin{aligned}
rt(t-1) &\leq \sum_{i=1}^{t} p_i, \text{ for } 1 \leq t \leq n \text{ with equality when} t=n, \\
&= \sum_{i=1}^{t} p_t \cdot 1, \text{ for } 1 \leq t \leq n \text{ with equality when} t=n, \\
&\leq \left(\sum_{i=1}^{t} p_i^g\right)^{\frac{1}{g}} \left(\sum_{i=1}^{t} 1^h\right)^{\frac{1}{h}}, \\
&\qquad \text{for } 1 \leq t \leq n \text{ with equality when } t=n \text{ and } p_1=p_2=\cdots=p_k, \\
&\qquad \text{(by Holders inequality)} \\
&= \left(\sum_{i=1}^{k} p_i^g\right)^{\frac{1}{g}} k^{\frac{1}{h}}, \\
&\qquad \text{for } 1 \leq t \leq n \text{ with equality when } t=n \text{ and } p_1=p_2=\cdots=p_t.
\end{aligned}$$

That is,

$$rt^{1-\frac{1}{h}}(t-1) \leq \left(\sum_{i=1}^{t} p_i^g\right)^{\frac{1}{g}},$$

for $1 \leq t \leq n$ with equality when $t=n$ and $p_1=p_2=\cdots=p_t$.

Hence

$$\sum_{i=1}^{t} p_i^g \geq t(rt-r)^g,$$

for $1 \leq t \leq n$ with equality when t=n, and p_1=$p_2=\cdots=p_t$, since $\dfrac{1}{g}+\dfrac{1}{h}=1$. ■

Theorem 2.6. *Let D be an r-digraph on n vertices with mark sequence $[p_i]_1^n$. Then, for each $t \geq 1$, there exists an r-digraph on tn vertices with mark sequence $[p_i+r(t-1)n]_1^{tn}$.*

Proof. For each $i, 1 \leq i \leq t$, let D^i be a copy of D with n vertices. Define an r-digraph D_1 as

$$D_1 = D^1 \cup D^2 \cup \cdots \cup D^t,$$

such that vertices and arcs of D_1 are that of D^i, and let there be no arc between the vertices of D^i and D^j $(i \neq j)$. Then D_1 is an r-digraph on tn vertices with mark sequence $[p_i+r(t-1)n]_1^{tn}$. ■

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IMPROVED EXPONENTIAL ESTIMATOR FOR POPULATION VARIANCE USING TWO AUXILIARY VARIABLES

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Abstract. In this paper, exponential ratio and exponential product type estimators using two auxiliary variables are proposed for estimating unknown population variance S_y^2. Problem is extended to the case of two-phase sampling. Theoretical results are supported by an empirical study.

AMS Subject Classification: 62D05.

Keywords: auxiliary information, exponential estimator, mean squared error.

1. Introduction

It is common practice to use the auxiliary variable for improving the precision of the estimate of a parameter. Out of many ratio and product methods of estimation are good examples in this context. When the correlation between the study variate and the auxiliary variate is positive (high) ratio method of estimation is quite effective. On the other hand, when this correlation is negative (high) product method of estimation can be employed effectively. Let y and (x, z) denote the study variate and auxiliary variates taking the values y_i and (x_i, z_i) respectively, on the unit U_i $(i = 1, 2, ..., N)$, where x is positively correlated with y and z is negatively correlated with y. To estimate $S_y^2 = \frac{1}{(N-1)}\sum_{N}^{i=1}(y_i - \bar{y})^2$, it is assumed that $S_x^2 = \frac{1}{(N-1)}\sum_{N}^{i=1}(x_i - \bar{X})^2$ and $S_z^2 = \frac{1}{(N-1)}\sum_{N}^{i=1}(z_i - \bar{Z})^2$ are known. Assume that population size N is large so that the finite population correction terms are ignored.

Assume that a simple random sample of size n is drawn without replacement (SRSWOR) from U. The usual unbiased estimator of S_y^2 is

$$s_y^2 = \frac{1}{(N-1)}\sum_{n}^{i=1}(y_i - \bar{y})^2, \tag{1.1}$$

where $\bar{y} = \frac{1}{n}\sum_{n}^{i=1} y_i$ is the sample mean of y.

When the population variance $S_x^2 = \frac{1}{(N-1)}\sum_{n}^{i=1}(x_i - \bar{X})^2$ is known, Isaki (1983) proposed a ratio estimator for S_y^2 as

$$t_k = s_y^2\ \frac{S_x^2}{s_x^2}, \tag{1.2}$$

where $s_x^2 = \frac{1}{(n-1)}\sum_{n}^{i=1}(x_i - \overline{X})^2$ is an unbiased estimator of S_x^2.

Upto the first order of approximation, the variance of S_y^2 and MSE of t_k (ignoring the finite population correction (fpc) term) are respectively given by

$$\mathrm{var}(s_y^2) = \left(\frac{S_y^4}{n}\right)[\partial_{400} - 1] \tag{1.3}$$

$$\mathrm{MSE}(t_k) = \left(\frac{S_y^4}{n}\right)[\partial_{400} + \partial_{040} - 2\partial_{220}] \tag{1.4}$$

where

$$\delta_{pqr} = \frac{\mu_{pqr}}{(\mu_{200}^{p/2}\mu_{020}^{q/2}\mu_{002}^{r/2})},$$

$$\mu_{pqr} = \frac{1}{N}\sum_{i=1}^{N}(y_i - \overline{Y})^p(x_i - \overline{X})^q(z_i - \overline{Z})^r;\ p, q, r \text{ being the non-negative integers.}$$

Following Bahl and Tuteja (1991), we propose exponential ratio type and exponential product type estimators for estimating population variance S_y^2 as

$$t_1 = s_y^2 \exp\left[\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right] \tag{1.5}$$

$$t_2 = s_y^2 \exp\left[\frac{s_z^2 - S_z^2}{s_z^2 + S_z^2}\right] \tag{1.6}$$

2. Bias and MSE of proposed estimators

To obtain the bias and MSE of t_1, we write

$$s_y^2 = S_y^2(1+e_0), \quad s_x^2 = S_x^2(1+e_1),$$

such that

$$E(e_0) = E(e_1) = 0$$

and

$$E(e_0^2) = \frac{1}{n}(\partial_{400} - 1), \;\; E(e_1^2) = \frac{1}{n}(\partial_{040} - 1), \;\; E(e_0e_1) = \frac{1}{n}(\partial_{220} - 1).$$

After simplification, we get the bias and MSE of t_1 as

$$B(t_1) \cong \frac{S_y^2}{n}\left[\frac{\partial_{040}}{8} - \frac{\partial_{220}}{2} + \frac{3}{8}\right], \tag{2.1}$$

$$\mathrm{MSE}(t_1) \cong \frac{S_y^4}{n}\left[\partial_{400} + \frac{\partial_{040}}{4} - \partial_{220} + \frac{1}{4}\right]. \tag{2.2}$$

To obtain the bias and MSE of t_2, we write

$$s_y^2 = S_y^2(1+e_0), \quad s_z^2 = S_z^2(1+e_2),$$

such that

$$E(e_0) = E(e_2) = 0, \quad E(e_2^2) = \frac{1}{n}(\partial_{004} - 1), \quad E(e_0e_2) = \frac{1}{n}(\partial_{202} - 1).$$

After simplification, we get the bias and MSE of t_2 as

$$B(t_2) \cong \frac{S_y^4}{n}\left[\frac{\partial_{040}}{8} - \frac{\partial_{220}}{2} + \frac{3}{8}\right], \tag{2.3}$$

$$\mathrm{MSE}(t_2) \cong \frac{S_y^2}{n}\left[\partial_{400} + \frac{\partial_{040}}{4} - \partial_{220} + \frac{1}{4}\right]. \tag{2.4}$$

3. Improved estimator

Following Kadilar and Cingi (2006) and Singh et.al. (2007), we propose an improved estimator for estimating population variance S_y^2 as

$$t = s_y^2\left[\alpha\exp\left\{\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right\} + (1-\alpha)\exp\left\{\frac{s_z^2 - S_z^2}{s_z^2 + S_z^2}\right\}\right], \tag{3.1}$$

where α is a real constant to be determined such that the MSE of t is minimum.

Expressing t in terms of e's, we have

$$t = S_y^2(1+e_0)\left[\alpha\exp\left\{-\frac{e_1}{2}\left(1+\frac{e_1}{2}\right)^{-1}\right\} + (1-\alpha)\exp\left\{\frac{e_2}{2}\left(1+\frac{e_2}{2}\right)^{-1}\right\}\right]. \tag{3.2}$$

Expanding the right hand side of (3.2) and retaining terms up to second power of e's, we have

$$\begin{aligned} t \cong S_y^2 \Bigg[1 + e_0 + \frac{e_2}{2} + \frac{e_2^2}{8} + \frac{e_0e_2}{2} + \alpha\left(e\frac{e_1}{2} + \frac{e_1^2}{8}\right) - \alpha\left(\frac{e_2}{2} + \frac{e_2^2}{8}\right) \\ + e_0\alpha\left(-\frac{e_1}{2} + \frac{e_1^2}{8}\right)\alpha e_0\left(\frac{e_2}{2} + \frac{e_2^2}{8}\right)\Bigg]. \end{aligned} \tag{3.3}$$

Taking expectations of both sides of (3.3) and then subtracting S_y^2 from both sides, we get the bias of the estimator t, up to the first order of approximation, as

$$\begin{aligned} B(t) = \frac{S_y^2}{n}\Bigg[\frac{\alpha}{8}(\partial_{040} - 1) + \frac{(1-\alpha)}{8}(\partial_{004} - 1) + \frac{(1-\alpha)}{2}(\partial_{202} - 1) \\ - \frac{\alpha}{2}(\partial_{220} - 1)\Bigg]. \end{aligned} \tag{3.4}$$

From (3.3), we have

$$(t - S_y^2) \cong S_7^2\left[e_0 - \frac{\alpha e_1}{2} + \frac{(1-\alpha)}{2}e_2\right]. \tag{3.5}$$

Squaring both the sides of (3.5) and then taking expectation, we get MSE of the estimator t, up to the first order of approximation, as

$$\begin{aligned} \text{MSE}(t) \cong \frac{S_y^4}{n}\Bigg[(\partial_{400} - 1) + \frac{\alpha^2}{4}(\partial_{040} - 1) + \frac{(1-\alpha^2)}{4}(\partial_{004} - 1) \\ + \alpha(\partial_{220} - 1) + (1-\alpha)(\partial_{202} - 1) - \frac{\alpha(1-\alpha)}{2}(\partial_{022} - 1)\Bigg]. \end{aligned} \tag{3.6}$$

Minimization of (3.6) with respect to αyields its optimum value as

$$\alpha = \frac{\{\partial_{004} + 2(\partial_{220} + \partial_{202}) + \partial_{022} - 6\}}{(\partial_{040} + \partial_{004} + 2\partial_{022} - 4)} = \alpha_0(say). \tag{3.7}$$

Substitution of α_0 from (3.7) into (3.6) gives minimum value of MSE of t.

4. Proposed estimators in two-phase sampling

In certain practical situations when S_x^2 is not known a priori, the technique of two-phase or double sampling is used. This scheme requires collection of information on x and z the first phase sample s of size n $(n < N)$ and on y for the second phase sample s of size n $(n < n)$ from the first phase sample.

The estimators t_1, t_2 and t in two-phase sampling will take the following form, respectively

$$t_{1d} = s_y^2 \exp\left[\frac{s_x^{\prime 2} - s_x^2}{s_x^{\prime 2} + s_x^2}\right] \tag{4.1}$$

$$t_{2d} = s_z^2 \exp\left[\frac{s_z^{\prime 2} - s_z^2}{s_z^{\prime 2} + s_z^2}\right] \tag{4.2}$$

$$t_d = s_y^2 \left[k \exp\left\{\frac{s_x^{\prime 2} - s_x^2}{s_x^{\prime 2} + s_x^2}\right\} + (1-k) \exp\left\{\frac{s_z^{\prime 2} - s_z^2}{s_z^{\prime 2} + s_z^2}\right\}\right] \tag{4.3}$$

To obtain the bias and MSE of t_{1d}, t_{2d} and t_d, we write

$$\begin{aligned} s_y^2 &= S_y^2(1+e_0), \quad & s_x^2 &= S_x^2(1+e_1), \quad & s_x^{\prime 2} &= S_x^2(1+e_1^{\prime}), \\ & & s_z^2 &= S_z^2(1+e_2), \quad & s_z^{\prime 2} &= S_z^2(1+e_2^{\prime}), \end{aligned}$$

where

$$s_x^{\prime 2} = \frac{1}{(n^{\prime}-1)}\sum_{i=1}^{n^{\prime}}(x_i - \bar{x}^{\prime})^2, \quad s_z^{\prime 2} = \frac{1}{(n^{\prime}-1)}\sum_{i=1}^{n^{\prime}}(z_i - \bar{z}^{\prime})^2,$$

$$\bar{x}^{\prime} = \frac{1}{n^{\prime}}\sum_{i=1}^{n^{\prime}} x_i, \quad \bar{z}^{\prime} = \frac{1}{n^{\prime}}\sum_{i=1}^{n^{\prime}} z_i.$$

Also,

$$\begin{aligned} E(e_1^{\prime}) &= E(e_2^{\prime}) = 0, \\ E(e_1^{\prime 2}) &= \frac{1}{n^{\prime}}(\partial_{040} - 1), \\ E(e_2^{\prime 2}) &= \frac{1}{n^{\prime}}(\partial_{004} - 1), \\ E(e_2^{\prime} e_2^{\prime}) &= \frac{1}{n^{\prime}}(\partial_{220} - 1). \end{aligned}$$

Expressing t_{1d}, t_{2d} and t_d in terms of es and following the procedure explained in Sections 2 and 3, we get the MSE of these estimators, respectively as

$$\begin{aligned}\mathrm{MSE}(t_{1d}) \;&\cong\; S_y^4\left[\frac{1}{n}(\partial_{400}-1)+\frac{1}{4}\left(\frac{1}{n}-\frac{1}{n'}\right)(\partial_{040}-1)\right.\\ &\left.+\left(\frac{1}{n'}-\frac{1}{n}\right)(\partial_{220}-1)\right].\end{aligned} \tag{4.4}$$

$$\begin{aligned}\mathrm{MSE}(t_{2d}) \;&\cong\; S_y^4\left[\frac{1}{n}(\partial_{400}-1)+\frac{1}{4}\left(\frac{1}{n}-\frac{1}{n'}\right)(\partial_{040}-1)\right.\\ &\left.-\left(\frac{1}{n'}-\frac{1}{n}\right)(\partial_{220}-1)\right].\end{aligned} \tag{4.5}$$

$$\begin{aligned}\mathrm{MSE}(t_{d}) \;&\cong\; S_y^4\left[\frac{1}{n}(\partial_{400}-1)+\frac{k^2}{4}\left(\frac{1}{n}-\frac{1}{n'}\right)(\partial_{040}-1)\right.\\ &+\frac{(k^2-1)}{4}\left(\frac{1}{n}-\frac{1}{n'}\right)(\partial_{004}-1)\\ &+k\left(\frac{1}{n}-\frac{1}{n'}\right)(\partial_{220}-1)+(k-1)\left(\frac{1}{n'}-\frac{1}{n}\right)(\partial_{202}-1)\\ &\left.-\frac{k(k-1)}{2}\left(\frac{1}{n'}-\frac{1}{n}\right)(\partial_{022}-1)\right].\end{aligned} \tag{4.6}$$

Minimization of (4.6) with respect to k yields its optimum value as

$$k=\frac{\{\partial_{004}+2(\partial_{220}-1)+\partial_{022}-6\}}{(\partial_{040}+\partial_{004}+2\partial_{022}-4)}=k_0(say). \tag{4.7}$$

Substitution of k_0 from (4.7) to (4.6) gives minimum value of MSE of t_d.

5. Empirical study

To illustrate the performance of various estimators of S_y^2, we consider the data given in Murthy (1967, p. 226). The variates are:

y: output, x: number of workers, z: fixed capital,

$N = 80$, $n' = 25$, $n = 10$.

$\partial_{400} = 2.2667$, $\partial_{040} = 3.65$, $\partial_{004} = 2.8664$,

$\partial_{220} = 2.3377$, $\partial_{202} = 2.2208$, $\partial_{400} = 3.14$.

The percent relative efficiency (PRE) of various estimators of S_y^2 with respect to conventional estimator s_y^2 has been computed and displayed in Table 5.1.

Table 5.1. PRE **of** s_y^2, t_1, t_2 **and** $\min.\mathrm{MSE}(t)$ **with respect to** s_y^2

Estimator	$\mathrm{PRE}(\cdot, s_y^2)$
s_y^2	100
t_1	214.35
t_2	42.90
t	215.47

In Table 5.2, PRE of various estimators of s_y^2 in two-phase sampling with respect to S_y^2 are displayed.

Table 5.2. PRE **of** s_y^2, t_{1d}, t_{2d} **and** $\min.\mathrm{MSE}(t_d)$ **with respect to** s_y^2

Estimator	$\mathrm{PRE}(\cdot, s_y^2)$
s_y^2	100
t_{1d}	1470.76
t_{2d}	513.86
t	513.86

6. Conclusion

From Tables 5.1 and 5.2, we infer that the proposed estimators t perform better than a conventional estimator s_y^2 and other mentioned estimators.

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ON SOME CLASSES OF SUBMANIFOLDS SATISFYING CHEN'S EQUALITY IN AN EUCLIDEAN SPACE

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Abstract. We study submanifolds satisfying Chen's equality in an Euclidean space. Firstly, we consider projectively semi-symmetric submanifolds satisfying Chen's equality in an Euclidean space. We also study submanifolds satisfying the condition $P \cdot P = 0$.

Keywords: Chen invariant, Chen's inequality, projectively semi-symmetric manifold, totally geodesic submanifold, minimal submanifold.
2000 Mathematics Subject Classification: 53C40, 53C25, 53C42.

1. Introduction

One of the basic problems in submanifold theory is to find simple relationships between the extrinsic and intrinsic invariants of a submanifold. In [1] and [4], B.Y. Chen established inequalities in this respect, called Chen inequalities. The main extrinsic invariant is the squared mean curvature and the main intrinsic invariants include the classical curvature invariants namely the scalar curvature and the Ricci curvature; and the well known modern curvature invariant namely Chen invariant [2]. In 1993, Chen obtained an interesting basic inequality for submanifolds in a real space form involving the squared mean curvature and the Chen invariant and found several of its applications. This inequality is now well known as Chen's inequality; and in the equality case it is known as Chen's equality.

In [6], Dillen, Petrovic and Verstraelen studied Einstein, conformally flat and semisymmetric submanifolds satisfying Chen's equality in Euclidean spaces. In [8], the first author and M.M. Tripathi studied the same problems for a submanifold of a real space form.

In this paper, we study submanifolds satisfying Chen's equality and the conditions $R \cdot P = 0$ and $P \cdot P = 0$ in an Euclidean space.

The paper is organized as follows. In Section 2, we give some known results about Riemannian submanifolds and Chen's inequality which will be used in the next sections. In Section 3, we study projectively semi-symmetric submanifolds satisfying Chen's equality in an Euclidean space. We also study submanifolds satisfying the condition $P \cdot P = 0$.

2. Chen's inequality

Let M be an n-dimensional submanifold of an $(n+m)$-dimensional Euclidean space $\mathbb{E}^{n+m}$. The Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y) \qquad \text{and} \qquad \widetilde{\nabla}_X N = -A_N X + \nabla_X^{\perp} N$$

for all $X, Y \in TM$ and $N \in T^{\perp}M$, where $\widetilde{\nabla}$, ∇ and $\nabla^{\perp}$ are respectively the Riemannian, induced Riemannian and induced normal connections in $\widetilde{M}$, M and the normal bundle $T^{\perp}M$ of M respectively, and σ is the second fundamental form related to the shape operator A by $\langle \sigma(X,Y), N \rangle = \langle A_N X, Y \rangle$. The equation of Gauss is given by

$$R(X,Y,Z,W) = \langle \sigma(X,W), \sigma(Y,Z) \rangle - \langle \sigma(X,Z), \sigma(Y,W) \rangle \tag{2.1}$$

for all $X, Y, Z, W \in TM$, where R is the curvature tensors of M.

The mean curvature vector H is given by $H = \frac{1}{n}\operatorname{trace}(\sigma)$. The submanifold M is *totally geodesic* in $\mathbb{E}^{m+n}$ if $\sigma = 0$, and *minimal* if $H = 0$ [3].

Let $\{e_1, ..., e_n\}$ be an orthonormal tangent frame field on M. For the plane section $e_i \wedge e_j$ of the tangent bundle TM spanned by the vectors e_i and e_j $(i \neq j)$ the scalar curvature of M is defined by $\kappa = \sum_{i,j=1}^{n} K(e_i \wedge e_j)$ where K denotes the sectional curvature of M. Consider the real function $\inf K$ on M^n defined for every $x \in M$ by

$$(\inf K)(x) := \inf\{K(\pi) \mid \pi \text{ is a plane in } T_x M^n\}.$$

Note that since the set of planes at a certain point is compact, this infimum is actually a minimum.

Lemma 2.1. [1] *Let M, $n \geq 2$, be any submanifold of $\mathbb{E}^{n+m}$. Then*

$$\inf K \geq \frac{1}{2}\left\{\kappa - \frac{n^2(n-2)}{n-1}|H|^2\right\}. \tag{2.2}$$

Equality holds in (2.2) *at a point* x *if and only if with respect to suitable local orthonormal frames* $e_1, ..., e_n \in T_xM^n$, *the Weingarten maps* A_t *with respect to the normal sections* $\xi_t = e_{n+t}$, $t = 1, ..., p$ *are given by*

$$A_1 = \begin{bmatrix} a & 0 & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & 0 & \cdots & 0 \\ 0 & 0 & \mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \mu \end{bmatrix},$$

$$A_t = \begin{bmatrix} c_t & d_t & 0 & \cdots & 0 \\ d_t & -c_t & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad (t > 1), \tag{2.3}$$

where $\mu = a + b$ *for any such frame,* $\inf K(x)$ *is attained by the plane* $e_1 \wedge e_2$.

The inequality (2.2) is well known as Chen's inequality. In case of equality, it is known as Chen's equality. For dimension $n = 2$, the Chen's equality is always true.

Let M be an n-dimensional ($n \geq 3$) submanifold of an Euclidean space $\mathbb{E}^{n+m}$ satisfying Chen's equality. Then, from Lemma 2.1 we immediately have the following

$$K_{12} = ab - \sum_{r=1}^{m}(c_r^2 + d_r^2), \tag{2.4}$$

$$K_{1j} = a\mu, \tag{2.5}$$

$$K_{2j} = b\mu, \tag{2.6}$$

$$K_{ij} = \mu^2, \tag{2.7}$$

$$S(e_1, e_1) = K_{12} + (n-2)a\mu, \tag{2.8}$$

$$S(e_2, e_2) = K_{12} + (n-2)b\mu, \tag{2.9}$$

$$S(e_i, e_i) = (n-2)\mu^2, \tag{2.10}$$

where $i, j > 2$. Furthermore, $R(e_i, e_j)e_k = 0$ if i, j and k are mutually different [6].

Projective curvature tensor of submanifolds satisfying Chen's equality

In this section, we consider projectively semi-symmetric submanifolds satisfying Chen's equality in an Euclidean space. We also consider submanifolds satisfying the condition $P\cdot P=0$.

The *projective curvature tensor* P *of an* n*-dimensional Riemannian manifold* (M,g) is defined by [9]

$$P(X,Y)Z=R(X,Y)Z-\frac{1}{n-1}[S(Y,Z)X-S(X,Z)Y]. \tag{3.1}$$

It is well-known that if the condition $R\cdot P=0$ holds on M, then M is said to be *projectively semi-symmetric.*

So from (2.4)-(2.10) we have the following corollary:

Corollary 3.1. *Let* M *be an* n*-dimensional* $(n\geq 3)$ *submanifold in an Euclidean space satisfying Chen's equality, then*

$$P_{122}=\frac{n-2}{n-1}\left(K_{12}-b\mu)\right)e_1, \tag{3.2}$$

$$P_{133}=\mu\left(a-\frac{n-2}{n-1}\mu\right)e_1, \tag{3.3}$$

$$P_{131}=\frac{1}{n-1}\left(K_{12}-a\mu\right)e_{3,} \tag{3.4}$$

$$P_{233}=\mu\left(b-\frac{n-2}{n-1}\mu\right)e_2, \tag{3.5}$$

$$P_{211}=\frac{n-2}{n-1}\left(K_{12}-a\mu\right)e_2 \tag{3.6}$$

$$P_{232}=\frac{1}{n-1}\left(K_{12}-b\mu\right)e_{3,} \tag{3.7}$$

and

$$P_{ijk}=0 \text{ if } i,j,k \text{ are mutually different.} \tag{3.8}$$

Theorem 3.2. *Let* M *be an* n*-dimensional* $(n\geq 3)$ *submanifold of an Euclidean space* $\mathbb{E}^{n+m}$ *satisfying Chen's equality. If* M *is projectively semi-symmetric then*

(i) M *is totally geodesic, or*

(ii) M *is minimal, or*

(iii) *M is a round hypercone in some totally geodesic subspace $\mathbb{E}^{n+1}$ of $\mathbb{E}^{n+m}$, or*

(iv) *$infK = 0$, or*

(v) *$a = b$, in this case if $n = 3$ then M is totally geodesic, if $n = 4$ then M is a pseudosymmetric hypersurface of $\mathbb{E}^5$ which has a shape operator of the form*

$$A_1 = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 2a & 0 \\ 0 & 0 & 0 & 2a \end{bmatrix}, \tag{3.9}$$

or

(vi) *M is a submanifold in some totally geodesic subspace $\mathbb{E}^{n+m-1}$ which has shape operators of the form (2.3).*

Proof. Assume that the condition $R \cdot P = 0$ holds on M. Then, we can write

$$\begin{aligned} \left(R(e_1, e_3) \cdot P\right)(e_2, e_3, e_1) &= R(e_1, e_3)P(e_2, e_3)e_1 \\ &-P(R(e_1, e_3)e_2, e_3)e_1 - P(e_2, R(e_1, e_3)e_3)e_1 \\ &-P(e_2, e_3)R(e_1, e_3)e_1 = 0 \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \left(R(e_2, e_3) \cdot P\right)(e_1, e_3, e_2) &= R(e_2, e_3)P(e_1, e_3)e_2 \\ &-P(R(e_2, e_3)e_1, e_3)e_2 - P(e_1, R(e_2, e_3)e_3)e_2 \\ &-P(e_1, e_3)R(e_2, e_3)e_2 = 0. \end{aligned} \tag{3.11}$$

Then, using (2.4)-(2.7) and (3.2)-(3.8), we get

$$a\mu \left[b\mu - (n-2)ab + (n-2)\sum_{r=1}^{m}(c_r^2 + d_r^2) \right] = 0 \tag{3.12}$$

and

$$b\mu \left[a\mu - (n-2)ab + (n-2)\sum_{r=1}^{m}(c_r^2 + d_r^2) \right] = 0. \tag{3.13}$$

Case I. If M is totally geodesic, the condition $R \cdot P = 0$ holds trivially.

Case II. If $\mu = 0$ then M is minimal.

Case III. If $\mu \neq 0$ and $a = 0$ then $\mu = b$. Hence, from (3.13), we get $(n-2)\sum_{r=1}^{m}(c_r^2 + d_r^2) = 0$. This gives us $c_r = d_r = 0$. So, by [5], M is a round hypercone in some totally geodesic subspace $\mathbb{E}^{n+1}$ of $\mathbb{E}^{n+m}$.

Case IV. If $\mu \neq 0$ and $b = 0$, then we obtain again the same result in Case III.

Case V. $a, b, \mu \neq 0$, then from (3.12) and (3.13) we obtain $a = b$, or $\mu = 0$, or $K_{12} = 0$. If $\mu = 0$, then M is minimal. If $K_{12} = 0$ then $\inf K = 0$. Assume that $a = b$. Then, from (3.13) we have

$$(4-n)a^2 + (n-2)\sum_{r=1}^{m}(c_r^2 + d_r^2) = 0.$$

In this case, if $n = 3$, then $c_r = d_r = 0$. Hence, M is totally geodesic. If $n = 4$, then $c_r = d_r = 0$, so by Theorem 2.12 of [7], M is a pseudosymmetric hypersurface in some totally geodesic subspace $\mathbb{E}^{n+1}$ of $\mathbb{E}^{n+m}$, which has a shape operator of the form (3.9).

Case VI. If $a = b = 0$, then M is a submanifold in some totally geodesic subspace $\mathbb{E}^{n+m-1}$, which has shape operators of the form (2.3).

This completes the proof of the theorem. ∎

Theorem 3.3. *Let M be an n-dimensional ($n \geq 3$) submanifold of an Euclidean space $\mathbb{E}^{n+m}$ satisfying Chen's equality. If the condition $P \cdot P = 0$ holds on M, then*

(i) *M is minimal, or*

(ii) *M is totally geodesic, or*

(iii) *M is 3-dimensional, or*

(iv) *$a = b$.*

Proof. Since the condition $P \cdot P = 0$ holds on M we have

$$\begin{aligned}(P(e_1,e_3)\cdot P)(e_2,e_3,e_1) &= P(e_1,e_3)P(e_2,e_3)e_1 \\ &\quad - P(P(e_1,e_3)e_2,e_3)e_1 - P(e_2,P(e_1,e_3)e_3)e_1 \\ &\quad - P(e_2,e_3)P(e_1,e_3)e_1 = 0\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}(P(e_2,e_3)\cdot P)(e_1,e_3,e_2) &= P(e_2,e_3)P(e_1,e_3)e_2 \\ &\quad - P(P(e_2,e_3)e_1,e_3)e_2 - P(e_1,P(e_2,e_3)e_3)e_2 \\ &\quad - P(e_1,e_3)P(e_2,e_3)e_2 = 0.\end{aligned} \tag{3.15}$$

So, in view of (3.2)-(3.8), we obtain

$$\mu\left[K_{12} - a\mu\right]\left[\left(a - \frac{n-2}{n-1}\mu\right)(n-2) + \left(b - \frac{n-2}{n-1}\mu\right)\right] = 0 \tag{3.16}$$

and

$$\mu\,[K_{12}-b\mu]\left[\left(b-\frac{n-2}{n-1}\mu\right)(n-2)+\left(a-\frac{n-2}{n-1}\mu\right)\right]=0 \tag{3.17}$$

Case I. If $\mu=0$, then M is minimal.

Case II. If $K_{12}-a\mu=0$, $K_{12}-b\mu=0$, then we obtain $a=b$. Since $K_{12}-a\mu=0$, from (2.4) we get $a^2+\sum_{r=1}^{m}(c_r^2+d_r^2)=0$, which gives us $a=c_r=d_r=0$. Hence, M is totally geodesic.

Case III. If

$$\left(a-\frac{n-2}{n-1}\mu\right)(n-2)+\left(b-\frac{n-2}{n-1}\mu\right)=0$$
$$\left(b-\frac{n-2}{n-1}\mu\right)(n-2)+\left(a-\frac{n-2}{n-1}\mu\right)=0,$$

and $\mu\neq 0$ and $K_{12}-a\mu\neq 0$, then either $n=3$ or $a=b$.

Case IV. If

$$K_{12}-a\mu=0 \text{ and } \left(b-\tfrac{n-2}{n-1}\mu\right)(n-2)+\left(a-\tfrac{n-2}{n-1}\mu\right)=0,$$

then, from (2.4), we get $a=c_r=d_r=0$, which gives us $a=c_r=d_r=0$. Hence, M is totally geodesic.

Case V. If

$$K_{12}-b\mu=0 \text{ and } \left(a-\tfrac{n-2}{n-1}\mu\right)(n-2)+\left(b-\tfrac{n-2}{n-1}\mu\right)=0,$$

then, from (2.4), we get $a=b=c_r=d_r=0$, which gives us M is totally geodesic.

This proves the theorem. ■

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AN APPROACH TO THE APPROXIMATION OF THE INVERSE OF A SQUARE MATRIX BY HE'S HOMOTOPY PERTURBATION METHOD

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Abstract. In this paper, we present an efficient numerical algorithm for approximating the inverse of a square matrix based on homotopy perturbation method. Some numerical illustrations are given to show the efficiency of the algorithm.

1. Introduction

Approximating the inverse of square matrices specially the coefficient matrix of a system of linear equations has widespread applications in applied mathematics. In 1992, Liao [11] employed the basic ideas of homotopy to propose a general method for nonlinear problems, and modified it step by step [12]–[16].

This method has been successfully applied to solve many types of nonlinear problems. Following Liao, an analytic approach based on the same theory in 1998, which is so called "homotopy perturbation method" (HPM), is provided by He [7]–[10], as well as the recent developments[4]–[6].

In most cases, using HPM, gives a very rapid convergence of the solution series, and usually only a few iterations leading to very accurate solutions, specially when the modified one is applied [1], [17].

In this article, HPM is applied to the equation $\mathbf{AX}-\mathbf{I}=\mathbf{0}$ and the convergence of the method is considered under certain conditions.

2. Analysis of the method

Consider the equation

$$\mathbf{AX}-\mathbf{I}=\mathbf{0}, \tag{1}$$

where

$$\mathbf{A}=[a_{ij}],\quad i=1,2,\cdots,n\quad ,j=1,2,\cdots,n.$$

Let

$$\begin{aligned}L(\mathbf{U})&=\mathbf{AU}-\mathbf{I},\\F(\mathbf{U})&=\mathbf{U}-\mathbf{W_0},\end{aligned}$$

we define homotopy $H(\mathbf{U},p)$ by

$$\begin{aligned}H(\mathbf{U},0)&=F(\mathbf{U}),\\H(\mathbf{U},1)&=L(\mathbf{U}),\end{aligned}$$

and typically, a convex homotopy as follows

$$H(\mathbf{U},p)=(1-p)F(\mathbf{U})+pL(\mathbf{U})=\mathbf{0},\tag{2}$$

where F is an operator with known solution W_0. HPM uses the homotopy parameter p as an expanding parameter[4,6] to obtain

$$\mathbf{U}=\mathbf{U_0}+p\mathbf{U_1}+p^2\mathbf{U_2}+\cdots,\tag{3}$$

and for $p\rightarrow 1$, it gives an approximation to the solution of (1) as follows

$$\mathbf{V}=\lim_{p\rightarrow 1}(\mathbf{U_0}+p\mathbf{U_1}+p^2\mathbf{U_2}+\cdots).$$

By substituting (3) in (2) and equating the terms with identical powers of p, we obtain

$$\begin{aligned}p^0:&\quad \mathbf{U}_0-\mathbf{W}_0=\mathbf{0},\\&\quad \mathbf{U}_0=\mathbf{W}_0,\\p^1:&\quad (\mathbf{A}-\mathbf{I})\mathbf{U}_0+\mathbf{U}_1-\mathbf{W}_0-\mathbf{I}=\mathbf{0},\\&\quad \mathbf{U}_1=\mathbf{I}-(\mathbf{A}-\mathbf{I})\mathbf{U}_0+\mathbf{W}_0,\\p^2:&\quad (\mathbf{A}-\mathbf{I})\mathbf{U}_1+\mathbf{U}_2=\mathbf{0},\\&\quad \mathbf{U}_2=-(\mathbf{A}-\mathbf{I})\mathbf{U}_1\end{aligned}$$

and in general

$$\mathbf{U}_{n+1}=-(\mathbf{A}-\mathbf{I})\mathbf{U}_n\quad ,\quad n=1,2,\cdots,$$

if we take $\mathbf{U}_0=\mathbf{W}_0=0$, then we have

$$\begin{aligned}\mathbf{U_1}&=\mathbf{I},\\\mathbf{U}_2&=-(\mathbf{A}-\mathbf{I})\mathbf{U}_1,\\&=(\mathbf{A}-\mathbf{I}),\\\mathbf{U}_3&=(\mathbf{A}-\mathbf{I})^2,\\&\vdots\\\mathbf{U}_{n+1}&=(-1)^n(\mathbf{A}-\mathbf{I})^n,\end{aligned}$$

hence, the solution can be of the form

$$\mathbf{U} = \mathbf{U}_0 + \mathbf{U}_1 + \mathbf{U}_2 + \cdots ,$$

or

$$\mathbf{U} = [\mathbf{I} - (\mathbf{A} - \mathbf{I}) + (\mathbf{A} - \mathbf{I})^2 - \cdots]. \tag{4}$$

Theorem 1. *The sequence*

$$\mathbf{U}^{[m]} = \left[\sum_{k=0}^{m}((\mathbf{A} - \mathbf{I})^k\right],$$

is a Chauchy sequence if

$$\| \mathbf{A} - \mathbf{I} \| < 1.$$

Proof. We must show that

$$\lim_{m\to\infty} \| \mathbf{U}^{[m+p]} - \mathbf{U}^{[m]} \| = 0,$$

so, for showing this we can write

$$\mathbf{U}^{[m+p]} - \mathbf{U}^{[m]} = \sum_{k=1}^{p}(-1)^{m+k}(\mathbf{A} - \mathbf{I})^{m+k},$$

or

$$\| \mathbf{U}^{[m+p]} - \mathbf{U}^{[m]} \| \leq \sum_{k=1}^{p} \| \mathbf{A} - \mathbf{I} \|^{m+k},$$

let $\gamma = \| \mathbf{A} - \mathbf{I} \|$, then

$$\begin{aligned} \| \mathbf{U}^{[m+p]} - \mathbf{U}^{[m]} \| &\leq \gamma^m \sum_{k=1}^{p} \gamma^k, \\ &\leq \left(\frac{\gamma^p - 1}{\gamma - 1}\right) \gamma^m, \end{aligned}$$

now if $\gamma < 1$, then we have

$$\lim_{m\to\infty} \| \mathbf{U}^{[m+p]} - \mathbf{U}^{[m]} \| \leq \left(\frac{\gamma^p - 1}{\gamma - 1}\right) \left(\lim_{m\to\infty} \gamma^m\right),$$

hence, we obtain

$$\lim_{m\to\infty} \| \mathbf{U}^{[m+p]} - \mathbf{U}^{[m]} \| = 0,$$

which completes the proof. ■

Lemma 1. *If* $\mathbf{A}$ *is diagonally dominated and*

$$\mathbf{D} = diag\left[\frac{1}{a_{ii}}\right],$$

then

$$\| \mathbf{DA} - \mathbf{I} \|_{\infty} < 1.$$

Proof. Let $\mathbf{C} = \mathbf{DA} - \mathbf{I}$, then it can be easily shown that

$$c_{ij} = \begin{cases} 0 & i = j \\ \dfrac{a_{ij}}{a_{ii}} & i \neq j \end{cases} ,$$

since $\mathbf{A}$ is diagonally dominated, then

$$| a_{ii} | > \sum_{j=1, j\neq i}^{n} | a_{ij} |,$$

or

$$\sum_{j=1, j\neq i}^{n} \frac{| a_{ij} |}{| a_{ii} |} < 1 \quad , \quad i = 1, 2, \cdots n,$$

hence

$$\sum_{j=1}^{n} | c_{ij} | < 1 \quad , \quad i = 1, 2 \cdots n,$$

which implies

$$\| \mathbf{C} \|_{\infty} = \| \mathbf{DA} - \mathbf{I} \|_{\infty} < 1.$$

If $\| \mathbf{A} - \mathbf{I} \|_{\infty} > 1$, by pre-multiplying the both sides of equation(1) by matrix $\mathbf{D}$, we rewrite the equation as follows

$$\mathbf{DAX} - \mathbf{D} = \mathbf{0}$$

and it can be easily shown that

$$\mathbf{U}_{n+1} = (-1)^n (\mathbf{DA} - \mathbf{I})^n \mathbf{D} \tag{5}$$

3. Numerical results

Example 1. Approximate the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & -1 \\ -1 & 6 & 1 \\ 0 & 1 & -3 \end{bmatrix}.$$

Since $\| \mathbf{A} - \mathbf{I} \|_\infty = 8$, and $\mathbf{A}$ is diagonally dominated, we can write

$$D = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix},$$

hence

$$\mathbf{DA} = \begin{bmatrix} 1 & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{6} & 1 & \frac{1}{6} \\ 0 & -\frac{1}{3} & 1 \end{bmatrix}$$

$$\mathbf{DA} - \mathbf{I} = \begin{bmatrix} 0 & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{6} & 0 & \frac{1}{6} \\ 0 & -\frac{1}{3} & 0 \end{bmatrix},$$

from (4)and (5) we have

$$\mathbf{U} = [\mathbf{I} - (\mathbf{DA} - \mathbf{I}) + (\mathbf{DA} - \mathbf{I})^2 - \cdots]\mathbf{D},$$

and using six terms, we approximate $\mathbf{A^{-1}} = \mathbf{U}$ as follows

$$\mathbf{U} \approx \mathbf{U}_0 + \mathbf{U}_1 + \cdots + \mathbf{U}_5,$$

or

$$\mathbf{U} \approx \begin{bmatrix} 0.2435 & -0.025 & -0.089 \\ 0.0385 & 0.1539 & 0.0387 \\ 0.0127 & 0.0514 & -0.32 \end{bmatrix},$$

where the exact solution is

$$\mathbf{A^{-1}} = \begin{bmatrix} 0.2435 & -0.025 & -0.089 \\ 0.0384 & 0.1538 & 0.0384 \\ 0.0128 & 0.0512 & -0.32 \end{bmatrix}$$

Example 2. Give an approximation to the inverse of the following matrix

$$\mathbf{A} = \begin{bmatrix} 0.5 & 0.5 & 0.2 \\ 0.1 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.3 \end{bmatrix}$$

Matrix **A** is nearly diagonally dominated, so by similar operations we obtain

$$\mathbf{DA} - \mathbf{I} = \begin{bmatrix} 0 & 1 & 0.4 \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix},$$

and using seven iterations, an approximation to $\mathbf{A^{-1}}$ will be as follows

$$\begin{bmatrix} 3.1956 & -4.4739 & -.15056 \\ -0.66063 & 5.19744 & -0.97097 \\ -0.66063 & 0.19973 & 4.02674 \end{bmatrix},$$

where the exact solution is

$$\mathbf{A^{-1}} = \begin{bmatrix} 3.07692 & -5 & -0.38462 \\ -0.76923 & 5 & -1.15385 \\ -0.76923 & 0 & 3.84615 \end{bmatrix}.$$

4. Conclusion

In this paper, we used homotopy perturbation method to approximate the inverse of a diagonally dominated matrix in terms of the subtraction of the given matrix (or its modification) and unit matrix. Solved problems show the convergence of the method increases as the the matrix becomes more strictly diagonally dominated.

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A NEW CHARACTERIZATION OF $L_2(q)$ WHERE $q = p^n < 125$[1]

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Abstract. It is a well-known topic to characterize a finite simple group by using two quantities, the order of G and $\pi_e(G)$ in the past 30 years, where $\pi_e(G)$ denotes the set of orders of elements in G. Recently this topic has been finished by V.D. Mazurov, et al. Here the authors will try to characterize some finite simple groups by using less quantities and have successfully characterized $L_2(q)$, where $q = p^n < 125$, by using the order of $L_2(q)$ and the three largest element orders of $L_2(q)$.

Key words: finite group, the largest element order, the second largest element order, the third largest element order, characterization.

AMS Subject Classification: 20D05, 20D60.

1. Introduction

It is a well-known topic to characterize a finite simple group by using two quantities, the order of G and $\pi_e(G)$ in the past 30 years, where $\pi_e(G)$ denotes the set of orders of elements in G. W.J. Shi characterized some finite simple groups by using $\pi_e(G)$ and $|G|$, for example, see [1]–[6]. Recently, this topic has been finished by V.D. Mazurov, et al. (see [7]). Now, the authors will try to characterize some finite simple groups by using less quantities and have successfully characterized

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simple K_3-groups, sporadic simple groups, $L_3(q)$ and $U_3(q)$, where q are some special powers of primes, by using three numbers: the order of a group, the largest and the second largest element orders in [8]–[10]. In this paper, we characterize $L_2(q)$ for $q = p^n < 125$, via the order of a group and the three largest element orders.

Notations: The groups mentioned are all finite groups, and the number in bracket "( )" behind a group is the order of the group, e.g., $L_2(7)(2^3 \cdot 3 \cdot 7)$ means that $L_2(7)$ is of order $2^3 \cdot 3 \cdot 7$. Let $\pi_e(G)$ denote the set of orders of elements in G, $\pi(G)$, the set of all prime divisors of $|G|$. Let $k_1(G)$ denote the largest element order of G, $k_2(G)$, the second largest element order and $k_3(G)$, the third largest element order. S_p-subgroup is a Sylow p-subgroup of G. We denote by $\Gamma(G)$ the prime graph of G and $t(G)$ is the number of connected components of $\Gamma(G)$. And we also denote the sets of vertex of the connected components of the prime graph by $\{\pi_i, i = 1, \dots, t(G)\}$. For convenience we call π_i $(1 \le i \le t(G))$ the connected components and if the order of G is even, denote the component containing 2 by π_1 (see [11]).

2. Preliminary results

Lemma 1. *Suppose that G has more than one prime graph component. Then one of the following holds:*

1. *G is a Frobenius group or a 2-Frobenius group;*
2. *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$.*

Proof. The lemma follows from Theorem A and Lemma 3 in [11].

Remark. A group G will be called a 2-Frobenius group provided G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that G/H and K are *Frobenius* groups with K/H and H as their *Frobenius* kernels respectively.

Lemma 2. *If G is a Frobenius group of even order with K the Frobenius kernel and H the Frobenius complement, then $t(G) = 2$ and $\Gamma(G) = \{\pi(H), \pi(K)\}$* (see [12]).

Lemma 3. *If G is a 2-Frobenius group of even order, then $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that $\pi(K/H) = \pi_2$, and $\pi(G/K) \cup \pi(H) = \pi_1$. Moreover, G/K and K/H are cyclic groups satisfying that $|G/K| \mid |Aut(K/H)|$, $(|G/K|, |K/H|) = 1$, and $|G/K| < |K/H|$. Particularly, G is solvable* (see [12]).

Lemma 4. *Let A be a π'-group of automorphisms of the π-group G, and suppose G or A is solvable. Then for each prime p in π, A leaves invariant some S_p-subgroup of G* (see [13], Theorem 6.22).

Lemma 5. *Let G be $L_2(q)$, where $q = p^n < 125$. Then $|G|$, $k_1(G)$ and $k_2(G)$are as in Table* 1:

Table 1

G	$\lvert G\rvert$	$k_1(G)$	$k_2(G)$
$L_2(5) \cong A_5$	$2^2 \cdot 3 \cdot 5$	5	3
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	7	4
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	9	7
$L_2(9) \cong A_6$	$2^3 \cdot 3^2 \cdot 5$	5	4
$L_2(11)$	$2^2 \cdot 3 \cdot 5 \cdot 11$	11	6
$L_2(13)$	$2^2 \cdot 3 \cdot 7 \cdot 13$	13	7
$L_2(16)$	$2^4 \cdot 3 \cdot 5 \cdot 17$	17	15
$L_2(17)$	$2^4 \cdot 3^2 \cdot 17$	17	9
$L_2(19)$	$2^2 \cdot 3^2 \cdot 5 \cdot 19$	19	10
$L_2(23)$	$2^3 \cdot 3 \cdot 11 \cdot 23$	23	12
$L_2(25)$	$2^3 \cdot 3 \cdot 5^2 \cdot 13$	13	12
$L_2(27)$	$2^2 \cdot 3^3 \cdot 7 \cdot 13$	14	13
$L_2(29)$	$2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 29$	29	15
$L_2(31)$	$2^5 \cdot 3 \cdot 5 \cdot 31$	31	16
$L_2(32)$	$2^5 \cdot 3 \cdot 11 \cdot 31$	33	31
$L_2(37)$	$2^2 \cdot 3^2 \cdot 19 \cdot 37$	37	19
$L_2(41)$	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 41$	41	21
$L_2(43)$	$2^2 \cdot 3 \cdot 7 \cdot 11 \cdot 43$	43	22
$L_2(47)$	$2^4 \cdot 3 \cdot 23 \cdot 47$	47	24
$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	25	24
$L_2(53)$	$2^2 \cdot 3^3 \cdot 13 \cdot 53$	53	27
$L_2(59)$	$2^2 \cdot 3 \cdot 5 \cdot 29 \cdot 59$	59	30
$L_2(61)$	$2^2 \cdot 3 \cdot 5 \cdot 31 \cdot 61$	61	31
$L_2(64)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	65	63
$L_2(67)$	$2^2 \cdot 3 \cdot 11 \cdot 17 \cdot 67$	67	34
$L_2(71)$	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 71$	71	36
$L_2(73)$	$2^3 \cdot 3^2 \cdot 37 \cdot 73$	73	37
$L_2(79)$	$2^4 \cdot 3 \cdot 5 \cdot 13 \cdot 79$	79	40
$L_2(81)$	$2^4 \cdot 3^4 \cdot 5 \cdot 41$	41	40
$L_2(83)$	$2^2 \cdot 3 \cdot 7 \cdot 41 \cdot 83$	83	42
$L_2(89)$	$2^3 \cdot 3^2 \cdot 5 \cdot 11 \cdot 89$	89	45
$L_2(97)$	$2^5 \cdot 3 \cdot 7^2 \cdot 97$	97	49
$L_2(101)$	$2^2 \cdot 3 \cdot 5^2 \cdot 17 \cdot 101$	101	51
$L_2(103)$	$2^3 \cdot 3 \cdot 13 \cdot 17 \cdot 103$	103	52
$L_2(107)$	$2^2 \cdot 3^2 \cdot 53 \cdot 107$	107	54
$L_2(109)$	$2^2 \cdot 3^3 \cdot 5 \cdot 11 \cdot 109$	109	55
$L_2(113)$	$2^4 \cdot 3 \cdot 7 \cdot 19 \cdot 113$	113	57
$L_2(121)$	$2^3 \cdot 3 \cdot 5 \cdot 11^2 \cdot 61$	61	60

Proof. The lemma follows from [14] and the properties of $L_2(q)$.

Lemma 6. *Let G be a finite group and M be one of the following simple groups: $L_2(5)$, $L_2(9)$, $L_2(11)$, $L_2(13)$, $L_2(16)$, $L_2(17)$, $L_2(19)$, $L_2(23)$, $L_2(25)$, $L_2(27)$, $L_2(29)$, $L_2(32)$, $L_2(37)$, $L_2(41)$, $L_2(43)$, $L_2(47)$, $L_2(53)$, $L_2(59)$, $L_2(61)$, $L_2(67)$, $L_2(71)$, $L_2(73)$, $L_2(79)$, $L_2(81)$, $L_2(83)$, $L_2(89)$, $L_2(97)$, $L_2(101)$, $L_2(103)$, $L_2(107)$, $L_2(109)$, $L_2(113)$, $L_2(121)$.*

Suppose that

(i) $k_1(G) = k_1(M)$;

(ii) $|G| = |M|$.

Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$.

Proof. We only need to prove the cases $L_2(5)$ $(2^2 \cdot 3 \cdot 5)$, $L_2(53)$ $(2^2 \cdot 3^3 \cdot 13 \cdot 53)$. For the other cases, we can prove them similarly.

1. Assume that $M = L_2(5)$ $(2^2 \cdot 3 \cdot 5)$. In such case, $|G| = 2^2 \cdot 3 \cdot 5$ and $k_1(G) = 5$. Because $k_1(G) = 5$, we get that 5 is an isolated point of $\Gamma(G)$, and therefore $t(G) \geq 2$. By Lemma 1, we know that G is either a Frobenius group or a 2-Frobenius group, or has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H is a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. Therefore, we only need to prove that G is neither a Frobenius group nor a 2-Frobenius group.

First, we suppose that G is a Frobenius group. Then, by Lemma 2 we get that $t(G) = 2$ and $\Gamma(G) = \{\ \pi(H), \pi(K)\ \}$, where K is the Frobenius kernel and H the Frobenius complement. As $k_1(G) = 5$, K is either a $\{2, 3\}$-Hall subgroup or a Sylow 5-subgroup of G. Since K is nilpotent, let S be a Sylow subgroup of K, one has that $|H| \mid (|S| - 1)$. We can find an suitable Sylow subgroup of K such that $|H| \nmid (|S| - 1)$, and then we get a contradiction. For this reason, K can't be a Sylow 5-subgroup of G. Hence K is a $\{2, 3\}$-Hall subgroup. Consider the Sylow 3-subgroup of K. We can get $5 \mid 2$, a contradiction. Therefore, G is not a Frobenius group.

Second, we suppose that G is a 2-Frobenius group. By Lemma 3, we know that $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(K/H) = \pi_2$ and $\pi(G/K) \cup \pi(H) = \pi_1$. Moreover, G/K and K/H are cyclic groups satisfying $|G/K| \mid |Aut(K/H)|$, and $|G/K| < |K/H|$. As 5 is an isolated point of $\Gamma(G)$, $\pi_2(G) = \{5\}$. Therefore, $\pi(G/K) \cup \pi(H) = \{2, 3\}$ and $|K/H| = 5$. Since $|G/K| \mid |Aut(K/H)| = 4$, we know that $3 \mid |H|$. Consider the action on H by the element of order 5. By Lemma 4, there exists a Sylow 3-subgroup L of H fixed by this action. Since $|L| = 3$, we have $5 \nmid |Aut(L)|$, which means that such action on L is trivial. Therefore, G has an element of order 15, a contradiction. So G is not a 2-Frobenius group.

Remark. This approach can be used to prove that G is not 2-Frobenius group for the most of the other cases. For a few exceptions, we only need to consider $\Omega_1(Z(L))$ of some special Sylow subgroup L to lead to a contradiction. The process can be seen in the case $M = L_2(53)$.

2. Assume that $M = L_2(53)$ $(2^2 \cdot 3^3 \cdot 13 \cdot 53)$. In such case, $|G| = 2^2 \cdot 3^3 \cdot 13 \cdot 53$ and $k_1(G) = 53$. Because $k_1(G) = 53$, we know that 53 is an isolated point of $\Gamma(G)$, and therefore $t(G) \geq 2$. By Lemma 1, we only need to prove that G is neither a Frobenius group nor a 2-Frobenius group.

Using the similar arguments in case $M = L_2(5)$, we can easily show that G is not a Frobenius group. Now we assert that G is not a 2-Frobenius group. Assume the contrary. Let G be a 2-Frobenius group. By Lemma 3, $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(K/H) = \pi_2$ and $\pi(G/K) \cup \pi(H) = \pi_1$. Moreover, G/K and K/H are cyclic groups satisfying $|G/K| \mid |Aut(K/H)|$, and $|G/K| < |K/H|$. As 53 is an isolated point of $\Gamma(G)$, $\pi_2(G) = \{53\}$. Therefore, $\pi(G/K) \cup \pi(H) = \{2, 3, 13\}$ and $|K/H| = 53$. Since $|G/K| \mid |Aut(K/H)| = 52$, we know that $3 \mid |H|$. Consider the action on H by the element y of order 53. Again by Lemma 4, there exists a Sylow 3-subgroup L of H fixed by this action. Obviously, $|L| = 3^3$. Clearly, $\Omega_1(Z(L))$ is an elementary abelian 3-group, and $|\Omega_1(Z(L))| \mid 3^3$. Because $\Omega_1(Z(L))$ is characteristic in L, y fixes $\Omega_1(Z(L))$ too. As $53 \nmid |Aut(\Omega_1(Z(L)))|$, the action on $\Omega_1(Z(L))$ by y is trivial, which implies that G has an element of order 159, a contradiction. So G is not a 2-Frobenius group.

3. Main results

Theorem 1. *Let G be a group and M be $L_2(q)$, where $q = p^n \neq 7, 31, 49, 64$, and $q < 125$. Then $G \cong M$ if and only if*

(i) $k_1(G) = k_1(M)$;

(ii) $|G| = |M|$.

Proof. We only need to prove the sufficiency. And the proof will be made through a case by case analysis.

When $q = 5,\ 8,\ 9,\ 17$, the theorem follows from Theorem 1 in [8]. Now, we assume that $q = 11,\ 13,\ 16,\ 19,\ 23,\ 25,\ 27,\ 29,\ 32,\ 37,\ 41,\ 43,\ 47,\ 53,\ 59,\ 61,\ 67,\ 71,\ 73,\ 79,\ 81,\ 83,\ 89,\ 97,\ 101,\ 103,\ 107,\ 109,\ 113,\ 121$.

Case 1. Since $q = 11,\ 13,\ 16,\ 19,\ 23,\ 25,\ 29,\ 32,\ 37,\ 41,\ 43,\ 47,\ 53,\ 59,\ 61,\ 67,\ 71,\ 73,\ 79,\ 81,\ 83,\ 89,\ 97,\ 101,\ 103,\ 107,\ 109,\ 113$ and 121 have similar proofs, we only consider a few of them.

Assume that $q = 11$. In such case, $M = L_2(11)$. By Lemma 5, we know that $|G| = 2^2 \cdot 3 \cdot 5 \cdot 11$ and $k_1(G) = 11$. Therefore, 11 is an isolated point of $\Gamma(G)$ and

$t(G) \geq 2$. By Lemma 6, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5\}$ and $11 \in \pi(K/H)$. From [14] we can know that K/H is isomorphic only to $L_2(11)$ $(2^2 \cdot 3 \cdot 5 \cdot 11)$. So $H = 1$, $K = G$, and therefore $G \cong L_2(11)$.

Assume that $q = 13$. In such case, $M = L_2(13)$. By Lemma 5, we know that $|G| = 2^2 \cdot 3 \cdot 7 \cdot 13$ and $k_1(G) = 13$. Therefore, 13 is an isolated point of $\Gamma(G)$ and $t(G) \geq 2$. By Lemma 6, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 7\}$ and $13 \in \pi(K/H)$. From [14] we can know that K/H is isomorphic only to $L_2(13)$ $(2^2 \cdot 3 \cdot 7 \cdot 13)$. So $H = 1$, $K = G$, and therefore $G \cong L_2(13)$.

Assume that $q = 16$. In such case, $M = L_2(16)$. By Lemma 5, we know that $|G| = 2^4 \cdot 3 \cdot 5 \cdot 17$ and $k_1(G) = 17$. Therefore, 17 is an isolated point of $\Gamma(G)$ and $t(G) \geq 2$. By Lemma 6, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5\}$ and $17 \in \pi(K/H)$. From [14] we can know that K/H is isomorphic only to $L_2(16)$ $(2^4 \cdot 3 \cdot 5 \cdot 17)$. So $H = 1$, $K = G$, and therefore $G \cong L_2(16)$.

Case 2. $q = 27$. In such case, $M = L_2(27)$. By Lemma 5, we know that $|G| = 2^2 \cdot 3^3 \cdot 7 \cdot 13$ and $k_1(G) = 14$. Therefore, 13 is an isolated point of $\Gamma(G)$ and $t(G) \geq 2$. By Lemma 6, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 7\}$ and $13 \in \pi(K/H)$. From [14] we can assume that K/H is isomorphic to $L_2(13)$ $(2^2 \cdot 3 \cdot 7 \cdot 13)$ or $L_2(27)$ $(2^2 \cdot 3^3 \cdot 7 \cdot 13)$.

Suppose that $K/H \cong L_2(13)$ $(2^2 \cdot 3 \cdot 7 \cdot 13)$. From [14] we know that $3 \nmid |Out(K/H))| = 2$, so $3 \mid |H|$ and $|H| = 3^2$. Consider the action on H by the element of order 13. Clearly, this action is trivial, which implies that G has an element of order 39, a contradiction.

Therefore, we have $K/H \cong L_2(27)$ $(2^2 \cdot 3^3 \cdot 7 \cdot 13)$. So $H = 1$, $K = G$, and therefore $G \cong L_2(27)$. This completes the proof.

Theorem 2. *Let G be a group and M be $L_2(q)$ for $q = 7, 31$. Then $G \cong M$ if and only if*

(i) $k_i(G) = k_i(M)$, *where* $i = 1,\ 2$

(ii) $|G| = |M|$.

Proof. It is enough to prove the sufficiency. Firstly, we suppose that $q = 7$. In such case, the theorem follows from the theorem 2 in [7]. Now, we assume that $q = 31$. In this case, $M = L_2(31)$ $(2^5 \cdot 3 \cdot 5 \cdot 31)$, and therefore $|G| = 2^5 \cdot 3 \cdot 5 \cdot 31$, $k_1(G) = 31$ and $k_2(G) = 16$. At first, we show that G is a non-solvable group. Assume the contrary. Let G be a solvable group. Then the minimal normal subgroup N of G is an elementary abelian p-group. If N is a 3-group, then $|N| = 3$ and $|Aut(N)| = 2$. Consider the action on N by an element of order 31. Obviously, this action is trivial. Therefore, G has an element of order 93, a contradiction. If N is a 5-group, then $|N| = 5$ and $|Aut(N)| = 4$. Consider the action on N by an element of order 31. Clearly, this action is also trivial. Therefore, G has an element of order 155, also a contradiction. If N is a 2-group, then $|N \mid |2^5$. Suppose that $|N| = 2^5$. Then N has an element with order 16 for $k_2(G) = 16$, which contradicts that N is an elementary abelian 2-group. So, $|N| \mid 2^4$, and hence $31 \nmid |Aut(N)|$. Consider the action on N by the element of order 31. We can see that this action is trivial. Therefore, G has an element of order 62, still a contradiction. Now assume that N is a 31-group. Then $|N| = 31$. Let $N =< a >$. Since $k_2(G) = 16$, G must have an element of order 4. Consider the action on N by an element x of order 4. There must exist a positive integer t such that $a^{x^t} = a$. Because $|Aut(N)| = 30$, we can conclude that $t = 1$, 2 or 3. If $t = 1$ or 3, then $|x^t| = 4$. In this case, G has an element of order 124, a contradiction. If $t = 2$, then $|x^t| = 2$. In such case, G has an element of order 62, also a contradiction. Therefore, G is a non-solvable group.

Because $k_1(G) = 31$, 31 is an isolated point of $\Gamma(G)$ and $t(G) \geq 2$. By Lemma 1, we know that G is either a Frobenius group or a 2-Frobenius group, or has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H is a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$.

First, we assume that G is a Frobenius group. By Lemma 2, we get that $t(G) = 2$ and $\Gamma(G) = \{\ \pi(H),\ \pi(K)\ \}$, where K is the Frobenius kernel and H the Frobenius complement. Since $k_1(G) = 31$, K is either a $\{2, 3, 5\}$-Hall subgroup or a Sylow 31-subgroup of G. If K is a Sylow 31-subgroup of G, then $|K| = 31$. Consider the action on K by an element x of order 4. Using the similar discussion in preceding paragraph, we can get that G has an element of order larger than 31, which is a contradiction. So we suppose that K is a $\{2, 3, 5\}$-Hall subgroup. Consider the action on K by the element of order 31. By Lemma 4 we can draw a conclusion that there exists a Sylow 5-subgroup L of K fixed by this action. Since $G = 2^5 \cdot 3 \cdot 5 \cdot 31$, we have $|L| = 5$ and thus $31 \nmid Aut(L)$, which implies that G has an element of order 155, a contradiction too. Therefore, G is not a Frobenius group.

Now we assume that G is a 2-Frobenius group. By Lemma 3, we know that G is solvable, also a contradiction. So G is not a 2-Frobenius group.

Therefore, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. Because $|G| = 2^5 \cdot 3 \cdot 5 \cdot 31$, and 31 is an isolated point of $\Gamma(G)$, we have $\pi(H) \cup \pi(G/K) \subseteq$

$\{2, 3, 5\}$ and $31 \in \pi(K/H)$. From [14] we know that K/H is isomorphic only to $L_2(31)(2^5 \cdot 3 \cdot 5 \cdot 31)$. Therefore, $H = 1$ and $K = G$, and thus $G \cong L_2(31)$. The proof is completed. ∎

Let $k_3(G)$ denote the third largest element order of G. We can easily know that $k_3(L_2(49)) = 12$ and $k_3(L_2(64)) = 21$ from the properties of $L_2(q)$.

For $q = 49$ and 64, we have the following result.

Theorem 3. *Let G be a group and M be $L_2(q)$ for $q = 49, 64$. Then $G \cong M$ if and only if*

(i) *$k_i(G) = k_i(M)$, where $i = 1, 2, 3$*

(ii) $|G| = |M|$.

Proof. We only need to prove the sufficiency.

Case 1. Assume that $M = L_2(49)$ $(2^4{\cdot}3{\cdot}5^2{\cdot}7^2)$. In such case, $|G| = 2^4{\cdot}3{\cdot}5^2{\cdot}7^2$ and $k_1(G) = 25$, $k_2(G) = 24$, $k_3(G)$=12, from which we know that 7 is an isolated point of $\Gamma(G)$, and therefore $t(G) \geq 2$. Using the similar discussion in Lemma 6, we can prove that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H is a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. Because $|G| = 2^4 \cdot 3 \cdot 5^2 \cdot 7^2$ and 7 is an isolated point of $\Gamma(G)$, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5\}$ and $7 \in \pi(K/H)$. From [14] we can suppose that $K/H \cong L_2(7)$ $(2^3 \cdot 3 \cdot 7)$ or $L_2(49)(2^4 \cdot 3 \cdot 5^2 \cdot 7^2)$.

First, we assume that $K/H{\cong}L_2(7)$. From [14] we know that $|Out(L_2(7))|$=2 and thus $|G/K| | 2$. Therefore, $7 \mid |H|$. Let L be a Sylow 7-subgroup of H. Then $|L| = 7$. As H is a nilpotent group, we have L is characteristic in H, and thus $L \trianglelefteq G$. Consider the action on L by the element of order 5. Clearly, this action is trivial. It implies that G has an element of order 35, a contradiction.

Therefore, we can get that $K/H \cong L_2(49)$. Because $|G| = 2^4 \cdot 3 \cdot 5^2 \cdot 7^2$, we can conclude that $H = 1$ and $K = G$, and thus $G \cong L_2(49)$.

Case 2. Assume that $M = L_2(64)$ $(2^6{\cdot}3^2{\cdot}5{\cdot}7{\cdot}13)$. In such case, $|G| = 2^6{\cdot}3^2{\cdot}5{\cdot}7{\cdot}13$ and $k_1(G) = 65$, $k_2(G) = 63$, $k_3(G)$=21, from which we know that 7 is an isolated point of $\Gamma(G)$, and therefore $t(G) \geq 2$. Using the similar discussion in Lemma 6, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H is a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. Because $|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$ and 7 is an isolated point of $\Gamma(G)$, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 13\}$ and $7 \in \pi(K/H)$. From [14] we can suppose that K/H is isomorphic to one of the following simple groups: $L_2(7)$ $(2^3 \cdot 3 \cdot 7)$, $L_2(8)$ $(2^3{\cdot}3^2{\cdot}7)$, A_7 $(2^3{\cdot}3^2{\cdot}5{\cdot}7)$, A_8 $(2^6{\cdot}3^2{\cdot}5{\cdot}7)$, $L_3(4)(2^6{\cdot}3^2{\cdot}5{\cdot}7)$, $Sz(8)$ $(2^6{\cdot}5{\cdot}7{\cdot}13)$ and $L_2(64)$ $(2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13)$.

Suppose that K/H is isomorphic to one of the simple groups mentioned above, except $Sz(8)$ $(2^6 \cdot 5 \cdot 7 \cdot 13)$, $L_2(64)$ $(2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13)$. From [14] we know that $13 \nmid |Out(K/H)|$ and thus $13 \mid |H|$. Let L be a Sylow 13-subgroup of H. Then

$|L| = 13$ and $L \trianglelefteq G$. Consider the action on L by the element of order 5. Clearly, this action is trivial. It implies that G has an element of order 65, a contradiction.

Suppose that $K/H \cong Sz(8)$. From [14] we know that $|Out(Sz(8))| = 3$ and hence $|G/K||3$. Therefore, we can get that $3 \mid |H|$ by comparing the order of G. Let L be a Sylow 3-subgroup of H. Then $|L| \mid 3^2$ and $L \trianglelefteq G$. Consider the action on L by the element of order 13. Clearly, this action is trivial. It implies that G has an element of order 39, a contradiction.

Therefore, we have $K/H \cong L_2(64)$. Because $|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$, we can get that $H = 1$ and $K = G$, and thus $G \cong L_2(64)$. This completes the proof. ■

As a corollary of the proceeding theorems, we have

Theorem 4. *Let G be $L_2(q)$, where $q = p^n < 125$. Then G can be uniquely determined by the order of G and $k_i(G)$, where $i \leq 3$.*

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IMPROVEMENT IN ESTIMATING POPULATION MEAN USING TWO AUXILIARY VARIABLES IN TWO-PHASE SAMPLING

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Abstract. This study proposes improved chain-ratio type estimator for estimating population mean using some known values of population parameter(s) of the second auxiliary character. The proposed estimators have been compared with two-phase ratio estimator and some other chain type estimators. The performances of the proposed estimators have been supposed with a numerical illustration.

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Keywords: auxiliary variables, chain ratio-type estimator, bias, mean squared error.

1. Introduction

The ratio method of estimation is generally used when the study variable Y is positively correlated with an auxiliary variable X whose population mean is known in advance. In the absence of the knowledge on the population mean of the auxiliary character we go for two-phase (double) sampling. The two-phase sampling happens to be a powerful and cost effective (economical) procedure for finding the reliable estimate in first phase sample for the unknown parameters of the auxiliary variable x and hence has eminent role to play in survey sampling, for instance, see Hidiroglou and Sarndal (1998). Consider a finite population $U = (U_1, U_2, ..., U_N)$. Let y and x be the study and auxiliary variable, taking values y_i and x_i, respectively, for the ith unit U_i. Allowing SRSWOR (Simple Random Sampling without Replacement) design in each phase, the two-phase sampling scheme is as follows:

(i) the first phase sample $s_{n'}$ $(s_{n'} \subset U)$ of a fixed size n' is drawn to measure only x in order to formulate a good estimate of a population mean $\overline{X}$,

(ii) given $s_{n'}$, the second phase sample s_n $(s_n \subset s_{n'})$ of a fixed size n is drawn to measure y only.

Let

$$\bar{x} = \frac{1}{n}\sum_{i \in s_n} x_i, \quad \bar{y} = \frac{1}{n}\sum_{i \in s_n} y_i, \quad \bar{x}' = \frac{1}{n'}\sum_{i \in s_{n'}} x_i,$$

The classical ratio estimator for $\overline{Y}$ is defined as

$$\bar{y}_r = \frac{\bar{y}}{\bar{x}}\,\overline{X}. \tag{1.1}$$

If $\overline{X}$ is not known, we estimate $\overline{Y}$ by the two-phase ratio estimator

$$\bar{y}_{rd} = \frac{\bar{y}}{\bar{x}}\,\bar{x}'. \tag{1.2}$$

Sometimes, even if $\overline{X}$ is not known, information on a cheaply ascertainable variable z, closely related to x but compared to x remotely related to y, is available on all units of the population. For instance, while estimating the total yield of wheat in a village, the yield and area under the crop are likely to be unknown, but the total area of each farm may be known from village records or may be obtained at a low cost. Then y, x and z are respectively yield, area under wheat and area under cultivation see Singh et.al. (2004). Assuming that the population mean $\overline{Z}$ of the variable z is known, Chand (1975) proposed a chain type ratio estimator as

$$t_1 = \frac{\bar{y}}{\bar{x}}\left(\frac{\bar{x}'}{\bar{z}'}\right)\overline{Z}. \tag{1.3}$$

Several authors have used prior value of certain population parameter(s) to find more precise estimates. Singh and Upadhyaya (1995) used coefficient of variation of z for defining modified chain type ratio estimator. In many situation the value of the auxiliary variable may be available for each unit in the population, for instance, see Das and Tripathi (1981). In such situations knowledge on $\overline{Z}$, C_z $\beta_1(z)$ (coefficient of skewness), $\beta_2(z)$ (coefficient of kurtosis) and possibly on some other parameters may be utilized. Regarding the availability of information on C_z, $\beta_1(z)$ and $\beta_2(z)$, the researchers may be referred to Searls (1964), Sen (1978), Singh et.al. (1973), Searls and Intarapanich (1990) and Singh et.al. (2007). Using the known coefficient of variation C_z and known coefficient of kurtosis $\beta_2(z)$ of the second auxiliary character z Upadhyaya and Singh (2001) proposed some estimators for $\overline{Y}$.

If the population mean and coefficient of variation of the second auxiliary character is known, the standard deviation σ_z is automatically known and it is more meaningful to use the σ_z in addition to C_z, see Srivastava and Jhajj (1980). Further, C_z, $\beta_1(z)$ and $\beta_2(z)$ are the unit free constants, their use in additive form is not much justified. Motivated with the above justifications and utilizing the known values of σ_z, $\beta_1(z)$ and $\beta_2(z)$, Singh (2001) suggested some modified estimators for $\overline{Y}$.

In this paper, under simple random sampling without replacement (SRSWOR), we have suggested improved chain ratio type estimator for estimating population mean using some known values of population parameter(s).

2. The suggested estimator

The work of authors discussed in Section 1 can be summarized by using following estimator

$$t = \bar{y}\left(\frac{\bar{x}'}{\bar{x}}\right)\left(\frac{a\overline{Z}+b}{a\bar{z}'+b}\right), \tag{2.1}$$

where a $(\neq 0)$, b are either real numbers or the functions of the known parameters of the second auxiliary variable z such as standard deviation (σ_z), coefficient of variation (C_z), skewness $(\beta_1(z))$ and kurtosis $(\beta_2(z))$.

The following scheme presents some of the important known estimators of the population mean which can be obtained by suitable choice of constants a and b.

Estimator	Values of	
	a	b
$t_1 = \bar{y}\left(\frac{\bar{x}'}{\bar{x}}\right)\left(\frac{\overline{Z}}{\bar{z}'}\right)$ Chand (1975) chain ratio type estimator	1	0
$t_2 = \bar{y}\left(\frac{\bar{x}'}{\bar{x}}\right)\left(\frac{\overline{Z}+C_z}{\bar{z}'+C_z}\right)$ Singh and Upadhyaya (1995) estimator	1	C_z
$t_3 = \bar{y}\left(\frac{\bar{x}'}{\bar{x}}\right)\left(\frac{\beta_2(z)\overline{Z}+C_z}{\beta_2(z)\bar{z}'+C_z}\right)$ Upadhyaya and Singh (2001) estimator	$\beta_2(z)$	C_z
$t_4 = \bar{y}\left(\frac{C_z\overline{Z}+\beta_2(z)}{C_z\bar{x}'+\beta_2(z)}\right)\left(\frac{\beta_2(z)\overline{Z}+C_z}{\beta_2(z)\bar{z}'+C_z}\right)$ Upadhyaya and Singh (2001) estimator	C_z	$\beta_2(z)$

$t_5 = \bar{y}\left(\frac{\bar{x}'}{\bar{x}}\right)\left(\frac{\overline{Z}+\sigma_z}{\beta_1(z)\bar{z}'+\sigma_z}\right)$ Singh (2001) estimator	1	σ_z
$t_6 = \bar{y}\left(\frac{\bar{x}'}{\bar{x}}\right)\left(\frac{\beta_1(z)\overline{Z}+\sigma_z}{\beta_1(z)\bar{z}'+\sigma_z}\right)$ Singh (2001) estimator	$\beta_1(z)$	σ_z
$t_7 = \bar{y}\left(\frac{\bar{x}'}{\bar{x}}\right)\left(\frac{\beta_2(z)\overline{Z}+\sigma_z}{\beta_2(z)\bar{z}'+\sigma_z}\right)$	$\beta_2(z)$	σ_z

In addition to these estimators, a large number of estimators can also be generated from the estimator t at (2.1) by putting suitable values of a and b.

Following Kadilar and Cingi (2006), we propose modified estimator combining t_1 and t_i $(i = 2, 3, ..., 7)$ as follows

$$t_i^* = \alpha t_1 + (1-\alpha)t_i, \quad (i = 2, 3, ..., 7), \tag{2.2}$$

where α is a real constant to be determined such that MSE of t_i^* is minimum and t_i $(i = 2, 3, ..., 7)$ are estimators listed above.

To obtain the bias and MSE of t_i^*, we write

$$\bar{y} = \overline{Y}(1+e_0),\ \bar{x} = \overline{X}(1+e_1),\ \bar{x}' = \overline{X}(1+e_1'),\ \bar{z}' = \overline{Z}(1+e_2')$$

such that

$$E(e_0) = E(e_1) = E(e_1') = E(e_2') = 0$$

and

$$\begin{array}{lll} E(e_0^2) = f_1 C_y^2, & E(e_1^2) = f_1 C_x^2, & E(e_1^{'2}) = f_2 C_x^2, \\ E(e_2^{'2}) = f_2 C_z^2, & E(e_0 e_1) = f_1 \rho_{xy} C_x C_y, & E(e_0 e_1') = f_2 \rho_{xy} C_x C_y, \\ E(e_0 e_2') = f_2 \rho_{yz} C_y C_z, & E(e_1 e_1') = f_2 C_x^2, & E(e_1 e_2') = f_2 \rho_{xz} C_x C_z, \\ E(e_1' e_2') = f_2 \rho_{xz} C_x C_z, & & \end{array}$$

where

$$f_1 = \left(\frac{1}{n} - \frac{1}{N}\right), \qquad f_2 = \left(\frac{1}{n'} - \frac{1}{N}\right),$$

$$C_y^2 = \frac{S_y^2}{\overline{Y}^2}, \quad C_x^2 = \frac{S_x^2}{\overline{X}^2}, \quad C_z^2 = \frac{S_z^2}{\overline{Z}^2},$$

$$\rho_{xy} = \frac{S_{xy}}{S_x S_y}, \quad \rho_{xz} = \frac{S_{xz}}{S_x S_z}, \quad \rho_{yz} = \frac{S_{yz}}{S_y S_z},$$

$$S_y^2=\frac{1}{(N-1)}\sum_{i\in U}(y_i-\overline{Y})^2,\quad S_x^2=\frac{1}{(N-1)}\sum_{i\in U}(x_i-\overline{X})^2,$$

$$S_z^2=\frac{1}{(N-1)}\sum_{i\in U}(z_i-\overline{Z})^2,\quad S_{xy}^2=\frac{1}{(N-1)}\sum_{i\in U}(x_i-\overline{X})(y_i-\overline{Y}),$$

$$S_{xz}^2=\frac{1}{(N-1)}\sum_{i\in U}(x_i-\overline{X})(z_i-\overline{Z}),\quad S_{yz}^2=\frac{1}{(N-1)}\sum_{i\in U}(y_i-\overline{Y})(z_i-\overline{Z}),$$

Expressing t_i^* in terms of e's, we have

$$\begin{aligned}t_i^* &= \overline{Y}(1+e_0)[\alpha(1+e_1')(1+e_1)^{-1}(1+e_2')^{-1}\\ &\quad+(1-\alpha)(1+e_1')(1+e_1)^{-1}(1+\theta e_2')^{-1}]\end{aligned}\tag{2.3}$$

where

$$\theta=\frac{a\overline{Z}}{a\overline{Z}+b}\,\cdot\tag{2.4}$$

Expanding the right hand side of (2.3) and retaining terms up to second power of e's, we have

$$t_i^*\cong\overline{Y}[1+e_0-e_1+e_1'-e_2'(\alpha+\theta-\alpha\theta)]\tag{2.5}$$

or

$$t_i^*-\overline{Y}\cong\overline{Y}[e_0-e_1+e_1'-e_2'(\alpha+\theta-\alpha\theta)].\tag{2.6}$$

Squaring both sides of (2.6) and then taking expectation, we get the MSE of the estimator t_i^*, up to the first order of approximation, as

$$\mathrm{MSE}(t_i^*)=\overline{Y}^2[f_1C_y^2+f_3C_x^2+(\alpha+\alpha\theta)^2f_2C_z^2],\tag{2.7}$$

where

$$f_3=\left(\frac{1}{n}-\frac{1}{n'}\right)\cdot$$

Minimization of (2.7) with respect to α yield its optimum value as

$$\alpha_{\mathrm{opt}}=\frac{K_{yz}-\theta}{1-\theta},\tag{2.8}$$

where

$$K_{yz}=\rho_{yz}\,\frac{C_y}{C_z}\,\cdot$$

Substitution of (2.8) in (2.7) yields the minimum value of $\mathrm{MSE}(t_i^*)$ as

$$\min.\mathrm{MSE}(t_i^*)=M_0=\overline{Y}^2[f_1C_y^2+f_3(C_x^2-2\rho_{yx}C_yC_x)-f_2\rho_{yz}^2C_y^2].\tag{2.9}$$

3. Efficiency comparisons

In this section, the conditions for which the proposed estimator is better than t_i $(i = 1, 2, ...7)$ have been obtained. The MSE's of these estimators up to the order $\circ(n) - 1$ are derived as

$$\text{MSE}(\bar{y}_{rd}) = \overline{Y}^2[f_1C_y^2 + f_3(C_x^2 - 2\rho_{yx}C_yC_x)] \tag{3.1}$$

$$\text{MSE}(t_1) = \overline{Y}^2[f_1C_y^2 + f_2(C_z^2 - 2\rho_{yz}C_yC_z) + f_3(C_x^2 - 2\rho_{yx}C_yC_x)] \tag{3.2}$$

$$\text{MSE}(t_2) = \overline{Y}^2[f_1C_y^2 + f_2(\theta_2^2C_z^2 - 2\theta_2\rho_{yz}C_yC_z) + f_3(C_x^2 - 2\rho_{yx}C_yC_x)] \tag{3.3}$$

$$\text{MSE}(t_3) = \overline{Y}^2[f_1C_y^2 + f_2(\theta_3^2C_z^2 - 2\theta_2\rho_{yz}C_yC_z) + f_3(C_x^2 - 2\rho_{yx}C_yC_x)] \tag{3.4}$$

$$\text{MSE}(t_4) = \overline{Y}^2[f_1C_y^2 + f_2(\theta_4^2C_z^2 - 2\theta_2\rho_{yz}C_yC_z) + f_3(C_x^2 - 2\rho_{yx}C_yC_x)] \tag{3.5}$$

$$\text{MSE}(t_5) = \overline{Y}^2[f_1C_y^2 + f_2(\theta_5^2C_z^2 - 2\theta_2\rho_{yz}C_yC_z) + f_3(C_x^2 - 2\rho_{yx}C_yC_x)] \tag{3.6}$$

$$\text{MSE}(t_6) = \overline{Y}^2[f_1C_y^2 + f_2(\theta_6^2C_z^2 - 2\theta_2\rho_{yz}C_yC_z) + f_3(C_x^2 - 2\rho_{yx}C_yC_x)] \tag{3.7}$$

and

$$\text{MSE}(t_7) = \overline{Y}^2[f_1C_y^2 + f_2(\theta_7^2C_z^2 - 2\theta_2\rho_{yz}C_yC_z) + f_3(C_x^2 - 2\rho_{yx}C_yC_x)] \tag{3.8}$$

where

$$\theta_2 = \frac{\overline{Z}}{\overline{Z} + C_z}, \quad \theta_3 = \frac{\beta_2(z)\overline{Z}}{\beta_2(z)\overline{Z} + C_z}, \quad \theta_4 = \frac{C_z\overline{Z}}{C_2\overline{Z} + \beta_2(z)},$$

$$\theta_5 = \frac{\overline{Z}}{\overline{Z} + \sigma_z}, \quad \theta_6 = \frac{\beta_1(z)\overline{Z}}{\beta_1(z)\overline{Z} + \sigma_z}, \quad \theta_7 = \frac{\beta_2(z)\overline{Z}}{\beta_2(z)\overline{Z} + (s)_z}.$$

From (2.9) and (3.1), we have

$$\text{MSE}(\bar{y}_{rd}) - M_0 = f_2\rho_{yz}^2C_y^2 \geq 0. \tag{3.9}$$

Also from (2.9) and (3.2)-(3.8), we have

$$\text{MSE}(t_i) - M_0 = f_2(\theta_iC_z - \rho_{yz}C_y)^2 \geq 0, \quad (i = 2, 3, ..., 7). \tag{3.10}$$

Thus, it follows from (3.9) and (3.10) that the suggested estimator under optimum condition is always better than the estimator t_i $(i = 1, 2, ...7)$.

4. Empirical study

To illustrate the performance of various estimators of $\overline{Y}$, we consider the data used by Anderson (1958). The variates are

y: Head length of second son,

x: Head length of first son,

z: Head breadth of first son,

$N = 25,\ \overline{Y} = 183.84,\ \overline{X} = 185.72,\ \overline{Z} = 151.12,\ \sigma_z = 7.224,$

$C_y = 0.0546,\ C_x = 0.0526,\ C_z = 0.0488,$

$\rho_{yx} = 0.7108,\ \rho_{yz} = 0.6932,\ \rho_{xz} = 0.7346,\ \beta_1(z) = 0.002,\ \beta_2(z) = 2.6519.$

Consider $n' = 10$ and $n = 7$. We have computed the percent relative efficiency (PRE) of different estimators of $\overline{Y}$ with respect to usual estimator $\bar{y}$ and compiled in the Table 4.1.

Table 4.1. PRE **of different estimators of** $\overline{Y}$ **with respect to** $\bar{y}$

Estimator	PRE
$\bar{y}$	100
$\bar{y}_{rd}$	122.5393
t_1	178.8189
t_2	178.8405
t_3	178.8277
t_4	186.3912
t_5	181.6025
t_6	122.5473
t_7	179.9636
t_i^*	186.6515

5. Conclusion

We have suggested modified estimators t_i^* $(i = 2, 3, ..., 7)$. From Table 4.1, we conclude that the proposed estimators are better than usual two-phase ratio estimator $\bar{y}_{rd}$, Chand (1975) chain type ratio estimator t_1, estimator t_2 proposed by Singh and Upadhyaya (1995), estimators t_i $(i = 3, 4)$ and than that of Singh (2001) estimators t_i $(i = 5, 6, 7)$. For practical purposes, the choice of the estimator depends upon the availability of the population parameter(s).

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A NOTE ON QUASI k-IDEALS AND BI k-IDEALS IN TERNARY SEMIRINGS

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Abstract. Dutta and Kar [3] have introduced the concept of ternary semiring. In this paper, the notion of quasi $k-$ideals and bi $k-$ideals of a ternary semiring is introduced and characterizations $k-$regular ternary semirings has been given.

Keywords:

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1. Introduction and preliminaries

Ternary semiring is a generalization of a ternary ring which is introduced by Lister [6]. T.K. Dutta and S. Kar have initiated the notion of ternary semiring and studied their properties. Recall ([3], [4]) the followings. A non-empty set S together with binary operation, called addition and ternary multiplication, denoted by juxtaposition, is said to be a ternary semiring if S is an additive commutative semigroup satisfying the following conditions:

(i) $abc \in S$,

(ii) $(abc)de = a(bcd)e = ab(cde)$,

(iii) $(a+b)cd = acd + bcd$,

(iv) $a(b+c)d = abd + acd$,

(v) $ab(c+d) = abc + abd$

for all $a, b, c, d, e \in S$.

Let S be a ternary semiring. An element $0 \in S$ such that $0 + x = x$ and $0xy = x0y = xy0 = 0$ for all $x, y \in S$, then $'0'$is called the zero element of the ternary semiring S. In this case, S is called ternary semiring S with zero. Through out this paper, S will always denote a ternary semiring with zero. An additive subsemigroup I of S is called left (right, lateral) ideal of S if s_1s_2a (respectively as_1s_2, s_1as_2) $\in I$ for all $s_1, s_2 \in S$ and $a \in I$. If I is a left, a right and a lateral ideal of S then I is called an ideal of S. An ideal I of S is called a $k-$ideal if $x + y \in I$, $x \in S$, $y \in I$ imply that $x \in I$. If A is an ideal of a ternary semiring S then $\overline{A}$=$\{a \in S : a + x \in A$ for some $x \in A\}$ is called $k-$closure of A. It can easily verified that an ideal A of S is $k-$ideal if and only if A=$\overline{A}$ and also $A \subseteq \overline{A}, A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$. Let A, B, C be three subsets of S. Then ABC denotes the set of all finite sums of the forms $\sum a_ib_ic_i$, where $a_i \in A, b_i \in B, c_i \in C$. An element $a \in S$ is called regular if there exists $x \in S$ such that $a = axa$. If all the elements of S are regular then S is called regular ternary semiring. An additive subsemigroup Q of a ternary semiring S is called a quasi-ideal of Sif $QSS \cap (SQS + SSQSS) \cap SSQ \subseteq Q$. Clearly, every quasi ideal of ternary semiring S is a ternary subsemiring of S.

2. Main results

Proposition 2.1. *Let S be a ternary semiring and X be a nonempty subset of S. Then*

(i) $\langle X\rangle_l = Z_0^+X + SSX$ *is the smallest left ideal generated by* X,

(ii) $\langle X\rangle_r = Z_0^+X + XSS$ *is the smallest right ideal generated by* X,

(iii) $\langle X\rangle_t = Z_0^+X + SSX + XSS + SSXSS$ *is the smallest two sided ideal generated by* X,

(iv) $\langle X\rangle_m = Z_0^+X + SXS + SSXSS$ *is the smallest lateral ideal generated by* X,

(v) $\langle X\rangle_i = Z_0^+X + SSX + XSS + SXS + SSXSS$ *is the smallest ideal generated by* X,

where $SSX, XSS, SXS, SSXSS, Z_0^+X$ *the set of all finite sum of the form* $\sum r_is_ix_i$, $\sum x_ip_iq_i, \sum u_ix_iv_i, \sum p_i'q_i'x_ir_i's_i', \sum n_ix_i$, *where* $r_i, s_i, p_i, q_i, u_i, v_i, p_i', q_i', r_i', s_i' \in S$, $x_i \in X$, $n_i \in Z_0^+$, *and* Z_0^+ *is the set of all positive integer with zero.*

The following corollary can be easily proved by the above proposition.

Corollary 2.2. *If X is a subsemigroup of $(S, +)$, then*

$$\begin{aligned}
\langle X\rangle_l &= X + SSX,\\
\langle X\rangle_r &= X + XSS,\\
\langle X\rangle_t &= X + SSX + XSS + SSXSS,\\
\langle X\rangle_m &= X + SXS + SSXSS,\\
\langle X\rangle_i &= X + SSX + XSS + SXS + SSXSS.
\end{aligned}$$

Proposition 2.3. [4] *Let S be a ternary semiring and $a \in S$. Then the principal*

(i) *left ideal generated by a is given by*

$$\langle a\rangle_l = \left\{\sum r_i s_i a + na : r_i, s_i \in S : n \in Z_0^+\right\}$$

(ii) *right ideal generated by a is given by*

$$\langle a\rangle_r = \left\{\sum a r_i s_i + na : r_i, s_i \in S : n \in Z_0^+\right\}$$

(iii) *two sided ideal generated by a is given by*

$$\langle a\rangle_t = \Big\{\sum r_i s_i a + \sum a p_i q_i + \sum p_k{}' q_k{}' a r_k{}' s_k{}' + na \\ : r_i, s_i, p_i, q_i, p_k{}', q_k{}', r_k{}', s_k{}' \in S : n \in Z_0^+\Big\}$$

(iv) *lateral ideal generated by a is given by*

$$\langle a\rangle_m = \left\{\sum r_i a s_i + \sum p_j q_j a r_j s_j + na : r_i, s_i, p_j, q_j, r_j, s_j \in S : n \in Z_0^+\right\}$$

(v) *ideal generated by a is given by*

$$\langle a\rangle_i = \Big\{\sum r_i s_i a + \sum a p_i q_i + \sum u_k a v_k + \sum p_k{}' q_k{}' a r_k{}' s_k{}' + na \\ : r_i, s_i, p_i, q_i, u_k, v_k, p_k{}', q_k{}', r_k{}', s_k{}' \in S : n \in Z_0^+\Big\}$$

where $\sum$ denote the finite sum and Z_0^+ is the set of all positive integer with zero.

Proposition 2.4. *If Q is a quasi-ideal of a ternary semiring S, then*

$$Q = Q + (SSQ \cap (SQS + SSQSS) \cap QSS).$$

Proof. The proof is obvious. ■

Let X be a non empty subset of a ternary semiring S. The smallest quasi-ideal containing X and generated by X is denoted by $(X)_q$, that is, the intersection of all quasi-ideal of S containing X.

Proposition 2.5. *Let S be a ternary semiring and X be nonempty subset of S. Then $(X)_q = Z_0^+X + (SSX \cap (SXS + SSXSS) \cap XSS)$.*

Proof. Let $Q = Z_0^+X + (SSX \cap (SXS + SSXSS) \cap XSS)$. Clearly, Q is a non empty additive subsemigroup of S. Now,

$$\begin{aligned}(SSQ \cap (SQS + SSQSS) \cap QSS) &\subseteq SSQ \\ &= SS(Z_0^+X + (SSX \cap (SXS + SSXSS) \cap XSS)) \\ &\subseteq Z_0^+SSX + SS(SSX) \subseteq SSX.\end{aligned}$$

Similarly, $(SSQ \cap (SQS + SSQSS) \cap QSS) \subseteq (SXS + SSXSS)$ and $(SSQ \cap (SQS + SSQSS) \cap QSS) \subseteq XSS$. Therefore

$$(SSQ \cap (SQS + SSQSS) \cap QSS) \subseteq (SSX \cap (SXS + SSXSS) \cap XSS) \subseteq Q.$$

Hence Q is a quasi-ideal of S containing X. Also, it is easy to show that Q is smallest quasi-ideal of S containing X. Hence

$$Q = (X)_q = Z_0^+ X + (SSX \cap (SXS + SSXSS) \cap XSS). \qquad \blacksquare$$

Theorem 2.6. *If Q is a quasi-ideal of a ternary semiring S and if $Q \subseteq QSS$ and $Q \subseteq SSQ$ and QSS, SSQ are $k-$ideals of S, then Q is the intersection of the left ideal $Q + SSQ$, lateral ideal $Q + SQS + SSQSS$, and right ideal $Q + QSS$.*

Proof. Let $D{=}(Q{+}QSS){\cap}(Q + SQS + SSQSS){\cap}(Q + SSQ)$. Clearly, $Q \subseteq D$. To show $D \subseteq Q$. Now, $Q \subseteq QSS$ and $Q \subseteq SSQ$. Therefore

$$D = QSS \cap (Q + SQS + SSQSS) \cap SSQ.$$

Let $d \in D$. Then $d \in QSS$, $d \in SSQ$ and

$$d = q + \Sigma s_i' q_i' s_i + \Sigma p_i' p_i'' q_i'' p_i''' p_i'''' \tag{2.1}$$

for $s_i', s_i, p_i', p_i'', p_i''', p_i'''' \in S$ and $q, q_i', q_i'' \in Q$.

Since $q \in Q \subseteq QSS$ and QSS is a $k-$ideal of S, therefore

$$\Sigma s_i' q_i' s_i + \Sigma p_i' p_i'' q_i'' p_i''' p_i'''' \in QSS.$$

Similarly,

$$\Sigma s_i' q_i' s_i + \Sigma p_i' p_i'' q_i'' p_i''' p_i'''' \in SSQ.$$

Therefore,

$$\Sigma s_i' q_i' s_i + \Sigma p_i' p_i'' q_i'' p_i''' p_i'''' \in QSS \cap (SQS + SSQSS) \cap SSQ \subseteq Q.$$

So,

$$\Sigma s_i' q_i' s_i + \Sigma p_i' p_i'' q_i'' p_i''' p_i'''' \in Q,$$

which implies that $d = q + \Sigma s_i' q_i' s_i + \Sigma p_i' p_i'' q_i'' p_i''' p_i'''' \in Q$. Thus $D \subseteq Q$. Hence $D = Q$. $\blacksquare$

Definition 2.7. An additive subsemigroup Q of a ternary semiring S is called a quasi $k-$ideal of Sif $\overline{QSS} \cap \overline{(SQS + SSQSS)} \cap \overline{SSQ} \subseteq Q$. Clearly, every quasi $k-$ideal is a quasi-ideal of S.

It is easy to see that if R be right $k-$ideal, M be lateral $k-$ideal and L be left $k-$ideal of S, then $Q = R \cap M \cap L$ is a quasi $k-$ideal of S, because $\overline{(R \cap M \cap L)SS} {\cap} \overline{SS(R \cap M \cap L)SS} {\cap} \overline{SS(R \cap M \cap L)} = \overline{RSS} {\cap} \overline{SSMSS} {\cap} \overline{SSL} = \overline{R} \cap \overline{M} \cap \overline{L} = R \cap M \cap L$.

Lemma 2.8. *Let S be a ternary semiring and $A, B, C \subseteq S$, then*

$$\overline{ABC} = \overline{\overline{A}\,\overline{B}\,\overline{C}}.$$

Proof. Since $A \subseteq \overline{A}$, $B \subseteq \overline{B}$ and $C \subseteq \overline{C}$, therefore $ABC \subseteq \overline{A}\,\overline{B}\,\overline{C}$. Hence, $\overline{ABC} \subseteq \overline{\overline{A}\,\overline{B}\,\overline{C}}$. Again, let $x \in \overline{A}$, $y \in \overline{B}$ and $z \in \overline{C}$. Then there exist $a_1, a_2 \in A$, $b_1, b_2 \in B$ and $c_1, c_2 \in C$ such that $x + a_1 = a_2$, $y + b_1 = b_2$ and $z + c_1 = c_2$. Now,

$$\begin{aligned}
&xyz + a_2b_2c_1 + a_2b_1c_2 + a_1b_2c_2 + a_1b_1c_1\\
&\quad = xyz + (x + a_1)(y + b_1)c_1 + a_2b_1c_2 + a_1b_2c_2 + a_1b_1c_1\\
&\quad = xyz + xyc_1 + xb_1c_1 + a_1yc_1 + a_1b_1c_1 + a_2b_1c_2\\
&\quad +a_1b_2c_2 + a_1b_1c_1\\
&\quad = xyc_2 + xb_1c_1 + a_1yc_1 + a_1b_1c_1 + a_2b_1c_2\\
&\quad +a_1b_2c_2 + a_1b_1c_1\\
&\quad = xyc_2 + xb_1c_1 + a_1yc_1 + a_1b_1c_1 + (x + a_1)b_1c_2\\
&\quad +a_1b_2c_2 + a_1b_1c_1\\
&\quad = x(y + b_1)c_2 + xb_1c_1 + a_1(y + b_1)c_1 + a_1b_1c_2\\
&\quad +a_1b_2c_2 + a_1b_1c_1\\
&\quad = xb_2c_2 + (x + a_1)b_1c_1 + a_1b_2c_1 + a_1b_1c_2 + a_1b_2c_2\\
&\quad = (x + a_1)b_2c_2 + a_2b_1c_1 + a_1b_2c_1 + a_1b_1c_2\\
&\quad = a_2b_2c_2 + a_2b_1c_1 + a_1b_2c_1 + a_1b_1c_2.
\end{aligned}$$

As $a_ib_jc_k(i, j, k = 1, 2) \in ABC$, therefore we can prove that $xyz \in \overline{ABC}$, for $x \in \overline{A}, y \in \overline{B}, z \in \overline{C}$. Suppose $t \in \overline{A}\,\overline{B}\,\overline{C}$. Then $t = \sum_{finite} a_ib_ic_i$ for some $a_i \in \overline{A}$, $b_i \in \overline{B}$ and $c_i \in \overline{C}$. Thus $t \in \overline{ABC}$, i.e.$\overline{A}\,\overline{B}\,\overline{C} \subseteq \overline{ABC}$. Therefore $\overline{\overline{A}\,\overline{B}\,\overline{C}} \subseteq \overline{\overline{ABC}} = \overline{ABC}$. Hence $\overline{\overline{A}\,\overline{B}\,\overline{C}} = \overline{ABC}$. ■

Definition 2.9. [4] Let S be a ternary semiring. Then S is called $k-$regular if for each $a \in S$ there exist $x, y \in S$ such that $a + axa = aya$.

Theorem 2.10. *If a ternary semiring S is $k-$regular, then every quasi $k-$ideal Q of S is of the form $Q = \overline{QSQSQ} = \overline{QSS} \cap \overline{SQS + SSQSS} \cap \overline{SSQ}$.*

Proof. Let Q be a quasi $k-$ideal of S. Then $\overline{QSS} \cap \overline{SQS + SSQSS} \cap \overline{SSQ} \subseteq Q$. Let $a \in Q$ and S is $k-$regular, then there exist $x, y \in S$ such that $a + axa = aya$. This implies that $axa + axaxa = ayaxa$. Since $axaxa, ayaxa \in QSQSQ$, therefore $axa \in \overline{QSQSQ}$. Similarly, $aya \in \overline{QSQSQ}$. Therefore $a \in \overline{\overline{QSQSQ}} = \overline{QSQSQ}$ (as $\overline{QSQSQ}$ is $k-$closed). Therefore $Q \subseteq \overline{QSQSQ}$. Again $QSQSQ \subseteq Q(SSS)S \subseteq QSS$ and $QSQSQ \subseteq SSQ$ and $QSQSQ \subseteq SSQSS$ which shows that $\overline{QSQSQ} \subseteq \overline{QSS}$, $\overline{QSQSQ} \subseteq \overline{SSQ}$ and $\overline{QSQSQ} \subseteq \overline{SQS + SSQSS}$ (as $0 \subseteq SQS$). Thus we have $Q \subseteq \overline{QSQSQ} \subseteq \overline{QSS} \cap \overline{SQS + SSQSS} \cap \overline{SSQ} \subseteq Q$ (as Q is a quasi $k-$ideal of S). Hence $Q = \overline{QSQSQ} = \overline{QSS} \cap \overline{SQS + SSQSS} \cap \overline{SSQ}$. ■

Theorem 2.11. *A ternary semiring S is $k-$regular if and only if $R \cap M \cap L = \overline{RML}$ holds for each right $k-$ideal R, lateral $k-$ideal M, and left $k-$ideal L of S.*

Proof. Since R is right $k-$ ideal, therefore $RML \subseteq RSS \subseteq R$ which shows that $\overline{RML}\subseteq\overline{R}=R$. Again, since M is a lateral $k-$ideal of S, then $RML\subseteq SMS\subseteq M$ and so $\overline{RML}\subseteq\overline{M}=M$. Similarly, we obtain that $\overline{RML} \subseteq L$. Therefore we have $RML \subseteq R \cap M \cap L$.

Again, suppose that $a \in R\cap M\cap L$. Since S is $k-$regular therefore there exist $x, y \in S$ such that $a + axa = aya$. This implies that $axa + a(xax)a = a(yax)a$. Since $a(xax)a, a(yax)a \in RML$ therefore $axa \in \overline{RML}$. Similarly, $aya \in \overline{RML}$. Therefore $a \in \overline{\overline{RML}} = \overline{RML}$(as $\overline{RMS}$ is $k-$closed. Hence $R \cap M \cap L \subseteq \overline{RML}$. Thus $R \cap M \cap L = \overline{RML}$.

Conversely, let $a \in S$. Then principal right $k-$ideal generated by a of S is given by $\overline{aSS + Z_0^+a}$. Now,

$$\begin{aligned}
\overline{aSS + Z_0^+a} &= \overline{aSS + Z_0^+a} \cap S \cap S \\
&= \overline{\overline{aSS + Z_0^+a}\,\overline{S}\,\overline{S}}\,\text{(as S is itself (lateral, left) $k-$ideal of S)} \\
&= \overline{(aSS + Z_0^+a)SS}\,\text{(by Lemma 2.8)} \\
&= \overline{aSS}.
\end{aligned}$$

Also, $a = a0S + 1.a \in aSS + Z_0^+a \subseteq \overline{aSS + Z_0^+a} = \overline{aSS}$. Therefore $a \in \overline{aSS}$. Similarly, it can easily be shown that $a \in \overline{SSa}$ and $a \in \overline{(SaS + SSaSS)}$.

Therefore, we have

$$\begin{aligned}
a \in \overline{aSS} \cap \overline{(SaS + SSaSS)} \cap \overline{SSa} &= \overline{\overline{aSS}\,\overline{(SaS + SSaSS)}\,\overline{SSa}} \\
&= \overline{aSS(SaS + SSaSS)SSa} = \overline{aSa}.
\end{aligned}$$

Therefore there exist $x, y \in S$ such that $a + axa = aya$. Thus S is $k-$regular. ∎

Definition 2.12. A ternary subsemiring B of a ternary semiring S is called a bi $k-$ideal of of S if $\overline{BSBSB} \subseteq B$.

Result 2.13. Every quasi $k-$ideal of a ternary semiring is a bi $k-$ideal of S.

Proof. It is obvious by Theorem 2.10. ∎

Result 2.14. Let S be a $k-$regular ternary semiring. Then every bi $k-$ideal B of a ternary semiring is a quasi $k-$ideal of S if $\overline{B} = B$ and $\overline{BSB} \subseteq B$.

Proof. Let B be a bi $k-$ideal of S. Then

$$\begin{aligned}
&\overline{BSS} \cap \overline{(SBS+SSBSS)} \cap \overline{SSB}\\
&\qquad = \overline{\overline{BSS}\,\overline{(SBS+SSBSS)}\,\overline{SSB}} \quad \text{(by Theorem 2.11)}\\
&\qquad = \overline{BSS(SBS+SSBSS)SSB} \quad \text{(by Lemma 2.8)}\\
&\qquad = \overline{B(SSS)B(SSS)B+B(SSS)SB(SSS)SB}\\
&\qquad \subseteq \overline{BSBSB+BSSBSSB}\\
&\qquad \subseteq \overline{\overline{BSBSB}+\overline{B(SSS)SSB}}\\
&\qquad \subseteq \overline{B+\overline{BSB}} \quad \text{(as B is a bi } k-\text{ideal of } S)\\
&\qquad \subseteq \overline{B+B} \quad \text{(by hypothesis).}\\
&\qquad \subseteq \overline{B} = B \quad \text{(by hypothesis).}
\end{aligned}$$

Hence B is a quasi $k-$ideal of S. ■

Theorem 2.15. *The following conditions in a ternary semiring S are equivalent*

(i) *S is $k-$regular.*

(ii) *for every bi $k-$ideal B of S, $B = \overline{BSBSB}$.*

(iii) *for every quasi $k-$ideal Q of S, $Q = \overline{QSQSQ}$.*

Proof. (i) ⇒ (ii) Let $a \in B$. Since S is $k-$regular, then there exist $x, y \in S$ such that $a+axa = aya$. This implies that $axa+axaxa = ayaxa$. Since $axaxa, ayaxa \in BSBSB$, therefore $axa \in \overline{BSBSB}$.

Similarly, $aya \in \overline{BSBSB}$. Therefore, $a \in \overline{BSBSB}$. Thus, $B \subseteq \overline{BSBSB}$. Since B is bi $k-$ideal of S therefore $\overline{BSBSB} \subseteq B$. Hence $B = \overline{BSBSB}$.

(ii) ⇒ (iii) By Result 2.13.

(iii) ⇒ (i) Let the condition (iii) holds. Suppose R be right $k-$ideal, M be lateral $k-$ ideal and L be left $k-$ideal of S, then $Q = R \cap M \cap L$ is a quasi$k-$ideal of S. By hypothesis, we have

$$\begin{aligned}
R \cap M \cap L &= \overline{(R \cap M \cap L)S(R \cap M \cap L)S(R \cap M \cap L)}\\
&\subseteq \overline{RSMSL} \subseteq \overline{RML}.
\end{aligned}$$

But $\overline{RML} \subseteq R \cap M \cap L$. Therefore, $R \cap M \cap L = \overline{RML}$. Therefore, by Theorem 2.11, we have S is $k-$regular. ■

Theorem 2.16. *Let S be a ternary semiring. Then the following are equivalent*

(i) *S is $k-$regular.*

(ii) *$B \cap M = \overline{BMBMB}$ for every lateral $k-$ideal M and bi$k-$ideal B of S.*

(iii) *$Q \cap M = \overline{QMQMQ}$ for every lateral $k-$ideal M and quasi$k-$ideal Q of S.*

Proof. (i) $\Rightarrow$ (ii) Let $a \in B \cap M$. Then $a \in B$ and $a \in M$. Since S is $k-$regular then there exist $x, y \in S$ such that $a + axa = aya$. This implies that $axa + axaxa = ayaxa$. Also, $axaxa + axaxaxa = ayaxaxa$ and $axaxaxa + axaxaxaxa = ayaxaxaxa$. Since $axaxaxaxa, ayaxaxaxa \in BSMSBSMSB \subseteq BMBMB$, therefore by property of $k-$closure it is easy to show that $a \in \overline{BMBMB}$. Therefore, $B \cap M \subseteq \overline{BMBMB}$.

Again, as $BMBMB \subseteq BSBSB$, therefore $\overline{BMBMB} \subseteq \overline{BSBSB} \subseteq B$ (as B is bi $k-$ideal of S). Therefore, $\overline{BMBMB} \subseteq B$. Also, $BMBMB \subseteq SM(SSS) \subseteq M, \overline{BMBMB} \subseteq \overline{M} = M$ (as M is lateral $k-$ideal of S). Whence $\overline{BMBMB} \subseteq B \cap M$. Therefore, $B \cap M = \overline{BMBMB}$.

(ii) $\Rightarrow$ (iii) By Result 2.13, (iii) holds.

(iii) $\Rightarrow$ (i) Since S is itself a lateral $k-$ideal of S, therefore $Q = Q \cap S = \overline{QSQSQ}$. Therefore, by the above theorem the result holds. ∎

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p-FUZZY HYPERGROUPOIDS ASSOCIATED TO THE PRODUCT OF p-FUZZY HYPERGRAPHS[1]

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Abstract. We construct fuzzy hyperoperations from the product of p-fuzzy hypergraphs and then we prove that the fuzzy hyperstructures determined by these hyperoperations are commutative p^2-fuzzy quasi-hypergroups. Some properties of these fuzzy hyperoperations are also listed.

Keywords: p-fuzzy hypergroupoid; product; p-fuzzy hypergraph.

1. Introduction and preliminaries

The connections between graphs and hypergroups had been looked into by several researchers (see, for instance, [4], [7]). Corsini studied the connections between hypergraphs and hypergroups in [5]. Ali studied the hypergroupoid associated to the product of hypergraphs in [1].

In this paper, we construct fuzzy hyperoperations from the product of p-fuzzy hypergraphs and then we prove that the fuzzy hyperstructures determined by these hyperoperations are commutative p^2-fuzzy quasi-hypergroups. Some properties of these fuzzy hyperoperations are also listed. This paper can be seen as a fuzzy version of [1].

We recall some notations of fuzzy hyperstructure theory. A fuzzy subset of a nonempty set H is a function $M : H \to [0,1]$. The collection of all fuzzy subsets of H is denoted by $F(H)$. The p-cut of a fuzzy subset M of H is defined by

$$M_p \doteq \{x \in H \mid M(x) \geq p\}.$$

Given a fuzzy hyperoperation $* : H \times H \to F(H)$, for all $a \in H$, $B \in F(H)$, the fuzzy subset $a * B$ of H is defined by

$$(a * B)(x) \doteq \bigvee_{B(b)>0} (a * b)(x).$$

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Given $A, B \in F(H)$, we give the following definitions

$$\begin{aligned}
A \subseteq B &\doteq A(x) \leq B(x), && \forall x \in H.\\
A = B &\doteq A(x) = B(x), && \forall x \in H.\\
(A \cup B)(x) &\doteq A(x) \vee B(x), && \forall x \in H.\\
(A \cap B)(x) &\doteq A(x) \wedge B(x), && \forall x \in H.
\end{aligned}$$

Proposition 1.1. ([8]) $\forall A, B, C \in F(H)$, *we have the following properties*

(1) $A \cup A = A,\ A \cap A = A$;
(2) $A \cup B = B \cup A,\ A \cap B = B \cap A$;
(3) $(A \cup B) \cup C = A \cup (B \cup C),\ (A \cap B) \cap C = A \cap (B \cap C)$;
(4) $A \cap (A \cup B) = A,\ A \cup (A \cap B) = A$;
(5) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C),\ (A \cap B) \cup C = (A \cup C) \cap (B \cup C)$;
(6) $A \cup \emptyset = A,\ A \cap \emptyset = \emptyset, A \cup H = H, A \cap H = A$.

Definition 1.2. Let A, B be fuzzy subsets of nonempty set H, $A \times B$ is defined to be a fuzzy subset of $H \times H$ such that

$$(A \times B)(a, b) = A(a) \cdot B(b), \quad \forall a, b \in H.$$

Proposition 1.3. *Let A, B, C be fuzzy subsets of nonempty set H, then*

$$A \times (B \cup C) = A \times B \cup A \times C.$$

Proof. For all $a, b \in H$ we have

$$\begin{aligned}
(A \times (B \cup C))(a, b) &= A(a) \cdot (B \cup C)(b)\\
&= A(a) \cdot (B(b) \vee C(b))\\
&= A(a) \cdot B(b) \vee A(a) \cdot C(b)\\
&= (A \times B)(a, b) \vee (A \times C)(a, b)\\
&= ((A \times B) \cup (A \times C))(a, b).
\end{aligned}$$

And so $A \times (B \cup C) = A \times B \cup A \times C$. ■

Proposition 1.4. *Let A, B be fuzzy subsets of a nonempty set H, then for all $p \in [0, 1]$ we have*

$$A_p \times B_p \subseteq (A \times B)_{p^2}.$$

Proof. For all

$$\begin{aligned}
x = (a, b) \in A_p \times B_p &\Rightarrow a \in A_p, b \in B_p\\
&\Rightarrow A(a) \geq p, B(b) \geq p\\
&\Rightarrow (A \times B)(a, b) = A(a)B(b) \geq p^2\\
&\Rightarrow x \in (A \times B)_{p^2}.
\end{aligned}$$

■

A fuzzy hypergroupoid $\langle H; *\rangle$ is a nonempty set H endowed with a fuzzy hyperoperation (i.e. a function $*$ from $H \times H$ to $F(H)$). A p-fuzzy quasi-hypergroup is a fuzzy hypergroupoid such that

$$(x * H)_p = H = (H * x)_p, \quad \forall x \in H.$$

The readers can consult [2], [3], [8] to learn more about hyperstructures and fuzzy sets.

2. Fuzzy hyperoperation $*$

We borrow some definitions from [6].

Definition 2.1. *H is a nonempty set, $\{A_i\}_i$ are fuzzy subsets of H, if there exists a $p \in (0, 1]$ such that*

$$\bigcup_i (A_i)_p = H,$$

then $\langle H; \{A_i\}_i\rangle$ is called a p-fuzzy hypergraph.

Definition 2.2. Let $\Gamma = \langle H; \{A_i\}_i\rangle$ be a p-fuzzy hypergraph, set $E_p(x) = \bigcup_{A_i(x)\geq p} A_i$. The fuzzy hypergroupoid $H_\Gamma = \langle H; *\rangle$ where the fuzzy hyperoperation $*$ is defined by

$$x * y \doteq E_p(x) \cup E_p(y), \quad \forall x, y \in H$$

is called a p-fuzzy hypergraph hypergroupoid or a p-f.h.g. hypergroupoid.

Proposition 2.3. ([6]) *The p-f.h.g. hypergroupoid H_Γ has the following properties for any $x, y \in H$:*

(1) $x * y = x * x \cup y * y$;

(2) $x \in (x * x)_p$;

(3) $y \in (x * x)_p \Leftrightarrow x \in (y * y)_p$;

(4) $\{x, y\} \subseteq (x * y)_p$;

(5) $x * y = y * x$;

(6) $(x * H)_p = H$;

(7) $\langle H; \{x * x\}_{x\in H}\rangle$ *is a p-fuzzy hypergraph;*

(8) $x * x * x = \bigcup_{(x*x)(z)>0} z * z$;

(9) $(x * x) * (x * x) = x * x * x$.

Proof. (1) $x*y = E_p(x)\cup E_p(y) = (E_p(x)\cup E_p(x))\cup(E_p(y)\cup E_p(y)) = x*x\cup y*y$.

(2) It is a special case of (4).

(3) Since $\bigcup\limits_i (A_i)_p = H$, then for any $x \in H$ there exists some $A_i \in F(H)$ such that $A_i(x) \geq p$.

We only prove the implication "$\Rightarrow$" . Since

$$\begin{aligned}(x*x)(y) &= (E_p(x)\cup E_p(x))(y) = (E_p(x))(y)\\ &= \left(\bigcup_{A_i(x)\geq p} A_i\right)(y) = \bigvee_{A_i(x)\geq p} A_i(y) \geq p,\end{aligned}$$

then there exists $A_i \in F(H)$ such that $A_i(x) \geq p$ and $A_i(y) \geq p$. So

$$(y*y)(x) = \bigvee_{A_j(y)\geq p} A_j(x) \geq p.$$

Thus $x \in (y*y)_p$.

(4) $(x*y)(x) = (E_p(x)\cup E_p(y))(x) = (E_p(x))(x)\vee(E_p(y))(x) \geq (E_p(x))(x) = \bigvee\limits_{A_i(x)\geq p} A_i(x) \geq p$. So $x \in (x*y)_p$. Similarly we can prove $y \in (x*y)_p$.

(5) $x*y = E_p(x)\cup E_p(y) = E_p(y)\cup E_p(x) = y*x$.

(6) for any $y \in H$,

$$\begin{aligned}(x*H)(y) &= \left(\bigcup_{t\in H} x*t\right)(y) = \left(\bigcup_{t\in H}(E_p(x)\cup E_p(t))(y)\right)\\ &\geq \left(\bigcup_{t\in H} E_p(t)\right)(y) \geq (E_p(y))(y) = \left(\bigcup_{A_i(y)\geq p} A_i\right)(y)\\ &= \bigvee_{A_i(y)\geq p} A_i(y) \geq p.\end{aligned}$$

So $H \subseteq (x*H)_p$ and thus $(x*H)_p = H$.

(7) From $x \in (x*x)_p$ we know $\bigcup\limits_{x\in H}(x*x)_p = H$. And then $\langle H; \{x*x\}_{x\in H}\rangle$ is a p-fuzzy hypergraph.

(8) $x*x*x = \bigcup\limits_{(x*x)(z)>0} z*x = \bigcup\limits_{(x*x)(z)>0} (z*z)\cup(x*x) = \bigcup\limits_{(x*x)(z)>0} z*z$.

(9) $(x*x)*(x*x) = \bigcup\limits_{(x*x)(a)>0,(x*x)(b)>0} a*b = \bigcup\limits_{(x*x)(a)>0,(x*x)(b)>0} (a*a \cup b*b) = \bigcup\limits_{(x*x)(a)>0} a*a = x*x*x$. ■

Remark 2.4. From (5), (6) of the previous proposition, we know that H_Γ is a commutative p-fuzzy quasi-hypergroup.

3. Fuzzy hyperoperations $\otimes$ and $\oplus$

Proposition 3.1. *Let* $\Gamma_1 = \langle H; \{A_i\}_i\rangle$, $\Gamma_2 = \langle H; \{B_j\}_j\rangle$ *be two p-fuzzy hypergraphs, then*

$$\Gamma_1 \times \Gamma_2 \doteq \langle H \times H; \{A_i\}_i \times \{B_j\}_j\rangle$$

is a p^2*-fuzzy hypergraph and called the product of* Γ_1 *and* Γ_2.

Proof. From Proposition 1.4 we know

$$\bigcup_{i,j}(A_i \times B_j)_{p^2} \supseteq \bigcup_{i,j}((A_i)_p \times (B_j)_p) = \left(\bigcup_i (A_i)_p\right) \times \left(\bigcup_j (B_j)_p\right) = H \times H.$$

And then $\bigcup_{i,j}(A_i \times B_j)_{p^2} = H \times H$. ∎

Definition 3.2. Let $H_{\Gamma_1} = \langle H; *\rangle$, $H_{\Gamma_2} = \langle H; \circ\rangle$ be two p-f.h.g. hypergroupoids associated to Γ_1, Γ_2 respectively. The product p^2-f.h.g. hypergroupoid of $\Gamma_1 \times \Gamma_2$ is defined by $H_{\Gamma_1\times\Gamma_2} = \langle H \times H; \otimes\rangle$ where the fuzzy hyperoperation $\otimes$ is defined by

$$(x_1, x_2) \otimes (y_1, y_2) \doteq (x_1 * y_1) \times (x_2 \circ y_2), \quad \forall x_1, x_2, y_1, y_2 \in H.$$

Proposition 3.3. *The* p^2*-f.h.g. hypergroupoid* $H_{\Gamma_1\times\Gamma_2}$ *has the following properties for any* $X, Y \in H \times H$:

(1) $X \otimes Y \supseteq X \otimes X \cup Y \otimes Y$;

(2) $X \in (X \otimes X)_{p^2}$;

(3) $Y \in (X \otimes X)_{p^2} \nRightarrow X \in (Y \otimes Y)_{p^2}$;

(4) $\{X, Y\} \subseteq (X \otimes Y)_{p^2}$;

(5) $X \otimes Y = Y \otimes X$;

(6) $(X \otimes (H \times H))_{p^2} = H \times H$;

(7) $\langle H; \{X \otimes X\}_{X\in H\times H}\rangle$ *is a* p^2*-fuzzy hypergraph;*

(8) $X \otimes X \otimes X \supseteq \bigcup_{(X\otimes X)(Z)>0} Z \otimes Z$;

(9) $(X \otimes X) \otimes (X \otimes X) \supseteq \bigcup_{(X\otimes X)(Z)>0} Z \otimes Z$.

Proof. Set $X = (x_1, x_2), Y = (y_1, y_2)$, then

(1) $X \otimes Y = (x_1, x_2) \otimes (y_1, y_2) = (x_1 * y_1) \times (x_2 \circ y_2) = ((x_1 * x_1) \cup (y_1 * y_1)) \times ((x_2 \circ x_2) \cup (y_2 \circ y_2)) \supseteq ((x_1 * x_1) \times (x_2 \circ x_2)) \cup ((y_1 * y_1) \times (y_2 \circ y_2)) = ((x_1, x_2) \otimes (x_1, x_2)) \cup ((y_1, y_2) \otimes (y_1, y_2)) = X \otimes X \cup Y \otimes Y$.

(2) It is a special case of (4).

(3) For example, let $\Gamma = \langle \{a,b,c,d\}; \{A_i\}_{i=1}^{3} \rangle$ where $A_1 = \dfrac{0.5}{a} + \dfrac{0}{b} + \dfrac{0.4}{c} + \dfrac{0}{d}$, $A_2 = \dfrac{0}{a} + \dfrac{0.5}{b} + \dfrac{0}{c} + \dfrac{0.8}{d}$, $A_3 = \dfrac{0}{a} + \dfrac{0}{b} + \dfrac{0.5}{c} + \dfrac{0}{d}$, then

$$\bigcup_{i=1}^{3} (A_i)_{0.5} = (A_1)_{0.5} \cup (A_2)_{0.5} \cup (A_3)_{0.5} = \{a,b,c,d\}.$$

Hence, Γ is a 0.5-fuzzy hypergraph.

Set $U = (a,b), V = (c,d)$, then

$$\begin{aligned}
(U \otimes U)(V) &= ((a,b) \otimes (a,b))(c,d) \\
&= ((a * a) \times (b \circ b))(c,d) \\
&= (a * a)(c) \cdot (b \circ b)(d) \\
&= \left(\bigvee_{A_i(a) \geq 0.5} A_i(c) \right) \cdot \left(\bigvee_{A_j(b) \geq 0.5} A_j(d) \right) \\
&= A_1(c) \cdot A_2(d) \\
&= 0.4 \times 0.8 \\
&= 0.32 \\
&\geq 0.25 \\
&= (0.5)^2.
\end{aligned}$$

But

$$\begin{aligned}
(V \otimes V)(U) &= ((c,d) \otimes (c,d))(a,b) \\
&= ((c * c) \times (d \circ d))(a,b) \\
&= (c * c)(a) \cdot (d \circ d)(b) \\
&= \left(\bigvee_{A_i(c) \geq 0.5} A_i(a) \right) \cdot \left(\bigvee_{A_j(d) \geq 0.5} A_j(b) \right) \\
&= A_3(a) \cdot A_2(b) \\
&= 0 \times 0.5 \\
&= 0 \\
&< 0.25 \\
&= (0.5)^2.
\end{aligned}$$

So $V \in (U \otimes U)_{0.5^2}$ but $U \overline{\in} (V \otimes V)_{0.5^2}$.

(4) $(X\otimes Y)(X) = ((x_1,x_2)\otimes(y_1,y_2))(x_1,x_2) = ((x_1*y_1)\times(x_2\circ y_2))(x_1,x_2) = (x_1*y_1)(x_1)\cdot(x_2\circ y_2)(x_2) \geq p\cdot p = p^2$. Thus $X\in(X\otimes Y)_{p^2}$. Similarly we can prove $Y\in(X\otimes Y)_{p^2}$.

(5) $X\otimes Y = (x_1,x_2)\otimes(y_1,y_2) = (x_1*y_1)\times(x_2\circ y_2) = (y_1*x_1)\times(y_2\circ x_2) = (y_1,y_2)\otimes(x_1,x_2) = Y\otimes X$.

(6) for any $T\in H\times H$,

$$\begin{aligned}(X\otimes(H\times H))(T) &= \left(\bigcup_{Y\in H\times H} X\otimes Y\right)(T)\\ &\geq (X\otimes T)(T)\\ &\geq ((X\otimes X)\cup(T\otimes T))(T)\\ &\geq (T\otimes T)(T)\\ &\geq p^2.\end{aligned}$$

So $H\times H\subseteq(X\otimes(H\times H))_{p^2}$ and thus $(X\otimes(H\times H))_{p^2} = H\times H$.

(7) From $\bigcup\limits_{X\in H\times H}(X\otimes X)_{p^2}\supseteq\bigcup\limits_{X\in H\times H}\{X\} = H\times H$ we know

$$\bigcup_{X\in H\times H}(X\otimes X)_{p^2} = H\times H.$$

Then $\langle H;\{X\otimes X\}_{X\in H\times H}\rangle$ is a p^2-fuzzy hypergraph.

(8) $X\otimes X\otimes X = \bigcup\limits_{(X\otimes X)(Z)>0} Z\otimes X \supseteq \bigcup\limits_{(X\otimes X)(Z)>0}((Z\otimes Z)\cup(X\otimes X)) = \bigcup\limits_{(X\otimes X)(Z)>0} Z\otimes Z$.

(9) $(X\otimes X)\otimes(X\otimes X) = \bigcup\limits_{(X\otimes X)(S)>0,(X\otimes X)(Z)>0} S\otimes Z \supseteq \bigcup\limits_{(X\otimes X)(S)>0,(X\otimes X)(Z)>0}(S\otimes S\cup Z\otimes Z) = \bigcup\limits_{(X\otimes X)(Z)>0} Z\otimes Z$. ∎

Remark 3.4. From (5), (6) of the above Proposition we know that $H_{\Gamma_1\times\Gamma_2}$ is a commutative p^2-fuzzy quasi-hypergroup.

Definition 3.5. The fuzzy hyperoperation $\oplus$ on $H_{\Gamma_1\times\Gamma_2}$ is defined by

$$(x_1,x_2)\oplus(y_1,y_2) \doteq (x_1*x_1)\times(x_2\circ x_2)\cup(y_1*y_1)\times(y_2\circ y_2),\quad \forall x_1,x_2,y_1,y_2\in H.$$

Proposition 3.6. *The p^2-f.h.g. hypergroupoid $H^{\oplus}_{\Gamma_1\times\Gamma_2} = \langle H\times H;\oplus\rangle$ has the following properties for any $X,Y\in H\times H$:*

(1) $X\oplus Y = X\oplus X\cup Y\oplus Y$;

(2) $X \in (X \oplus X)_{p^2}$;

(3) $Y \in (X \oplus X)_{p^2} \nRightarrow X \in (Y \oplus Y)_{p^2}$;

(4) $\{X, Y\} \subseteq (X \oplus Y)_{p^2}$;

(5) $X \oplus Y = Y \oplus X$;

(6) $(X \oplus (H \times H))_{p^2} = H \times H$;

(7) $\langle H; \{X \oplus X\}_{X \in H \times H} \rangle$ *is a* p^2*-fuzzy hypergraph;*

(8) $X \oplus X \oplus X = \bigcup_{(X \oplus X)(Z) > 0} Z \oplus Z$;

(9) $(X \oplus X) \oplus (X \oplus X) = X \oplus X \oplus X$.

Proof. Set $X = (x_1, x_2), Y = (y_1, y_2)$, then

(1) Since $X \oplus Y = (x_1, x_2) \oplus (y_1, y_2) = (x_1 * x_1) \times (x_2 \circ x_2) \cup (y_1 * y_1) \times (y_2 \circ y_2)$ and $X \oplus X \cup Y \oplus Y = (x_1, x_2) \oplus (x_1, x_2) \cup (y_1, y_2) \oplus (y_1, y_2) = (x_1 * x_1) \times (x_2 \circ x_2) \cup (y_1 * y_1) \times (y_2 \circ y_2)$. Then $X \oplus Y = X \oplus X \cup Y \oplus Y$.

(2) It is a special case of (4).

(3) For example, set $\Gamma = \langle \{a, b, c, d\}; \{A_i\}_{i=1}^3 \rangle$ where $A_1 = \frac{0.5}{a} + \frac{0}{b} + \frac{0.4}{c} + \frac{0}{d}$, $A_2 = \frac{0}{a} + \frac{0.5}{b} + \frac{0}{c} + \frac{0.8}{d}$, $A_3 = \frac{0}{a} + \frac{0}{b} + \frac{0.5}{c} + \frac{0}{d}$, then from Proposition 3.3 we know that Γ is a 0.5-fuzzy hypergraph.

Set $U = (a, b), V = (c, d)$, then

$$
\begin{aligned}
(U \oplus U)(V) &= ((a, b) \oplus (a, b))(c, d) \\
&= ((a * a) \times (b \circ b))(c, d) \\
&= (a * a)(c) \cdot (b \circ b)(d) \\
&= \left(\bigvee_{A_i(a) \geq 0.5} A_i(c) \right) \cdot \left(\bigvee_{A_j(b) \geq 0.5} A_j(d) \right) \\
&= A_1(c) \cdot A_2(d) \\
&= 0.4 \times 0.8 \\
&= 0.32 \\
&\geq 0.25 \\
&= (0.5)^2.
\end{aligned}
$$

But

$$\begin{aligned}
(V\oplus V)(U) &= ((c,d)\oplus(c,d))(a,b)\\
&= ((c*c)\times(d\circ d))(a,b)\\
&= (c*c)(a)\cdot(d\circ d)(b)\\
&= \left(\bigvee_{A_i(c)\geq 0.5} A_i(a)\right)\cdot\left(\bigvee_{A_j(d)\geq 0.5} A_j(b)\right)\\
&= A_3(a)\cdot A_2(b)\\
&= 0\times 0.5\\
&= 0\\
&< 0.25\\
&= (0.5)^2.
\end{aligned}$$

So $V\in(U\oplus U)_{0.5^2}$ but $U\overline{\in}(V\oplus V)_{0.5^2}$.

(4) $(X\oplus Y)(X)=((x_1,x_2)\oplus(y_1,y_2))(x_1,x_2)=((x_1*x_1)\times(x_2\circ x_2)\cup(y_1*y_1)\times(y_2\circ y_2))(x_1,x_2)\geq((x_1*x_1)\times(x_2\circ x_2))(x_1,x_2)=(x_1*x_1)(x_1)\cdot(x_2\circ x_2)(x_2)\geq p\cdot p=p^2$. Thus $X\in(X\oplus Y)_{p^2}$. Similarly we can prove $Y\in(X\oplus Y)_{p^2}$.

(5) $X\oplus Y=(x_1,x_2)\oplus(y_1,y_2)=((x_1*x_1)\times(x_2\circ x_2))\cup((y_1*y_1)\times(y_2\circ y_2))=((y_1*y_1)\times(y_2\circ y_2))\cup((x_1*x_1)\times(x_2\circ x_2))=(y_1,y_2)\oplus(x_1,x_2)=Y\oplus X$.

(6) for any $T\in H\times H$,

$$\begin{aligned}
(X\oplus(H\times H))(T) &= \left(\bigcup_{Y\in H\times H} X\oplus Y\right)(T)\\
&\geq (X\oplus T)(T)\\
&= ((X\oplus X)\cup(T\oplus T))(T)\\
&\geq (T\oplus T)(T)\\
&\geq p^2.
\end{aligned}$$

So $H\times H\subseteq(X\oplus(H\times H))_{p^2}$ and thus $(X\oplus(H\times H))_{p^2}=H\times H$.

(7) From $\bigcup\limits_{X\in H\times H}(X\oplus X)_{p^2}\supseteq\bigcup\limits_{X\in H\times H}\{X\}=H\times H$ we know

$$\bigcup_{X\in H\times H}(X\oplus X)_{p^2}=H\times H.$$

Then $\langle H;\{X\oplus X\}_{X\in H\times H}\rangle$ is a p^2-fuzzy hypergraph.

(8) $X\oplus X\oplus X=\bigcup\limits_{(X\oplus X)(Z)>0}T\oplus X=\bigcup\limits_{(X\otimes X)(Z)>0}((Z\otimes Z)\cup(X\otimes X))$
$=\bigcup\limits_{(X\otimes X)(Z)>0}Z\otimes Z.$

$$(9)\ (X \oplus X) \oplus (X \oplus X) = \bigcup_{(X\oplus X)(S)>0, (X\oplus X)(Z)>0} S \oplus Z$$

$$= \bigcup_{(X\oplus X)(S)>0, (X\oplus X)(Z)>0} (S \oplus S \cup Z \oplus Z) = \bigcup_{(X\oplus X)(Z)>0} Z \oplus Z = X \oplus X \oplus X. \qquad \blacksquare$$

Remark 3.7. From (5), (6) of the previous Proposition we know that $H^{\oplus}_{\Gamma_1\times\Gamma_2} = \langle H \times H; \oplus\rangle$ is a commutative p^2-fuzzy quasi-hypergroup.

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A NOTE ON TESTING OF HYPOTHESIS

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Abstract. In this paper problem of testing of hypothesis is discussed when the samples have been drawn from normal distribution. The study of hypothesis testing is also extended to Bayes set up.

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Keywords: hypothesis, level of significance, Bayes rule.

Let the random variable (r.v.) X have a normal distribution $N(\theta, \sigma_2)$, where σ_2 is assumed to be known.

The hypothesis $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$, $\theta_1 > \theta_0$ is to be tested.

Let $X_1, X_2, ..., X_n$ be a random sample from $N(\theta, \sigma^2)$ population.

Let $\overline{X}$ $\left(= \frac{1}{n} \sum_{n}^{i=1} X_i \right)$ be the sample mean.

By Neyman–Pearson lemma, the most powerful test rejects H_0 at α % level of significance, if $\dfrac{\sqrt{n}\ (\overline{X}-\theta_0)}{\sigma} \geq d_\alpha$, where d_α is such that

$$\int_{d_\alpha}^{\infty} \frac{1}{\sqrt{2\pi}}\ e^{-\dfrac{Z^2}{2}}\, dZ = \alpha.$$

If the sample is such that H_0 is rejected, then will it imply that H_1 will be accepted?

In general, this will not be true for all values of θ_1, but will be true for some specific value of θ_1, i.e., when θ_1 is at a specific distance from θ_0.

H_0 is rejected

$$\text{(1)} \qquad \text{if } \frac{\sqrt{n}\ (\overline{X}-\theta_0)}{\sigma} \geq d_\alpha, \text{ i.e., } \overline{X} \geq \theta_0 + d_\alpha\ \frac{\sigma}{\sqrt{n}}.$$

Similarly, the Most Powerful Test will accept H_1 against H_0

$$\text{(2)} \qquad \text{if } \frac{\sqrt{n}\ (\overline{X}-\theta_0)}{\sigma} \geq d_\alpha, \text{ i.e., } \overline{X} \geq \theta_1 - d_\alpha\ \frac{\sigma}{\sqrt{n}}.$$

Rejecting H_0 will mean accepting H_1

$$\text{(3)} \qquad \begin{aligned} &\text{if} && (1) \Longrightarrow (2) \\ &\text{i.e.,} && \overline{X} \geq \theta_0 + d_\alpha\ \frac{\sigma}{\sqrt{n}} \Longrightarrow \overline{X} \geq \theta_1 - d_\alpha\ \frac{\sigma}{\sqrt{n}} \\ &\text{i.e.,} && \theta_1 - d_\alpha\ \frac{\sigma}{\sqrt{n}} \leq \theta_0 + d_\alpha\ \frac{\sigma}{\sqrt{n}}. \end{aligned}$$

Similarly, accepting H_1 will mean rejecting H_0

$$\text{(4)} \qquad \begin{aligned} &\text{if} && (2) \Longrightarrow (1) \\ &\text{i.e.,} && \theta_0 + d_\alpha\ \frac{\sigma}{\sqrt{n}} \leq \theta_1 - d_\alpha\ \frac{\sigma}{\sqrt{n}}. \end{aligned}$$

From (3) and (4) we have

$$\text{(5)} \qquad \theta_0 + d_\alpha\ \frac{\sigma}{\sqrt{n}} = \theta_1 - d_\alpha\ \frac{\sigma}{\sqrt{n}} \text{ i.e., } \theta_1 - \theta_0 = 2d_\alpha\ \frac{\sigma}{\sqrt{n}}\cdot$$

Thus,

$$d_\alpha\ \frac{\sigma}{\sqrt{n}} = \frac{\theta_1-\theta_0}{2} \quad \text{and} \quad \theta_1 = \theta_0 + 2d_\alpha\ \frac{\sigma}{\sqrt{n}}\cdot$$

From (1),

$$\text{Reject } H_0 \text{ if } \overline{X} > \theta_0 + \frac{\theta_1-\theta_0}{2} = \frac{\theta_0+\theta_1}{2}$$

and from (2),

Accept H_1 if $\overline{X} > \theta_1 - \dfrac{\theta_1 - \theta_0}{2} = \dfrac{\theta_0 + \theta_1}{2}\cdot$

Thus, rejecting H_0 will mean accepting H_1 when

$$\overline{X} > \frac{\theta_0 + \theta_1}{2}\cdot$$

From (5), this will be true only when

$$\theta_1 = \theta_0 + 2d_\alpha \frac{\sigma}{\sqrt{n}}\cdot$$

For other values of $\theta_1 \neq \theta_0 + 2d_\alpha \dfrac{\sigma}{\sqrt{n}}$ rejecting H_0 will not mean accepting H_1.

Therefore, it is recommended that, instead of testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$, $\theta_1 > \theta_0$, it is more appropriate to test $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_0$.

In this situation, rejecting H_0 will mean $\theta >= \theta_0$ and is not equal to some given value $= \theta_1$.

But in Baye's setup, rejecting H_0 means accepting H_1 whatever may be H_0 and H_1.

In this set up, the level of significance is not a preassigned constant, but depends on H_0, H_1, σ_2 and n.

Consider $(0, 1)$ loss function and equal prior probabilities $1/2$ for θ_0 and θ_1. The Baye's test rejects H_0 (accept H_1)

$$\text{if } \overline{X} > \frac{\theta_0 + \theta_1}{2}$$

and accepts H_0 (rejects H_1)

$$\text{if } \overline{X} < \frac{\theta_0 + \theta_1}{2}\cdot$$

[See Rohatagi, p.463, Example 2].

The level of significance is given by

$$P_{H_0}\left[\overline{X} > \frac{\theta_0 + \theta_1}{2}\right] = P_{H_0}\left[\frac{(\overline{X} - \theta_0)\sqrt{n}}{\sigma} > \frac{(\theta_1 - \theta_0)\sqrt{n}}{2\sigma}\right] = 2 - \Phi\left(\frac{\sqrt{n}(\theta_1 - \theta_0)}{2\sigma}\right),$$

where

$$\Phi(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{Z^2}{2}} dZ.$$

Thus, the level of significance depends on $\theta_0, \theta_1, \sigma^2$ and n.

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SOME PROPERTIES OF n-ISOCLINISM IN LIE ALGEBRAS

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Abstract. In 1940, P. Hall introduced the concept of isoclinism of groups and it was generalized to n-isoclinism and isologism with respect to a given variety of groups.

In the present article this notion is studied in Lie algebras and give some results similar to N.S. Hekster in 1986. In particular, it is shown that every family of n-isoclinism of Lie algebras contains an n-stem Lie algebra of minimal dimension.

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Keywords: n-isoclinism, Lie algebra, n-stem Lie algebra.

1. Introduction and Preliminaries

Let L be a Lie algebra, then the *lower* and *upper central series* of L are defined as follows:

$$L = L^1 \supseteq L^2 \supseteq \cdots \supseteq L^{n+1} \supseteq \cdots ,$$

and

$$(0) = Z_0(L) \subseteq Z_1(L) = Z(L) \subseteq Z_2(L) \subseteq \cdots \subseteq Z_n(L) \subseteq \cdots ,$$

respectively, where $L^{n+1} = [L, L^n]$ and

$$\frac{Z_n(L)}{Z_{n-1}(L)} = Z\left(\frac{L}{Z_{n-1}(L)}\right).$$

The following definition is vital in our investigation and it is similar to the case for groups (see [1] or [2]).

Definition 1.1. Let L and K be two Lie algebras, $\alpha : \frac{L}{Z_n(L)} \longrightarrow \frac{K}{Z_n(K)}$ and $\beta : L^{n+1} \longrightarrow K^{n+1}$ be Lie algebra homomorphisms such that the following diagram is commutative

$$\begin{tikzcd}
\frac{L}{Z_n(L)} \times \cdots \times \frac{L}{Z_n(L)} \arrow[r, "\varphi"] \arrow[d, "\underbrace{\alpha \times \cdots \times \alpha}_{(n+1)-times}"'] & L^{n+1} \arrow[d, "\beta"] \\
\underbrace{\frac{K}{Z_n(K)} \times \cdots \times \frac{K}{Z_n(K)}}_{(n+1)-times} \arrow[r, "\psi"] & K^{n+1}
\end{tikzcd}$$

where $\varphi : (\bar{l_1}, \bar{l_2}, ..., \bar{l}_{n+1}) \mapsto [l_1, l_2, ..., l_{n+1}]$, for all $\bar{l_i} \in \frac{L}{Z_n(L)}$, $i = 1, ..., n+1$ and similarly for ψ. In fact, α and β are defined in such a way that they are compatible, i.e., for all $l_i \in L$, $\beta([l_1, l_2, \cdots, l_{n+1}]) = [k_1, k_2, ..., k_{n+1}]$, in which $k_i \in \alpha(l_i + Z_n(L)), 1 \le i \le n+1$.

The pair (α, β) is called *n-homoclinism* and if they are both isomorphisms, then (α, β) is called *n-isoclinism*. In this case, L and K are said to be *n-isoclinic*, which is denoted by $L \underset{n}{\sim} K$. If $n = 1$, then it will be the notion of isoclinism, which was first introduced by P. Hall [1] in 1940. The kernel and the image of (α, β) are defined as follows:

$$Ker(\alpha, \beta) = Ker\beta \quad \text{and} \quad Im(\alpha, \beta) = I \subseteq K,$$

where $\alpha\left(\frac{L}{Z_n(L)}\right) = \frac{I}{Z_n(K)}$.

Now, the above definition gives the following

Theorem 1.2. *Let (α, β) be an n-homoclinism of Lie algebras L into K, then*

(i) $Ker(\alpha, \beta) \trianglelefteq L$;

(ii) $\frac{L}{Ker(\alpha, \beta)} \underset{n}{\sim} Im(\alpha, \beta)$.

Proof. It is obvious that $Ker(\alpha,\beta) = Ker\beta$ is an ideal of L. To prove the second part, assume that

$$\begin{aligned} \alpha\left(\frac{L}{Z_n(L)}\right) &= \frac{I}{Z_n(K)}, \\ Z_n\left(\frac{L}{Ker\beta}\right) &= \frac{J}{Ker\beta}. \end{aligned}$$

Clearly, $J \trianglelefteq L$, $Z_n(L) + Ker\beta \trianglelefteq J$, $Z_n(K) \trianglelefteq Z_n(I)$ and $Im\beta = I^{n+1}$. Thus

$$\frac{L/Ker\beta}{Z_n(L/Ker\beta)} \cong \frac{L}{J} \cong \frac{L/Z_n(L)}{J/Z_n(L)}.$$

Now, the homomorphism α induces the homomorphism

$$\begin{aligned} \bar{\alpha} : \frac{L}{Z_n(L)} &\longrightarrow \frac{I/Z_n(K)}{Z_n(I)/Z_n(K)} \\ l + Z_n(L) &\longmapsto \alpha(l + Z_n(L)) + \frac{Z_n(I)}{Z_n(K)} \end{aligned}$$

Clearly, $Ker\bar{\alpha} = \dfrac{J}{Z_n(L)}$ and $\bar{\alpha}$ is surjective. We also have $\dfrac{L}{J} \cong \dfrac{I}{Z_n(I)}$, which implies that

$$\frac{L/Ker\beta}{Z_n(L/Ker\beta)} \cong \frac{I}{Z_n(I)}.$$

Since $(L/Ker\beta)^{n+1} = L^{n+1}/Ker\beta$, the homomorphism β induces the homomorphism $\bar{\beta} : L^{n+1}/Ker\beta \longrightarrow I^{n+1}$. Therefore, a pair $(\bar{\alpha}, \bar{\beta})$ is an n-isoclinism from $L/Ker(\alpha,\beta)$ onto $Im(\alpha,\beta)$ and the proof is complete. ∎

2. Some properties of n-isoclinism of Lie algebras

In this section, some of the basic properties of n-isoclinism in Lie algebras are discussed.

The following lemma is useful in proving the next result and its proof is straightforward.

Lemma 2.1. *Let L_1 and L_2 be two Lie algebras and $L = L_1 \oplus L_2$, then, for all $n \geq 1$,*

$$L^n = L_1^n \oplus L_2^n \quad \text{and} \quad Z_n(L) = Z_n(L_1) \oplus Z_n(L_2).$$

Theorem 2.2. *Let L be a Lie algebra and M an abelian Lie algebra. Then, for all $n > 1$,*

$$L \underset{n}{\sim} L \oplus M.$$

Proof. Clearly, since M is abelian we have $M^{n+1} = 0$ and $Z_n(M) = M$, for some $n \geq 1$. Hence

$$Z_n(L \oplus M) = Z_n(L) \oplus M,\ (L \oplus M)^{n+1} = L^{n+1}.$$

Now, we define

$$\begin{aligned} \alpha : \frac{L}{Z_n(L)} &\longrightarrow \frac{L \oplus M}{Z_n(L \oplus M)} \\ x + Z_n(L) &\longmapsto x + Z_n(L \oplus M) \end{aligned}$$

and β is assumed to be the identity on L^{n+1}. Then α and β are both isomorphisms and the following diagram is commutative

$$\begin{tikzcd}
(x_1 + Z_n(L), ..., x_{n+1} + Z_n(L)) \arrow[r, mapsto] \arrow[d, "\alpha \times \cdots \times \alpha"'] & {[x_1, ..., x_{n+1}]} \arrow[d, "\beta"] \\
(x_1 + Z_n(L \oplus M), ..., x_{n+1} + Z_n(L \oplus M) \arrow[r, mapsto] & {[x_1, ..., x_{n+1}]},
\end{tikzcd}$$

where $x_i \in \alpha(x_i + Z_n(L))$, for $i = 1, \cdots, n+1$. Thus

$$L \underset{n}{\sim} L \oplus M.$$

■

The following result is a criterion for two Lie algebras being n-isoclinic.

Theorem 2.3. *Two Lie algebras L and K are n-isoclinic if and only if there exist ideals L_1 and K_1 of L and K contained in $Z_n(L)$ and $Z_n(K)$, respectively, and the isomorphisms $\alpha : L/L_1 \longrightarrow K/K_1$ and $\beta : L^{n+1} \longrightarrow K^{n+1}$ such that α induces β.*

Proof. If $L \underset{n}{\sim} K$, then choosing $L_1 = Z_n(L)$ and $K_1 = Z_n(K)$ will do the job.

Conversely, consider the pair of isomorphisms (α, β) satisfying the assumption. We show that $\alpha\left(\frac{Z_n(L)}{L_1}\right) = \frac{Z_n(K)}{K_1}$. To do this, let the elements $x \in Z_n(L)$, $y \in \alpha(x + L_1)$, $l_1, ..., l_n \in L$ and $k_1, ..., k_n \in \alpha(l_i + L_1)$ such that

$$[y, k_1, ..., k_n] = \beta[x, l_1, ..., l_n] = \beta(0) = 0,$$

which implies that $y \in Z_n(K)$ and hence

$$\alpha\left(\frac{Z_n(L)}{L_1}\right) \subseteq \frac{Z_n(K)}{K_1}.$$

The reverse inclusion follows similarly, by using the property of β. Thus

$$\alpha\left(\frac{Z_n(L)}{L_1}\right) = \frac{Z_n(K)}{K_1},$$

which gives the following isomorphism

$$\bar{\alpha} : \frac{L}{Z_n(L)} \longrightarrow \frac{K}{Z_n(K)}.$$

Now, the pair of isomorphisms $(\bar{\alpha}, \beta)$ implies that $L \underset{n}{\sim} K$. ■

The following corollary is an immediate consequence of the above theorem.

Corollary 2.4. *Let L and K be Lie algebras. If $L \underset{n}{\sim} K$, then $L \underset{m}{\sim} K$, for all $m \geq n$.*

Theorem 2.5. *Let (α, β) be the pair of n-isoclinism between two Lie algebras L_1 and L_2. Then, for all $n \geq 1$*

(a) *if M_1 is a subalgebra of L_1 containing $Z_n(L_1)$ such that $\alpha(M_1/Z_n(L_1)) = M_2/Z_n(L_2)$, then $M_1 \underset{n}{\sim} M_2$.*

(b) *if M_1 is an ideal of L_1 and $M_1 \subseteq L_1^{n+1}$, then $M_2 = \beta(M_1) \lhd L_2$ and $\dfrac{L_1}{M_1} \underset{n}{\sim} \dfrac{L_2}{M_2}$.*

Proof. (a) Clearly, $Z_n(L_1) \subseteq Z_n(M_1)$ and similarly $Z_n(L_2) \subseteq Z_n(M_2)$.

Now, as in Theorem 2.3, by choosing $L_1 = Z_n(L_1)$, $K_1 = Z_n(L_2)$, $\bar{\alpha} = \alpha\big|_{\frac{M_1}{Z_n(L_1)}}$ and $\bar{\beta} = \beta\big|_{M_1^{n+1}}$, then $(\bar{\alpha}, \bar{\beta})$ is the required n-isoclinim pair for $M_1 \underset{n}{\sim} M_2$.

(b) We first show that $M_2 = \beta(M_1)$ is an ideal of L_2, i.e., $[\beta(m_1), y] \in M_2$ for all $m_1 \in M_1$ and $y \in L_2$. By the assumption $M_1 \subseteq L_1^{n+1}$, and hence

$$\alpha(m_1 + Z_n(L_1)) = \beta(m_1) + Z_n(L_2).$$

Thus there exists $x \in L_1$ so that for any $y \in \alpha(x + Z_n(L_1))$,

$$\beta[m_1, x] = [\beta(m_1), y] \in M_2,$$

which implies that $M_2 \lhd L_2$.

Clearly, $\left(\dfrac{L_i}{M_i}\right)^{n+1} = \dfrac{L_i^{n+1}}{M_i}$, $i = 1, 2$ and $\beta(M_1) = M_2$. Therefore, β induces

$$\bar{\beta} : \left(\frac{L_1}{M_1}\right)^{n+1} \longrightarrow \left(\frac{L_2}{M_2}\right)^{n+1}.$$

Now, since $\alpha\left(\dfrac{Z_n(L_1) + M_1}{Z_n(L_1)}\right) = \dfrac{Z_n(L_2) + M_2}{Z_n(L_2)}$, the following isomorphism

$$\frac{L_1/Z_n(L_1)}{(Z_n(L_1)+M_1)/Z_n(L_1)} \cong \frac{L_2/Z_n(L_2)}{(Z_n(L_2)+M_2)/Z_n(L_2)},$$

implies $\dfrac{L_1}{Z_n(L_1)+M_1} \cong \dfrac{L_2}{Z_n(L_2)+M_2}$. So they give the following isomorphisms

$$\bar{\alpha} : \frac{L_1/M_1}{(Z_n(L_1)+M_1)/M_1} \cong \frac{L_1}{Z_n(L_1)+M_1} \cong \frac{L_2}{Z_n(L_2)+M_2} \cong \frac{L_2/M_2}{(Z_n(L_2)+M_2)/M_2}$$

One can easily see that, using Theorem 2.3, $\dfrac{L_1}{M_1} \underset{n}{\sim} \dfrac{L_2}{M_2}$, via the n-isoclinism pair $(\bar{\alpha}, \bar{\beta})$. ■

Theorem 2.6. *Let K and M be subalgebra and ideal of a Lie algebra L, respectively. Then for all $n \geq 0$,*

(a) *$K \underset{n}{\sim} K + Z_n(L)$. In particular, if $L = K + Z_n(L)$ then $L \underset{n}{\sim} K$.*

Conversely, if $\dfrac{L}{Z_n(L)}$ is finite dimensional and $L \underset{n}{\sim} K$, then $L = K + Z_n(L)$.

(b) *$\dfrac{L}{M} \underset{n}{\sim} \dfrac{L}{M \cap L^{n+1}}$. In particular, if $L \cap L^{n+1} = 0$, then $\dfrac{L}{M} \underset{n}{\sim} L$.*

Conversely, if L^{n+1} is of finite dimension and $L \underset{n}{\sim} \dfrac{L}{M}$, then $L^{n+1} \cap M = 0$.

Proof. (a) Clearly, $Z_n(K + Z_n(L)) = Z_n(K) + Z_n(L)$, for all $n \geq 0$. Now, consider

$$\alpha : \frac{K}{Z_n(K)} \longrightarrow \frac{K + Z_n(L)}{Z_n(K + Z_n(L))}$$

given by $\alpha : k + Z_n(K) \mapsto k + Z_n(K + Z_n(L))$, for all $k \in K$ and $\beta = id_{K^{n+1}}$. Then the pair of (α, β), guaranties the property $K \underset{n}{\sim} K + Z_n(L)$. In particular, if $L = K + Z_n(L)$ then $L \underset{n}{\sim} K$.

Conversely, suppose that $dim\left(\dfrac{L}{Z_n(L)}\right) < \infty$ and $J = K + Z_n(L)$. Then, we show that $L = J$. Clearly, $L \underset{n}{\sim} K \underset{n}{\sim} J$, which imply that $L \underset{n}{\sim} J$ and hence $\dfrac{L}{Z_n(L)} \cong \dfrac{J}{Z_n(J)}$ and $Z_n(L) \subseteq Z_n(J)$. Thus, $dim\left(\dfrac{J}{Z_n(J)}\right)$ is finite and it follows that

$$dim\left(\frac{J}{Z_n(J)}\right) \leq dim\left(\frac{J}{Z_n(L)}\right).$$

Hence, $dim\left(\dfrac{L}{Z_n(L)}\right) \leq dim\left(\dfrac{J}{Z_n(L)}\right) \leq dim\left(\dfrac{L}{Z_n(L)}\right)$, which follow that $J = L$.

(b) Put $\bar{L} = L/M$ and $\tilde{L} = \dfrac{L}{M \cap L^{n+1}}$. Then one can easily check that $\bar{l} \in Z_n(\bar{L})$ if and only if $\tilde{l} \in Z_n(\tilde{L})$. Therefore, the following maps

$$\begin{array}{rcl} \alpha : \dfrac{\bar{L}}{Z_n(\bar{L})} & \longrightarrow & \dfrac{\tilde{L}}{Z_n(\tilde{L})} \\ \bar{l} + Z_n(\bar{L}) & \longmapsto & \tilde{l} + Z_n(\tilde{L}) \end{array}$$

and

$$\begin{array}{rcl} \beta : \bar{L}^{n+1} & \longrightarrow & \tilde{L}^{n+1} \\ \bar{l} & \longmapsto & \tilde{l} \end{array}$$

are the required n-isoclinism pair.

Now, if $M \cap L^{n+1} = 0$, then $\dfrac{L}{M} \underset{n}{\sim} L$.

To prove the converse, since $L \underset{n}{\sim} \dfrac{L}{M} \underset{n}{\sim} \dfrac{L}{M \cap L^{n+1}}$, we have

$$L^{n+1} \cong \left(\frac{L}{M \cap L^{n+1}} \right)^{n+1} = \frac{L^{n+1}}{M \cap L^{n+1}},$$

which implies that

$$dimL^{n+1} = dimL^{n+1} + dim(M \cap L^{n+1})$$

and so $dim(M \cap L^{n+1}) = 0$. Hence $M \cap L^{n+1} = 0$. ■

Theorem 2.7. *Let $f : L \longrightarrow K$ be a homomorphism of Lie algebras such that $f(Z_n(L)) \subseteq Z_n(K)$, for some $n \geq 1$. If*

$$\begin{array}{rcl} \alpha : \dfrac{L}{Z_n(L)} & \longrightarrow & \dfrac{K}{Z_n(K)} \\ l + Z_n(L) & \longmapsto & f(l) + Z_n(K) \end{array}$$

and

$$\begin{array}{rcl} \beta : L^{n+1} & \longrightarrow & K^{n+1} \\ [l_1, ..., l_{n+1}] & \longmapsto & [f(l_1), ..., f(l_{n+1})], \end{array}$$

for all $l, l_1, ..., l_{n+1} \in L$. Then, the following statements hold:

(a) *(α, β) is an n-homoclinism of L into K;*

(b) *$L/Kerf \underset{n}{\sim} L/Ker(\alpha, \beta)$;*

(c) *$Imf \underset{n}{\sim} Im(\alpha, \beta)$.*

Proof. (a) is obvious.

(b) Clearly, $Ker(\alpha,\beta) = Ker\beta = Kerf \cap L^{n+1}$. Then, using Theorem 2.6(b), it implies that

$$\frac{L}{Kerf} \underset{n}{\sim} \frac{L}{Kerf \cap L^{n+1}} = \frac{L}{Ker(\alpha,\beta)}.$$

(c) By Theorem 1.2,

$$Imf \cong \frac{L}{Kerf} \underset{n}{\sim} \frac{L}{Ker(\alpha,\beta)} \underset{n}{\sim} Im(\alpha,\beta). \qquad \blacksquare$$

The following corollary gives a sufficient condition that two Lie algebras are n-isoclinic.

Corollary 2.8. *If $f : L \longrightarrow K$ is an epimorphism of Lie algebras such that $Kerf \cap L^{n+1} = 0$, then $L \underset{n}{\sim} K$.*

Proof . Since f is surjective, it follows that $f(Z_n(L)) \subseteq Z_n(K)$. Hence the pair (α,β) is defined in the above theorem is an n-homoclinism such that $Im(\alpha,\beta) = K$. Thus

$$L \cong \frac{L}{Kerf \cap L^{n+1}} \underset{n}{\sim} \frac{L}{Kerf} \cong K. \qquad \blacksquare$$

Theorem 2.9. *Let K be a subalgebra of a Lie algebra L and $\frac{L}{Z_n(L)}$ satisfies the descending chain condition on subalgebras. Then the following are equivalent:*

(a) $L = K + Z_n(L)$;

(b) $L \underset{n}{\sim} K$;

(c) $\frac{L}{Z_n(L)} \cong \frac{K}{Z_n(K)}$.

Proof. The conclusions (a)$\Rightarrow$ (b) and (b)$\Rightarrow$ (c) follow by Theorem 2.6 and Definition 1.1, respectively.

(c)$\Rightarrow$(a). Put $J = K + Z_n(L)$ and so $K \underset{n}{\sim} J$. Hence $\alpha : \frac{L}{Z_n(L)} \longrightarrow \frac{J}{Z_n(J)}$ is an isomorphism, which implies that

$$\frac{L}{Z_n(L)} \cong \frac{K}{Z_n(K)} \cong \frac{J}{Z_n(J)}.$$

Clearly, $\frac{J}{Z_n(L)} \subseteq \frac{L}{Z_n(L)}$ and so assume $\alpha\left(\frac{J}{Z_n(L)}\right) = \frac{J_1}{Z_n(J)}$, for some $J_1 \subseteq J$.

Now, if $J_1 = J$, then

$$\alpha\left(\frac{J}{Z_n(L)}\right) = \frac{J}{Z_n(J)} \cong \frac{L}{Z_n(L)},$$

which gives $J = L$.

Conversely, we have $Z_n(L) \subseteq Z_n(J) \subseteq J_1$ and hence $\frac{J_1}{Z_n(L)} \subseteq \frac{J}{Z_n(L)}$. So, there exists a subalgebra J_2 of J contained in J_1 such that

$$\alpha\left(\frac{J_1}{Z_n(L)}\right) = \frac{J_2}{Z_n(J)}.$$

Thus $J_2 = J_1 \Leftrightarrow J_1 = J \Leftrightarrow J = L$. Clearly, using this process we obtain a descending chain of subalgebras of $\frac{J}{Z_n(J)}$ as follows:

$$\frac{J}{Z_n(J)} \supseteq \frac{J_1}{Z_n(J)} \supseteq \frac{J_2}{Z_n(J)} \supseteq \cdots .$$

By the assumption, $\frac{J}{Z_n(J)}$ satisfies the descending chain condition, and hence there exists a natural number n such that for all $m > n$,

$$\frac{J_n}{Z_n(L)} = \frac{J_m}{Z_n(L)} \Rightarrow J_n = J_m, \quad \forall m > n.$$

Thus $J = K$, which completes the proof. ■

The following result gives some equivalence conditions on n-isoclinism of Lie algebras, which can be proved using Theorem 2.6(b).

Theorem 2.10. *Let M be an ideal of a Lie algebra L. If L^{n+1} satisfies the ascending chain condition on ideals, for all $n \geq 1$, then the following conditions are equivalent:*

(a) $M \cap L^{n+1} = 0$;

(b) $L \underset{n}{\sim} \frac{L}{M}$;

(c) $L^{n+1} \cong \left(\frac{L}{M}\right)^{n+1}$.

The above theorems have the following corollary.

Corollary 2.11. *Let K and M be a subalgebra and an ideal of a Lie algebra L, respectively.*

(i) *If N is an ideal of L and $M \cap L^{n+1} = 0$, then $L \underset{n}{\sim} \frac{L}{M \cap N}$.*

(ii) *If* $M \cap L^{n+1} = 0$, *then* $\dfrac{K+M}{M} \underset{n}{\sim} K$.

(iii) *If* $L = K + Z_n(L)$, *then* $\dfrac{K+M}{M} \underset{n}{\sim} \dfrac{L}{M}$.

(iv) *If* $J \leq L$ *and* $L = K + Z_n(L)$, *then* $L \underset{n}{\sim} K + J$.

Proof. (i) Clearly, $(M \cap N) \cap L^{n+1} = 0$. Now, the result follows by Theorem 2.8.

(ii) By the assumption, we have $(M \cap K) \cap K^{n+1} = 0$ and so

$$\frac{K+M}{M} \cong \frac{K}{M \cap K} \underset{n}{\sim} K.$$

(iii) One observes that

$$\frac{L}{M} = \frac{K + Z_n(L)}{M} = \frac{K+M}{M} + \frac{Z_n(L)+M}{M} \text{ and } \frac{Z_n(L)+M}{M} \leq Z_n\left(\frac{L}{M}\right).$$

Thus

$$\frac{L}{M} = \frac{K+M}{M} + Z_n\left(\frac{L}{M}\right).$$

Now, the result follows, using Theorem 2.7.

(iv) Clearly, $L = (K + L) + Z_n(L)$ and hence

$$L \underset{n}{\sim} K + J$$

■

In the next section, we study the concept of n-stem Lie algebras.

3. The structure of n-stem Lie algebras

In this section, we introduce the concept of n-stem Lie algebras and similar to group theory case (see [1], [2]), it is shown that every family of n-isoclinic Lie algebras contains an n-stem Lie algebra of minimum dimension.

Definition 3.1. A Lie algebra L is said to be an *n-stem Lie algebra*, if $Z_n(L) \leq L^{n+1}$, for some $n \geq 1$.

Theorem 3.2. *Let* $\mathcal{C}$ *be a family of* n*-isoclinic Lie algebras, then*

(a) $\mathcal{C}$ *contains an* n*-stem Lie algebra;*

(b) *Suppose* T *is a finite dimensional Lie algebra in* $\mathcal{C}$, *then* T *is an* n*-stem Lie algebra if and only if*

$$dim T = \min\{dim L \mid L \in \mathcal{C}\}.$$

Proof. Let $L \in \mathcal{C}$ and S be a vector space complement of $Z_n(L) \cap L^{n+1}$ in $Z_n(L)$, i.e., $Z_n(L) = S \oplus (Z_n(L) \cap L^{n+1})$. It follows that $S \cap L^{n+1} = 0$ and $S \lhd L$. By Theorem 2.6, $T = \dfrac{L}{S} \underset{n}{\sim} L$ and so $T \in \mathcal{C}$. Thus, we have

$$\begin{aligned} Z_n(T) = Z_n\left(\frac{L}{S}\right) &= \frac{Z_n(L)+S}{S} = \frac{S+Z_n(L)\cap L^{n+1}}{S} \subseteq \frac{S+L^{n+1}}{S} \\ &= \left(\frac{L}{S}\right)^{n+1} = T^{n+1}, \end{aligned}$$

which implies that T is an n-stem Lie algebra in $\mathcal{C}$.

(b) Let L be a finite dimension Lie algebra in $\mathcal{C}$ and T be the n-stem Lie algebra of $\mathcal{C}$. Then

$$\begin{aligned} \frac{L^{n+1}}{L^{n+1}\cap Z_n(L)} \cong \frac{L^{n+1}+Z_n(L)}{Z_n(L)} &= \left(\frac{L}{Z_n(L)}\right)^{n+1} \cong \left(\frac{T}{Z_n(T)}\right)^{n+1} \\ &= \frac{T^{n+1}+Z_n(T)}{Z_n(T)} = \frac{T^{n+1}}{Z_n(T)}. \end{aligned}$$

Now, since $L^{n+1} \cong T^{n+1}$, it follows that

$$dim(Z_n(T)) = dim(Z_n(L) \cap L^{n+1}) \leq dim Z_n(L).$$

Clearly, $\dfrac{L}{Z_n(L)} \cong \dfrac{T}{Z_n(T)}$ and hence $dimT \leq dimL$.

Conversely, Let T be a Lie algebra in the family $\mathcal{C}$ of minimum dimension. As in the first part, there exists an ideal S of T such that $\dfrac{T}{S} \underset{n}{\sim} T$. Now, since T is of minimum dimension, it implies that

$$Z_n(T) = Z_n(T) \cap T^{n+1},$$

i.e., $S = 0$. Therefore $Z_n(T) \subseteq T^{n+1}$ and hence T is an n-stem Lie algebra. ■

The following lemma shortens the proof of our final result.

Lemma 3.3. *Let L and M be two Lie algebras such that L* $\underset{n}{\sim}$ *M with the isomorphisms pair* (α, β). *Then, for all* $x \in L^{n+1}$

(a) $\alpha(x + Z_n(L)) = \beta(x) + Z_n(M)$;

(b) $\beta([x, y]) = [\beta(x), m], \quad \forall y \in L, \ m \in \alpha(y + Z_n(L))$.

Theorem 3.4. *If L and M are n-stem Lie algebras. Then* $Z_n(L) \cong Z_n(M)$, *for* $n \geq 1$.

Proof . Suppose $L \underset{n}{\sim} M$ with the isomorphisms pair (α, β). By the assumption, we have $Z_n(L) \subseteq L^{n+1}$. Then using Lemma 3.3(a), for all $z \in Z_n(L)$

$$\alpha(z + Z_n(L)) = \beta(z) + Z_n(M),$$

which implies that $\beta(z) \in Z_n(M)$ and hence $\beta(Z_n(L)) \subseteq Z_n(M)$.

Now, take an arbitrary element $x \in Z_n(M)$. Then, there exists $l \in L$ with $\alpha(l + Z_n(L)) = x + Z_n(M) = Z_n(M)$. Clearly, using Lemma 3.3(b),

$$0 = [x, m_1, ..., m_n] = [\beta(l), l_1, ..., l_n],$$

where $\beta(l) = x \in Z_n(M)$ and $m_i \in \alpha(l_i + Z_n(L))$, for all $1 \leq i \leq n$. Thus $Z_n(M) = \beta(Z_n(L))$, which implies that $Z_n(L) \cong Z_n(M)$. ■

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THE DIRICHLET BVP FOR THE SECOND ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATION AT RESONANCE

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Abstract. Efficient sufficient conditions are established for the solvability of the Dirichlet problem

$$u''(t) = p(t)u(t) + f(t, u(t)) + h(t) \quad \text{for} \quad a \le t \le b,$$

$$u(a) = 0, \quad u(b) = 0,$$

where $h, p \in L([a, b]; R)$ and $f \in K([a, b] \times R; R)$, in the case where the linear problem

$$u''(t) = p(t)u(t), \quad u(a) = 0, \quad u(b) = 0$$

has nontrivial solutions.

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1. Introduction

Consider on the set $I = [a, b]$ the second order nonlinear ordinary differential equation

$$u''(t) = p(t)u(t) + f(t, u(t)) + h(t) \quad \text{for} \quad t \in I \tag{1.1}$$

with the boundary conditions

$$u(a) = 0, \quad u(b) = 0, \tag{1.2}$$

where $h, p \in L(I; R)$ and $f \in K(I \times R; R)$.

By a solution of the problem (1.1), (1.2) we understand a function $u \in \widetilde{C}'(I, R)$, which satisfies the equation (1.1) almost everywhere on I and satisfies the conditions (1.2).

Along with (1.1), (1.2) we consider the homogeneous problem

$$w''(t) = p(t)w(t) \quad \text{for} \quad t \in I, \tag{1.3}$$

$$w(a) = 0, \quad w(b) = 0. \tag{1.4}$$

At present, the foundations of the general theory of two-point boundary value problems are already laid and problems of this type are studied by many authors and investigated in detail (see, for instance, [1], [4], [5], [8], [12],[13], [14]-[16], [17] and references therein). On the other hand, in all of these works, only the case when the homogeneous problem (1.3), (1.4) has only a trivial solution is studied. The case when the problem (1.3), (1.4) has also the nontrivial solution is still little investigated and in the majority of articles, the authors study the case with p constant in the equation (1.1), i.e., when the problem (1.1), (1.2) and the equation (1.3) are of type

$$u''(t) = -\lambda^2 u(t) + f(t, u(t)) + h(t) \quad \text{for} \quad t \in [0, \pi], \tag{1.5}$$

$$u(0) = 0, \quad u(\pi) = 0, \tag{1.6}$$

and

$$w''(t) = -\lambda^2 w(t) \quad \text{for} \quad t \in [0, \pi] \tag{1.7}$$

respectively, with $\lambda = 1$. (see, for instance, [2], [3], [4], [6]-[11], [14]-[16], and references therein).

In the present paper, we study solvability of the problem (1.1), (1.2) in the case when the function $p \in L(I; R)$ is not necessarily constant, under the assumption that the homogeneous problem (1.3), (1.4) has the nontrivial solution with an arbitrary number of zeroes. For the equation (1.7), this is the case when λ is not necessarily the first eigenvalue of the problem (1.7), (1.4), with $a = 0$, $b = \pi$.

The obtained results are new and generalize some well-known results (see,[2], [3], [4], [6], [10]).

The following notation is used throughout the paper: N is the set of all natural numbers. R is the set of all real numbers, $R_+ = [0, +\infty[$. $C(I; R)$ is the Banach space of continuous functions $u : I \to R$ with the norm $\|u\|_C = \max\{|u(t)| : t \in I\}$. $\widetilde{C}'(I; R)$ is the set of functions $u : I \to R$ which are absolutely continuous together with their first derivatives. $L(I; R)$ is the Banach space of Lebesgue integrable functions $p : I \to R$ with the norm $\|p\|_L = \int_a^b |p(s)| ds$.

$K(I \times R; R)$ is the set of functions $f : I \times R \to R$ satisfying the Carathéodory conditions, i.e., $f(\cdot, x) : I \to R$ is a measurable function for all $x \in R$, $f(t, \cdot) : R \to R$ is a continuous function for almost all $t \in I$, and for every $r > 0$ there exists $q_r \in L(I; R_+)$ such that $|f(t,x)| \le q_r(t)$ for almost all $t \in I$, $|x| \le r$.

Having $w : I \to R$, we put: $N_w \overset{def}{=} \{t \in]a, b[: w(t) = 0\}$,

$$\Omega_w^+ \overset{def}{=} \{t \in I : w(t) > 0\},$$
$$\Omega_w^- \overset{def}{=} \{t \in I : w(t) < 0\},$$

and $[w(t)]_+ = (|w(t)| + w(t))/2$, $[w(t)]_- = (|w(t)| - w(t))/2$ for $t \in I$.

Definition 1.1. Let A be a finite (eventually empty) subset of I. We say that $f \in E(A)$, if $f \in K(I \times R; R)$ and, for any measurable set $G \subseteq I$ and an arbitrary constant $r > 0$, we can choose $\varepsilon > 0$ such that if

$$\int_G |f(s,x)| ds \ne 0 \quad \text{for} \quad x \ge r \ (x \le -r)$$

then

$$\int_{G \setminus U_\varepsilon} |f(s,x)| ds - \int_{U_\varepsilon} |f(s,x)| ds \ge 0 \quad \text{for} \quad x \ge r \ (x \le -r),$$

where $U_\varepsilon = I \cap \left(\bigcup_{k=1}^{n}]t_k - \varepsilon/2n,\ t_k + \varepsilon/2n[\right)$ if $A = \{t_1, t_2, ..., t_n\}$, and $U_\varepsilon = \emptyset$ if $A = \emptyset$.

Remark 1.1. If $f \in K(I \times R; R)$ then $f \in E(\emptyset)$.

Remark 1.2. It is clear that if $f_1 \in L(I; R)$ and $f(t,x) \overset{def}{\equiv} f_1(t)$ then $f \in E(A)$ for every finite set $A \subset I$.

Remark 1.3. It is clear that if $f(t,x) \overset{def}{\equiv} f_0(t) g_0(x)$, where $f_0 \in L(I; R)$ and $g_0 \in C(I; R)$, then $f \in E(A)$ for every finite set $A \subset I$.

The example below shows that there exists a function $f \in K(I \times R; R)$ such that $f \notin E(\{t_1, ..., t_k\})$ for some points $t_1, ..., t_k \in I$.

Example 1.1. Let $f(t,x) = |t|^{-1/2} g(t,x)$ for $t \in [-1, 0[\cup]0, 1]$, $x \in R$, and $f(0, .) \equiv 0$, where $g(-t, x) = g(t, x)$ for $t \in]-1, 1]$, $x \in R$, and

$$g(t,x) = \begin{cases} x & \text{for} \quad x \le 1/t,\ t > 0 \\ 1/t & \text{for} \quad x > 1/t,\ t > 0 \end{cases}.$$

Then $f \in K([0,1] \times R; R)$ and it is clear that $f \notin E(\{0\})$ because, for every $\varepsilon > 0$, if $x \ge 1/\varepsilon$ then $\int_\varepsilon^1 f(s,x) ds - \int_0^\varepsilon f(s,x) ds = 4(\varepsilon^{-1/2} - x^{1/2}) - 2 < 0$.

2. Main results

Theorem 2.1. *Let* w *be a nonzero solution of the problem* (1.3), (1.4),

$$N_w = \emptyset, \tag{2.1}$$

there exist a constant $r > 0$, *functions* $f^-, f^+ \in L(I; R_+)$ *and* $g, h_0 \in L(I;]0, +\infty[\,)$ *such that*

$$f(t,x)\mathrm{sgn}x \le g(t)|x| + h_0(t) \quad \textit{for} \quad |x| \ge r \tag{2.2}$$

and

$$\begin{gathered} f(t,x) \le -f^-(t) \quad \textit{for} \quad x \le -r, \\ f^+(t) \le f(t,x) \quad \textit{for} \quad x \ge r \end{gathered} \tag{2.3}$$

on I. *Let, moreover, there exist* $\varepsilon > 0$ *such that*

$$-\int_a^b f^-(s)|w(s)|ds + \varepsilon||\gamma_r||_L \le -\int_a^b h(s)|w(s)|ds \le$$

$$\le \int_a^b f^+(s)|w(s)|ds - \varepsilon||\gamma_r||_L, \tag{2.4$_1$}$$

where

$$\gamma_r(t) = \sup\{|f(t,x)| : |x| \le r\}. \tag{2.5}$$

Then the problem (1.1), (1.2) *has at least one solution.*

Example 2.2. It follows from Theorem 2.1 that the equation

$$u''(t) = -\lambda^2 u(t) + \sigma|u(t)|^\alpha \mathrm{sgn}u(t) + h(t) \quad \text{for} \quad 0 \le t \le \pi \tag{2.6}$$

where $\sigma = 1$, $\lambda = 1$, and $\alpha \in]0,1]$, with the conditions (1.6) has at least one solution for every $h \in L([0,\pi], R)$.

Theorem 2.2. *Let* w *be a nonzero solution of the problem* (1.3), (1.4), *condition* (2.1) *hold, there exist a constant* $r > 0$, *functions* $f^-, f^+ \in L(I; R_+)$ *and* $q \in K(I \times R; R_+)$ *such that* q *is non-decreasing in the second argument,*

$$|f(t,x)| \le q(t,x) \quad \textit{for} \quad |x| \ge r, \tag{2.7}$$

$$\begin{gathered} f^-(t) \le f(t,x) \quad \textit{for} \quad x \le -r, \\ f(t,x) \le -f^+(t) \quad \textit{for} \quad x \ge r \end{gathered} \tag{2.8}$$

on I, *and*

$$\lim_{|x| \to +\infty} \frac{1}{x} \int_a^b q(s,x)ds = 0. \tag{2.9}$$

Let, moreover, there exist $\varepsilon > 0$ *such that*

$$-\int_a^b f^-(s)|w(s)|ds + \varepsilon||\gamma_r||_L \le \int_a^b h(s)|w(s)|ds \le$$

$$\le \int_a^b f^+(s)|w(s)|ds - \varepsilon||\gamma_r||_L, \tag{2.4$_2$}$$

where γ_r *is defined by* (2.5). *Then the problem* (1.1), (1.2) *has at least one solution.*

Example 2.3. From Theorem 2.2 it follows that the problem (2.6), (1.6) with $\sigma = -1$, $\lambda = 1$, and $\alpha \in]0,1[$ has at least one solution for every $h \in L([0,\pi]; R)$.

Remark 2.4. If $f \not\equiv 0$ the condition (2.4$_i$) of Theorem 2.i ($i = 1,2$) can be replaced by

$$\begin{gathered} -\int_a^b f^-(s)|w(s)|ds < (-1)^i \int_a^b h(s)|w(s)|ds < \\ < \int_a^b f^+(s)|w(s)|ds, \end{gathered} \tag{2.10$_i$}$$

because, from (2.10$_i$) there follows the existence of a constant $\varepsilon > 0$ such that the condition (2.4$_i$) is satisfied.

Theorem 2.3. *Let* $i \in \{0,1\}$, w *be a nonzero solution of the problem* (1.3), (1.4), $f \in E(N_w)$, *there exist a constant* $r > 0$ *such that the function* $(-1)^i f$ *is non-decreasing in the second argument for* $|x| \ge r$,

$$(-1)^i f(t,x)\mathrm{sgn}x \ge 0 \quad \text{for} \quad t \in I, \ |x| \ge r, \tag{2.11}$$

$$\int_{\Omega_w^+} |f(s,r)|ds + \int_{\Omega_w^-} |f(s,-r)|ds \ne 0, \tag{2.12}$$

and

$$\lim_{|x|\to+\infty} \frac{1}{|x|} \int_a^b |f(s,x)|ds = 0. \tag{2.13}$$

Then there exists $\delta > 0$ *such that the problem* (1.1), (1.2) *has at least one solution for every* h *satisfying the condition*

$$\left| \int_a^b h(s)w(s)ds \right| < \delta. \tag{2.14}$$

Corollary 2.1. *Let the assumptions of Theorem* 2.3 *be satisfied and*

$$\int_a^b h(s)w(s)ds = 0. \tag{2.15}$$

Then the problem (1.1)*,* (1.2) *has at least one solution.*

Example 2.4. From Theorem 2.3 it follows that the problem (2.6), (1.6) with $\sigma \in \{-1, 1\}$, $\lambda \in N$, and $\alpha \in]0, 1[$ has at least one solution if $h \in L([0, \pi], R)$ is such that $\int_0^\pi h(s) \sin \lambda s ds = 0$.

Theorem 2.4. *Let* $i \in \{0, 1\}$*,* w *be a nonzero solution of the problem* (1.3),(1.4)*,* $f(t, x) \overset{def}{\equiv} f_0(t)g_0(x)$ *with* $f_0 \in L(I; R_+)$*,* $g_0 \in C(R; R)$*, there exist a constant* $r > 0$ *such that* $(-1)^i g_0$ *is non-decreasing for* $|x| \geq r$ *and*

$$(-1)^i g_0(x)\mathrm{sgn}x \geq 0 \qquad \textit{for} \quad |x| \geq r. \tag{2.16}$$

Let, moreover,

$$|g_0(r)| \int_{\Omega_w^+} f_0(s)ds + |g_0(-r)| \int_{\Omega_w^-} f_0(s)ds \neq 0 \tag{2.17}$$

and

$$\lim_{|x| \to +\infty} |g_0(x)| = +\infty, \qquad \lim_{|x| \to +\infty} \frac{g_0(x)}{x} = 0. \tag{2.18}$$

Then, for every $h \in L(I; R)$*, the problem* (1.1)*,* (1.2) *has at least one solution.*

Example 2.5. From Theorem 2.4 it follows that the equation

$$u''(t) = p_0(t)u(t) + p_1(t)|u(t)|^\alpha \mathrm{sgn}u(t) + h(t) \quad \text{for} \quad t \in I, \tag{2.19}$$

where $\alpha \in]0, 1[$ and $p_0, p_1, h \in L(I; R)$, with the conditions (1.2) has at least one solution provided that $p_1(t) > 0$ for $t \in I$.

Theorem 2.5. *Let* $i \in \{0, 1\}$ *and* w *be a nonzero solution of the problem* (1.3)*,* (1.4)*. Let, moreover, there exist constants* $r > 0$*,* $\varepsilon > 0$*, and functions* $\alpha, f^+, f^- \in L(I; R_+)$ *such that the conditions*

$$\begin{gathered} (-1)^i f(t, x) \leq -f^-(t) \quad \textit{for} \quad x \leq -r, \\ f^+(t) \leq (-1)^i f(t, x) \quad \textit{for} \quad x \geq r, \end{gathered} \tag{2.20$_i$}$$

$$\sup\{|f(t, x)| : x \in R\} \leq \alpha(t) \tag{2.21}$$

hold on I, and let

$$
\begin{aligned}
&-\int_a^b (f^+(s)[w(s)]_- + f^-(s)[w(s)]_+)ds + \varepsilon||\alpha||_L \leq \\
(2.22_i)\qquad &\leq (-1)^{i+1}\int_a^b h(s)w(s)ds \leq \\
&\leq \int_a^b (f^-(s)[w(s)]_- + f^+(s)[w(s)]_+)ds - \varepsilon||\alpha||_L.
\end{aligned}
$$

Then the problem (1.1), (1.2) *has at least one solution.*

Remark 2.5. If $f \not\equiv 0$ then the condition (2.22_i) $(i = 1, 2)$ of Theorem 2.5 can be replaced by

$$
\begin{aligned}
&-\int_a^b (f^+(s)[w(s)]_- + f^-(s)[w(s)]_+)ds < \\
(2.23_i)\qquad &< (-1)^{i+1}\int_a^b h(s)w(s)ds < \\
&< \int_a^b (f^-(s)[w(s)]_- + f^+(s)[w(s)]_+)ds.
\end{aligned}
$$

because from (2.23_i) there follows the existence of a constant $\varepsilon > 0$ such that the condition (2.22_i) is satisfied.

Remark 2.6. If $\widetilde{f}(t) = \min\{f^+(t), f^-(t)\}$ then the condition (2.22_i) of Theorem 2.5 can be replaced by

$$\left|\int_a^b h(s)w(s)ds\right| \leq \int_a^b \widetilde{f}(s)|w(s)|ds - \varepsilon||\alpha||_L.$$

Example 2.6. From Theorem 2.5 it follows that the equation

$$u''(t) = -\lambda^2 u(t) + \frac{|u(t)|^\alpha}{1 + |u(t)|^\alpha}\mathrm{sgn}u(t) + h(t) \quad \text{for} \quad 0 \leq t \leq \pi, \tag{2.24}$$

where $\lambda \in N$ and $\alpha \in]0, +\infty[$, with the conditions (1.6) has at least one solution if $h \in L([0, \pi], R)$ is such that $|h(t)| < 1$ for $0 \leq t \leq \pi$.

3. Problem (1.5), (1.6).

Throughout this section we will assume that $a = 0$, $b = \pi$, and $I = [0, \pi]$. Since the functions $\beta \sin \lambda t$ $(\beta \in R)$ are nontrivial solutions of the problem (1.7), (1.4), from Theorems 2.1–2.5 it immediately follows:

Corollary 3.2. *Let $\lambda = 1$ and all the assumptions of Theorem* 2.1 (*resp. Theorem* 2.2) *except* (2.1) *be fulfilled with $w(t) = \sin t$. Then the problem* (1.5), (1.6) *has at least one solution.*

Now, note that

$$N_{\sin\lambda t}=\begin{cases}\emptyset & \text{for} \quad \lambda=1\\ \{\pi n/\lambda : n=1,...,\lambda-1\} & \text{for} \quad \lambda\geq 2\end{cases}.$$

Corollary 3.3. *Let* $i\in\{0,1\}$, $\lambda\in N$, $f\in E(N_{\sin\lambda t})$, *there exist a constant* $r>0$ *such that the function* $(-1)^i f$ *is non-decreasing in the second argument for* $|x|\geq r$, *and let the conditions* (2.11)–(2.13) *be fulfilled with* $w(t)=\sin\lambda t$. *Then there exists* $\delta>0$ *such that the problem* (1.5), (1.6) *has at least one solution for every* $h\in L(I;R)$ *satisfying the condition* $|\int_0^\pi h(s)\sin\lambda s ds|<\delta$.

Corollary 3.4. *Let* $i\in\{0,1\}$, $\lambda\in N$, *and let all the assumptions of Theorem* 2.4 *be fulfilled with* $w(t)=\sin\lambda t$. *Then, for any* $h\in L(I;R)$, *the problem* (1.5), (1.6) *has at least one solution.*

Corollary 3.5. *Let* $i\in\{0,1\}$, $\lambda\in N$ *and let there exist a constant* $r>0$ *such that* (2.20_i)–(2.22_i) *be fulfilled with* $w(t)=\sin\lambda t$. *Then the problem* (1.5), (1.6) *has at least one solution.*

Remark 3.7. If $f\not\equiv 0$ then in Corollary 3.2 (resp. Corollary 3.5), the condition (2.4_i) (resp. (2.22_i)) can be replaced by the condition (2.10_i) (resp. (2.23_i)) with $w(t)=\sin t$ (resp. $w(t)=\sin\lambda t$).

4. Auxiliary propositions

Let $u_n\in\widetilde{C}'(I;R)$, $\|u_n\|_C\neq 0$ $(n\in N)$, w be an arbitrary solution of the problem (1.3), (1.4), and $r>0$. Then, for every $n\in N$, we define:

$$A_{n,1}\overset{def}{=}\{t\in I:|u_n(t)|\leq r\},\qquad A_{n,2}\overset{def}{=}\{t\in I:|u_n(t)|>r\},$$

$$B_{n,i}\overset{def}{=}\{t\in A_{n,2}:\mathrm{sgn}u_n(t)=(-1)^{i-1}\mathrm{sgn}w(t)\}\ \ (i=1,2),$$

$$C_{n,1}\overset{def}{=}\{t\in A_{n,2}:|w(t)|\geq 1/n\},\qquad C_{n,2}\overset{def}{=}\{t\in A_{n,2}:|w(t)|<1/n\},$$

$$D_n\overset{def}{=}\{t\in I:|w(t)|>r||u_n||_C^{-1}+1/2n\},$$

$$A^{\pm}_{n,2}\overset{def}{=}\{t\in A_{n,2}:\pm u_n(t)>r\},\quad B^{\pm}_{n,i}\overset{def}{=}A^{\pm}_{n,2}\cap B_{n,i},$$

$$C^{\pm}_{n,i}\overset{def}{=}A^{\pm}_{n,2}\cap C_{n,i}\ \ (i=1,2),\ D^{\pm}_n\overset{def}{=}\{t\in I:\pm w(t)>r||u_n||_C^{-1}+1/2n\},$$

From these definitions it is clear that, for any $n\in N$, we have

$$A_{n,1}\cap A_{n,2}=\emptyset, A^+_{n,2}\cap A^-_{n,2}=\emptyset,\ \ B_{n,1}\cap B_{n,2}=\emptyset,\ C_{n,1}\cap C_{n,2}=\emptyset,$$

$$D^+_n\cap D^-_n=\emptyset,\ B^+_{n,2}\cap B^-_{n,2}=\emptyset,\ C^+_{n,i}\cap C^-_{n,i}=\emptyset\ (i=1,2), \tag{4.1}$$

and

$$A_{n,1}\cup A_{n,2}=I,\ A^+_{n,2}\cup A^-_{n,2}=A_{n,2},\ B_{n,1}\cup B_{n,2}=A_{n,2}\setminus N_w,$$

$$\begin{gathered}C_{n,1}\cup C_{n,2}=A_{n,2},\ B^+_{n,2}\cup B^-_{n,2}=B_{n,2},\ C^{\pm}_{n,1}\cup C^{\pm}_{n,2}=A^{\pm}_{n,2},\\ C^+_{n,i}\cup C^-_{n,i}=C_{n,i}\ (i=1,2),\ D^+_n\cup D^-_n=D_n.\end{gathered} \tag{4.2}$$

Lemma 4.1. *Let* $u_n \in \widetilde{C}'(I;R)$ $(n \in N)$, $r > 0$, w *be an arbitrary nonzero solution of the problem* (1.3), (1.4), *and*

$$||u_n||_C \geq 2rn \quad for \quad n \in N, \tag{4.3}$$

$$||v_n - w||_C \leq 1/2n \quad for \quad n \in N, \tag{4.4}$$

where $v_n(t) = u_n(t)||u_n||_C^{-1}$. *Then, for any* $n_0 \in N$, *we have*

$$D^+_{n_0} \subset A^+_{n,2}, \quad D^-_{n_0} \subset A^-_{n,2} \quad for \quad n \geq n_0, \tag{4.5}$$

$$C^+_{n_0,1} \subset D^+_n \quad C^-_{n_0,1} \subset D^-_n \quad for \quad n \geq n_0. \tag{4.6}$$

Moreover

$$\lim_{n\to+\infty} \mathrm{mes} A_{n,1} = 0, \qquad \lim_{n\to+\infty} \mathrm{mes} A_{n,2} = \mathrm{mes} I, \tag{4.7}$$

$$C_{n,1} \subset B_{n,1}, \qquad B_{n,2} \subset C_{n,2}, \tag{4.8}$$

$$B^+_{n,2} \subset C^+_{n,2}, \qquad B^-_{n,2} \subset C^-_{n,2}, \tag{4.9}$$

$$C^+_{n,1} \subset B^+_{n,1}, \qquad C^-_{n,1} \subset B^-_{n,1}, \tag{4.10}$$

$$\begin{gathered} \lim_{n\to+\infty} \mathrm{mes} C_{n,1} = \lim_{n\to+\infty} \mathrm{mes} B_{n,1} = \mathrm{mes} I, \\ \lim_{n\to+\infty} \mathrm{mes} C_{n,2} = \lim_{n\to+\infty} \mathrm{mes} B_{n,2} = 0, \end{gathered} \tag{4.11}$$

$$r < |u_n(t)| \leq ||u_n||_C/2n \qquad for \quad t \in B_{n,2}, \tag{4.12}$$

$$|u_n(t)| \geq ||u_n||_C/2n > r \qquad for \quad t \in C_{n,1}, \tag{4.13}$$

$$C^{\pm}_{n,2} = \{t \in A_{n,2} : 0 \leq \pm w(t) < 1/n\}, \tag{4.14$_1$}$$

$$C^{\pm}_{n,1} \subset \Omega^{\pm}_w, \quad \lim_{n\to+\infty} \mathrm{mes} C^{\pm}_{n,1} = \mathrm{mes} \Omega^{\pm}_w. \tag{4.15}$$

Proof. From the unique solvability of the Cauchy problem for the equation (1.3) it follows that the set N_w is finite. Consequently, we can assume that $N_w = \{t_1, ..., t_k\}$. Let also $t_0 = a$, $t_{k+1} = b$ and $T_n \overset{def}{=} I \cap \left(\bigcup_{i=0}^{k+1} [t_i - 1/n, \ t_i + 1/n] \right)$. We first show that, for every $n_0 \in N$, there exists $n_1 > n_0$ such that

$$A_{n,1} \subseteq T_{n_0} \quad \text{for} \quad n \geq n_1. \tag{4.16}$$

Suppose on the contrary that, for some $n_0 \in N$, there exists the sequence $t'_{n_j} \in A_{n_j,1}$ $(j \in N)$ with $n_j < n_{j+1}$, such that $t'_{n_j} \notin T_{n_0}$ for $j \in N$. Without loss of generality we can assume that $\lim\limits_{j\to+\infty} t'_{n_j} = t'_0$. Then from the conditions (4.3), (4.4), the definition of the set $A_{n,1}$ and the equality $w(t'_0) = (w(t'_0) - w(t'_{n_j})) + (w(t'_{n_j}) - v_{n_j}(t'_{n_j})) + v_{n_j}(t'_{n_j})$, we get $|w(t'_0)| = 0$, i.e., $t'_0 \in \{t_0, t_1, ..., t_{k+1}\}$. But this contradicts the condition $t'_{n_j} \notin T_{n_0}$ and thus (4.16) is true. Since $\lim\limits_{n\to+\infty} \mathrm{mes} T_n = 0$, it follows from (4.2) and (4.16) that (4.7) is valid.

Let $t_0 \in D^+_{n_0}$. Then from (4.4) it follows that

$$\frac{u_n(t_0)}{||u_n||_C} \geq w(t_0) - |v_n(t_0) - w(t_0)| > \frac{r}{||u_{n_0}||_C} + \frac{1}{2n_0} - \frac{1}{2n} \geq \frac{r}{||u_{n_0}||_C}$$

for $n \geq n_0$, and thus $t_0 \in A^+_{n,2}$ for $n \geq n_0$, i.e., $D^+_{n_0} \subset A^+_{n,2}$ for $n \geq n_0$. The second relation of (4.5) can be proved analogously. Now suppose that $t_0 \in C_{n,1}$ and $t_0 \notin B_{n,1}$. Then, in view of (4.1) and (4.2), it is clear that $t_0 \in B_{n,2}$, and thus

$$|v_n(t_0) - w(t_0)| = |v_n(t_0)| + |w(t_0)| > 1/n \,, \tag{4.17}$$

which contradicts (4.4). Consequently, $C_{n,1} \subset B_{n,1}$ for $n \in N$. This, together with the relations $C_{n,2} = A_{n,2} \setminus C_{n,1}$, $B_{n,2} \subseteq A_{n,2} \setminus B_{n,1}$, implies $B_{n,2} \subset C_{n,2}$, i.e., (4.8) holds. The conditions (4.9) and (4.10) follow immediately from (4.8). In view of the fact that $\lim\limits_{n\to+\infty} \mathrm{mes} C_{n,i} = (2-i)\mathrm{mes} I$, from (4.8) we get (4.11). Now, let $t_0 \in B_{n,2}$ and suppose that $|v_n(t_0)| > 1/2n$. Then from (4.4) we obtain the contradiction $1/2n \geq |v_n(t_0) - w(t_0)| = |v_n(t_0)| + |w(t_0)| > 1/2n$. Thus $\dfrac{|u_n(t_0)|}{||u_n||_C} = |v_n(t_0)| \leq \dfrac{1}{2n}$ and using the definitions of the sets $B_{n,2}$ and $A_{n,2}$ we obtain (4.12). Also, from the inequality $\dfrac{|u_n(t)|}{||u_n||_C} = |v_n(t)| \geq |w(t)| - |v_n(t) - w(t)|$ by (4.3), (4.4) and the definition of the sets $C_{n,1}$ and $A_{n,2}$ we obtain (4.13).

Let there exist $t_0 \in C^+_{n,2}$ such that $t_0 \notin \{t \in A_{n,2} : 0 \leq w(t) \leq 1/n\}$. Then from the definition of the sets $C_{n,2}$ and the inclusion $C^+_{n,2} \subset C_{n,2}$ we get $-1/n < w(t) < 0$ and $t_0 \in A^+_{n,2}$. In this case the inequality (4.17) is fulfilled, which contradicts (4.4). Therefore $C^+_{n,2} \subset \{t \in A_{n,2} : 0 \leq w(t) \leq 1/n\}$. Let now $t_0 \in \{t \in A_{n,2} : 0 \leq w(t) \leq 1/n\}$ and $t_0 \notin C^+_{n,2}$. Then from the definition of the set $C_{n,2}$ and (4.2) it is clear that $t_0 \in C^-_{n,2}$, i.e., $t_0 \in A^-_{n,2}$, and that the inequality (4.17) holds, which contradicts (4.4). Therefore $\{t \in A_{n,2} : 0 \leq w(t) \leq 1/n\} \subset C^+_{n,2}$. From the last two inclusions it follows that (4.14_1) holds for $C^+_{n,2}$. From (4.2) and (4.14_1) for $C^+_{n,1}$ it is clear that (4.14_1) is true for $C^-_{n,1}$ too. Analogously one can prove that

$$C^{\pm}_{n,1} = \{t \in A_{n,2} : \pm w(t) \geq 1/n\} \quad \text{for} \quad n \in N. \tag{4.14_2}$$

From (4.14_2), the definition of the sets $D^{\pm}_n$ and (4.3) we obtain (4.6). From the definition of the set $\Omega^{\pm}_w$ and (4.14_2) we have $C^{\pm}_{n,1} \subset \Omega^{\pm}_w$. Hence

$$\mathrm{mes} C^{\pm}_{n,1} \leq \mathrm{mes} \Omega^{\pm}_w.$$

On the other hand $C^{\pm}_{n,1} = \{t \in I : \pm w(t) \geq 1/n\} \setminus (I \setminus A_{n,2})$ and thus

$$\text{mes}C^{\pm}_{n,1} \geq \text{mes}\Omega^{\pm}_{w} - \text{mes}(I \setminus A_{n,2}).$$

In view of (4.7) from last two inequalities we conclude that (4.15) holds. ■

Lemma 4.2. *Let* $i \in \{1, 2\}$, $r > 0$, $k \in N$, w_0 *be a nonzero solution of the problem* (1.3), (1.4), $N_{w_0} = \{t_1, ..., t_k\}$, *the function* $f_1 \in E(N_{w_0})$ *be non-decreasing in the second argument for* $|x| \geq r$, *and*

$$f_1(t, x)\text{sgn}x \geq 0 \qquad \textit{for} \quad t \in I,\ |x| \geq r. \tag{4.18}$$

Then:

a) If $G \subset I$ *and*

$$\int_G |f_1(s, (-1)^i r)w_0(s)|ds \neq 0, \tag{4.19}$$

then there exist $\delta_0 > 0$ *and* $\varepsilon_1 > 0$ *such that*

$$\mathbb{I}(G, U_\varepsilon, x) \overset{def}{\equiv} \int\limits_{G \setminus U_\varepsilon} |f_1(s, x)w_0(s)|ds - \int\limits_{U_\varepsilon} |f_1(s, x)w_0(s)|ds \geq \delta_0 \tag{4.20}$$

for $(-1)^i x \geq r$ *and* $0 < \varepsilon \leq \varepsilon_1$, *where* $U_\varepsilon = I \cap \Big(\cup_{j=1}^{k} [t_j - \varepsilon/2k,\ t_j + \varepsilon/2k]\Big)$.

b) If $u_n \in \widetilde{C}'(I; R)$ $(n \in N)$, $r > 0$, w *is an arbitrary nonzero solution of the problem* (1.3), (1.4), *and the condition* (4.3) *holds, then there exist* $\varepsilon_2 \in]0, \varepsilon_1]$ *and* $n_0 \in N$ *such that*

$$\mathbb{I}(D^+_n, U^+_\varepsilon, x) \geq -\frac{\delta_0}{2} \quad \textit{for} \quad x \geq r, \tag{4.21$_1$}$$

$$\mathbb{I}(D^-_n, U^-_\varepsilon, x) \geq -\frac{\delta_0}{2} \quad \textit{for} \quad x \leq -r \tag{4.21$_2$}$$

for $n \geq n_0$ *and* $0 < \varepsilon \leq \varepsilon_2$, *where* $U^{\pm}_\varepsilon = \{t \in U_\varepsilon : \pm w(t) \geq 0\}$.

Proof. First note that, for any nonzero solution w of the problem (1.3), (1.4), there exists $\beta \neq 0$ such that $w(t) = \beta w_0(t)$ and thus $N_w = N_{w_0}$.

a) For any $\alpha \in R_+$, we put $G_1 = ([a, a + \alpha] \cup [b - \alpha, b]) \cap G$. In view of the condition (4.19), we can choose $\alpha \in]0, (b - a)/2[$ such that if $G_2 = G \setminus G_1$, $t_a = \inf\{G_2\}$ and $t_b = \sup\{G_2\}$, then

$$a < t_a, \quad t_b < b, \tag{4.22}$$

and $\int_{G_1} |f_1(s, (-1)^i r)w_0(s)|ds \neq 0$, $\int_{G_2} |f_1(s, (-1)^i r)|ds \neq 0$. From these inequalities, by virtue of conditions (4.18) and $f_1 \in E(N_{w_0})$, where f_1 is non-decreasing in the second argument, there follows the existence of $\delta_0 > 0$ and $\varepsilon^* > 0$ such that

$$\int_{G_2 \setminus U_{\varepsilon^*}} |f_1(s, x)|ds - \int_{U_{\varepsilon^*}} |f_1(s, x)|ds \geq 0 \quad \text{for} \quad (-1)^i x \geq r, \tag{4.23}$$

$$\int_{G_1\setminus U_{\varepsilon^*}} |f_1(s,x)w_0(s)|ds \geq \delta_0 \quad \text{for} \quad (-1)^i x \geq r. \tag{4.24}$$

Now we put $I^* = [t_a^*,\ t_b^*]$, where $t_a^* = \dfrac{a+\min(t_a,t_1)}{2}$ and $t_b^* = \dfrac{\max(t_k,t_b)+b}{2}$. In view of (4.22), we obtain

$$G_2 \subset I^*, \quad N_{w_0} \subset I^*, \quad w_0(t_a^*) \neq 0, \ w_0(t_b^*) \neq 0. \tag{4.25}$$

Then it is clear that there exists $\gamma_1 > 0$ such that, for any $\gamma \in]0,\gamma_1[$, the equation $|w_0(t)| = \gamma$ has only $t_{\gamma,i}, t_{\gamma,i}^* \in I^*$ $(i=1,...,k)$ solutions such that

$$t_{\gamma,i} < t_i < t_{\gamma,i}^* \quad (i=1,...,k), \tag{4.26}$$

$$|w_0(t)| \leq \gamma \quad \text{for} \quad t \in H_\gamma, \qquad |w_0(t)| > \gamma \quad \text{for} \quad t \in I^* \setminus H_\gamma, \tag{4.27}$$

where $H_\gamma = \bigcup\limits_{i=1}^{k} [t_{\gamma,i}, t_{\gamma,i}^*]$, and

$$\lim_{\gamma\to+0} t_{\gamma,i} = \lim_{\gamma\to+0} t_{\gamma,i}^* = t_i \quad (i=1,...,k). \tag{4.28}$$

The relations (4.26) and (4.28) imply that there exist $\gamma \in]0,\gamma_1]$ and $\varepsilon_1 \in]0,\varepsilon^*]$ such that

$$U_{\varepsilon_1} \subset H_\gamma \subset U_{\varepsilon^*}. \tag{4.29}$$

Moreover, in view of the inclusion $G_1 \subset G$, it is clear that

$$G \setminus U_{\varepsilon_1} = \big[\big(G\setminus G_1\big)\setminus U_{\varepsilon_1}\big] \cup \big(G_1 \setminus U_{\varepsilon_1}\big), \ \big[\big(G\setminus G_1\big)\setminus U_{\varepsilon_1}\big] \cap \big(G_1 \setminus U_{\varepsilon_1}\big) = \emptyset,$$

and thus

$$\mathbb{I}(G, U_{\varepsilon_1}, x) = \int_{G_1\setminus U_{\varepsilon_1}} |f_1(s,x)w_0(s)|ds + \mathbb{I}(G_2, U_{\varepsilon_1}, x) \quad \text{for} \quad (-1)^i x \geq r.$$

By virtue of (4.23), (4.25), (4.27), and (4.29), we get

$$\mathbb{I}(G_2, U_{\varepsilon_1}, x) \geq \gamma\Big(\int_{G_2\setminus H_\gamma} |f_1(s,x)|ds - \int_{H_\gamma} |f_1(s,x)|ds\Big) \geq$$

$$\geq \gamma\Big(\int_{G_2\setminus U_{\varepsilon^*}} |f_1(s,x)|ds - \int_{U_{\varepsilon^*}} |f_1(s,x)|ds\Big) \geq 0$$

for $(-1)^i x \geq r$. In view of the last two relations, (4.24), (4.29), and the fact that $U_\varepsilon \subset U_{\varepsilon_1}$ for $\varepsilon \leq \varepsilon_1$, we conclude that the inequality (4.20) holds.

b) First consider the case when

$$\int_{D_n^+} |f_1(s,x)w_0(s)|ds = 0 \ \text{ for } \ x \geq r, \ n \in N. \tag{4.30}$$

From (4.3) and the definitions of the sets $D_n^{\pm}$ and $U_\varepsilon^{\pm}$ we get

$$\lim_{n\to+\infty} \operatorname{mes}(U_\varepsilon^{\pm} \setminus D_n^{\pm}) = 0. \tag{4.31}$$

Then, in view of (4.30) and the fact that for any $\varepsilon > 0$ and $n \in N$

$$U_\varepsilon^{\pm} = (U_\varepsilon^{\pm} \cap D_n^{\pm}) \cup (U_\varepsilon^{\pm} \setminus D_n^{\pm}), \quad (U_\varepsilon^{\pm} \cap D_n^{\pm}) \cap (U_\varepsilon^{\pm} \setminus D_n^{\pm}) = \emptyset, \tag{4.32}$$

we have $\int_{U_\varepsilon^+} |f_1(s,x)w_0(s)|ds = \int_{U_\varepsilon^+ \setminus D_n^+} |f_1(s,x)w_0(s)|ds$ for $x \ge r$, $n \in N$, and $\varepsilon > 0$. Thus by virtue of (4.31), we get $\int_{U_\varepsilon^+} |f_1(s,x)w_0(s)|ds = 0$. From the last equality and (4.30) we conclude that

$$I(D_n^+, U_\varepsilon^+, x) = 0 \quad \text{for} \quad x \ge r, \ n \in N, \ \varepsilon > 0. \tag{4.33}$$

Therefore, in this case the condition (4.21_1) is true.

Now consider the case when for some $r_1 \ge r$ there exists $n_0 \in N$ such that

$$\int_{D_n^+} |f_1(s,x)w_0(s)|ds \ne 0 \ \text{ for } \ x \ge r_1, \ n \ge n_0. \tag{4.34}$$

It is clear that there exist $\eta > 0$ and $\varepsilon_2 \in]0, \varepsilon_1]$ such that

$$\int_{U_\varepsilon^+} |f_1(s,x)w_0(s)|ds \le \frac{\delta_0}{2} \ \text{ for } \ r \le x \le r_1 + \eta, \ \varepsilon \le \varepsilon_2,$$

and thus

$$I(D_n^+, U_\varepsilon^+, x) \ge -\frac{\delta_0}{2} \quad \text{for} \quad r \le x \le r_1 + \eta, \ n \ge n_0, \ \varepsilon \le \varepsilon_2. \tag{4.35}$$

On the other hand, from (4.34) it is clear that $\int_{D_{n_0}^+} |f_1(s, r_1 + \eta)w_0(s)|ds \ne 0$. Therefore, from the item *a*) of our lemma with $G = D_n^+$, and the inclusions $D_{n_0}^+ \subset D_n^+$, $U_\varepsilon^+ \subset U_\varepsilon$ for $n \ge n_0$, $\varepsilon > 0$, we get $I(D_n^+, U_\varepsilon^+, x) \ge \delta_0$ for $x \ge r_1 + \eta$, $n \ge n_0$, $0 < \varepsilon \le \varepsilon_2$. From this inequality and (4.35) we obtain (4.21_1) in second case too.

Analogously one can prove (4.21_2). ∎

Lemma 4.3. *Let all the conditions of Lemma* 4.1 *be fulfilled and there exist* $r > 0$ *such that the condition* (4.18) *holds, where* $f_1 \in K(I \times R; R)$. *Then*

$$\liminf_{n\to+\infty} \int_s^t f_1(\xi, u_n(\xi)) \operatorname{sgn} u_n(\xi) d\xi \ge 0 \quad \textit{for} \quad a \le s < t \le b. \tag{4.36}$$

Proof. Let

$$\gamma_r^*(t) \overset{def}{=} \sup\{|f_1(t,x)| : |x| \le r\} \quad \text{for} \quad t \in I. \tag{4.37}$$

Then, according to (4.1), (4.2), and (4.18), we obtain the estimate

$$\int_s^t f_1(\xi, u_n(\xi)) \operatorname{sgn} u_n(\xi) d\xi \ge$$

$$\ge -\int_{[s,t]\cap A_{n,1}} \gamma_r^*(\xi) d\xi + \int_{[s,t]\cap A_{n,2}} |f_1(\xi, u_n(\xi))| d\xi$$

for $a \le s < t \le b$, $n \in N$. This estimate and (4.7) imply (4.36). ∎

Lemma 4.4. *Let* $r > 0$, *the functions* $f_1 \in K(I \times R; R)$, $h_1 \in L(I; R)$, $f^+, f^- \in L(I; R_+)$ *be such that*

$$f_1(t,x) \le -f^-(t) \quad for \quad x \le -r,$$
$$f^+(t) \le f_1(t,x) \quad for \quad x \ge r \tag{4.38}$$

on I, *and there exist a nonzero solution* w_0 *of the problem* (1.3), (1.4) *and* $\varepsilon > 0$ *such that*

$$N_{w_0} = \emptyset \tag{4.39}$$

and

$$-\int_a^b f^-(s)|w_0(s)|ds + \varepsilon||\gamma_r^*||_L \le -\int_a^b h_1(s)|w_0(s)|ds \le$$

$$\le \int_a^b f^+(s)|w_0(s)|ds - \varepsilon||\gamma_r^*||_L, \tag{4.40}$$

where γ_r^* *is defined by* (4.37). *Then, for every nonzero solution* w *of the problem* (1.3), (1.4), *and functions* $u_n \in \widetilde{C}'(I; R)$ $(n \in N)$ *such that the conditions* (4.3),

$$|v_n^{(i)}(t) - w^{(i)}(t)| \le 1/2n \quad for \quad t \in I, \ n \in N, \ (i = 0, 1) \tag{4.41}$$

where $v_n(t) = u_n(t)||u_n||_C^{-1}$ *for* $t \in I$ *and*

$$u_n(a) = 0, \quad u_n(b) = 0 \tag{4.42}$$

are fulfilled, there exists $n_1 \in N$ *such that*

$$\mathbb{M}_n(w) \overset{def}{\equiv} \int_a^b (h_1(s) + f_1(s, u_n(s)))w(s)ds \ge 0 \quad for \quad n \ge n_1. \tag{4.43}$$

Proof. First note that, for any nonzero solution w of the problem (1.3), (1.4), there exists $\beta \ne 0$ such that $w(t) = \beta w_0(t)$. Also, it is not difficult to verify that all the assumptions of Lemma 4.1 are satisfied for the function $w(t) = \beta w_0(t)$. From the unique solvability of the Cauchy problem for the equation (1.3) and the conditions (1.4) we conclude that $w'(a) \ne 0$ and $w'(b) \ne 0$. Therefore, in view of (4.41) and (4.42), there exists $n_2 \in N$ such that

$$u_n(t)\mathrm{sgn}\beta w_0(t) > 0 \qquad \text{for} \quad n \ge n_2, \ a < t < b. \tag{4.44}$$

Moreover, by (4.1) and (4.2) we get the estimate

$$\frac{\mathbb{M}_n(w)}{|\beta|} \ge -\int_{A_{n,1}} \gamma_r^*(s)|w_0(s)|ds + \sigma \int_a^b h_1(s)w_0(s)ds +$$
$$+\sigma \int_{A_{n,2}} f_1(s, u_n(s))w_0(s)ds, \tag{4.45}$$

where γ_r^* is given by (4.37) and $\sigma = \operatorname{sgn}\beta$. Now note that $f^- \equiv 0$, $f^+ \equiv 0$ if $f_1(t,x) \equiv 0$. Then by virtue of (4.7), we see that there exist $\varepsilon > 0$ and $n_1 \in N$ $(n_1 \geq n_2)$ such that $\int_a^b f^{\pm}(s)|w_0(s)|ds - \frac{\varepsilon}{2}||\gamma_r^*||_L \leq \int_{A_{n,2}} f^{\pm}(s)|w_0(s)|ds$ and $\frac{\varepsilon}{2}||\gamma_r^*||_L \geq \int_{A_{n,1}} \gamma_r^*(s)|w_0(s)|ds$ for $n \geq n_1$. By these inequalities, (4.3), (4.38) and (4.44), from (4.45) we obtain

$$\frac{\mathbb{M}_n(w)}{|\beta|} \geq -\varepsilon||\gamma_r^*||_L + \int_a^b h_1(s)|w_0(s)|ds + \int_a^b f^+(s)|w_0(s)|ds$$

if $n \geq n_1$, $\sigma w_0(t) \geq 0$, and

$$\frac{\mathbb{M}_n(w)}{|\beta|} \geq -\varepsilon||\gamma_r^*||_L - \int_a^b h_1(s)|w_0(s)|ds + \int_a^b f^-(s)|w_0(s)|ds$$

if $n \geq n_1$, $\sigma w_0(t) \leq 0$. From the last two estimates in view of (4.40) it follows that (4.43) is valid. ■

Lemma 4.5. *Let w_0 be a nonzero solution of the problem* (1.3), (1.4), $r > 0$, *the function $f_1 \in E(N_{w_0})$ be non-decreasing in the second argument for $|x| \geq r$, condition* (4.18) *hold, and*

$$\int_{\Omega_{w_0}^+} |f_1(s,r)|ds + \int_{\Omega_{w_0}^-} |f_1(s,-r)|ds \neq 0. \tag{4.46}$$

Then there exist $\delta > 0$ and $n_1 \in N$ such that if

$$\left| \int_a^b h_1(s)w_0(s)ds \right| < \delta \tag{4.47}$$

then, for every nonzero solution w of the problem (1.3), (1.4) *and the functions $u_n \in \widetilde{C}'(I;R)$ $(n \in N)$ fulfilling the conditions* (4.3), (4.41), (4.42), *the inequality* (4.43) *holds.*

Proof. It is not difficult to verify that all the assumption of Lemma 4.1 are satisfied. Then, by the definition of the sets $B_{n,1}$, $B_{n,2}$, the conditions (4.1), (4.2), and (4.18), we obtain the estimate

$$\int_a^b f_1(s,u_n(s))w(s)ds \geq -\int_{A_{n,1}} \gamma_r^*(s)|w(s)|ds + \widehat{\mathbb{M}}_n(w), \tag{4.48}$$

where

$$\widehat{\mathbb{M}}_n(w) \overset{def}{\equiv} -\int_{B_{n,2}} |f_1(s,u_n(s))w(s)|ds + \int_{B_{n,1}} |f_1(s,u_n(s))w(s)|ds.$$

On the other hand, from the unique solvability of the Cauchy problem for the equation (1.3) it is clear that

$$w'(a) \neq 0, \quad w'(b) \neq 0, \quad w'(t_i) \neq 0 \qquad \text{for } \ i = 1, ..., k \tag{4.49}$$

if $N_{w_0} = \{t_1, ..., t_k\}$. Now note that, for any nonzero solution w of the problem (1.3), (1.4), there exists $\beta \neq 0$ such that $w(t) = \beta w_0(t)$. Consequently,

$$\Omega_w^{\pm} = \Omega_{w_0}^{\pm} \quad \text{if} \quad \beta > 0 \quad \text{and} \quad \Omega_w^{\mp} = \Omega_{w_0}^{\pm} \quad \text{if} \quad \beta < 0. \tag{4.50}$$

Then in view of (4.15) and (4.46), there exists $n_2 \geq n_0$ such that

$$\int_{C^+_{n_2,1}} |f_1(s, r) w_0(s)| ds \neq 0 \ \text{ and/or } \int_{C^-_{n_2,1}} |f_1(s, -r) w_0(s)| ds \neq 0. \tag{4.51}$$

From (4.51), in view of (4.6), it follows that

$$\int_{D_n^+} |f_1(s, r) w_0(s)| ds \neq 0 \quad \text{for} \quad n \geq n_2 \tag{4.52$_1$}$$

and/or

$$\int_{D_n^-} |f_1(s, -r) w_0(s)| ds \neq 0 \quad \text{for} \quad n \geq n_2. \tag{4.52$_2$}$$

Consequently, all the assumptions of Lemma 4.2 are satisfied with $G = D_n^+$ and/or $G = D_n^-$. Therefore, there exist $\varepsilon_0 \in]0, \varepsilon_2[$, $n_3 \geq n_2$, and $\delta_0 > 0$ such that

$$\begin{aligned} \mathbb{I}(D_n^+, U_{\varepsilon_0}^+, x) &\geq \delta_0 \ \text{ for } \ x \geq r, \quad n \geq n_3, \\ \mathbb{I}(D_n^-, U_{\varepsilon_0}^-, x) &\geq -\delta_0/2 \ \text{ for } \ x \leq -r, \ \ n \geq n_3 \end{aligned} \tag{4.53}$$

if (4.52$_1$) holds, and

$$\begin{aligned} \mathbb{I}(D_n^-, U_{\varepsilon_0}^-, x) &\geq \delta_0 \quad \text{for} \quad x \leq -r, \ n \geq n_3, \\ \mathbb{I}(D_n^+, U_{\varepsilon_0}^+, x) &\geq -\delta_0/2 \quad \text{for} \quad x \geq r, \ \ n \geq n_3 \end{aligned} \tag{4.54}$$

if (4.52$_2$) holds.

On the other hand, the definition of the set U_ε and (4.14$_1$), imply that there exists $n_4 > n_3$, such that

$$C_{n,2}^+ \subset U_{\varepsilon_0}^+, \quad C_{n,2}^- \subset U_{\varepsilon_0}^- \quad \text{for} \quad n \geq n_4. \tag{4.55}$$

By these inclusions, (4.2), and (4.5) we obtain

$$C_{n,1}^+ = A_{n,2}^+ \setminus C_{n,2}^+ \supset D_{n_4}^+ \setminus U_{\varepsilon_0}^+, \ C_{n,1}^- = A_{n,2}^- \setminus C_{n,2}^- \supset D_{n_4}^- \setminus U_{\varepsilon_0}^+ \tag{4.56}$$

for $n \geq n_4$. First suppose that $N_{w_0} \neq \emptyset$ and there exists $n \geq n_4$ such that

$$B_{n,2} \neq \emptyset. \tag{4.57}$$

Then, by taking into account that f_1 is non-decreasing in the second argument for $|x| \geq r$, (4.3), (4.12), (4.18) and the definitions of the sets $B^+_{n,2}, B^-_{n,2}$, we get

$$\begin{gathered}|f_1(t,u_n(t))| = f_1(t,u_n(t)) \leq \\ \leq f_1\left(t, \frac{\|u_n\|_C}{2n}\right) = \left|f_1\left(t, \frac{\|u_n\|_C}{2n}\right)\right| \quad \text{for } t \in B^+_{n,2}, \\ |f_1(t,u_n(t))| = -f_1(t,-u_n(t)) \leq \\ \leq -f_1(t, -\frac{\|u_n\|_C}{2n}) = \left|f_1\left(t, -\frac{\|u_n\|_C}{2n}\right)\right| \quad \text{for } t \in B^-_{n,2}.\end{gathered} \tag{4.58}$$

Analogously, from (4.3), (4.13), (4.18), and the definitions of the sets $C^+_{n,1}, C^-_{n,1}$, we obtain the estimates

$$\begin{gathered}|f_1(t,u_n(t))| \geq \left|f_1\left(t, \frac{\|u_n\|_C}{2n}\right)\right| \quad \text{for} \quad t \in C^+_{n,1}, \\ |f_1(t,u_n(t))| \geq \left|f_1\left(t, -\frac{\|u_n\|_C}{2n}\right)\right| \quad \text{for} \quad t \in C^-_{n,1}.\end{gathered} \tag{4.59}$$

Then from (4.1), (4.2), (4.9), (4.58) and respectively from (4.1), (4.2), (4.8), and (4.59) we have

$$\int_{B_{n,2}} |f_1(s,u_n(s))w(s)|ds \leq$$

$$\leq \int_{B^+_{n,2}} \left|f_1\left(s, \frac{\|u_n\|_C}{2n}\right) w(s)\right| ds + \int_{B^-_{n,2}} \left|f_1\left(s, -\frac{\|u_n\|_C}{2n}\right) w(s)\right| ds \leq \tag{4.60}$$

$$\leq \int_{C^+_{n,2}} \left|f_1\left(s, \frac{\|u_n\|_C}{2n}\right) w(s)\right| ds + \int_{C^-_{n,2}} \left|f_1\left(s, -\frac{\|u_n\|_C}{2n}\right) w(s)\right| ds$$

and respectively

$$\int_{B_{n,1}} |f_1(s,u_n(s))w(s)|ds \geq \int_{C_{n,1}} |f_1(s,u_n(s))w(s)|ds \geq$$

$$\geq \int_{C^+_{n,1}} \left|f_1\left(s, \frac{\|u_n\|_C}{2n}\right) w(s)\right| ds + \int_{C^-_{n,1}} \left|f_1\left(s, -\frac{\|u_n\|_C}{2n}\right) w(s)\right| ds. \tag{4.61}$$

If the condition (4.57) holds, from (4.60) and (4.61) we obtain

$$\frac{\widehat{\mathbb{M}}_n(w)}{|\beta|} \geq \left(\int_{C^+_{n,1}} \left|f_1\left(s, \frac{\|u_n\|_C}{2n}\right) w_0(s)\right| ds - \int_{C^+_{n,2}} \left|f_1\left(s, \frac{\|u_n\|_C}{2n}\right) w_0(s)\right| ds\right)$$

$$+\left(\int_{C^-_{n,1}} \left|f_1\left(s, -\frac{\|u_n\|_C}{2n}\right) w_0(s)\right| ds - \int_{C^-_{n,2}} \left|f_1\left(s, -\frac{\|u_n\|_C}{2n}\right) w_0(s)\right| ds\right),$$

Whence, by (4.55) and (4.56) we get

$$\frac{\widehat{\mathbb{M}}_n(w)}{|\beta|} \geq \mathbb{I}\left(D^+_{n_4}, U^+_{\varepsilon_0}, \frac{\|u_n\|_C}{2n}\right) + \mathbb{I}\left(D^-_{n_4}, U^-_{\varepsilon_0}, -\frac{\|u_n\|_C}{2n}\right) \tag{4.62}$$

for $n \geq n_4$. From (4.62) by (4.53) and (4.54) we obtain

$$\widehat{\mathbb{M}}_n(w) \geq \frac{\delta_0|\beta|}{2} \quad \text{for} \quad n \geq n_4. \tag{4.63}$$

On the other hand, in view of (4.10), (4.18), the definition of the sets $A_{n,2}, B_{n,1}$, and the fact that f_1 is non-decreasing in the second argument, we obtain the estimate

$$\int_{B_{n,1}} |f_1(s, u_n(s))w(s)|ds \geq$$

$$\geq \int_{B^+_{n,1}} |f_1(s, r)w(s)|ds + \int_{B^-_{n,1}} |f_1(s, -r)w(s)|ds \geq \tag{4.64}$$

$$\geq \int_{C^+_{n,1}} |f_1(s, r)w(s)|ds + \int_{C^-_{n,1}} |f_1(s, -r)w(s)|ds.$$

Now suppose that there exists $n \geq n_4$ such that

$$B_{n,2} = \emptyset. \tag{4.65}$$

Then from (4.51) and (4.64), (4.65) there follows the existence of $\delta^* > 0$ such that $\widehat{\mathbb{M}}_n(w) \geq |\beta|\delta^*$. From this inequality and (4.63) it follows that, in both cases when (4.57) or (4.65) are fulfilled, the inequality

$$\widehat{\mathbb{M}}_n(w) \geq |\beta|\delta \quad \text{for} \quad n \geq n_4 \tag{4.66}$$

holds with $\delta = \min\{\delta_0/2, \delta^*\}$. From (4.48) by (4.7) and (4.66), we see that for any $\varepsilon \in]0, \delta[$ there exists $n_1 > n_4$ such that

$$\int_a^b f_1(s, u_n(s))w(s)ds \geq |\beta|(\delta - \varepsilon) \quad \text{for} \quad n \geq n_1,$$

and thus

$$\frac{\mathbb{M}_n(w)}{|\beta|} \geq \delta - \varepsilon - \left|\int_a^b h_1(s)w_0(s)ds\right| \quad \text{for} \quad n \geq n_1. \tag{4.67}$$

If $N_{w_0} = \emptyset$ then $|w(t)| > 0$ for $a < t < b$ and in view of (4.3), (4.41), (4.42) and (4.49), the condition (4.65) holds, i.e., the inequality (4.67) also holds.

Consequently, since $\varepsilon > 0$ is arbitrary, the inequality (4.43) from (4.67) and (4.47) follows. ∎

Lemma 4.6. *Let* w_0 *be a nonzero solution of the problem* (1.3), (1.4), $r>0$, *and the conditions* (4.18), (4.47) *hold with* $f_1(t,x)\overset{def}{\equiv}f_0(t)g_1(x)$, *where* $f_0\in L(I;R_+)$, $\int_a^b|f_0(s)|ds\neq 0$ *and a non-decreasing function* $g_1\in C(R;R)$ *be such that*

$$\lim_{|x|\to+\infty}|g_1(x)|=+\infty. \tag{4.68}$$

Then, for every nonzero solution w *of the problem* (1.3), (1.4) *and functions* $u_n\in\widetilde{C}'(I;R)$ $(n\in N)$ *fulfilling the conditions* (4.3), (4.41), (4.42), *the inequality* (4.43) *holds.*

Proof. From the assumptions of our lemma it is clear that the relations (4.48)–(4.56), (4.58)-(4.61) and (4.64) with $f_1(t,x)=f_0(t)g_1(x)$ and $w(t)=\beta w_0(t)$ $(\beta\neq 0)$ are fulfilled.

Assuming $\int_{C^+_{n_2,1}}|f_1(s,r)w_0(s)|ds\neq 0$, the condition (4.52_1) is satisfied i.e., (4.53) holds.

Now notice that from (4.15) and the equality $C^+_{n,1}=\Omega^+_w\setminus(\Omega^+_w\setminus C^+_{n,1})$ it follows that there exist $\varepsilon>0$ and $n_0\in N$ such that

$$\int_{C^+_{n,1}}|f_0(s)w_0(s)|ds\geq\int_{\Omega^+_w}|f_0(s)w_0(s)|ds-\varepsilon>0 \tag{4.69}$$

for $n\geq n_0$.

First consider the case when there exists $n\geq n_4$ such that the condition (4.65) holds. Without loss of generality we can assume that $n_4>n_0$. Then by (4.50), (4.64), (4.65) and (4.69), we obtain

$$\widehat{\mathbb{M}}_n(w)\geq|\beta||g_1(r)|\left(\int_{\Theta_\beta}|f_0(s)w_0(s)|ds-\varepsilon\right)>0, \tag{4.70}$$

where $\Theta_\beta=\Omega^+_{w_0}$ if $\beta>0$ and $\Theta_\beta=\Omega^-_{w_0}$ if $\beta<0$.

Consider now the case when there exists $n\geq n_4$ such that (4.57) holds. From (4.3) and the definition of the set D^+_n it follows that $D^+_n\subset D^+_{n+1}$, and since g_1 is non-decreasing, from (4.53) we obtain $\mathbb{I}(D^+_n,U^+_{\varepsilon_0},x)\geq|g_1(r)|\mu=\mathbb{I}(D^+_{n_4},U^+_{\varepsilon_0},r)\geq\delta_0$ for $x\geq r$, with $\mu=\int_{D^+_{n_4}\setminus U^+_{\varepsilon_0}}|f_0(s)w_0(s)|ds-\int_{U^+_{\varepsilon_0}}|f_0(s)w_0(s)|ds$. By the last inequality, (4.3), (4.53), and (4.62) we get $\mu>0$ and

$$\widehat{\mathbb{M}}_n(w)\geq|\beta|(|g_1(r)|\mu-\delta_0/2). \tag{4.71}$$

Applying (4.70), (4.71) in (4.48) and taking (4.7) into account, we conclude that there exist $\varepsilon_1>0$ and $n_1\geq n_4$ such that

$$|\beta|\left(|g_1(r)|\mu_1-\frac{\delta_0}{2}-\varepsilon_1\right)\leq\int_a^b f_1(s,u_n(s))w(s)ds\quad\text{for}\quad n\geq n_1$$

with $\mu_1=\min(\mu,\int_{\Omega^+_{w_0}}|f_0(s)w_0(s)|ds-\varepsilon)$. From (4.68) and the last inequality it is clear that, for any function h_1, we can choose $r>0$ such that the inequality (4.43) will be true. Analogously one can prove (4.43) in the case when $\int_{C^-_{n_2,1}}|f_1(s,r)w_0(s)|ds\neq 0$. ■

Lemma 4.7. *Let* $r > 0$, *there exist functions* $\alpha, f^-, f^+ \in L(I, R_+)$ *such that the condition* (4.38) *is satisfied,*

$$\sup\{|f_1(t,x)| : x \in R\} = \alpha(t) \quad for \quad t \in I, \tag{4.72}$$

and there exist a nonzero solution w_0 *of the problem* (1.3), (1.4) *and* $\varepsilon > 0$ *such that*

$$\begin{gathered} -\int_a^b (f^+(s)[w_0(s)]_- + f^-(s)[w_0(s)]_+)ds + \varepsilon||\alpha||_L \leq \\ \leq -\int_a^b h_1(s)w_0(s)ds \leq \\ \leq \int_a^b (f^-(s)[w_0(s)]_- + f^+(s)[w_0(s)]_+)ds - \varepsilon||\alpha||_L. \end{gathered} \tag{4.73}$$

Then, for every nonzero solution w *of the problem* (1.3), (1.4) *and functions* $u_n \in \widetilde{C}'(I;R)$ $(n \in N)$ *fulfilling the conditions* (4.3), (4.41), *and* (4.42), *there exists* $n_1 \in N$ *such that the inequality* (4.43) *holds.*

Proof. First note that, for any nonzero solution w of the problem (1.3), (1.4), there exists $\beta \neq 0$ such that $w(t) = \beta w_0(t)$. Moreover, it is not difficult to verify that all the assumptions of Lemma4.1 are satisfied for the function $w(t) = \beta w_0(t)$. From (4.1), (4.2), and (4.72) we get

$$\begin{gathered} \mathbb{M}_n(w) \geq -\int_{A_{n,1}\cup B_{n,2}} \alpha(s)|w(s)|ds + \int_{B_{n,1}} f_1(s,u_n)w(s)ds + \\ + \int_a^b h_1(s)w(s)ds. \end{gathered} \tag{4.74}$$

On the other hand, by the definition of the set $B_{n,1}$ we obtain

$$\operatorname{sgn} u_n(t) = \operatorname{sgn} w(t) \quad \text{for} \quad t \in B^+_{n,1} \cup B^-_{n,1}. \tag{4.75}$$

Hence, by (4.1), (4.2), (4.10), (4.38), and (4.75), from (4.74) we obtain the estimate

$$\begin{gathered} \mathbb{M}_n(w) - \int_a^b h_1(s)w(s)ds \geq -\int_{A_{n,1}\cup B_{n,2}} \alpha(s)|w(s)|ds + \\ + \int_{B^+_{n,1}} f^+(s)|w(s)|ds + \int_{B^-_{n,1}} f^-(s)|w(s)|ds \geq \\ \geq -\int_{A_{n,1}\cup B_{n,2}} \alpha(s)|w(s)|ds + \int_{C^+_{n,1}} f^+(s)|w(s)|ds + \int_{C^-_{n,1}} f^-(s)|w(s)|ds. \end{gathered} \tag{4.76}$$

Now, note that $f^- \equiv 0$ and $f^+ \equiv 0$ if $f_1(t,x) \equiv 0$. Therefore by (4.7), (4.11), (4.15), and the inclusions $C^+_{n,1} \subset \Omega^+_w$, $C^-_{n,1} \subset \Omega^-_w$, we see that there exist $\varepsilon > 0$

and $n_1 \in N$ such that

$$\begin{aligned}\frac{1}{3}\varepsilon\|\alpha\|_L &\geq \int_{A_{n,1}\cup B_{n,2}} \alpha(s)|w_0(s)|ds \\ \int_{\Omega_w^{\pm}} f^{\pm}(s)|w_0(s)|ds - \frac{1}{3}\varepsilon\|\alpha\|_L &\leq \int_{C_{n,1}^{\pm}} f^{\pm}(s)|w_0(s)|ds\end{aligned} \tag{4.77}$$

for $n \geq n_1$. By virtue of (4.76) and (4.77), we obtain

$$\frac{\mathbb{M}_n(w)}{|\beta|} \geq -\varepsilon\|\alpha\|_L + \int_{\Omega_w^+} f^+(s)|w_0(s)|ds +$$

$$+ \int_{\Omega_w^-} f^-(s)|w_0(s)|ds + \sigma \int_a^b h_1(s)w_0(s)ds$$

for $n \geq n_1$, where $\sigma = \operatorname{sgn}\beta$. Now, by taking into account that

$$\int_{\Omega_w^{\pm}} l(s)|w_0(s)|ds = \int_{\Omega_{w_0}^{\pm}} l(s)|w_0(s)|ds = \int_a^b l(s)[w_0(s)]_{\pm}ds$$

if $\beta > 0$ and

$$\int_{\Omega_w^{\pm}} l(s)|w_0(s)|ds = \int_{\Omega_{w_0}^{\mp}} l(s)|w_0(s)|ds = \int_a^b l(s)[w_0(s)]_{\mp}ds$$

if $\beta < 0$ for an arbitrary $l \in L(I, R)$, from the last inequalities we get

$$\frac{\mathbb{M}_n(w)}{|\beta|} \geq -\varepsilon\|\alpha\|_L + \int_a^b (f^+(s)[w_0(s)]_+ + f^-(s)[w_0(s)]_-)ds +$$

$$+ \int_a^b h_1(s)w_0(s)ds \quad \text{for} \quad n \geq n_1$$

if $\sigma = 1$, and

$$\frac{\mathbb{M}_n(w)}{|\beta|} \geq -\varepsilon\|\alpha\|_L + \int_a^b (f^+(s)[w_0(s)]_- + f^-(s)[w_0(s)]_+)ds -$$

$$- \int_a^b h_1(s)w_0(s)ds \quad \text{for} \quad n \geq n_1$$

if $\sigma = -1$. From the last inequalities and (4.73) we immediately obtain (4.43). ■

Now we consider the definitions of the sets $V_{10}((a,b))$ introduced and described in [12] (see [Definition 1.3, p. 2350])

Definition 4.2. We say that the function $p \in L([a,b])$ belongs to the set $V_{10}((a,b))$ if for any function p^* satisfying the inequality $p^*(t) \geq p(t)$ for $t \in I$ the unique solution of the initial value problem

$$u''(t) = p^*(t)u(t) \quad \text{for} \quad t \in I, \quad u(a) = 0, \quad u'(a) = 1, \tag{4.78}$$

has no zeros in the set $]a,b]$.

Lemma 4.8. *Let* $i \in \{1,2\}$, $p \in L(I;R)$, $p_n(t) = p(t) + (-1)^i/n$, *and* $w_n \in \widetilde{C}'(I;R)$ $(n \in N)$ *be a solution of the problem*

$$w_n''(t) = p_n(t)w_n(t) \quad \text{for} \quad t \in I, \quad w_n(a) = 0, \quad w_n(b) = 0. \tag{4.79$_n$}$$

Then:

a) There exists $n_0 \in N$ *such that the problem* (4.79_n) *has only the zero solution for* $n \geq n_0$.

b) If $i = 2$ *and* $N_w = \emptyset$, *where* w *is a solution of the problem* (1.3), (1.4), *then the inclusion* $p_n \in V_{10}((a,b))$ *for every* $n \in N$ *holds.*

Proof. *a*) Let $N^*_{w_n}$ be the number of zeros of the function w_n on I. Assume on the contrary that there exists a sequence $\{w_n\}^{+\infty}_{n \geq n_0}$ of nonzero solutions of the problem (4.79_n).

If $i = 1$ then from the facts that $p_n(t) < p_{n+1}(t)$ and $w_n \not\equiv 0$, by Sturm's comparison theorem, we obtain $N^*_{w_n} - N^*_{w_{n+1}} \geq 1$ $(n \in N)$. Now notice that, in view of (4.79_n), the inequality $N^*_{w_n} \geq 2$ holds. Hence there exist $k_0 \geq 2$ and $n_0 \geq 2$ such that $N^*_{w_{n_0}} = k_0$. Therefore, we obtain the contradiction $k_0 = N^*_{w_{n_0}} > N^*_{w_{n_0}} - N^*_{w_{n_0+k_0}} = (N^*_{w_{n_0}} - N^*_{w_{n_0+1}}) + (N^*_{w_{n_0+1}} - N^*_{w_{n_0+2}}) + \ldots + (N^*_{w_{n_0+k_0-1}} - N^*_{w_{n_0+k_0}}) \geq k_0$.

If $i = 2$, from the fact that $p_{n-1}(t) > p_n(t) > p(t)$ and $w_n \not\equiv 0$, by Sturm's comparison theorem, we obtain $N^*_{w_n} - N^*_{w_{n-1}} \geq 1$ and $N^*_w \geq N^*_{w_n} - 1$ $(n \in N)$ if w is a nonzero solution of the equation (1.3). Now notice that, in view of (4.79_n), the inequality $N^*_{w_n} \geq 2$ holds for every $n \in N$. Therefore, if we denote $N^*_w = k_0$, we obtain the contradiction $k_0 = N^*_w \geq N^*_{w_{n+k_0}} - 1 > N^*_{w_{n+k_0}} - N^*_{w_n} \geq k_0$.

The contradiction obtained proves the item *a*) of our lemma.

b) Assume on the contrary that there exists $n \in N$ such that $p_n \notin V_{10}([a,b])$. If $p^*(t) \geq p_n(t)$ and u is a solution of the problem (4.78), then there exists $t_0 \in]a,b]$ such that $u(t_0) = 0$. Since $p(t) < p^*(t)$, by Sturm's comparison theorem, we obtain that w, the solution of the problem (1.3), (1.4), has a zero in the interval $]a, t_0[$, which contradicts our assumption $N_w = \emptyset$. The contradiction obtained proves the item *b*) of our lemma. ■

5. Proof of the main results

Proof of Theorem 2.1. Let $p_n(t) = p(t) + 1/n$ and, for any $n \in N$, consider the problem

$$u_n''(t) = p_n(t)u_n(t) + f(t, u_n(t)) + h(t) \quad \text{for} \quad t \in I, \tag{5.1}$$

$$u_n(a) = 0, \quad u_n(b) = 0. \tag{5.2}$$

In view of the condition (2.1) and Lemma 4.8, the inclusion $p_n \in V_{10}((a,b))$ holds for every $n \in N$. On the other hand, from the conditions (2.2) and (2.3) we find

$$0 \leq f(t,x)\mathrm{sgn}x \leq g(t)|x| + h_0(t) \quad \text{for } t \in I, \ |x| \geq r. \tag{5.3}$$

Then the inclusion $p_n \in V_{10}((a,b))$, as is well-known (see [12, Theorem 2.2, p.2367]), guarantees that the problem (5.1), (5.2) has at least one solution, suppose u_n. In view of the condition (2.2), without loss of generality we can assume that there exists $\varepsilon^* > 0$ such that $h_0(t) \geq \varepsilon^*$ on I. Then $g(t)|x| + h_0(t) \geq \varepsilon^*$ for $x \in R$, $t \in I$. Consequently, it is not difficult to verify that u_n is also a solution of the equation

$$u_n''(t) = (p_n(t) + p_0(t, u_n(t))\mathrm{sgn} u_n(t))u_n(t) + p_1(t, u_n(t)) \tag{5.4}$$

with $p_0(t,x) = \dfrac{f(t,x)g(t)}{g(t)|x| + h_0(t)}$, $p_1(t,x) = h(t) + \dfrac{f(t,x)h_0(t)}{g(t)|x| + h_0(t)}$.

Now assume that

$$\lim_{n \to +\infty} ||u_n||_C = +\infty \tag{5.5}$$

and $v_n(t) = u_n(t)||u_n||_C^{-1}$. Then

$$v_n''(t) = (p_n(t) + p_0(t, u_n(t))\mathrm{sgn} u_n(t))v_n(t) + \frac{1}{||u_n||_C} p_1(t, u_n(t)), \tag{5.6}$$

$$v_n(a) = 0 \qquad v_n(b) = 0, \tag{5.7}$$

and

$$||v_n||_C = 1 \tag{5.8}$$

for any $n \in N$. In view of the condition (5.3), the functions $p_0, p_1 \in K(I \times R; R)$ are bounded respectively by the functions $g(t)$ and $h(t) + h_0(t)$. Therefore, from (5.6), by virtue of (5.5), (5.7) and (5.8), we see that there exists $r_0 > 0$ such that $||v_n'||_C \leq r_0$. Consequently in view of (5.8), by Arzela-Ascoli lemma, without loss of generality we can assume that there exists $w \in \widetilde{C}'(I, R)$ such that $\lim\limits_{n \to +\infty} v_n^{(i)}(t) = w^{(i)}(t)$ $(i = 0, 1)$ uniformly on I. From the last equality and (5.5) there follows the existence of an increasing sequence $\{\alpha_k\}_{k=1}^{+\infty}$ of a natural numbers, such that $||u_{\alpha_k}||_C \geq 2rk$ and $||v_{\alpha_k}^{(i)} - w^{(i)}||_C \leq 1/2k$ for $k \in N$. Without loss of generality we can suppose that $u_n \equiv u_{\alpha_n}$ and $v_n \equiv v_{\alpha_n}$. In this case we see that u_n and v_n are the solutions of the problems (5.1), (5.2) and (5.6), (5.7) respectively with $p_n(t) = p(t) + 1/\alpha_n$ for $t \in I$, $n \in N$, and that the inequalities

$$||u_n||_C \geq 2rn, \quad ||v_n^{(i)} - w^{(i)}||_C \leq 1/2n \quad \text{for} \quad n \in N \tag{5.9}$$

are fulfilled. Analogously, since the functions $p_0, p_1 \in K(I \times R; R)$ are bounded, in view of (5.5), we can assume without loss of generality that there exists a function $\widetilde{p} \in L(I; R)$ such that

$$\lim_{n \to +\infty} \frac{1}{||u_n||_C^j} \int_a^t p_j(s, u_n(s))\mathrm{sgn} u_n(s) ds = (1 - j) \int_a^t \widetilde{p}(s) ds \tag{5.10$_j$}$$

uniformly on I for $j = 0, 1$. By virtue of (5.8)–(5.10$_j$) $(j = 0, 1)$, from (5.6) we obtain

$$w''(t) = (p(t) + \widetilde{p}(t))w(t), \tag{5.11}$$

$$w(a) = 0, \quad w(b) = 0, \tag{5.12}$$

and

$$||w||_C = 1. \tag{5.13}$$

From the conditions (2.3) and (5.9) it is clear that all the assumptions of Lemma 4.3 with $f_1(t, x) = f(t, x)$ are satisfied, and thus we obtain from (5.10$_j$) $(j = 0)$ the relation $\int_s^t \widetilde{p}(\xi)d\xi \geq 0$ for $a \leq s < t \leq b$, i.e.,

$$\widetilde{p}(t) \geq 0 \quad \text{for} \quad t \in I. \tag{5.14}$$

Now assume that $\widetilde{p} \not\equiv 0$ and w_0 is a solution of the problem (1.3), (1.4). Then using Sturm's comparison theorem for the equations (1.3) and (5.11), from (5.14) we see that there exists a point $t_0 \in]a, b[$ such that $w_0(t_0) = 0$, which contradicts (2.1). This contradiction proves that $\widetilde{p} \equiv 0$. Consequently, w is a solution of the problem (1.3), (1.4). Multiplying the equations (5.1) and (1.3) respectively by w and $-u_n$, and therefore integrating their sum from a to b, in view of the conditions (5.2) and (1.4), we obtain

$$-\frac{1}{\alpha_n}\int_a^b w(s)u_n(s)ds = \int_a^b (h(s) + f(s, u_n(s)))w(s)ds \tag{5.15}$$

for $n \geq n_0$. Therefore by virtue of (5.9) we get

$$\int_a^b (h(s) + f(s, u_n(s)))w(s)ds < 0 \quad \text{for} \quad n \geq n_0. \tag{5.16}$$

On the other hand, in view the conditions (2.1)-(2.4$_1$), (5.2), and (5.9) it is clear that all the assumption of Lemma 4.4 with $f_1(t, x) = f(t, x)$, $h_1(t) = h(t)$ are fulfilled. Therefore, the inequality (4.43) is true, which contradicts (5.16). This contradiction proves that (5.5) does not hold and thus there exists $r_1 > 0$ such that $||u_n||_C \leq r_1$ for $n \in N$. Consequently, from (5.1) and (5.2) it is clear that there exists $r_1' > 0$ such that $||u_n'||_C \leq r_1'$ and $|u_n''(t)| \leq \sigma(t)$ for $t \in I$, $n \in N$, where $\sigma(t) = (1+|p(t)|)r_1+|h(t)|+\gamma_{r_1}(t)$. Hence, by Arzela-Ascoli lemma, without loss of generality we can assume that there exists a function $u_0 \in \widetilde{C}'(I; R)$ such that $\lim\limits_{n\to+\infty} u_n^{(i)}(t) = u_0^{(i)}(t)$ $(i = 0, 1)$ uniformly on I. Therefore, it follows from (5.1) and (5.2) that u_0 is a solution of the problem (1.1), (1.2). ∎

Proof of Theorem 2.2. Let $p_n(t) = p(t) - 1/n$ and, for any $n \in N$, consider the problems (5.1), (5.2) and (4.79_n). In view of Lemma 4.8, the problem (4.79_n) has only the zero solution for every $n \geq n_0$. Therefore, as is well-known (see [9, Theorem 1.1, p.345]), from the conditions (2.7), (2.9) it follows that the problem (5.1), (5.2) has at least one solution, suppose u_n.

Now assume that (5.5) holds and put $v_n(t) = u_n(t)||u_n||_C^{-1}$. Then the conditions (5.7) and (5.8) are fulfilled, and

$$v_n''(t) = p_n(t)v_n(t) + \frac{1}{||u_n||_C}(f(t, u_n(t))) + h(t)). \tag{5.17}$$

In view the conditions (2.7) and (2.9), from (5.17) there follows the existence of $r_0 > 0$ such that $||v_n'||_C \leq r_0$. Consequently, in view (5.8) by Arzela-Ascoli lemma, without loss of generality we can assume that there exists a function $w \in \widetilde{C}'(I, R)$ such that $\lim\limits_{n\to+\infty} v_n^{(i)}(t) = w^{(i)}(t) \quad (i = 0, 1)$ uniformly on I. Analogously as in the proof of Theorem 2.1, we can find a sequence $\{\alpha_k\}_{n=1}^{+\infty}$ of natural numbers such that, if we suppose $u_n = u_{\alpha_n}$ then the conditions (5.9) will by true when the functions u_n and v_n are the solutions of the problems (5.1), (5.2) and (5.17), (5.7) respectively with $p_n(t) = p(t) - 1/\alpha_n$ for $t \in I,\ n \in N$. From (5.17), by virtue of (5.7), (5.9) and (2.9), we obtain that w is a solution of the problem (1.3), (1.4). In a similar manner as the condition (5.15) in the proof of Theorem 2.1, we show that

$$\frac{1}{\alpha_n}\int_a^b w(s)u_n(s)ds = \int_a^b (h(s) + f(s, u_n(s)))w(s)ds \tag{5.18}$$

for $n \geq n_0$. Now note that, in view of the conditions (2.1), (2.8), (2.4_2), (5.2), and (5.9), all the assumptions of Lemma 4.4 with $f_1(t, x) = -f(t, x),\ h_1(t) = -h(t)$ are satisfied. Hence, analogously as in the proof of Theorem 2.1, from (5.18) we show that the problem (1.1), (1.2) has at least one solution. ■

Proof of Theorem 2.3. Let $p_n(t) = p(t) + (-1)^i/n$ and for any $n \in N$, consider the problems (5.1), (5.2) and (4.79_n). In view of the condition (2.13) and the fact that $(-1)^i f(t, x)$ is non-decreasing in the second argument for $|x| \geq r$, we obtain

$$\lim_{n\to+\infty} \frac{1}{||z_n||_C}\int_a^b |f(s, z_n(s))|ds = 0 \tag{5.19}$$

for an arbitrary sequence $z_n \in C(I; R)$ with $\lim\limits_{n\to+\infty} ||z_n||_C = +\infty$. Moreover, in view of Lemma 4.8, the problem (4.79_n) has only the zero solution for every $n \geq n_0$. Therefore, as it is well-known (see [9, Theorem 1.1, p. 345]), from the inequality (5.19) it follows that the problem (5.1), (5.2) has at least one solution, suppose u_n.

Now assume that (5.5) is fulfilled and put $v_n(t) = u_n(t)||u_n||_C^{-1}$. Then (5.7), (5.8) and (5.17) are also fulfilled. Hence, by the conditions (5.8) and (5.19), from (5.17) we get the existence of $r_0 > 0$ such that $||v_n'||_C \leq r_0$. Consequently, in view of (5.8) by the Arzela-Ascoli lemma, without loss of generality we can assume that there exists a function $w \in \widetilde{C}'(I, R)$ such that $\lim\limits_{n\to+\infty} v_n^{(i)}(t) = w^{(i)}(t) \quad (i = 0, 1)$ uniformly on I. Analogously as in the proof of Theorem 2.1, we can find a sequence $\{\alpha_k\}_{n=1}^{+\infty}$ of natural numbers such that, assuming $u_n = u_{\alpha_n}$, the conditions (5.9) is true and the functions u_n and v_n are the solutions of the problems (5.1), (5.2) and (5.17), (5.7) respectively with $p_n(t) = p(t) + (-1)^i/\alpha_n$ for $t \in I,\ n \in N$. From (5.17), by virtue of (5.7), (5.9) and (2.13), we obtain that w is a solution of the problem (1.3), (1.4). In a similar manner as the condition (5.15) in the proof of Theorem 2.1, we show

$$-\frac{1}{\alpha_n}\int_a^b w(s)u_n(s)ds = (-1)^i \int_a^b (h(s) + f(s, u_n(s)))w(s)ds \tag{5.20}$$

for $n \in N \geq n_0$. Now note that, in view the conditions (2.11), (2.12), (2.14), (5.2), and (5.9), all the assumptions of Lemma 4.5 with $f_1(t, x) = (-1)^i f(t, x),\ h_1(t) = (-1)^i h(t)$ are satisfied. Hence, analogously as in the proof of Theorem 2.1, from (5.20) by Lemma 4.5 we obtain that the problem (1.1), (1.2) has at least one solution. ∎

Proof of Corollary 2.1. From the condition (2.15) we immediately obtain (2.14). Therefore all the conditions of Theorem 2.3 are fulfilled. ∎

Proof of Theorem 2.4. The proof is the same as the proof of Theorem 2.3. The only difference is that we use Lemma 4.6 instead of Lemma 4.5. ∎

Proof of Theorem 2.5. From (2.21) it is clear that, for an arbitrary sequence $z_n \in C(I; R)$ such that $\lim\limits_{n\to+\infty} ||z_n||_C = +\infty$, the equality (5.19) is holds. From (5.19) and Lemma 4.7, analogously as in the proof of Theorem 2.3, we show that the problem (1.1), (1.2) has at least one solution. ∎

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GRÜSS' TYPE INEQUALITIES FOR FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. Some inequalities of Grüss' type for functions of selfadjoint operators in Hilbert spaces, under suitable assumptions for the involved operators, are given.

Keywords: selfadjoint operators, Grüss inequality, functions of selfadjoint operators.

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1. Introduction

In 1935, G. Grüss [19] proved the following integral inequality which gives an approximation of the integral of the product in terms of the product of the integrals as follows:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma),$$

where f, $g : [a,b] \to \mathbb{R}$ are integrable on $[a,b]$ and satisfy the condition

$$\phi \leq f(x) \leq \Phi,\ \gamma \leq g(x) \leq \Gamma \tag{1.1}$$

for each $x \in [a,b]$, where $\phi, \Phi, \gamma, \Gamma$ are given real constants.

Moreover, the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

In 1950, M. Biernacki, H. Pidek and C. Ryll-Nardjewski [22, Chapter X] established the following discrete version of Grüss' inequality:

Let $a=(a_1,...,a_n)$, $b=(b_1,...,b_n)$ be two $n-$tuples of real numbers such that $r\leq a_i\leq R$ and $s\leq b_i\leq S$ for $i=1,...,n$. Then one has

$$\left|\frac{1}{n}\sum_{i=1}^{n}a_ib_i-\frac{1}{n}\sum_{i=1}^{n}a_i\cdot\frac{1}{n}\sum_{i=1}^{n}b_i\right|\leq\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)(R-r)(S-s), \tag{1.2}$$

where $[x]$ denotes the integer part of x, $x\in\mathbb{R}$.

For a simple proof of (1.1) as well as for some other integral inequalities of Grüss type, see Chapter X of the recent book [22]. For other related results see the papers [1]-[3], [4]-[6], [7]-[9], [10]-[17], [18], [25], [27] and the references therein.

2. Operator inequalities

Let A be a selfadjoint linear operator on a complex Hilbert space $(H;\langle .,.\rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, an the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [20, p. 3]):

For any $f,g\in C(Sp(A))$ and any $\alpha,\beta\in\mathbb{C}$ we have

(i) $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;

(ii) $\Phi(fg)=\Phi(f)\Phi(g)$ and $\Phi(\bar{f})=\Phi(f)^*$;

(iii) $\|\Phi(f)\|=\|f\|:=\sup_{t\in Sp(A)}|f(t)|$;

(iv) $\Phi(f_0)=1_H$ and $\Phi(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$, for $t\in Sp(A)$.

With this notation we define

$$f(A):=\Phi(f)\text{ for all }f\in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t)\geq 0$ for any $t\in Sp(A)$ implies that $f(A)\geq 0$, *i.e.* $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$f(t)\geq g(t)\text{ for any }t\in Sp(A)\text{ implies that }f(A)\geq g(A) \tag{P}$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [20] and the references therein. For other results, see [24], [21] and [26].

The following operator version of the Grüss inequality was obtained by Mond & Pečarić in [23]:

Theorem 2.1 (Mond-Pečarić, 1993, [23]) *Let* C_j, $j \in \{1,...,n\}$ *be selfadjoint operators on the Hilbert space* $(H, \langle .,. \rangle)$ *and such that* $m_j \cdot 1_H \leq C_j \leq M_j \cdot 1_H$ *for* $j \in \{1,...,n\}$,, *where* 1_H *is the identity operator on* H. *Further, let* $g_j, h_j : [m_j, Mj] \to \mathbb{R}$, $j \in \{1,...,n\}$ *be functions such that*

$$\varphi \cdot 1_H \leq g_j(C_j) \leq \Phi \cdot 1_H \text{ and } \gamma \cdot 1_H \leq h_j(C_j) \leq \Gamma \cdot 1_H \tag{2.1}$$

for each $j \in \{1,...,n\}$.

If $x_j \in H$, $j \in \{1,...,n\}$ *are such that* $\sum_{j=1}^{n} \|x_j\|^2 = 1$, *then*

$$\begin{aligned}\left| \sum_{j=1}^{n} \langle g_j(C_j) h_j(C_j) x_j, x_j \rangle - \sum_{j=1}^{n} \langle g_j(C_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle h_j(C_j) x_j, x_j \rangle \right| \\ \leq \frac{1}{4} (\Phi - \varphi)(\Gamma - \gamma).\end{aligned} \tag{2.2}$$

If $C_j, j \in \{1,...,n\}$ are selfadjoint operators such that $Sp(C_j) \subseteq [m, M]$ for $j \in \{1,...,n\}$ and for some scalars $m < M$ and if $g, h : [m, M] \longrightarrow \mathbb{R}$ are continuous then by the Mond-Pečarić inequality we deduce the following version of the Grüss inequality for operators

$$\begin{aligned}\left| \sum_{j=1}^{n} \langle g(C_j) h(C_j) x_j, x_j \rangle - \sum_{j=1}^{n} \langle g(C_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle h(C_j) x_j, x_j \rangle \right| \\ \leq \frac{1}{4} (\Phi - \varphi)(\Gamma - \gamma),\end{aligned} \tag{2.3}$$

where $x_j \in H$, $j \in \{1,...,n\}$ are such that $\sum_{j=1}^{n} \|x_j\|^2 = 1$ and $\varphi = \min_{t \in [m,M]} g(t)$, $\Phi = \max_{t \in [m,M]} g(t)$, $\gamma = \min_{t \in [m,M]} h(t)$ and $\Gamma = \max_{t \in [m,M]} h(t)$.

In particular, if the selfadjoint operator C satisfy the condition $Sp(C) \subseteq [m, M]$ for some scalars $m < M$, then

$$|\langle g(C) h(C) x, x \rangle - \langle g(C) x, x \rangle \cdot \langle h(C) x, x \rangle| \leq \frac{1}{4} (\Phi - \varphi)(\Gamma - \gamma), \tag{2.4}$$

for any $x \in H$ with $\|x\| = 1$.

We say that the functions $f, g : [a, b] \longrightarrow \mathbb{R}$ are *synchronous (asynchronous)* on the interval $[a, b]$ if they satisfy the following condition:

$$(f(t) - f(s))(g(t) - g(s)) \geq (\leq)\, 0 \text{ for each } t, s \in [a, b].$$

It is obvious that, if f, g are monotonic and have the same monotonicity on the interval $[a, b]$, then they are synchronous on $[a, b]$ while if they have opposite monotonicity, they are asynchronous.

In the recent paper [15] the following *Čebyšev type inequality* for operators has been obtained:

Theorem 2.2 (Dragomir, 2008, [15]) *Let* A *be a selfadjoint operator on the Hilbert space* $(H, \langle .,. \rangle)$ *with the spectrum* $Sp(A) \subseteq [m, M]$ *for some real numbers* $m < M$. *If* $f, g : [m, M] \longrightarrow \mathbb{R}$ *are continuous and synchronous (asynchronous) on* $[m, M]$, *then*

$$\langle f(A) g(A) x, x \rangle \geq (\leq) \langle f(A) x, x \rangle \cdot \langle g(A) x, x \rangle \tag{2.5}$$

for any $x \in H$ *with* $\|x\| = 1$.

This can be generalized for n operators as follows:

Theorem 2.3 (Dragomir, 2008, [15]) *Let* A_j *be selfadjoint operators with* $Sp(A_j) \subseteq [m, M]$ *for* $j \in \{1, ..., n\}$ *and for some scalars* $m < M$. *If* $f, g : [m, M] \longrightarrow \mathbb{R}$ *are continuous and synchronous (asynchronous) on* $[m, M]$, *then*

$$\sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle \geq (\leq) \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle, \tag{2.6}$$

for each $x_j \in H, j \in \{1, ..., n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

Another version for n operators is incorporated in:

Theorem 2.4 (Dragomir, 2008, [15]) *Let* A_j *be selfadjoint operators with* $Sp(A_j) \subseteq [m, M]$ *for* $j \in \{1, ..., n\}$ *and for some scalars* $m < M$. *If* $f, g : [m, M] \longrightarrow \mathbb{R}$ *are continuous and synchronous (asynchronous) on* $[m, M]$, *then*

$$\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle \geq (\leq) \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle, \tag{2.7}$$

for any $p_j \geq 0, j \in \{1, ..., n\}$ *with* $\sum_{j=1}^{n} p_j = 1$ *and* $x \in H$ *with* $\|x\| = 1$.

Motivated by the above results we investigate in this paper other Grüss' type inequalities for selfadjoint operators in Hilbert spaces. Some of the obtained results improve the inequalities (2.3) and (2.4) derived from the Mond-Pečarić inequality. Others provide different operator versions for the celebrated Grüss' inequality mentioned above. Examples for power functions and the logarithmic function are given as well.

3. An inequality of Grüss' type for one operator

The following result may be stated:

Theorem 3.1 *Let A be a selfadjoint operator on the Hilbert space $(H;\langle .,.\rangle)$ and assume that $Sp(A) \subseteq [m,M]$ for some scalars $m<M$. If f and g are continuous on $[m,M]$ and $\gamma := \min_{t\in[m,M]} f(t)$ and $\Gamma := \max_{t\in[m,M]} f(t)$ then*

$$\begin{aligned}&\left|\langle f(A)g(A)y,y\rangle - \langle f(A)y,y\rangle\cdot\langle g(A)x,x\rangle - \frac{\gamma+\Gamma}{2}\left[\langle g(A)y,y\rangle - \langle g(A)x,x\rangle\right]\right| \\ &\quad\le \frac{1}{2}\cdot(\Gamma-\gamma)\left[\|g(A)y\|^2 + \langle g(A)x,x\rangle^2 - 2\langle g(A)x,x\rangle\langle g(A)y,y\rangle\right]^{1/2}\end{aligned} \tag{3.1}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$.

Proof. First of all, observe that, for each $\lambda\in\mathbb{R}$ and $x,y\in H$, $\|x\|=\|y\|=1$ we have the identity

$$\begin{aligned}&\langle (f(A)-\lambda\cdot 1_H)(g(A)-\langle g(A)x,x\rangle\cdot 1_H)y,y\rangle \\ &\quad= \langle f(A)g(A)y,y\rangle - \lambda\cdot[\langle g(A)y,y\rangle - \langle g(A)x,x\rangle] \\ &\quad- \langle g(A)x,x\rangle\langle f(A)y,y\rangle.\end{aligned} \tag{3.2}$$

Taking the modulus in (3.2) we have

$$\begin{aligned}&|\langle f(A)g(A)y,y\rangle - \lambda\cdot[\langle g(A)y,y\rangle - \langle g(A)x,x\rangle] \\ &\quad- \langle g(A)x,x\rangle\langle f(A)y,y\rangle| \\ &\quad= |\langle (g(A)-\langle g(A)x,x\rangle\cdot 1_H)y,(f(A)-\lambda\cdot 1_H)y\rangle| \\ &\quad\le \|g(A)y-\langle g(A)x,x\rangle y\|\,\|f(A)y-\lambda y\| \\ &\quad= \left[\|g(A)y\|^2 + \langle g(A)x,x\rangle^2 - 2\langle g(A)x,x\rangle\langle g(A)y,y\rangle\right]^{1/2} \\ &\quad\times\|f(A)y-\lambda y\| \\ &\quad\le \left[\|g(A)y\|^2 + \langle g(A)x,x\rangle^2 - 2\langle g(A)x,x\rangle\langle g(A)y,y\rangle\right]^{1/2} \\ &\quad\times\|f(A)-\lambda\cdot 1_H\|\end{aligned} \tag{3.3}$$

for any $x,y\in H$, $\|x\|=\|y\|=1$.

Now, since $\gamma = \min_{t\in[m,M]} f(t)$ and $\Gamma = \max_{t\in[m,M]} f(t)$, then by the property (P) we have that $\gamma \le \langle f(A)y,y\rangle \le \Gamma$ for each $y\in H$ with $\|y\|=1$ which is clearly equivalent with

$$\left|\langle f(A)y,y\rangle - \frac{\gamma+\Gamma}{2}\|y\|^2\right| \le \frac{1}{2}(\Gamma-\gamma)$$

or with

$$\left|\left\langle\left(f(A)-\frac{\gamma+\Gamma}{2}1_H\right)y,y\right\rangle\right| \le \frac{1}{2}(\Gamma-\gamma)$$

for each $y\in H$ with $\|y\|=1$.

Taking the supremum in this inequality we get

$$\left\|f(A)-\frac{\gamma+\Gamma}{2}\cdot 1_H\right\| \le \frac{1}{2}(\Gamma-\gamma),$$

which together with the inequality (3.3) applied for $\lambda = \frac{\gamma+\Gamma}{2}$ produces the desired result (3.1). ∎

As a particular case of interest we can derive from the above theorem the following result of Grüss' type that improves (2.4):

Corollary 3.1 *With the assumptions in Theorem* 3.1 *we have*

$$|\langle f(A)\, g(A)\, x, x\rangle - \langle f(A)\, x, x\rangle \cdot \langle g(A)\, x, x\rangle| \\ \leq \frac{1}{2}\cdot(\Gamma-\gamma)\left[\|g(A)\,x\|^2 - \langle g(A)\,x,x\rangle^2\right]^{1/2}\left(\leq \frac{1}{4}(\Gamma-\gamma)(\Delta-\delta)\right) \tag{3.4}$$

for each $x \in H$ *with* $\|x\| = 1$, *where* $\delta := \min_{t\in[m,M]} g(t)$ *and* $\Delta := \max_{t\in[m,M]} g(t)$.

Proof. The first inequality follows from (3.1) by putting $y = x$.

Now, if we write the first inequality in (3.4) for $f = g$ we get

$$0 \leq \|g(A)\,x\|^2 - \langle g(A)\,x,x\rangle^2 = \langle g^2(A)\,x,x\rangle - \langle g(A)\,x,x\rangle^2 \\ \leq \frac{1}{2}(\Delta-\delta)\left[\|g(A)\,x\|^2 - \langle g(A)\,x,x\rangle^2\right]^{1/2}$$

which implies that

$$\left[\|g(A)\,x\|^2 - \langle g(A)\,x,x\rangle^2\right]^{1/2} \leq \frac{1}{2}(\Delta-\delta)$$

for each $x \in H$ with $\|x\| = 1$.

This together with the first part of (3.4) proves the desired bound. ∎

The following particular cases that hold for power function are of interest:

Example 3.1 Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some scalars $m < M$.

If A is positive ($m \geq 0$) and $p, q > 0$, then

$$(0 \leq) \left\langle A^{p+q}x, x\right\rangle - \langle A^p x, x\rangle \cdot \langle A^q x, x\rangle \\ \leq \frac{1}{2}\cdot(M^p - m^p)\left[\|A^q x\|^2 - \langle A^q x, x\rangle^2\right]^{1/2}\left[\leq \frac{1}{4}\cdot(M^p-m^p)(M^q-m^q)\right] \tag{3.5}$$

for each $x \in H$ with $\|x\| = 1$.

If A is positive definite ($m > 0$) and $p, q < 0$, then

$$(0 \leq) \left\langle A^{p+q}x, x\right\rangle - \langle A^p x, x\rangle \cdot \langle A^q x, x\rangle \\ \leq \frac{1}{2}\cdot\frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}}\left[\|A^q x\|^2 - \langle A^q x, x\rangle^2\right]^{1/2}\left[\leq \frac{1}{4}\cdot\frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}}\frac{M^{-q}-m^{-q}}{M^{-q}m^{-q}}\right] \tag{3.6}$$

for each $x \in H$ with $\|x\| = 1$.

If A is positive definite ($m > 0$) and $p < 0$, $q > 0$ then

$$(3.7)\quad (0 \leq) \langle A^p x, x\rangle \cdot \langle A^q x, x\rangle - \left\langle A^{p+q} x, x\right\rangle \\ \leq \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left[\|A^q x\|^2 - \langle A^q x, x\rangle^2\right]^{1/2} \left[\leq \frac{1}{4} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} (M^q - m^q)\right]$$

for each $x \in H$ with $\|x\| = 1$.

If A is positive definite ($m > 0$) and $p > 0$, $q < 0$ then

$$(3.8)\quad (0 \leq) \langle A^p x, x\rangle \cdot \langle A^q x, x\rangle - \left\langle A^{p+q} x, x\right\rangle \\ \leq \frac{1}{2} \cdot (M^p - m^p) \left[\|A^q x\|^2 - \langle A^q x, x\rangle^2\right]^{1/2} \left[\leq \frac{1}{4} \cdot (M^p - m^p) \frac{M^{-q} - m^{-q}}{M^{-q} m^{-q}}\right]$$

for each $x \in H$ with $\|x\| = 1$.

We notice that the positivity of the quantities in the left hand side of the above inequalities (3.5)-(3.8) follows from the Theorem 2.2.

The following particular cases when one function is a power while the second is the logarithm are of interest as well:

Example 3.2 Let A be a positive definite operator with $Sp(A) \subseteq [m, M]$ for some scalars $0 < m < M$.

If $p > 0$ then

$$(3.9)\quad (0 \leq) \langle A^p \ln A x, x\rangle - \langle A^p x, x\rangle \cdot \langle \ln A x, x\rangle \\ \leq \begin{cases} \frac{1}{2} \cdot (M^p - m^p) \left[\|\ln A x\|^2 - \langle \ln A x, x\rangle^2\right]^{1/2} \\ \\ \ln\sqrt{\frac{M}{m}} \cdot \left[\|A^p x\|^2 - \langle A^p x, x\rangle^2\right]^{1/2} \end{cases} \left[\leq \frac{1}{2} \cdot (M^p - m^p) \ln\sqrt{\frac{M}{m}}\right]$$

for each $x \in H$ with $\|x\| = 1$.

If $p < 0$ then

$$(3.10)\quad (0 \leq) \langle A^p x, x\rangle \cdot \langle \ln A x, x\rangle - \langle A^p \ln A x, x\rangle \\ \leq \begin{cases} \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left[\|\ln A x\|^2 - \langle \ln A x, x\rangle^2\right]^{1/2} \\ \\ \ln\sqrt{\frac{M}{m}} \cdot \left[\|A^p x\|^2 - \langle A^p x, x\rangle^2\right]^{1/2} \end{cases} \left[\leq \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \ln\sqrt{\frac{M}{m}}\right]$$

for each $x \in H$ with $\|x\| = 1$.

4. An inequality of Grüss' type for n operators

The following multiple operator version of Theorem 3.1 holds:

Theorem 4.1 *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m,M]$ for $j \in \{1,...,n\}$ and for some scalars $m < M$. If $f,g : [m,M] \longrightarrow \mathbb{R}$ are continuous and $\gamma := \min_{t\in[m,M]} f(t)$ and $\Gamma := \max_{t\in[m,M]} f(t)$ then*

$$\begin{aligned}
&\Bigg|\sum_{j=1}^{n} \langle f(A_j)\, g(A_j)\, y_j, y_j\rangle - \sum_{j=1}^{n} \langle f(A_j)\, y_j, y_j\rangle \cdot \sum_{j=1}^{n} \langle g(A_j)\, x_j, x_j\rangle \\
&\qquad - \frac{\gamma+\Gamma}{2}\left[\sum_{j=1}^{n} \langle g(A_j)\, y_j, y_j\rangle - \sum_{j=1}^{n} \langle g(A_j)\, x_j, x_j\rangle\right]\Bigg| \\
&\qquad \le \frac{1}{2}(\Gamma-\gamma)\left[\sum_{j=1}^{n} \|g(A_j)\, y_j\|^2 + \left(\sum_{j=1}^{n} \langle g(A_j)\, x_j, x_j\rangle\right)^2\right. \\
&\qquad \left. - 2\sum_{j=1}^{n} \langle g(A_j)\, x_j, x_j\rangle \sum_{j=1}^{n} \langle g(A_j)\, y_j, y_j\rangle\right]^{1/2}
\end{aligned} \tag{4.1}$$

for each $x_j, y_j \in H, j \in \{1,...,n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = \sum_{j=1}^{n} \|y_j\|^2 = 1.$

Proof. As in [20, p. 6], if we put

$$\widetilde{A} := \begin{pmatrix} A_1 & . & . & . & 0 \\ & . & & & \\ & & . & & \\ & & & . & \\ 0 & . & . & . & A_n \end{pmatrix} \text{ and } \widetilde{x} = \begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix}, \widetilde{y} = \begin{pmatrix} y_1 \\ . \\ . \\ . \\ y_n \end{pmatrix}$$

then we have $Sp\left(\widetilde{A}\right) \subseteq [m,M]$, $\|\widetilde{x}\| = \|\widetilde{y}\| = 1$

$$\left\langle f\left(\widetilde{A}\right) g\left(\widetilde{A}\right) \widetilde{y}, \widetilde{y}\right\rangle = \sum_{j=1}^{n} \langle f(A_j)\, g(A_j)\, y_j, y_j\rangle, \left\langle g\left(\widetilde{A}\right) \widetilde{x}, \widetilde{x}\right\rangle = \sum_{j=1}^{n} \langle g(A_j)\, x_j, x_j\rangle,$$

$$\left\langle f\left(\widetilde{A}\right) \widetilde{y}, \widetilde{y}\right\rangle = \sum_{j=1}^{n} \langle f(A_j)\, y_j, y_j\rangle,\ \left\langle g\left(\widetilde{A}\right) \widetilde{y}, \widetilde{y}\right\rangle = \sum_{j=1}^{n} \langle g(A_j)\, y_j, y_j\rangle,$$

and

$$\left\|g\left(\widetilde{A}\right) \widetilde{y}\right\|^2 = \sum_{j=1}^{n} \|g(A_j)\, y_j\|^2, \left\langle g\left(\widetilde{A}\right) \widetilde{x}, \widetilde{x}\right\rangle^2 = \left(\sum_{j=1}^{n} \langle g(A_j)\, x_j, x_j\rangle\right)^2.$$

Applying Theorem 3.1 for $\widetilde{A}$, $\widetilde{x}$ and $\widetilde{y}$ we deduce the desired result (4.1). ∎

The following particular case provides a refinement of the Mond–Pečarić result from (2.3).

Corollary 4.1 *With the assumptions of Theorem* 4.1 *we have*

$$\begin{aligned}(4.2)\quad &\left|\sum_{j=1}^{n}\langle f(A_j)\,g(A_j)\,x_j,x_j\rangle-\sum_{j=1}^{n}\langle f(A_j)\,x_j,x_j\rangle\cdot\sum_{j=1}^{n}\langle g(A_j)\,x_j,x_j\rangle\right|\\ &\le\frac{1}{2}\cdot(\Gamma-\gamma)\left[\sum_{j=1}^{n}\|g(A_j)\,x_j\|^2-\left(\sum_{j=1}^{n}\langle g(A_j)\,x_j,x_j\rangle\right)^2\right]^{1/2}\left(\le\frac{1}{4}(\Gamma-\gamma)(\Delta-\delta)\right)\end{aligned}$$

for each $x_j\in H, j\in\{1,...,n\}$ *with* $\sum_{j=1}^{n}\|x_j\|^2=1$ *where* $\delta:=\min_{t\in[m,M]}g(t)$ *and* $\Delta:=\max_{t\in[m,M]}g(t)$.

Example 4.1 Let A_j, $j\in\{1,...,n\}$ be a selfadjoint operators with $Sp(A_j)\subseteq[m,M], j\in\{1,...,n\}$ for some scalars $m<M$.

If A_j are positive $(m\ge0)$ and $p,q>0$, then

$$\begin{aligned}(4.3)\quad (0\le)&\sum_{j=1}^{n}\left\langle A_j^{p+q}x_j,x_j\right\rangle-\sum_{j=1}^{n}\left\langle A_j^{p}x_j,x_j\right\rangle\cdot\sum_{j=1}^{n}\left\langle A_j^{q}x_j,x_j\right\rangle\\ &\le\frac{1}{2}\cdot(M^p-m^p)\left[\sum_{j=1}^{n}\left\|A_j^{q}x_j\right\|^2-\left(\sum_{j=1}^{n}\left\langle A_j^{q}x_j,x_j\right\rangle\right)^2\right]^{1/2}\\ &\qquad\left[\le\frac{1}{4}\cdot(M^p-m^p)(M^q-m^q)\right]\end{aligned}$$

for each $x_j\in H, j\in\{1,...,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2=1.$

If A_j are positive definite $(m>0)$ and $p,q<0$, then

$$\begin{aligned}(4.4)\quad (0\le)&\sum_{j=1}^{n}\left\langle A_j^{p+q}x_j,x_j\right\rangle-\sum_{j=1}^{n}\left\langle A_j^{p}x_j,x_j\right\rangle\cdot\sum_{j=1}^{n}\left\langle A_j^{q}x_j,x_j\right\rangle\\ &\le\frac{1}{2}\cdot\frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}}\left[\sum_{j=1}^{n}\left\|A_j^{q}x_j\right\|^2-\left(\sum_{j=1}^{n}\left\langle A_j^{q}x_j,x_j\right\rangle\right)^2\right]^{1/2}\\ &\qquad\left[\le\frac{1}{4}\cdot\frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}}\frac{M^{-q}-m^{-q}}{M^{-q}m^{-q}}\right]\end{aligned}$$

for each $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

If A_j are positive definite $(m > 0)$ and $p < 0$, $q > 0$ then

$$
\begin{aligned}
&(0 \le) \sum_{j=1}^{n} \langle A_j^p x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle A_j^q x_j, x_j \rangle - \sum_{j=1}^{n} \langle A_j^{p+q} x_j, x_j \rangle \\
&\le \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left[\sum_{j=1}^{n} \left\| A_j^q x_j \right\|^2 - \left(\sum_{j=1}^{n} \langle A_j^q x_j, x_j \rangle \right)^2 \right]^{1/2} \\
&\qquad \left[\le \frac{1}{4} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} (M^q - m^q) \right]
\end{aligned}
\tag{4.5}
$$

for each $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

If A_j are positive definite $(m > 0)$ and $p > 0$, $q < 0$ then

$$
\begin{aligned}
&(0 \le) \sum_{j=1}^{n} \langle A_j^p x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle A_j^q x_j, x_j \rangle - \sum_{j=1}^{n} \langle A_j^{p+q} x_j, x_j \rangle \\
&\le \frac{1}{2} \cdot (M^p - m^p) \left[\sum_{j=1}^{n} \left\| A_j^q x_j \right\|^2 - \left(\sum_{j=1}^{n} \langle A_j^q x_j, x_j \rangle \right)^2 \right]^{1/2} \\
&\qquad \left[\le \frac{1}{4} \cdot (M^p - m^p) \frac{M^{-q} - m^{-q}}{M^{-q} m^{-q}} \right]
\end{aligned}
\tag{4.6}
$$

for each $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

We notice that the positivity of the quantities in the left hand side of the above inequalities (4.3)-(4.6) follows from the Theorem 2.3.

The following particular cases when one function is a power while the second is the logarithm are of interest as well:

Example 4.2 Let A_j be positive definite operators with $Sp(A_j) \subseteq [m, M]$, $j \in \{1, ..., n\}$ for some scalars $0 < m < M$.

If $p > 0$ then

$$(4.7)\quad (0\le)\sum_{j=1}^{n}\left\langle A_j^p\ln A_j x_j,x_j\right\rangle-\sum_{j=1}^{n}\left\langle A_j^p x_j,x_j\right\rangle\cdot\sum_{j=1}^{n}\left\langle \ln A_j x_j,x_j\right\rangle$$

$$\le\begin{cases}\dfrac{1}{2}\cdot\left(M^p-m^p\right)\left[\displaystyle\sum_{j=1}^{n}\left\|\ln A_j x_j\right\|^2-\left(\sum_{j=1}^{n}\left\langle \ln A_j x_j,x_j\right\rangle\right)^2\right]^{1/2}\\[2ex] \ln\sqrt{\dfrac{M}{m}}\cdot\left[\displaystyle\sum_{j=1}^{n}\left\|A_j^p x_j\right\|^2-\left(\sum_{j=1}^{n}\left\langle A_j^p x_j,x_j\right\rangle\right)^2\right]^{1/2}\end{cases}$$

$$\left[\le\frac{1}{2}\cdot\left(M^p-m^p\right)\ln\sqrt{\frac{M}{m}}\right]$$

for each $x_j\in H, j\in\{1,...,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2=1$.

If $p<0$ then

$$(4.8)\quad (0\le)\sum_{j=1}^{n}\left\langle A_j^p x_j,x_j\right\rangle\cdot\sum_{j=1}^{n}\left\langle \ln A_j x_j,x_j\right\rangle-\sum_{j=1}^{n}\left\langle A_j^p\ln A_j x_j,x_j\right\rangle$$

$$\le\begin{cases}\dfrac{1}{2}\dfrac{M^{-p}-m^{-p}}{M^{-p}m^{-p}}\left[\displaystyle\sum_{j=1}^{n}\left\|\ln A_j x_j\right\|^2-\left(\sum_{j=1}^{n}\left\langle \ln A_j x_j,x_j\right\rangle\right)^2\right]^{1/2}\\[2ex] \ln\sqrt{\dfrac{M}{m}}\cdot\left[\displaystyle\sum_{j=1}^{n}\left\|A_j^p x_j\right\|^2-\left(\sum_{j=1}^{n}\left\langle A_j^p x_j,x_j\right\rangle\right)^2\right]^{1/2}\end{cases}$$

$$\left[\le\frac{1}{2}\cdot\frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}}\ln\sqrt{\frac{M}{m}}\right]$$

for each $x_j\in H, j\in\{1,...,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2=1$.

5. Another inequality of Grüss' type for n operators

The following different result for n operators can be stated as well:

Theorem 5.1 *Let A_j be selfadjoint operators with $Sp(A_j)\subseteq[m,M]$ for $j\in\{1,...,n\}$ and for some scalars $m<M$. If f and g are continuous on $[m,M]$ and $\gamma:=\min_{t\in[m,M]}f(t)$ and $\Gamma:=\max_{t\in[m,M]}f(t)$ then for any $p_j\ge 0, j\in\{1,...,n\}$ with $\sum_{j=1}^{n}p_j=1$ we have*

$$\begin{aligned}(5.1)\quad & \Bigg| \left\langle \sum_{k=1}^{n} p_k f(A_k) g(A_k) y, y \right\rangle \\ & - \frac{\gamma+\Gamma}{2} \cdot \left[\left\langle \sum_{k=1}^{n} p_k g(A_k) y, y \right\rangle - \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \right] \\ & - \left\langle \sum_{k=1}^{n} p_k f(A_k) y, y \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \Bigg| \\ & \le \frac{\Gamma-\gamma}{2} \left[\sum_{k=1}^{n} p_k \|g(A_k) y\|^2 - 2 \left\langle \sum_{k=1}^{n} p_k g(A_k) y, y \right\rangle \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \right. \\ & \qquad \left. + \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle^2 \right]^{1/2},\end{aligned}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

Proof. Follows from Theorem 4.1 on choosing $x_j = \sqrt{p_j} \cdot x$, $y_j = \sqrt{p_j} \cdot y$, $j \in \{1, ..., n\}$, where $p_j \ge 0, j \in \{1, ..., n\}$, $\sum_{j=1}^{n} p_j = 1$ and $x, y \in H$, with $\|x\| = \|y\| = 1$. The details are omitted. ∎

Remark 5.1 The case $n = 1$ (therefore $p = 1$) in (5.1) provides the result from Theorem 3.1.

As a particular case of interest we can derive from the above theorem the following result of Grüss' type:

Corollary 5.1 *With the assumptions of Theorem* 5.1 *we have*

$$\begin{aligned}(5.2)\quad & \left| \left\langle \sum_{k=1}^{n} p_k f(A_k) g(A_k) x, x \right\rangle - \left\langle \sum_{k=1}^{n} p_k f(A_k) x, x \right\rangle \cdot \left\langle \sum_{k=1}^{n} p_k g(A_k) x, x \right\rangle \right| \\ & \le \frac{\Gamma-\gamma}{2} \left(\sum_{k=1}^{n} p_k \|g(A_k) x\|^2 - \left\langle \sum_{k=1}^{n} p_k g(A_k) x, x \right\rangle^2 \right)^{1/2} \\ & \qquad \left[\le \frac{1}{4} \cdot (\Gamma-\gamma)(\Delta-\delta) \right]\end{aligned}$$

for each $x \in H$ with $\|x\| = 1$, where $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$.

Proof. It is similar with the proof from Corollary 3.1 and the details are omitted. ∎

The following particular cases that hold for power function are of interest:

Example 5.1 Let A_j, $j \in \{1,...,n\}$ be a selfadjoint operators with $Sp(A_j) \subseteq [m,M]$, $j \in \{1,...,n\}$ for some scalars $m < M$ and $p_j \geq 0, j \in \{1,...,n\}$ with $\sum_{j=1}^{n} p_j = 1$.

If A_j, $j \in \{1,...,n\}$ are positive ($m \geq 0$) and $p, q > 0$, then

$$\begin{aligned}(5.3)\quad (0 \leq) &\left\langle \sum_{k=1}^{n} p_k A_k^{p+q} x, x \right\rangle - \left\langle \sum_{k=1}^{n} p_k A_k^{p} x, x \right\rangle \cdot \left\langle \sum_{k=1}^{n} p_k A_k^{q} x, x \right\rangle \\ &\leq \frac{1}{2} \cdot (M^p - m^p) \left[\sum_{k=1}^{n} p_k \left\| A_k^q x \right\|^2 - \left\langle \sum_{k=1}^{n} p_k A_k^{q} x, x \right\rangle^2 \right]^{1/2} \\ &\qquad \left[\leq \frac{1}{4} \cdot (M^p - m^p)(M^q - m^q) \right]\end{aligned}$$

for each $x \in H$ with $\|x\| = 1$.

If A_j, $j \in \{1,...,n\}$ are positive definite ($m > 0$) and $p, q < 0$, then

$$\begin{aligned}(5.4)\quad (0 \leq) &\left\langle \sum_{k=1}^{n} p_k A_k^{p+q} x, x \right\rangle - \left\langle \sum_{k=1}^{n} p_k A_k^{p} x, x \right\rangle \cdot \left\langle \sum_{k=1}^{n} p_k A_k^{q} x, x \right\rangle \\ &\leq \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left[\sum_{k=1}^{n} p_k \left\| A_k^q x \right\|^2 - \left\langle \sum_{k=1}^{n} p_k A_k^{q} x, x \right\rangle^2 \right]^{1/2} \\ &\qquad \left[\leq \frac{1}{4} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \frac{M^{-q} - m^{-q}}{M^{-q} m^{-q}} \right]\end{aligned}$$

for each $x \in H$ with $\|x\| = 1$.

If A_j, $j \in \{1,...,n\}$ are positive definite ($m > 0$) and $p < 0$, $q > 0$ then

$$\begin{aligned}(5.5)\quad (0 \leq) &\left\langle \sum_{k=1}^{n} p_k A_k^{p} x, x \right\rangle \cdot \left\langle \sum_{k=1}^{n} p_k A_k^{q} x, x \right\rangle - \left\langle \sum_{k=1}^{n} p_k A_k^{p+q} x, x \right\rangle \\ &\leq \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left[\sum_{k=1}^{n} p_k \left\| A_k^q x \right\|^2 - \left\langle \sum_{k=1}^{n} p_k A_k^{q} x, x \right\rangle^2 \right]^{1/2} \\ &\qquad \left[\leq \frac{1}{4} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} (M^q - m^q) \right]\end{aligned}$$

for each $x \in H$ with $\|x\| = 1$.

If A_j, $j \in \{1, ..., n\}$ are positive definite $(m > 0)$ and $p > 0$, $q < 0$ then

$$(0 \leq) \left\langle \sum_{k=1}^{n} p_k A_k^p x, x \right\rangle \cdot \left\langle \sum_{k=1}^{n} p_k A_k^q x, x \right\rangle - \left\langle \sum_{k=1}^{n} p_k A_k^{p+q} x, x \right\rangle \tag{5.6}$$
$$\leq \frac{1}{2} \cdot (M^p - m^p) \left[\sum_{k=1}^{n} p_k \left\| A_k^q x \right\|^2 - \left\langle \sum_{k=1}^{n} p_k A_k^q x, x \right\rangle^2 \right]^{1/2}$$
$$\left[\leq \frac{1}{4} \cdot (M^p - m^p) \frac{M^{-q} - m^{-q}}{M^{-q} m^{-q}} \right]$$

for each $x \in H$ with $\|x\| = 1$.

We notice that the positivity of the quantities in the left hand side of the above inequalities (5.3)-(5.6) follows from the Theorem 2.4.

The following particular cases when one function is a power while the second is the logarithm are of interest as well:

Example 5.2 Let A_j, $j \in \{1, ..., n\}$ be positive definite operators with $Sp(A_j) \subseteq [m, M]$, $j \in \{1, ..., n\}$ for some scalars $0 < m < M$ and $p_j \geq 0, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$.

If $p > 0$ then

$$(0 \leq) \left\langle \sum_{k=1}^{n} p_k A_k^p \ln A_k x, x \right\rangle - \left\langle \sum_{k=1}^{n} p_k A_k^p x, x \right\rangle \cdot \left\langle \sum_{k=1}^{n} p_k \ln A_k x, x \right\rangle \tag{5.7}$$
$$\leq \begin{cases} \frac{1}{2} \cdot (M^p - m^p) \cdot \left[\sum_{k=1}^{n} p_k \left\| \ln A_k x \right\|^2 - \left\langle \sum_{k=1}^{n} p_k \ln A_k x, x \right\rangle^2 \right]^{1/2} \\ \\ \ln \sqrt{\frac{M}{m}} \cdot \left[\sum_{k=1}^{n} p_k \left\| A_k^p x \right\|^2 - \left\langle \sum_{k=1}^{n} p_k A_k^p x, x \right\rangle^2 \right]^{1/2} \end{cases}$$
$$\left[\leq \frac{1}{2} \cdot (M^p - m^p) \ln \sqrt{\frac{M}{m}} \right]$$

for each $x \in H$ with $\|x\| = 1$.

If $p < 0$ then

$$(0 \le) \left\langle \sum_{k=1}^{n} p_k A_k^p x, x \right\rangle \cdot \left\langle \sum_{k=1}^{n} p_k \ln A_k x, x \right\rangle - \left\langle \sum_{k=1}^{n} p_k A_k^p \ln A_k x, x \right\rangle \tag{5.8}$$

$$\le \begin{cases} \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left[\sum_{k=1}^{n} p_k \left\| \ln A_k x \right\|^2 - \left\langle \sum_{k=1}^{n} p_k \ln A_k x, x \right\rangle^2 \right]^{1/2} \\ \\ \ln \sqrt{\frac{M}{m}} \cdot \left[\sum_{k=1}^{n} p_k \left\| A_k^p x \right\|^2 - \left\langle \sum_{k=1}^{n} p_k A_k^p x, x \right\rangle^2 \right]^{1/2} \end{cases}$$

$$\left[\le \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \ln \sqrt{\frac{M}{m}} \right]$$

for each $x \in H$ with $\|x\| = 1$.

The following norm inequalities may be stated as well:

Corollary 5.2 *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, ..., n\}$ and for some scalars $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous, then for each $p_j \ge 0, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$ we have the norm inequality:*

$$\left\| \sum_{j=1}^{n} p_j f(A_j) g(A_j) \right\| \le \left\| \sum_{j=1}^{n} p_j f(A_j) \right\| \cdot \left\| \sum_{j=1}^{n} p_j g(A_j) \right\| + \frac{1}{4} (\Gamma - \gamma)(\Delta - \delta), \tag{5.9}$$

where $\gamma := \min_{t \in [m,M]} f(t)$, $\Gamma := \max_{t \in [m,M]} f(t)$, $\delta := \min_{t \in [m,M]} g(t)$ and $\Delta := \max_{t \in [m,M]} g(t)$.

Proof. Utilising the inequality (5.2) we deduce the inequality

$$\left| \left\langle \sum_{k=1}^{n} p_k f(A_k) g(A_k) x, x \right\rangle \right| \le \left| \left\langle \sum_{k=1}^{n} p_k f(A_k) x, x \right\rangle \right| \cdot \left| \left\langle \sum_{k=1}^{n} p_k g(A_k) x, x \right\rangle \right| + \frac{1}{4} (\Gamma - \gamma)(\Delta - \delta)$$

for each $x \in H$ with $\|x\| = 1$. Taking the supremum over $\|x\| = 1$ we deduce the desired inequality (5.9). ∎

Example 5.3 a. Let A_j, $j \in \{1, ..., n\}$ be a selfadjoint operators with $Sp(A_j) \subseteq [m, M], j \in \{1, ..., n\}$ for some scalars $m < M$ and $p_j \ge 0, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$.

If A_j, $j\in\{1,...,n\}$ are positive $(m\geq 0)$ and $p,q>0$, then

$$\left\|\sum_{k=1}^{n}p_kA_k^{p+q}\right\|\leq\left\|\sum_{k=1}^{n}p_kA_k^{p}\right\|\cdot\left\|\sum_{k=1}^{n}p_kA_k^{q}\right\|+\frac{1}{4}\cdot\left(M^p-m^p\right)\left(M^q-m^q\right). \tag{5.10}$$

If A_j, $j\in\{1,...,n\}$ are positive definite $(m>0)$ and $p,q<0$, then

$$\left\|\sum_{k=1}^{n}p_kA_k^{p+q}\right\|\leq\left\|\sum_{k=1}^{n}p_kA_k^{p}\right\|\cdot\left\|\sum_{k=1}^{n}p_kA_k^{q}\right\|+\frac{1}{4}\cdot\frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}}\frac{M^{-q}-m^{-q}}{M^{-q}m^{-q}}. \tag{5.11}$$

b. Let A_j, $j\in\{1,...,n\}$ be positive definite operators with $Sp\left(A_j\right)\subseteq[m,M]$, $j\in\{1,...,n\}$ for some scalars $0<m<M$ and $p_j\geq 0, j\in\{1,...,n\}$ with $\sum_{j=1}^{n}p_j=1$.

If $p>0$ then

$$\left\|\sum_{k=1}^{n}p_kA_k^{p}\ln A_k\right\|\leq\left\|\sum_{k=1}^{n}p_kA_k^{p}\right\|\cdot\left\|\sum_{k=1}^{n}p_k\ln A_k\right\|+\frac{1}{2}\cdot\left(M^p-m^p\right)\ln\sqrt{\frac{M}{m}}. \tag{5.12}$$

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SOME DIVISIBLE MATRIX GROUPS

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Abstract. A group G is said to be *divisible* if for any $x \in G$ and any positive integer n, there exists an element $y \in G$ such that $x = y^n$. For a positive integer $n > 1$, let $G_n(\mathbb{R})$ be the group under multiplication of all invertible $n \times n$ matrices over $\mathbb{R}$. For distinct positive integers $p, q \leq n$, let $G(n, p, q)$ be the subgroup of $G_n(\mathbb{R})$ consisting of all $A \in G_n(\mathbb{R})$ with $A_{ii} > 0$ for all $i \in \{1, \ldots, n\}$ and $A_{ij} = 0$ for distinct i, j such that $(i, j) \neq (p, q)$. Also, let $U(n)[L(n)]$ be the subgroup of $G_n(\mathbb{R})$ consisting of all upper [lower] triangular matrices $A \in G_n(\mathbb{R})$ with $A_{ii} > 0$ for all $i \in \{1, \ldots, n\}$. The purpose of this paper is to show that the matrix groups $G(n, p, q), U(n)$ and $L(n)$ are all divisible.

Keywords: matrix groups, divisible groups.

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1. Introduction

Let $\mathbb{N}$ and $\mathbb{R}$ be respectively the set of all positive integers and the set of all real numbers. The set of all positive real numbers is denoted by $\mathbb{R}^+$.

A semigroup S is called *divisible* if for every $x \in S$ and $n \in \mathbb{N}$, $x = y^n$ for some $y \in S$. Note that the commutative groups $(\mathbb{R}, +)$ and $(\mathbb{R}^+, \cdot)$ are divisible but $(\mathbb{R}\backslash\{0\}, \cdot)$ is not divisible. Certain divisible semigroups have long been studied. For example, see [6], [1], [2], [9] and [8]. In [8], the authors gave characterizations determining when some periodic semigroups are divisible. Divisible commutative groups are characterized in terms of injectivity. This can be seen in [4], pp.195-196.

For $n \in \mathbb{N}$, let $M_n(\mathbb{R})$ be the multiplicative semigroup of all $n \times n$ matrices over $\mathbb{R}$. The entry of $A \in M_n(\mathbb{R})$ in the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column will be denoted by A_{ij}. Let $G_n(\mathbb{R})$ be the unit group of $M_n(\mathbb{R})$, that is, $G_n(\mathbb{R})$ is the multiplicative group of all invertible $n \times n$ matrices over $\mathbb{R}$, or equivalently, $G_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\}$. Clearly, $M_n(\mathbb{R})$ is not a divisible semigroup and $G_n(\mathbb{R})$ is not a divisible group. In [7] and [5], the authors introduced a skew-semifield SK_n which is neither a semifield nor a skew-field as follows : Let $n \in \mathbb{N}$ and $n > 1$. If SK_n is the set of all $n \times n$ matrices of the form

$$\begin{bmatrix} a_1 & 0 & 0 & \dots & b \\ 0 & a_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_n \end{bmatrix}$$

where $a_1, \dots, a_n \in \mathbb{R}^+$ and $b \in \mathbb{R}$, then SK_n with the $n \times n$ zero matrix under the usual addition $+$ and multiplication $\cdot$ of matrices is a skew-semifield which is neither a skew-field nor a semifield. Recall that a semiring $(S, +, \cdot)$ is called a *skew-semifield* if $(S, +)$ is a commutative semigroup with identity 0 and $(S, \cdot)$ is a group with zero 0. A *semifield* is a commutative skew-semifield. Note that SK_n is a noncommutative subgroup of the group $G_n(\mathbb{R})$ and

$$\begin{bmatrix} a_1 & 0 & 0 & \dots & b \\ 0 & a_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_n \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{a_1} & 0 & 0 & \dots & \frac{-b}{a_1 a_n} \\ 0 & \frac{1}{a_2} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{1}{a_n} \end{bmatrix}$$

Since $(\mathbb{R}^+, \cdot)$ is a divisible group and every entry of any $A \in SK_n$ in the diagonal is positive, it is natural to ask whether the group SK_n is divisible. We first generalize the group SK_n as follows : For distinct positive integers $p, q \leq n$, let $G(n, p, q)$ be the set of all $A \in M_n(\mathbb{R})$ with $A_{ii} > 0$ for all $i \in \{1, \dots, n\}$ and $A_{ij} = 0$ for distinct $i, j \in \{1, \dots, n\}$ such that $(i, j) \neq (p, q)$. Then $G(n, 1, n) = SK_n$. It is straightforward to check that $G(n, p, q)$ is a noncommutative semigroup. If $A \in G(n, p, q)$, then $B \in G(n, p, q)$ defined

$$B_{ij} = \begin{cases} \dfrac{1}{A_{ij}} & \text{if } i = j, \\ -\dfrac{A_{pq}}{A_{pp}A_{qq}} & \text{if } i = p \text{ and } j = q, \\ 0 & \text{otherwise} \end{cases}$$

is an inverse of A in $G_n(\mathbb{R})$. Therefore $G(n, p, q)$ is a noncommutative subgroup of $G_n(\mathbb{R})$. The first main interest in this paper is to show that the group $G(n, p, q)$ is always divisible.

It can be seen from [3], page 410 that the set $\{A \in G_n(\mathbb{R}) \mid A \text{ is upper [lower] triangular}\}$ forms a subgroup of $G_n(\mathbb{R})$. This fact and the divisibility of the group $G(n, p, q)$ motivate us to consider the set $U(n)[L(n)]$ of all upper [lower] triangular matrices $A \in G_n(\mathbb{R})$ with $A_{ii} > 0$ for all $i \in \{1, \dots, n\}$. Note that for $A, B \in U(n)[L(n)]$, $\det A = A_{11}A_{22} \dots A_{nn} > 0$, $(AB)_{ii} = A_{ii}B_{ii} > 0$ and $(A^{-1})_{ii} = \frac{1}{A_{ii}} > 0$ for all $i \in \{1, \dots, n\}$. Therefore we deduce that both $U(n)$ and $L(n)$ are subgroups of $G_n(\mathbb{R})$. Also, $U(n)$ and $L(n)$ are noncommutative.

Moreover, $G(n,p,q) \subseteq U(n)$ if $p < q \leq n$ and $G(n,p,q) \subseteq L(n)$ if $q < p \leq n$. Our second purpose is to show that the groups $U(n)$ and $L(n)$ are also divisible.

In the remainder of this paper, let $n \in \mathbb{N}$ with $n > 1$, $G(n,p,q)$ where $p \neq q$ and $p, q \leq n$, $U(n)$ and $L(n)$ be the matrix groups over $\mathbb{R}$ mentioned above. Recall that if A is an element of $G(n,p,q)$, $U(n)$ or $L(n)$, then

$$(A^m)_{ii} = A_{ii}^m \text{ for all } m \in \mathbb{N} \text{ and } i \in \{1, \ldots, n\}.$$

2. The matrix group $G(n,p,q)$

To show that $G(n,p,q)$ is a divisible group, the following lemma is a main tool.

Lemma 2.1. *If $A \in G(n,p,q)$ and $m \in \mathbb{N}$, then*

$$(A^m)_{pq} = \Big(\sum_{i=0}^{m-1} A_{pp}^i A_{qq}^{m-1-i}\Big) A_{pq}.$$

Proof. First, assume that $p < q$. Since $A_{ij} = 0$ for all $i, j \in \{1, ..., n\}$ with $i > j$, we have

$$(A^m)_{pq} = \sum_{p=t_1 \leq t_2 \leq \ldots \leq t_{m+1}=q} A_{t_1t_2} A_{t_2t_3} \ldots A_{t_mt_{m+1}}.$$

By the property of A, $A_{pi} = A_{iq} = 0$ for all $i \in \{1, ...s, n\}$ with $p < i < q$. It follows that

$$\begin{aligned}(A^m)_{pq} &= \sum_{\substack{p=t_1 \leq t_2 \leq \ldots \leq t_{m+1}=q \\ t_r \in \{p,q\} \text{ for all } r}} A_{t_1t_2} A_{t_2t_3} \ldots A_{t_mt_{m+1}} \\ &= \sum_{k=0}^{m-1} (A_{pp}^k A_{pq} A_{qq}^{m-1-k}) = \Big(\sum_{k=0}^{m-1} A_{pp}^k A_{qq}^{m-1-k}\Big) A_{pq}.\end{aligned}$$

It can be proved analogously for the case that $p > q$. ■

Theorem 2.2. *The group $G(n,p,q)$ is divisible.*

Proof. Let $A \in G(n,p,q)$ and $m \in \mathbb{N}$. Recall that $A_{ii} > 0$ for every $i \in \{1, ..., n\}$. Define $B \in G(n,p,q)$ by

$$\begin{aligned} B_{ii} &= A_{ii}^{\frac{1}{m}} \text{ for all } i \in \{1, \ldots, n\}, \\ B_{pq} &= \frac{A_{pq}}{\displaystyle\sum_{k=0}^{m-1} B_{pp}^k B_{qq}^{m-1-k}} \end{aligned}$$

Then for every $i \in \{1, \ldots, n\}$, $(B^m)_{ii} = B_{ii}^m = (A_{ii}^{\frac{1}{m}})^m = A_{ii}$. From Lemma 2.1 and the definition of B_{pq}, we have

$$(B^m)_{pq} = \Big(\sum_{k=0}^{m-1} B_{pp}^k B_{qq}^{m-1-k}\Big) B_{pq} = A_{pq}.$$

This shows that $A = B^m$, as desired. ■

3. The matrix groups $U(n)$ and $L(n)$

The following lemma is needed to show that the matrix group $U(n)$ is divisible.

Lemma 3.1. *Let $A \in U(n)$ and $m \in \mathbb{N}$. Then the following statements hold.*

(i) *For $i \in \{1, ..., n-1\}$,*

$$(A^m)_{i,i+1} = \Big(\sum_{k=0}^{m-1} A_{ii}^k A_{i+1,i+1}^{m-1-k}\Big) A_{i,i+1}.$$

(ii) *If $n > 2$ and $2 \leq l < n$, then, for $i \in \{1, \ldots, n-l\}$,*

$$(A^m)_{i,i+l} = \Big(\sum_{k=0}^{m-1} A_{ii}^k A_{i+l,i+l}^{m-1-k}\Big) A_{i,i+l} + \sum_{\substack{i=t_1 \leq t_2 \leq \ldots \leq t_{m+1} = i+l \\ t_{r+1}-t_r < l \text{ for all } r}} A_{t_1 t_2} A_{t_2 t_3} \ldots A_{t_m t_{m+1}}.$$

Proof. (i) Since A is upper triangular, we have that for $i \in \{1, \ldots, n-1\}$,

$$\begin{aligned}(A^m)_{i,i+1} &= \sum_{i=t_1 \leq t_2 \leq \ldots \leq t_{m+1} = i+1} A_{t_1 t_2} A_{t_2 t_3} \ldots A_{t_m t_{m+1}} \\ &= \sum_{k=0}^{m-1} \big(A_{ii}^k A_{i,i+1} A_{i+1,i+1}^{m-1-k}\big) = \Big(\sum_{k=0}^{m-1} A_{ii}^k A_{i+1,i+1}^{m-1-k}\Big) A_{i,i+1}\end{aligned}$$

(ii) Assume that $n > 2$ and $2 \leq l < n$. Then for $i \in \{1, \ldots, n-l\}$,

$$\begin{aligned}(A^m)_{i,i+l} &= \sum_{i=t_1 \leq t_2 \leq \ldots \leq t_{m+1} = i+l} A_{t_1 t_2} A_{t_2 t_3} \ldots A_{t_m t_{m+1}} \\ &= \sum_{\substack{i=t_1 \leq t_2 \leq \ldots \leq t_{m+1} = i+l \\ t_{r+1}-t_r = l \text{ for some } r}} A_{t_1 t_2} A_{t_2 t_3} \ldots A_{t_m t_{m+1}} \\ &\quad + \sum_{\substack{i=t_1 \leq t_2 \leq \ldots \leq t_{m+1} = i+l \\ t_{r+1}-t_r < l \text{ for all } r}} A_{t_1 t_2} A_{t_2 t_3} \ldots A_{t_m t_{m+1}} \\ &= \sum_{k=0}^{m-1} A_{ii}^k A_{i,i+l} A_{i+l,i+l}^{m-1-k} + \sum_{\substack{i=t_1 \leq t_2 \leq \ldots \leq t_{m+1} = i+l \\ t_{r+1}-t_r < l \text{ for all } r}} A_{t_1 t_2} A_{t_2 t_3} \ldots A_{t_m t_{m+1}} \\ &= \Big(\sum_{k=0}^{m-1} A_{ii}^k A_{i+l,i+l}^{m-1-k}\Big) A_{i,i+l} + \sum_{\substack{i=t_1 \leq t_2 \leq \ldots \leq t_{m+1} = i+l \\ t_{r+1}-t_r < l \text{ for all } r}} A_{t_1 t_2} A_{t_2 t_3} \ldots A_{t_m t_{m+1}}.\end{aligned}$$ ■

Theorem 3.2. *The matrix group $U(n)$ is divisible.*

Proof. Let $A \in U(n)$ and $m \in \mathbb{N}$. Define $B \in U(n)$ by defining B_{ij} with $i \leq j$ recursively on entries of the lines parallel to the diagonal as follows : Let

$$B_{ii} = A_{ii}^{\frac{1}{m}} \quad \text{for all } i \in \{1, \dots, n\}$$

and let

$$B_{i,i+1} = \frac{A_{i,i+1}}{\sum\limits_{k=0}^{m-1} B_{ii}^k B_{i+1,i+1}^{m-1-k}} \quad \text{for all } i \in \{1, \dots, n-1\}.$$

If $i \in \{1, \dots, n-2\}$, let

$$B_{i,i+2} = \frac{1}{\sum\limits_{k=0}^{m-1} B_{ii}^k B_{i+2,i+2}^{m-1-k}} \Big(A_{i,i+2} - \sum_{\substack{i=t_1 \leq t_2 \leq \dots \leq t_{m+1}=i+2 \\ t_{r+1}-t_r<2 \text{ for all } r}} B_{t_1t_2} B_{t_2t_3} \dots B_{t_mt_{m+1}} \Big).$$

Let $2 < l < n$ and assume that B_{ij} is defined for all $i, j \in \{1, \dots, n\}$ with $i \leq j$ and $j - i < l$. If $i \in \{1, \dots, n-l\}$, define $B_{i,i+l}$ by

$$B_{i,i+l} = \frac{1}{\sum\limits_{k=0}^{m-1} B_{ii}^k B_{i+l,i+l}^{m-1-k}} \Big(A_{i,i+l} - \sum_{\substack{i=t_1 \leq t_2 \leq \dots \leq t_{m+1}=i+l \\ t_{r+1}-t_r<l \text{ for all } r}} B_{t_1t_2} B_{t_2t_3} \dots B_{t_mt_{m+1}} \Big).$$

We claim that $B^m = A$. If $i \in \{1, \dots, n\}$, then $(B^m)_{ii} = B_{ii}^m = (A_{ii}^{\frac{1}{m}})^m = A_{ii}$. Let $i, j \in \{1, \dots, n\}$ be such that $i < j$ and let $l = j - i$.

Case 1 : $l = 1$. It follows from Lemma 3.1(i) that

$$(B^m)_{ij} = (B^m)_{i,i+1} = \Big(\sum_{k=0}^{m-1} B_{ii}^k B_{i+1,i+1}^{m-1-k} \Big) B_{i,i+1},$$

so by the definition of $B_{i,i+1}$, $(B^m)_{ij} = (B^m)_{i,i+1} = A_{i,i+1} = A_{ij}$.

Case 2 : $l > 1$. By Lemma 3.1(ii) and the definition of $B_{i,i+l}$, we have

$$\begin{aligned}(B^m)_{ij} = (B^m)_{i,i+l} = {} & \Big(\sum_{k=0}^{m-1} B_{ii}^k B_{i+l,i+l}^{m-1-k} \Big) B_{i,i+l} \\ & + \sum_{\substack{i=t_1 \leq t_2 \leq \dots \leq t_{m+1}=i+l \\ t_{r+1}-t_r<l \text{ for all } r}} B_{t_1t_2} B_{t_2t_3} \dots B_{t_mt_{m+1}}.\end{aligned}$$

The last equality and the definition of $B_{i,i+l}$ yield the equality $(B^m)_{i,i+l} = A_{i,i+l}$. Thus $(B^m)_{ij} = A_{ij}$.

Therefore $B^m = A$ and hence the proof of the theorem is complete. ■

For a matrix A, let A^t be the transpose of A. Then

$$L(n) = \{A^t \mid A \in U(n)\},$$

and hence the following result is obtained obviously from Theorem 3.2 and the facts that $(A^t)^t = A$ and $(A^t)^m = (A^m)^t$ for all $A \in M_n(F)$ and $m \in \mathbb{N}$.

Corollary 3.3. *The matrix group $L(n)$ is divisible.*

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FUZZY SUBNEXUSES

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Abstract. The notion of fuzzy prime subnexus of a nexus is defined, and some related results are obtained. In particular, by considering the concept of homomorphism, some theorems about the coimage and preimage are proved. Finally, the notion of Quotient nexus induced by a fuzzy subnexus is introduced and some its properties are investigated.

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1. Introduction and preliminaries

The space Structure Research Center of the University of Surrey was founded by Z.S. Makoswski as part of Department of Civil Engineering in 1963.

The aim of the Center is to carry out research into the design and analysis of space structures. Space structures include structural forms such as single and double layer girds, barrel vaults, shells and various forms of tension structures.

The basic idea of a nexus has been further developed as a mathematical object for general use. The aim of recent study has been to evolve a mathematical object that allows complex processes on groups of mathematical objects to be formulated with ease of elegance. This notion is very useful for study of space structure.

In this paper the notion of a fuzzy subnexus of a nexus is defined and some related results are obtained.

Definition 1.1. [1]

(i) An address is a sequence of $\aleph^* = \aleph \cup \{0\}$ such that $a_k = 0$ implies that $a_i = 0$ for all $i \geq k$. The sequence of zero is called the empty address and is denoted by (). In other word, every nonempty address is of the form

$$(a_1, a_2, ..., a_n, 0, 0, ...),$$

where n, belongs to $\aleph$. Hereafter, this address will be denoted by

$$(a_1, a_2, ..., a_n).$$

(ii) A nexus N is the set of address with the following properties:

$$\begin{aligned} (a_1, a_2, ..., a_n) \in N &\implies (a_1, a_2, ..., a_{n-1}, t) \in N,\ \forall 0 \leq t \leq a_n. \\ \{a_i\}_{i=1}^{\infty} \in N, a_i \in \aleph &\implies \forall n \in \aleph, (a_1, a_2, ..., a_n) \in N. \end{aligned}$$

For example the set:

$$N = \{(), (1), (2), (1,1), (1,2), (2,1), (2,2), (2,3)\}$$

is a nexus.

Remark 1.1. Condition (ii) in Definition 1.1 implies that

$$(a_1, a_2, ..., a_n) \in N \Rightarrow (a_1, a_2, ..., a_{n-1}) \in N,\ \ \forall\, n \geq 2.$$

Definition 1.2. [1] Let $w \in N$. The level of w is said to be:

(i) n, if $w = (a_1, a_2, ..., a_n)$, for some $a_n \in \aleph$,

(ii) ∞, if w is an infinite sequence of $\aleph$,

(iii) 0, if $w = ()$.

The level of w is denoted by $l(w)$.

Definition 1.3. [1] Let $w = \{a_i\},\ i \in \aleph$ and $v = \{b_i\}\ i \in \aleph$ be addresses. Then $w \leq v$ if $l(w) = 0$ or one of the following cases are satisfied:

Case 1. If $l(w) = 1$, that is $w = (a_1)$, for all $a_1 \in \aleph$ and $a_1 \leq b_1$.

Case 2. If $1 < l(w) < \infty$, then $l(w) \leq l(v)$ and $a_{l(w)} \leq b_{l(v)}$ and for any $1 \leq i \leq l(w)$, $a_i = b_i$.

Case 3. If $l(w) = \infty$, then $w = v$. For example, consider the nexus:

$$N = \{(), (1), (2), (1,1), (1,2), (1,3), (1,3,1), (1,3,2)\}.$$

Therefore

$$(1) \leq (2), (1,2) \leq (1,3,1), (1,3,1) \leq (1,3,2).$$

Proposition 1.1. [1] $(N, \leq)$ *is a lower semi-lattice.*

Proof. Clearly, $\leq$ is reflexive. Let $w, v \in N, w \leq v$ and $v \leq w$. Now one can show that $\leq$ is anti-symmetric. For this consider three cases:

Case 1. If $l(w) = 0$, then $l(v) \leq l(w) = 0$, hence $v = () = w$.

Case 2. If $l(w) = 1$, then $1 \leq l(v) \leq l(w)$ and hence, $l(v) = 1$. suppose that $v = (a)$ and $w = (b)$, for some $a, b \in \aleph$. Thus, $a \leq b$ and $b \leq a$. Therefore $a = b$ and $v = w$.

Case 3. Let $n = l(w) > 1$. That is, $w = (a_1, a_2, ..., a_n)$. Then $l(v) = n$. Assume that $v = (b_1, b_2, ..., b_n)$. Therefore, $a_n \leq b_n$, $b_n \leq a_n$ and $a_i = b_i$, for all $1 \leq i \leq n$. Hence $a_j = b_j$ for all $1 \leq j \leq n$. In other words, $w = v$.

Clearly, if $l(w) = \infty$, then $w = v$.

Now, the transitive property of $\leq$ must be proved. To this aim, let $v, w, t \in N$, $w \leq v$ and $v \leq t$.

If $l(t) = 0$, then $l(v) = l(w) = 0$ and hence $w = t = ()$.

If $l(t) = 1$, then $l(v) = 0$ or $l(v) = 1$. Firstly, assume that $l(v) = 0$. It is easy to see that $l(w) = 0$, therefore $w = () \leq t$.

Now suppose that $l(v) = 1$ and $l(w) \neq 0$. Then $w = (a), v = (b)$, and $t = (c)$ for some $a, b, c \in \aleph$, and $a \leq b \leq c$. Therefore, $w \leq t$. At last, suppose that $1 < l(t) = k < \infty$ and $t = (c_1, c_2, ..., c_k)$ for some $c_i \in \aleph$. Then, $v = (b_1, b_2, ..., b_m)$ and $w = (a_1, a_2, ..., a_n)$ where that $b_i, a_i \in \aleph, n \leq m \leq k, a_n \leq b_n, b_m \leq c_m, a_i = b_i$ and $b_j = c_j$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Therefore, $a_n \leq c_n$ and $a_r \leq c_r$ for all $1 \leq r \leq n$. Consequently, $w \leq t$. It is easy to see that, if $l(t) = \infty$, then $t = v = w$.

Now, for the last part of the proof of the theorem, it must be shown that the greatest lower bounded of v and w that $v \wedge w$ belongs to N. Suppose that $v = \{a_i\}_{i \in \aleph}$ and $w = \{b_i\}_{i \in \aleph}$ are two addresses. Now, consider three cases:

Case 1. If v or w is empty address, then $v \wedge w = ()$.

Case 2. If v and w are not empty addresses and $a_1 \neq b_1$, then $v \wedge w = (a_1 \wedge b_1)$.

Case 3. Assume that v and w are not empty addresses and $a_1 = b_1$. In this case, suppose that n is the greatest element of $\aleph$ such that $a_n = b_n$. Put $s = (a_1, a_2, ..., a_n, a_{n+1} \wedge b_{n+1})$. It is easy to see that $v \wedge w = s$ and s belongs to N.

Proposition 1.2. [1] *Suppose that N is a set of addresses. Then N is a nexus if and only if, $v \in N$ and $w \leq v$ implies that,$w \in N$.*

Definition 1.4. [1] A nonempty subset S of N is called a subnexus of N provided that S itself is a nexus. The set of all subnexus of N is denoted by $SUB(N)$.

Definition 1.5. [1] Let $\emptyset \neq X \subseteq N$. Then the smallest subnexus of N containing X is called the subnexus of N generated by X and denoted $\langle X\rangle$. If $\mid X \mid= 1$, then $\langle X\rangle$ is called a cyclic subnexus of N.

Definition 1.6. [5] A proper subnexus P of a nexus N is said to be a prime subnexus of N if $a \wedge b \in P$ implies that $a \in P$ or $b \in P$ for any $a, b \in N$.

Definition 1.7. [7] A fuzzy subset of a set S is a function μ from S into $[0, 1]$.

2. Fuzzy subnexus

Definition 2.1. Let $\widetilde{P}$ be fuzzy subset of nexus N. Then $\widetilde{P}$ is called a fuzzy subnexus of N, if $w \leq v$ implies that $\widetilde{P}(v) \leq \widetilde{P}(w)$, for all $v, w \in N$.

The set of all fuzzy subnexus of N is denoted by $FSUB(N)$.

Remark 2.1. If $\widetilde{P} \in FSUB(N)$, then $\widetilde{P}(0) \geq \widetilde{P}(v)$, for all $v \in N$.

Example 2.1. Consider the nexus $N = \{(), (1), (2), (2, 1), (2, 2)\}$ define the fuzzy subset $\widetilde{P}$ of N as follows,

$$\widetilde{P}(()) = \alpha_1, \widetilde{P}((1)) = \alpha_2, \widetilde{P}((2, 1)) = \alpha_4, \widetilde{P}((2, 2)) = \alpha_5,$$

where that $\alpha_5 \leq \alpha_4 \leq \alpha_3 \leq \alpha_2 \leq \alpha_1$. Therefore, $\widetilde{P} \in FSUB(N)$.

Example 2.2. Let N be an arbitrary nexus. If $0 \neq w = (a_1, a_2, ..., a_n) \in N$, then $a_1a_2...a_n$ we mean a number with n, digits. Define the fuzzy subset $\widetilde{P}$ of N as follows if $w = (a_1, a_2, ..., a_n) \in N$ is of level n, then

$$\widetilde{P}(w) = \frac{1}{a_1a_2...a_n}.$$

Moreover,

$$\widetilde{P}(v) = \begin{cases} 1, & v = 0, \\ 0, & l(v) = \infty. \end{cases}$$

Clearly, $\widetilde{P} \in FSUB(N)$.

Definition 2.2. Let A is a subset of N. A function $\chi_A : N \longrightarrow [0, 1]$ is called a characteristic function of A if

$$\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin a. \end{cases}$$

Theorem 2.1. *Let A be a nonempty subset of nexus N, $A \in N$, if and only if*

$$\chi_A \in FSUB(N),$$

where that χ_A is the characteristic function of A.

Proof. Let A be a subnexus of N and $v \leq w$, for some $v, w \in N$. If $w \in A$, then $v \in A$. Hence $\chi_A(w) = \chi_A(v)$ and if $w \notin A$, then $\chi_A(w) \leq \chi_A(v)$.

Conversely, suppose $w \in N$, $v \in A$ and $w \leq v$. Then, $\chi_A(v) \leq \chi_A(w)$. Since $\chi_A(v) = 1$, we have $\chi_A(w) = 1$ and $w \in A$. Hence, $A \in SUB(N)$.

Definition 2.3. [3] Let $\widetilde{P}$ be a fuzzy subset of a set S.

For $t \in [0,1]$, the set $\widetilde{P}_t = \{s \in S \mid \widetilde{P}(s) \geq t\}$ is called a level subset of $\widetilde{P}$.

Theorem 2.2. *$\widetilde{P} \in FSUB(N)$ if and only if $\widetilde{P}_t \in SUB(N)$, for all $t \in [0,1]$, where $\widetilde{P}_t \neq \emptyset$.*

Proof. Suppose $\widetilde{P} \in SUB(N)$ and $\widetilde{P}_t \neq \emptyset$, for $t \in [0,1]$ and let $v \in N, w \in \widetilde{P}_t$ such that, $v \leq w$. Then, $t \leq \widetilde{P}(w) \leq \widetilde{P}(v)$. Hence, $v \in \widetilde{P}_t$.

Conversely, suppose $v, w \in N$ and $v \leq w$ and suppose $\widetilde{P}(w) = t$, for $t \in [0,1]$.

Since $\widetilde{P}_t \in SUB(N)$, then $v \in \widetilde{P}_t$ i.e $\widetilde{P}(v) \geq \widetilde{P}(w) = t$.

Theorem 2.3. *Let N be a nexus and $\mathcal{A} = \{\widetilde{P}_\alpha \mid \widetilde{P}_\alpha \in FSUB(N)\}$. Then*

(i) $\bigcap\limits_{\alpha \in I} \widetilde{P}_\alpha \in FSUB(N)$.

(ii) $\bigcup\limits_{\alpha \in I} \widetilde{P}_\alpha \in FSUB(N)$.

Proof. Suppose $v, w \in N$ and $w \leq v$. Now we have:

$$\left(\bigcap_{\alpha \in I} \widetilde{P}_\alpha\right)(v) = \inf_{\alpha \in I}(\widetilde{P}_\alpha(v)) \leq \inf_{\alpha \in I}(\widetilde{P}_\alpha(w)) = \left(\bigcap_{\alpha \in I} \widetilde{P}_\alpha(v)\right).$$

And, similarly,

$$\left(\bigcup_{\alpha \in I} \widetilde{P}_\alpha\right)(v) = \sup_{\alpha \in I}(\widetilde{P}_\alpha(v)) \leq \sup_{\alpha \in I}(\widetilde{P}_\alpha(w)) = \left(\bigcup_{\alpha \in I} \widetilde{P}_\alpha\right)(w).$$

3. Prime fuzzy subnexus

Definition 3.1. Let N be a nontrivial nexus, (i.e., $N \neq \{()\}$). A fuzzy subnexus $\widetilde{P}$ of N is called a *prime fuzzy subnexus* if

$$\widetilde{P}(a \wedge b) \leq \max\{\widetilde{P}(a), \widetilde{P}(b)\},$$

for all $a, b \in N$.

The set of all prime fuzzy subnexus of N is denoted by $FPSUB(N)$.

Remark 3.1. If N is a nontrivial nexus and $\widetilde{P}$ a prime fuzzy subnexus of N, then

$$\widetilde{P}(a \wedge b) = \widetilde{P}(a) \text{ or } \widetilde{P}(b).$$

Example 3.1. Let $N = \{(), (1), (2), (2, 1), (2, 2)\}$.

Consider the fuzzy subnexus $\widetilde{P}$ of as follows,

$$\widetilde{P}(()) = \alpha_1, \widetilde{P}(1) = \alpha_2, \widetilde{P}(2) = \alpha_3, \widetilde{P}(2, 1) = \alpha_4, \widetilde{P}(2, 2) = \alpha_4,$$

such that,

$$\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4.$$

The $\widetilde{P}$ is a prime fuzzy subnexus of N.

Example 3.2. Let

N=$\{(), (1), (2), (3), (2,1), (2, 2), (2, 3), (3,1), (3, 2), (3,1,1), (3,1, 2), (3, 2,1), (3, 2, 2)\}$.

Now define the fuzzy subnexus $\widetilde{P}$ of N as follows, $\widetilde{P}(()) = 1$ and, if

$$w = (a_1, a_2, ..., a_n) \in N,$$
$$\widetilde{P}(w) = \frac{1}{a_1 a_2 ... a_n},$$

then $\widetilde{P}$ is a fuzzy subnexus of N, but $\widetilde{P}$ is not prime, because

$$\widetilde{P}((3, 2) \wedge (3, 1, 1)) = \widetilde{P}(3, 1) = \frac{1}{31} > \max\{\widetilde{P}(3, 2), \widetilde{P}(3, 1, 1)\} = \frac{1}{32}.$$

Example 3.3. Let $N = \{(), (1), (2), (1, 1), (1, 2), (1, 3), (1, 3, 1), (1, 3, 2)\}$.

Define

$$\widetilde{P}(()) = \alpha_1, \widetilde{P}(1) = \widetilde{P}(2) = \alpha_2,\ \widetilde{P}(1, 1) = \alpha_3,$$
$$\widetilde{P}(1, 2) = \alpha_4, \widetilde{P}(1, 3) = \alpha_5,\ \widetilde{P}(1, 3, 1) = \alpha_6,\ \widetilde{P}(1, 3, 2) = \alpha_7,$$

such that,

$$\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_7.$$

Then, $\widetilde{P}$ is a prime fuzzy subnexus of N.

Proposition 3.1. *Let N be a cyclic subnexus and $\widetilde{P}$ is a fuzzy subnexus of N. Then, $\widetilde{P} \in FPSUB(N)$.*

Proof. Since N is cyclic, then for all $a, b \in N$, $a \leq b$ or $b \leq a$ without loss of generality suppose $a \leq b$. Then, we have

$$\widetilde{P}(b) \leq \widetilde{P}(a)$$

and it is obvious that

$$\widetilde{P}(a \wedge b) \leq \max\{\widetilde{P}(a), \widetilde{P}(b)\}.$$

Definition 3.2. Let F be a mapping from M onto N, if $\widetilde{P}$ is a fuzzy subset of M. Then, the fuzzy subset $\widetilde{Q}$ of N defined by

$$\widetilde{Q}(y) = \inf_{x \in F^{-1}(y)} \widetilde{P}(x),$$

for all $y \in N$, is called the coimage of M under F.

Similarly, if $\widetilde{Q}$ is a fuzzy subset of N, then the fuzzy subset $\widetilde{P} = \widetilde{Q}F$ of M (i.e., the fuzzy subset defined by

$$\widetilde{P}(x) = \widetilde{Q}(F(x))$$

for all $x \in X$) is called the preimage of v under F.

Definition 3.3. A fuzzy subnexus $\widetilde{P}$ of N has inf-property if, for any subset T of N, there exists $t_0 \in T$ such that

$$\widetilde{P}(t_0) = \inf_{t \in T} \widetilde{P}(t).$$

Example 3.4. In Example 2.2, if N is infinite, then $\widetilde{P}$ has inf-property.

Definition 3.4. Let M, N be two nexus and $F : M \longrightarrow N$ be a function. Then

(i) F is called homomorphism of nexuses if

$$w \leq v \text{ implies } F(w) \leq F(v), \text{ for all } w, v \in M.$$

(ii) F is called a semi-lattice homomorphism of nexuses if

$$F(w \wedge v) = F(w) \wedge F(v), \text{ for all } w, v \in M.$$

Remark 3.2. Clearly every semi-lattice homomorphism of nexuses is a homomorphism of nexuses.

Example 3.5. Let $M = \{(), (1), (2), (1,1)\}$, $N = \{(), (1), (2), (1,1), (2,1)\}$ and $F : M \longrightarrow N$ be a function such that,

$$F(()) = F((1)) = (1),\ F((1,1)) = F((2)) = (2).$$

Then

(i) F is a homomorphism of nexuses.

(ii) F is not a semi-lattice homomorphism of nexuses.

(iii) $F(()) \neq ()$.

Theorem 3.1. *Let $F : M \longrightarrow N$ be an onto homomorphism of nexuses.*

(i) $F(()) = ()$.

(ii) *If $F((v)) = ()$, then $F((w)) = ()$, for all $w \leq v$.*

Proof. It is obvious.

Remark 3.3. Let $F : M \longrightarrow N$ be an onto homomorphism of nexuses and $M \neq ()$. If $F((1)) \neq ()$, then $F((v)) \neq ()$, for all $() \neq v \in M$.

Theorem 3.2. *An onto semi-lattice homomorphism coimage of a one to one prime fuzzy subnexus with inf-property is a prime fuzzy subnexus.*

Proof. Let $F : M \longrightarrow N$ be an onto semi-lattice homomorphism of nexuses and $\widetilde{P}$ a one to one prime fuzzy subnexus of M with inf-property and so $\widetilde{Q}$ the coimage of $\widetilde{P}$ under F and suppose $w', v' \in N$ are arbitrary. Consider the following cases.

Case 1. If $w' \leq v'$ and $w' \neq v'$.

Since F is onto there exists $w \in F^{-1}(w')$, $v \in F^{-1}(v')$ such that,

$$\widetilde{P}(w) = \inf_{t \in F^{-1}(w')} \widetilde{P}(t),$$

$$\widetilde{P}(v) = \inf_{t \in F^{-1}(v')} \widetilde{P}(t).$$

If $\widetilde{P}(w) < \widetilde{P}(v)$, since $\widetilde{P}$ is prime, then $\widetilde{P}(w \wedge v) \leq \widetilde{P}(v)$. Hence, $v = w \wedge v$, i.e., $v \leq w$ implies that $F(v) \leq F(w)$. That is a contradiction.

Now, we have

$$\widetilde{Q}(v') = \inf_{t \in F^{-1}(v')} \widetilde{P}(t) = \widetilde{P}(v) \leq \widetilde{P}(w) = \inf_{t \in F^{-1}(w')} \widetilde{P}(t) = \widetilde{Q}(w').$$

Case 2. If $w' = v'$, then $\widetilde{Q}(v') = \widetilde{Q}(w')$. Therefore, $\widetilde{Q} \in FSUB(N)$.

Finally,

$$\begin{aligned} \widetilde{Q}(v' \wedge w') &= \inf_{t \in F^{-1}(v' \wedge w')} \widetilde{P}(t) \\ &\leq \widetilde{P}(v \wedge w) \\ &\leq \max\{\widetilde{P}(v), \widetilde{P}(w)\} \\ &= \max\left\{\inf_{t \in F^{-1}(v')} \widetilde{P}(t), \inf_{t \in F^{-1}(w')} \widetilde{P}(t)\right\} \\ &= \max\{\widetilde{Q}(v'), \widetilde{Q}(w')\}. \end{aligned}$$

Theorem 3.3. *Any onto homomorphism of nexuses preimage of a fuzzy subnexus is also a fuzzy subnexus.*

Proof. Suppose that $F : M \longrightarrow N$ is a homomorphism of nexuses and $\widetilde{Q} \in FSUB(N)$ and $\widetilde{P}$ the preimage of $\widetilde{Q}$ under F.

Now, for all $v, w \in M$, if $v \leq w$, then $F(v) \leq F(w)$. Hence,

$$\widetilde{Q}(F(w)) \leq \widetilde{Q}(F(v)),$$

i.e., $\widetilde{P}(w) \leq \widetilde{P}(v)$ implies that $\widetilde{P} \in FSUB(N)$.

Theorem 3.4. *Any onto semi-lattice homomorphism of nexuses preimage of a prime fuzzy subnexus is also a prime fuzzy subnexus.*

Proof. Let $F : M \longrightarrow N$ be an onto semi-lattice homomorphism of nexus and $\widetilde{Q} \in FSUB(N)$ and $\widetilde{P}$ the preimage of $\widetilde{Q}$ under F.

Now for all $a, b \in M$, $\widetilde{Q}(F(a) \wedge F(b)) \leq \max\{\widetilde{Q}(F(a)), \widetilde{Q}(F(b))\}$ implies

$$\widetilde{Q}(F(a \wedge b)) \leq \max\{(\widetilde{Q}F)(a), (\widetilde{Q}F)(b)\}.$$

Hence, by the previous theorem, $\widetilde{P} \in FPSUB(M)$.

Proposition 3.2. *Suppose N be a nexus and $\widetilde{P}$ a fuzzy subnexus arbitrary of N.*

(i) *If N is a chain, then $\widetilde{P}$ is a prime fuzzy subnexus.*

(ii) *If $\widetilde{P}$ is a prime fuzzy subnexus and one to one, then N is a chain.*

Proof. (i) Let $v, w \in N$ and $v \leq w$ implies $\widetilde{P}(v) \geq \widetilde{P}(w)$. Then

$$\widetilde{P}(w \wedge v) = \max\{\widetilde{P}(w), \widetilde{P}(v)\}.$$

(ii) Let $v, w \in N$ and $v \wedge w = u$ such that $u \neq v$, $u \neq w$. Hence,

$$\widetilde{P}(w) < \widetilde{P}(u), \;\; \widetilde{P}(v) < \widetilde{P}(u).$$

Then,

$$\widetilde{P}(u) > \max\{\widetilde{P}(w), \widetilde{P}(v)\}.$$

That is a contradiction.

Proposition 3.3. *Let B be a subnexus of nexus N. Then, $B \in PSUB(N)$ if and only if $\chi_B \in FPSUB(N)$, where that χ_B is a characteristic function of B.*

Proof. Let $a, b \in N$. If $a \wedge b \notin B$, then

$$\chi_B(a \wedge b) \leq \max\{\chi_B(a), \chi_B(b)\}.$$

If $a \wedge b \in B$, then by assumption $a \in B$ or $b \in B$. Thus,

$$\chi_B(a \wedge b) \leq \max\{\chi_B(a), \chi_B(b)\}.$$

Conversely, let $a \wedge b \in B$. Since $\chi_B(a \wedge b) \leq \max\{\chi_B(a), \chi_B(b)\}$, then,

$$\max\{\chi_B(a), \chi_B(b)\} = 1.$$

Hence, $a \in B$ or $b \in B$.

Theorem 3.5. *Suppose that N is a nexus. Then, $\widetilde{P} \in FPSUB(N)$ if and only if $\widetilde{P}_t \in PSUB(N)$, for all $t \in [0,1]$, whenever $\widetilde{P}_t \neq \emptyset$.*

Proof. Let $\widetilde{P} \in FPSUB(N)$, $t \in [0,1]$, $\widetilde{P}_t \neq \emptyset$ and suppose $a \wedge b \in \widetilde{P}_t$.

Hence, $t \leq \widetilde{P}(a \wedge b) \leq \max\{\widetilde{P}(a), \widetilde{P}(b)\}$ implies that $a \in \widetilde{P}_t$ or $b \in \widetilde{P}_t$.

Conversely, suppose $a, b \in N$ is arbitrary and suppose $\widetilde{P}(a \wedge b) = t$, for $t \in [0,1]$. Then, $a \wedge b \in \widetilde{P}_t$ implies that $a \in \widetilde{P}_t$ or $b \in \widetilde{P}_t$.

Hence,

$$\widetilde{P}(a \wedge b) \leq \max\{\widetilde{P}(a), \widetilde{P}(b)\}.$$

Theorem 3.6. *Suppose that N is a nexus and $\mathbf{B} = \{\widetilde{P} \mid \widetilde{P} \in FPSUB(N)\}$.*

(i) $\bigcap_{i \in I} \widetilde{P}_i \in FPSUB(N)$.

(ii) $\bigcap_{i \in I} \widetilde{P}_i \in FPSUB(N)$.

Proof. (i) Let $a, b \in N$.

$$\begin{aligned}
\bigcap_{i\in I} \widetilde{P}_i(a \wedge b) &= \inf_{i\in I} \widetilde{P}_i(a \wedge b) \\
&\leq \inf_{i\in I} \left[\max\{\widetilde{P}_i(a), \widetilde{P}_i(b)\}\right] \\
&= \max\left\{\inf_{i\in I}\{\widetilde{P}_i(a), \widetilde{P}_i(b)\}\right\} \\
&= \max\left\{\inf_{i\in I} \widetilde{P}_i(a), \inf_{i\in I} \widetilde{P}_i(b)\right\} \\
&= \max\left\{\left(\bigcap_{i\in I} \widetilde{P}_i\right)(a), \left(\bigcap_{i\in I} \widetilde{P}_i\right)(b)\right\}.
\end{aligned}$$

(ii) Similarly, for all $a, b \in N$. We have:

$$\begin{aligned}
\bigcup_{i\in I} \widetilde{P}_i(a \wedge b) &= \sup_{i\in I} \widetilde{P}_i(a \wedge b) \\
&\leq \sup_{i\in I} \left[\max\{\widetilde{P}_i(a), \widetilde{P}_i(b)\}\right] \\
&= \max\left\{\sup_{i\in I}\{\widetilde{P}_i(a), \widetilde{P}_i(b)\}\right\} \\
&= \max\left\{\sup_{i\in I} \widetilde{P}_i(a), \sup_{i\in I} \widetilde{P}_i(b)\right\} \\
&= \max\left\{\left(\bigcup_{i\in I} \widetilde{P}_i\right)(a), \left(\bigcup_{i\in I} \widetilde{P}_i\right)(b)\right\}.
\end{aligned}$$

4. Quotient nexuses induced by a fuzzy subnexus

Definition 4.1. Let $N \neq \{()\}$ be a nexus and $\widetilde{P}$ a fuzzy subnexus of N such that, $\widetilde{P}(x) \neq 0$, for some $x \neq ()$. Define the relation " $\sim$ " as follow.

$$v \sim w \;\Leftrightarrow\; \exists\, t \in N - \{()\}$$

such that, $\min\{\widetilde{P}(v \wedge t), \widetilde{P}(w \wedge t)\} > 0$, for all $v, w \in N$.

Theorem 4.1. " $\sim$ " *is a an equivalence relation.*

Proof. First we show that " $\sim$ " is reflexive. To do this let v be an arbitrary element of N. If $v = ()$, then it is obvious that $v \sim v$. Now let $v \neq ()$. If $\widetilde{P}(v) > 0$, then put $t = v$ and hence $v \sim v$. If $\widetilde{P}(v) = 0$, then there exists $w \in N - \{()\}$ such that $\widetilde{P}(w) > 0$. Choose $t = v \wedge w$ and hence

$$\widetilde{P}(v \wedge w) \geq \widetilde{P}(w) > 0,$$

which implies that $v \sim v$. The symmetric proof of " $\sim$ " is easy.

To prove the transitivity of " $\sim$ ", we assume that $v, w, z \in N$ and $v \sim w$, $w \sim z$. Thus,

$$\min\{\widetilde{P}(v \wedge t), \widetilde{P}(w \wedge t)\} > 0 \text{ and } \min\{\widetilde{P}(w \wedge t'), \widetilde{P}(z \wedge t')\} > 0,$$

for some $t \in N - \{()\}$, $t' \in N - \{()\}$.

Choose $t'' = t \wedge t'$ and hence

$$\min\{\widetilde{P}(v \wedge t''), \widetilde{P}(z \wedge t'')\} > 0.$$

Remark 4.1. The following example shows that the condition

$$\widetilde{P}(x) \neq 0, \text{ for some } () \neq x \in N$$

is necessary.

Example 4.1. Let $N = \{(), (1)\}$ and $\widetilde{P}(()) = 1, \widetilde{P}(1) = 0$. Then $(1) \not\sim (1)$, because $\min\{\widetilde{P}((1) \wedge (1)), \widetilde{P}((1) \wedge (1))\} = 0$.

Remark 4.2. Let $N \neq \{()\}$ be a nexus and $\widetilde{P}$ a fuzzy subnexus of N such that $\widetilde{P}(x) \neq 0$, for some $x \neq ()$. The equivalence class of v and the set of all equivalence classes of $\widetilde{P}$ are denoted by $\widetilde{P}_v$ and $\dfrac{N}{\widetilde{P}}$ respectively.

Definition 4.2. We define " $\leq$ " on $\dfrac{N}{\widetilde{P}}$ as follows:

$$\widetilde{P}_v \leq \widetilde{P}_w \Leftrightarrow v \leq w.$$

Lemma 4.1. $\left(\frac{N}{\widetilde{P}}, \leq\right)$ *is a $\wedge$-semi-lattice.*

Proof. Let $v, w \in N$. Since $v \wedge w \leq v$, $v \wedge w \leq w$. Then $\widetilde{P}_{v\wedge w} \leq \widetilde{P}_v$, $\widetilde{P}_{v\wedge w} \leq \widetilde{P}_w$. Now if $\widetilde{P}_s \leq \widetilde{P}_v$, $\widetilde{P}_s \leq \widetilde{P}_w$, for all $s \in N$, then it is obvious that $\widetilde{P}_s \leq \widetilde{P}_{v\wedge w}$. Hence

$$\inf\{\widetilde{P}_v, \widetilde{P}_w\} = \widetilde{P}_{v\wedge w}.$$

$\widetilde{P}_{v\wedge w}$ is denoted by $\widetilde{P}_v \widetilde{\wedge} \widetilde{p}_w$.

Theorem 4.2. *Let $M \neq \emptyset$, $F : M \longrightarrow N$ be an onto semi-lattices homomorphism of nexuses such that, $F((1)) \neq ()$ and $\widetilde{Q}$ a fuzzy subnexus of N such that $\widetilde{Q}(x) \neq 0$ for some $x \neq ()$. Then,*

$$\begin{array}{rcl} \Psi : \dfrac{M}{\widetilde{Q}F} & \longrightarrow & \dfrac{N}{\widetilde{Q}} \\ (\widetilde{Q}F)_w & \longmapsto & \widetilde{Q}_{F(w)} \end{array}$$

is an isomorphism of semi-lattices.

Proof. Ψ is well defined, because for all $w_1, w_2 \in M$, if $(\widetilde{Q}F)_{w_1} = (\widetilde{Q}F)_{w_2}$, then $w_1 \sim w_2$. Hence

$$\begin{aligned} &\min\{\widetilde{Q}F(w_1 \wedge t), \widetilde{Q}F(w_2 \wedge t)\} > 0, \text{ for some } t \in M - \{()\}. \\ &\Rightarrow \min\{\widetilde{Q}(F(w_1 \wedge t)), \widetilde{Q}(F(w_2 \wedge t))\} > 0 \\ &\Rightarrow \min\{\widetilde{Q}(F(w_1) \wedge F(t)), \widetilde{Q}(F(w_2) \wedge F(t))\} > 0 \\ &\Rightarrow F(w_1) \sim F(w_2) \Rightarrow \widetilde{Q}_{F(w_1)} = \widetilde{Q}_{F(w_2)}. \end{aligned}$$

To prove one to one of Ψ, let $w_1, w_2 \in M$ and $\widetilde{Q}_{F(w_1)} = \widetilde{Q}_{F(w_2)}$. Then

$$F(w_1) \sim F(w_2).$$

Hence,

$$\min\{\widetilde{Q}(F(w_1 \wedge t')), \widetilde{Q}(F(w_2) \wedge t')\} > 0, \text{ for some } t' \in N - \{()\}.$$

Then, since F is onto, there exists $t \in M - \{()\}$ such that

$$\begin{aligned} &\min\{\widetilde{Q}(F(w_1) \wedge F(t)), \widetilde{Q}(F(w_2) \wedge F(t))\} > 0 \\ &\Rightarrow \min\{\widetilde{Q}(F(w_1 \wedge t)), \widetilde{Q}(F(w_2 \wedge t))\} > 0 \\ &\Rightarrow \min\{\widetilde{Q}F(w_1 \wedge t), \widetilde{Q}F(w_2 \wedge t)\} > 0. \end{aligned}$$

Hence, $w_1 \sim w_2$ and $(\widetilde{Q}F)_{w_1} = (\widetilde{Q}F)_{w_2}$.

The proof onto of Ψ is obvious.

Now, for all $w_1, w_2 \in M$, we have:

$$\begin{aligned}\Psi((\widetilde{Q}F)_{w_1}\widetilde{\wedge}(\widetilde{Q}F)_{w_2}) &= \Psi((\widetilde{Q}F)_{w_1 \wedge w_2})\\ &= \widetilde{Q}_{F(w_1 \wedge w_2)} = \widetilde{Q}_{F(w_1)\wedge F(w_2)} = \widetilde{Q}_{F(w_1)}\widetilde{\wedge}\widetilde{Q}_{F(w_2)}\\ &= \Psi((\widetilde{Q}F)_{w_1})\widetilde{\wedge}\Psi((\widetilde{Q}F)_{w_2}).\end{aligned}$$

Hence Ψ is isomorphism.

Remark 4.3. The following example shows that in the above theorem the condition $F((1)) \neq ()$ is necessary.

Example 4.2. Let $M = \{(), (1), (1, 1)\}$, $N = \{(), (1)\}$ and $F : M \longrightarrow N$ be an function such that,

$$F(()) = F((1)) = () \ , \ F((1, 1)) = (1).$$

Also, suppose $\widetilde{Q}$ to be a fuzzy subnexus of N such that,

$$\widetilde{Q}(()) = 1, \ \widetilde{Q}((1)) = 0.$$

Now, F is a semi-lattice homomorphism and

$$\begin{aligned}\frac{M}{\widetilde{Q}F} &= \{(), (1)\},\\ \frac{N}{\widetilde{Q}} &= \{\{(), (1), (1, 1)\}\}.\end{aligned}$$

Hence, $\dfrac{M}{\widetilde{Q}F}$ and $\dfrac{N}{\widetilde{Q}}$ are not isomorphic.

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ON A CLASS OF CHINESE HYPERRINGS

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Abstract. In this paper we have shown one construction of the Chinese hyperrings, using the class of multiendomorphisms of the starting ring $(R, +, \cdot)$ or the class of multiendomorphisms of its additive group $(R, +)$.

Key words: hyperring, hypergroup, multiendomorphism.

MSC200. 35A25.

1. Introduction

This paper deals with a class of Chinese hyperrings. Some Chinese mathematicians have developed an interesting theory of HX-groups. P. Corsini [4] showed that from every HX-group, a hypergrupoid is obtained which is, in some cases, a hypergroup (Chinese hypergroup). Similarly, we show that, if we start from the ring $(R, +, \circ)$, then each family $\mathcal{R} \subset P_0(R)$, that satisfies conditions of Theorem 3.1, generates a Chinese hyperring $(\mathcal{R}^*, \widehat{\oplus}, \widehat{\odot})$. Theorem 3.5 shows that each multiendomorphism of the additive group $(R, +)$, under certain conditions, generates the family $\mathcal{R} \subset P_0(R)$ that satisfies the conditions of Theorem 3.1. So, there exists a class of multiendomorphisms of the additive group $(R, +)$, that generates a class of Chinese hyperrings. Theorem 3.10 shows that between the starting ring $(R, +, \circ)$ and the appropriate Chinese hyperring $(\mathcal{R}^*, \widehat{\oplus}, \widehat{\odot})$, we can establish a good multihomomorphism.

2. Preliminaries

Definition 2.1. A hypergroupoid $(H, \circ)$ is a non-empty set H equipped with a function $o : H \times H \to P^*(H)$ called a hyperoperation, where $P^*(H)$ denotes the set of all non-empty subsets of H. If A, B are subsets of H, then $A \circ B$ denotes the set $\bigcup\limits_{\substack{a \in A \\ b \in B}} a \circ b$.

Definition 2.2.

1) An H_v-semigroup is a hypergroupoid $(H, \circ)$ such that:

$$\forall\,(x,y,z) \in H^3, \quad (x \circ y) \circ z \cap x \circ (y \circ z) \neq \emptyset.$$

2) An H_v-semigroup is called an H_v-group if

$$\forall\, a \in H, \quad H \circ a = a \circ H = H.$$

3) A hyperstructure $(H, +, \circ)$ is called an H_v ring if:
   - i) $(H, +)$ is an H_v-group.
   - ii) $(H, \circ)$ is an H_v-semigroup.
   - iii) For all x, y, z in H we have:

$$x \circ (y + z) \cap (x \circ y + x \circ z) \neq \emptyset \text{ and } \quad (y + z) \circ x \cap (y \circ x + z \circ x) \neq \emptyset.$$

Definition 2.3. A hypergroupoid $\langle H; o\rangle$ is called a semihypergroup if:

$$\forall\,(x,y,z) \in H^3 : (x \circ y) \circ z = x \circ (y \circ z).$$

Definition 2.4. A semi-hypergroup $\langle H; o\rangle$ is called a hypergroup (or also a multigroup) if

$$(\forall\, a \in H) \;\; H \circ a = a \circ H = H.$$

The previous axiom is called the *reproduction axiom.*

Definition 2.5. A multivalued system $(R, +, \circ)$ is a hyperring if:

1) $(R, +)$ is a hypergroup.

2) $(R, \circ)$ is a semihypergroup.

3) For all x, y, z in R, we have:
   - (a) $x \circ (y + z) \subseteq x \circ y + x \circ z$
     and
   - (b) $(y + z) \circ x \subseteq y \circ x + z \circ x$.

If in the conditions 3a) and 3b) the equality is valid, the hyperring is called *strongly distributive.*

Definition 2.6. Let $(A, +, \cdot)$ and $(B, +', \cdot')$ be two hyperrings. The map $f : A \to B$ is called an *inclusion homomorphism* if for all $x, y \in A$, the following relations hold:

$$f(x + y) \subseteq f(x) +' f(y) \;\text{ and }\; f(x \cdot y) \subseteq f(x) \cdot' f(y).$$

The map f is called a *good homomorphism* if for all $x, y \in A$, we have:

$$f(x+y) = f(x) +' f(y) \quad \text{and} \quad f(x \cdot y) = f(x) \cdot' f(y).$$

Definition 2.7. Let $(A, +, \cdot)$ and $(B, +', \cdot')$ be two hyperrings. The map $\Phi : A \to P^*(B)$ is called a *multihomomorphism from the hyperring A to the hyperring B* if, for all $x, y \in A$, the following relations hold:

1) $\displaystyle\bigcup_{u \in x+y} \Phi(u) \subseteq \Phi(x) +' \Phi(y)$

2) $\displaystyle\bigcup_{u \in x \cdot y} \Phi(u) \subseteq \Phi(x) \cdot' \Phi(y).$

If in the conditions 1) and 2) the equality is valid, then the map Φ is called a *good multihomomorphism.*

Let us notice that, if $(A, +, \cdot)$ and $(B, +', \cdot')$ are the rings, then the conditions 1) and 2) means:

$$(\forall\, x, y \in A) \;\; \Phi(x+y) = \Phi(x) +' \Phi(y) \;\; \text{and} \;\; \Phi(xy) = \Phi(x) \cdot' \Phi(y).$$

Let $(G, \cdot)$ be an arbitrary group and $P(G)$ the powerset of G. Under the subset multiplication:

$$(\forall\, A, B \in P_0(G))) \; A \cdot B = \{ab \mid a \in A, \;\, b \in B\}.$$

$P_0(G) = P(G) \setminus \{\emptyset\}$ forms a semigroup which has an identity.

Definition 2.8. [2] A subgroup $(\mathcal{G}, \cdot)$ of $P_0(G)$ is called an *HX-group* on G and $(G, \cdot)$ the *generating group* (or the *support*) of $(\mathcal{G}, \cdot)$. The identity in $(\mathcal{G}, \cdot)$ is denoted by E.

Definition 2.9. [2] Let $(\mathcal{G}, \cdot)$ be an HX-group on $(G, \cdot)$. $(\mathcal{G}, \cdot)$ is called a *regular HX-group* if $e \in E$ (e is the identity of G).

Let $A \in \mathcal{G}$ and A^{-1} be the inverse element of A. The set $A^{(-1)} = \{x^{-1} \mid x \in A\}$ is called the *inverse set* of A.

Definition 2.10. [2] An HX-group $(\mathcal{G}, \cdot)$ is called a *uniform HX-group* if for all $A \in \mathcal{G}$ it holds:

$$A^{-1} = A^{(-1)}.$$

Theorem 2.11. [2] *An HX-group $(\mathcal{G}, \cdot)$ is a uniform HX-group iff E is a subgroup of G.*

Definition 2.12. [4] Let $(\mathcal{G},\cdot)$ be an HX-group with $(G,\cdot)$ as support and E as identity. We call a *Chinese hypergroupoid* the hyperstructure $\langle G^*,\widehat{\circ}\rangle$, where $G^*=\bigcup\limits_{A\in\mathcal{G}} A$ and

$$\forall\,(x,y)\in G^*\times G^*,\quad x\widehat{\circ}y=\bigcup_{\substack{x\in A,\ y\in B\\ \{A,B\}\subset\mathcal{G}}} A\cdot B.$$

Let $(\forall\,x\in G^*)$, $\alpha(x)=\{A\mid A\in\mathcal{G},\ A\ni x\}$ and $A(x)=\bigcup\limits_{A\in\alpha(x)} A.$

Lema 2.13. [4] $\forall\,(x,y)\in G^*\times G^*$, *we have:*

$$x\widehat{\circ}y=A(x)\cdot A(y).$$

Theorem 2.14. [4] *The hypergroupoid* $\langle G^*,\widehat{\circ}\rangle$ *is an* H_v*-group.*

Theorem 2.15 [4] *If* $(\mathcal{G},\cdot)$ *is an* HX*-group such that*

$$\forall\,(A,B)\in\mathcal{G}\times\mathcal{G},\quad A\cap B\neq\emptyset\Longrightarrow A=B,$$

then $\langle G^*,\widehat{\circ}\rangle$ *is a hypergroup.*

3. Chinese hyperrings

Theorem 3.1.

1) Let $(R,+,\cdot)$ be a ring and $\mathcal{R}\subset P_0(R)$ additive HX-group i.e. $\mathcal{R}$ is a group with respect to the operation:

$$(\forall\,A,B\in\mathcal{R})\ \ A+B=\{a+b\mid a\in A,\ b\in B\}.$$

Let $(\mathcal{R},\cdot)$ be a semigroup with respect to the operation:

$$(\forall\,A,B\in\mathcal{R})\ \ A\cdot B=\{ab\mid a\in A,\ b\in B\}.$$

On the set $\mathcal{R}^*=\bigcup\limits_{A\in\mathcal{R}} A$, we define the operations $\widehat{\oplus}$ and $\widehat{\odot}$:

$$(\forall\,x,y\in\mathcal{R}^*)\ \ x\widehat{\oplus}y=A(x)+A(y)\ \ \text{and}\ \ x\widehat{\odot}y=A(x)\cdot A(y),$$

where

$$A(x)=\bigcup_{\substack{A\ni x\\ A\in\mathcal{R}}} A.$$

The structure $(\mathcal{R}^*,\widehat{\oplus},\widehat{\odot})$ is an H_v-ring.

2) If $\mathcal{R} \subset P_0(R)$ beside previous assumptions satisfies the condition:

$$(\forall\, A, B \in \mathcal{R}) \;\; A \cap B \neq \emptyset \Longrightarrow A = B \tag{$**$}$$

then the structure $(\mathcal{R}^*, \widehat{\oplus}, \widehat{\odot})$ is a hyperring. (This hyperring we will call the Chinese hyperring).

Proof. Theorem 2.15 shows that $(\mathcal{R}^*, \widehat{\oplus})$ is an H_v-group.

We can show that $(\mathcal{R}^*, \widehat{\odot})$ is an H_v-semigroup using the method of P. Corsini [4]. Indeed, if $z \in x\widehat{\odot}y = A(x) \cdot A(y) = \left(\bigcup\limits_{\substack{x\in A\\ A\in\mathcal{R}}} A\right) \cdot \left(\bigcup\limits_{\substack{y\in B\\ B\in\mathcal{R}}} B\right)$, then A, B exist in $\mathcal{R}$ such that $x \in A$, $y \in B$ and $z \in A \cdot B$. Since $(\mathcal{R}, \cdot)$ is a semigroup, it is clear that $A \cdot B \in \mathcal{R}$, i.e., $z \in A \cdot B \subseteq \mathcal{R}^*$.

Let $x, y, z \in \mathcal{R}^*$. Then, we have:

$$\begin{aligned}
(x\widehat{\odot}y)\widehat{\odot}z &= (A(x) \cdot A(y))\widehat{\odot}z = \bigcup_{u\in A(x)\cdot A(y)} u\widehat{\odot}z \\
&= \bigcup_{u\in A(x)\cdot A(y)} A(u) \cdot A(z) = \bigcup_{\substack{u\in A(x)\cdot A(y)\\ z\in A_3}} A(u) \cdot A_3 \\
&= \bigcup_{\substack{A_u \ni u \in \dot{A}_1 \dot{A}_2\\ x\in\dot{A}_1 \cdot y\in\dot{A}_2 \cdot z\in A_3}} A_u \cdot A_3 = (*).
\end{aligned}$$

So, we proved: $\forall\,(A_1, A_2, A_3) \in \mathcal{R}^3$ such that $x \in A_1$, $y \in A_2$, $z \in A_3$, it holds:

$$(*) \supseteq (A_1A_2)A_3 \;\text{ i.e. }\; (x\widehat{\odot}y)\widehat{\odot}z \supseteq (A_1A_2)A_3.$$

Similarly, we can prove: $\forall\,(A_1, A_2, A_3) \in \mathcal{R}^3$ such that $x \in A_1$, $y \in A_2$, $z \in A_3$, it holds:

$$x\widehat{\odot}(y\widehat{\odot}z) \supseteq A_1(A_2A_3).$$

Therefore, $(x\widehat{\odot}y)\widehat{\odot}z \cap x\widehat{\odot}(y\widehat{\odot}z) \neq \emptyset$, i.e. $(\mathcal{R}^*, \widehat{\odot})$ is an H_v-semigroup.

Let us prove now that $(\mathcal{R}^*, \widehat{\oplus}, \widehat{\odot})$ satisfies the condition

$$(x\widehat{\oplus}y)\widehat{\odot}z \cap ((x\widehat{\odot}z)\widehat{\oplus}(y\widehat{\odot}z)) \neq \emptyset$$

for all $x, y, z \in \mathcal{R}^*$.

Let $x, y, z \in \mathcal{R}^*$. Then we have:

$$L = (x\widehat{\oplus}y)\widehat{\odot}z = (A(x)+A(y))\widehat{\odot}z = \bigcup_{u\in A(x)+A(y)} u\widehat{\odot}z = \bigcup_{\substack{A_u \ni u \in \dot{A}_1+\dot{A}_2\\ x\in\dot{A}_1,\; y\in\dot{A}_2,\; z\in A_3}} A_u{\cdot}A_3 = (*)'.$$

We can notice that for arbitrary $(A_1, A_2, A_3) \in \mathcal{R}^3$, where $x \in A_1$, $y \in A_2$ and $z \in A_3$, it holds:

$$(*)' \supseteq (A_1 + A_2) \cdot A_3, \;\text{ i.e. }\; L \supseteq (A_1 + A_2) \cdot A_3. \tag{1}$$

Also we have:

$$D=(x\widehat{\odot}z)\widehat{\oplus}(y\widehat{\odot}z)=(A(x)\cdot A(z))\widehat{\oplus}(A(y)\cdot A(z))=\bigcup u\widehat{\oplus}v,$$

where,

$$u\in A(x)\cdot A(z)=\bigcup_{\substack{x\in A_1\\ z\in A_3}} A_1\cdot A_3 \quad\text{and}\quad v\in A(y)\cdot A(z)=\bigcup_{\substack{y\in A_2\\ z\in a_3}} A_2\cdot A_3.$$

Thus,

$$D=\bigcup_{\substack{A_u\ni u\in A_1\cdot A_3,\ x\in A_1,\ z\in A_3\\ A_v\ni v\in A_2\cdot A_3,\ y\in A_2,\ z\in A_3}} A_u+A_v.$$

Therefore, for arbitrary $(A_1,A_2,A_3)\in\mathcal{R}^3$, where $x\in A_1$, $y\in A_2$ and $z\in A_3$, it holds:

$$D\supseteq A_1\cdot A_3+A_2\cdot A_3\supseteq(A_1+A_2)\cdot A_3. \tag{2}$$

From (1) and (2), it follows that $L\cap D\neq\emptyset$.

Similarly, we can show that for arbitrary $x,y,z\in\mathcal{R}^*$ it holds:

$$x\widehat{\odot}(y\widehat{\oplus}z)\cap((x\widehat{\odot}y)\widehat{\oplus}(x\widehat{\odot}z))\neq\emptyset.$$

So, we proved that $(\mathcal{R}^*,\widehat{\oplus},\widehat{\odot})$ is an H_v-hyperring.

2) It is enough to remark that the condition $(**)$ implies $(\forall\, x\in\mathcal{R}^*)\ |\alpha(x)|=1$, where

$$\alpha(x)=\{A\mid A\in\mathcal{R},\ A\ni x\}.$$

Let $(G,\cdot)$ and $(G_1,\circ)$ be two arbitrary groups, and let e be an identity element of $(G,\cdot)$.

Definition 3.2. [3] A multivalued mapping $f:G\to P_0(G_1)=\{B\subset G_1\mid B\neq\emptyset\}$ such that:

$$(\forall\, x,y\in G)\quad f(x\cdot y)=f(x)\circ f(y)=\{a\circ b\mid a\in f(x),b\in f(y)\}$$

is called a *multihomorphism* of group G into a group G_1.

It is easy to see, that for arbitrary $x,y\in G$ it holds:

1) $f(x)\circ f(y)=f(xy)$

2) $(f(x)\circ f(y))\circ f(z)=f(xyz)=f(x)\circ(f(y)\circ f(z))$

3) $f(x)=f(x)\circ f(e)=f(e)\circ f(x)$

4) $f(e)=f(x)\circ f(x^{-1})=f(x^{-1})\circ f(x)$.

Theorem 3.3. [3] *Let f be a multihomomorphism of the group $(G,\cdot)$ into the group $(G_1,\circ)$, such that $f(e)$ is subgroup of G_1. Then it holds:*

$$(\forall\, x,y\in G)\quad f(x)\cap f(y)\neq\emptyset\Longrightarrow f(x)=f(y).$$

We prove this theorem using the theory of HX-groups.

Proof. Let $\mathcal{G}_1=\{f(x)\mid x\in G\}$. Then it is easy to see that $\mathcal{G}_1$ is an HX-group on G_1, with identity $E=f(e)$, (e is the identity of G). Since, $E=f(e)$ is subgroup of G_1, then from Theorem 2.11 it follows that $\mathcal{G}_1$ is a uniform HX-group, i.e, it holds:

$$(f(x))^{(-1)}=(f(x))^{-1}=f(x^{-1}).\tag{$*$}$$

Now, suppose that $z\in f(x)\cap f(y)$.

From $z\in f(y)$, it follows that $z^{-1}\in(f(y))^{(-1)}\overset{(*)}{=}f(y^{-1})$.

Let e_1 be the identity of $(G_1,\circ)$. From

$$e_1=z\circ z^{-1}\in f(x)\circ f(y^{-1})=f(xy^{-1}),$$

it follows $f(e)=e_1\circ f(e)\subseteq f(xy^{-1})\circ f(e)=f(xy^{-1}e)=f(xy^{-1})$. Therefore,

$$f(e)\subseteq f(xy^{-1})\tag{1}$$

Similarly,

$$f(e)\subseteq f(yx^{-1}).\tag{2}$$

From (1) and (2) it follows that:

$$f(xy^{-1})\supseteq f(e)=f(xy^{-1}yx^{-1})=f(xy^{-1})\circ f(yx^{-1})\supseteq f(xy^{-1})\circ f(e)=f(xy^{-1}),$$

i.e. $f(xy^{-1})\supseteq f(e)\supseteq f(xy^{-1})$. Thus, $f(xy^{-1})=f(e)$. Therefore,

$$f(x)=f(xy^{-1}y)=f(xy^{-1})f(y)=f(e)f(y)=f(y).$$

Corollary 3.4. *Let f be a multihomomorphism of the group $(G,\cdot)$ into the group $(G_1,\circ)$ such that $f(e)$ is subgroup of G_1. Then, for arbitrary $x\in G$ it holds:*

$$f(x)=a\circ f(e)=f(e)\circ a,$$

where a is an arbitrary element of $f(x)$.

Proof. Let $a\in f(x)$. Then:

$$a\circ f(e)\subseteq f(x)\circ f(e)=f(xe)=f(x)$$

and

$$f(e)\circ a\subseteq f(e)\circ f(x)=f(ex)=f(x).$$

Conversely, if $b \in f(x)$, then it holds:

$$b = a \circ (a^{-1} \circ b).$$

Let us prove that $a^{-1} \circ b \in f(e)$. From $a \in f(x)$ it follows that $a^{-1} \in (f(x))^{(-1)} = f(x^{-1})$ and thus $a^{-1} \circ b \in f(x^{-1}) \circ f(x) = f(e)$ and for this reason $b \in a \circ f(e)$.

We proved that $f(x) \subseteq a \circ f(e)$. Therefore, $a \circ f(e) = f(x)$. Similarly, we can prove that $f(e) \circ a = f(x)$.

Theorem 3.5. *Let* $(R, +, \cdot)$ *be a ring and* $f : R \to P_0(R)$ *a multiendomorphism of the additive group* $(R, +)$ *such that* $f(0)$ *is a subgroup of a group* $(R, +)$*. Let* $\mathcal{R}_f = \{f(x) \mid x \in R\}$ *and* $\mathcal{R}_f^* = \bigcup\limits_{x \in R} f(x)$*. If the following condition is valid:*

$$(\forall\, x, y \in R) \ \ (\exists z \in R) \ \ f(x) \cdot f(y) = f(z) \tag{1}$$

and if we define on $\mathcal{R}_f^*$ *the hyperoperations* $\widehat{\oplus}$ *and* $\widehat{\odot}$ *as in Theorem* 3.1*, then the structure* $(\mathcal{R}_f^*, \widehat{\oplus}, \widehat{\odot})$ *is a hyperring.*

Proof. Let us notice that $(\mathcal{R}_f, +)$ is an HX-group on $(R, +)$ and from the condition (1) it follows that $(\mathcal{R}_f, \cdot)$ is a semigroup. Since the multiendomorphism f fulfills the conditions of Theorem 3.3, then for arbitrary $f(x), f(y) \in \mathcal{R}_f$ it holds:

$$f(x) \cap f(y) \neq \emptyset \Longrightarrow f(x) = f(y).$$

Thus, the family $\mathcal{R}_f \subseteq P_0(R)$ fulfils the conditions of Theorem 3.1 and for that reason $(\mathcal{R}_f^*, \widehat{\oplus}, \widehat{\odot})$ is a hyperring.

Corollary 3.6. *Let* $f : R \to P_0(R)$ *be a multiendomorphism of a ring* $(R, +, \cdot)$ *such that* $f(0)$ *is a subgroup of a group* $(R, +)$*. Then, the structure* $(\mathcal{R}_f^*, \widehat{\oplus}, \widehat{\odot})$ *is a hyperring.*

Proof. As for all $x, y \in R$ it holds

$$f(x) \cdot f(y) = f(xy),$$

it is easy to see that the multiendomorphism f fulfils the conditions of the previous theorem.

Corollary 3.7. *Let* $(R, +, \cdot)$ *be a ring and* $f : R \to P_0(R)$ *multiendomorphism of a group* $(R, +)$*, such that* $f(0)$ *is an ideal of a ring* R*, which satisfies the condition* $f(0) \cdot f(0) = f(0)$*. If* $\bigcup\limits_{x \in R} f(x) = R$*, then the structure* $(\mathcal{R}_f^* = R, \widehat{\oplus}, \widehat{\odot})$ *is a hyperring.*

Proof. Let $x, y \in R$. From Corollary 3.4, for arbitrary $a \in f(x)$ and $b \in f(y)$, it holds:

$$f(x) = a + f(0) \ \text{ and } \ f(y) = b + f(0)$$

and thus:

$$\begin{aligned} f(x)\cdot f(y) &= (a+f(0))(b+f(0)) = ab+f(0)b+af(0)+f(0)\cdot f(0) \\ &\subseteq ab+f(0)+f(0)+f(0) = ab+f(0). \end{aligned}$$

On the other hand,

$$ab+f(0) = ab+f(0)\cdot f(0) \subseteq ab+f(0)b+af(0)+f(0)\cdot f(0)$$

since $0 \in a\cdot f(0)$ and $0 \in f(0)b$. Thus, $f(x)\cdot f(y) = ab+f(0)$. From $\bigcup\limits_{x\in R} f(x) = R$, it follows that there exists $z \in R$ such that $ab \in f(z)$, and by Corollary 3.4, we have:

$$f(z) = ab+f(0).$$

Therefore,

$$f(x)\cdot f(y) = ab+f(0) = f(z).$$

From Theorem 3.5, it follows that $(\mathcal{R}^*_f = R, \widehat{\oplus}, \widehat{\odot})$ is a hyperring.

Example 3.8. Let $(R,+,\cdot) = (Z_6, +_6, \cdot_6)$ and $f : Z_6 \to P_0(Z_6)$ be a multimapping defined by $f(x) = (3\cdot_6 x) +_6 A$, where $A = \{0,2,4\}$. Then:

$$\begin{aligned} f(x+y) &= (3\cdot_6 (x+_6 y)) +_6 A = (3\cdot_6 x) +_6 (3\odot_6 y) +_6 A \\ &= ((3\cdot_6 x) +_6 A) +_6 ((3\cdot_6 y) +_6 A) = f(x) +_6 f(y), \end{aligned}$$

since $A +_6 A = A$.

2) $f(0) = A$ and A is obviously an ideal of a ring $(Z_6, +_6, \cdot_6)$.

3) $A\cdot A = A$.

4) $\bigcup\limits_{x\in Z_6} f(x) = Z_6$.

Thus, f satisfied the conditions of the previous corollary. Therefore, the family $\mathcal{R}_f = \{f(x) \mid x \in Z_6\}$ generates the hyperring $(Z_6, \widehat{\oplus}, \widehat{\odot})$.

Generally, if A is an ideal of a ring $(R,+,\cdot)$ such that $A\cdot A = A$, then for arbitrary homomorphism $h : R \to R$ of additive group $(R,+)$, such that $h(R)+A = R$, there exists a multiendomorphism $f : R \to P_0(R)$ of a group $(R,+)$, defined by $f(x) = h(x)+A$, that satisfied the conditions of Theorem 3.5.

Theorem 3.9. *Let $f : R \to P_0(R)$ be a multiendomorphism of a ring $(R,+,\cdot)$ such that $f(0)$ is subgroup of a group $(R,+)$. Then, $f(0)$ is an ideal of a ring $(f(R),+,\cdot)$ and there exists a homomorphism $\bar f$ of a ring R into the quotient ring $f(R)/f(0)$ such that $f(x) = \bar f(x)$ for all $x \in R$. (In a quotient ring $f(R)/f(0)$ we have ordinary operations $\oplus$ and $\odot$ defined by*

$$\begin{aligned} (a+f(0)) \oplus (b+f(0)) &= a+b+f(0), \\ (a+f(0)) \odot (b+f(0)) &= ab+f(0)). \end{aligned}$$

Proof. Let us prove that $f(R)$ is a subring of a ring $(R,+,\cdot)$. Let $a,b\in f(R)$. Then, there exist $x,y\in R$ such that $a\in f(x)$, $b\in f(y)$. Thus,

$$a+b\in f(x)+f(y)=f(x+y)\subseteq f(R) \text{ and } ab\in f(x)\cdot f(y)=f(xy)\subseteq f(R).$$

If $a\in f(x)$, then

$$(-a)\in -(f(x))\overset{Th.2.11}{=\!=\!=}f(-x)\subseteq f(R).$$

Therefore, $f(R)$ is a subring of a ring R.

Let us prove that $f(0)$ is an ideal of a ring $f(R)$. Since $f(0)$ is a subgroup of a group $(f(R),+)$, it is sufficient to notice that for arbitrary $a\in f(R)$ it holds:

$$a\cdot f(0)\subseteq f(R)\cdot f(0)=f(0)$$

and

$$f(0)\cdot a\subseteq f(0)\cdot f(R)=f(0).$$

As by the Theorem 3.3 for arbitrary $x,y\in R$ it holds $f(x)\cap f(y)\neq\emptyset\Longrightarrow f(x)=f(y)$, we have that the mapping $\bar{f}:R\to f(R)/f(0)$ defined by:

$$\bar{f}(x)=f(x)$$

is well defined. (It is clear that, for arbitrary $x\in R$, from Corollary 3.4 it holds $f(x)=a+f(0)$ for arbitrary $a\in f(x)\subseteq f(R)$ and, therefore, $f(x)\in f(R)/f(0)$).

Let us prove that $\bar{f}$ is a homomorphism. Let $x,y\in R$. Then it holds:

$$\bar{f}(x+y)=f(x+y)=f(x)+f(y)=(*)$$

Let $a\in f(x)$ and $b\in f(y)$. Then:

$$\begin{aligned}(*)=(a+f(0))+(b+f(0)) &= a+b+f(0)=(a+f(0))\oplus(b+f(0))\\ &= f(x)\oplus f(y)=\bar{f}(x)\oplus\bar{f}(y).\end{aligned}$$

Thus, $\bar{f}(x+y)=\bar{f}(x)\oplus\bar{f}(y)$.

Similarly,

$$\begin{aligned}\bar{f}(xy)=f(xy) &= f(x)\cdot f(y)=(a+f(0))\cdot(b+f(0))\\ &= ab+f(0)\cdot b+a\cdot f(0)+f(0)\cdot f(0)\\ &= ab+f(0)\cdot b+a\cdot f(0)+f(0\cdot 0)\\ &= ab+f(0)\cdot b+a\cdot f(0)+f(0).\end{aligned}$$

Since, $f(0)\cdot b+a\cdot f(0)\subseteq f(0)$, it holds:

$$f(0)\cdot b+a\cdot f(0)+f(0)=f(0).$$

Therefore,

$$\bar{f}(xy)=ab+f(0)=(a+f(0))\odot(b+f(0))=f(x)\odot f(y)=\bar{f}(x)\odot\bar{f}(y).$$

Theorem 3.10. *Let $(R,+,\cdot)$ be a ring and let $f:R\to P_0(R)$ be a multiendomorphism of a group $(R,+)$ such that $f(0)$ is an ideal of a ring R which satisfies the conditions:*

1) $f(0)\cdot f(0)=f(0)$

2) $\bigcup\limits_{x\in R} f(x)=R$.

Then the family $\mathcal{R}_f=\{f(x)\mid x\in R\}$ *generates the hyperring* $(R_f^*=R,\widehat{\oplus},\widehat{\odot})$ *and the mapping* $\Phi: R\to P_0(R)$ *defined by* $\Phi(x)=x+f(0)$ *is a good multihomomorphism of a ring* $(R,+,\cdot)$ *into a hyperring* $(R,\widehat{\oplus},\widehat{\odot})$.

Proof. Let $x,y\in R$. Then:

$$\Phi(x+y)=x+y+f(0)$$

and

$$\Phi(x)\widehat{\oplus}\Phi(y)=(x+f(0))\widehat{\oplus}(y+f(0))=\bigcup_{\substack{u\in x+f(0)\\ v\in y+f(0)}} u\widehat{\oplus}v=x+y+f(0).$$

Indeed, as $\bigcup\limits_{z\in R} f(z)=R$, it follows by Theorem 3.3 that there exists only one element $f(a)$ of the family $\mathcal{R}_f=\{f(z)\mid z\in R\}$ such that $x\in f(a)$ and, by Corollary 3.4, it holds $f(a)=x+f(0)$. Similarly, there exists only one element $f(b)$ of the family $\mathcal{R}_f$ such that $f(b)=y+f(0)$. Thus, for arbitrary $u\in f(a)$ and $v\in f(b)$, we have

$$u\widehat{\oplus}v=\bigcup_{\substack{u\in f(a')\\ v\in f(b')}} f(a')+f(b')=f(a)+f(b)=x+y+f(0).$$

Similarly,

$$\Phi(xy)=xy+f(0),$$

and

$$\Phi(x)\widehat{\odot}\Phi(y)=\bigcup_{\substack{u\in x+f(0)\\ v\in y+f(0)}} u\widehat{\odot}v\overset{(**)}{=}\bigcup_{\substack{u\in x+f(0)\\ v\in y+f(0)}} uv+f(0)\overset{(***)}{=}xy+f(0).$$

The equality $(**)$ is valid since, for $u\in x+f(0)=f(a)$ and $v\in y+f(0)=f(b)$, it holds:

$$\begin{aligned} u\widehat{\odot}v &= \bigcup_{\substack{u\in f(a')\\ v\in f(b')}} f(a')\cdot f(b')=f(a)\cdot f(b)\\ &= (u+f(0))(v+f(0))\\ &= uv+f(0)\cdot v+u\cdot f(0)+f(0)\cdot f(0)\\ &= uv+f(0)\cdot v+u\cdot f(0)+f(0)\\ &= uv+f(0). \end{aligned}$$

The equality $(***)$ is valid since $x\in x+f(0)$, $y\in y+f(0)$, and thus:

$$xy+f(0)\subseteq\bigcup_{\substack{u\in x+f(0)\\ v\in y+f(0)}} uv+f(0),$$

while, for arbitrary $u \in x + f(0)$ and $v \in y + f(0)$, it holds:

$$u + f(0) = x + f(0) \text{ and } v + f(0) = y + f(0)$$

and thus:

$$uv + f(0) = (u + f(0))(v + f(0)) = (x + f(0))(y + f(0)) = xy + f(0).$$

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SOME RESULTS ON ANALOGOUS CONTINUED FRACTION OF RAMANUJAN

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Abstract. We give a differential for $\frac{1}{C(q)}$ and prove an identity which is analogous to Ramanujan's Entry 3.2.7. We also give a simpler proof for Entry 9(v).

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1. Introduction

The Rogers-Ramanujan continued fraction, defined by

$$(1.1)\qquad R(q) = \frac{q^{\frac{1}{5}}}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\cdots, \quad |q| < 1.$$

first appeared in a paper by L.J. Rogers [6] in 1894. In his first two letters to G.H. Hardy [5, pp xxvii, xxviii], Ramanujan communicated several theorems on $R(q)$. Ramanujan's major work on continued fraction centres around the continued fraction $R(q)$. I find the analogous continued fraction $C(q)$ defined by

$$(1.2)\qquad C(q) = \frac{1}{1+}\frac{1+q}{1+}\frac{q^2}{1+}\frac{q+q^3}{1+}\frac{q^4}{1+}\cdots = \frac{(q;q^4)_\infty(q^3;q^4)_\infty}{(q^2;q^4)_\infty^2}$$

equally interesting. In this paper, we have given some results for $C(q)$ notably a differential for the reciprocal for $C(q)$ which is analogous to Entry 9(v) [4, p.258] for $R(q)$. In [2, p. 259], Berndt has proved this Entry. I have given a simpler proof of this Entry. Further I have considered two sets of relations which are equivalent and have proved an identity which is analogous to Ramanujan's Entry 3.2.7 [1, p.89].

2. Notations

We shall be using the customary q-product notation. Thus

For $|q| < 1$

$$(a)_0 = (a;q)_0 = 1 \text{ and for } n \geq 1,$$
$$(a)_n = (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k).$$

Furthermore,

$$(a)_\infty = (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \quad |q|<1.$$

If the base q is understood we use $(a)_n$ and $(a)_\infty$ instead of $(a;q)_n$ and $(a;q)_\infty$, respectively.

Ramanujan's general theta function $f(a,b)$,

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = (-a;ab)_\infty\,(-b;ab)_\infty(ab;ab)_\infty, \quad |ab|<1 \tag{2.1}$$

and

$$f(-q) = f(-q,-q^2) = (q;q)_\infty, \quad |q|<1. \tag{2.2}$$

3. A result for the differential of $\dfrac{1}{C(q)}$

By definition,

$$C(q) = \frac{1}{1+}\,\frac{1+q}{1+}\,\frac{q^2}{1+}\,\frac{q+q^3}{1+}\,\frac{q^4}{1+\ldots.} = \frac{(q;q^4)_\infty(q^3;q^4)_\infty}{(q^2;q^4)^2_\infty} \tag{3.1}$$

Let $F(q) = q^{-\frac{1}{8}}C(q)$ and $A(q) = \dfrac{1}{F(q)}$. Then

$$8q\frac{d}{dq}\left(\log\frac{(q^2;q^4)^2_\infty}{(q;q^4)_\infty(q^3;q^4)_\infty}\right) = 1 - \frac{(q;q)^4_\infty}{(-q;q)^4_\infty}. \tag{3.2}$$

Proof. Now

$$A(q) = q^{\frac{1}{8}}\frac{(q^2;q^4)^2_\infty}{(q;q^4)_\infty(q^3;q^4)_\infty}\,.$$

Taking logarithmic differentiation with respect to q, we get

$$\frac{8q}{A(q)}\frac{d}{dq}A(q) = 1+8\sum_{n=0}^{\infty}\left\{\frac{(4n+1)q^{4n+1}}{1-q^{4n+1}} - \frac{2(4n+2)q^{4n+2}}{1-q^{4n+2}} + \frac{(4n+3)q^{4n+3}}{1-q^{4n+3}}\right\}. \tag{3.3}$$

Using the result Srivastava [7],

$$1-8\sum_{n=0}^{\infty}\left\{\frac{(4n+1)q^{4n+1}}{1-q^{4n+1}} - \frac{2(4n+2)q^{4n+2}}{1-q^{4n+2}} + \frac{(4n+3)q^{4n+3}}{1-q^{4n+3}}\right\} = \frac{(q;q)^4_\infty}{(-q;q)^4_\infty},$$

equation (3.3) becomes

$$\frac{8q}{A(q)}\frac{d}{dq}\left(q^{\frac{1}{8}}\frac{(q^2;q^4)^2_\infty}{(q;q^4)_\infty(q^3;q^4)_\infty}\right)=1+1-\frac{(q;q)^4_\infty}{(-q;q)^4_\infty}$$

or

$$\frac{8q}{A(q)}\left[\frac{1}{8q}+\frac{d}{dq}\left(\log\frac{(q^2;q^4)^2_\infty}{(q;q^4)_\infty(q^3;q^4)_\infty}\right)\right]A(q)=1+1-\frac{(q;q)^4_\infty}{(-q;q)^4_\infty}$$

or

$$8q\frac{d}{dq}\left(\log\frac{(q^2;q^4)^2_\infty}{(q;q^4)_\infty(q^3;q^4)_\infty}\right)=1-\frac{(q;q)^4_\infty}{(-q;q)^4_\infty},$$

which proves (3.2).

Simpler proof of Entry 9(v) [4, p. 258]

Entry 9(v):

If

$$R(q)=\frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+...},$$

then

$$1-\frac{f^5(-q)}{f(-q^5)}=5q\frac{d}{dq}\log\frac{f(-q^2,-q^3)}{f(-q,-q^4)}. \tag{3.4}$$

Proof. By [1, eq. (1.1.1), p. 9],

$$R(q)=\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}=\frac{f(-q,-q^4)}{f(-q^2,-q^3)}. \tag{3.5}$$

Let $G(q)=q^{-\frac{1}{5}}R(q)$ and $B(q)=\dfrac{1}{G(q)}$, then

$$B(q)=q^{\frac{1}{5}}\frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty}.$$

Taking logarithmic differentiation of (3.5) with respect to q, we have

$$\begin{aligned}\frac{5q}{B(q)}\frac{d}{dq}[B(q)] &= 1+5\sum_{n=0}^{\infty}\left\{\frac{(5n+1)q^{5n+1}}{1-q^{5n+1}}\right.\\ &\quad\left.-\frac{(5n+2)q^{5n+2}}{1-q^{5n+2}}-\frac{(5n+3)q^{5n+3}}{1-q^{5n+3}}+\frac{(5n+4)q^{5n+4}}{1-q^{5n+4}}\right\}.\end{aligned} \tag{3.6}$$

Using the following formula of Ramanujan [2]

$$\begin{aligned}&1-5\sum_{n=0}^{\infty}\left\{\frac{(5n+1)q^{5n+1}}{1-q^{5n+1}}-\frac{(5n+2)q^{5n+2}}{1-q^{5n+2}}\right.\\ &\quad\left.-\frac{(5n+3)q^{5n+3}}{1-q^{5n+3}}+\frac{(5n+4)q^{5n+4}}{1-q^{5n+4}}\right\}=\frac{(q;q)^5_\infty}{(q^5;q^5)_\infty}=\frac{f^5(-q)}{f(-q^5)},\end{aligned} \tag{3.7}$$

equation (3.6) can be written as

$$\frac{5q}{B(q)}\frac{d}{dq}\left[B(q)\right]=1+1-\frac{f^5(-q)}{f(-q^5)} \tag{3.8}$$

or

$$\frac{5q}{B(q)}\left[\frac{1}{5q}+\frac{d}{dq}\log\frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty}\right]B(q)=1+1-\frac{f^5(-q)}{f(-q^5)}$$

or

$$5q\frac{d}{dq}\log\frac{f(-q^2,-q^3)}{f(-q,-q^4)}=1-\frac{f^5(-q)}{f(-q^5)}$$

we have the Entry 9(v)[4, p.258].

4. A transformation formula for $C(q)$

Ramanujan in the "lost" note book has stated the formula

$$\frac{f(-\lambda^2q^3,-\lambda q^6)+qf(-\lambda,-\lambda^2q^9)}{f(-\lambda q^3)}=\frac{f(-q^2,-\lambda q)}{f(-q,-\lambda q^2)}.$$

This formula is extensively used by Ramanujan in his note book, but the statement is found only in the "lost" note book.

Take $\lambda = q$

$$\frac{f(-q^5,-q^7)+qf(-q,-q^{11})}{f(-q^4)}=\frac{f(-q^2,-q^2)}{f(-q,-q^3)}. \tag{4.1}$$

Using (2.1) and then (1.2), we get

$$\frac{f(-q^5,-q^7)+qf(-q,-q^{11})}{f(-q^4)}=\frac{(q^2;q^4)_\infty^2}{(q;q^4)_\infty(q^3;q^4)_\infty}=\frac{1}{C(q)}. \tag{4.2}$$

5. Two equivalent relations

Let $u = C(q)$ and $v = C(q^4)$.

(i) $$2u=B+2^8q\frac{f^8(-q^4)}{f^8(-q)} \tag{5.1}$$

and

$$2u=B+\frac{f^8(-q)}{qf^8(-q^4)} \tag{5.2}$$

are equivalent.

(ii) $$2v = K + 4q^{\frac{5}{8}}\frac{f(-q^{16})}{f(-q)} \tag{5.3}$$

and

$$2v = K + \frac{f(-q^{\frac{1}{4}})}{q^{\frac{5}{32}}f(-q^4)} \tag{5.4}$$

are equivalent.

Proof of (i)

We shall use Ramanujan's transformation formula [2, p. 270, eq. 12.10]

$$e^{-\frac{a}{12}}a^{\frac{1}{4}}f(-e^{-2a}) = e^{-\frac{b}{12}}b^{\frac{1}{4}}f(-e^{-2b}) \tag{5.5}$$

twice, first by taking $a = \dfrac{\alpha}{2}, b = \dfrac{\beta}{2}$ and then by taking $a = 2\alpha,\ b = \dfrac{\beta}{8}$ to get

$$e^{-\frac{\alpha}{24}}\left(\frac{\alpha}{2}\right)^{\frac{1}{4}}f(-e^{-a}) = e^{-\frac{\beta}{24}}\left(\frac{\beta}{2}\right)^{\frac{1}{4}}f(-e^{-\beta}) \tag{5.6}$$

and

$$e^{-\frac{4\alpha}{24}}\left(\frac{4\alpha}{2}\right)^{\frac{1}{4}}f(-e^{-4a}) = e^{-\frac{\beta}{96}}\left(\frac{\beta}{8}\right)^{\frac{1}{4}}f(-e^{-\frac{\beta}{4}}). \tag{5.7}$$

By (5.6) and (5.7), we have

$$e^{\frac{\alpha}{8}}\frac{f(-e^{-\alpha})}{f(-e^{-4\alpha})} = 2e^{-\frac{\beta}{32}}\frac{f(-e^{-\beta})}{f(-e^{-\frac{\beta}{4}})}. \tag{5.8}$$

Putting $q = e^{-\alpha}$ and $Q = e^{-\frac{\beta}{4}}$ in (5.8), we have

$$\frac{f(-q)}{q^{\frac{1}{8}}f(-q^4)} = 2Q^{\frac{1}{8}}\frac{f(-Q^4)}{f(-Q)}. \tag{5.9}$$

or

$$\frac{f^8(-q)}{qf^8(-q^4)} = 2^8Q\frac{f^8(-Q^4)}{f^8(-Q)} \tag{5.10}$$

Writing Q for q in (5.1), we have (5.2). Thus (5.1) and (5.2) are equivalent.

Proof of (ii)

Applying transformation formula (5.5) twice, first with $a = \dfrac{\alpha}{8},\ b = 2\beta$ and then with $a = 2\alpha,\ b = \dfrac{\beta}{8}$ and then taking $q = e^{-\alpha}, Q^4 = e^{-\beta}$, we get

$$q^{-\frac{5}{32}}\frac{f(-q^{\frac{1}{4}})}{f(-q^4)} = 4Q^{\frac{5}{8}}\frac{f(-Q^{16})}{f(-Q)}. \tag{5.11}$$

Replacing q by Q in (5.3), we have (5.4). Thus (5.3) and (5.4) are equivalent.

6. Another identity

We prove the following identity which is analogous to Ramanujan's Entry 3.2.7 (p 364) [1, p 89].

If

$$q^{-1}C^8(q) - \frac{2}{q^{-1}C^8(q)} = -2\mu, \tag{6.1}$$

then

$$q^{-\frac{1}{8}}C(q) = \left[\left(\mu^2+2\right)^{\frac{1}{2}} - \mu\right]^{\frac{1}{8}}. \tag{6.2}$$

Proof. Let

$$J = q^{-1}C^8(q),$$

then (6.1) is

$$J - \frac{2}{J} = -2\mu, \quad \text{or} \quad J^2 + 2\mu J - 2 = 0.$$

Solving for J

$$J = \left(\mu^2+2\right)^{\frac{1}{2}} - \mu$$

$$\left[q^{-\frac{1}{8}}C(q)\right]^8 = \left(\mu^2+2\right)^{\frac{1}{2}} - \mu$$

$$q^{-\frac{1}{8}}C(q) = \left[\left(\mu^2+2\right)^{\frac{1}{2}} - \mu\right]^{\frac{1}{8}}.$$

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ON THE HYPERBANACH SPACES

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Abstract. In this paper we are going to define hyperBanach spaces and prove some interesting theorems such as open mapping theorem, closed graph theorem, and uniform boundedness principal in these spaces. Also we define a quasinorm over hypervector spaces that converts a factor hypervector space into a normed hyper vector space.

Keywords: norm; open map; banach space; hypervector space.

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1. Introduction

In 1934 Marty [3] introduced a new mathematical structure as a generalization of groups and called it hypergroup. Subsequently, many authors worked on this new field and constructed some other generalizations such as hyperrings, hypermodules, and hyperfields. In 1988 the notion of hypervector space was given by Tallini [11]. She studied some algebraic properties of this new structure in [8], [9], and [10]. A wealth of applications of these new constructions in geometry, hypergraphs, binary relations, combinatorics, codes, cryptography, probability, and etc. can be found in [2]. Recently, we studied hypervector spaces in the viewpoint of analysis and generalized some definitions and proved many interesting theorems about them in [5], [6], and [7]. In this paper we are going to define hyperBanach space and prove open mapping theorem, closed graph theorem, and uniform boundedness principal which are very important and have key roles in

Banach space theory for this new space. Also we define a quasinorm over hypervector spaces that converts a factor hypervector space into a normed hyper vector space.

Let $P(X)$ be the power set of a set X, $P^\star(X) = P(X)\backslash\{\varnothing\}$, and K a field. A *hypervector space* over K that is defined in [8], is a quadruplet $(X, +, \circ, K)$ such that $(X, +)$ is an abelian group and

$$\circ : K \times X \longrightarrow P^\star(X)$$

is a mapping that for all a, $b \in K$ and x, $y \in X$ the following properties holds:

(i) $(a + b) \circ x \subseteq (a \circ x) + (b \circ x)$,

(ii) $a \circ (x + y) \subseteq (a \circ x) + (a \circ y)$,

(iii) $a \circ (b \circ x) = (ab) \circ x$, where $a \circ (b \circ x) = \{\, a \circ y \;:\; y \in b \circ x \,\}$,

(iv) $(-a) \circ x = a \circ (-x)$

(v) $x \in 1 \circ x$.

Note that every vector space is a hypervector space and specially, every field is a hypervector space over itself.

A non-empty subset of a hypervector space X over a field K is called a *subspace* of X if the following holds:

(i) $H - H \subseteq H$,

(ii) $a \circ H \subseteq H$, for every $a \in K$.

Note that $a \circ A = \bigcup\limits_{x \in A} a \circ x$, for every $k \in K$ and $A \subseteq X$.

If H is a subspace of X, the *factor hypervector space* of X with respect to H that is defined in [8], is denoted by $(X/Y,\ +,\ *,\ K)$ and is a hypervector space with the elements

$$\{\, [x] = x + Y \;:\; x \in X \,\},$$

and for every $a \in K$,

$$a * [x] = [a \circ x] = \{\, [y] \;:\; y \in a \circ x \,\}.$$

Let $(X, +, \circ, K)$ be a hypervector space, where K is a valued field. Suppose that for every $a \in K$, $|a|$ denoted the valuation of a in K. A *pseudonorm* on X that is defined in [9], is a mapping

$$|| \cdot || : X \longrightarrow \mathbb{R}$$

that for all $a \in K$ and x, $y \in X$ has the following properties:

(i) $||0|| = 0$,

(ii) $||x + y|| \leqslant ||x|| + ||y||$,

(iii) $\sup ||a \circ x|| = |a|\, ||x||$.

A pseudonorm on X is called a *norm*, if:

$$||x|| = 0 \quad \Longleftrightarrow \quad x = 0.$$

Let $(X, +_1, \circ_1, K)$ and $(Y, +_2, \circ_2, K)$ be two hypervector spaces. A *strong homomorphism* between X and Y is a mapping

$$f : X \longrightarrow Y$$

such that for all $a \in K$ and x, $y \in X$ the following hold:

(i) $f(x +_1 y) = f(x) +_2 f(y)$,

(ii) $f(a \circ_1 x) = a \circ_2 f(x)$.

A strong homomorphism $f : X \longrightarrow Y$, where $X = (X, +_1, \circ_1, ||\ .\ ||_1, K)$ and $Y = (Y, +_2, \circ_2, ||\ .\ ||_2, K)$ are two normed hypervector spaces is called *bounded* if there exists $M \geqslant 0$ such that $||f(x)||_2 \leqslant M||x||_1$, for every $x \in X$.

As we define in [7], a subset A of X is called *convex* if $t \circ x + (1 - t) \circ y \subseteq A$, for every x, $y \in A$ and $0 \leqslant t \leqslant 1$. If $kA \subseteq A$, for every k, $|k| \leqslant 1$, then A is called *balanced*. Also the set A is *absorbing* if for each $x \in X$, there is a positive number s_x, such that $x \in t \circ A$ whenever $t > s_x$.

Let $(X, +, \circ, ||\ .\ ||, K)$ be a normed hypervector space. For $x \in X$ and $\epsilon > 0$ the *open ball* $B_\epsilon(x)$ is defined as

$$B_\epsilon(x) = \{\, y \in X \ \ : \ \ ||x - y|| < \epsilon \,\},$$

and the unit ball is the open ball with radius equals to 1. Furthermore, the *closed ball*, $C_\epsilon(x)$ is defined as

$$C_\epsilon(x) = \{\, y \in X \ \ : \ \ ||x - y|| \leqslant \epsilon \,\}.$$

The $\{\, B_\epsilon(x) \ : \ x \in X, \epsilon > 0 \,\}$ is a basis for a topology on X which is the topology induced by this norm. The set of all interior points of A is denoted by A°. Also the closure of A denoted by $\overline{A}$.

As we define in [5], a sequence $\{x_n\}$ in X is said to be a *Cauchy sequence* if for every $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $||x_n - x_m|| < \epsilon$, for every m, $n \geqslant N$.

2. HyperBanach spaces

Throughout this section K will denote either the real field, $\mathbb{R}$, or the complex field, $\mathbb{C}$.

Definition 2.1. A normed hypervector space $X = (X, +, \circ, ||\cdot||, K)$ is called a hyperBanach space if every Cauchy sequence in X is convergent.

Example 2.2. Consider the following hypervector space that is defined in [8]:

Let $(\mathbb{R}^n, +)$ be the classical additive group over $\mathbb{R}^n$ and for every $a \in \mathbb{R}$ let

$$a \circ x = \{\, tax \,:\, 0 \leqslant t \leqslant 1\,\},$$

where tax is the classical multiplication of $\mathbb{R}$ over $\mathbb{R}^n$.

Now, let $||x||$ be the distance of x from the origin in $\mathbb{R}^n$. Then it is easily seen that $(\mathbb{R}^n, +, \circ, ||\cdot||, \mathbb{R})$ is a hyperBanach space.

Theorem 2.3. *Let $X = (X, +, \circ, ||\,.\,||, K)$ be a hyperBanach space such that $k \circ 0 = \{0\}$, for every $k \in K$. Then every closed, convex, and absorbing subset of X includes a neighborhood of the origin.*

Proof. Let C be a closed, convex, and absorbing subset of a hyperBanach space X and let $D = C \cap (-C)$, where $-C$ denotes $\{\,-x : x \in C\}$. It is enough to show that D includes a neighborhood of the origin. If A is a non-empty subset of D, then we have

$$0 \in \frac{1}{2} \circ (A - A) \subseteq \frac{1}{2} \circ A + \frac{1}{2} \circ (-A) \subseteq \frac{1}{2} \circ D + \frac{1}{2} \circ (-D) = \frac{1}{2} \circ D + \frac{1}{2} \circ D \subseteq D,$$

because D is convex. Since the neighborhood $\frac{1}{2} \circ D^\circ + \frac{1}{2} \circ (-D^\circ)$ of the origin must be included in D, so it is enough to prove that $D^\circ \neq \varnothing$.

By contradiction, suppose that $D^\circ = \varnothing$. For each $n \in \mathbb{N}$, the set nD is closed and has empty interior, where

$$nD = \underbrace{D + \cdots + D}_{n \text{ times}},$$

and so $X \setminus nD$ is an open set that is dense in X. Suppose that B_1 is a closed ball in $X \setminus D$ with radius no more than 1. Since $(X \setminus 2D) \cap B_1^\circ$ is a non-empty open set, there is a closed ball B_2 in $B_1 \setminus 2D$ with radius no more than $\frac{1}{2}$. There is a closed ball B_3 in $B_2 \setminus 3D$ with radius no more than $\frac{1}{3}$. Continuing in the obvious way, we find a sequence $\{B_n\}$ of closed balls such that for every $n \in \mathbb{N}$, $B_n \cap nD = \varnothing$, the radius of B_n is no more than $\frac{1}{n}$, and $B_m \subseteq B_n$ if $n \leqslant m$. It follows that the centers of the balls form a Cauchy sequence whose limit x is in each of the balls and hence is in $X \setminus nD$, for every n. Since C is absorbing, there is a positive real number s such that if $t > s$ then x, $-x \in t \circ C$ and therefore $x \in t \circ D$. It implies that $x \in nD$, for some $n \in \mathbb{N}$, a contradiction. This proves the theorem. ■

Definition 2.4. Let $X = (X, +, \circ, K)$ be a hypervector space. A prenorm on X is a positive real valued function p on X such that the following conditions are satisfied by all members x and y of X and each scalar α:

(i) $p(0) = 0$,

(ii) $\sup p(\alpha \circ x) \leqslant |\alpha| p(x)$,

(iii) $p(x + y) \leqslant p(x) + p(y)$,

(iv) $p(x - y) = p(y - x)$.

Definition 2.5. A function f from a normed hypervector space X into the nonnegative reals is countably subadditive if

$$f\left(\sum_{n=1}^{\infty} x_n\right) \leqslant \sum_{n=1}^{\infty} f(x_n),$$

for each convergent series $\sum\limits_{n=1}^{\infty} x_n$ in X.

Theorem 2.6. *Let* $X = (X, +, \circ, ||\ .\ ||, K)$ *be a hyperBanach space such that* $k \circ 0 = \{0\}$, *for every* $k \in K$, $y \in k^{-1} \circ x$, *for* $x \in k \circ y$, $0 \neq k \in K$, *and* $x, y \in X$, *and* $k \circ \overline{A} \subseteq \overline{k \circ A}$, *for every* $k \in K$, , *and* $A \subseteq X$. *Then every countably subadditive prenorm on* X *is continuous.*

Proof. Let p be a countably prenorm on X. Suppose that $x, y \in X$. Then $p(x) \leqslant p(x - y) + p(y)$, and $p(y) \leqslant p(y - x) + p(x) = p(x - y) + p(x)$. So

$$|p(x) - p(y)| = \max\{p(x) - p(y), p(y) - p(x)\} \leqslant p(x - y) = |p(x - y) - p(0)|.$$

Therefore if p is continuous at 0 and x is an element of X, then p is continuous at x, too. Thus it is enough to show that p is continuous at 0.

Let $G = \{x : x \in X,\ p(x) < 1\}$. If $t > 0$, then

$$t \circ G = \bigcup_{x \in G} t \circ x \subseteq \{x : x \in X,\ p(x) < t\}.$$

Thus G is absorbing. If $x, y \in G$ and $0 \leqslant t \leqslant 1$, then

$$\sup p(t \circ x + (1 - t) \circ y) \leqslant \sup p(t \circ x) + \sup p((1 - t) \circ y) \leqslant tp(x) + (1 - t)p(y) < 1,$$

so G is convex. Therefore $\overline{G}$ is a closed, convex, and absorbing subset of X and by Theorem 2.3, it includes an open ball U centered at 0 with some positive radius ϵ. Suppose that there is a positive real number s such that $p(x) < s$ whenever $||x|| < \epsilon$. Now, if $x \in X$ is such that $||X|| < s^{-1}t\epsilon$, then $\sup ||(st^{-1}) \circ x|| < \epsilon$. It shows that for every $y \in (st^{-1}) \circ x$, $p(y) < s$, and therefore $\sup p((s^{-1}t) \circ y) \leqslant (s^{-1}t)p(y) < t$, and so $p(x) < t$. It implies the continuity of p at 0. Thus to complete the proof it is enough to show that such an s exists.

Fix an x in X such that $||x|| < \epsilon$. Since $x \in U \subseteq \overline{G}$, there is $x_1 \in G$ such that $||x - x_1|| < 2^{-1}\epsilon$. Since

$$x - x_1 \in 2^{-1} \circ U \subseteq 2^{-1} \circ \overline{G} \subseteq \overline{2^{-1} \circ G},$$

there is $x_2 \in 2^{-1} \circ G$ such that $||x - x_1 - x_2|| < 2^{-2}\epsilon$. Similarly, there is $x_3 \in 2^{-2} \circ G$ such that $||x - x_1 - x_2 - x_3|| < 2^{-3}\epsilon$. Continuing in this way, we find a sequence $\{x_n\}$ such that $x_n \in 2^{-n+1} \circ G$ and $\left\|x - \sum_{i=1}^{n} x_i\right\| < 2^{-n}\epsilon$, for every $n \in \mathbb{N}$. It follows that $p(x_n) < 2^{-n+1}$, for every $n \in \mathbb{N}$, and $x = \sum_{n=1}^{\infty} x_n$, and so the countable subadditivity of p implies that

$$p(x) = p\left(\sum_{n=1}^{\infty} x_n\right) \leqslant \sum_{n=1}^{\infty} p(x_n) < 2.$$

Put $s = 2$ and the proof is complete. ∎

Example 2.7. In Example 2.2 it is not hard to see that $a \circ 0 = \{0\}$, for every $a \in \mathbb{R}$, $y \in a^{-1} \circ x$, for $x \in a \circ y$ whenever $x, y \in \mathbb{R}^n$ and $0 \neq a \in \mathbb{R}$. Now, we show that $\alpha \circ \overline{A} \subseteq \overline{\alpha \circ A}$, for every $\alpha \in \mathbb{R}$ and $A \subseteq \mathbb{R}^n$. Suppose that $\alpha \in \mathbb{R}$ and $A \subseteq \mathbb{R}^n$ are arbitrary. Let $x \in \alpha \circ \overline{A}$. So there is $y \in \overline{A}$ such that $x \in \alpha \circ y$. It means that there is t_0, $0 \leqslant t_0 \leqslant 1$, such that $x = \alpha t_0 y$. Let $r_0 > 0$ be arbitrary. Since $y \in \overline{A}$, then there is $a \in A$ such that

$$||y - a|| < \frac{r_0}{|\alpha||t_0|}.$$

Put $z = \alpha t_0 a \in \alpha \circ A$. We have

$$||x - z|| = ||\alpha t_0 y - \alpha t_0 a|| = |\alpha t_0|||y - a|| < r_0.$$

Therefore $z \in B_{r_0}(x) \cap \alpha \circ A$, and $x \in \overline{\alpha \circ A}$. So $(\mathbb{R}^n, +, \circ, || \cdot ||, \mathbb{R})$ is a normed hypervector space satisfying the hypothesis of Theorem 2.6.

We say the series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent if $\sum_{n=1}^{\infty} ||x_n||$ is a convergent series. The following lemma can be proved similar as the normed vector spaces. So we omit its proof.

Lemma 2.8. *Let $X = (X, +, \circ, || \cdot ||, K)$ be a normed hypervector space. Then X is a hyperBanach space if and only if every absolutely convergent series in X is convergent.*

A function f from a topological space X into a topological space Y is an open mapping if $f(U)$ is an open subset of Y, for every open subset U of X.

Theorem 2.9. (Open Mapping Theorem) *Let $X = (X, +_1, \circ_1, || \,.\, ||_1, K)$ be a hyperBanach space and $Y = (Y, +_2, \circ_2, || \,.\, ||_2, K)$ be a hyperBanach space such that $k \circ_2 0 = \{0\}$, for every $k \in K$, $y \in k^{-1} \circ_2 x$, for $x \in k \circ_2 y$, $0 \neq k \in K$, and $x, y \in Y$, and $k \circ_2 \overline{A} \subseteq \overline{k \circ_2 A}$, for every $k \in K$, and $A \subseteq Y$. Then every bounded strong homomorphism from X onto Y is an open mapping.*

Proof. Let $T : X \longrightarrow Y$ be an onto bounded strong homomorphism. Suppose that the image under T of the unit ball U of X is open. Let V be an open subset of X. If $x \in V$, then $x +_1 r \circ_1 U \subseteq V$, for some $r > 0$, and so $T(V)$ includes the neighborhood $T(x) +_2 r \circ_2 T(U)$ of $T(x)$. Thus it is enough to show that $T(U)$ is an open set.

For each $y \in Y$, let $p(y) = \inf\{\, ||x||_1 \,:\, x \in X, T(x) = y\,\}$. For $y \in Y$ and $\alpha \in K$, we have if $w \in X$ such that $T(w) = y$, then

$$\{\inf\{||x||_1 \,:\, x \in X, T(x) = z\} \,:\, z \in \alpha \circ_2 T(w) = T(\alpha \circ_1 w)\} \subseteq ||\alpha \circ_1 w||_1.$$

Hence

$$\begin{aligned} \sup p(\alpha \circ_2 y) &= \sup\{\inf\{||x||_1 : x \in X, T(x) = z\} : z \in \alpha \circ_2 T(w) = T(\alpha \circ_1 w)\} \\ &\leqslant \sup ||\alpha \circ_1 w||_1 = |\alpha|||w||_1, \end{aligned}$$

and therefore

$$\sup p(\alpha \circ_2 y) \leqslant |\alpha| \inf\{\, ||w||_1 \,:\, w \in X, T(w) = y\,\} = |\alpha| p(y).$$

Now, let $\sum_{n=1}^{\infty} y_n$ converges in Y. For an arbitrary $\epsilon > 0$, let $\{x_n\}$ be a sequence in X that $T(x_n) = y_n$ and $||x_n|| < p(y_n) + 2^{-n}\epsilon$, for every $n \in \mathbb{N}$. Then $\sum_{n=1}^{\infty} ||x_n||_1 < \sum_{n=1}^{\infty} p(y_n) + \epsilon$, a finite number. Since X is a hyperBanach space, the absolutely convergent series $\sum_{n=1}^{\infty} x_n$ is convergent. Now,

$$T\left(\sum_{n=1}^{\infty} x_n\right) = \sum_{n=1}^{\infty} T(x_n) = \sum_{n=1}^{\infty} y_n,$$

and so

$$p\left(\sum_{n=1}^{\infty} y_n\right) \leqslant \left\|\sum_{n=1}^{\infty} x_n\right\|_1 \leqslant \sum_{n=1}^{\infty} ||x_n||_1 < \sum_{n=1}^{\infty} p(y_n) + \epsilon.$$

So, p is countably subadditive.

The other properties of the prenorm can easily be checked.

Thus, p is a countably subadditive prenorm on Y, and by Theorem 2.6, it is continuous. Finally,

$$T(U) = \{y \,:\, y \in Y, T(x) = y, \text{for some}\, x \in U\,\} = \{\, y \,:\, y \in Y, p(y) < 1\,\},$$

so $T(U)$ is open and the proof is complete. ■

A strongly homomorphism T between two normed hypervector spaces is called and *isomorphism* if it is one-to-one and continuous and its inverse mapping T^{-1} is continuous on the range of T.

Corollary 2.10. *Let* $X = (X, +_1, \circ_1, || \cdot ||_1, K)$ *be a hyperBanach space and* $Y = (Y, +_2, \circ_2, || \cdot ||_2, K)$ *a hyperBanach space be such that* $k \circ_2 0 = \{0\}$*, for every* $k \in K$*,* $y \in k^{-1} \circ_2 x$*, for* $x \in k \circ_2 y$*,* $0 \neq k \in K$*, and* $x, y \in Y$*, and* $k \circ_2 \overline{A} \subseteq \overline{k \circ_2 A}$*, for every* $k \in K$*, and* $A \subseteq Y$*. Then every one-to-one bounded strongly homomorphism from* X *onto* Y *is an isomorphism.*

Theorem 2.11. (The Uniform Boundedness Principle Theorem) *Suppose that* $X = (X_1, +_1, \circ_1, || \, . \, ||_1, K)$ *is a hyperBanach space such that* $k \circ_1 0 = \{0\}$*, for every* $k \in K$*,* $y \in k^{-1} \circ_1 x$*, for* $x \in k \circ_1 y$*,* $0 \neq k \in K$*, and* x*,* $y \in X$*, and* $k \circ_1 \overline{A} \subseteq \overline{k \circ_1 A}$*, for every* $k \in K$*, and* $A \subseteq X$*, and* $Y = (Y, +_2, \circ_2, || \cdot ||_2, K)$ *a normed hypervector space. Let* $\mathfrak{F}$ *be a non-empty family of bounded strong homomorphisms from* X *into* Y*. If* $\sup\{\, ||T(x)|| \, : \, T \in \mathfrak{F}\, \}$ *is finite for every* $x \in X$*, then* $\sup\{||T|| : T \in \mathfrak{F}\}$ *is finite.*

Proof. Let $p(x) = \sup\{||T(x)||_2 : T \in \mathfrak{F}\}$, for every $x \in X$. Then we have

$$\begin{aligned}
\sup p(\alpha \circ_1 x) &= \sup\{\sup\{||T(y)||_2 : T \in \mathfrak{F}\} : \, y \in \alpha \circ_1 x\} \\
&= \sup\{|\alpha| ||T(x)||_2 : T \in \mathfrak{F}, \} = |\alpha| \sup\{||T(x)||_2 : T \in \mathfrak{F}\} \\
&= |\alpha| p(x).
\end{aligned}$$

If $\sum\limits_{n=1}^{\infty} x_n$ is a convergent series in X and $T \in \mathfrak{F}$, then

$$\left\| T\left(\sum_{n=1}^{\infty} x_n\right) \right\|_2 = \left\| \sum_{n=1}^{\infty} T(x_n) \right\|_2 \leqslant \sum_{n=1}^{\infty} ||T(x_n)||_2 \leqslant \sum_{n=1}^{\infty} p(x_n),$$

from which it follows that

$$p\left(\sum_{n=1}^{\infty} x_n\right) \leqslant \sum_{n=1}^{\infty} p(x_n).$$

So p is countably subadditive. Also we have

$$||T(x - Y)||_2 = \sup ||(-1) \circ_2 \, T(y - x)||_2 = ||T(y - x)||_2,$$

for every x, $y \in X$. So $p(x - y) = p(y - x)$. Since $p(0) = 0$, then p is a countably subadditive prenorm on X. Therefore, by Theorem 2.6, p is continuous and there is $\delta > 0$ such that $p(x) \leqslant 1$ whenever $||x||_1 < \delta$. It follows that $p(x) \leqslant \delta^{-1}$ whenever $||x||_1 < 1$, and therefore $||T(x)||_2 \leqslant \delta^{-1}$ whenever $T \in \mathfrak{F}$ and $||x||_1 < 1$, that is $||T||_2 \leqslant \delta^{-1}$, for each $T \in \mathfrak{F}$. The proof is complete. ∎

Theorem 2.12. (Closed Graph Theorem) *Let* $X = (X_1, +_1, \circ_1, || \, . \, ||_1, K)$ *be a hyperBanach space such that* $k \circ_1 0 = \{0\}$*, for every* $k \in K$*,* $y \in k^{-1} \circ_1 x$*, for* $x \in k \circ_1 y$*,* $0 \neq k \in K$*, and* x*,* $y \in X$*, and* $k \circ_1 \overline{A} \subseteq \overline{k \circ_1 A}$*, for every* $k \in K$*, and* $A \subseteq X$*,* $(Y, +_2, \circ_2, || \, . \, ||_2, K)$ *a hyperBanach space, and* T *a strong homomorphism* *from X into Y. Suppose that whenever a sequence $\{x_n\}$ in X converges to some x in X and $\{T(x_n)\}$ converges to some y in Y, it follows that $y = T(x)$. Then T is bounded.*

Proof. Let $p(x) = ||T(x)||_2$, for every $x \in X$. If we prove that p is continuous, then there is a neighborhood U of 0 such that the set $p(U)$ is bounded and therefore $T(U)$ is bounded. Let $r > 0$ be small enough that the closed ball of radius r and center 0, $C_r(0)$, is include in U, and let $M_0 = \sup\{\, ||T(x)||_2 \, : \, x \in C_r(0)\,\}$. If $0 \neq x \in X$, then

$$(r||x||_1^{-1}) \circ x \subseteq C_r(0),$$

and so

$$(r||x||_1^{-1})||T(x)||_2 = \sup ||T((r||x||_1^{-1}) \circ_1 x)|| \leqslant M_0,$$

and therefore $||T(x)||_2 \leqslant r^{-1}M_0||x||_1$. So T is bounded. Hence by Theorem 2.6, it is enough to show that p is a countably subadditive prenorm on X. Clearly, $\sup p(k \circ_1 x) = |k|p(x)$, $p(0) = 0$, and by the proof of the previous theorem, $p(x-y) = p(y-x)$. So, let $\sum_{n=1}^{\infty} x_n$ be a convergent series in X. Without loss of generality, we may assume that

$$\sum_{n=1}^{\infty} ||T(x_n)||_2 < \infty.$$

Since Y is a hyperBanach space, then $\sum_{n=1}^{\infty} T(x_n)$ converges. Also by the hypothesis,

$$\lim_{n \longrightarrow \infty} \sum_{n=1}^{m} x_n = \sum_{n=1}^{\infty} x_n,$$

and

$$\lim_{n \longrightarrow \infty} T\left(\sum_{n=1}^{m} x_n\right) = \sum_{n=1}^{m} T(x_n) = \sum_{n=1}^{\infty} T(x_n)$$

imply that

$$\sum_{n=1}^{\infty} T(x_n) = T\left(\sum_{n=1}^{\infty} x_n\right).$$

Therefore

$$\left\| T\left(\sum_{n=1}^{\infty} x_n\right)\right\|_2 = \left\| \sum_{n=1}^{\infty} T(x_n)\right\|_2 \leqslant \sum_{n=1}^{\infty} ||T(x_n)||_2,$$

which shows that p is countably subadditive and the proof is complete. ■

3. Quasi norm and factor hypervector spaces

Definition 3.1. Let $X = (X, +, \circ, K)$ be a hypervector space. Suppose that for every $a \in K$, $|a|$ denoted the valuation of a in K. A quasinorm in X is a mapping

$$||\cdot|| : X \longrightarrow \mathbb{R}$$

that for all $a \in K$ and x, $y \in X$ has the following properties:

(i) $||x|| = 0 \Longleftrightarrow x = 0$,

(ii) $||x + y|| \leqslant ||x|| + ||y||$,

(iii) $\sup ||a \circ x|| \leqslant |a|\, ||x||$.

Definition 3.2. A quasinormed hypervector space $X = (X, +, \circ, ||\ .\ ||, K)$ is called well if for every $x \in X$ and a, $b \in K$,

$$\inf ||a \circ x|| \leqslant \sup ||b \circ x||.$$

Example 3.3. It is easily seen that $(\mathbb{R}^n, +, \circ, ||\cdot||, \mathbb{R})$ that is defined in Example 2.2, is a quasinormed hypervector space that is well, since for every $x \in \mathbb{R}^n$ and $a \in \mathbb{R}$, $\inf ||a \circ x|| = 0$.

Theorem 3.4. *Let $X = (X, +, \circ, ||\cdot||, K)$ be a well normed hypervector space and Y be a closed subspace of X. Then $(X/Y, +, \circ, ||\cdot||, K)$ is a quasinormed hypervector space, where for every $[x] \in X/Y$,*

$$||[x]|| = \inf\{||x + y|| : y \in Y\}.$$

Proof. Let $\lambda \in K$ and $[x]$, $[y] \in X/Y$. If $[x] = [0]$, then $x \in Y$ and therefore

$$0 \leqslant ||[x]|| \leqslant ||x - x|| = 0.$$

Conversely, $||[x]|| = 0$ implies that there is a sequence $\{x_n\}$ in Y such that $\lim_{n\to\infty} ||x + x_n|| = 0$. Since Y is closed, it follows that x is in Y so $[x] = [0]$. Further, we have

$$\begin{aligned}
||[x] + [y]|| &= ||[x + y]|| = \inf\{\, ||x + y + z|| \,:\, z \in Y\,\} \\
&\leqslant \inf\{\, ||x + z_1 + y + z_2|| \,:\, z_1,\, z_2 \in Y\,\} \\
&\leqslant \inf\{\, ||x + z_1|| \,:\, z_1 \in Y\,\} + \inf\{\, ||y + z_2|| \,:\, z_2 \in Y\,\} \\
&= ||[x]|| + ||[y]||.
\end{aligned}$$

At last,

$$\begin{aligned}\sup||\lambda\circ[x]|| &= \sup\{\,||[z]||\,:\,z\in\lambda\circ x\,\}\\ &= \sup\{\inf\{||z+y||\,:\,y\in Y\,:\,z\in\lambda\circ x\,\}\\ &\leqslant \sup\{\inf\{\,||\lambda\circ x+\lambda\circ y||\,:\,y\in Y\,\}\\ &\leqslant \inf\{\sup\{\,||\lambda\circ x+\lambda\circ y||\,:\,y\in Y\,\}\\ &\leqslant \inf\{\lambda||x+y||\,:\,y\in Y\,\}=\lambda||[x]||.\end{aligned}$$

Therefore $||\ .\ ||$ is a quasinorm on X/Y and the proof is complete. ■

Finally, the following lemmas can be proved easily similar as the normed vector spaces. So, we omit their proofs.

Theorem 3.5. *Let Y be a closed subspace of a well normed hypervector space $X=(X,+,\circ,||\ .\ ||,K)$ and $F:X\longrightarrow X/Y$ the quotient map defined by $F(x)=x+Y$. Then F is continuous and maps open sets in X onto open sets in X/Y.*

Theorem 3.6. *Let Y be a closed subspace of a well normed hypervector space $X=(X,+,\circ,||\ .\ ||,K)$. If X is a hyperBanach space, then so is X/Y.*

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ON THE QUALITATIVE BEHAVIORS OF SOLUTIONS TO A KIND OF NONLINEAR THIRD ORDER DIFFERENTIAL EQUATIONS WITH RETARDED ARGUMENT

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Abstract. In this paper, with use of a Lyapunov functional, we discuss stability and boundedness of solutions to a kind of nonlinear third order differential equation with retarded argument:

$$\begin{aligned}&x'''(t)+h(x(t),x'(t),x''(t),x(t-r(t)),x'(t-r(t)),x''(t-r(t)))x''(t)\\&\quad+g(x(t-r(t)),x'(t-r(t)))+f(x(t-r(t)))\\&\quad=p(t,x(t),x'(t),x(t-r(t)),x'(t-r(t)),x''(t)),\end{aligned}$$

when $p(t,x(t),x'(t),x(t-r(t)),x'(t-r(t)),x''(t))=0$ and $\neq 0$, respectively. Our results include and improve some well-known results in the literature. An example is also given to illustrate the importance of results obtained and the topic.

Keywords: stability, boundedness, Lyapunov functional, nonlinear third order differential equations, retarded argument.

AMS (MOS) Subject Classification: 34K20.

1. Introduction

In a recent paper, Afuwape and Omeike [1] discussed stability and boundedness of solutions to nonlinear third order delay differential equation:

$$\begin{aligned}&x'''(t)+h(x'(t))x''(t)+g(x(t-r(t)),x'(t-r(t)))+f(x(t-r(t)))\\&\quad=p(t,x(t),x'(t),x''(t)),\end{aligned}$$

when $p(t,x(t),x'(t),x''(t))\equiv 0$ and $\neq 0$, respectively.

In this paper, we consider nonlinear third order differential equation with retarded argument, $r(t)$:

$$\begin{aligned} & x'''(t)+h(x(t),x'(t),x''(t),x(t-r(t)),x'(t-r(t)),x''(t-r(t)))x''(t) \\ & \quad +g(x(t-r(t)),x'(t-r(t)))+f(x(t-r(t))) \\ & \quad =p(t,x(t),x'(t),x(t-r(t)),x'(t-r(t)),x''(t)), \end{aligned} \tag{1}$$

which is equivalent to the system

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= z(t), \\ z'(t) &= -h(x(t),y(t),z(t),x(t-r(t)),y(t-r(t)),z(t-r(t)))z(t) \\ & \quad -g(x(t),y(t))-f(x(t))+\int_{t-r(t)}^{t} g_x(x(s),y(s))y(s)ds \\ & \quad +\int_{t-r(t)}^{t} g_y(x(s),y(s))z(s)ds+\int_{t-r(t)}^{t} f'(x(s))y(s)ds \\ & \quad +p(t,x(t),y(t),x(t-r(t)),y(t-r(t)),z(t)), \end{aligned} \tag{2}$$

where $0 \le r(t) \le \gamma$, γ is a positive constant which will be determined later, and $r'(t) \le \beta$, $0<\beta<1$; the primes in equation (1) denote differentiation with respect to t, $t \in \Re^+$, $\Re^+=[0,\infty)$; h, g, f and p are continuous functions in their respective arguments on $\Re^6$, $\Re^2$, $\Re$ and $\Re^+ \times \Re^5$, respectively, with $g(x,0)=f(0)=0$, in the statement of Theorem 1. The continuity of functions h, g, f and p guarantees the existence of the solution of equation (1) (see [3, pp.14]). In addition, it is also supposed that the derivatives $g_x(x,y) \equiv \frac{\partial}{\partial x}g(x,y)$, $g_y(x,y) \equiv \frac{\partial}{\partial y}g(x,y)$ and $f'(x) \equiv \frac{df}{dx}$ exist and are continuous; all solutions of (1) are real valued and the functions h, g, f and p satisfy a Lipschitz condition in x, y, z, $x(t-r(t))$, $y(t-r(t))$ and $z(t-r(t))$. Then the solution is unique (see [3, pp.14]). Throughout the paper $x(t)$, $y(t)$ and $z(t)$ are abbreviated as x, y and z, respectively.

The motivation for the present work has been inspired basically by the paper of Afuwape and Omeike [1], Sadek [9] and Tunç ([11], [12]). Our aim here is to extend and improve the results established by Afuwape and Omeike [1] to nonlinear differential equation with retarded argument (1) for the asymptotic stability of trivial solution and boundedness of all solutions of this equation, when $p \equiv 0$ and $p \neq 0$ in (1), respectively. We also give an explanatory example for the illustration of the subject. All aforementioned papers have been published without including an explanatory example on the stability and boundedness of solutions of third order nonlinear differential equations with retarded argument. In addition, to the best of our knowledge, so far throughout all the papers published on the subject of this paper, the second term in (1) has only consisted of $ax''(t)$, $a_1x''(t)$, $\alpha\, x''(t)$, (a, a_1 and α are constants), $h(x'(t))x''(t)$, $\varphi(x(t),x'(t))x''(t)$, $a(t)x''(t)$, $f(x(t),x'(t),x''(t))x''(t)$ or $a(t)f(x(t),x'(t))x''(t)$ (see [1], [9], [11], [12] and the references registered thereof). But, our second term has the form

$$h(x(t),x'(t),x''(t),x(t-r(t)),x'(t-r(t)),x''(t-r(t)))x''(t).$$

This case, clearly, is an improvement.

2. Preliminaries

Now, we will give some basic information for the general non-autonomous differential system with retarded argument (see, also, the books of Èl'sgol'ts [3], Hale [4], Kolmanovskii and Myshkis [5], Kolmanovskii and Nosov [6], Krasovskii [7] and Yoshizawa [13]). Consider the general non-autonomous differential system with retarded argument:

$$\dot{x} = f(t, x_t), x_t = x(t+\theta), -r \le \theta \le 0, t \ge 0, \tag{3}$$

where $f : [0,\ \infty) \times C_H \to \Re^n$ is a continuous mapping, $f(t, 0) = 0$, and we suppose that f takes closed bounded sets into bounded sets of $\Re^n$. Here $(C,\ \|.\ \|)$ is the Banach space of continuous function $\phi : [-r,\ 0] \to \Re^n$ with supremum norm, $r > 0$; C_H is the open H-ball in C; $C_H := \{\phi \in (C\,[-r, 0]\,,\ \Re^n) : \|\phi\| < H\}$.

Definition 1. (See [13].) A function $x(t_0, \phi)$ is said to be a solution of (3) with the initial condition $\phi \in C_H$ at $t = t_0$, $t_0 \ge 0$, if there is a constant $A > 0$ such that $x(t_0, \phi)$ is a function from $[t_0 - r,\ \ t_0 + A]$ into $\Re^n$ with the properties:

(i) $x_t(t_0, \phi) \in C_H$ for $t_0 \le t < t_0 + A$,

(ii) $x_{t_0}(t_0, \phi) = \phi$,

(iii) $x(t_0, \phi)$ satisfies (3) for $t_0 \le t < t_0 + A$.

Standard existence theory, see Burton [2], shows that if $\phi \in C_H$ and $t \ge 0$, then there is at least one continuous solution $x(t, t_0, \phi)$ such that on $[t_0,\ t_0 + \alpha)$ satisfying equation (3) for $t > t_0$, $x_t(t, \phi) = \phi$ and α is a positive constant. If there is a closed subset $B \subset C_H$ such that the solution remains in B, then $\alpha = \infty$. Further, the symbol $|\cdot|$ will denote a convenient norm in $\Re^n$ with $|x| = \max\limits_{1 \le i \le n} |x_i|$.

Definition 2. (See [2].) A continuous function $W : [0,\ \infty) \to [0,\ \infty)$ with $W(0) = 0$, $W(s) > 0$ if $s > 0$, and W strictly increasing is a wedge. (We denote wedges by W or W_i, where i an integer.)

Definition 3. (See [2].) Let D be an open set in $\Re^n$ with $0 \in D$. A function $V : [0,\ \infty) \times D \to [0,\ \infty)$ is called positive definite if $V(t, 0) = 0$ and if there is a wedge W_1 with $V(t, x) \ge W_1(|x|)$, and is called decrescent if there is a wedge W_2 with $V(t, x) \le W_2(|x|)$.

Definition 4. (See [2].) Let $f(t, 0) = 0$. The zero solution of equation (3) is:

(i) stable if for each $\varepsilon > 0$ and $t_1 \ge t_0$ there exists $\delta > 0$ such that $[\phi \in C(t_1),\ \ \|\phi\| < \delta, t \ge t_1]$ implies that $|x(t, t_1, \phi)| < \varepsilon$.

(ii) asymptotically stable if it is stable and if for each $t_1 \ge t_0$ there is an $\eta > 0$ such that $[\phi \in C(t_1),\ \ \|\phi\| < \delta]$ implies that $x(t, t_0, \phi) \to 0$ as $t \to \infty$.

Definition 5. (See [2].) Let $V(t,\phi)$ be a continuous functional defined for $t \geq 0$, $\phi \in C_H$. The derivative of V along solutions of (3) will be denoted by $\dot{V}$ and is defined by the following relation:

$$\dot{V}(t,\phi) = \limsup_{h\to 0} \frac{V(t+h, x_{t+h}(t_0,\phi)) - V(t, x_t(t_0,\phi))}{h},$$

where $x(t_0,\phi)$ is the solution of (3) with $x_{t_0}(t_0,\phi) = \phi$.

For the general autonomous delay differential system

$$\dot{x} = f(x_t), \tag{4}$$

which is a special case of (3), the following lemma is given:

Lemma. (See[10].) *Suppose* $f(0) = 0$. *Let* V *be a continuous functional defined on* $C_H = C$ *with* $V(0) = 0$, *and let* $u(s)$ *be a function, non-negative and continuous for* $0 \leq s < \infty$, $u(s) \to \infty$ *as* $u \to \infty$ *with* $u(0) = 0$. *If for all* $\phi \in C$, $u(|\phi(0)|) \leq V(\phi)$, $V(\phi) \geq 0$, $\dot{V}(\phi) \leq 0$, *then the solution* $x_t = 0$ *of* (4) *is stable.*

If we define $Z = \left\{ \phi \in C_H : \dot{V}(\phi) = 0 \right\}$, then the solution $x_t = 0$ of (4) is asymptotically stable, provided that the largest invariant set in Z is $Q = \{0\}$.

3. Main result

In this section, we establish two theorems, which are the main results of this paper.

First, for the case

$$p(t, x, y, x(t - r(t)), y(t - r(t)), z) \equiv 0,$$

the following result is introduced.

Theorem 1. *In addition to the basic assumptions imposed on the functions* h, g *and* f *that appearing in* (1), *we assume that there are positive constants* a, b, c, ε, ρ, μ, K, L *and* M *such that the following conditions hold:*

(i) $ab - c > 0$.

(ii) $f(x)\operatorname{sgn} x > 0$, $(x \neq 0)$, $\sup\{f'(x)\} = c$, $|f'(x)| \leq L$.

(iii) $\dfrac{g(x,y)}{y} \geq b + \varepsilon$, $(y \neq 0)$, $|g_x(x,y)| \leq K$, $|g_y(x,y)| \leq M$.

(iv) $\rho \leq h(x, y, z, x(t - r(t)), y(t - r(t)), z(t - r(t))) - a \leq 2(\varepsilon\rho\mu^{-1})^{\frac{1}{2}}$.

Then the zero solution of equation (1) *is asymptotically stable, provided that*

$$\gamma < \min\left\{\frac{2(\mu b - c)}{\mu(K+L+M)+2\lambda}, \ \frac{2(a-\mu)}{K+L+M+2\delta}\right\}$$

with $\mu = \dfrac{ab+c}{2b}$, *where* γ *is the bound on* $r(t)$.

Proof. To verify Theorem 1, we define the following Lyapunov functional $V_1 = V_1(x_t, y_t, z_t)$:

$$\begin{aligned} V_1(x_t,y_t,z_t) = \ & \mu\int_0^x f(\xi)d\xi + yf(x) + \frac{1}{2}\mu a y^2 + \int_0^y g(x,\eta)d\eta + \mu yz \\ & + \frac{1}{2}z^2 + \lambda\int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds + \delta\int_{-r(t)}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds, \end{aligned} \tag{5}$$

where λ and δ are positive constants which will be determined later in the proof.

Now, it is obvious that $V_1(0,0,0) = 0$.

We also have, by the assumption $\dfrac{g(x,y)}{y} \geq b + \varepsilon$, $(y \neq 0)$,

$$\begin{aligned} V_1(x_t,y_t,z_t) &= \mu\int_0^x f(\xi)d\xi + yf(x) + \frac{1}{2}\mu a y^2 + \int_0^y \frac{g(x,\eta)}{\eta}\eta d\eta + \mu yz + \frac{1}{2}z^2 \\ &+\lambda\int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds + \delta\int_{-r(t)}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds \\ &\geq \mu\int_0^x f(\xi)d\xi + yf(x) + \frac{\mu a}{2}y^2 + \frac{b}{2}y^2 + \frac{\varepsilon}{2}y^2 + \mu yz + \frac{1}{2}z^2 \\ &+\lambda\int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds + \delta\int_{-r(t)}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds \\ &= \frac{1}{2b}\left[by + f(x)\right]^2 + \mu\int_0^x f(\xi)d\xi + \frac{\mu a}{2}y^2 + \frac{\varepsilon}{2}y^2 - \frac{1}{2b}f^2(x) \\ &+\mu yz + \frac{1}{2}z^2 + \lambda\int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds + \delta\int_{-r(t)}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds \\ &= \frac{1}{2by^2}\left[4\int_0^x f(\xi)\left\{\int_0^y(\mu b - f'(\xi))\eta d\eta\right\}d\xi\right] \\ &+\frac{\varepsilon}{2}y^2 + \frac{1}{2}(\mu y + z)^2 + \frac{1}{2}\mu(a-\mu)y^2 + \frac{1}{2b}\left[by + f(x)\right]^2 \\ &+\lambda\int_{-r(t)}^{0}\int\limits_{t+s}^{t} y^2(\theta)d\theta ds + \delta\int_{-r(t)}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds. \end{aligned} \tag{6}$$

By using the assumptions $a - \mu = \dfrac{ab-c}{2b} > 0$ and $\mu b - f'(x) \geq \dfrac{ab-c}{2} > 0$, it follows from (6) that there exist sufficiently small positive constants D_i, $(i = 1,\ 2,\ 3)$, such that

$$
\begin{aligned}
V_1(x_t,y_t,z_t) &\geq D_1x^2+D_2y^2+D_3z^2\\
&+\lambda\int_{-r(t)}^{0}\int_{t+s}^{t}y^2(\theta)d\theta ds+\delta\int_{-r(t)}^{0}\int_{t+s}^{t}z^2(\theta)d\theta ds\\
&\geq D_4(x^2+y^2+z^2)+\lambda\int_{-r(t)}^{0}\int_{t+s}^{t}y^2(\theta)d\theta ds+\delta\int_{-r(t)}^{0}\int_{t+s}^{t}z^2(\theta)d\theta ds\\
&\geq D_4(x^2+y^2+z^2),
\end{aligned}
\tag{7}
$$

since the integrals $\int_{-r(t)}^{0}\int_{t+s}^{t}y^2(\theta)d\theta ds$ and $\int_{-r(t)}^{0}\int_{t+s}^{t}z^2(\theta)d\theta ds$ are non-negative, where $D_4=\min\{D_1,\ D_2,\ D_3\}$. Now, we can deduce that there exists a continuous function u with $u(|\phi(0)|)\geq 0$ such that $u(|\phi(0)|)\leq V(\phi)$.

Next, by a straightforward calculation from (5) and (2), we compute the total derivative of $V_1(x_t,y_t,z_t)$ with respect to t :

$$
\begin{aligned}
\frac{d}{dt}V_1(x_t,y_t,z_t) &= f'(x)y^2+\mu z^2-\mu yg(x,y)+y\int_0^y g_x(x,\eta)d\eta\\
&-\mu\left\{h(x,y,x(t-r(t)),y(t-r(t)),z(t-r(t)))-a\right\}yz\\
&-h(x,y,x(t-r(t)),y(t-r(t)),z(t-r(t)),z)z^2\\
&+(\mu y+z)\int_{t-r(t)}^{t}f'(x(s))y(s)ds+(\mu y+z)\int_{t-r(t)}^{t}g_x(x(s),y(s))y(s)ds\\
&+(\mu y+z)\int_{t-r(t)}^{t}g_y(x(s),y(s))z(s)ds+\lambda y^2r(t)+\delta z^2r(t)\\
&-\lambda(1-r'(t))\int_{t-r(t)}^{t}y^2(s)ds-\delta(1-r'(t))\int_{t-r(t)}^{t}z^2(s)ds.
\end{aligned}
\tag{8}
$$

By use of the assumptions of Theorem 1 and the inequality $2|uv|\leq u^2+v^2$, we obtain

$$-h(x,y,x(t-r(t)),y(t-r(t)),z(t-r(t)),z)z^2\leq-(a+\rho)z^2,\tag{9}$$

$$-\left(\mu\frac{g(x,y)}{y}-f'(x)\right)y^2\leq-(\mu b+\mu\varepsilon-c)y^2,\tag{10}$$

$$
\begin{aligned}
\mu y\int_{t-r(t)}^{t}f'(x(s))y(s)ds &\leq \frac{\mu Lr(t)}{2}y^2+\frac{\mu L}{2}\int_{t-r(t)}^{t}y^2(s)ds\\
&\leq \frac{\mu L\gamma}{2}y^2+\frac{\mu L}{2}\int_{t-r(t)}^{t}y^2(s)ds,
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
z\int_{t-r(t)}^{t}f'(x(s))y(s)ds &\leq \frac{Lr(t)}{2}z^2+\frac{L}{2}\int_{t-r(t)}^{t}y^2(s)ds\\
&\leq \frac{L\gamma}{2}z^2+\frac{L}{2}\int_{t-r(t)}^{t}y^2(s)ds,
\end{aligned}
\tag{12}
$$

$$\begin{aligned}\mu y\int_{t-r(t)}^{t} g_x(x(s),y(s))y(s)ds &\le \frac{\mu K r(t)}{2}y^2+\frac{\mu K}{2}\int_{t-r(t)}^{t} y^2(s)ds\\ &\le \frac{\mu K\gamma}{2}y^2+\frac{\mu K}{2}\int_{t-r(t)}^{t} y^2(s)ds,\end{aligned}\tag{13}$$

$$\begin{aligned}z\int_{t-r(t)}^{t} g_x(x(s),y(s))y(s)ds &\le \frac{K r(t)}{2}z^2+\frac{K}{2}\int_{t-r(t)}^{t} y^2(s)ds\\ &\le \frac{K\gamma}{2}z^2+\frac{K}{2}\int_{t-r(t)}^{t} y^2(s)ds,\end{aligned}\tag{14}$$

$$\begin{aligned}\mu y\int_{t-r(t)}^{t} g_y(x(s),y(s))z(s)ds &\le \frac{\mu M r(t)}{2}y^2+\frac{\mu M}{2}\int_{t-r(t)}^{t} z^2(s)ds\\ &\le \frac{\mu M\gamma}{2}y^2+\frac{\mu M}{2}\int_{t-r(t)}^{t} z^2(s)ds,\end{aligned}\tag{15}$$

$$\begin{aligned}z\int_{t-r(t)}^{t} g_y(x(s),y(s))z(s)ds &\le \frac{M r(t)}{2}z^2+\frac{M}{2}\int_{t-r(t)}^{t} z^2(s)ds\\ &\le \frac{M\gamma}{2}z^2+\frac{M}{2}\int_{t-r(t)}^{t} z^2(s)ds,\end{aligned}\tag{16}$$

$$\lambda y^2 r(t)\le \lambda\gamma y^2, \delta z^2 r(t)\le \delta\gamma z^2.\tag{17}$$

Combining the inequalities (9)-(17) into (8), we obtain

$$\begin{aligned}&\frac{d}{dt}V_1(x_t,y_t,z_t)-\left(\mu b-c-\frac{\mu K}{2}\gamma-\frac{\mu L}{2}\gamma-\frac{\mu M}{2}\gamma-\lambda\gamma\right)y^2\\ &-\left(a-\mu-\frac{K}{2}\gamma-\frac{L}{2}\gamma-\frac{M}{2}\gamma-\delta\gamma\right)z^2\\ &-(\mu\varepsilon)y^2-\mu\left\{h(x,y,z,x(t-r(t)),y(t-r(t)),z(t-r(t)))-a\right\}yz\\ &-\rho z^2+\left[\frac{K}{2}+\frac{L}{2}+\frac{\mu K}{2}+\frac{\mu L}{2}-(1-\beta)\lambda\right]\int_{t-r(t)}^{t} y^2(s)ds\\ &+\left[\frac{M}{2}+\frac{\mu M}{2}-(1-\beta)\delta\right]\int_{t-r(t)}^{t} z^2(s)ds.\end{aligned}\tag{18}$$

Now, we consider the terms

$$W=:(\mu\varepsilon)y^2+\mu\left\{h(x,y,z,x(t-r(t)),y(t-r(t)),z(t-r(t)))-a\right\}yz+\rho z^2,$$

which are contained in (18). Clearly, by the assumption (iv), we get

$$\begin{aligned}W &\ge (\mu\varepsilon)y^2-2\mu(\varepsilon\rho\mu^{-1})^{\frac{1}{2}}\,|y|\;\;|z|+\rho z^2\\ &=(\mu\varepsilon)y^2-2(\varepsilon\rho\mu)^{\frac{1}{2}}\,|y|\;\;|z|+\rho z^2\\ &=[\sqrt{\mu\varepsilon}\;\;|y|-\sqrt{\rho}\;\;|z|]^2\ge 0.\end{aligned}$$

This estimate implies that

$$
\begin{aligned}
\frac{d}{dt}V_1(x_t,y_t,z_t) \leq & -\left(\mu b - c - \frac{\mu K}{2}\gamma - \frac{\mu L}{2}\gamma - \frac{\mu M}{2}\gamma - \lambda\gamma\right) y^2 \\
& -\left(a - \mu - \frac{K}{2}\gamma - \frac{L}{2}\gamma - \frac{M}{2}\gamma - \delta\gamma\right) z^2 \\
& + \left[\frac{K}{2} + \frac{L}{2} + \frac{\mu K}{2} + \frac{\mu L}{2} - (1-\beta)\lambda\right] \int_{t-r(t)}^{t} y^2(s)ds \\
& + \left[\frac{M}{2} + \frac{\mu M}{2} - (1-\beta)\delta\right] \int_{t-r(t)}^{t} z^2(s)ds.
\end{aligned} \tag{19}
$$

By taking $\lambda = \frac{1}{2(1-\beta)}(K+L)(1+\mu)$ and $\delta = \frac{1}{2(1-\beta)}M(1+\mu)$, we get from (19) that

$$
\begin{aligned}
\frac{d}{dt}V_1(x_t,y_t,z_t) \leq & -\left(\mu b - c - \frac{\mu K}{2}\gamma - \frac{\mu L}{2}\gamma - \frac{\mu M}{2}\gamma - \lambda\gamma\right) y^2 \\
& -\left(a - \mu - \frac{K}{2}\gamma - \frac{L}{2}\gamma - \frac{M}{2}\gamma - \delta\gamma\right) z^2.
\end{aligned} \tag{20}
$$

The above inequity, that is, (20), yields

$$\frac{d}{dt}V_1(x_t,y_t,z_t) \leq -k_1y^2 - k_2z^2 \leq 0 \tag{21}$$

for some positive constants k_1 and k_2 provided that

$$\gamma < \min\left\{\frac{2(\mu b - c)}{\mu(K+L+M)+2\lambda}, \frac{2(a-\mu)}{K+L+M+2\delta}\right\}.$$

It is also clear that the largest invariant set in Z is $Q = \{0\}$, where

$$Z = \left\{\phi \in C_H : \dot{V}_1(\phi) = 0\right\}.$$

Namely, the only solution of equation (1) for which $\frac{d}{dt}V_1(x_t,y_t,z_t) = 0$ is the solution $x_t \equiv 0$. Thus, under the above discussion, we conclude that the trivial solution of equation (1) is asymptotically stable. This fact completes the proof of Theorem 1.

In the case

$$p(t,x,y,x(t-r(t)),y(t-r(t)),z) \neq 0,$$

we establish the following result.

Theorem 2. *We assume that the assumptions* (i)-(iv) *of Theorem* 1 *and the following condition hold:*

$$|p(t,x,y,x(t-r(t)),y(t-r(t)),z)| \leq q(t),$$

where $q \in L^1(0,\infty)$, $L^1(0,\infty)$ *is space of integrable Lebesgue functions. Then, there exists a finite positive constant* K *such that the solution* $x(t)$ *of equation* (1) *defined by the initial functions*

$$x(t) = \phi(t), x'(t) = \phi'(t), x''(t) = \phi''(t)$$

satisfies the inequalities

$$|x(t)| \leq \sqrt{K},\ \ |x'(t)| \leq \sqrt{K},\ \ |x''(t)| \leq \sqrt{K}$$

for all $t \geq t_0$, *where* $\phi \in C^2\left([t_0 - r, t_0]\,,\ \Re\right)$, *provided that*

$$\gamma < \min\left\{\frac{2(\mu b - c)}{\mu(K+L+M)+2\lambda},\ \ \frac{2(a-\mu)}{K+L+M+2\delta}\right\}$$

with $\mu = \dfrac{ab+c}{2b}$.

Proof. To prove Theorem 2, we use the Lyapunov functional, $V_1 = V_1(x_t, y_t, z_t)$, defined by (5). Now, taking into account the assumptions of Theorem 2, the result of Theorem 1 and (21), a straightforward calculation from (5) and (2) gives that

$$\frac{d}{dt}V_1(x_t,y_t,z_t) \leq -k_1y^2 - k_2z^2 + (\mu y + z)p(t, x(t), y(t), x(t-r(t)), y(t-r(t)), z(t)).$$

Hence

$$\begin{aligned}\frac{d}{dt}V_1(x_t,y_t,z_t) &\leq (\mu\,|y| + |z|)\,|p(t, x(t), y(t), x(t-r(t)), y(t-r(t)), z(t))| \\ &\leq (\mu\,|y| + |z|)q(t) \leq D_5(|y| + |z|)q(t),\end{aligned}$$

where $D_5 = \max\{1,\ \ \mu\}$.

Now, by using the inequalities $|y| < 1 + y^2$ and $|z| < 1 + z^2$, we have

$$\frac{d}{dt}V_1(x_t,y_t,z_t) \leq D_5(2 + y^2 + z^2)q(t).$$

The estimate (7) implies that

$$y^2 + z^2 \leq D_4^{-1}\ V_1(x_t,y_t,z_t).$$

The last two inequalities lead to

$$\begin{aligned}\frac{d}{dt}V_1(x_t,y_t,z_t) &\leq D_5(2 + D_4^{-1}V_1(x_t,y_t,z_t))q(t) \\ &= 2D_5q(t) + D_5D_4^{-1}V_1(x_t,y_t,z_t)q(t).\end{aligned} \tag{22}$$

Now, we integrate (22) from 0 to t and use the assumption $q \in L^1(0,\infty)$, and Gronwall-Reid-Bellman inequality to make the next estimate:

$$\begin{aligned}V_1(x_t,y_t,z_t) &\leq V_1(x_0,y_0,z_0) + 2D_5A + D_5D_4^{-1}\int_0^t (V_1(x_s,y_s,z_s))\,q(s)ds \\ &\leq (V_1(x_0,y_0,z_0) + 2D_5A)\exp\left(D_5D_4^{-1}\int_0^t q(s)ds\right) \\ &\leq (V_1(x_0,y_0,z_0) + 2D_5A)\exp\left(D_5D_4^{-1}A\right) = K_1 < \infty,\end{aligned} \tag{23}$$

where $K_1 > 0$ is a constant, $K_1 = (V_1(x_0, y_0, z_0) + 2D_5A)\ \exp\left(D_5D_4^{-1}A\right)$ and $A = \int_0^\infty q(s)ds$.

In view of (7) and (23), it follows that

$$x^2 + y^2 + z^2 \le D_4^{-1}V_1(x_t, y_t, z_t) \le K,$$

where $K = K_1D_4^{-1}$. Thus, we can deduce

$$|x(t)| \le \sqrt{K},\ |y(t)| \le \sqrt{K},\ |z(t)| \le \sqrt{K},$$

for all $t \ge t_0$. That is,

$$|x(t)| \le \sqrt{K},\ |x'(t)| \le \sqrt{K},\ |x''(t)| \le \sqrt{K},$$

for all $t \ge t_0$. The proof of Theorem 2 is now complete.

Example. We consider the following third order nonlinear differential equation with retarded argument

$$\begin{aligned} x''' &+ \left(4 + \frac{1}{1+x^2+(x')^2+(x'')^2+x^2(t-r(t))+(x'(t-r(t)))^2+(x''(t-r(t)))^2}\right) x'' \\ &+7x'(t-r(t)) + \sin x'(t-r(t)) + x(t-r(t)) + \operatorname{arctg} x(t-r(t)) \\ &= \frac{1}{1+t^2+x^2+(x')^2+(x(t-r(t)))^2+(x'(t-r(t)))^2+(x'')^2}. \end{aligned}$$

This equation can be stated as the following equivalent system:

$$\begin{aligned} x' &= y, \\ y' &= z, \\ z' &= -\left(4 + \frac{1}{1+(x^2+y^2+z^2+x^2(t-r(t))+y^2(t-r(t))+z^2(t-r(t))}\right) z \\ &\quad -(7y + \sin y) - x - \operatorname{arctg} x + \int_{t-r(t)}^{t} (1 + \frac{1}{1+(x(s))^2})y(s)ds \\ &\quad + \int_{t-r(t)}^{t} (7 + \cos y(s))\, z(s)ds \\ &\quad + \frac{1}{1+t^2+x^2+y^2+x^2(t-r(t))+y^2(t-r(t))+z^2}. \end{aligned} \tag{24}$$

So, we have

$$\begin{aligned} &h(x, y, z, x(t-r(t)), y(t-r(t)), z(t-r(t))) \\ &\quad = 4 + \frac{1}{1+x^2+y^2+z^2+x^2(t-r(t))+y^2(t-r(t))+z^2(t-r(t))}, \end{aligned}$$

$$4 \leq 4 + \frac{1}{1 + x^2 + y^2 + z^2 + x^2(t - r(t)) + y^2(t - r(t)) + z^2(t - r(t))} \leq 5,$$

that is,

$$\begin{aligned}
&4 \leq h(x, y, z, x(t - r(t)), y(t - r(t)), z(t - r(t))) \leq 5,\\
&g(x, y) = 7y + \sin y, g(x, 0) = 0,\\
&\frac{g(x, y)}{y} = 7 + \frac{\sin y}{y} \geq 6, (y \neq 0, |y| < \pi),\\
&g_y(x, y) = 7 + \cos y, |g_y(x, y)| = |7 + \cos y| \leq 8,\\
&f(x) = x + \operatorname{arctg} x, f(0) = 0,\\
&f'(x) = 1 + \frac{1}{1 + x^2}, |f'(x)| \leq 2,\\
&p(t, x, y, x(t - r(t)), y(t - r(t)), z)\\
&\qquad = \frac{1}{1 + t^2 + x^2 + y^2 + x^2(t - r(t)) + y^2(t - r(t)) + z^2},\\
&|p(t, x, y, x(t - r(t)), y(t - r(t)), z)| \leq \frac{1}{1 + t^2} = q(t),
\end{aligned}$$

and hence

$$\int_0^\infty q(s)ds = \int_0^\infty \frac{1}{1 + s^2} ds = \frac{\pi}{2} < \infty, \text{ that is, } q \in L^1(0, \infty).$$

Clearly, in cases of appropriate choice of the constants a, b and c, one can easily show that all the assumptions Theorem 1 and Theorem 2 hold for (24). That is, for the case $p(t, x, y, x(t-r(t)), y(t-r(t)), z) = 0$, the trivial solution of our equation is asymptotically stable, and for the case $p(t, x, y, x(t - r(t)), y(t - r(t)), z) \neq 0$, all solutions of the equation are bounded.

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HYPERACTION OF SEMIGROUPS AND MONOIDS

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Abstract. The purpose of this note is the study of hyper action as a generalization of action of a monoid on a set. In this regards, first we introduce the notion of hyperactions of type 1 and type 2 and then we study the basic properties of this notion. In particular, we investigate the relationship between hyperactions and non-deterministic automata.

Keywords: hyperaction, non-deterministic finite automata, congruence.

1. Introduction

Hyperstructure theory was born in 1934, when Marty defined hypergroups ([6]), began to analysis their properties and applied them to groups, rational algebraic functions. Now, they are widely studied from theoretical point of view and for their applications to many subjects of pure and applied mathematics. Since then, many researchers have studied in this field and developed it, for example, see [2].

The notion of a fuzzy subset of a nonempty set was introduced by L.A. Zadeh in 1965 [10] as a function from a nonempty set X to interval $I = [0, 1]$. Rosenfeld defined the concept of a fuzzy subgroup of a given group G [8] and then many researchers has developed it in all subjects of algebra. In [7], Malik, Mordeson and Sen have studied some properties of S-set in the name poly transformation semigroup. In the literature, we find that Wee [2] first introduced the concept of fuzzy automata. In this note, we introduce the notion of a hyperactions of types 1 and 2 as a generalization of action of a monoid on a set and obtain basic results of them. In this regards, we study the relationship between non-deterministic automata and hyperactions. In addition to the above mentioned concept of hyperaction, some other directions have been pursued in the literature. A. Madanshekaf and A.R. Ashrafi have introduced a notion of generalized action of a hypergroup on a set in [5], which is different from the concept we present in this paper.

2. Preliminaries

Let H be a nonempty set and $P(H)$ the family of all nonempty subsets of H.

A map $\cdot : H \times H \longrightarrow P(H)$ is called *hyperoperation* or join operation (see [2]).

The join operation is extended to subsets of H in natural way, so that $A \cdot B$ or AB is given by

$$AB = \bigcup \{ab \mid a \in A \ \text{ and } \ b \in B\}.$$

The relational notation $A \approx B$ (read A meets B) is used to asserts that A and B have an element in common, that is, $A \cap B \neq \emptyset$. The notations aA and Aa are used for $\{a\}A$ and $A\{a\}$ respectively. Generally, the singleton $\{a\}$ is identified by its element a.

Throughout the paper, S denotes a monoid and Q denotes a nonempty set.

A (left) action of S on Q is a function $f : S \times Q \longrightarrow Q$ (usually denoted by $f(x, q) \longrightarrow xq$) for all $x \in X$ and $q \in Q$. Q is called an S-set [3] if there exists an action of S on Q such that

(i) $(xy)q = x(yq)$

(ii) $1q = q$, for all $x, y \in X$ and $q \in Q$.

This concept of S-set also plays an important role in the theory of Deterministic finite automata [1]. Considering the theory of Non-deterministic finite automata [1], one can introduce the concept of the hyperaction of S on Q. We denote the set of all nonempty subsets of Q by $P(Q)$.

3. Hyper action

Definition 3.1 A (left) hyper action of S on Q is a function $\circ : S \times Q \mapsto P(Q)$ (usually denoted by $\circ(x, q) \mapsto x \circ q$) for all $x \in S$ and $q \in Q$.

Let $A \in P(Q)$ and $x \in S$. We define $x \circ A \in P(Q)$ by

$$x \circ A = \begin{cases} \bigcup\limits_{a \in A} (x \circ a), & \text{if } A \neq \emptyset \\ \emptyset, & \text{if } A = \emptyset \end{cases}$$

Q is called an S-hyper set of type 1 if there exists an hyper action $\circ$ of S on Q such that

(i) $(xy) \circ q = x \circ (y \circ q)$

(ii) $q \in 1 \circ q$, for all $x, y \in S$ and $q \in Q$.

Q is called an S-hyper set of type 2 if there exists an hyper action $\circ$ of S on Q such that

(i) $(xy) \circ q = x \circ (y \circ q)$

(ii) $1 \circ q = \{q\}$, for all $x, y \in S$ and $q \in Q$.

Example 3.2 Let Q be a non-empty set. Let B_Q denotes the set of all binary relations on Q. Let $\alpha, \beta \in B_Q$. Now B_Q is a semigroup where the binary operation o is defined by $\alpha o \beta = \{(x, y) \in Q \times Q : (x, z) \in \beta,\ (z, y) \in \alpha, \text{ for some } z \in Q\}$. One can show that B_Q is a semigroup with identity.

Theorem 3.3 *Q is a B_Q-hyper set of type* 2.

Proof. We define $\circ : B_Q \times Q \mapsto P(Q)$ by

$$\alpha \circ x = \{y \in Q : (x, y) \in \alpha\ \}, \text{ for all } \alpha \in B_Q \text{ and } x \in Q.$$

Clearly, this is a hyper action of B_Q on Q. $(x, x) \in i$ (identity relation) $\in B_Q$, for all $x \in Q$ i.e. $i \circ x = \{x\}$, for all $x \in Q$.

$$\alpha \circ (\beta \circ x) = \bigcup \{\alpha \circ y : y \in \beta \circ x\} = \bigcup \{\alpha \circ y : (x, y) \in \beta\} = \{z \in Q : (y, z) \in \alpha$$

and

$$(x, y) \in \beta\} = \{z \in Q : (x, z) \in \alpha \circ \beta\} = (\alpha \circ \beta) \circ x, \text{ for all } \alpha \in B_Q \text{ and } x \in Q.$$

Hence the result. ■

Example 3.4 Let Q be a non-empty set. Let $f : Q \mapsto P(Q)$. Let $S = \{f : (f : Q \mapsto P(Q))\}$. Clearly, $i : Q \mapsto P(Q)$, where $i(q) = \{q\}$ for all $q \in Q$, is a member of S. Let $f, g \in S$. We define $f \bullet g : Q \mapsto P(Q)$ by

$$f \bullet g(q) = \begin{cases} \bigcup \{f(p) : p \in g(q)\}, & \text{if } g(q) \neq \emptyset \\ \emptyset, & \text{if } g(q) = \emptyset, \end{cases}$$

for all $q \in Q$.

Then, $f \bullet g \in S$.

Let $f, g, h \in S$. Let $h(q) = \emptyset$, then $(f \bullet g) \bullet h(q) = \emptyset$ and $f \bullet (g \bullet h)(q) = \bigcup\{f(p) : p \in g \bullet h(q)\} = \bigcup\{f(p) : p \in \emptyset\} = \emptyset$, i.e., $(f \bullet g) \bullet h(q) = f \bullet (g \bullet h)(q)$.

Let $h(q) \neq \emptyset$, then $f \bullet (g \bullet h)(q) = \bigcup\{p : p \in f(r), \text{ where } r \in g \bullet h(q)\} = \bigcup\{p : p \in f(r), \text{ where } r \in g(t) \text{ and } t \in \bullet h(q)\} = \bigcup\{p : p \in f(g(t)), \text{ where } t \in h(q)\} = \bigcup\{p : p \in f \bullet g(t), \text{ where } t \in h(q)\} = (f \bullet g) \bullet h(q)$.

Therefore, S is a monoid.

Now, we define $\circ : S \times Q \mapsto P(Q)$ (described as $(f, q) \mapsto f \circ q$) by

$$f \circ q = f(q),$$

where

$$f \circ A = \begin{cases} \bigcup\{f \circ p : p \in A\}, & \text{if } A \neq \emptyset \\ \emptyset, & \text{if } A = \emptyset \end{cases}$$

Here $i \circ q = i(q) = \{q\}$, for all $q \in Q$.

Now, it can be shown that $f \circ (g \circ q) = fg \circ q$, where fg stands for $f \bullet g$, in short.

Let $g \circ q = g(q) = \emptyset$. Then $f \circ (g \circ q) = \emptyset$ (by definition of $\circ$). In this case, $(fg) \circ q = \emptyset$ (by definition of $\bullet$ on S). Thus, $f \circ (g \circ q) = fg \circ q$, whenever $g \circ q = \emptyset$.

Now, let $g \circ q \neq \emptyset$, then we have

$$f \circ (g \circ q) = \bigcup\{f \circ p : p \in (g \circ q)\} = \bigcup\{f(p) : p \in g(q)\} = fg(q).$$

Therefore, Q is an S-hyper set of type 2.

Example 3.5 Let G be a group and H be a normal subgroup of G. Define a hyper action of G on G by

$$a \circ b = aHb, \text{ for all } a, b \in G.$$

We can show that G is a G-hyper set of type 1.

Proof. Let e be the identity element of G. Then $e \circ q = eHq = Hq$. This implies that $q \in Hq = e \circ q$, for all $q \in G$.

$$ab \circ q = abHq = abHHq = aHbHq = \bigcup_{c \in bHq} a \circ c = \bigcup_{c \in b \circ q} a \circ c = a \circ (b \circ q),$$

for all $a, b, q \in G$. Hence the result. ■

Theorem 3.6 *S is an S-hyper set of type 2.*

Proof. We define $\circ : S \times S \mapsto P(S)$ by $x \circ q = \{xq\}$

$$\lambda \circ q = \{\lambda q\} = \{q\}, \quad \text{for all } q \in S.$$

$$x \circ (y \circ q) = x \circ yq = \{xyq\} = xy \circ q, \text{ for all } x, y, q \in S.$$

Therefore, S is an S-hyper set. ■

Theorem 3.7 *Let Σ and Q be two non-empty finite sets. $\Sigma^\star$ denotes the free monoid generated by Σ. Suppose for all $x \in \Sigma^\star$, $q \in Q$, $x \circ q \in P(Q)$ such that $xa \circ q = x \circ (a \circ q)$ for all $x \in \Sigma^\star$, $a \in \Sigma$, $q \in Q$ and $\lambda \circ q = \{q\}$, for all $q \in Q$. Then Q is an $\Sigma^\star$-hyper set of type 2.*

Proof. We have to show that $xy \circ q = x \circ (y \circ q)$ for all $x, y \in \Sigma^\star$ and $q \in Q$.

We prove the result by induction on $|y| = n$.

If $n = 0$, then $y = \lambda$.

$x \circ (y \circ q) = x \circ (\lambda \circ q) = x \circ \{q\} = x \circ q = x\lambda \circ q = xy \circ q$. Given that the result is true for $|y| = 1$, assume the result is true for $|y| = n - 1$, $n > 1$.

Let $y = ua$, where $|u| = n - 1$, $n > 1$ and $a \in \Sigma$. Now,

$$\begin{aligned} x \circ (y \circ q) &= x \circ (ua \circ q) = x \circ (u \circ (a \circ q)) \\ &= \bigcup \{x \circ t : t \in u \circ (a \circ q)\} \\ &= \bigcup \{x \circ t : t \in u \circ r, \text{ for some } r \in a \circ q\} \\ &= \bigcup \{x \circ (u \circ r) : r \in a \circ q\} \\ &= \bigcup \{xu \circ r : r \in a \circ q\} \\ &= xu \circ (a \circ q) = xua \circ q = xy \circ q. \end{aligned}$$

Therefore, Q is a $\Sigma^\star$-hyper set of type 2. ■

Theorem 3.8 *Let $\mathcal{M} = (\Sigma, Q, \delta)$ be a non-deterministic finite automata. Then, Q is a $\Sigma^\star$-hyper set of type 2.*

Proof. We define $\circ : \Sigma^\star \times Q \mapsto P(Q)$ by

$$\begin{aligned} x \circ q = \delta^\star(x, q) & \qquad \text{for all } x \in \Sigma, q \in Q. \\ \lambda \circ q = \delta^\star(\lambda, q) = \{q\}, & \qquad \text{for all } q \in Q. \end{aligned}$$

By definition of $\delta^\star$, we have

$$\delta^\star(xa, q) = \bigcup_{r \in \delta(a,q)} \delta^\star(x, r).$$

We have to show that

$$x \circ (y \circ q) = xy \circ q, \text{ for all } x, y \in \Sigma^\star, q \in Q.$$

We prove the result by induction on $|y| = n$.

If $n = 0$, then $y = \lambda$.

$$x \circ (y \circ q) = x \circ (\lambda \circ q) = x \circ \delta^\star(\lambda, q) = x \circ \{q\} = x \circ q = x\lambda \circ q = xy \circ q.$$

Assume the result is true for $|y| = n - 1, n > 0$.

Let $y = ua$, where $|u| = n-1,\ n > 0,\ a \in \Sigma$.

$$\begin{aligned} x \circ (y \circ q) &= \bigcup \{x \circ r : r \in y \circ q\} = \bigcup \{x \circ r : r \in ua \circ q\} \\ &= \bigcup \{x \circ r : r \in u \circ (a \circ q)\} = \bigcup \{x \circ r : r \in u \circ p, \end{aligned}$$

where $p \in a \circ q\} = \bigcup \{x \circ (u \circ p) : p \in a \circ q\} = \bigcup \{xu \circ p : p \in a \circ q\} = xu \circ (a \circ q) = xua \circ q = xy \circ q$.

Hence the result. ■

Theorem 3.9 *Let Σ and Q be two non-empty sets. Suppose that Q is $\Sigma^\star$-hyper set of type 2. Then, there exists a non-deterministic finite automata $\mathcal{M} = (\Sigma, Q, \delta)$ such that*

$$\delta^\star(\lambda, q) = \{q\}, \text{ for all } q \in Q \text{ and}$$

$$\delta^\star(xy, q) = \bigcup_{r \in \delta^\star(y,q)} \delta^\star(x, r).$$

Proof. We define $\delta^\star : \Sigma^\star \times Q \mapsto P(Q)$ by

$$\delta^\star(x, q) = x \circ q, \text{ for all } x \in \Sigma^\star, q \in Q.$$

If $x \in \Sigma$, then $\delta^\star(x, q) = \delta(x, q)$.

$$\delta^\star(\lambda, q) = \lambda \circ q = \{q\},$$

for all $q \in Q$.

$$\delta^\star(xy, q) = xy \circ q = x \circ (y \circ q) = x \circ \delta^\star(y, q) = \bigcup_{r \in \delta^\star(y,q)} x \circ r = \bigcup_{r \in \delta^\star(y,q)} \delta^\star(x, r),$$

for all $x \in \Sigma^\star, q \in Q$. ■

Theorem 3.10 *Let Q_1 and Q_2 be two S-hyper sets of type 1 (or type 2), then $Q_1 \times Q_2$ is also an S-hyper set of type 1 (or type 2).*

Proof. We define $\circ : S \times (Q_1 \times Q_2) \mapsto P(Q_1 \times Q_2)$ by

$$x \circ (q_1, q_2) = (x \circ q_1) \times (x \circ q_2),$$

for all $x \in S, q_1 \in Q_1, q_2 \in Q_2$.

$$(q_1, q_2) \in (1 \circ q_1) \times (1 \circ q_2) = 1 \circ (q_1, q_2) \ \ [1 \circ (q_1, q_2) = (1 \circ q_1) \times (1 \circ q_2) = \{(q_1, q_2)\}],$$

for all $(q_1, q_2) \in (Q_1 \times Q_2)$.

$$\begin{aligned} x \circ (y \circ (q_1, q_2)) &= \bigcup_{(r_1, r_2) \in y \circ (q_1, q_2)} x \circ (r_1, r_2) = \bigcup_{r_1 \in y \circ q_1, r_2 \in y \circ q_2} (x \circ r_1) \times (x \circ r_2) \\ &= (x \circ (y \circ q_1)) \times (x \circ (y \circ q_2)) = xy \circ q_1 \times xy \circ q_2 = xy \circ (q_1, q_2), \end{aligned}$$

for all $x, y \in S$ and $(q_1, q_2) \in (Q_1 \times Q_2)$. ■

Definition 3.11 Let S and T be two semigroups. Then, their cartesian product $S \times T$ will be a semigroup with respect to the binary operation $\bullet$ defined by

$$(x, y) \bullet (s, t) = (xs, yt), \text{ for all } x, s \in S \text{ and } y, t \in T.$$

The semigroup $(S \times T, \bullet)$ is called the direct product of S and T, written by $S \times T$.

Theorem 3.12 *Let P be an S-hyper set of type 1 (type 2) and Q be a T-hyper set of type 1 (type 2), then $P \times Q$ is $S \times T$-hyper set of type 1 (type 2).*

Proof. We define $\circ : (S \times T) \times (P \times Q) \mapsto P(P \times Q)$ by

$$(x, y) \circ (p, q) = \begin{cases} (x \circ p) \times (y \circ q), & \text{if } (x \circ p) \neq \emptyset \text{ and } (y \circ q) \neq \emptyset \\ \emptyset, & \text{otherwise,} \end{cases}$$

for all $(x, y) \in (S \times T)$ and $(p, q) \in (P \times Q)$.

Let $(x, y) \circ ((s, t) \circ (p, q)) = \emptyset$, then either $(s, t) \circ (p, q) = \emptyset$ or $(x, y) \circ (p_1, q_1) = \emptyset$, for all $(p_1, q_1) \in (s, t) \circ (p, q)$.

If $(s, t) \circ (p, q) = \emptyset$, then either $s \circ p = \emptyset$ or $t \circ q = \emptyset$, i.e., either $x \circ (s \circ p) = \emptyset$ or $y \circ (t \circ q) = \emptyset$, i.e., either $xs \circ p = \emptyset$ or $yt \circ q = \emptyset$, i.e., $(xs, yt) \circ (p, q) = \emptyset$, i.e., $((x, y)(s, t)) \circ (p, q) = \emptyset$.

If $(x, y) \circ (p_1, q_1) = \emptyset$, for all $(p_1, q_1) \in (s, t) \circ (p, q)$, then either $x \circ p_1 = \emptyset$ or $y \circ q_1 = \emptyset$, for all $p_1 \in s \circ p$ and for all $q_1 \in t \circ q$, i.e., either $x \circ (s \circ p) = \emptyset$ or $y \circ (t \circ q) = \emptyset$, i.e., either $xs \circ p = \emptyset$ or $yt \circ q = \emptyset$, i.e., $(xs, yt) \circ (p, q) = \emptyset$, i.e., $((x, y)(s, t)) \circ (p, q) = \emptyset$.

Let $(x, y) \circ ((s, t) \circ (p, q)) \neq \emptyset$, then

$$\begin{aligned} (x, y) \circ ((s, t) \circ (p, q)) &= \bigcup \{(x, y) \circ (p_1, q_1) : (p_1, q_1) \in (s, t) \circ (p, q)\} \\ &= \bigcup \{x \circ p_1 \times y \circ q_1 : p_1 \in s \circ p \end{aligned}$$

and

$$\begin{aligned} q_1 \in t \circ q\} &= x \circ (s \circ p) \times y \circ (t \circ q) = xs \circ p \times yt \circ q \\ &= (xs, yt) \circ (p, q) = ((x, y)(s, t)) \circ (p, q). \end{aligned}$$

$$(p, q) \in (1 \circ p) \times (1 \circ q) = 1 \circ (p, q) \;\; [1 \circ (p, q) = (1 \circ p) \times (1 \circ q) = \{(p, q)\}],$$

for all $(x, y), (s, t) \in (S \times T)$ and $(p, q) \in (P \times Q)$. Hence the result. ■

Definition 3.13 Let Q_1 and Q_2 be two S-hyper sets. A mapping $f : Q_1 \mapsto Q_2$ is said to be an S-hyper homomorphism if $f(x \circ q) = x \circ f(q)$, for all $x \in S$ and $q \in Q_1$ where $f(x \circ q) = \{f(t) : t \in x \circ q\}$.

Definition 3.14 Let σ be an equivalence relation on Q(S-hyper set). Let $A, B \in P(Q)$. We say that $A \sigma B$ if for each $a \in A$ there exists $b \in B$ such that $a \sigma b$ and for each $y \in B$ there exists $x \in A$ such that $x \sigma y$.

The equivalence relation σ on Q is said to be S-hyper congruence if $q \sigma p$ implies $x \circ q \sigma x \circ p$, for all $x \in S$ and for all $q, p \in Q$.

If σ be an equivalence relation on Q then we define $Q/\sigma = \{q\sigma : q \in Q\}$.

Theorem 3.15 *If σ be an S-hyper congruence on an S-hyper set Q, then Q/σ is an S-hyper set.*

Proof. We define $\star : S \times Q/\sigma \mapsto P(Q/\sigma)$ by

$$x \star q\sigma = \begin{cases} \{p\sigma \in Q/\sigma : p \in x \circ q\}, & \text{if } x \circ q \neq \emptyset \\ \emptyset, & \text{otherwise,} \end{cases}$$

for all $x \in S$ and $q\sigma \in Q/\sigma$.

Let $q\sigma = r\sigma$, then $q\sigma r$ implies $x \circ q \sigma x \circ r$, for all $x \in S$.

Let $p\sigma \in x \star q\sigma$ implies $p \in x \circ q$ implies there exists $t \in x \circ r$ such that $p\sigma t$ implies $p\sigma = t\sigma$, for some $t \in x \circ r$ implies $p\sigma = t\sigma$, $t\sigma \in x \star r\sigma$ implies $p\sigma \in x \star r\sigma$.

This implies that $x \star q\sigma \subseteq x \star r\sigma$. Similarly we can show that $x \star r\sigma \subseteq x \star q\sigma$. Therefore $x \star q\sigma = x \star r\sigma$.

If $x \star q\sigma = \emptyset$, then $x \circ q = \emptyset$ and so $x \circ r = \emptyset$ and therefore $x \star r\sigma = \emptyset$, i.e., $x \star q\sigma = x \star r\sigma$.

Hence, $\star$ is well-defined.

$q \in 1 \circ q$ $[1 \circ q = \{q\}]$ implies $q\sigma \in 1 \star q\sigma$ $[1 \star q\sigma = \{q\sigma\}]$, for all $q\sigma \in Q/\sigma$.

Let $x \star (y \star q\sigma) = \emptyset$, then either $y \star q\sigma = \emptyset$ or $x \star p\sigma = \emptyset$, for all $p\sigma \in y \star q\sigma$. If $y \star q\sigma = \emptyset$, then $y \circ q = \emptyset$, implies $x \circ (y \circ q) = \emptyset$, implies $xy \circ q = \emptyset$, i.e., $xy \star q\sigma = \emptyset$. Again if $x \star p\sigma = \emptyset$, for all $p\sigma \in y \star q\sigma$, then $x \circ p = \emptyset$, for all $p \in y \circ q$ i.e. $x \circ (y \circ q) = \emptyset$, i.e., $xy \circ q = \emptyset$, i.e., $xy \star q\sigma = \emptyset$. Similarly if $xy \star q\sigma = \emptyset$, then we can prove $x \star (y \star q\sigma) = \emptyset$.

Now let $r\sigma \in x \star (y \star q\sigma)$ then $r\sigma \in x \star p\sigma$, for some $p \in y \star q\sigma$ implies $r \in x \circ p\sigma$ for some $p \in y \circ q$ implies $r \in x \circ (y \circ q)$ implies $r \in xy \circ q$ implies $r\sigma \in xy \star q\sigma$.

This implies that $x \star (y \star q\sigma) \subseteq xy \star q\sigma$.

Similarly, we can show that $xy \star q\sigma \subseteq x \star (y \star q\sigma)$.

Therefore, $x \star (y \star q\sigma) = xy \star q\sigma$. Hence, Q/σ is an S-hyper set. ∎

Theorem 3.16 *If $f : Q_1 \mapsto Q_2$ be an S-hyper homomorphism, then $\sigma = \{(q,p) : f(q) = f(p)\}$ is an S-hyper congruence on Q_1.*

Proof. Clearly, σ is an equivalence relation on Q_1. Let $q\sigma p$ then $f(q) = f(p)$ implies $x \circ f(q) = x \circ f(p)$ i.e. $f(x \circ q) = f(x \circ p)$, for all $x \in S$

Let $r \in x \circ q$ then $f(r) \in f(x \circ q) = f(x \circ p)$ implies there exists $t \in x \circ p$ such that $f(r) = f(t)$ implies there exists $t \in x \circ p$ such that $t\sigma r$. Similarly, we can show that if $t \in x \circ p$ then there exists $r \in x \circ q$ such that $t\sigma r$.

Therefore, $x \circ q \sigma x \circ p$. Hence, σ is an S-hyper congruence on Q_1. ∎

Theorem 3.17 *If σ be an S-hyper congruence on an S-hyper set Q then there is an S-hyper homomorphism $f : Q \mapsto Q/\sigma$ such that* $\ker f = \sigma$.

Proof. We define $f : Q \mapsto Q/\sigma$ by $f(q) = q\sigma$, for all $q \in Q$.

Now $f(x \circ q) = \{f(p) : p \in x \circ q\} = \{p\sigma : p \in x \circ q\} = x \star q\sigma = x \star f(q)$.

Therefore, f is S-hyper homomorphism.

Let $(q,p) \in \ker f$, then $f(q) = f(p)$, implies $q\sigma = p\sigma$, implies $q\sigma p$, i.e., $(q,p) \in \sigma$. Similarly, we can show that, if $(q,p) \in \sigma$, then $(q,p) \in \ker f$. Therefore, $\ker f = \sigma$. Hence the result. ■

Theorem 3.18 *Let $f : Q_1 \mapsto Q_2$ be a surjective S-hyper homomorphism and σ be the S-hyper congruence on Q_1 induced by f. There exists a bijective S-hyper homomorphism $g : Q_1/\sigma \mapsto Q_2$ such that $g(q\sigma) = f(q)$, for all $q \in Q_1$.*

Proof. Let $q\sigma = p\sigma$ in Q_1/σ, then $q\sigma p$ implies $f(q) = f(p)$. Therefore, $g(q\sigma) = f(q)$ is well-defined as well as injective.

Let $q_2 \in Q_2$ then there exists $q_1 \in Q_1$ such that $f(q_1) = q_2$. Then there exists $q_1\sigma \in Q_1/\sigma$ such that $g(q_1\sigma) = f(q_1) = q_2$. Hence g is surjective.

$g(x \star q\sigma) = \{g(p\sigma) : p\sigma \in x \star q\sigma\} = \{f(p) : p \in x \circ q\} = f(x \circ q) = x \circ f(q) = x \star g(q\sigma)$.

Therefore, g is a bijective S-hyper homomorphism. ■

Definition 3.19 A subset T of an S-hyper set Q is said to be an S-subhyper set if $x \circ T \subseteq T$ for all $x \in S$.

Corollary 3.20 *If T be an S-subhyper set of Q, then $1 \circ T = T$.*

Proof. T is an S-subhyper set of Q, then $x \circ T \subseteq T$, for all $x \in S$. In particular, $1 \circ T \subseteq T$. Again $q \in 1 \circ q$, for all $q \in Q$. In particular, $t \in 1 \circ t$, for all $t \in T$, implies $T \subseteq 1 \circ T$. Therefore, $T = 1 \circ T$. □

Corollary 3.21 *If $f : Q_1 \mapsto Q_2$ be an S-hyper homomorphism then $f(Q_1)$ is an S-subhyper set.*

Proof. $$x \circ f(Q_1) = \bigcup_{f(p) \in f(Q_1)} x \circ f(p) = \bigcup_{f(p) \in f(Q_1)} f(x \circ p) = f\left(\bigcup_{f(p) \in f(Q_1)} x \circ p\right)$$
$$= f\left(\bigcup_{p \in Q_1} x \circ p\right) \subseteq f(Q_1), \text{ for all } x \in S.$$ ■

Theorem 3.22 *Let Q be an S-hyper set. Then*

(i) *Q is itself an S-subhyper set.*

(ii) *$S \circ q$ is an S-subhyper set.*

(iii) *Union and Intersection of two S-subhyper sets are subhyper sets.*

Proof. (i) $x \circ Q \subseteq Q$, for all $x \in S$.

(ii) $x \circ (S \circ q) = \bigcup\{x \circ p : p \in S \circ q\} = \bigcup\{x \circ p : p \in y \circ q$, where $y \in S\} = \bigcup\{x \circ (y \circ q) : y \in S\} = \bigcup\{xy \circ q : y \in S\} = xS \circ q \subseteq S \circ q$, for all $x \in S$. Hence $S \circ q$ is an S-subhyper set.

(iii) Let Q_1 and Q_2 be two S-subhyper sets of Q.

$x \circ (Q_1 \cup Q_2) = \bigcup\{x \circ t : t \in Q_1 \cup Q_2\} = \bigcup\{x \circ t : t \in Q_1\} \cup \bigcup\{x \circ t : t \in Q_2\} = x \circ Q_1 \cup x \circ Q_2 \subseteq Q_1 \cup Q_2$, for all $x \in S$.

$x \circ (Q_1 \cap Q_2) = \bigcup\{x \circ t : t \in Q_1 \cap Q_2\} \subseteq x \circ Q_1 \cap x \circ Q_2 \subseteq Q_1 \cap Q_2$, for all $x \in S$. ■

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UNE REMARQUE SUR CERTAINES FORMES DE WEIERSTRASS

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Abstract. Dans cette note, on s'intéresse à la recherche d'une forme de Weierstrass de courbes de genre 1. On souligne sur des exemples l'intérêt de la méthode pour mettre en évidence les points de torsion ou d'ordre infini et pour obtenir des formes de Weierstrass tempérées.

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1. Introduction

La transformation classique exposée dans Cassels [4] et les algorithmes de van Hoeij [6] faisant passer de l'équation d'une courbe de genre 1 de bidegré $(2,2)$ à une équation de Weierstrass utilisent un point rationnel sur un corps K. Lorsque la courbe a plusieurs points définis dans un corps, il est parfois souhaitable d'obtenir une équation de Weierstrass où les autres points apparaissent de façon évidente.

Par exemple, lorsque la courbe C de genre 1 est donnée par une équation $F(x,y)=0$ du second degré en chaque variable dont les coefficients sont dans un corps K, les pôles ou les zéros des fonctions x et y peuvent être définis dans K. Des exemples seront donnés dans les sections 4 et 5.

Nous nous intéresserons également aux polynômes des faces des polygones de Newton de F et de l'équation de Weierstrass donnée par l'algorithme. Dans certains cas les relations entre ces polynômes sont simples et permettent de construire, si F est *tempéré*, des polynômes de Weierstrass *tempérés*.

2. L'algorithme

Rappelons que si

$$Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6$$

est un modèle de Weierstrass pour une courbe elliptique E, la fonction X sur E possède deux zéros et un pôle double au point à l'infini de la courbe tandis que la fonction Y possède trois zéros et un pôle triple en ce même point à l'infini. En outre $\frac{Y^2}{X^3} = 1$ en ce point. Lorsque la courbe de genre 1 est donnée par une équation $F(x, y)$ du second degré en chaque variable, chercher une forme de Weierstrass revient donc à trouver deux fonctions X et Y sur la courbe E, avec X (resp. Y) possédant un pôle double (resp. triple) en l'infini.

Soit $F(x, y)$ un polynôme de degré 2 en chacune des variables qui est une équation pour une courbe de genre 1 (c'est-à -dire le discriminant par rapport à l'une des variables est de degré 4 ou 3).

En ordonnant le polynôme F par rapport à y puis par rapport à x, on obtient l'écriture suivante.

$$\begin{aligned} F(x, y) &= y^2(ax^2 + bx + c) + y(a'x^2 + b'x + c') + a''x^2 + b''x + c'' \\ &= x^2(ay^2 + a'y + a'') + x(by^2 + b'y + b'') + cy^2 + c'y + c''. \end{aligned}$$

On note

$$\begin{array}{lll} M(x) = ax^2 + bx + c, & N(x) = a'x^2 + b'x + c', & R(x) = a''x^2 + b''x + c'', \\ M_1(y) = ay^2 + a'y + a'', & N_1(y) = by^2 + b'y + b'', & R_1(y) = cy^2 + c'y + c''. \end{array}$$

On pose

$$\begin{array}{lll} \tilde{M}(x) = x^2M\left(\frac{1}{x}\right), & \tilde{N}(x) = x^2N\left(\frac{1}{x}\right), & \tilde{R}(x) = x^2R\left(\frac{1}{x}\right), \\ \tilde{M_1}(y) = y^2M_1\left(\frac{1}{y}\right), & \tilde{N_1}(y) = y^2N_1\left(\frac{1}{y}\right), & \tilde{R_1}(y) = y^2R_1\left(\frac{1}{y}\right). \end{array}$$

I. Un des huit polynômes M, R, M_1, R_1, $\tilde{M}$, $\tilde{R}$, $\tilde{M_1}$, $\tilde{R_1}$ a un degré nul

En changeant au besoin x en y ou x en $\frac{1}{x}$, y en $\frac{1}{y}$, on peut supposer que F s'écrit:

$$\begin{aligned} &(1) \qquad F(x, y) = a''x^2 + \left(by^2 + b'y + b''\right)x + cy^2 + c'y + c'' \\ &(2) \qquad = (bx + c)\,y^2 + (b'x + c')\,y + a''x^2 + b''x + c''. \end{aligned}$$

Comme la courbe est de genre 1, on a $b \neq 0$.

En considérant la forme (2) de F on voit que y a deux pôles simples, l'un pour $x = -c/b$ et l'autre noté A qui est un pôle de x; en considérant la forme (1) de F on voit que x a un pôle double en A.

Donc y a un unique pôle simple qui est un pôle double de x. Par suite, la fonction $U = y(bx + c)$ possède un pôle triple en A. Donc, en multipliant (2) par $(bx + c)$, en posant $X = -ba''x$ et $Y = ba''U$, l'équation liant X et Y

$$Y^2 + (-b'X + bc'a'')Y - X^3 + (ca'' + bb'')X^2 - ba''(cb'' + bc'')X + (ba'')^2cc'' = 0$$

est une équation de Weierstrass de la courbe.

II. Aucun des huit polynômes n'est constant mais l'un d'eux est de degré 1

On peut alors écrire

$$\begin{aligned} F(x, y) &= y^2(bx + c) + y(a'x^2 + b'x + c') + a''x^2 + b''x + c'' \\ &= x^2(a'y + a'') + x(by^2 + b'y + b'') + cy^2 + c'y + c'', \end{aligned}$$

avec $ba' \neq 0$. Dans ce cas, x et y ont chacun 2 pôles distincts; de plus x et y ont un pôle simple commun A.

La fonction $X = (bx + c)(a'y + a'')$ aura donc un pôle double en A qui est un pôle simple de y. L'équation en X et y sera par suite du type **I**.

$$\begin{aligned} G(X, y) &= (b^2X + c^2a'^2 - bb'ca' + c'b^2a')y^2 + \ldots \\ &= X^2 + \ldots \end{aligned}$$

III. Cas général: les huit polynômes sont de degré 2

a) La courbe affine définie par $F(x, y)$ a un point $(x_0, y_0) \in K^2$.
Après une translation, on peut supposer $x_0 = y_0 = 0$ et on obtient $c'' = 0$. En posant $U = \dfrac{1}{x}$ et $V = \dfrac{1}{y}$, on obtient

$$\begin{aligned} &(c + c'U)V^2 + (b + b'U + b''U^2)V + a + a'U + a''U^2 = 0 \\ &(a'' + b''V)U^2 + (a' + b'V + c'V^2)U + a + bV + cV^2 = 0 \end{aligned}$$

et on est dans le cas **I** ou **II**.

b) Si la courbe projective d'équation affine $F(x, y) = 0$ a un point K-rationnel à l'infini, alors c'est le point double $(1, 0, 0)$ (ou $(0, 1, 0)$) et les tangentes au point $(1, 0, 0)$ (ou $(0, 1, 0)$) sont rationnelles. Il en résulte que le polynôme M a une racine $x_1 \in K$ (ou le polynôme M_1 a une racine $y_1 \in K$). Soit

$$U = \frac{1}{x - x_1} \qquad \left(\text{ou } V = \frac{1}{y - y_1}\right).$$

L'équation devient alors:

$$\begin{aligned} &y^2(a + U(2ax_1 + b)) + y(a' + U(2a'x_1 + b') + U^2(a'x_1^2 + b'x_1 + c')) \\ &+ a'' + U(2a''x_1 + b'') + U^2(a''x_1^2 + b''x_1 + c'') \\ &= U^2(a''x_1^2 + b''x_1 + c'' + y(a'x_1^2 + b'x_1 + c')) \\ &+ U((2ax_1 + b)y^2 + (2a'x_1 + b')y + 2a''x_1 + b'') + ay^2 + a'y + a'' \end{aligned}$$

et on est dans le cas **I** ou **II** de l'algorithme.

3. Le cas d'une cubique

Le cas d'une courbe de genre 1 définie par un polynôme de degré 3 en x et y avec un point rationnel $(x_0, y_0) \in K^2$ se ramène au calculs précédent.

Supposons que F s'écrive

$$F(x,y) = x^3 + ax^2y + bxy^2 + cy^3 + dx^2 + exy + fy^2 + gx + hy + i.$$

Après une translation, on peut supposer que $i = 0$. Comme la courbe est lisse g et h ne sont pas tous deux nuls, on peut se ramener par un changement de variables à ce que la tangente en $(0,0)$ soit la droite $y = 0$ i.e. $g = 0$.

En posant

$$x = \frac{u}{w},\ y = \frac{1}{w}$$

on obtient l'équation

$$u^3 + (a + dw)u^2 + (b + ew)u + fw + c + hw^2 = 0.$$

Si $d = 0$, on a la forme de Weierstrass, sinon en posant $w_1 = w + \dfrac{u}{d}$ on se ramème au cas **I** de l'algorithme.

4. Exemples et applications

4.1. Constructions de courbes elliptiques sur $\mathbb{Q}(t)$ avec point de 7-torsion rationnel et rang ≥ 1 sur $\mathbb{Q}(t)$. La recherche de familles de courbes elliptiques sur $\mathbb{Q}$ avec N-torsion et de rang ≥ 1 sur $\mathbb{Q}$ conduit à l'étude de la surface elliptique modulaire S_N.

Par exemple pour $N = 7$, toute courbe elliptique sur $\mathbb{Q}$ ayant un point d'ordre 7 rationnel peut être définie par l'équation suivante

$$Y^2 - (d^2 - d - 1)XY - d^2(d-1)Y = X^3 - d^2(d-1)X^2,$$

avec $d \in \mathbb{Q}$, le point $(0,0)$ est d'ordre 7.

Si on considére d comme variable alors on note S_7 la surface d'équation l'équation précédente.

Dans le cas de torsion $N = 7, 8$ et 2×6, la surface elliptique S_N est une surface $K3$ de nombre de Picard $\rho = 20$. Une possibilité pour cette recherche est de construire d'autres fibrations elliptiques de la surface S_N (voir [7], [5] et [9]).

Pour $N = 7$, on montre que la surface S_7 est birationnellement équivalente à la surface

$$-d(d-1)xy + (xy - x - y)(1 + d(xy - x - y)) = 0.$$

On sait construire des fibrations de S_7

$$S_7 \longrightarrow B \simeq \mathbb{P}^1_t$$

où B est la base de la fibration et t un générateur du corps des fonctions de B, si t est dans l'ensemble

$$\left\{\frac{d-1}{x-1}, \frac{d}{1-y}, \frac{dy-1}{x-1}, \frac{dy-1}{x}, \frac{d-1}{(x-1)(y-1)}\right\};$$

dans ce cas les fibrations construites ont des mauvaises fibres de type I_n, I_n^* et il est facile d'obtenir une équation bidegré $(2,2)$.

Nous détaillons le calcul pour $t = \dfrac{dy-1}{x}$.

On pose donc $t = \dfrac{dy-1}{x}$ et on élimine d entre les deux équations on obtient l'équation $F(x,y) = 0$ avec

$$\begin{aligned} F(x,y) &= (x-1)(tx-t+1)y^2 + (-2tx^2 + (3t-2)x + 2)y \\ &\quad - (tx+1)((t-1)x+1) \\ &= t(y^2 - 2y - t + 1)x^2 + ((-2t+1)y^2 + (3t-2)y \\ &\quad - 2t + 1)x + (t-1)y^2 + 2y - 1. \end{aligned}$$

On est dans le cas **III** de l'algorithme.

En posant $U = \dfrac{1}{x-1}$, l'équation $F(x,y) = 0$ devient $G(U,y) = 0$ avec

$$\begin{aligned} G(U,y) &= -(U+t)y^2 - (tU^2 - (t+2)U - 2t)y + ((t+1)U + t)(tU + t - 1)) \\ &= (-ty + t^2 + t)U^2 + (-y^2 + (t+2)y + 2t^2 - 1)U - t(y^2 - 2y - t + 1). \end{aligned}$$

On est alors dans le cas **II**.

Le changement de variables $X = -(U+t)(-ty + t^2 + t)$ donne l'équation $G_1(X,y) = 0$ avec

$$\begin{aligned} G_1(X,y) &= (X + t^3(t+1))y^2 - (2t^2 + t + 2)(X + t^3)y \\ &\quad + (X + t^3)(X + t^3 + 1) \\ &= X^2 + X(y^2 - (2t^2 + t + 2)y + 2t^3 + 1) \\ &\quad + t^3(-y + t + 1)(-(t+1)y + t^2 - t + 1). \end{aligned}$$

et on est dans le cas le cas **I** de l'algorithme.

On pose $Y = (X + t^3(t+1))y$ et on a l'équation $G_2(X,Y) = 0$ avec

$$\begin{aligned} G_2(X,Y) = &Y^2 - (2t^2 + t + 2)(X + t^3)Y + X^3 + (t^4 + 3t^3 + 1)X^2 \\ &+ (t+1)(2t^3 + t^2 - t + 2)t^3X + (t+1)^2(t^2 - t + 1)t^6. \end{aligned}$$

En changeant enfin X par $-X$, on aura l'équation de Weierstrass W_t

$$Y^2 + (2t^2 + t + 2)(X - t^3)Y = (X - t^3)(X - t^3 - 1)(X - t^3(t+1)).$$

Cette forme de Weierstrass permet d'obtenir le rang du groupe de Mordell-Weil de la courbe ([5]).

Tout d'abord, on a 8 points évidents sur la courbe d'équation $F(x,y)=0$:

$$\begin{array}{ll} A_1=(x=1,\infty) & A_2=\left(\dfrac{t-1}{t},\infty\right) \\ A_1'=(1,t+1) & A_2'=\left(\dfrac{t-1}{t},\dfrac{t^2-t+1}{t+1}\right) \\ A_3=\left(-\dfrac{1}{t},0\right) & A_4=\left(-\dfrac{1}{t},\dfrac{1}{t+1}\right) \\ A_5=\left(-\dfrac{1}{t-1},0\right) & A_6=\left(-\dfrac{1}{t-1},\dfrac{t+1}{t^2-t+1}\right) \end{array}$$

Le point A_1 donne le point $(X=\infty, Y=\infty)$ de W_t.

En résumé, le passage de l'équation de F à W_t est donné par les transformations

$$U=\frac{1}{x-1}, X=(U+t)(-ty+t+t^2), Y=(-X+t^3(t+1))y$$

ce qui permet de calculer les coordonnées des points suivants dans le modèle W_t

$$\begin{array}{ll} A_1'=(X=-t(2t+1), Y=t(t+1)(t^3+t^2+2t+1)) & A_2'=(0,t^3(t^2-t+1)) \\ A_3=(t^3,0) & A_5=(t^3+1,0) \\ A_4=\left(-\dfrac{t^4(t+2)}{(t+1)^2},\dfrac{t^3(t^3+2t^2+t+1)}{(t+1)^3}\right) & A_6=(-t(t^2-1),t(t+1)^2) \end{array}$$

Enfin le diviseur de la fonction y étant égal à $-A_1-A_2+A_3+A_5$ il en résulte que sur W_t les points $-A_2, A_3, A_5$ sont alignés. On obtient alors

$A_2=(X=t^3+t^4, Y=-t^4(2t^2+t+2))$.

On vérifie que, sur la courbe elliptique W_t le point $A_3=(t^3,0)$ est d'ordre 2 et que c'est le seul point d'ordre 2 sur $\mathbb{C}(t)$. On vérifie à l'aide d'un logiciel que pour $t=1$ les deux points A_2' et A_5 sont d'ordre infini et indépendants.

Nous allons montrer qu'en fait le rang du groupe de Mordell-Weil de la courbe elliptique W_t sur $\mathbb{Q}(t)$ est 2.

Les fibres singulières de la fibration

$$\begin{array}{rcl} S_7 & \longrightarrow & \mathbb{P}^1 \\ (x,y,d) & \longmapsto & t \end{array}$$

sont de type I_4^*, I_4^*, I_1,I_1,I_1,I_1. Par suite, d'après la formule [12],

$$\rho=r+2+\sum_t(m_t-1)$$

où r désigne le rang du groupe de Mordell-Weil de W_t sur $\mathbb{C}(t)$, m_t le nombre de composantes irréductibles des fibres singulières de S_7 on trouve $r=2$.

Pour déterminer le groupe de torsion on utilise le résultat suivant:

On sait que l'application de spécialisation, pour $t_0\in\mathbb{C}$

$$W_t^0(\mathbb{C}(t))_{tor}\to W_{t_0}^{ns}(\mathbb{C})$$

est injective, où $W_t^0(\mathbb{C}(t))$ est le sous-groupe de $W_t(\mathbb{C}(t))$ formé des points qui se spécialisent en des points lisses de $W_{t_0}(\mathbb{C})$ [8]. Ceci est vrai même pour les mauvaises fibres et dans ce cas $W_{t_0}^{ns}(\mathbb{C})$ est la composante connexe de zéro de la fibre du modéle de Néron de W_t.

Considérant les mauvaises fibres I_4^* et I_1 on voit que $W_t(\mathbb{C}(t))_{tor}$ est un sous groupe de $\mathbb{C} \times (\mathbb{Z})^2$ et de $\mathbb{C}^*$, ce qui implique que $W_t(\mathbb{C}(t))_{tor}$ est d'ordre 2. Il en résulte que $W_t(\mathbb{Q}(t))_{tor} = \langle A_3 \rangle$.

La forme de Weierstrass ainsi obtenue met en évidence deux points sur $\mathbb{Q}(t)$ d'ordre infini et indépendants, ce qui montre l'existence d'une infinité de courbes rationnelles sur S_7. D'autre part ajouter un point d'ordre infini sur la fibration en t définit un automorphisme d'ordre infini sur la surface S_7. On obtient ainsi le théor̀eme.

Théorème 4.1. *L'ensemble des points rationnels de la surface S_7 est Zariski dense et le groupe des automorphismes de cette surface est infini, ce groupe contenant un sous groupe isomorphe à $\mathbb{Z}^2$.*

Remarque 1.

1. On peut donner explicitement ces automorphismes en utilisant les formules habituelles d'addition sur la forme de Weierstrass W_t.
2. En utilisant en plus le paragraphe 3 on peut aussi considérer le cas $t = \dfrac{d}{1-y}$.

4.2. Constructions de courbes elliptiques sur $\mathbb{Q}$ avec point de 7-torsion rationnel et rang$\geq$ 2 sur $\mathbb{Q}$. Pour la recherche des courbes elliptiques sur $\mathbb{Q}$ avec point de 7-torsion rationnel et rang$\geq$ 2 sur $\mathbb{Q}$, on amené à résoudre des équations diophantiennes qui se présentant sous la forme $\dfrac{A(X)}{B(X)} = \dfrac{C(Y)}{D(Y)}$ $(= d)$ avec A, B, C, D des polynômes de degré ≤ 2 de $\mathbb{Q}(t)$ ou bien $A = C$ et $B = D$ deux polynômes de degré ≤ 3 sans facteurs communs. Le cas de polynômes de degré ≤ 2 a été traité dans [7]. Nous nous intéressons ici au cas $A = C$, $B = D$ polynômes de degré ≤ 3 avec $A = X^3 + pX^2 + qX + r$ et $B(X) = X\,(X-d)\,(X-e)$ (voir [5]). Nous supposerons aussi que A et B sont premiers entre eux, ce qui entraîne que leur résultant n'est pas nul.

Théorème 4.2. *Soit $A = X^3 + pX^2 + qX + r$ et $B = X\,(X-e)\,(X-d)$, avec $d\,e \neq 0$, deux polynômes premiers entre eux de $K\,[X]$. La courbe elliptique sur K définie par*

$$F(X,Y) = \frac{A(X)B(Y) - A(Y)B(X)}{X-Y}$$

a une équation de Weierstrass en Z et T

$$Z^2 + (edp + (e+d)\,q + 3r)\,ZT + SZ = T^3$$

où $S = rA(d)A(e)$ *est le résultant des polynômes* A *et* B. *De plus, le point* $(T = 0, Z = 0)$ *est d'ordre* 3 *et le point* $\left(T = \dfrac{d(d-e)rA(e)}{e^2}, Z = \dfrac{r^2(d-e)^3A(e)}{e^3}\right)$ *est en général d'ordre infini.*

Proof. Le changement de variables $x = 1/X$, $y = 1/Y$ nous donne

$$\begin{aligned} & r(ey-1)(dy-1)x^2-(r(d+e)y^2+(ped+q(e+d)-r)y+de-q)x+ry^2 \\ &\quad -(ed-q)y+e+d+p \\ &\quad = r(ex-1)(dx-1)y^2-(r(d+e)x^2+(ped+q(e+d)-r)x+de-q)y+rx^2 \\ &\quad -(ed-q)x+e+d+p. \end{aligned}$$

Puis le changement de variable $U = \dfrac{1}{ex-1}$ nous ramène au cas **II**,

$$\begin{aligned} & A(e)(dy-1)U^2+(re(e-d)y^2+(e^2(dp+q)+e(dq+r)+2rd)y-2r-qe+de^2)U \\ &\quad -r(ey-1)(dy-1) \\ &\quad = re((e-d)U-d)y^2+(dA(e)U^2+((pd+q)e^2+(qd+r)e+2rd)U+r(e+d))y \\ &\quad -A(e)U^2+(de^2-qe-2r)U-r. \end{aligned}$$

Cas $d \neq e$: Enfin le changement de variable

$$T_1 = (dy-1)(U(e-d)-d)$$

nous conduit au cas **I**. On obtient donc

$$\begin{aligned} & reT_1^2+(d^2A(e)U^2+(2re^2+pe^2d^2+dqe(e+d)-red+2rd^2)U+rd(d-e))T_1 \\ &\quad +e^2A(d)U((e-d)U-d) \\ &\quad = (d^2A(e)T_1+e^2(e-d)A(d))U^2 \\ &\quad +(((pd^2+qd+2r)e^2+(qd^2-rd)e+2rd^2)T_1-de^2A(d))U-rT_1(-eT_1-d(d-e)). \end{aligned}$$

On pose alors

$$Z_1 = (d^2A(e)T_1+e^2(e-d)A(d))U.$$

Par suite, il vient

$$\begin{gathered} -Z_1^2-(pe^2d^2+qed(d+e)+r(2d^2+2e^2-de))Z_1T_1+de^2A(d)Z_1 \\ +rT_1(eT_1+d(d-e))(-d^2A(e)T_1+e^2(d-e)A(d)) = 0 \end{gathered}$$

Quelques changements de variables nous donnent enfin la forme de Weierstrass proposée:

$$Z_1 = -\frac{Z_2}{red^2A(e)} \qquad T_1 = -\frac{T_2}{red^2A(e)}$$

$$Z_2 = Z_3 - r(d-e)^2T_2$$

$$Z = \frac{Z_3}{(de)^3} \qquad T = \frac{T_2}{(de)^2}$$

Le modèle de Weierstrass

$$Z^2+(edp+(e+d)\,q+3r)\,ZT+rA(d)A(e)Z=T^3$$

est bien défini sur K. On voit également sur ce modèle que le point $(T=0,Z=0)$ est de 3-torsion et le point $\left(T_1=-\dfrac{d(d-e)}{e},Z_1=0\right)$ soit $\left(T=\dfrac{d(d-e)rA(e)}{e^2},Z=\dfrac{r^2(d-e)^3A(e)}{e^3}\right)$ est K-rationnel en général d'ordre infini.

Cas $d=e$: On se trouve déjà au cas **I**. L'algorithme nous donne alors la même forme de Weierstrass que précédemment avec $e=d$. ∎

Remarque 2.

- Si $e=0$, $d\neq 0$ (ou si $e\neq 0$, $d=0$) le changement de variables $x=1/X$, $y=1/Y$ nous donne

  $$\begin{aligned}&r(dy-1)x^2+(rdy^2+(dq-r)y-q)x-ry^2-qy-(d+p)\\&\quad=r(dx-1)y^2+(rdx^2+(dq-r)x-q)y-rx^2-qx-(d+p)\end{aligned}$$

  c'est-à-dire le cas **II** de l'algorithme. Le changement de variables

  $$\begin{aligned}&T=-r^2(dx-1)(dy-1),\\&Z=r(dy-1)T\end{aligned}$$

  nous donne la forme de Weierstrass

  $$Z^2+(dq+3r)TZ+r^2A(d)Z=T^3.$$

- Si le polynôme A se factorise sur K, on a des points supplmentaires sur la courbe pouvant donner au plus deux points indpendants.
- Si l'on prend pour polynômes $A=X^3+aX+b$ et $B=X(X-e)$, par un calcul analogue on peut pour certaines valeurs de b et e obtenir des points de 2-torsion sur le corps contenant les coefficients des polynômes.

5. Forme de Weierstrass tempérée

Soit $P\in\mathbb{C}[x^{\pm 1},y^{\pm 1}]$. On note

$$P(x,y)=\sum_{(n,m)\in\mathbb{Z}^2}a_{(n,m)}x^ny^m.$$

On appelle polygone de Newton Δ_P associé au polynôme P, l'enveloppe convexe de l'ensemble des points $\{(n,m)\in\mathbb{Z}^2/a_{(n,m)}\neq 0\}$.

A une face τ du polygone de Newton on associe un polynôme P_τ d'une seule variable dont le degré est égal au nombre de points du réseau des entiers situés sur la face moins 1. On définit alors P_τ par

$$P_\tau = \sum_{k=0}^{\infty} a_{\tau(k)} t^k, \qquad \tau \in \Delta$$

où la paramétrisation de la face est définie dans le sens indirect sur Δ de façon à noter $\tau(0), \tau(1), ...$, les points consécutifs du réseau des entiers sur Δ.

Un polynôme de deux variables est dit tempéré si les polynômes associés aux faces de son polygone de Newton n'ont pour racines que des racines de l'unité. Si P est à coefficients rationnels et définit une courbe elliptique E, le fait d'être tempéré garantit l'appartenance du symbole de Steinberg $\{x, y\}$ au second groupe de K-théorie $K_2(E)$ [10] .

Si E possède en outre un modèle de Weierstrass tempéré, cela permet de calculer le régulateur elliptique de E donc de donner une expression de $L(E, 2)$ en terme d'une combinaison linéaire de dilogarithmes elliptiques de points de la courbe elliptique.

Le régulateur elliptique permet en outre de comparer les mesures de Mahler de polynômes définissant les mêmes courbes elliptiques [11] et dans certains cas de démontrer des relations *exotiques* sur le dilogarithme elliptique [1], [13].

Nous pouvons montrer le résultat suivant.

Proposition 1. *Soit $P \in \mathbb{Z}[x, y]$ de bidegré $(2, 2)$ définissant une courbe elliptique. On suppose P tempéré et vérifiant la condition I) de l'algorithme. Alors la forme de Weierstrass donnée par l'algorithme est tempérée.*

Proof. Soit P tempéré; il s'écrit donc

$$P(x, y) = (x + \epsilon)y^2 + (b'x + c')y + \epsilon' x^2 + b''x + \epsilon''$$

avec $\epsilon = \pm 1$, $\epsilon' = \pm 1$, $\epsilon'' = \pm 1$, $c' = 0$ ou $c' = \pm 2, \pm 1$ si $\epsilon\epsilon'' = 1$, $b'' = 0$ ou $b'' = \pm 2, \pm 1$ si $\epsilon'\epsilon'' = 1$, de sorte que les polynômes des faces $t + \epsilon$, $\epsilon t^2 + c't + \epsilon''$, $\epsilon'' t^2 + b''t + \epsilon'$, $\epsilon' t + 1$ ne possèdent que des racines de l'unité.

On a donc

$$\begin{aligned} P(x, y) &= (x + \epsilon)y^2 + (b'x + c')y + \epsilon' x^2 + b''x + \epsilon'' \\ &= \epsilon' x^2 + (y^2 + b'y + b")x + \epsilon y^2 + c'y + \epsilon''. \end{aligned}$$

On pose alors $Y = (x + \epsilon)y$ et l'on obtient, au besoin en posant $x = -X$, la forme de Weierstrass à partir de l'équation

$$Y^2 + Y(b'x + c') + (\epsilon' x^2 + b''x + \epsilon'')(x + \epsilon) = 0.$$

On vérifie aisément que ce dernier polynôme est tempéré. ∎

Corollaire 5.1. *Soit P un polynôme tempéré. On suppose qu'après une transformation convenable, le polynôme obtenu soit tempéré et satisfasse le cas I) de l'algorithme. Alors P possède une forme de Weierstrass tempérée.*

On peut donner de nombreux exemples de polynômes satisfaisant la proposition ou son corollaire.

5.1. Familles de polynômes tempérés définissant des courbes elliptiques ayant une forme de Weierstrass tempérée

1) La famille
$$y^2x + y(kx+1) + x^2$$
a la forme de Weierstrass tempérée
$$Y^2 + Y(-kX+1) = X^3.$$
L'isomorphisme est donné par
$$x = -X, \qquad y = -Y/X.$$
Le point $P = (X = 0, Y = 0)$ image du point $(x = 0, y = 0)$ est un point de 3-torsion tel que $2P = (X = 0, Y = -1)$ soit l'image du point $(x = 0, y = \infty)$.

2) La famille
$$y^2x + y(kx+1) + x^2 + x$$
a la forme de Weierstrass tempérée
$$Y^2 + kXY + Y = X^3 - X^2.$$
L'isomorphisme est donné par
$$x = -X, \qquad y = -Y/X.$$
Les zéros de x sont ceux de X, i.e. les points $P = (X = 0, Y = 0)$ et $P_1 = (X = 0, Y = -1)$. Les zéros de y à savoir $(x = 0, y = 0)$ et $(x = -1, y = 0)$ donnent les points $P = (X = 0, Y = 0)$ et $P_2 = (X = 1, Y = 0)$. Le pôle de y donne le point P_1. On vérifie facilement, avec PARI par exemple, que pour $k \neq 0, 1$, la courbe elliptique correspondante a un groupe de torsion trivial et un rang 1 avec P d'ordre infini.

Pour $k = 0$, le point P est d'ordre 5 (la courbe correspondante est la courbe modulaire $X_1(11)$).

Pour $k = 1$, les points P et P_1 sont d'ordre 4, le point P_2 est d'ordre 2.

3) La famille E_k, $k \neq 1$ ([13])
$$y^2x + y(x^2 + kx + 1) + x^2 + x$$
a la forme de Weierstrass tempérée
$$Y^2 - kXY + Y = X(X-1)^2.$$

L'isomorphisme est donné par

$$x = \frac{X(X-1)}{Y-X+1}, \qquad y = \frac{-Y}{X-1}$$

et l'isomorphisme inverse par

$$X = -x(y+1) \qquad Y = y(xy+x+1).$$

En effet, les équations

$$\begin{aligned} &y^2x + (x^2+kx+1)y + x(x+1) = 0 \\ &x^2(y+1) + x(y^2+ky+1) + y = 0 \end{aligned}$$

montrent que l'on est dans le cas II). La transformation $x = \dfrac{-X}{y+1}$ nous ramène au cas I) avec le modèle tempéré

$$\begin{aligned} &X^2 - (y^2+ky+1)X + y^2 + y = 0 \\ &(-X+1)y^2 + (-kX+1)y + X^2 - X = 0. \end{aligned}$$

La transformation $Y = -(X+1)y$ va alors donner le modèle de Weierstrass tempéré W_k

$$Y^2 - kXY + Y = X(X-1)^2.$$

Il résulte de l'isomorphisme précédent que les zéros (resp. pôles) de x dans E_k, à savoir $(x=0, y=0)$, $(x=0, y=\infty)$ (resp. $(x=\infty, y=-1)$, $(x=\infty, y=\infty)$) donnent les points $(X=0, Y=0)$, $(X=1, Y=k-1)$ (resp. $(2-k, 1-k)$,(0)) dans le modèle de Weierstrass W_k.

De même, les zéros (resp. pôles) de y dans E_k, à savoir $(x=0, y=0)$, $(x=-1, y=0)$ (resp. $(x=0, y=\infty)$, $(x=\infty, y=\infty)$) donnent les points $(X=0, Y=0)$, $(X=1, Y=0)$ (resp. $(1, k-1)$,(0)) dans le modèle de Weierstrass W_k.

Supposons $k \neq 2, 3$ et posons $P = (X=1, Y=0)$.

On a alors

$$\begin{aligned} &P = (1,0) && -P = (1, k-1) \\ &2P = (0,-1) && -2P = (0,0) \\ &3P = (2-k, -(k-1)(k-2)) && -3P = (-k+2, -k+1) \\ &4P = (3-k, 3-k) && -4P = (-k+3, -k^2+4k-4) \\ &5P = \left(\frac{1}{(k-2)^2}, \frac{k-1}{(k-2)^3}\right) \\ &-5P = \left(\frac{1}{(k-2)^2}, -\frac{(k-1)(k-3)^2}{(k-2)^3}\right) \\ &6P = \left(\frac{k^2-5k+7}{(k-3)^2}, \frac{(k-2)^2(k^2-5k+7)}{(k-3)^3}\right) \\ &-6P = \left(\frac{k^2-5k+7}{(k-3)^2}, -\frac{1}{(k-3)^3}\right). \end{aligned}$$

Donc si $k = 2$, $3P = -2P$ et le point P est un point de 5-torsion.

Et si $k = 3$, $4P = -2P$ et le point P est un point de 6-torsion.

Si $k \neq 2, 3, 1$ (cas non elliptique), le point P est d'ordre infini.

Par ailleurs, les zéros de x s'envoient par l'isomorphisme sur les points $-2P$ et $-P$; les pôles de x s'envoient sur $-3P$ et (0). Les zéros de y s'envoient par l'isomorphisme sur les points $-2P$ et P; les pôles de y s'envoient sur $-P$ et (0).

5.2. Les modèles tempérés de la courbe $21A$.

La courbe $21A$ des tables de Cremona ayant pour modèle de Weierstrass tempéré

$$Y^2 + XY = X^3 + X$$

possède un autre modèle de Weierstrass tempéré

$$Y_1^2 + 3X_1Y_1 = X_1(X_1 - 1)^2.$$

Ce dernier modèle est obtenu avec l'algorithme précédent à partir du modèle [3] réciproque

$$y^2 + y(x^2 + 3x + 1) + x^2 = 0$$

Il existe en outre deux autres modèles réciproques de la courbe $21A$ [3], le modèle

$$(x + 1)^2y^2 + xy + (x + 1)^2 = 0$$

et le modèle

$$y^2(x + 1)^2 + y(2(x + 1)^2 - 9x) + (x + 1)^2 = 0$$

auquels s'appliquent la proposition ou le corollaire; mais on obtient dans les deux cas le modèle tempéré de Cremona.

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NON PERIODIC SOLUTIONS OF FOURTH ORDER NONLINEAR EQUATIONS

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Abstract. In this paper we obtain a result of existence of solutions for a 4th order nonlinear equation subject to non periodic boundary conditions.

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1. Introduction

The purpose of this paper is to provide sufficient and/or necessary conditions on $f(x) \in C^o(\overline{I})$ $(\overline{I} = [0,1])$ for the existence of a solution to equations of the type:

$$\text{(i)} \qquad \begin{cases} \dfrac{d^4u(x)}{dx^4} + g(x, u(x)) = f(x) \\ \dfrac{d^3u(0)}{dx^3} = \dfrac{d^2u(0)}{dx^2} = 0 \\ \dfrac{d^3u(1)}{dx^3} = \dfrac{d^2u(1)}{dx^2} = 0 \end{cases}$$

where $g(x, \xi) \in C^o(\overline{I} \times \Re)$.

The conditions on g will be conditions of sign and not conditions of growth. More exactly, we assume $g(x, \xi) \geq 0$ for all $\xi \geq 0$ and for every $x \in \overline{I}$ and that $g(x, \xi) \leq 0$ for all $\xi \leq 0$ and for every $x \in \overline{I}$. An important example is the jumping nonlinearities (i.e. $g(x, \xi) = \mu\xi^+ - \nu\xi^-$ for all $x \in \overline{I}$, where: $\mu \geq 0$ and $\nu \geq 0$)

Boundary value problems associated to fourth order equations raised a lot of interest in these years.

The assumptions that we are going to introduce on the nonlinearity do not prevent the linear or the sub-linear cases. Therefore they seem not directly to be tackled by Mountain Pass Theorem (MPL) (recall that in [1] the authors considered the superlinear situation).

The fourth derivative makes methods such sub or supersolutions not easily applicable. Indeed, these methods need a maximum principle for the following linear inequality [2]:

$$\text{(ii)} \qquad \begin{cases} \dfrac{d^4u(x)}{dx^4} + \omega.u(x) \geq 0 \\ \dfrac{d^3u(0)}{dx^3} = \dfrac{d^2u(0)}{dx^2} = 0 \\ \dfrac{d^3u(1)}{dx^3} = \dfrac{d^2u(1)}{dx^2} = 0 \end{cases}$$

or for some related inequalities. That's true for a second order equation, if $\omega \geq 0$, and if the second derivative coefficient is negative but is in general false for a fourth order differential equation.

The linear term has a bidimensional kernel. The non invertibility (resonance) of this term raises a lot of problems [13].

Abstract methods that use the surjectivity of a nonlinear operator $T\colon V \to V^*$, where V is a Banach space, are applicable with difficulty in our situation. For instance, the results in [3] require a one-dimensional kernel for a linear application L associated to T. Again, this is usually true for a second order differential equation, whereas, in general, it is false for a fourth order equation. It seems also hard to generalize the use of these theorems to bidimensional kernels. With respect to some methods relying on the application of the Schauder fixed point theorem, we observe that the usual approach based on the consideration of the Green function and the evaluation of the norm of the inverse, appears rather complicated, due to the presence of a fourth derivative and the difficulty to estimate the norm of the inverse operator [4].

In this work we use an abstract Theorem of Hess [5] which is very suitable for ours boundary conditions.

Several articles deal with similar problems, even if the case of the *periodic* boundary conditions is more often considered. In some cases, the authors considered also more general nonlinearities by taking into account the presence of lower order derivatives or higher order systems. However, the nonlinearities and boundary conditions treated in this work, appear not yet fully investigated.

We present now a brief discussion of some related results appeared in the literature.

Ward [7], using some a priori bounds, proved that *periodic* solutions exist for a second order differential equation with a nonnull linear term in the first derivative.

Besides, Bates and Ward [8] studied *periodic* solutions for a n-order system and allowed nonlinearities in the derivative until the $n-1$-th degree, too.

By means of a continuation theorem of Leray-Schauder [9], they showed the existence of periodic solutions under assumptions of sub-linear growth (at infinity) or super-linear growth (at zero) for the function. He also considered the possibility of nonlinearities in lower derivative which depend from nonlinear components of the vector g. If the nonlinearity in the function is supposed to be superlinear at infinity (i.e. $\frac{\|g(\xi)\|}{\xi} \to +\infty$ if $\xi \to +\infty$), these results don't apply.

Further general results were obtained by Ward [10] using topological degree. He studied *periodic* solutions of $2n$-order differential equations where nonlinearities until the n-order derivative are allowed. Ward requires that the nonlinearity satisfies: $g(t,\xi)\text{sign}\xi \geq \alpha(t)$ if $|\xi| \to +\infty$ for every $t \in \overline{I}$, where $\alpha(t) \not\equiv 0$ and $\alpha(t) \geq 0$. Note that, the function $g(t,\xi) = \xi^+$ for every $t \in \overline{I}$ doesn't satisfy the above condition but it satisfies ours (see Theorems 4 and 4A in the cap. 4).

Finally, we recall that Ward in [11] proved the existence of periodic solutions for a class of $n-$th order scalar differential equations, using only sign conditions and one-sided growth restrictions on the nonlinearity.

Fučík[12] has got other results about periodic solutions in two variables for fourth order equations. In his conference [12] the author, by using Leray-Schauder topological degree, treats fourth order nonlinear problems (only in the function), when the function depends also from a time-variable; he obtains results if g is bounded or at most linear. Our work gives a solution to the *Open problem* at page 335 if the function u doesn't depend from t (stationary case). Moreover it gives a solution to the problems for the case of jumping nonlinearities (cap. 7) with $\nu = 0$ that isn't studied in Fučík's article ("TERRA INCOGNITA" for the author).

In conclusion, we also mention the papers of V. Khoi Le and K. Schmitt in [14] and [15], where the authors have used a method based on elliptic regularization in order to find several results of existence of solutions for partial and ordinary differential equations.

We recall the following nonlinear boundary value problem:

$$\begin{cases} \dfrac{d^4u(x)}{dx^4} = \alpha \left(k|u'|^{\alpha-2}u'\right)' \quad \text{in } (0,T) \\ \dfrac{d^2u(0)}{dx^2} = \dfrac{d^2u(T)}{dx^2} = 0 \\ \alpha k(0)|u'(0)|^{\alpha-2}u'(0) = \dfrac{d^3u(0)}{dx^3} \\ \alpha k(T)|u'(T)|^{\alpha-2}u'(T) = \dfrac{d^3u(T)}{dx^3} \end{cases}$$

where $k \in C^1[0,T]$, which was considered by Khoi Le and Schmitt and has some relations with our problem.

In this case, it is clear that constants are trivial solutions. To obtain nontrivial solutions the authors use a method based on elliptic regularization. They treat with polynomial nonlinearities on the first derivative. The boundary conditions are the same as those of (i) if $k(0) = k(1) = 0$ ($T = 1$). Also, in this case, our results are not contained in that work.

2. The weak problem

Before stating our main results, we put together some notations and definitions we employ. $I = (0,1)$. The Hilbert space $W^{2,2}(I)$ $(W^{2,2})$ is endowed with the inner product

$$\langle u, v\rangle = \int_I \left(\frac{d^2u(x)}{dx^2}\frac{d^2v(x)}{dx^2} + u(x)v(x)\right) dx$$

and the norm $\|w\| = \langle w, w\rangle^{\frac{1}{2}}$, while (u, v) indicates the inner product in $L^2(I)$, i.e., $(u, v) = \displaystyle\int_I uvdx$.

We will consider the problem:

$$(1) \qquad \begin{cases} \dfrac{d^4u(x)}{dx^4} + g(x, u(x)) = f(x) \\ \dfrac{d^3u(0)}{dx^3} = \dfrac{d^2u(0)}{dx^2} = 0 \\ \dfrac{d^3u(1)}{dx^3} = \dfrac{d^2u(1)}{dx^2} = 0 \end{cases}$$

where $f(x) \in C^0(\overline{I})$ and $g(x, \xi) \in C^0(\overline{I} \times \Re)$.

We will impose suitable conditions on $g(x, \xi)$ which will allow us to find necessary and sufficient conditions on $f(x)$ so that there exists at least one solution for equation (1).

We make the following hypothesis:

(A) $f(x) \in L^1(I)$.

(B) $g(x, \xi)$ of Carathéodory.

(C) $\forall\, \tau > 0, \displaystyle\int_I \sup_{|\xi|\le\tau} g(x, \xi)dx < \infty$.

Remark A. The assumption (C) doesn't put conditions on the "functional" growth of $g(x, \xi)$, but restricts only to the x dependence, thanks to the validity of the following Sobolev (compact) embedding in one-dimension:

(C1) $W^{2,2}(I) \overset{\hookrightarrow}{\hookrightarrow} C^0(\overline{I})$.

The conditions (B), (C) and (C1) ensure that

(D) $g(x, u(x)) \in L^1(I)$, $\forall u \in W^{2,2}(I)$.

Moreover, from (D)+(C1) we have:

(E) $g(x, u(x))v(x) \in L^1(I)$, $\forall v \in W^{2,2}$.

Therefore, if u satisfies (1), then:

$$\int_I \Big(\frac{d^2u(x)}{dx^2}\frac{d^2v(x)}{dx^2} + g(x,u(x))\Big)v(x)dx = \int_I f(x)v(x)dx,\ \forall v \in W^{2,2} \tag{2}$$

for (E)+(C1), equation (2) is well defined in $W^{2,2}$.

3. Abstract results

For the proof of our result, we will rely on an abstract theorem of P. Hess [4].

Theorem 1. *Let* $T : \Re^n \to \Re^n$ *be a continuous map. We suppose* $\exists R > 0$*:*

$$\langle Tw, w\rangle > 0,\ \forall w : \|w\| = R,$$

then $\exists x_0$ *with* $\|x_0\| < R$ *:*

$$Tx_0 = 0.$$

To generalize this theorem to arbitrary Banach (possibly infinite dimensional) spaces, we must impose some conditions on the operator T [5].

Let V be a reflexive separable Banach space. We impose on $T : V \to V^*$ the following conditions:

(i) T is continuous from each finite-dimensional subspace F of V.

(ii) to the weak topology on V^*, if $v_n \rightharpoonup v$ and $Tv_n \rightharpoonup g$, then $Tv = g$.

(iii) T maps bounded sets of V to bounded sets of V^*

The following theorem has been proved by Hess under conditions lightly more general than ours. Our proof simplifies that in [4].

Theorem 2. *If* $T : V \to V^*$*, satisfies* (i), (ii) *and* (iii) *and* $\exists R > 0$:

(iv) $\langle Tw, w\rangle > 0,\ \forall\ w : \|w\| = R,$

then $\exists\ x_0 \in V : \|x_0\| < R$ *and*:

$$Tx_0 = 0.$$

Proof. Let V_n be spaces of finite dimension such that $V_1 \subset V_2 \subset V_3 \subset ...$ and $\bigcup_{n=1}^{\infty} V_n$ is dense in V.

Let $j_n : V_n \to V$ be the canonical immersion and $j_n^* : V^* \to V_n^*$ denotes the adjoint. We put:

$$T_n = j_n^* T j_n : V_n \to V_n^*.$$

Let $w \in V_n$ be such that $\|w\| = R$. Since:

$$\langle T_n w, w\rangle = \langle Tw, w\rangle > 0$$

for the theorem (1) $\exists\ v_n : \|v_n\| < R$ and $T_n v_n = 0$.

By reflexivity, we can suppose that $\{v_n\}_{n\in\mathbf{N}}$ converges weakly to v.

Therefore, v_n is bounded in V. For (iii), $\{Tv_n\}_{n\in\mathbf{N}}$ is bounded in V^*.

If $w \in \displaystyle\bigcup_{n=1}^{\infty} V_n$, then there exists n_0 such that $w \in V_n$, $\forall n \geq n_0$. For these n, we have:

$$\langle Tv_n, w\rangle = \langle T_n v_n, w\rangle = 0. \tag{3}$$

Since $\displaystyle\bigcup_{n=1}^{\infty} V_n$ is dense in V and $\{Tv_n\}_{n\in\mathbf{N}}$ is bounded from (3) we have $Tv_n \rightharpoonup 0$ in V^*. From (ii), we conclude that:

$$Tv = 0.$$

4. Existence of solutions for the weak problem

In addition to (B) and (C), we suppose that $g(x,\xi)$ satisfies:

$$g(x,\xi)\ \mathrm{sign}\ \xi \geq 0, \quad \forall \text{ a.e. } x \in I,\ \forall\ \xi \in \Re. \tag{F}$$

We study the following problem:

$$\begin{cases} \displaystyle\int_I \Big(\frac{d^2u(x)}{dx^2}\frac{d^2v(x)}{dx^2} + u(x)v(x)\Big) - \lambda_1 \int_I u(x)v(x) \\ \qquad \displaystyle + \int_I g(x,u(x))v(x) = \int f(x)v(x), \quad \forall\ v \in W^{2,2}(I), \end{cases} \tag{4}$$

where λ_1 is the lower eigenvalue of the compact linear operator $B\colon W^{2,2} \to W^{2,2}$ defined as follows:

The application

$$v \to \int_I uv$$

is, for each *fixed* $u \in W^{2,2}$, a linear and continuous functional on $W^{2,2}$, therefore by the Riesz theorem, there exists a unique $Bu \in W^{2,2}$ such that:

$$(u,v) = \int_I uv = \langle Bu, v\rangle\,,\ \forall\ v \in W^{2,2}.$$

It is easy to prove that $B \in \mathcal{L}\,(W^{2,2}, W^{2,2})$.

For the immersion (C1), we have $B \in \mathcal{K}\left(W^{2,2}, W^{2,2}\right)$ and B is self-adjoint.

Using Freedholm's theory for compact operators, we have the existence of a first eigenvalue $\lambda_1 > 0$, and there exists a diverging sequence of eigenvalues of B such that: $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_4 \ldots \rightarrow \infty$, dim $V_{\lambda_i} < \infty$ for every $i \in \mathbf{N}$, where V_{λ_i} is the eigenspace associated to the eigenvalue λ_i. We define:

$$L := I - \lambda_1 B$$

so that $\mathrm{KerL} = \mathrm{V}_{\lambda_1}$.

Since $v \rightarrow \int_I fv$ is a linear and continuous functional on $W^{2,2}$, by the Riesz theorem there exists a unique $\mathcal{F} = F(u) \in W^{2,2}$ such that:

$$(f, v) = \int_I fv = \langle \mathcal{F}, v \rangle = \langle F(u), v \rangle\,, \ \forall\ v \in W^{2,2}.$$

In this case, $F : W^{2,2} \longrightarrow W^{2,2}$ is the (affine and continuous) constant application

$$F(u) = \mathcal{F},\ \forall\ u \in W^{2,2}.$$

With respect to the nonlinearity, it is easy to prove that the linear form $v \rightarrow \int_I g(x, u(x))v(x)$ is, for each *fixed* $u \in W^{2,2}$, continuous (by using (C1), the hypothesis (B) and the Lebesgue theorem). So that, using again the Riesz theorem, there exists a non linear application $G : W^{2,2} \rightarrow W^{2,2}$ such that:

$$\int_I g\left(x, u(x)\right) v(x) = \langle G(u), v \rangle \quad \forall\ v \in W^{2,2}.$$

It is standard to prove that such application is continuous.

Therefore, our problem becomes a problem formulated in a Hilbert space. More precisely, we define the operator:

$$T = L + G - F : W^{2,2} \rightarrow W^{2,2}. \tag{5}$$

The zeroes of the operator T defined in (5) are the solutions of (4).

Now, we must prove that T satisfies (i), (ii), (iii) and (iv).

Hypothesis (i) is immediate (T is continuous from $W^{2,2} \rightarrow W^{2,2}$).

Hypothesis (ii) is easily proved if we decompose T into its own linear part L and into its nonlinear part G ((ii) is clearly true for the constant part F).

For the linear part we use the following [6]:

Theorem. *Let E and F be two Banach spaces. If L is a linear operator, continuous from* $(E, \|\cdot\|) \rightarrow (F, \|\cdot\|)$, *then L is continuous from* $(E, \mathrm{weak}) \rightarrow (F, \mathrm{weak})$.

For the non linear part, we use the compactness of the immersion (C1) and hypothesis (B).

The hypothesis (iii) is verified by use of (C) and by following the equivalent definition of the norm $\|Tu\| = \sup\limits_{\|v\| \leq 1} \langle Tu, v \rangle$.

The (iv) is verified by imposing on constant part F a condition that will be formulated later on.

Therefore, we can apply Theorem 2 and obtain:

Theorem 3. *If g satisfies* (B) *and* (C) *and there exists $R > 0$ such that:*

$$\text{(5A)} \quad \begin{cases} \displaystyle\int_I \left(\frac{d^2 w(x)}{dx^2} \frac{d^2 w(x)}{dx^2} + w^2(x) \right) - \lambda_1 \int_I w^2(x) \\ \displaystyle\qquad + \int_I g\,(x, w(x))\, w(x) - \int_I fw > 0, \\ \qquad\qquad\qquad \forall w \in W^{2,2} : \|w\| = R, \end{cases}$$

then equation (4) *has at least one solution $u_o \in W^{2,2}$ such that $\|u_o\| < R$.*

We note that, in Theorem 3, the hypothesis (F) has not been used.

From the equality:

$$\int_I \left(\frac{d^2 u(x)}{dx^2} \right)^2 = (\lambda_1 - 1) \int_I u^2, \quad \forall u \in \mathrm{KerL} = \mathrm{V}_{\lambda_1},$$

we have that all eigenvalues of B are ≥ 1. Indeed, $\lambda_1 = 1$ as:

$$\dim V_{\lambda_1} = \dim \mathrm{Ker} L = 2$$

and the eigenvectors of λ_1 are given from $ax + b$ where a and $b \in \Re$. Therefore, the solutions of equation (4) are the same of the equation (2).

Since B is self-adjoint and compact, it can be represented as:

$$Bv = \sum_{i=1}^{+\infty} \mu_i < w_i, v > w_i, \ \forall \ v \in W^{2,2},$$

where w_i is the eigenvector of B, which corresponds to the characteristics numbers $\mu_i = \dfrac{1}{\lambda_i}$.

Such representation allows to obtain the following valuation:

if $v \in Im(L) = Ker(L)^{\perp}$ then:

$$\begin{aligned} \langle v, (I - \lambda_1 B) v \rangle = & \ \|v\|^2 - \lambda_1 \langle v, Bv \rangle = \|v\|^2 - \lambda_1 \sum_{i=1}^{+\infty} \mu_i \langle w_i, v \rangle w_i \\ \geq & \ \|v\|^2 - \lambda_1 \mu_2 \|v\|^2 = \|v\|^2 (1 - \lambda_1 \mu_2) > 0. \end{aligned}$$

Hence, we have proved that $\exists m > 0$ such that:

$$\text{(6)} \qquad a(v, v) \geq m \|v\|^2, \ \forall \ v \in Ker(L)^{\perp},$$

where $a(v, v) = \langle v, Lv \rangle = (\ddot{v}, \ddot{v}) + (v, v) - \lambda_1 (v, v) \ \forall \ \ v \in W^{2,2}$.

Therefore, on the subspace $Ker(L)^{\perp}$ the norm of $W^{2,2}$ is equivalent to the one induced from the bilinear form a.

Decomposing every vector of $W^{2,2}$ in the component in KerL = $\mathrm{V}_{\lambda_1}$and in the normal component ($\in KerL^{\perp} = ImL$), we see that the bilinear form a is semidefinite positive and is, therefore, l.s.c. for the weak topology, i.e., if $v_n \rightharpoonup v$, then:

$$\liminf_{n\to\infty} a(v_n, v_n) \geq a(v, v). \tag{7}$$

Now, we search a condition on $f(x)$ more "applicable" of the $(5A)$.

We put:

(I) $\liminf_{\xi\to+\infty} g(x,\xi) = d(x), \quad \forall$ a.e. $x \in I$

(II) $\limsup_{\xi\to-\infty} g(x,\xi) = c(x) \quad \forall$ a.e. $x \in I$.

We prove the following (cfr. Fučík [4]):

Theorem 4. *Suppose that g satisfies* (B), (C) *and:*

(F) $g(x,\xi)$ sign $\xi \geq 0, \quad \forall$ a.e. $x \in I, \ \forall\ \xi \in \Re$.

Suppose further that $f(x) \in L^1(I)$ satisfies:

$$\int_I f\phi < \int_I (d\phi_+ - c\phi_-), \ \forall\ \phi \not\equiv 0 \in Ker(L), \tag{8}$$

then problem (2) *has at least one solution.*

Proof. By contradiction, assume (see Theorem 3) that there exists a sequence $\{w_n\}_{n\in\mathbf{N}}$ with $\|w_n\| \to \infty$ such that:

$$\begin{cases} \displaystyle\int_I \Big(\frac{d^2w_n(x)}{dx^2}\frac{d^2w_n(x)}{dx^2} + {w_n}^2(x)\Big) - \lambda_1 \int_I {w_n}^2(x) \\ \qquad\displaystyle + \int_I g\left(x, w_n(x)\right) w_n(x) - \int_I f(x) w_n(x) \leq 0 \end{cases} \tag{8A}$$

If we put $v_n = \dfrac{w_n}{\|w_n\|}$, we may suppose that $v_n \rightharpoonup v$. We wish to show that $v \not\equiv 0$ and $v \in \mathrm{KerL} = \mathrm{V}_{\lambda_1}$, i.e., $a(v,w) = 0, \ \forall w \in W^{2,2}$. We can suppose that $v_n \xrightarrow{L^2} v$ and $v_n \longrightarrow v$ a.e.

From (8A), we obtain:

$$a(v_n, v_n) + \frac{1}{\|w_n\|}\Big(\int_I g(x, w_n(x))v_n(x) - \int_I f v_n\Big) \leq 0. \tag{8B}$$

Hence, the following relations are true:

$$\frac{1}{\|w_n\|}\int_I f v_n \overset{n\to\infty}{\longrightarrow} 0$$

$$\frac{1}{\|w_n\|}\Big(\int_I g(x, w_n(x)) v_n(x)\Big) \geq 0, \ \forall\, n \in \mathbf{N}.$$

The last, by relation (F). We have, therefore, obtained:

$$\limsup_{n\to\infty} a(v_n, v_n) \leq 0. \tag{9}$$

From the weak l.s.c. of the form $a(\cdot,\cdot)$, we obtain:

$$a(v,v) \leq \liminf_{n\to\infty} a(v_n, v_n) \leq \limsup_{n\to\infty} a(v_n, v_n) \leq 0$$

and, since a is positive semidefinite, we get:

$$a(v,v) = 0.$$

This relation implies that $v \in \mathrm{KerL} = \mathrm{V}_{\lambda_1}$.

To see this, put $v = (v - \phi) + \phi$, where $\phi \in \mathrm{KerL} = \mathrm{V}_{\lambda_1}$. Since $a(\phi, w) = 0$ for all $w \in W^{2,2}$, we obtain:

$$a(v - \phi, v - \phi) = 0.$$

From (6), we get:

$$\|v - \phi\| = 0.$$

Therefore, $v \in \mathrm{KerL} = \mathrm{V}_{\lambda_1}$.

Now, we calculate $\|v_n - v\|$.

If P is the projection $P : W^{2,2}(I) \to \mathrm{KerL} = \mathrm{V}_{\lambda_1}$and P^c is the projection $P^c : W^{2,2} \to ImL$, we have that $P^c + P = I$ and then:

$$\|v_n - v\| \leq \|P(v_n - v)\| + \|P^c(v_n - v)\|.$$

Then, since $\|P(v_n - v)\|^2 = \lambda_1(P(v_n - v), P(v_n - v))$ and $v_n \overset{L^2}{\longrightarrow} v$, one has:

$$\|P(v_n - v)\| \to 0.$$

Moreover, for (6) , (9) and since $v \in \mathrm{KerL} = \mathrm{V}_{\lambda_1}$, we have:

$$\|P^c(v_n - v)\|^2 \leq \frac{1}{m} a(v_n - v, v_n - v) = \frac{1}{m} a(v_n, v_n) \leq 0.$$

Then $v_n \overset{W^{2,2}}{\longrightarrow} v$ and, therefore, $v \not\equiv 0$.

By passing to limit in (8B), since $\lim\limits_{n\to\infty} a(v_n, v_n) = 0$, we have:

$$\limsup_{n\to\infty} \frac{1}{\|w_n\|}\int_I g(x, w_n(x)) v_n(x) \leq \liminf_{n\to\infty} \frac{1}{\|w_n\|}\int_I f v_n$$

i.e.

$$\limsup_{n\to\infty}\int_I g(x,w_n(x))v_n(x)\le\liminf_{n\to\infty}\int_I fv_n. \tag{10}$$

By Fatou's lemma we have:

$$\liminf_{n\to\infty}\int_I g(x,w_n(x))v_n(x)\ge\int_I d(x)v(x)-c(x)v(x). \tag{11}$$

Since $v\in\mathrm{KerL}=\mathrm{V}_{\lambda_1}$and $v\not\equiv 0$, (10) and (11) contradict hypothesis (8). ■

If (I) becomes:

$$\liminf_{\xi\to+\infty} g(x,\xi)=+\infty,\quad\forall\text{ a.e. } x\in I, \tag{M}$$

then (8) is not trivial only in the case $\phi_+=0$, i.e., $\phi=-\phi_-=ax+b$, where a and b are two real numbers such that $ax+b\le 0$ for every $x\in\overline{I}$.

If $\phi_{-b}=-b(1-x)$, where $b<0$, then we have: $\phi_{-b}\le\phi_-$ for every ϕ_- such that $\phi_-(0)=-b=\phi_{-b}(0)$. Since, for (F), $c(x)\le 0$ for every $x\in\overline{I}$, we have that $\phi_{-b}c\ge\phi_-c$ for every ϕ_- con $\phi_-(0)=-b$. In (8), we replace therefore only cases with: 1) $a<0$ and $b=0$; 2) $a=0$ e $b<0$; 3) and the worst case $a=-b>0$ i.e. $\phi_{-b}=\phi_-$ and we get:

$$\int_I xf(x)>\int_I xc(x) \tag{12}$$

$$\int_I f(x)>\int_I c(x) \tag{13}$$

$$\int_I f(x).(1-x)>\int_I c(x).(1-x). \tag{14}$$

Therefore, we have proved the following:

Theorem 4A. *Let $g(x,\xi)$ be a function of Carathéodory that satisfies the conditions* (C), (M) *and* (F). *If $f\in L^1(I)$ satisfies the hypothesis* (12), (13) *and* (14), *then equation* (2) *has at least one solution in $W^{2,2}$.*

5. Existence of one solution for the classical problem

The following theorem will be basic in our further consideration:

Theorem 5. *If:*

$$\begin{aligned}&f(x)\in C^0(\overline{I})\\&g(x,\xi)\in C^0(\overline{I}\times\Re)\end{aligned} \tag{N}$$

then every weak solution of (4) (*or of* (2)) *is in $C^4(\overline{I})$ and satisfies the classical equation* (1).

Proof. See the appendix. ■

Remark 1. If $g(x, u(x))$ satisfies hypothesis (N), then it verifies automatically hypothesis (C). Therefore, in this case, the only non trivial hypothesis is (F).

Corollary 6. *If* $g(x, \xi) \in C^0(\overline{I} \times \Re)$ *satisfies* (F) *and* (M) (*or* (I)), *while* $f(x) \in C^0(\overline{I})$ *satisfies* (12), (13) *and* (14) (*or* (8)), *then equation* (1) *has at least one solution.*

Proof. We need only to apply Theorem 4A, Remark 1 and Theorem 5. ■

Corollary 7. *If* $c(x) \equiv -\infty$, $\forall x \in \overline{I}$ (*for example, the linear case*), (M) *is true and* $g(x, \xi) \in C^0(\overline{I} \times \Re)$, *then the condition* (8) *is trivially verified, and therefore equation* (1) *has at least a solution for every* $f \in C^0(\overline{I})$ (*well known result in the linear case*).

We assume, now, the following hypothesis in the study of the necessary conditions on $f(x) \in C^0(\overline{I})$:

$$c(x) \leq g(x, \xi), \ \ \forall x \in \overline{I} \ \text{ and } \ \forall \xi \in \Re. \tag{O}$$

We note that for hypothesis (F) we have that: $c(x) \leq 0$ for every $x \in \overline{I}$.

If u satisfies equation (2), then, choosing $v = ax + b$ so that segment $ax + b$ belongs to the first quadrant for every $x \in \overline{I}$, using hypothesis (O) we obtain that the following conditions are necessary for the existence of solution of equation (2)

$$\int_I x f(x) \geq \int_I x c(x) \tag{15}$$

$$\int_I f(x) \geq \int_I c(x) \tag{16}$$

$$\int_I f(x).(1-x) \geq \int_I c(x).(1-x) \tag{17}$$

We summarize:

Theorem 8. *If* $g(x, \xi) \in C^0(\overline{I} \times \Re)$ *satisfies* (F), (M) *and* (O), *and* $f(x) \in C^0(\overline{I})$, *then conditions* (12), (13) *and* (14) *are sufficient, while conditions* (15), (16) *and* (17) *are necessary for the existence of at least one solution for the equation* (1).

Remark 1A. This theorem is true for the weak solutions of (2) if $f \in L^1(I)$ and if $g\,(x, \xi)$ is of Carathéodory and satisfies (C), (M), (O) and (F), where condition (O) is for a.e. $x \in I$ and $\forall\ \xi \in \Re$.

Now, what happens when there is an equality in at least one of the conditions (15), (16) or (17)?

Do they suffice for the existence of at least one solution for equation (1)?

If $g(x, \xi) \in C^0(\overline{I} \times \Re)$, instead of the hypothesis (O), satisfies the following (more restrictive) condition:

(P) $g(x,\xi) \equiv 0, \quad \forall x \in I$ and $\forall \xi \leq 0$,

we shall show that (P) implies the next result:

Theorem 9. *Let* $g(x,\xi) \in C^0(\overline{I} \times \Re)$ *be a function that verifies hypotheses* (F), (M) *and* (P), *and* $f(x) \in C^0(\overline{I})$.

If

$$\int_I f(x)dx = 0, \tag{18}$$

then the further condition:

$$\int_I xf(x)dx = 0 \tag{18A}$$

is necessary and sufficient so that equation (1) *had infinitely many non-positive solutions.*

If

$$\int_I f(x)dx > 0, \tag{19}$$

then the further conditions:

$$\int_I xf(x)dx > 0 \tag{19A}$$

$$\int f(x)dx > \int_I xf(x)dx \tag{19B}$$

are necessary and sufficient so that equation (1) *has at least one solution.*

Proof. Suppose first that (1) has at least one solution. Then, for Theorem 8, one has that conditions (18) and (18A) are necessary.

Now, let us assume (18) and (18A). It follows that this linear

$$\begin{cases} \dfrac{d^4u(x)}{dx^4} = f(x) \\ \dfrac{d^3u(0)}{dx^3} = \dfrac{d^2u(0)}{dx^2} = 0 \\ \dfrac{d^3u(1)}{dx^3} = \dfrac{d^2u(1)}{dx^2} = 0 \end{cases} \tag{20}$$

has a solution given by:

$$u(x) = \alpha + \beta x + \int_0^x \left(\int_0^s \left(\int_0^t \left(\int_0^r f(m)dm \right) dr \right) dt \right) ds$$

where α and $\beta \in \Re$.

Now, we choose α and β so that:

$$\max_{x\in\overline{I}} u(x) \leq 0.$$

For the hypothesis (P), we have found infinitely many not positive solutions of equation (1).

Now, suppose that (19), (19A) and (19B) are true. Then, for Theorem 8, equation (1) has at least one solution.

In view of Theorem 8, the last step of the proof concerns with the fact that there exist no solution of equation (1) in the following cases below:

1^o Case:

$$\int_I xf = 0$$

$$\int_I f > 0.$$

2^o Case:

$$\int_I xf = \int_I f > 0. \tag{21}$$

About the first case, we assume to the contrary that there exists one solution $u(x)$ of problem (1). If we integrate (1) on I, we have:

$$\int_I g\,(t, u(t))\,dt > 0. \tag{22}$$

If we multiply the same equation for x and we integrate on I, we get:

$$\int_I tg\,(t, u(t))\,dt = 0 \tag{23}$$

since $g(x,\xi) \geq 0$ for all $x \in \overline{I}$ and for every $\xi \in \Re$ relation (22) contradicts (23).

About the second case, we suppose by contradiction that there exists one solution $u(x)$ of problem (1). Hypothesis (21) is equivalent to:

$$\int_I \left(\int_0^x f(t)dt\right) dx = 0. \tag{24}$$

If we integrate first equation (1) from 0 to x and then on I, we get:

$$\int_I \left(\int_0^x g\,(t, u(t))\,dt\right) dx = \int_I \left(\int_0^x f(t)dt\right) dx = 0. \tag{25}$$

Since the function:

$$q(x) = \int_0^x g(t, u(t))\,dt$$

is monotone nondecreasing with $q(0) = 0$ and $q(1) = \int_I g(t, u(t))\, dt = \int_I f > 0$, the relation (25) is the needed contradiction. ∎

Remark 1B. If the equality (P) means for a.e. $x \in I$ and $\forall\ \xi \leq 0$, then this theorem is true for the weak solutions of (2) if $f(x) \in L^1(I)$ and if $g\,(x, \xi)$ is of Carathéodory and satisfies hypothesis (C), (M), (P) and (F).

Remark 2. As it was already observed in the introduction, an example particularly significant to which it is possible to apply Theorem 9 is the following:

$$g(x, \xi) = q(x)\xi^+$$

where $q(x) \in C^0(\overline{I})$ and $q(x) \geq 0$.

Appendix

There is a proof of Theorem 5 much simpler and more direct than the following version which, nevertheless, can be applied to more general situations. (See Fučík[4].)

We start with the proof of the following:

Lemma A. *Let* $R(x) \in L^2(I)$ *be such that:*

$$\text{(a)} \qquad \int_I R(x) \frac{d^2 v(x)}{dx^2} = 0, \quad \forall\ v \in W_0^{2,2}(I),$$

then there exists two numbers α_1 *and* α_2 *such that:*

$$\text{(b)} \qquad R(x) = \alpha_1 + \alpha_2 x, \quad \text{for a.e.} \quad x \in I.$$

Proof. The relation (a) is equivalent to the following:

$$\text{(c)} \qquad \frac{d^2 R(x)}{dx^2} = 0,$$

where the equality in (c) is in $\mathcal{D}'(\mathcal{I})$.

We obtain relation (b) if we remark that also in $\mathcal{D}'(\mathcal{I})$ the solutions of (c) are only first grade polynomial. ∎

Proof of Theorem 5. If $u(x) \in W^{2,2}(I)$ is a weak solution of equation (2), then we get:

$$\int_I \left(\frac{d^2 u(x)}{dx^2} \cdot \frac{d^2 v(x)}{dx^2} + g\,(x, u(x)) - f(x) \right) v(x) = 0,$$
$$\forall\ v \in W_0^{2,2}(I).$$

Integrating by parts the above relation and remembering that $v \in W_0^{2,2}(I)$, we obtain:

$$\int_I \left(\frac{d^2u(x)}{dx^2} + \int_0^x \left(\int_0^t \left(g\left(m, u(m)\right) - f(m) \right) dm \right) dt \right) .v(x)dx = 0,$$
$$\forall\ v \in W_0^{2,2}(I).$$

For Lemma A, we have that there exists two constants α_1 and α_2 such that:

$$\frac{d^2u(x)}{dx^2} + \int_0^x \left(\int_0^t \left(g\left(m, u(m)\right) - f(m) \right) dm \right) dt + \alpha_1 + \alpha_2 x = 0,$$
$$\text{for a.e. } \ x \in I.$$

Now, we put:

$$F(x,z) = z + \int_0^x \left(\int_0^t \left(g\left(m, u(m)\right) - f(m) \right) dm \right) dt + \alpha_1 + \alpha_2 x.$$

Of course, $F(x,z) \in C^2(\overline{I})$ and $\dfrac{\partial F(x,z)}{\partial z} = 1$ for all $x \in \overline{I}$. Applying the implicit function theorem we get that there exists a unique function $z(x) \in C^2(\overline{I})$ such that:

$$F(x, z(x)) = 0.$$

For the uniqueness of that function, we have:

$$\frac{d^2u(x)}{dx^2} = z(x)$$

and, therefore, $u \in C^4(\overline{I})$. We can, therefore, integrate by parts equation (2) and obtain:

$$(d) \qquad \begin{cases} \dfrac{d^2u(1)}{dx^2}\dfrac{dv(1)}{dx} - \dfrac{d^2u(0)}{dx^2}\dfrac{dv(0)}{dx} - \dfrac{d^3u(1)}{dx^3}v(1) + \dfrac{d^3u(0)}{dx^3}v(0) \\ \qquad + \displaystyle\int_I \left(\frac{d^4u(x)}{dx^4} + g\left(x, u(x)\right) - f(x) \right) .v(x) = 0, \\ \hfill \forall\ v \in W^{2,2}(I). \end{cases}$$

In the relation (d), we can choose in particular $v \in W_0^{2,2}(I)$. Since this space is dense in $W^{2,2}$, we have:

$$\frac{d^4u(x)}{dx^4} + g\left(x, u(x)\right) = f(x), \quad \forall\ x \in \overline{I}.$$

Therefore, relation (d) becomes:

$$\frac{d^2u(1)}{dx^2}\frac{dv(1)}{dx} - \frac{d^2u(0)}{dx^2}\frac{dv(0)}{dx} - \frac{d^3u(1)}{dx^3}v(1) + \frac{d^3u(0)}{dx^3}v(0) = 0,$$
$$\forall\ v \in W^{2,2}.$$

For the arbitrary choice of $v \in W^{2,2}$, we get:

$$\begin{cases} \dfrac{d^3u(0)}{dx^3} = \dfrac{d^2u(0)}{dx^2} = 0 \\ \dfrac{d^3u(1)}{dx^3} = \dfrac{d^2u(1)}{dx^2} = 0. \end{cases}$$

Therefore, every weak solution of equation (2) is a classical solution of equation (1). ■

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ON EIGHT ORDER MOCK THETA FUNCTION

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Abstract. In the present paper we have introduced partial mock theta functions of orders eight. We have established certain identities relating these functions with those of different orders.

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1. Introduction

The discovery of mock theta functions, the last gift of Ramanujan to the world of mathematics, was communicated by him in his last letter to Hardy [13, 14]. Ramanujan wrote that he has discovered some very interesting functions, called them as mock theta functions, defined by a q-series and which for $q \to e^{2\pi ir/s}$ along some radius vector of the unit circle $|q| = 1$, has precisely same behavior as that of one of Jacobi's theta functions. He gave a list of seventeen such functions classifying them as of order three, five and seven, but did not say what he meant by order. So far there is no widely acceptable definition of the order, although, several definitions of the order have been given [1, 11]. The recent work of Bringmann and Ono [4, 5, 6] also clarifies the concept of order. Further, after the discovery of Ramanujan's 'Lost' Notebook more mock theta functions were identified and studied by Andrews and Hickerson [6] and Choi [9] who have designated them as mock theta functions of order six and ten respectively. Recently, Gordon and McIntosh [10] have defined the following eighth mock theta functions of order eight

$$S_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n} \qquad S_1(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_n}{(-q^2;q^2)_n}$$

$$T_0(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}} \qquad T_1(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}}$$

$$U_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(-q^4;q^4)_n} \qquad U_1(q)=\sum_{n=0}^{\infty}\frac{q^{(n+1)^2}(-q;q^2)_n}{(-q^2;q^4)_{n+1}}$$

$$V_0(q)=-1+2\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n} \qquad V_1(q)=\sum_{n=0}^{\infty}\frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}}$$

where $(a;q^k)_n$ is q-shifted factorial, defined as

$$\left(a;q^k\right)_n=\prod_{r=0}^{n-1}(1-aq^{kr}),\,(a;q^k)_0=1,$$

Later, Gordon and McIntosh [11] gave a definition of the order of mock theta function and in [12], the mock theta functions $U_0(q)$, $U_1(q)$, $V_0(q)$ and $V_1(q)$ have been reclassified as of second order. In the last one decade the truncated (finite) series of mock theta functions have been studied by several mathematicians [2, 7, 8] who have established the identities connecting functions of different orders. In the present paper, we have considered the partial series of eighth order mock theta functions, i.e. the series of first $m+1$ terms, and have designated them as partial mock theta functions of order eight. Therefore, an eighth order partial mock theta function corresponding to $S_0(q)$ is

$$S_{0m}(q)=\sum_{n=0}^{m}\frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n}.$$

We have established certain identities connecting these function with following mock theta function of various other orders.

Third Order [16]

$$\Phi(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q^2;q^2)_n} \qquad \nu(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(-q;q^2)_{n+1}}.$$

Fifth Order [16]

$$\Psi_0(q)=\sum_{n=0}^{\infty}q^{(n+1)(n+2)/2}(-q)_n \qquad \Psi_1(q)=\sum_{n=0}^{\infty}q^{n(n+1)/2}(-q)_n$$

$$\varphi_0(q)=\sum_{n=0}^{\infty}q^{n^2}(-q;q^2)_n \qquad \varphi_1(q)=\sum_{n=0}^{\infty}q^{(n+1)^2}(-q;q^2)_n$$

Sixth Order [3]

$$\varphi(q)=\sum_{n=0}^{\infty}\frac{(-1)^nq^{n^2}(q;q^2)_n}{(-q)_{2n}} \qquad \psi(q)=\sum_{n=0}^{\infty}\frac{(-1)^nq^{(n+1)^2}(q;q^2)_n}{(-q)_{2n+1}}$$

$$\chi(q)=\sum_{n=0}^{\infty}\frac{q^{(n+1)(n+2)}(-q)_n}{(q;q^2)_{(n+1)}}\qquad\qquad X(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(-q)_n}{(q;q^2)_{(n+1)}}$$

2. Main Results

In this section we give the following identities, which show the relations of eighth order (partial) mock theta functions and mock theta functions of various other orders.

$$(-q;q^2))_\infty\sum_{m=0}^{\infty}(-q^3;q^2)_m^{-1}q^{2m}S_{0m}(q)=(-q;q^2)_\infty\Phi(q)-S_0(q)\tag{2.1}$$

$$q^2\sum_{m=0}^{\infty}(-q;q^2)_m q^{2m}S_{0m}(q)=(-q^2;q^2)_\infty S_0(q)-\varphi_0(q)\tag{2.2}$$

$$(q;q^2))_\infty\sum_{m=0}^{\infty}(q^3;q^2)_m^{-1}q^{2m}S_{0m}(q)=(q;q^2)_\infty\varphi(-q)-S_0(q)\tag{2.3}$$

$$q^2(1+q)(q^3;q^2)_\infty\sum_{m=0}^{\infty}\frac{(-q;q^2)_m q^{2m}}{(q^3;q^2)_m}S_{0m}(q)=(q^2;q^2)_\infty S_0(q)-(q;q^2)_\infty\overline{V}_0(q)\tag{2.4}$$

where $\overline{V}_q=(1+V_0(q))/2$.

$$q^3\sum_{m=0}^{\infty}(-q^2;q^2)_m q^{2m}S_{1m}(q)=\varphi_1(q)-q(-q^2;q^2)_\infty S_1(q)\tag{2.5}$$

$$q^4(q^5;q^2)_\infty\sum_{m=0}^{\infty}(q^5;q^2)_m^{-1}q^{2m}S_{1m}(q)=(q^3;q^2)_\infty\psi(-q)+S_1(q)\tag{2.6}$$

$$q^2(q;q^2)_\infty\sum_{m=0}^{\infty}\frac{(-q^2;q^2)_m q^{2m}}{(q^3;q^2)_m}S_{1m}(q)=q(q^2;q^2)_\infty S_1(q)-(q;q^2)_\infty V_1(q)\tag{2.7}$$

$$q^3(1+q)\sum_{m=0}^{\infty}(-q^3;q^2)_m q^{2m}T_{0m}(q)=(-q;q^2)_\infty T_0(q)-\Psi_0(q^2)\tag{2.8}$$

$$q^3(q^5;q^2)_\infty\sum_{m=0}^{\infty}(q^5;q^2)_m^{-1}q^{2m}T_{0m}(q)=T_0(q)-(q^3;q^2)_\infty\chi(q^2)\tag{2.9}$$

$$q^2(-q^4;q^2)_\infty\sum_{m=0}^{\infty}(-q^4;q^2)_m^{-1}q^{2m}T_{1m}(q)=T_1(q)-(-q^2;q^2)_\infty\nu(q) \tag{2.10}$$

$$q^3(1+q)\sum_{m=0}^{\infty}(-q^3;q^2)_m q^{2m}T_{1m}(q)=(-q;q^2)_\infty T_1(q)-\Psi_1(q^2) \tag{2.11}$$

$$q^3(q^5;q^2)_\infty\sum_{m=0}^{\infty}(q^5;q^2)_m^{-1}q^{2m}T_{1m}(q)=T_1(q)-(q^3;q^2)_\infty X(q^2) \tag{2.12}$$

$$q\sum_{m=0}^{\infty}(q;q^2)_m q^{2m}V_{0m}(q)=\varphi_0(q)-(q;q^2)_\infty V_0(q) \tag{2.13}$$

$$q^2(-q^4;q^2)_\infty\sum_{m=0}^{\infty}(-q^4;q^2)_m^{-1}q^{2m}V_{0m}(q)=(-q^2;q^2)_\infty\varphi(-q)-V_0(q) \tag{2.14}$$

$$q(1+q)(-q^4;q^2)_\infty\sum_{m=0}^{\infty}(q;q^2)(-q^4;q^2)_m^{-1}q^{2m}V_{0m}(q)=(q;q^2)_\infty V_0(q)+S_0(q) \tag{2.15}$$

$$q^3\sum_{m=0}^{\infty}(q^3;q^2)_m q^{2m}V_{1m}(q)=\varphi_1(q)-(q^3;q^2)_\infty V_1(q) \tag{2.16}$$

$$q^2(-q^4;q^2)_\infty\sum_{m=0}^{\infty}(-q^4;q^2)_m^{-1}q^{2m}V_{1m}(q)=(-q^2;q^2)_\infty\psi(-q)-V_1(q) \tag{2.17}$$

$$(1-q^2)\sum_{m=0}^{\infty}q^{2m}S_{0m}(q)=S_1(q) \tag{2.18}$$

$$q^2(1-q^2)\sum_{m=0}^{\infty}q^{2m}T_{1m}(q)=T_0(q) \tag{2.19}$$

Proof. To prove identities (2.1) - (2.17), we consider a simple series identity, subject to the convergence of the series involved

$$A(q)\sum_{m=0}^{\infty}B_m(q)\sum_{r=0}^{m}\alpha_r+C_\infty(q)\sum_{m=0}^{\infty}\alpha_m=\sum_{m=0}^{\infty}C_m(q)\alpha_m, \tag{2.20}$$

where

$$A(q)=\frac{(aq-e)(e-bq)}{(q-e)(e-abq)},\ B_m(q)=\frac{(a,b;q)_m(q)^m}{(e,abq^2/e;q)_m},\ C_m(q)=\frac{(a,b;q)_m}{(e/q,abq/e;q)_m}.$$

The Series identity (2.20) can easily be proved [1] by using the summation formula

$${}_3\Phi_2(a,b,q;e,f;q)_n=\frac{(q-e)(e-abq)}{(aq-e)(e-bq)}\left[1-\frac{(a,b;q)_{n+1}}{(e/q,abq/e;q)_{n+1}}\right],$$

for $ef=abq^2$ which is a particular case of a transformation between two Saalschutzian terminating ${}_3\Phi_2(..)$ due to Sears [15].

Taking $\alpha_r=\dfrac{q^{r(r+2)}(-q;q^2)_r}{(-q^2;q^2)_r}$, $a=0,\ b=-q^2$ and then $e=0$, we get (2.1).

The identities (2.2) - (2.17) can now easily be proved with a proper choice of the sequence α_r and the parameters $a,\ b$ and e in (2.20).

To prove (2.18) and (2.19), we consider the identity

$$\sum_{m=0}^{p}\beta_m\sum_{r=0}^{m}\alpha_r=\sum_{r=0}^{p}\alpha_r\sum_{m=0}^{p}\beta_m-\sum_{r=1}^{p}\alpha_r\sum_{m=0}^{r-1}\beta_m. \tag{2.21}$$

Taking $\beta_m=q^{\lambda m}$, after some simplification we get

$$\sum_{r=0}^{\infty}q^{\lambda r}\sum_{m=0}^{r}\alpha_m q^m=(1-q^{\lambda})^{-1}\sum_{m=0}^{\infty}\alpha_m(q)^{m(1+\lambda)}. \tag{2.22}$$

Now for $\alpha_m=\dfrac{q^{m^2-m}(-q;q^2)_m}{(-q^2;q^2)_m}$, $\lambda=2$ and for $\alpha_m=\dfrac{q^{m^2}(-q^2;q^2)_m}{(-q;q^2)_m}$, $\lambda=2$, (2.22) gives (2.18) and (2.19) respectively.

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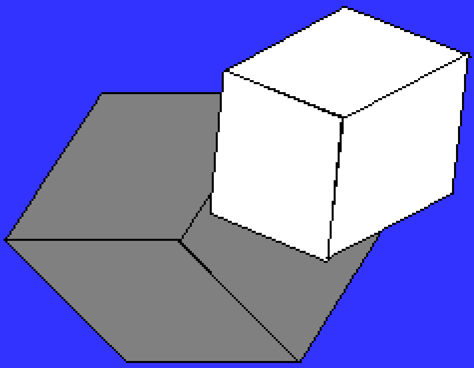

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