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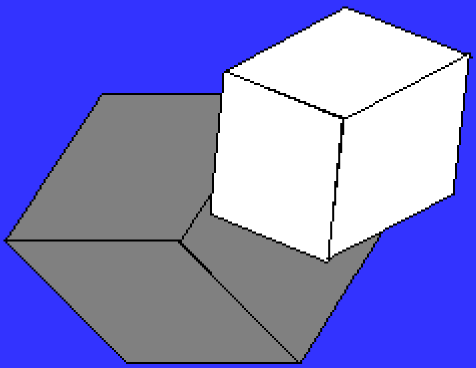

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Editorial

Number 27 of this journal concludes a stage of the *Italian Journal of Pure and Applied Mathematics*, namely the stage of the hard copy, but on the other hand it opens a new stage, that of the on-line journal. It seems the moment has come when an evaluation is called for of its activity so far and its prospects.

The journal was founded in 1987 (its title back then was in Italian) based on an idea I had which was supported by Dr. LIESC, the then Head of the University Consortium, and approved of by the Rector, Prof. FRILLI. The growing international success of the journal was due to several reasons, the most important of which was the fact that the journal was open to any scientific research, independent of fashion or prejudice.

The sole prerequisite for the acceptance of a manuscript has always been the original nature of the topic and the correctness of the results, hence their actual contribution to the progress of Science. Starting 1997, the title has been changed to its English version; it was a successful idea that belonged to Prof. STRASSOLDO, then a rector who felt enthusiastic about our publication.

Beginning with 2002, the printing house has changed; the press that has typed the journal ever since is the printing house Panfilius in Iasi, Romania. The better quality of the print and the better prices have made it possible to have a larger number of papers published each year; thus, the journal was made even more successful.

The financial support and the ideas of Prof. HONSELL, the Rector, made a great difference for the journal in the years that followed.

A few words now about those who contributed with their intelligence and dedication to the running of the journal:

Professor Violeta LEOREANU has been a competent reviewer for many articles and has been instrumental in maintaining the contact with the Panfilius printing press; she updated the address list of the journals with which the "*Italian Journal*" has permanent exchanges, she has revised each volume.

PhD. Giovanni FALCONE created the web page of the journal and has maintained it for many years.

PhD. Irina CRISTEA has been a constant and precious support as a reviewer, especially as she has searched for the international experts to assess the papers submitted for publication. Given the great variety of the topics, this is a demanding task.

Dr. Domenico (Nico) CHILLEMI, IBM Executive IT Specialist, has had and will continue to have – especially at this stage – an essential role in the creation and maintenance of the online version of the journal.

Mrs. Elena MOCANU has had a major role in typing and editing the papers in their final form, ready for publication.

Also through exchanges, contacts with universities in several countries have been established:

EUROPE

Belgium, Bosnia Herzegovina, Bulgaria, The Czech Republic, Croatia, Finland, France, Germany, Great Britain, Greece, Italy, Macedonia, Montenegro, Holland, Hungary, Poland, Portugal, Republic of Moldova, Romania, Russia, Serbia, Slovak Republic, Slovenia, Spain, Sweden

ASIA

India, China, Iran, Thailand, Vietnam, South Korea, North Korea, Malaysia, Jordan, Taiwan, Ukraine, Uzbekistan, Japan, Indonesia, Israel, Tajikistan

AMERICA

Canada, U.S.A., Argentina, Brazil, Mexico, Cuba, Colombia, Chile, Venezuela, Paraguay, Uruguay

AFRICA

Egypt, Nigeria, South Africa

OCEANIA

Australia, New Zealand

Further papers from the following countries were published:

EUROPE

Romania, Greece, The Czech Republic, Italy, Poland, France, Germany, Serbia, Bosnia, Herzegovina, Montenegro, Russia, Denmark, Great Britain

ASIA

Iran, India, China, Thailand, Turkey, Korea, Saudi Arabia, Georgia, Jordan, Pakistan, Nepal, Vietnam, Israel, Tajikistan, Belarus, Japan, Kuwait, Oman, Bahrain

AFRICA

Egypt, Morocco, Nigeria, Tunisia, Mali, Madagascar

OCEANIA

Australia

We all hope that the on-line edition of the "*Italian Journal*" will be as successful as the hard copy version.

The "Italian Journal of Pure and Applied Mathematics" cannot more take advantage of the precious collaboration of Prof. Jaroslav Jezek, who passed away recently.
The Chief-Editor and the members of Editorial Board express their deep sorrow for the loss of a first-class scientist and a very dear friend.

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Exchanges

Up to January 2011 this journal is exchanged with the following periodicals:

1.	Acta Cybernetica - Szeged	H
2.	Acta Mathematica et Informatica Universitatis Ostraviensis	CZ
3.	Acta Mathematica Vietnamica – Hanoi	VN
4.	Acta Mathematica Sinica, New Series – Beijing	RC
5.	Acta Scientiarum Mathematicarum – Szeged	H
6.	Acta Universitatis Lodziensis – Lodz	PL
7.	Acta Universitatis Palackianae Olomucensis, Mathematica – Olomouc	CZ
8.	Actas del tercer Congreso Dr. Antonio A.R. Monteiro - Universidad Nacional del Sur Bahía Blanca	AR
9.	Algebra Colloquium - Chinese Academy of Sciences, Beijing	PRC
10.	Alxebra - Santiago de Compostela	E
11.	Analele Ştiinţifice ale Universităţii “Al. I Cuza” - Iaşi	RO
12.	Analele Universităţii din Timişoara - Universitatea din Timişoara	RO
13.	Annales Academiae Scientiarum Fennicae Mathematica - Helsinki	SW
14.	Annales de la Fondation Louis de Broglie - Paris	F
15.	Annales Mathematicae Silesianae – Katowice	PL
16.	Annales Scientif. Université Blaise Pascal - Clermont II	F
17.	Annales sect. A/Mathematica – Lublin	PL
18.	Annali dell’Università di Ferrara, Sez. Matematica	I
19.	Annals of Mathematics - Princeton - New Jersey	USA
20.	Applied Mathematics and Computer Science -Technical University of Zielona Góra	PL
21.	Archivium Mathematicum - Brno	CZ
22.	Atti del Seminario di Matematica e Fisica dell’Università di Modena	I
23.	Atti dell’Accademia delle Scienze di Ferrara	I
24.	Automatika i Telemekhanika - Moscow	RU
25.	Boletim de la Sociedade Paranaense de Matematica - San Paulo	BR
26.	Bolétin de la Sociedad Matemática Mexicana - Mexico City	MEX
27.	Bollettino di Storia delle Scienze Matematiche - Firenze	I
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39.	Bulletin of the Australian Mathematical Society - St. Lucia - Queensland	AUS
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44.	Bulletin of the Korean Mathematical Society - Seoul	ROK
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46.	Bulletin of the Transilvania University of Braşov - Braşov	RO
47.	Bulletin of the USSR Academy of Sciences - San Pietroburgo	RU
48.	Bulletin for Applied Mathematics - Technical University Budapest	H
49.	Busefal - Université P. Sabatier - Toulouse	F
50.	Calculus CNR - Pisa	I
51.	Chinese Annals of Mathematics - Fudan University – Shanghai	PRC
52.	Chinese Quarterly Journal of Mathematics - Henan University	PRC
53.	Classification of Commutative FPF Ring - Universidad de Murcia	E
54.	Collectanea Mathematica - Barcelona	E
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STEADY THREE-DIMENSIONAL HYDROMAGNETIC STAGNATION POINT FLOW TOWARDS A STRETCHING SHEET WITH HEAT GENERATION

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Abstract. An analysis is made of the steady hydromagnetic laminar three dimensional stagnation point flow of an incompressible viscous fluid impinging on a permeable stretching sheet with heat generation or absorption. A uniform magnetic field is applied normal to the plate which is maintained at a constant temperature. Numerical solution for the governing nonlinear momentum and energy equations is obtained. The effect of the strength of the uniform magnetic field, the surface stretching velocity, and the heat generation/absorption coefficient on both the flow and heat transfer is presented and discussed.

Introduction

The axisymmetric three-dimensional stagnation point flow was studied by Homann [1] who demonstrated that the Navier-Stokes equations governing the flow can be reduced to an ordinary differential equation of third order using similarity transformation. Later the problem of stagnation point flow either in the two- or three-dimensional cases [1], [2] has been extended in numerous ways to include various physical effects. The results of these studies are of great technical importance, for example in the prediction of skin-friction as well as heat/mass transfer near stagnation regions of bodies in high speed flows and also in the design of thrust bearings and radial diffusers, drag reduction, transpiration cooling and thermal oil recovery. In hydromagnetics, the problem of Hiemenz flow was chosen by Na [3] to illustrate the solution of a third-order boundary value problem using the technique of finite differences. An approximate solution of the same problem has been provided by Ariel [4]. The effect of an externally applied uniform magnetic field on the two or three-dimensional stagnation point flow was given, respectively, by Attia in [5] and [6] in the presence of uniform suction or injection. The study of heat transfer in boundary layer flows is of importance in many engineering applications such as the design of thrust bearings and radial diffusers, transpiration cooling, drag reduction, thermal recovery of oil, etc. [7]. Massoudi and Ramezan [7] used a perturbation technique to solve for the stagnation point flow and heat transfer of a non-Newtonian fluid of second grade. Their analysis is valid only for small values of the parameter that determines the behavior of the non-Newtonian fluid. Later Massoudi and Ramezan [8] extended the problem to nonisothermal surface. Garg [9] improved the solution obtained by Massoudi [8] by computing numerically the flow characteristics for any value of the non-Newtonian parameter using a pseudo-similarity solution.

Flow of an incompressible viscous fluid over stretching surface has important applications in polymer industry. For instance, a number of technical processes concerning polymers involves the cooling of continuous strips (or filaments) extruded from a die by drawing them through a stagnant fluid with controlled cooling system and in the process of drawing these strips are sometimes stretched. The quality of the final product depends on the rate of heat transfer at the stretching surface. Crane [10] gave a similarity solution in closed analytical form for steady two-dimensional incompressible boundary layer flow caused by the stretching of a sheet which moves in its own plane with a velocity varying linearly with the distance from a fixed point. Carragher and Crane [11] investigated heat transfer in the above flow in the case when the temperature difference between the surface and the ambient fluid is proportional to a power of distance from the fixed point. Temperature distribution in the flow over a stretching surface subject to uniform heat flux was studied by Dutta et al. [12]. Recently, Chiam [13] analyzed steady two-dimensional stagnation-point flow of an incompressible viscous fluid towards a stretching surface. Temperature distribution in the steady plane stagnation-point flow of a viscous fluid towards a stretching surface was investigated by Ray Mahapatra and Gupta [14]. Steady flow of a non-Newtonian viscoelastic fluid [15]-[16] or micropolar fluid [17] past a stretching sheet was investigated with zero vertical velocity at the surface.

In the present paper the steady hydromagnetic laminar axisymmetric three dimensional stagnation point flow of an incompressible viscous fluid impinging on a permeable stretching surface is studied with heat generation/absorption. A uniform magnetic field directed normal to the plate is applied where the induced magnetic field is neglected [18]. The wall and stream temperatures are assumed to be constants. A numerical solution is obtained for the governing momentum and energy equations using finite difference approximations which takes into account the asymptotic boundary conditions. The numerical solution computes the flow and heat characteristics for the whole range of the uniform magnetic field, the surface stretching velocity, the heat generation/absorption coefficient and Prandtl number.

Formulation of the problem

Consider the steady three-dimensional stagnation point flow of a viscous incompressible fluid near a stagnation point at a surface coinciding with the plane $z = 0$, the flow being in a region $z > 0$. Two equal and opposing forces are applied along the radial direction so that the surface is stretched keeping the origin fixed. We use the cylindrical coordinates r, φ, z and assume that the wall is at $z = 0$, the stagnation point is at the origin and that the flow is in the direction of the negative z-axis. We denote the radial and axial velocity components in frictionless flow by U and W, respectively, whereas those in viscous flow will be denoted by $u = u(r, z)$ and $w = w(r, z)$ where the component in the φ direction vanishes [19]. A uniform magnetic field B_0 is applied normal to the plate where the induced magnetic field is neglected by assuming very small magnetic Reynolds number [18]. For three-dimensional flow let the fluid far from the plate, as z tends from infinity, be driven by the potential flow

$$U = ar, \quad W = -2az,$$

where a (> 0) is a constant characterizing the velocity of the mainstream flow. Then, from Euler equation the pressure distribution will be [19]

$$p = p_0 - \frac{\rho a^2}{2}\,(r^2 + 4z^2),$$

where ρ is the density of the fluid and p_0 is the pressure at the stagnation point. The continuity and momentum equations for the three dimensional steady state flows, using the usual boundary layer approximations [19], reduce to

$$(1) \quad \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0,$$

$$(2) \quad \rho\left(u\,\frac{\partial u}{\partial r} + w\,\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\,\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right) + \sigma B_0^2(U(r) - u),$$

$$(3) \quad \rho\left(u\,\frac{\partial w}{\partial r} + w\,\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\,\frac{\partial w}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 w}{\partial z^2}\right),$$

where μ is the coefficient of viscosity of the fluid and σ is the electrical conductivity of the fluid. The boundary conditions for the above flow situation are

$$(4a) \qquad z = 0 : u = cr,\ w = 0,$$

$$(4b) \qquad z \to \infty : u \to ar,$$

where c is a positive constant related to the stretching velocity.

The boundary layer equations (1)-(3) admit a similarity solution

$$(5) \qquad u = crf'(\eta),\ w = -2\sqrt{c\nu}\ f(\eta),\ \eta = \sqrt{c/\nu z},$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid and prime denotes differentiation with respect to η. If we now substitute u and w from Eq. (5) into the Navier-Stokes equations (1)-(3), we find that Eq. (3) yields simply the relation

$$\frac{\partial^2 p}{\partial r \partial z} = 0. \tag{6}$$

Using Eq. (5), Eqs. (1)-(2) and (4) lead to

$$f'^2 - f''' - 2ff'' - C^2 - Ha^2(C - f') = 0, \tag{7}$$

$$f(0) = 0,\ f'(0) = 1,\ f'(\infty) = C, \tag{8}$$

where $Ha^2 = \sigma B_0^2/c\rho$, Ha is the modified Hartmann number [18] and $C = a/c$ is the stretching parameter.

The governing boundary layer equation of energy, neglecting the dissipation, with temperature dependent heat generation or absorption is [19]

$$\rho c_p \left(u\, \frac{\partial \theta}{\partial r} + w\, \frac{\partial \theta}{\partial z} \right) = k\, \frac{\partial^2 T}{\partial z^2} + Q(T - T_\infty), \tag{9}$$

where θ is the temperature of the fluid, c_p is the specific heat capacity at constant pressure of the fluid, k is the thermal conductivity of the fluid, T_∞ the constant temperature of the fluid far away from the sheet, Q is the volumetric rate of heat generation/absorption, and T is the temperature profile. A similarity solution exists if the wall and stream temperatures, T_w and T_∞ are constants – a realistic approximation in typical stagnation point heat transfer problems [19].

The thermal boundary conditions are

$$z = 0 : T = T_w, \tag{10a}$$

$$z \to \infty : T \to T_\infty. \tag{10b}$$

By introducing the non-dimensional variable

$$\theta = \frac{T - T_\infty}{T_w - T_\infty},$$

and using Eq. (5), we find that Eqs. (9) and (10) reduce to,

$$\theta'' + 2\mathrm{Pr}\, f\theta' + \mathrm{Pr}\, B\theta = 0, \tag{11}$$

$$\theta(0) = 1,\ \theta(\infty) = 0, \tag{12}$$

where $\mathrm{Pr} = \mu c_p/k$ is the Prandtl number and $B = Q/c\rho c_p$ is the dimensionless heat generation/absorption coefficient.

The flow Eqs. (7) and (8) are decoupled from the energy Eqs. (11) and (12), and need to be solved before the latter can be solved. The flow Eq. (7)

constitutes a non-linear, non-homogeneous boundary value problem (BVP). In the absence of an analytical solution of a problem, a numerical solution is indeed an obvious and natural choice. The boundary value problem given by Eqs. (7) and (8) may be viewed as a prototype for numerous other situations which are similarly characterized by a boundary value problem having a third order differential equation with an asymptotic boundary condition at infinity. Therefore, its numerical solution merits attention from a practical point of view. The flow Eqs. (7) and (8) are solved numerically using finite difference approximations. A quasi-linearization technique is first applied to replace the non-linear terms at a linear stage, with the corrections incorporated in subsequent iterative steps until convergence. Then, Crank-Nicolson method is used to replace the different terms by their second order central difference approximations. An iterative scheme is used to solve the quasi-linearized system of difference equations. The solution for the Newtonian case is chosen as an initial guess and the iterations are continued till convergence within prescribed accuracy. Finally, the resulting block tri-diagonal system was solved using generalized Thomas' algorithm.

The energy Eq. (11) is a linear second order ordinary differential equation with variable coefficient, $f(\eta)$, which is known from the solution of the flow Eqs. (7) and (8) and the Prandtl number Pr is assumed constant. Equation (11) is solved numerically under the boundary condition (12) using central differences for the derivatives and Thomas' algorithm for the solution of the set of discritized equations. The resulting system of equations has to be solved in the infinite domain $0 < \eta < \infty$. A finite domain in the η-direction can be used instead with η chosen large enough to ensure that the solutions are not affected by imposing the asymptotic conditions at a finite distance. Grid-independence studies show that the computational domain $0 < \eta < \eta_\infty$ can be divided into intervals each is of uniform step size which equals 0.02. This reduces the number of points between $0 < \eta < \eta_\infty$ without sacrificing accuracy. The value $\eta_\infty = 10$ was found to be adequate for all the ranges of parameters studied here. Convergence is assumed when the ratio of every one of f, f', f'', or f''' for the last two approximations differed from unity by less than 10^{-5} at all values of η in $0 < \eta < \eta_\infty$.

Results and discussion

Figures 1 and 2 present the velocity profiles of f and f', respectively, for various values of C and Ha. The figures show that increasing the parameter C increases both f and f'. The effect of Ha on both f and f' depends on C. For $C < 1$, increasing Ha decreases f and f' while for $C > 1$, increasing Ha increases them. The figures indicate also that the effect of C on f and f' is more pronounced for smaller values of Ha. Also, increasing C decreases the velocity boundary layer thickness. Figure 3 presents the profile of temperature θ for various values of C and Ha and for $\mathrm{Pr} = 0.7$ and $B = 0.1$. It is clear that increasing C decreases θ and its effect on θ becomes more apparent for smaller values of Ha. The figure indicates that the thermal boundary layer thickness decreases when C increases. Increasing Ha decreases θ for all C and its effect is more clear for smaller C.

Figure 4 presents the temperature profiles for various values of C and Pr and for $Ha = 1$ and $B = 0.1$. Figure 4 brings out clearly the effect of the Prandtl number on the thermal boundary layer thickness. As shown in Fig. 4, increasing Pr decreases the thermal boundary layer thickness for all C. Increasing C decreases θ and its effect is more apparent for smaller Pr. Figure 5 presents the temperature profiles for various values of C and B and for $Ha = 0.5$ and $\mathrm{Pr} = 0.7$. Increasing B increases the temperature θ and the boundary layer thickness. The effect of B on θ is more pronounced for smaller C. However, the effect of C on θ is more apparent for higher B.

Tables 1 and 2 present the variation of the dimensionless wall shear stress $f''(0)$ and the dimensionless heat transfer rate at the wall $-\theta'(0)$, respectively, for various values of C and Ha and for $\mathrm{Pr} = 0.7$ and $B = 0.1$. Increasing C increases $f''(0)$ for all Ha and its effect becomes more pronounced for higher Ha. Increasing Ha increases the magnitude of $f''(0)$ and its effect is more apparent for smaller C. It is of interest to see the reversal of the sign of $f''(0)$ for $C < 1$ for all Ha. Table 2 shows that, increasing C increases $-\theta'(0)$ for all Ha. The effect of C on $-\theta'(0)$ is more pronounced for higher Ha. For $C < 1$, increasing Ha decreases $-\theta'(0)$, however, for $C > 1$, increasing Ha increases $-\theta'(0)$.

Table 1. Variation of the wall shear stress $f''(0)$ with C and Ha

Ha	$C = 0.1$	$C = 0.2$	$C = 0.5$	$C = 1$	$C = 1.1$	$C = 1.2$	$C = 1.5$
0	−1.1246	−1.0556	−0.7534	0	0.1821	0.3735	1.0009
1	−1.4334	−1.3179	−0.9002	0	0.2070	0.4004	1.1157
2	−2.1138	−1.9080	−1.2456	0	0.2691	0.5445	1.4080
3	−2.9174	−2.6141	−1.6724	0	0.3494	0.7037	1.7954

Table 2. Variation of the wall heat transfer rate $-\theta'(0)$ with C and Ha $(\mathrm{Pr} = 0.7,\ B = 0.1)$

Ha	$C = 0.1$	$C = 0.2$	$C = 0.5$	$C = 1$	$C = 1.1$	$C = 1.2$	$C = 1.5$
0	0.6454	0.6819	0.7773	0.9109	0.9354	0.9588	1.0263
1	0.5974	0.6493	0.7653	0.9109	0.9365	0.9612	1.0309
2	0.5112	0.5901	0.7421	0.9109	0.9392	0.9661	1.0412
3	0.4402	0.5405	0.7211	0.9109	0.9419	0.9713	1.0522

Table 3 presents the effect of C on $-\theta'(0)$ for various values of Pr and for $Ha = 1$ and $B = 0.1$. Increasing C increases $-\theta'(0)$ for all Pr and its effect is more pronounced for higher Pr. Increasing Pr increases $-\theta'(0)$ for all C and its effect is more apparent for higher C. Table 4 presents the effect of the parameters

C and B on $-\theta'(0)$ for $Ha = 0.5$ and $\mathrm{Pr} = 0.7$. Increasing C increases $-\theta'(0)$ for all B. But, increasing B decreases $-\theta'(0)$ for all C.

Table 3. Variation of the wall heat transfer rate $-\theta'(0)$ with C and Pr ($Ha = 1$, $B = 0.1$)

Pr	$C = 0.1$	$C = 0.2$	$C = 0.5$	$C = 1$	$C = 1.1$	$C = 1.2$	$C = 1.5$
0.05	0.1273	0.1421	0.1845	0.2439	0.2545	0.2632	0.2919
0.1	0.1618	0.1911	0.2615	0.3343	0.3581	0.3700	0.4080
0.5	0.4691	0.5223	0.6345	0.7699	0.7933	0.8136	0.8793
1	0.7657	0.8152	0.9332	1.0888	1.1166	1.1408	1.2200

Table 4. Variation of the wall heat transfer rate $-\theta'(0)$ with C and B ($Ha = 0.5$, $\mathbf{Pr} = 0.7$)

B	$C = 0.1$	$C = 0.2$	$C = 0.5$	$C = 1$	$C = 1.1$	$C = 1.2$	$C = 1.5$
-0.1	0.1273	0.1421	0.1845	0.2439	0.2545	0.2632	0.2919
0	0.1618	0.1911	0.2615	0.3343	0.3581	0.3700	0.4080
0.1	0.4691	0.5223	0.6345	0.7699	0.7933	0.8136	0.8793

Conclusions

The three dimensional hydromagnetic stagnation point flow of a viscous incompressible fluid impinging on a permeable stretching surface is studied in the presence of uniform magnetic field with heat generation/absorption. A numerical solution for the governing equations is obtained which allows the computation of the flow and heat transfer characteristics for various values of the modified Hartmann number Ha, the stretching velocity, the heat generation/absorption parameter and the Prandtl number Pr. The results indicate that increasing the stretching velocity increases the velocity components but decreases the velocity boundary layer thickness. On the other hand, increasing the stretching velocity decreases the temperature as well as the thermal boundary layer thickness. The effect of the stretching parameter on the velocity and temperature is more apparent for smaller values of the magnetic field. The variation of velocity components as well as the rate of heat transfer at the wall with the magnetic field depends on the magnitude of the stretching velocity. The sign of the wall shear stress was shown to depend on the stretching velocity. The effect of the heat generation/absorption parameter B on the rate of heat transfer at the wall becomes more apparent for smaller C.

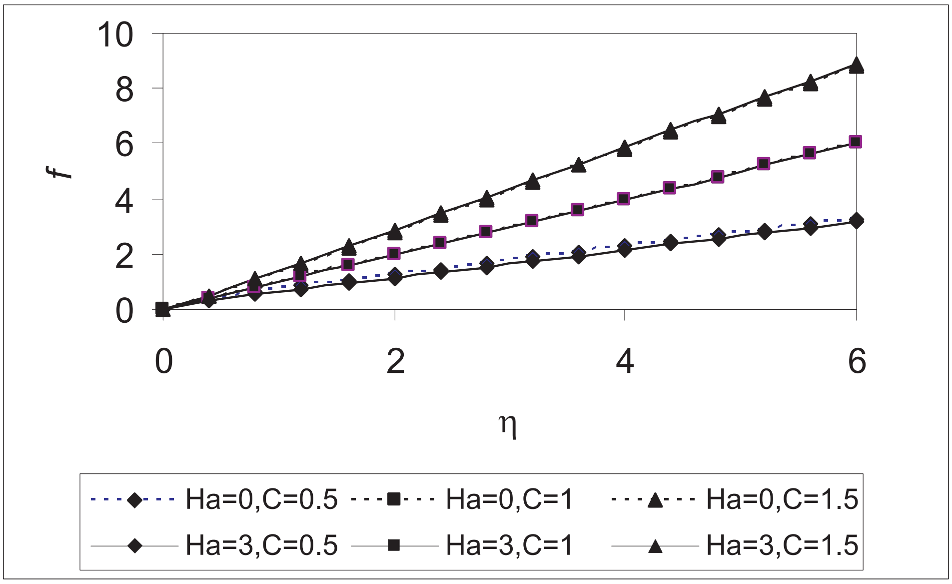

Figure 1: Effect of the parameters C and Ha on the profile of f

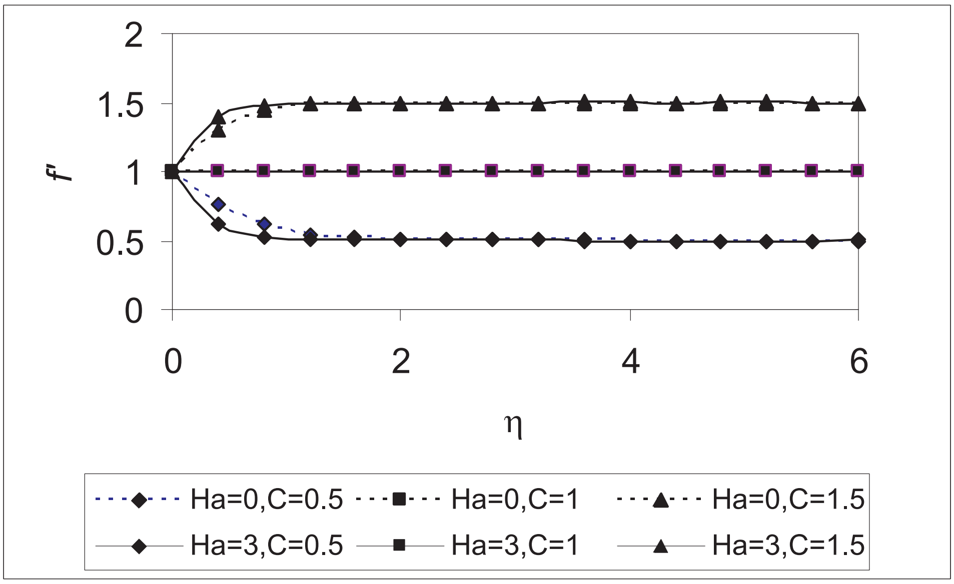

Figure 2: Effect of the parameters C and Ha on the profile of f'

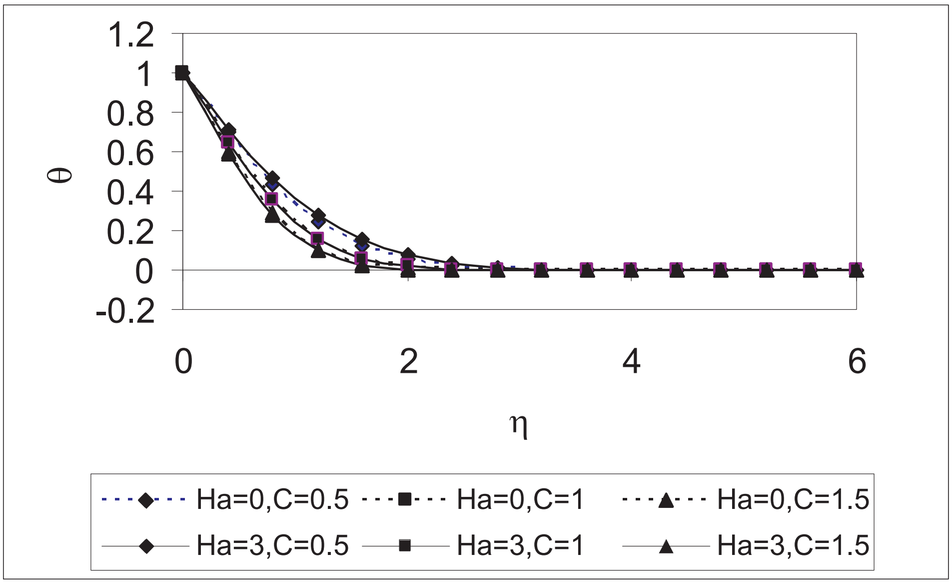

Figure 3: Effect of the parameters C and Ha on the profile of θ (Pr=0.7, B=0.1)

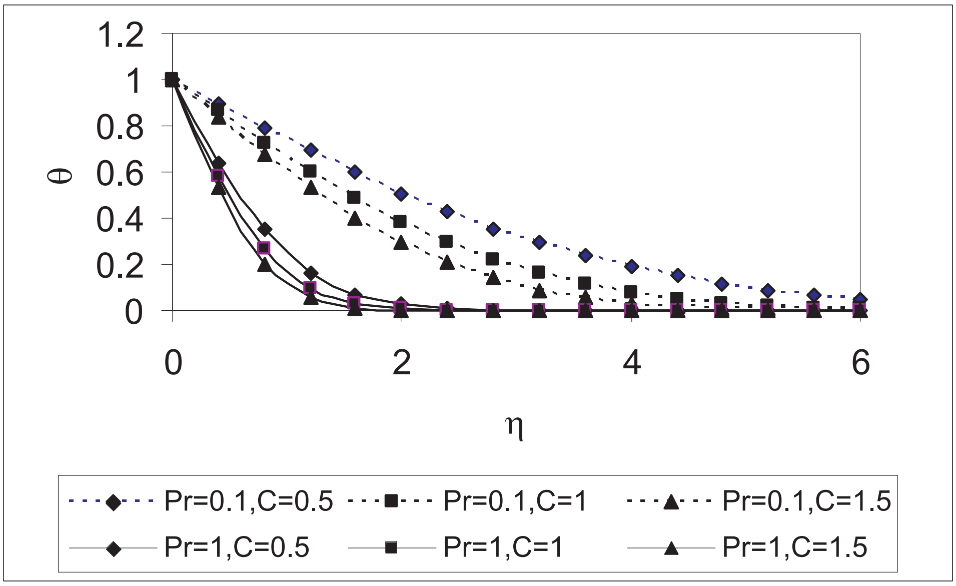

Figure 4: Effect of the parameters C and Pr on the profile of θ $(Ha = 1,\ B = 0.1)$

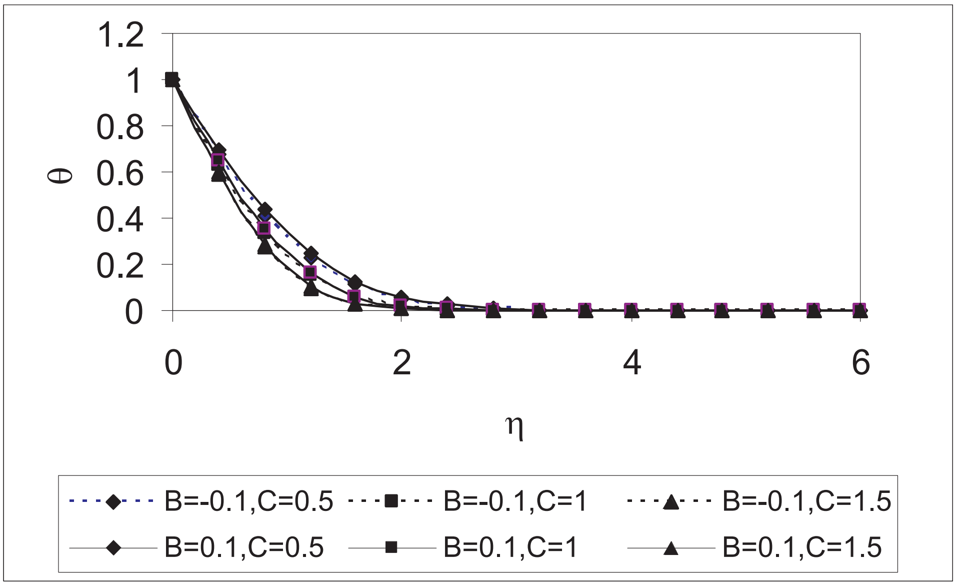

Figure 5: Effect of the parameters C and B on the profile of θ (Ha=$0, 5$, Pr=0.7)

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TRANSIENT MHD COUETTE FLOW OF A CASSON FLUID BETWEEN PARALLEL PLATES WITH HEAT TRANSFER

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Abstract. The unsteady magnetohydrodynamic flow of an electrically conducting viscous incompressible non-Newtonian Casson fluid bounded by two parallel non-conducting porous plates is studied with heat transfer considering the Hall effect. An external uniform magnetic field is applied perpendicular to the plates and the fluid motion is subjected to a uniform suction and injection. The lower plate is stationary and the upper plate is suddenly set into motion and simultaneously suddenly isothermally heated to a temperature other than the lower plate temperature. Numerical solutions are obtained for the governing momentum and energy equations taking the Joule and viscous dissipations into consideration. The effect of the Hall term, the parameter describing the non-Newtonian behavior, and the velocity of suction and injection on both the velocity and temperature distributions are studied.

Keywords: MHD flow, heat transfer, non-Newtonian fluids, Hall effect, numerical solution.

1. Introduction

The study of Couette flow in a rectangular channel of an electrically conducting viscous fluid under the action of a transversely applied magnetic field has immediate applications in many devices such as magnetohydrodynamic (MHD) power generators, MHD pumps, accelerators, aerodynamics heating, electrostatic precipitation, polymer technology, petroleum industry, purification of crude oil and fluid droplets sprays. Channel flows of a Newtonian fluid with heat transfer have been studied with or without Hall currents by many authors [1]-[10]. These results are important for the design of the duct wall and the cooling arrangements. The most important non-Newtonian fluid possessing a yield value is the Casson fluid, which has significant applications in polymer processing industries and biomechanics. Casson fluid is a shear thinning liquid which has an infinite viscosity at a zero rate of shear, a yield stress below which no flow occurs and a zero viscosity at an infinite rate of shear. Casson's constitute equation represents a nonlinear relationship between stress and rate of strain and has been found to be accurately applicable to silicon suspensions, suspensions of bentonite in water and lithographic varnishes used for printing inks [11]-[13]. Many authors [14]-[16] studied the flow or/and heat transfer of a Casson fluid in different geometries. Attia [10] has studied the influence of the Hall current on the velocity and temperature fields of an unsteady Hartmann flow of a conducting Newtonian fluid between two infinite non-conducting horizontal parallel and porous plates. The extension of such problem to the case of Couettee flow of non-Newtonian Casson fluid is done in the present study. The upper plate is moving with a uniform velocity while the lower plate is stationary. The fluid is acted upon by a constant pressure gradient, a uniform suction from above, and a uniform injection from below and is subjected to a uniform magnetic field perpendicular to the plates. The Hall current is taken into consideration while the induced magnetic field is neglected by assuming a very small magnetic Reynolds number [5]. The two plates are kept at two different but constant temperatures. This configuration is a good approximation of some practical situations such as heat exchangers, flow meters, and pipes that connect system components. The Joule and viscous dissipations are taken into consideration in the energy equation. The governing momentum and energy equations are solved numerically using the finite difference approximations. The inclusion of the Hall current, the suction and injection, and the non-Newtonian fluid characteristics lead to some interesting effects on both the velocity and temperature fields.

2. Formulation of the problem

The geometry of the problem is shown in Fig. 1. The fluid is assumed to be laminar, incompressible and obeying a Casson model and flows between two infinite horizontal plates located at the $y = \pm h$ planes and extend from $x = 0$ to ∞ and from $z = 0$ to ∞. The upper plate is suddenly set into motion and moves with a uniform velocity U_0 while the lower plate is stationary. The upper plate is simultaneously subjected to a step change in temperature from T_1 to T_2. Then, the upper and lower plates are kept at two constant temperatures T_2 and T_1 respectively, with $T_2 > T_1$. The fluid is acted upon by a constant pressure gradient dp/dx in the x-direction, and a uniform suction from above and injection from below which are applied at $t = 0$. A uniform magnetic field $\mathbf{B}_0$ is applied in the positive y-direction and is assumed undisturbed as the induced magnetic field is neglected by assuming a very small magnetic Reynolds number. The Hall effect is taken into consideration and consequently a z-component for the velocity is expected to arise. The uniform suction implies that the y-component of the velocity is constant. Thus, the fluid velocity vector is given by,

$$\mathbf{v}(y,t) = u(y,t)\mathbf{i} + \nu_0\mathbf{j} + w(y,t)\mathbf{k}.$$

The fluid motion starts from rest at $t = 0$, and the no-slip condition at the plates implies that the fluid velocity has neither a z nor an x-component at $y = \pm h$. The initial temperature of the fluid is assumed to be equal to T_1. Since the plates are infinite in the x and z-directions, the physical quantities do not change in these directions.

The flow of the fluid is governed by the momentum equation

$$\rho \frac{D\mathbf{v}}{Dt} = \nabla \cdot (\mu \nabla \mathbf{v}) - \nabla p + \mathbf{J} \times \mathbf{B}_0, \tag{1}$$

where ρ is the density of the fluid and μ is the apparent viscosity of the model and is given by

$$\mu = \left[K_c + \left(\tau_0 \Big/ \sqrt{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2} \right)^{1/2} \right]^2, \tag{2}$$

where K_c^2 is the Casson's coefficient of viscosity and τ_0 is the yield stress. If the Hall term is retained, the current density $\mathbf{J}$ is given by

$$\mathbf{J} = \sigma[\mathbf{v} \times \mathbf{B}_0 - \beta(\mathbf{J} \times \mathbf{V}_0)], \tag{3}$$

where σ is the electric conductivity of the fluid and β is the Hall factor [5]. Equation (3) may be solved in $\mathbf{J}$ to yield

$$\mathbf{J} \times \mathbf{B}_0 = -\frac{\sigma B_0^2}{1+m^2} [(u + mw)\mathbf{i} + (w - mu)\mathbf{k}], \tag{4}$$

where m is the Hall parameter and $m = \sigma\beta\mathbf{B}_0$. Thus, the two components of the momentum Eq. (1) read

$$\rho \frac{\partial u}{\partial t} + \rho \nu_0 \frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) - \frac{\sigma B_0^2}{1+m^2}(u + mw), \tag{5}$$

$$\rho \frac{\partial w}{\partial t} + \rho \nu_0 \frac{\partial w}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial w}{\partial y}\right) - \frac{\sigma B_0^2}{1+m^2}(w - mu), \tag{6}$$

The energy equation with viscous and Joule dissipations is given by

$$\rho c_p \frac{\partial T}{\partial t} + \rho c_p \nu_0 \frac{\partial T}{\partial y} = k \frac{\partial^2 T}{\partial y^2} + \mu \left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \right] + \frac{\sigma B_0^2}{1+m^2}(u^2 + w^2), \tag{7}$$

where c_p and k are, respectively, the specific heat capacity and the thermal conductivity of the fluid. The second and third terms on the right-hand side represent the viscous and Joule dissipations respectively. We notice that each of these terms has two components. This is because the Hall effect brings about a velocity w in the z-direction. The initial and boundary conditions of the problem are given by

$$u = w = 0 \text{ at } t < 0, \text{ and } w = 0 \text{ at } y = -h \text{ and } y = h \text{ for } t > 0,$$

$$u = 0 \text{ at } y = -h \text{ for } t \leq 0, \; u = U_0 \text{ at } y = h \text{ for } t > 0, \tag{8}$$

$$T = T_1 \text{ at } t \leq 0, \; T = T_2 \text{ at } y = h \text{ and } T = T_1 \text{ at } y = -h \text{ for } t > 0. \tag{9}$$

It is expedient to write the above equations in the non-dimensional form. To do this, we introduce the following non-dimensional quantities

$$\bar{x} = \frac{x}{h}, \; \bar{y} = \frac{y}{h}, \; \bar{t} = \frac{tU_0}{h}, \; \bar{u} = \frac{u}{U_0}, \; \bar{w} = \frac{w}{U_0}, \; \bar{p} = \frac{p}{\rho U_0^2}, \; \theta = \frac{T - T_1}{T_2 - T_1}, \; \bar{\mu} = \frac{\mu}{K_c^2},$$

$\tau_D = \dfrac{\tau_0 h}{K_c^2 U_0}$ is the Casson number (dimensionless yield stress)

$\mathrm{Re} = \dfrac{\rho U_0 h}{K_c^2}$ is the Reynolds number,

$\$ = \dfrac{\rho \nu_0 h}{K_c^2}$ is the suction parameter,

$\mathrm{Pr} = \dfrac{\rho c_p U_0 h}{k}$ is the Prandtl number,

$\mathrm{Ec} = \dfrac{U_0 K_c^2}{\rho c_p h (T_2 - T_1)}$ is the Eckert number,

$Ha^2 = \dfrac{\sigma B_0^2 h^2}{K_c^2}$ is the Hartmann number squared

In terms of the above non-dimensional variables and parameters Eqs.(5)-(9) and (2) are, respectively, written as (where the hats are dropped for convenience);

$$\frac{\partial u}{\partial t} + \frac{\$}{\mathrm{Re}} \frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{1}{\mathrm{Re}} \left[\frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) - \frac{Ha^2}{1 + m^2} (u + mw) \right], \tag{10}$$

$$\frac{\partial w}{\partial t} + \frac{\$}{\mathrm{Re}} \frac{\partial w}{\partial y} = \frac{1}{\mathrm{Re}} \left[\frac{\partial}{\partial y} \left(\mu \frac{\partial w}{\partial y} \right) - \frac{Ha^2}{1 + m^2} (w - mu) \right], \tag{11}$$

$$\frac{\partial \theta}{\partial t} + \frac{\$}{\mathrm{Re}} \frac{\partial \theta}{\partial y} = \frac{1}{\mathrm{Pr}} \frac{\partial^2 \theta}{\partial y^2} + \mathrm{Ec}\, \mu \left[\left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] + \frac{Ha^2 \mathrm{Ec}}{1 + m^2} (u^2 + w^2), \tag{12}$$

$$u = w = 0 \text{ for } t \leq 0 \text{ and } u = w = 0 \text{ at } y = -1, \tag{13}$$

$$w = 0, \; u = 1 \text{ at } y = 1 \text{ for } t > 0,$$

$$\theta = 0 \text{ for } t \leq 0 \text{ and } \theta = 0 \text{ at } y = -1, \; \theta = 1 \text{ at } y = 1 \text{ for } t > 0, \tag{14}$$

$$\mu = \left[1 + \left(\tau_D \Big/ \sqrt{ \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 } \right)^{1/2} \right]. \tag{15}$$

The shear stress at the two walls is given by

$$\tau_w = \left[\left(\left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right)^{1/4} + \tau_D^{1/2} / \left(\left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right)^{1/4} \right]^2 \Big| y = \pm 1.$$

The Nusselt number at the stationary wall is given by

$$Nu_1 = \frac{2 \frac{\partial \theta}{\partial y} \Big| y = -1}{-\theta_m} \cdot$$

The Nusselt number at the upper moving wall is given by

$$Nu_2 = \frac{2 \frac{\partial \theta}{\partial y} \Big| y = 1}{1 - \theta_m} \cdot$$

3. Numerical solution

Equations (10), (11) and (15) represent coupled system of non-linear partial differential equations which are solved numerically under the initial and boundary conditions (13) using the finite difference approximations. A linearization technique is first applied to replace the nonlinear terms at a linear stage, with the corrections incorporated in subsequent iterative steps until convergence is reached. Then the Crank-Nicolson implicit method is used at two successive time levels [17]. An iterative scheme is used to solve the linearized system of difference equations. The solution at a certain time step is chosen as an initial guess for next time step and the iterations are continued till convergence, within a prescribed accuracy. Finally, the resulting block tridiagonal system is solved using the generalized Thomas-algorithm [17]. The energy Eq. (12) is a linear inhomogeneous second-order ordinary differential equation whose right-hand side is known from the solutions of the flow Eqs. (10), (11) and (15) subject to the conditions (13). The values of the velocity components are substituted in the right-hand side of Eq. (12) which is solved numerically with the initial and boundary conditions (14) using central differences for the derivatives and Thomas-algorithm for the solution of the set of discritized equations. Finite difference equations relating the variables are obtained by writing the equations at the mid point of the computational cell and then replacing the different terms by their second order central difference approximations in the y-direction. The diffusion terms are replaced by the average of the central differences at two successive time-levels. The computational domain is divided into meshes each of dimension Δt and Δy in time and space respectively as shown in Fig. 2. We define the variables $\nu = u_y$, $B = w_y$,

$H = \theta_y$ and $\mu' = \mu_y$ to reduce the second order differential Eqs. (10), (11) and (12) to first order differential equations. The finite difference representations for the resulting first order differential Eqs. (10) and (11) together the equations defining the new variables take the form

$$\begin{aligned}
&\left(\frac{u_{i+1,j+1}-u_{i,j+1}+u_{i+1,j}-u_{i,j}}{2\Delta t}\right)+\frac{\$}{\mathrm{Re}}\left(\frac{v_{i+1,j+1}+v_{i,j+1}+v_{i+1,j}+v_{i,j}}{2\Delta t}\right)\\
&=-\frac{dp}{dx}+\left(\frac{\bar{\mu}_{i,j+1}+\bar{\mu}_{i,j}}{2\,\mathrm{Re}}\right)\left(\frac{(v_{i+1,j+1}+v_{i,j+1})-(v_{i+1,j}+v_{i,j})}{2\Delta y}\right)\\
&+\left(\frac{\bar{\mu}'_{i,j+1}+\bar{\mu}'_{i,j}}{2\mathrm{Re}}\right)\left(\frac{v_{i+1,j+1}+v_{i,j+1}+v_{i+1,j}+v_{i,j}}{4}\right)\\
&-\frac{Ha^2}{1+m^2}\left(\frac{u_{i+1,j+1}+u_{i,j+1}+u_{i+1,j}-u_{i,j}}{4\mathrm{Re}}\right.\\
&\left.+\,m\,\frac{w_{i+1,j+1}+w_{i,j+1}+w_{i+1,j}+w_{i,j}}{4\mathrm{Re}}\right),
\end{aligned}\tag{16}$$

$$\begin{aligned}
&\left(\frac{w_{i+1,j+1}-w_{i,j+1}+w_{i+1,j}-w_{i,j}}{2\Delta t}\right)+\frac{\$}{\mathrm{Re}}\left(\frac{B_{i+1,j+1}+B_{i,j+1}+B_{i+1,j}+B_{i,j}}{4}\right)\\
&=\left(\frac{\bar{\mu}_{i,j+1}+\bar{\mu}_{i,j}}{2\,\mathrm{Re}}\right)\left(\frac{(B_{i+1,j+1}+B_{i,j+1})-(B_{i+1,j}+B_{i,j})}{2\Delta y}\right)\\
&+\left(\frac{\bar{\mu}'_{i,j+1}+\bar{\mu}'_{i,j}}{2\mathrm{Re}}\right)\left(\frac{B_{i+1,j+1}+B_{i,j+1}+B_{i+1,j}+B_{i,j}}{4}\right)\\
&+\frac{Ha^2}{1+m^2}\left(m\,\frac{u_{i+1,j+1}+u_{i,j+1}+u_{i+1,j}+u_{i,j}}{4\mathrm{Re}}\right.\\
&\left.-\frac{w_{i+1,j+1}+w_{i,j+1}+w_{i+1,j}+w_{i,j}}{4\mathrm{Re}}\right).
\end{aligned}\tag{17}$$

The variables with bars are given initial guesses from the previous time steps and an iterative scheme is used at every time to solve the linearized system of difference equations. Then the finite difference form for the energy Eq. (12) can be written as

$$\begin{aligned}
&\left(\frac{\theta_{i+1,j+1}-\theta_{i,j+1}+\theta_{i+1,j}-\theta_{i,j}}{2\Delta t}\right)+\frac{\$}{\mathrm{Re}}\left(\frac{H_{i+1,j+1}+H_{i,j+1}+H_{i+1,j}+H_{i,j}}{4}\right)\\
&=\frac{1}{\mathrm{Pr}}\left[\frac{(H_{i+1,j+1}+H_{i,j+1})-(H_{i+1,j}+H_{i,j})}{2\Delta y}\right]+DISP,
\end{aligned}\tag{18}$$

$$(19)\quad \left(\frac{v_{i+1,j+1}+v_{i,j+1}+v_{i+1,j}+v_{i,j}}{4}\right)=\frac{(u_{i+1,j+1}+u_{i,j+1})-(u_{i+1,j}+u_{i,j})}{2\Delta y},$$

$$(20)\quad \left(\frac{B_{i+1,j+1}+B_{i,j+1}+B_{i+1,j}+B_{i,j}}{4}\right)=\frac{(w_{i+1,j+1}+w_{i,j+1})-(w_{i+1,j}+w_{i,j})}{2\Delta y},$$

$$(21)\quad \left(\frac{H_{i+1,j+1}+H_{i,j+1}+H_{i+1,j}+H_{i,j}}{4}\right)=\frac{(\theta_{i+1,j+1}+\theta_{i,j+1})-(\theta_{i+1,j}+\theta_{i,j})}{2\Delta y},$$

where $DISP$ represents the Joule and viscous dissipation terms which are known from the solution of the momentum equations and can be evaluated at the mid point $(i+1/2, j+1/2)$ of the computational cell. Computations have been made for $dp/dx = 5$, Pr=1, Re=1, and Ec=0.2. Grid-independence studies show that the computational domain $0 < t < \infty$ and $-1 < y < 1$ can be divided into intervals with step sizes $\Delta t = 0.0001$ and $\Delta y = 0.005$ for time and space respectively. Smaller step sizes do not show any significant change in the results. Convergence of the scheme is assumed when all of the unknowns u, v, w, B, θ and H for the last two approximations differ from unity by less than 10^{-6} for all values of y in $-1 < y < 1$ at every time step. Less than 7 approximations are required to satisfy this convergence criteria for all ranges of the parameters studied here. In order to examine the accuracy and correctness of the solutions, the results for the non-magnetic and Newtonian cases are compared and shown to have complete agreement with those reported by Attia [10]. This ensures the satisfaction of all the governing equations; mass continuity, momentum and energy equations.

4. Results and discussion

Figures 3, 4 and 5 present the profiles of the velocity components u and w and the temperature θ respectively for various values of time t and for $\tau_D = 0.0$, 0.05, and 0.1. The figures are evaluated for $Ha = 3$, $m = 3$, and \$ = 1. It is clear from Figs. 3 and 4 that increasing the yield stress τ_D decreases the velocity components u and w and the time at which they reach their steady state values as a result of increasing the viscosity. The figures show also that the velocity components u and w do not reach their steady state monotonically. Both u and w increase with time up till a maximum value and then decrease up to the steady state. This behaviour is more pronounced for small values of the parameter τ_D and it is more clear for u than for w. Figure 5 shows that the temperature profile reaches its steady state monotonically. It is observed also that the velocity component u reaches the steady state faster than w which, in turn, reaches the steady state faster than θ . This is expected as u is the source of w, while both u and w act as sources for the temperature.

Figures 6, 7, and 8 depict the variation of the velocity components u and w and the temperature θ at the centre of the channel $(y = 0)$ with time respectively for various values of the Hall parameter m and for $\tau_D = 0.0$, 0.05, and 0.1. In these figures, $Ha = 3$ and $\$ = 1$. Figure 6 shows that u increases with increasing m for all values of τ_D as the effective conductivity $(= \sigma/(1+m^2))$ decreases with increasing m which reduces the magnetic damping force on u . It is observed also from the figure that the time at which u reaches its steady state value increases with increasing m while it decreases when τ_D increases. The effect of τ_D on u becomes more pronounced for large values of m. In Fig. 7, the velocity component w increases with increasing m as w is a result of the Hall effect. On the other hand, at small times, w decreases when m increases. This happens due to the fact that, at small times w is very small and then the source term of w is proportional to $(mu/(1+m^2))$ which decreases with increasing m $(m > 1)$. This accounts for the crossing of the curves of w with t for all values of τ_D. Figures 6 and 7 indicate also that the influence of τ_D on u and w depends on m and becomes more clear when m is large. An interesting phenomenon is observed in Figs. 6 and 7, which is that, when m has a nonzero value the component u and, sometimes, w overshoot. For some times they exceed their steady state values and then go down towards steady state. This may be explained by stating that with the progress of time, u increases and consequently w increases according to Eq. (11) until w reaches its maximum value. The increase in w results in a small decrease in u according to Eq. (10). This reduction in u may, in turn, result in a decrease in w according to Eq. (11) which explains the reduction after the peaks. The time at which overshooting occurs decreases with increasing τ_D. Figure 8 shows that the influence of m on θ depends on t . Increasing m decreases θ at small times and increases it at large times. This is due to the fact that, for small times, u and w are small and an increase in m increases u but decreases w. Then, the Joule dissipation which is also proportional to $(1/1+m^2)$ decreases. For large times, increasing m increases both u and w and, in turn, increases the Joule and viscous dissipations. This accounts for the crossing of the curves of θ with time for all values of τ_D. It is also observed that increasing τ_D increases the temperature θ for small t, but decreases it for large t (see also Table 1 for more clear presentation of these results). This is because increasing τ_D decreases both u and w and their gradients which decreases the Joule and viscous dissipations. Tables 2 and 3 present the effect of the parameters m and τ_D, respectively, on the steady state time of the temperature θ. It is clear that increasing m increases the steady state time of θ while increasing τ_D decreases it. The figure shows also that the time at which θ reaches its steady state value increases with increasing m while it is not greatly affected by changing τ_D.

Table 1. The development of the temperature θ with time t for $Ha = 3$, $m = 5$ and $\$ = 1$

τ_D	t=0.1	t=0.3	t=0.5	t=0.7	t=0.9	t=1.1	t=1.3	t=1.5	t=1.7
0.0	.0172	.1419	.2634	.3686	.4488	.5035	.5376	.5574	.5680
0.05	.0299	.1675	.2920	.3874	.4530	.4949	.5199	.5345	.5426
0.1	.0384	.1776	.2998	.3899	.4503	.4880	.5105	.5235	.5309

Table 2. The steady state temperature θ at $y = 0$ and its corresponding time for $\tau_D = 0$, $\$ = 1$ and $Ha = 3$.

m	0	1	3
θ	0.440	0.450	0.500
t	2.0	2.25	2.5

Table 3. The steady state temperature θ at $y = 0$ and its corresponding time for $m = 5$, $\$ = 1$ and $Ha = 3$.

τ_D	0.0	0.05	0.1
θ	0.580	0.549	0.536
t	2.45	2.20	2.00

Figures 9, 10, and 11 show the effect of the suction parameter $\$$ on the time development of the velocity components u and w and the temperature θ at $y = 0$ with time respectively for various values for $\tau_D = 0.0$, 0.05, and 0.1. In these figures, $Ha = 3$ and $m = 3$. Figure 9 shows that u at the centre of the channel decreases with increasing $\$$ for all values of τ_D due to the convection of the fluid from regions in the lower half to the centre, which has higher fluid speed. Figure 10 shows that w decreases with increasing $\$$ for all values of τ_D as a result of decreasing u which affects the source term of w . The figure presents also the influence of $\$$ on the reduction of the overshooting in w especially for small values of τ_D. Figure 11 indicates that increasing $\$$ decreases the temperature at the centre of the channel for all values of τ_D. This is due to the influence of the convection in pumping the fluid from the cold lower half towards the centre of the channel.

5. Conclusions

The transient Couette flow of a Casson non-Newtonian fluid under the influence of an applied uniform magnetic field is studied considering the Hall effect. The effects of the Casson number τ_D, the Hall parameter m , and the suction parameter $\$$ on the velocity and temperature distributions are studied. The Hall term affects the main velocity component u in the x-direction and gives rise to another velocity component w in the z-direction. An overshooting in the velocity components u and w with time due to the Hall effect is observed for all values of τ_D. The flow index τ_D has an apparent effect in controlling the overshooting in u or w and the time at which it occurs. The results show that the influence of the parameter τ_D on u and w depends on m and becomes more apparent when m is large. It is found also that the effect of m on w depends on t for all values of τ_D which accounts for a crossover in the $w - t$ graph for various values of m. The effect of m on the magnitude of θ depends on n and becomes more pronounced in case of small τ_D. The time at which u and w reach the steady state increases with increasing m, but decreases when τ_D increases. The time at which θ reaches its steady state increases with increasing m while it is not greatly affected by changing τ_D.

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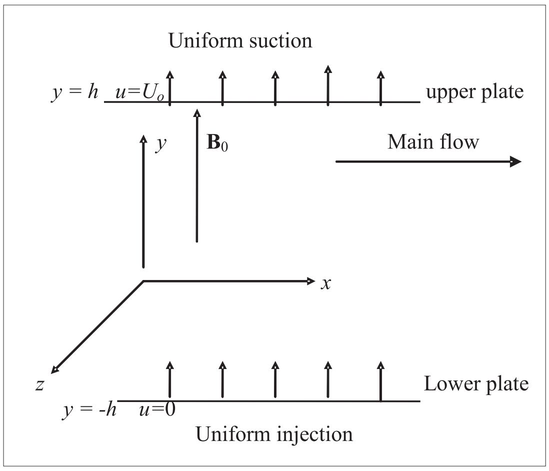


Figure 1: The geometry of the problem

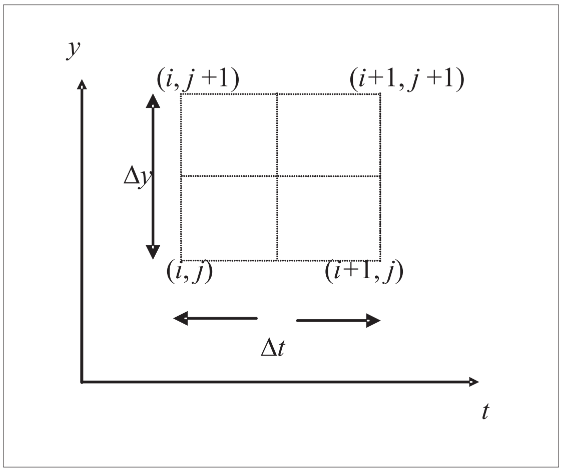
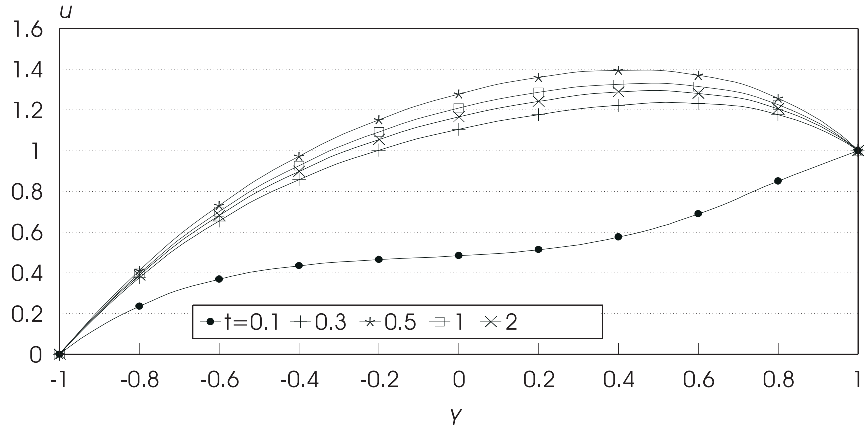
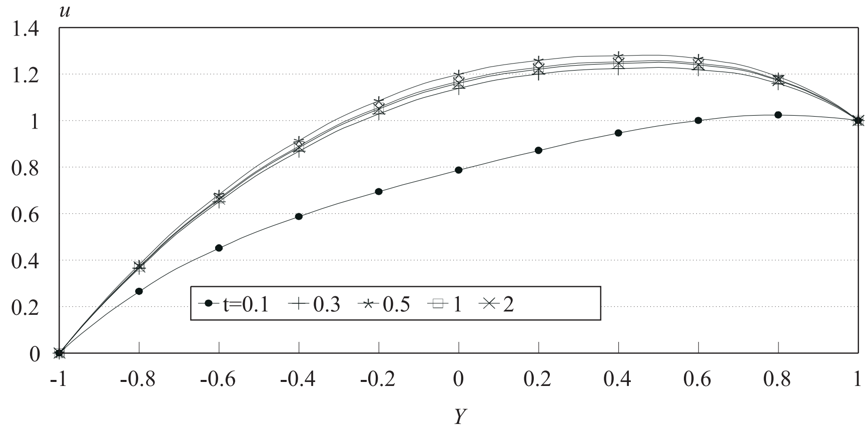


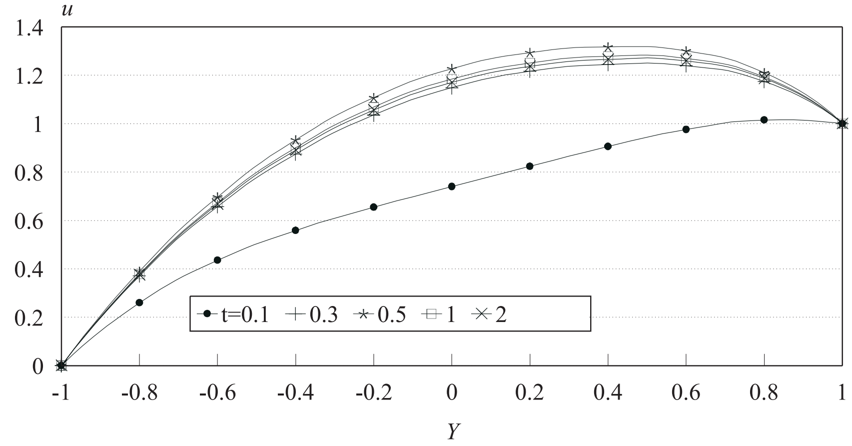
Figure 2: Mesh network



(a) $\tau_D = 0.0$

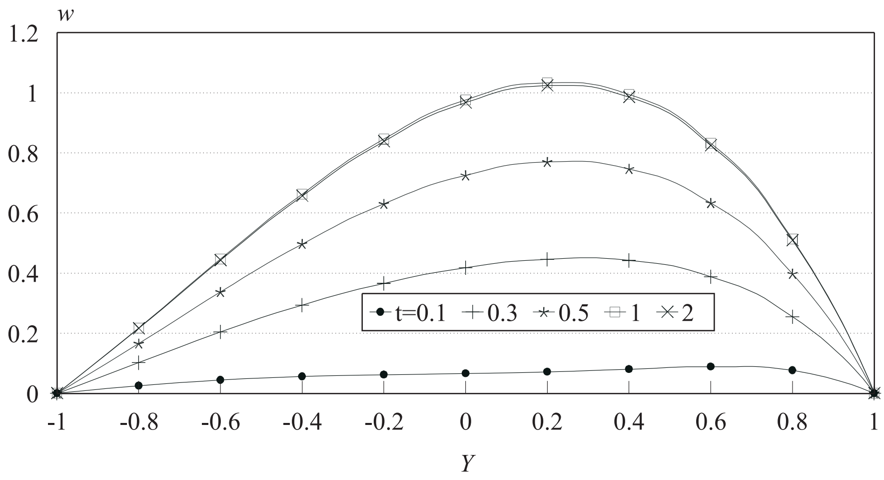


(b) $\tau_D = 0.05$

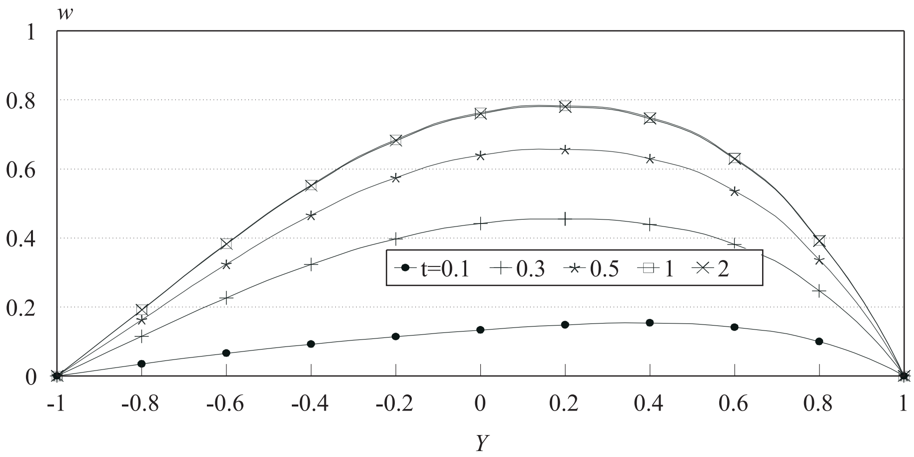


(c) $\tau_D = 0.1$

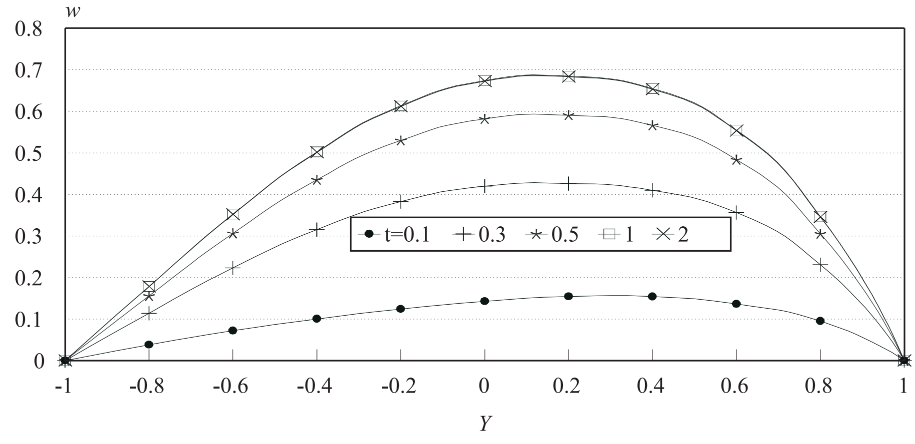
Figure 3: Time Development of the velocity component u for S=1, Ha=3, m=3.



(a) $\tau_D = 0.0$

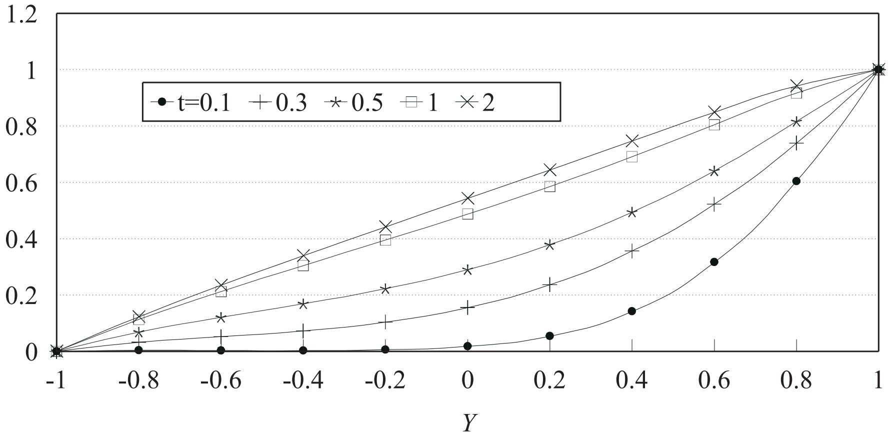


(b) $\tau_D = 0.05$

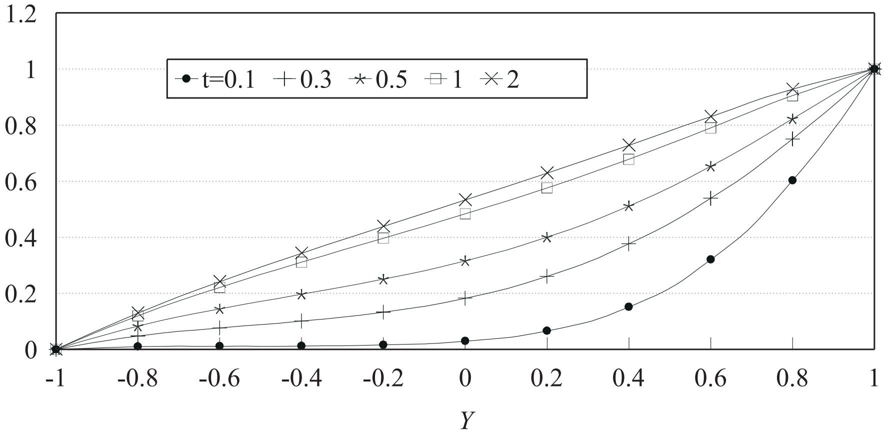


(c) $\tau_D = 0.1$

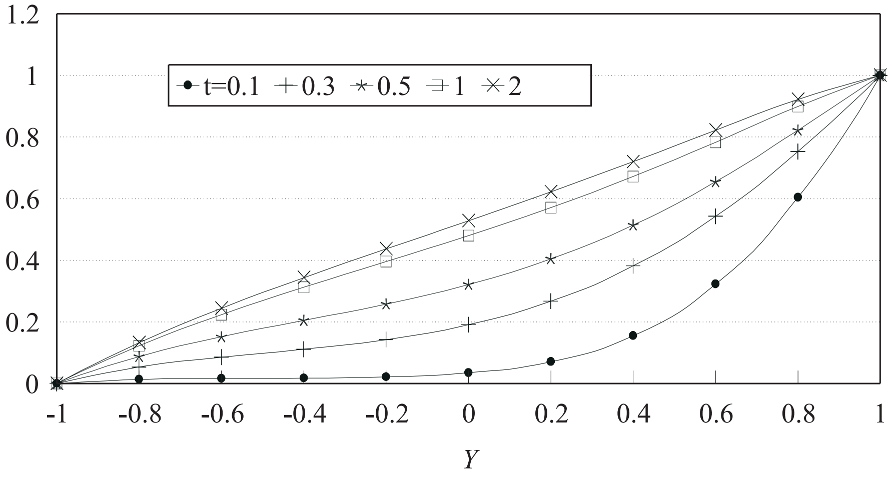
Figure 4: Time Development of the velocity component w for $S = 1,\ Ha = 3,\ m = 3$.



(a) $\tau_D = 0.0$

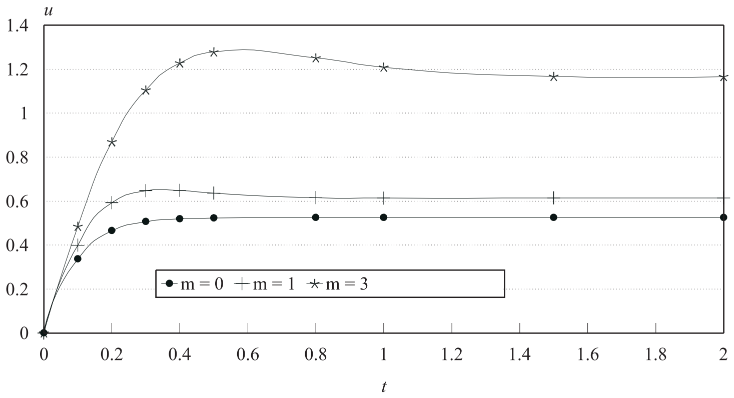


(b) $\tau_D = 0.05$

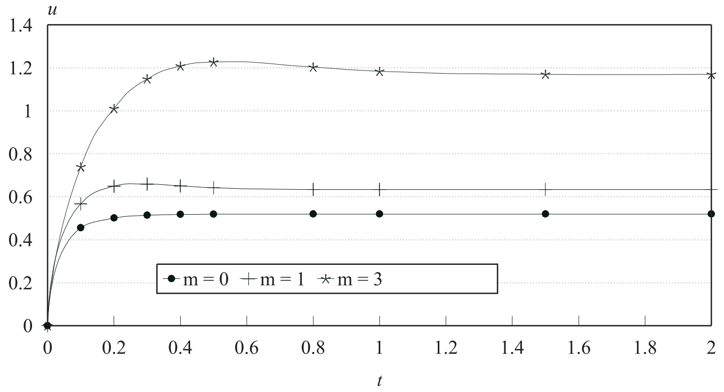


(c) $\tau_D = 0.1$

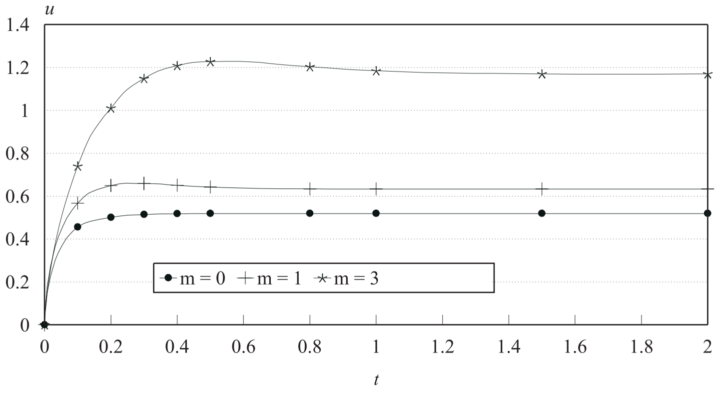
Figure 5: Time Development of the velocity component θ for $S = 1,\ Ha = 3,\ m = 3$.



(a) $\tau_D = 0.0$

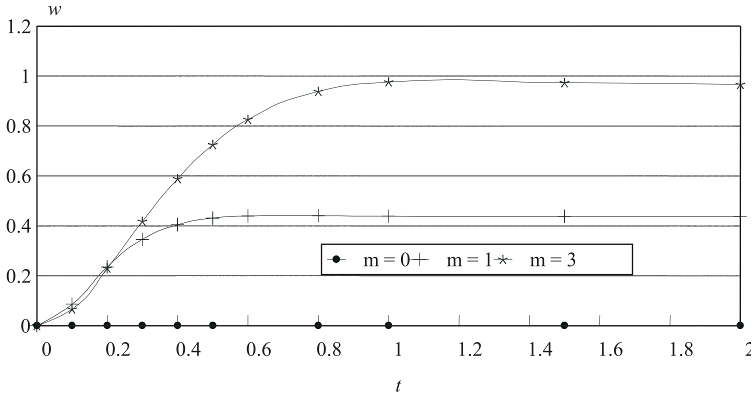


(b) $\tau_D = 0.05$

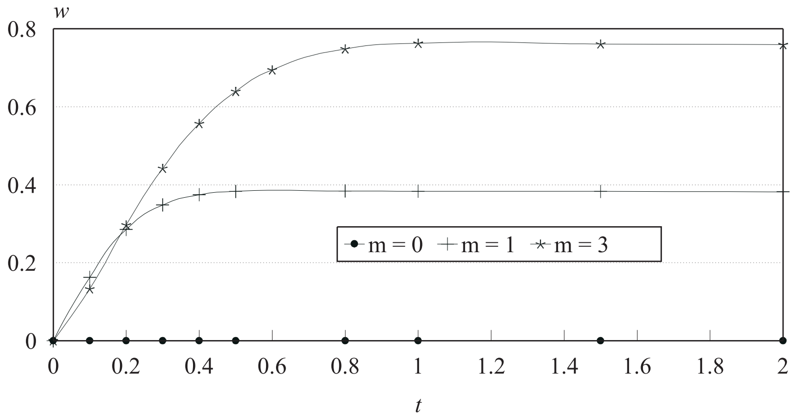


(c) $\tau_D = 0.1$

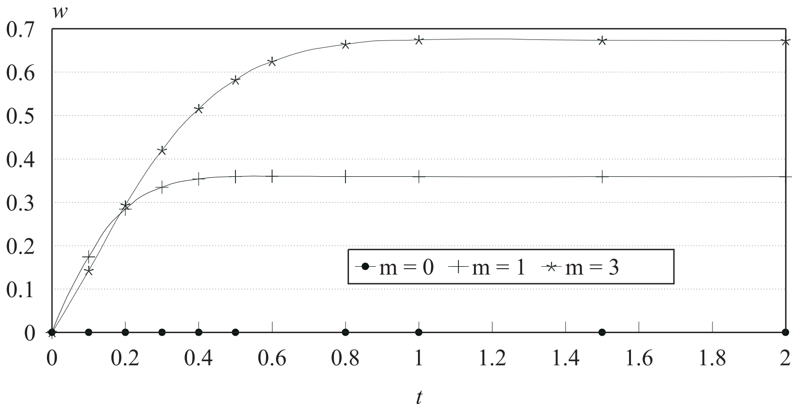
Figure 6: Effect of the Hall parameter m on the time development of u at $y = 1$ for $S = 1,\ Ha = 3$.



(a) $\tau_D = 0.0$

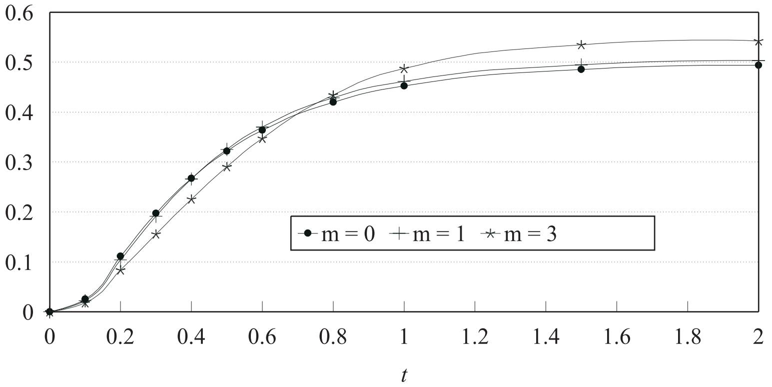


(b) $\tau_D = 0.05$

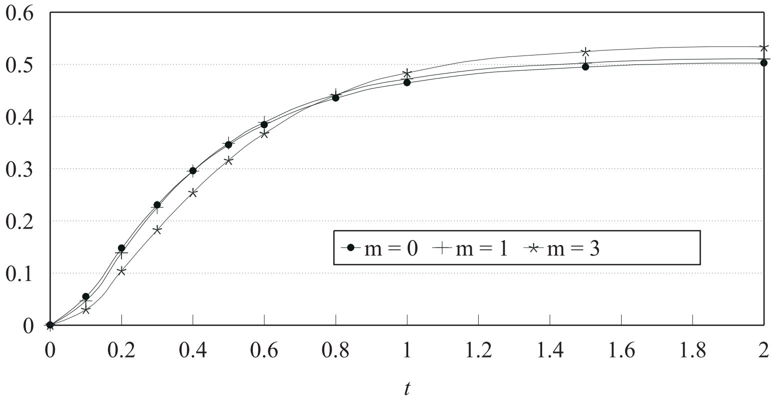


(c) $\tau_D = 0.1$

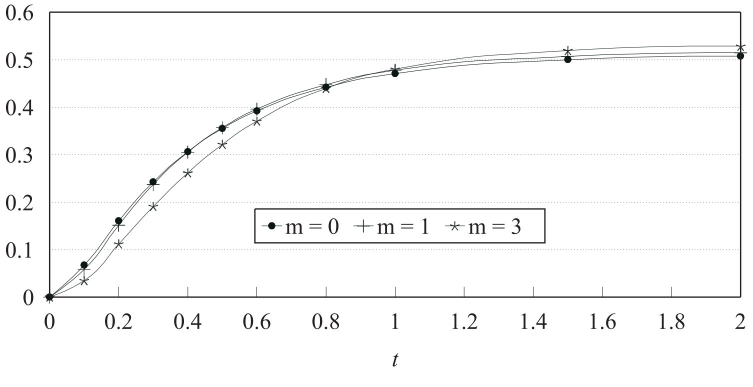
Figure 7: Effect of the Hall parameter m on the time development of w at $y = 1$ for $S = 1,\ Ha = 3$.



(a) $\tau_D = 0.0$

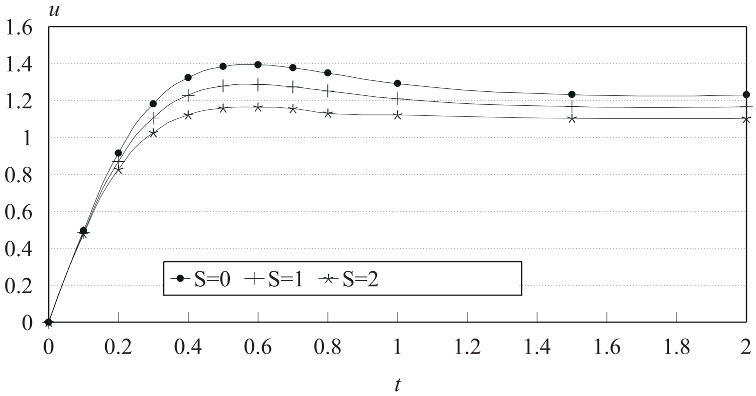


(b) $\tau_D = 0.05$

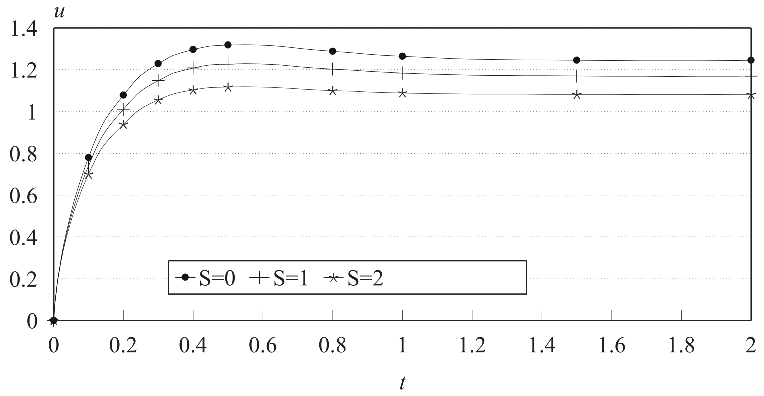


(c) $\tau_D = 0.1$

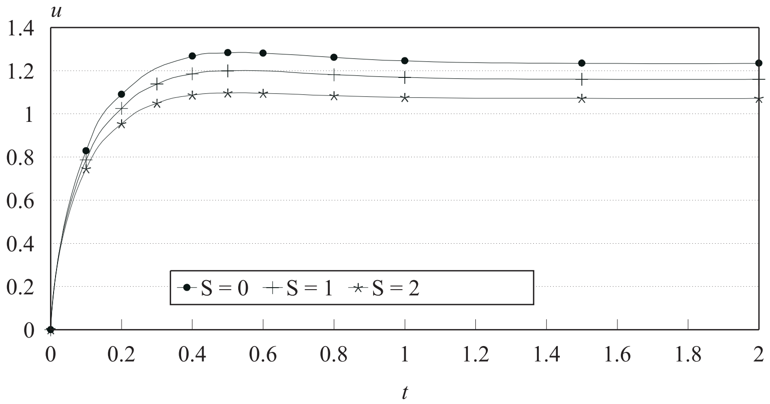
Figure 8: Effect of the Hall parameter m on the time development of θ at $y = 1$ for $S = 1,\ Ha = 3$.



(a) $\tau_D = 0.0$

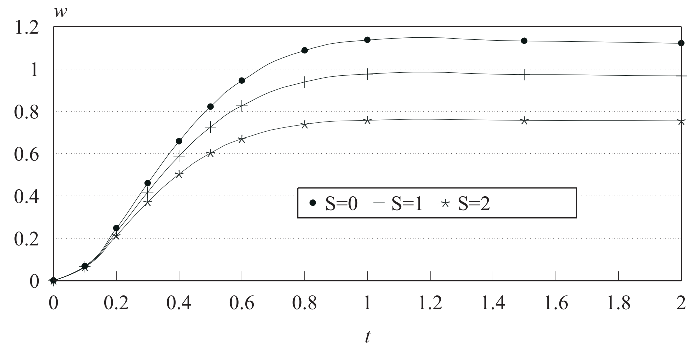


(b) $\tau_D = 0.05$

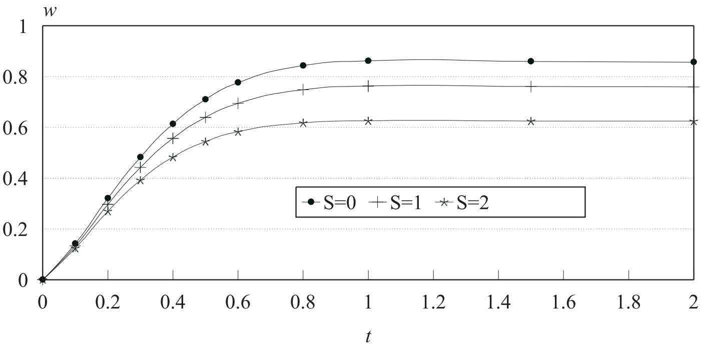


(c) $\tau_D = 0.1$

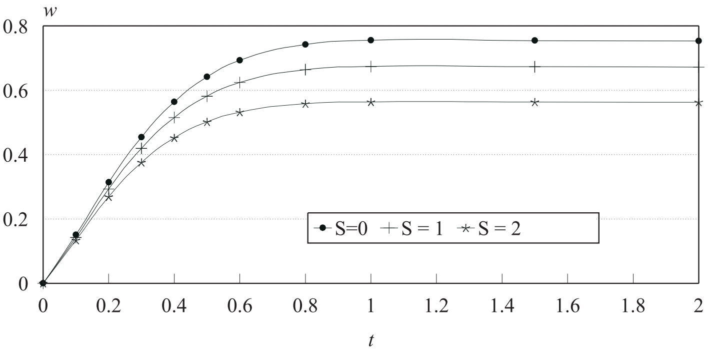
Figure 9: Effect of the suction parameter S on the time development of u at $y = 1$ for $S = 1,\ Ha = 3$.



(a) $\tau_D = 0.0$

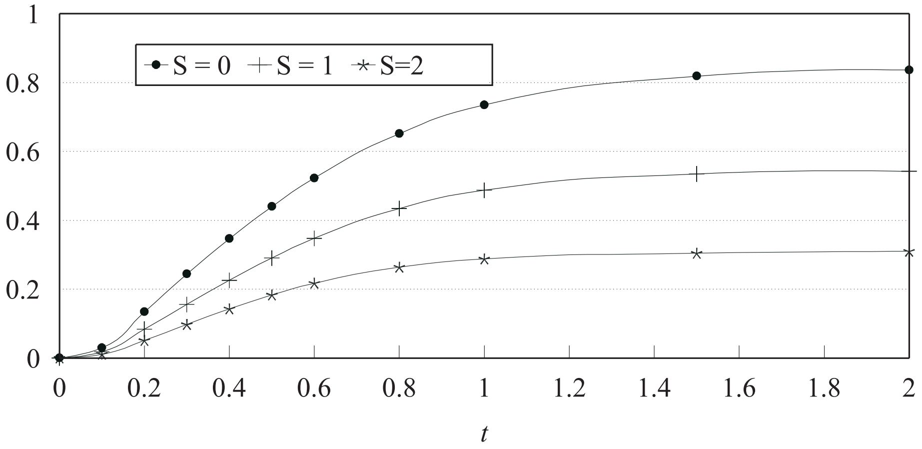


$\tau_D = 0.05$

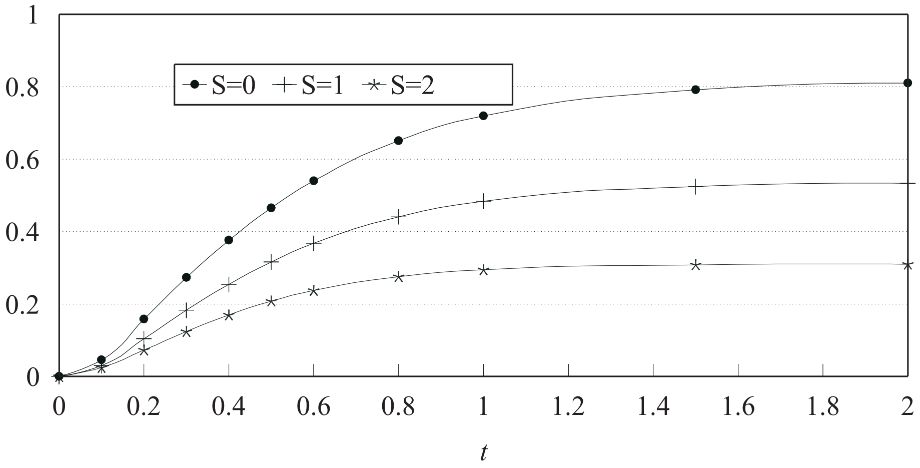


$\tau_D = 0.1$ (c)

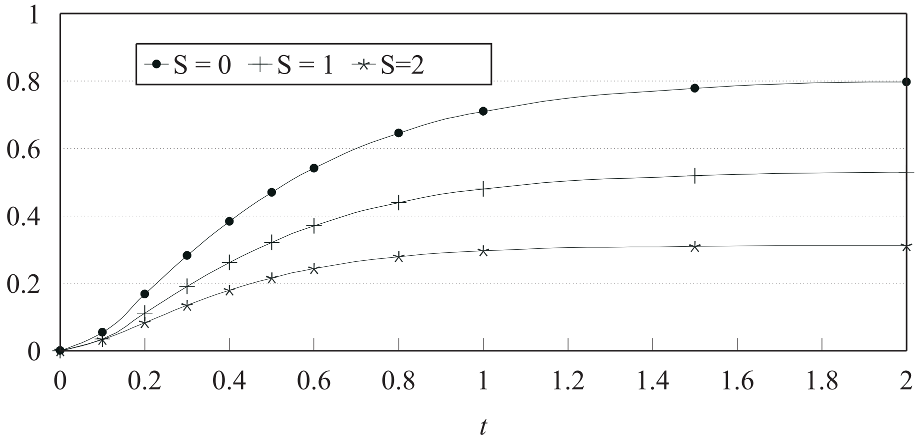
Figure 10: Effect of the suction parameter S on the time development of w at $y = 1$ for $S = 1,\ Ha = 3$.



(a) $\tau_D = 0.0$



(b) $\tau_D = 0.05$



(c) $\tau_D = 0.1$

Figure 11: Effect of the suction parameter S on the time development of θ at $y = 1$ for $S = 1,\ Ha = 3$.

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TWO-DIMENSIONAL WAVELETS FOR NONLINEAR AUTOREGRESSIVE MODELS WITH AN APPLICATION IN DYNAMICAL SYSTEM

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Abstract. In this note we introduce a new estimator for estimating autoregressive model function based on two-dimensional wavelet expansion of joint density function. We investigate some asymptotic properties of the proposed estimator. We also added the problem of estimating of derivative of autoregressive estimator through new approach. Finally, we apply our method in dynamical systems. In particular, we estimate a chaotic map from a noisy data and filter entropy of the chaotic map.

Keywords: two-dimensional wavelet, multiresolution analysis, Random design, Besov space, wavelets.

1. Introduction

Autoregressive models form important class of processes in time series theory. A nonparametric version of these models was first introduced by Jones [21]. Let $(X, \mathcal{F}, P)$ be a probability space and $\{X_i\}_{i\geq 0}$ be a random process associated with $(X, \mathcal{F}, P)$. We observe the series $\{X_0, X_1, \ldots, X_n\}$ that follow the nonlinear autoregressive model $X_{i+1} = \tau(X_i) + \epsilon_i$, where τ is a transformation and ϵ_i is an error. For theoretical purposes, we consider *iid* perturbations ϵ_i with $\mathbb{E}(\epsilon_i) = 0$, $\mathbb{E}(\epsilon_i^2) = \sigma_i^2$, not necessarily gaussian.

Several authors dealt with the problem of estimating the autoregressive function τ nonparametrically. See Frank et al.[15], Hardle and Tsybakov [19], Robinson [27], Masry and Tjosthein [22], Hafner [18], Tjosthein [31], Buhlmann and McNeil [4], Delouille et al. [7], Delouille and Von Sachs [8] and Delouille et al. [9]. However, very little is known about 'wavelet' estimation for autoregressive designs. The result of Hoffmann [20] that treats autoregressive models using a wavelet estimator is concerned with asymptotical results only, and does not provide an efficient algorithm in practice. Delouille et al. [10] estimated nonlinear autoregressive models using a design-adapted wavelet estimator. We closely follow the work of Doosti et al. [13]. In their note [13], they dealt with two-dimensional wavelets for stochastic regression.

The organization of the paper is as follows. After some preliminaries given in Section 2, we introduce our proposed estimators in Section 3. Asymptotic properties of our proposed estimator are discussed in Section 4. Derivative of of wavelet estimator is found in Section 5. In Section 6, we apply our wavelet estimator to dynamical systems; we estimate a chaotic map from noisy data and entropy for this chaotic map.

2. Preliminaries

In this section, we first introduce one-dimensional wavelet density function estimator and then we introduce multiresolution analysis in two-dimensional case for joint density function estimation.

2.1. Wavelet linear density estimator

Let $\{X_i\}_{i \geq 0}$ be a sequence of real-valued random variables on the probability space $(X, \mathcal{F}, P)$. We suppose that X_i has a bounded and compactly supported marginal density f with respect to Lebesgue measure which does not depend on i. We estimate this density from N observation X_i, $i = 1, ..., N$. For any function $f \in \mathbb{L}_2(\mathbb{R})$, we can write a formal expansion (see Daubechies [5]):

$$f = \sum_{k \in Z} \alpha_{j_0,k} \phi_{j_0,k} + \sum_{j \geq j_0} \sum_{k \in Z} \delta_{j,k} \psi_{j,k} = P_{j_0} f + \sum_{j \geq j_0} D_j f,$$

where the functions

$$\phi_{j0,k}(x) = 2^{j_0/2} \phi(2^{j_0} x - k) \quad \text{and} \quad \psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k)$$

constitute an (inhomogeneous) orthonormal basis of $\mathbb{L}_2(\mathbb{R})$ and $\phi(x)$ and $\psi(x)$ are the scale function and the orthogonal wavelet respectively. Wavelet coefficients are given by the integrals

$$\alpha_{j_0,k} = \int f(x) \phi_{j_0,k}(x) dx, \quad \delta_{j,k} = \int f(x) \psi_{j,k} dx.$$

We suppose that both ϕ and $\psi \in \mathbb{C}^{r+1}$, $r \in \mathbb{N}$, and have compact supports in $[-\delta, \delta]$. Note that, by Corollary 5.5.2 in (Daubechies [6]), ψ is orthogonal to polynomials of degree $\leq r$, i.e., $\int \psi(x)x^l dx = 0$, $l = 0, 1, ..., r$. We suppose that f belongs to the Besov class (see Meyer [23], section VI.10)

$$F_{s,p,q} = \{f \in B^s_{p,q}, \|f\|_{B^s_{p,q}} \leq M\}$$

for some $0 \leq s \leq r+1$, $p \geq 1$, $q \geq 1$, where $\|f\|_{B^s_{p,q}} = \|P_{j_0} f\|_p + \Big(\sum_{j \geq j_0} (\|D_j f\|_p 2^{js})^q\Big)^{1/q}$.

We say that $f \in B^s_{p,q}$ if and only if

$$\text{(2.1)} \qquad \|\alpha_{j_0,.}\|_{l_p} < \infty, \quad \text{and} \quad \left(\sum_{\mathrm{j} \geq \mathrm{j_0}} (\|\delta_{\mathrm{j},.}\|_{\mathrm{l_p}} 2^{\mathrm{j(s+1/2-1/p)}})^{\mathrm{q}}\right)^{1/\mathrm{q}} < \infty,$$

where $\|\gamma_{j,.}\|_{l_p} = \Big(\sum_{k \in Z} \gamma^p_{j,k}\Big)^{1/p}$. We consider Besov spaces essentially because of their executional expressive power (see Triebel [33]) and the discussion in Donoho et al. [12]. We construct the density estimator

$$\text{(2.2)} \qquad \hat{f}_1 = \sum_{k \in K_{j_0}} \hat{\alpha}_{j_0,k} \phi_{j_0,k}, \quad \text{with} \quad \hat{\alpha}_{\mathrm{j_0,k}} = \frac{1}{\mathrm{n}} \sum_{\mathrm{i=1}}^{\mathrm{n}} \phi_{\mathrm{j_0,k}}(\mathrm{X_i}),$$

where K_{j_0} is the set of k such that supp(f) $\cap$ supp $\phi_{\mathrm{j_0,k}} \neq \emptyset$. The fact that ϕ has a compact support implies that K_{j_0} is finite and card $K_{j_0} = O(2^{j_0})$. Wavelet density estimators aroused much interest in the recent literature (see Donoho et al. [1] and Doukhan and Leon [14]). In the case of independent samples, the properties of the linear estimator in (2.2) was studied for a variety of error measures and density classes by Kerkyacharian and Picard [24] and Tribouley [32]). It was shown, for example, that these estimators are minimax when the L_p-risk is concerned and the density belongs to Besov space $B^s_{p,q}$. When the error of estimation is measures in $L_{\acute{p}} - norm$, with $\acute{p} \geq p$, the linear wavelet estimators are not optimal any more, although they are still minimax in the class of linear estimators (see Donoho et al. [11]), Kerkyacharian and Picard [24]). In "weak dependent" cases, Leblanc [16] investigated some asymptotic property of estimator (2.2). The estimator in (2.2) is a special case of a kernel density estimator with kernel $K(x, y) = \sum_k \phi_{j_0,k}(x)\phi_{j_0,k}(y)$. In terms of kernel, (2.2) can be expressed as

$$\hat{f}_1(x) = \frac{1}{n} \sum_{i=1}^{n} K_{j_0}(x, X_i),$$

where the orthogonal projection kernels are $K_{j_0}(x, y) = 2^{j_0} K(2^{j_0} x, 2^{j_0} y)$. Huang [20] studied asymptotic bias and variance of linear wavelet density estimation. Define

$$b_m(x) = x^m - \int_{-\infty}^{\infty} K(x, y) y^m dy.$$

The functions $b_m(x)$ are important in expressing the asymptotic bias of linear estimators and finding their efficiencies with respect to the standard kernel density estimators. Theorem 2.1 below gives the bias for our linear density function estimator 2.2.

Theorem 2.1 (Huang [20]) *Assume that the density f belongs to the Holder space* $\mathbb{C}^{m+\alpha}$, $0 \le \alpha \le 1$, *and the wavelet-kernel* $K(x,y)$ *satisfies the following localization property* $\int_{-\infty}^{\infty} K(x,y)(y-x)^{m+\alpha}dy \le C$, *for some positive* C. *Let* $j \to \infty$ *and* $n2^{-j} \to \infty$, *as* $n \to \infty$. *Then, for fixed* x,

$$\mathbb{E}\hat{f}_1(x) - f(x) = -\frac{1}{m!} f^{(m)}(x) b_m(2^j x) 2^{-mj} + O(2^{-j(m+\alpha)}).$$

The asymptotic variance of $\hat{f}_1$ is given in Theorem 2.2 below. This theorem is a generalization of a theorem proved by Huang [20].

Theorem 2.2 *Let* $f \in \mathbb{C}^1$, $\acute{f}$ *be the first derivative of* f, f *and* $\acute{f}$ *be uniformly bounded and the mixing rate* α *satisfies* $\sum_{k=1}^{\infty} \alpha(k) < \infty$. *Then, for* x *fixed,*

$$Var\hat{f}(x) = \frac{2^j}{n} f(x) V(2^j x) + O(n^{-1}),$$

where $V(x) = \int_{-\infty}^{\infty} K^2(x,y)dy = K(x,x)$.

Proof.

$$\begin{aligned} Var\hat{f}(x) &= Var\left\{\frac{1}{n}\sum_{i=1}^{n} K_h(x,X_i)\right\} = \frac{1}{n^2}\sum_{i=1}^{n} Var\{K_h(x,X_i)\} \\ &+ \frac{2}{n^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} Cov(K_h(x,X_i),K_h(x,X_j)) = T_1 + T_2. \end{aligned} \tag{2.3}$$

Now, we have

$$\begin{aligned} T_1 &= \frac{1}{n}\int_{-\infty}^{\infty} K_h^2(x,y)f(y)dy - \frac{1}{n}\left(\int_{-\infty}^{\infty} K_h(x,y)f(y)dy\right)^2 \\ &= \frac{1}{n}f(x)\int_{-\infty}^{\infty} K_h^2(x,y)dy + \frac{1}{n}\int_{-\infty}^{\infty} K_h^2(x,y)(f(y)-f(x))dy \\ &\quad -\frac{1}{n}\left(\int_{-\infty}^{\infty} K_h(x,y)f(y)dy\right)^2 \\ &= \frac{1}{nh}f(x)V(x/h) + \frac{1}{n}\int_{-\infty}^{\infty} K_h^2(x,y)(f(y)-f(x))dy \\ &\quad -\frac{1}{n}\left(\int_{-\infty}^{\infty} K_h(x,y)f(y)dy\right)^2. \end{aligned}$$

Below, we show that the second and the third terms in the last equality are of order $O(n^{-1})$

$$\begin{aligned}|\frac{1}{n}\int_{-\infty}^{\infty} K_h^2(x,y)(f(y)-f(x))dy| &\leq \frac{1}{n}\sup_x |\acute{f}(x)|\frac{1}{h^2}\int_{-\infty}^{\infty} K^2(x/h,y/h)|y-x|dy\\ &\leq \frac{1}{n}\sup_x |\acute{f}(x)| \sup_{s,t\in\mathbb{R}} |K(s,t)| \int_{-\infty}^{\infty} |K(x/h,t)(t-x/h)|dt = O(n^{-1}).\end{aligned}$$

By the uniform boundedness of $f(x)$, it is easy to see that

$$\frac{1}{n}\left(\int_{-\infty}^{\infty} K_h(x,y)f(y)dy\right)^2 = O(\frac{1}{n}).$$

Thus,

$$T_1 = \frac{2^j}{n} f(x)V(2^j x) + O(n^{-1}) \tag{2.4}$$

To complete the proof, it is enough to prove $T_2 = O(n^{-1})$. Now,

$$\begin{aligned}Cov(K_h(x,X_i),K_h(x,X_j)) &= \mathbb{E}(K_h(x,X_i)K_h(x,X_j)) - \mathbb{E}K_h(x,X_i)\mathbb{E}K_h(x,X_j)\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_h(x,y)K_h(x,z)f_{X,Y}(y,z)dydz\\ &- \left(\int_{-\infty}^{\infty} K_h(x,y)f(y)dy\right)^2 \leq \left(\int_{-\infty}^{\infty} K_h(x,y)dy\right)^2 \alpha(j-i).\end{aligned}$$

Thus

$$T_2 \leq \frac{2}{n^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\phi(j-i) = \frac{2}{n}\sum_{k=1}^{n}(1-k/n)\alpha(k) \leq \frac{2}{n}\sum_{k=1}^{n}\alpha(k). \tag{2.5}$$

By assumption, the above sum is finite. Hence, (2.4) and (2.5) complete the proof. ∎

2.2 Multiresolution analysis in two-dimension

The simplest approach consists in building a 2-D multiresolution analysis by taking the direct (tensor) product of two such structure in 1-D, one for the x direction, one for the y direction. If $V_j \in \mathbb{Z}$ is a multiresolution analysis of $L^2(\mathbb{R})$, then $V_j^{(2)} = V_j \bigotimes V_j, j \in \mathbb{Z}$ is a multiresolution analysis of $L^2(\mathbb{R}^2)$. Writing again $V_j^{(2)} \bigoplus W_j^{(2)} = V_{j+1}^{(2)}$, where $W_j^{(2)}$ is orthogonal complement of $V_j^{(2)}$, it is easy to see that this 2-D analysis requires one scaling function: $\phi(x,y) = \phi(x)\phi(y)$, and three wavelets:

$$\psi^h(x,y) = \phi(x)\psi(y), \quad \psi^v(x,y) = \psi(x)\phi(y), \quad \psi^d(x,y) = \psi(x)\psi(y). \tag{2.6}$$

ψ^h detects preferentially horizontal edges, that is, discontinuities in the vertical direction, whereas ψ^v and ψ^d detect vertical and oblique edges, respectively. Indeed, for j=1, the relation $V_1 = V_0 \bigoplus W_0$ yields:

$$V_1^{(2)} = V_1^{(x)} \bigotimes V_1^{(y)} = (V_0^{(x)} \bigoplus W_0^{(x)}) \bigotimes (V_0^{(y)} \bigoplus W_0^{(y)}),$$

where $V_0^{(2)} = V_0^{(x)} \bigotimes V_0^{(y)} \ni \phi(x)\phi(y)$ is the direct sum of three other products, generated by three wavelets given in (2.6), respectively. Based on the two-multiresolution analysis discussed in Vidakovic [34] and Antoine et al. [3], we introduce two-variate wavelet density estimators. Let f be a density from $\mathbb{L}_2(\mathbb{R}^2)$. The wavelet series is

$$\begin{aligned} f_{X,Y}(x,y) &= \sum_{\underline{k}} \alpha_{j_0,\underline{k}} \phi_{j_0,k_1}(x)\phi_{j_0,k_2}(y) + \sum_{j \geq j_0} \sum_{\underline{k}} (d_{j,\underline{k}}^{(1)} \phi_{j,k_1}(x)\psi_{j,k_2}(y) \\ &\quad + d_{j,\underline{k}}^{(2)} \psi_{j,k_1}(x)\phi_{j,k_2}(y) + d_{j,\underline{k}}^{(3)} \psi_{j,k_1}(x)\psi_{j,k_2}(y)), \end{aligned}$$

where

$$\begin{aligned} \alpha_{j_0,\underline{k}} &= \int\int \phi_{j,k_1}(x)\phi_{j,k_2}(y) f_{X,Y}(x,y)dxdy, \\ d_{j,\underline{k}}^{(1)} &= \int\int \phi_{j,k_1}(x)\psi_{j,k_2}(y) f_{X,Y}(x,y)dxdy, \\ d_{j,\underline{k}}^{(2)} &= \int\int \psi_{j,k_1}(x)\phi_{j,k_2}(y) f_{X,Y}(x,y)dxdy, \\ d_{j,\underline{k}}^{(3)} &= \int\int \psi_{j,k_1}(x)\psi_{j,k_2}(y) f_{X,Y}(x,y)dxdy \end{aligned}$$

and their estimators are

$$\hat{\alpha}_{j_0,\underline{k}} = \frac{1}{n}\sum_{i=1}^{n} \phi_{j,k_1}(X_i)\phi_{j,k_2}(Y_i), \quad \hat{d}_{j,\underline{k}}^{(1)} = \frac{1}{n}\sum_{i=1}^{n} \phi_{j,k_1}(X_i)\psi_{j,k_2}(Y_i)$$
$$\hat{d}_{j,\underline{k}}^{(2)} = \frac{1}{n}\sum_{i=1}^{n} \psi_{j,k_1}(X_i)\phi_{j,k_2}(Y_i), \quad \hat{d}_{j,\underline{k}}^{(3)} = \frac{1}{n}\sum_{i=1}^{n} \psi_{j,k_1}(X_i)\psi_{j,k_2}(Y_i)$$

3. Two-dimensional wavelet for density function estimation

Let the process $\{X_i\}$ be strongly mixing, i.e.,

$$|Pr\{X_i \in A_1, X_{i+k} \in B_1\} - Pr\{X_i \in A_1\}Pr\{X_{i+k} \in B_1\}| \leq \alpha(k),$$

where $\alpha(k) \to 0$ as $k \to \infty$. Let (X_i, X_{i+1}), $i = 0, ..., n$ has unknown joint density function $f_{X_i,X_{i+1}}$ on $\mathbb{R}^2$. We calculate marginal density function by integrating X_{i+1},

$$\begin{aligned} f_{X_i}(x) &= \int_0^1 f_{X_i,X_{i+1}}(x,y)dy = \sum_{\underline{k}} \alpha_{j0,\underline{k}}\phi_{j0,k_1}(x)\int_0^1 \phi_{j0,k_2}(y)dy \\ &\quad + \sum_{j\geq j0}\sum_{\underline{k}} \Bigg[d^{(1)}_{j,\underline{k}}\phi_{j,k_1}(x)\int_0^1 \psi_{j,k_2}(y)dy \\ &\quad + d^{(2)}_{j,\underline{k}}\psi_{j,k_1}(x)\int_0^1 \phi_{j,k_2}(y)dy + d^{(3)}_{j,\underline{k}}\psi_{j,k_1}(x)\int_0^1 \psi_{j,k_2}(y)dy \Bigg]. \end{aligned}$$

Now,

$$\int_0^1 \phi_{j,k_2}dy = 2^{j/2}\int_0^1 \phi(2^j y - k_2)dy.$$

Let $t = 2^j y - k_2$. Then

$$\int_0^1 \phi_{j,k_2}dy = 2^{-j/2}\int_{-k_2}^{2^j-k_2} \phi(t)dt.$$

If $0 \leq k2 \leq 2^j - 1$ then we have

$$\int_0^1 \phi_{j,k_2}dy = 2^{-j/2}\int_0^1 \phi(t)dt = 2^{-j/2}.$$

Similarly, for $0 \leq k_2 \leq 2^j - 1$, we have

$$\int_0^1 \psi_{j,k_2}(y)dy = 2^{-j/2}\int_0^1 \psi(t)dt = 0.$$

By our assumption X_i has density function independent of i. Thus,

$$f_X(x) = \sum_{k_1} \beta^1_{j0,k_1}\phi_{j0,k_1}(x) + \sum_{j\geq j0}\sum_{k_1} \gamma^1_{j,k_1}\psi_{j,k_1}(x), \tag{3.1}$$

where

$$\beta^1_{j0,k_1} = 2^{-j0/2}\sum_{k_2=0}^{2^j 0-1} \alpha_{j0,\underline{k}} \text{ and } \gamma^1_{j,k_1} = 2^{-j/2}\sum_{k_2=0}^{2^j 0-1} d^{(2)}_{j,\underline{k}}.$$

Now, using (3.1), we propose the following density function estimator:

$$\hat{f}_2(x) = \sum_{k_1\in K_{j_0}} \hat{\beta}^1_{j_0,k_1}\phi_{j_0,k_1}(x), \tag{3.2}$$

where

$$\hat{\beta}^1_{j_0,k_1} = 2^{-j_0/2}\sum_{k_2=0}^{2^{j_0}-1} \hat{\alpha}_{j_0,\underline{k}} = \frac{1}{n}\sum_{i=1}^{n}\left[\phi_{j_0,k_1}(X_{i-1})(2^{-j_0/2}\sum_{k_2=0}^{2^{j_0}-1}\phi_{j_0,k_2}(X_i))\right].$$

Since ϕ has compact support, $\sum_{k_2=0}^{2^{j_0}-1} 2^{-j_0/2}\phi_{j_0,k_2}(X_i)$ is a finite sum. Thus, $\hat{f}_1$ and $\hat{f}_2$ are close to each other. We can investigate general $L_{\acute{p}}$ convergence rates. The following two lemmas will be useful later.

Lemma 3.3 (Meyer [23]) *Let ϕ be a piecewise continuous function such that for any $i \in \mathbb{N}$ the set of functions $\{\phi_{j,k} = 2^{j/2}\phi(2^j x - k),\ k \in \mathbb{Z}\}$ is an orthogonal family of $L_2(\mathbb{R})$.*

Moreover, suppose that $\theta(x) = \sum_{k\in\mathbb{Z}} |\phi(x-k)| < \infty$. Let $f(x) = \sum_{k\in\mathbb{Z}} \lambda_k \phi_{j,k}$. Then, for $1 \le p \le \infty, \dfrac{1}{p} + \dfrac{1}{q} = 1$, we have

$$2^{j(1/2-1/p)} \|\lambda\|_{l_p} \frac{1}{\|\theta\|_1^{1/q} \|\theta\|_\infty^{1/p}} \le \|f\|_p \le 2^{j(1/2-1/p)} \|\lambda\|_{l_p} \|\theta\|_p$$

Lemma 3.4 (Leblanc [16]) *Let $\infty > p \ge 2$ and $\xi_1, ..., \xi_n$ be a sequence of real-valued random variable such that $\mathbb{E}(\xi_i) = 0$, $\|\xi_i\|_\infty < S$, and $\mathbb{E}(\xi_i^2) \le \sigma^2$. Then, there exists C such that*

$$\mathbb{E}\left(\left|\sum_{i=1}^n \xi_i\right|^p\right) \le C\left\{\left(\frac{n}{l}\right)^{p/2} \sigma_l^p + \frac{n}{l}\sigma_l^2 (lS)^{p-2} + S^p n^p \alpha(l)\right\},$$

where $l \in \mathbb{N}$, $2 \le l \le \dfrac{n}{2}$,

$$\sigma_l^2 = \max\left\{\max_{1\le u\le n} \sigma_u^2(l), \max_{1\le u\le n} \sigma_u^2(l-1)\right\} \quad \text{and} \quad \sigma_u^2(l) = \mathbb{E}\left(\sum_{i=u}^{u+l-1} \xi_i\right)^2.$$

The following two theorems are proved for $\hat{f}_1$ by Leblanc [16]. We prove these theorems for our proposed estimator $\hat{f}_2$.

Theorem 3.5 *Let $f_X \in F_{s,p,q}$ with $s \ge \dfrac{1}{p}$, $p \ge 1$, and $q \ge 1$. Suppose that there exist constants $\alpha > 1$ and c_α such that for any l, $\alpha(l) \le c_\alpha \alpha^{-1}$. Furthermore, suppose that there is a function g with $g(l) \ge G$ (G is a positive constant), such that for any $l = O(\ln(n))$, $\sigma_l^2 \le lg(l)$. Then, for $\acute{p} \ge \max(2, p)$, there exists a constant C such that*

$$\mathbb{E}\|f_X - \hat{f}_2\|_{\acute{p}}^2 \le C\left[\frac{n}{g(\ln(n))}\right]^{-\frac{2\acute{s}}{1+2\acute{s}}},$$

where $\acute{s} = s + \dfrac{1}{\acute{p}} - \dfrac{1}{p}$ and $2^{j_0} = \left[\dfrac{n}{g(\ln(n))}\right]^{\frac{1}{1+2\acute{s}}}$.

Theorem 3.6 *Let $f_X \in F_{s,p,q}$ with $s \ge \dfrac{1}{p}$, $p \ge 1$, and $q \ge 1$. Suppose that $\alpha(l) \le c_\alpha l^{-\alpha}$, $\alpha \ge \acute{p}(1+\acute{s})/\acute{s}$ for any $l \in \mathbb{N}, 2 \le l \le n/2$. Let us set $\mu = \acute{p}(\acute{s}+1)/[\alpha(1+2\acute{s})]$ and suppose that there is a function g with $g(l) \ge G$ (G is a positive constant), such that for any $l = O(\ln(n))$, $\sigma_l^2 \le lg(l)$. Then, for $\acute{p} \ge \max(2, p)$, there exists a constant C such that*

$$\mathbb{E}\|f_X - \hat{f}_2\|_{\acute{p}}^2 \le C\left[\frac{n}{g(n^\mu)}\right]^{-\frac{2\acute{s}}{1+2\acute{s}}},$$

where $\acute{s} = s + 1/\acute{p} - 1/p$.

Theorem 3.3 and 3.4 are corollaries of the following lemmas:

Lemma 3.7 *Let* $f_X \in F_{s,p,q}$ *with* $s \geq \frac{1}{p}$, $p \geq 1$, $q \geq 1$ *and* $\acute{p} \geq \max(2,p)$. *Then, there exists a constant* C *such that*

$$\mathbb{E}\|f_X - \hat{f}_2\|_{\acute{p}}^2 \leq C\left\{2^{-2j_0\acute{s}} + \frac{2^{j_0}}{n}\frac{\sigma_l^2}{l} + \left(\frac{2^{j_0}}{n}\right)^{2-2/\acute{p}} l^{2/\acute{P}(\acute{p}-3)}\sigma_l^{4/\acute{p}} + 2^{2j_0}\alpha(l)^{2/\acute{p}}\right\},$$

where $l \in \mathbb{N}$, $2 \leq l \leq \frac{n}{2}$, *and* $\acute{s} = \frac{s+1}{\acute{p}} - \frac{1}{p}$.

Proof. First, we decompose $\mathbb{E}\|f_X - \hat{f}_2\|_{\acute{p}}^2$ into a bias term and a stochastic term

$$\mathbb{E}\|f_X - \hat{f}_2\|_{\acute{p}}^2 \leq 2(\|f_X - P_{j_0}f_X\|_{\acute{p}}^2 + \mathbb{E}\|\hat{f}_2 - P_{j_0}f\|_{\acute{p}}^2) = 2(T_1 + T_2) \tag{3.3}$$

Now, we want to find upper bounds for T_1 and T_2.

$$\begin{aligned}\sqrt{T_1} &= \left\|\sum_{j\geq j_0} D_j f_X\right\|_{\acute{p}} \leq \sum_{j\geq j_0}(\|D_j f\|_{\acute{p}}2^{j\acute{s}})2^{-j\acute{s}} \\ &\leq \left\{\sum_{j\geq j_0}(\|Djf\|_{\acute{p}}2^{j\acute{s}})^q\right\}^{1/q}\left\{\sum_{j\geq j_0}2^{-j\acute{s}\acute{q}}\right\}^{1/\acute{q}}.\end{aligned}$$

By Hölder inequality, with $\frac{1}{q} + \frac{1}{\acute{q}} = 1$,

$$C\|f_X\|_{B^{\acute{s}}_{\acute{p},q}}2^{-\acute{s}j_0} \leq C\|f_X\|_{B^{s}_{p,q}}2^{-\acute{s}j_0}.$$

The last inequality holds, because the continuity of Sobolev injection (see Triebel [33] and the discussion in Donoho et al. [12]), $B^s_{p,q} \subset B^{\acute{s}}_{\acute{p},q}$ implies that $\|f_X\|_{B^{\acute{s}}_{\acute{p},q}} \leq \|f_X\|_{B^s_{p,q}}$. Thus,

$$T_1 \leq K2^{-2\acute{s}j_0}. \tag{3.4}$$

Now,

$$T_2 = \mathbb{E}\|\hat{f}_2 - P_{j_0}f\|_{\acute{p}}^2 = \mathbb{E}\left\|\sum_{k\in K_{j_0}}(\hat{\beta}^1_{j_0,k} - \alpha_{j_0,k})\phi_{j_0,k}(x)\right\|_{\acute{p}}^2.$$

By Lemma 3.1,

$$T_2 \leq C\mathbb{E}\{\|\hat{\beta}^1_{j_0,k} - \alpha_{j_0,k}\|^2_{l_{\acute{p}}}\}2^{2j_0(1/2-1/\acute{p})}.$$

Using Jensen inequality we obtain,

$$T_2 \leq C2^{2j_0(1/2-1/\acute{p})}\left\{\sum_{k\in K_{j_0}}\mathbb{E}|\hat{\beta}^1_{j_0,k} - \alpha_{j_0,k}|^{\acute{p}}\right\}^{2/\acute{p}}. \tag{3.5}$$

To complete the proof, it is enough to estimate $\mathbb{E}|\hat{\beta}^1_{j_0,k} - \alpha_{j_0,k}|^{\acute{p}}$. We know

$$\hat{\beta}^1_{j_0,k} - \alpha_{j_0,k} = \frac{1}{n}\sum_{i=1}^{n}\left\{\left[\phi_{j_0,k}(X_{i-1})\sum_{k_2=0}^{2^{j_0}-1} 2^{-j_0/2}\phi_{j_0,k_2}(X_i) - \alpha_{j_0,k}\right]\right\}.$$

Denote $\xi_i = \left[\phi_{j_0,k}(X_{i-1})\sum_{k_2=0}^{2^{j_0}-1} 2^{-j_0/2}\phi_{j_0,k_2}(X_i) - \alpha_{j_0,k}\right]$. Since ϕ has compact support, $\sum_{k_2=0}^{2^{j_0}-1} 2^{-j_0/2}\phi_{j_0,k_2}(X_i)$ is a finite sum, and we have

$$\|\xi_i\|_\infty \le K.2^{j_0/2}\|\phi\|_\infty, \quad \mathbb{E}\xi_i = 0, \quad \mathbb{E}\xi_i^2 \le \|f_{X_i,X_{i+1}}\|_\infty$$

and

$$|\hat{\beta}^1_{j_0,k} - \alpha_{j_0,k}| = \frac{1}{n}\left|\sum_{i=1}^{n}\xi_i\right|.$$

Hence, applying Lemma 3.2 and using card $K_{j_0} = O(2^{j_0})$ we get,

$$\begin{aligned}
&\left\{\sum_{k\in K_{j_0}} \mathbb{E}|\hat{\beta}^1_{j_0,k} - \alpha_{j_0,k}|^{\acute{p}}\right\}^{2/\acute{p}} \\
&\quad\le \left\{C2^{j_0}\frac{1}{n^{\acute{p}}}\left(\left(\frac{n}{l}\right)^{\acute{p}/2}\sigma_l^{\acute{p}} + \frac{n}{l}\sigma_l^2 l^{\acute{p}-2}2^{j_0/2(\acute{p}-2)} + 2^{j_0\acute{p}/2}n^{\acute{p}}\alpha(l)\right)\right\}^{2/\acute{p}} \\
&\quad\le K\left\{\frac{\sigma_l^2}{l}\frac{2^{2j_0/\acute{p}}}{n} + \sigma_l^{4/\acute{p}}\frac{2^{j_0}l^{2/\acute{p}(\acute{p}-3)}}{n^{2/\acute{p}(\acute{p}-1)}} + 2^{2j_0/\acute{p}(\acute{p}/2+1)}\alpha^{2/\acute{p}}(l)\right\}.
\end{aligned}$$

Now, substituting above inequality in (3.5) we get

$$T_2 \le K2^{2j_0(1/2-1/\acute{p})}\left\{\frac{\sigma_l^2}{l}\frac{2^{2j_0/\acute{p}}}{n} + \sigma_l^{4/\acute{p}}\frac{2^{j_0}l^{2/\acute{p}(\acute{p}-3)}}{n^{2/\acute{p}(\acute{p}-1)}} + 2^{2j_0/\acute{p}(\acute{p}/2+1)}\alpha^{2/\acute{p}}(l)\right\}$$

or

$$T_2 \le K\left\{\frac{2^{j_0}}{n}\frac{\sigma_l^2}{l} + \left(\frac{2^{j_0}}{n}\right)^{2-2/\acute{p}} l^{2/\acute{P}(\acute{p}-3)}\sigma_l^{4/\acute{p}} + 2^{2j_0}\alpha(l)^{2/\acute{p}}\right\}. \tag{3.6}$$

By substituting (3.4) and (3.6) in (3.3) completes the proof of the lemma. ■

In the case of independent variables, $\sigma_l^2 = O(l)$. Moreover, in the dependent case a rough bound $\sigma_l^2 = O(l^2)$ can be easily obtained. If some additional conditions are imposed on the process $\{X_i\}$, the bound $\sigma_l^2 = O(l)$ can be achieved. If $\sigma_l^2 = O(l)$, then the same rate as for the independent case, $n^{-\frac{2\acute{s}}{1+2\acute{s}}}$, is attained using lemma 3.5. If the process is α-mixing, we obtain:

Lemma 3.8 *Let* $\{X_n,\ n \geq 1\}$ *be a stochastic process on* $\mathbb{R}$. *Suppose that* X_n *admits a bounded marginal density which is common for all* n. *If* $\sum_{k=1}^{\infty} \alpha(k) < \infty$, *then there exists a constant* G *such that for any* $l \in \mathbb{N}$, $2 \leq l \leq n/2$, $\sigma_l^2 \leq G \cdot l$.

Proof. First, we define

$$Y_i = \sum_{k_2=0}^{2^{j_0}-1} 2^{-j_0/2} \phi_{j_0,k_2}(X_i).$$

Now, we use the decomposition

$$\begin{aligned}\sigma_{k,u}^2(l) = &\sum_{i=k}^{u+k-1} \mathbb{E}(\phi_{j_0,k}(X_{i-1})Y_i - \alpha_{j_0,k})^2 \\ &+ 2 \sum_{u \leq i < t \leq l+u-1} \mathbb{E}(\phi_{j_0,k}(X_{i-1})Y_i - \alpha_{j_0,k})(\phi_{j_0,k}(X_{t-1})Y_t - \alpha_{j_0,k}) = T_1 + T_2.\end{aligned}$$

Now, we prove T_1 and T_2 are $O(l)$.

$$T_1 \leq l \max_{u \leq i \leq l+u-1} (\phi_{j_0,k}(X_{i-1})Y_i - \alpha_{j_0,k})^2 \leq l ||f_{X_{i-1},X_i}||_{\infty}^2.$$

Proposition 2 of Babu et al. [2] implies that the process $\{X_i, X_{i-1}\}$ is strongly mixing with the same order of speed as $\{X_i\}$. Thus

$$T_2 \leq K \sum_{u \leq i < t \leq l+u-1} \alpha(t-i) = Kl \sum_{k=1}^{l} (1 - k/l)\alpha(k) \leq Kl \sum_{k=1}^{l} \alpha(k).$$

By assumption the above series is finite. Hence, the proof is completed. ■

To compare bias and variance of two density function estimation, $\hat{f}_1$ and $\hat{f}_2$, we prove the following lemma:

Lemma 3.9 *For* x *fixed, under assumptions of Theorem* 2.2*:*

(i) $\mathbb{E}\hat{f}_2 = \mathbb{E}\hat{f}_1$;

(ii) $Bias\hat{f}_2 = Bias\hat{f}_1$;

(iii) $Var\hat{f}_2 = O\left(\frac{2^{j_0}}{n}\right)$.

Proof. (i) and (ii) are obvious, because $\hat{f}_1$ and $\hat{f}_2$ are unbiased estimators for $P_{j_0}f_X$. To prove (iii) note that ϕ has compact support. Thus

$$\sum_{k_2=0}^{2^{j_0}-1} 2^{-j_0/2} \phi_{j_0,k_2}(X_i)$$

is a finite sum and hence substituting in $\hat{f}_2$ and using the result of Theorem 2.2, completes the proof. ■

4. Wavelet autoregressive estimators

We consider the nonparametric regression model which is given below. Suppose that we observe the time series $X_0, X_1, ..., X_n$ following the nonlinear autoregressive model

$$X_i = \tau(X_{i-1}) + \epsilon_i, \qquad i = 1, ..., n. \tag{4.1}$$

In this section our main objective is to estimate τ. Observe that

$$\tau(x) = \mathbb{E}(X_i | X_{i-1} = x).$$

We closely follow the method of Delouilie et al. [10]. We will obtain our estimator of τ by taking the ratio of wavelet estimators of g and f, where $g(x) = \tau(x) \cdot f(x)$. One uses the following estimator

$$\hat{g}_1(x) = \sum_{k=-\infty}^{\infty} \left[\frac{1}{n} \sum_{i=1}^{n} X_i \phi_{j,k}(X_{i-1}) \right] \phi_{j,k}(x). \tag{4.2}$$

We propose new wavelet estimator for g as follows: if we have confine our attention to the wavelet basis of $L_2[0,1]$, we know $g(x) = \int_0^1 y f_{X_{i-1},X_i}(x,y)dy$. Now,

$$\begin{aligned} g(x) &= \sum_{\underline{k}} \alpha_{j_0,\underline{k}} \phi_{j_0,k_1}(x) \int_0^1 y \phi_{j_0,k_2}(y) dy \\ &\quad + \sum_{j \geq j_0} \sum_{\underline{k}} d^{(1)}_{j,\underline{k}} \phi_{j,k_1}(x) \int_0^1 y \psi_{j,k_2}(y) dy \\ &\quad + d^{(2)}_{j,\underline{k}} \psi_{j,k_1}(x) \int_0^1 y \phi_{j,k_2}(y) dy + d^{(3)}_{j,\underline{k}} \psi_{j,k_1}(x) \int_0^1 y \psi_{j,k_2}(y) dy. \end{aligned} \tag{4.3}$$

We have

$$\int_0^1 y \phi_{j,k_2} dy = 2^{j/2} \int_0^1 y \phi(2^j y - k_2) dy.$$

Let $t = 2^j y - k_2$. Then

$$\int_0^1 y \phi_{j,k_2} dy = 2^{-3/2j} \int_{-k_2}^{2^j - k_2} (t + k_2) \phi(t) dt.$$

For $0 \leq k2 \leq 2^j - 1$, we get

$$2^{-3/2j} \int_0^1 (t + k_2) \phi(t) dt = (k_2 + c_0) 2^{-3/2j},$$

where $c_0 = \displaystyle\int_0^1 t \phi(t) dt$. Similarly for $0 \leq k2 \leq 2^j - 1$, we have

$$\int_0^1 y \psi_{j,k_2}(y) dy = 2^{-3/2j} \int_0^1 (t + k_2) \psi(t) dt = \acute{c}_0 2^{-3/2j},$$

where $\acute{c}_0 = \int_0^1 t\psi(t)dt$. For simplicity we can choose other wavelet such that $\acute{c}_0 = 0$. Then the expansion of g is as follow:

$$g(x) = \sum_{k_1} \beta_{j_0,k_1}\phi_{j_0,k_1}(x) + \sum_{j\geq j_0}\sum_{k_1} \gamma_{j,k_1}\psi_{j,k_1}(x), \tag{4.4}$$

where

$$\beta_{j_0,k_1} = 2^{-3/2j_0} \sum_{k_2=0}^{2^{j_0}-1} (k_2 + c_0)\alpha_{j_0,\underline{k}}$$

and

$$\gamma_{j,k_1} = 2^{-3/2j} \sum_{k_2=0}^{2^{j_0}-1} (k_2 + c_0)d^{(2)}_{j,\underline{k}}.$$

Now, using estimators of coefficient (2.7) we obtain our estimator:

$$\hat{g}(x) = \sum_{k_1} \hat{\beta}_{j0,k_1}\phi_{j0,k_1}(x), \tag{4.5}$$

where

$$\begin{aligned}\hat{\beta}_{j0,k_1} &= 2^{-3/2j0} \sum_{k_2=0}^{2^j0-1} (k_2 + c_0)\hat{\alpha}_{j0,\underline{k}} \\ &= \frac{1}{n}\sum_{i=1}^{n}\left[\phi_{j_0,k_1}(X_{i-1})(2^{-3j_0/2}\sum_{k_2=0}^{2^{j_0}-1}(k_2+c_0)\phi_{j_0,k_2}(X_i))\right].\end{aligned}$$

Wavelet estimator of autoregressive model. We propose the following estimator for our autoregressive model (4.1)

$$\hat{r} = \frac{\hat{g}}{\hat{f}_2}, \tag{4.6}$$

where $\hat{g}$ and $\hat{f}_2$ are given by the equations (4.5) and (3.2) respectively.

Below, we study some properties of our proposed estimator (4.6). First, we prove the following useful lemma.

Lemma 4.10 *If marginal density* $f_X \in F_{s,p,q}$, *then* $g \in F_{s,p,q}$.

Proof. By (2.1), we need to prove the following two inequalities:

$$\|\beta_{j0,.}\|_{l_p} < \infty, \qquad \left[\sum_{j\geq j0}(\|\gamma_{j,.}\|_{l_p}2^{j(s+1/2-1/p)})q\right]^{1/q} < \infty. \tag{4.7}$$

We know $f_X \in F_{s,p,q}$. Thus, (2.1) holds for f_X and hence

$$\|\beta^1_{j0,.}\|^p_{l_p} = \sum_{k_1}\left[2^{-j0/2}\sum_{k_2=0}^{2^{j0}-1}\alpha_{j0,\underline{k}}\right]^p < \infty.$$

Now, since $0 \le k_2 + c_0 < 2^j$, $\sum_{k_1}\left[2^{-3/2j0}\sum_{k_2=0}^{2^{j0}-1}(k_2+c_0)\alpha_{j0,\underline{k}}\right]^p < \infty$ or $\|\beta_{j0,.}\|_{l_p} < \infty$. Similarly, we prove $\left[\sum_{j\ge j0}(\|\gamma_{j,.}\|_{l_p}2^{j(s+1/2-1/p)^q})\right]^{1/q} < \infty.$ ■

Using the following lemma, we can apply Theorems 3.3 and 3.4 to find similar convergence rate for $\hat{g}_1(x)$.

Lemma 4.11 *For every $\acute{p} \ge 2$ we have:*

$$\|\hat{g} - g\|^2_{\acute{p}} \le \|\hat{f}_2 - f_X\|^2_{\acute{p}}. \tag{4.8}$$

Proof. We have

$$\|\hat{g} - g\|^2_{\acute{p}} \le 2(\|g - P_{j_0}g\|^2_{\acute{p}} + \|\hat{g} - P_{j_0}g\|^2_{\acute{p}})$$

and

$$\|\hat{f}_2 - f_X\|^2_{\acute{p}} \le 2(\|f_X - P_{j_0}f_x\|^2_{\acute{p}} + \|\hat{f}_2 - P_{j_0}f_X\|^2_{\acute{p}}).$$

Now, by (3.7), we have:

$$\begin{aligned}\|f_X - P_{j_0}f_X\|^2_{\acute{p}} &= \left\|\sum_{j\ge j_0}\sum_{k_1}\gamma^1_{j,k_1}\psi_{j,k_1}(x)\right\|^2_{\acute{p}} \\ &= \left\|\sum_{j\ge j_0}\sum_{k_1}2^{-j/2}\sum_{k_2=0}^{2^j-1}d^{(2)}_{j,\underline{k}}\psi_{j,k_1}(x)\right\|^2_{\acute{p}}.\end{aligned}$$

Since $0 \le k_2 + c_0 < 2^j$, we have

$$\begin{aligned}\|f_X - P_{j_0}f_X\|^2_{\acute{p}} &\ge \left\|\sum_{j\ge j_0}\sum_{k_1}2^{-3/2j}\sum_{k_2=0}^{2^j-1}(k_2+c_0)\psi_{j,k_1}(x)\right\|^2_{\acute{p}} \\ &= \left\|\sum_{j\ge j_0}\sum_{k_1}\gamma_{j,k_1}\psi_{j,k_1}(x)\right\|^2_{\acute{p}} = \|g - P_{j_0}g\|^2_{\acute{p}}.\end{aligned}$$

Similarly, we can prove $\|\hat{f}_2 - P_{j_0}f_X\|^2_{\acute{p}}) \ge \|\hat{g} - P_{j_0}g\|^2_{\acute{p}}$. This proves (4.7). ■

Using the following lemma, we compare bias and variance of $\hat{g}$ and $\hat{f}_2$. The proof of the following lemma is similar to the proof of Lemma 4.2.

Lemma 4.12 $|\mathbb{E}\hat{g}(x) - g(x)| \le |\mathbb{E}\hat{f}_2(x) - f_X(x)|$ *and* $Var\hat{g}(x) \le Var\hat{f}_2(x)$.

These results allow us to control the convergence rate of the estimator $\hat{\tau} = \dfrac{\hat{g}}{\hat{f}_2}$. Using Rosenblatt's expansion (2.6) ([28], P13), we have

$$\begin{aligned}\hat{\tau}(x) &= \frac{\mathbb{E}\hat{g}}{\mathbb{E}\hat{f}_2} + \frac{\hat{g}(x) - \mathbb{E}\hat{g}(x)}{\mathbb{E}\hat{f}_2(x)} - \frac{\hat{f}_2(x) - \mathbb{E}\hat{f}_2(x)}{[\mathbb{E}\hat{f}_2(x)]^2} \\ &\quad + O_p([\hat{g}(x) - \mathbb{E}\hat{g}(x)]^2) + O_p([\hat{f}_2(x) - \mathbb{E}\hat{f}_2(x)]^2).\end{aligned}$$

Then it follows that

$$\mathbb{E}\hat{\tau}(x) = \frac{\mathbb{E}\hat{g}}{\mathbb{E}\hat{f}_2} + O(Var\hat{g}(x)) + O(Var\hat{f}_2(x)) \le \frac{\mathbb{E}\hat{g}}{\mathbb{E}\hat{f}_2} + O\left(\frac{2^{j_0}}{n}\right),$$

using Theorem 2.1 and Lemmas 3.4, 4.3.

Now, by equation (2.7) of Rosenblatt [28],

$$\begin{aligned}\frac{\mathbb{E}\hat{g}}{\mathbb{E}\hat{f}_2} &= \tau(x) + \frac{\mathbb{E}\hat{g}(x) - g(x)}{f_X(x)} - \frac{\mathbb{E}\hat{f}_2(x) - f_X(x)}{f_X(x)}\tau(x) \\ &\quad + O_p([g(x) - \mathbb{E}\hat{g}(x)]^2) + O_p([f_X(x) - \mathbb{E}\hat{f}_2(x)]^2).\end{aligned}$$

By Lemmas 3.4, 4.3 and Theorem 2.1, it follows that

$$\frac{\mathbb{E}\hat{g}}{\mathbb{E}\hat{f}_2} \le \tau(x) + O(2^{-j_0 m}).$$

Therefore, the bias of the estimator $\hat{\tau}$, by (3.9), is

$$\text{bias } (\hat{\tau}(\mathrm{x})) = \mathrm{O}(2^{-\mathrm{j_0 m}}) + \mathrm{O}\left(\frac{2^{\mathrm{j_0}}}{\mathrm{n}}\right). \tag{4.9}$$

For variance of $\tau(x)$ we have,

$$\begin{aligned}Var(\hat{\tau}(x)) &\le \frac{Var\hat{g}(x)}{[\mathbb{E}\hat{f}(x)]^2} + \frac{[\mathbb{E}\hat{g}(x)]^2}{[\mathbb{E}\hat{f}(x)]^4} Var\hat{f}(x) \\ &\quad + O_p(\mathbb{E}[g(x) - \mathbb{E}\hat{g}(x)]^4) + O_p(\mathbb{E}[f_X(x) - \mathbb{E}\hat{f}(x)]^4).\end{aligned}$$

Assuming that $f(x) > 0$ for all x and given the asymptotic biases and variance of $\hat{g}(x)$ and $\hat{f}(x)$ one can easily, using Theorem 2.2 and Lemmas 3.4, 4.3, obtain

$$Var(\hat{\tau}(x)) \le O\left(\frac{2^{j_0}}{n}\right). \tag{4.10}$$

5. Derivative of wavelet autoregressive estimators

In this section we restrict our attention to the space $X = [0, 1]$. Prakasa Rao [25] studied estimation of a derivative of a density using the method of wavelets. Let

ϕ be a scaling function generating an r-regular multiresoulution analysis and let $f^{(d)} \in \mathbb{L}_2(\mathbb{R})$. Assume that there exist $C_m \geq 0$ and $\beta_m \geq 0$ such that

$$|f^{(m)}(x)| \leq C_m(1+|x|)^{-\beta_m}, 0 \leq m \leq d. \tag{5.1}$$

He showed that projection of $f^{(d)}$ on V_{j_0} is

$$f_{n,d}^{(d)}(x) = \sum_k a_{j_0,k}\phi_{j_0,k}(x),$$

where $a_{j_0,k} = (-1)^d \int \phi_{j_0,k}^{(d)}(x) f_X(x)dx$. So, its estimator is

$$\hat{f}_{n,d}^{(d)}(x) = \sum_k \hat{a}_{j_0,k}\phi_{j_0,k}(x), \tag{5.2}$$

where $\hat{a}_{j_0,k} = \frac{(-1)^d}{n}\sum_{i=1}^{n} \phi_{j_0,k}^{(d)}(X_i)$.

Define the kernel $E(u,v)$ by $E(u,v) = \sum_k \phi(u,k)\acute{\phi}(y-k)$. We rewrite the above estimator, in a special case $d=1$,

$$\hat{\acute{f}}(x) = \frac{-2^{2j_0}}{n}\sum_{i=1}^{n} E(2^{j_0}x, 2^{j_0}X_i).$$

Note that $\frac{\partial}{\partial y}K(u,y) = E(u,y)$. By using results of Prakasa Rao [26], we see that there exist constants G_j such that

$$\int |E(x,y)|^j dy \leq G_j, j \geq 1. \tag{5.3}$$

As above, we can show that the projection of $\acute{g}$ on V_{j_0} is

$$\acute{g}(x) = \sum_k b_{j_0,k}\phi_{j_0,k}(x),$$

where $b_{j_0,k} = (-1)\int \acute{\phi}_{j_0,k}(x)g(x)dx$. Thus, its estimator is

$$\hat{\acute{g}}_(x) = \sum_k \hat{b}_{j_0,k}\phi_{j_0,k}(x), \tag{5.4}$$

where $\hat{b}_{j_0,k} = \frac{-1}{n}\sum_{i=1}^{n} X_{i+1}\acute{\phi}_{j_0,k}(X_i)$.

Now, we want to find the derivative of estimated dynamical system τ. We have

$$\acute{\tau}(x) = \frac{g^*(x)}{f(x)},$$

where $g^*(x) = \acute{g}(x) - \tau(x)\acute{f}(x)$. Thus, we propose the derivative as follows

$$\hat{\tau}(x) = \frac{\hat{\acute{g}}(x) - \hat{\tau}(x)\hat{\acute{f}}(x)}{\hat{f}(x)}. \tag{5.5}$$

To control the convergence rate of our proposed estimator, we need bias and variance of $\hat{g}^*$ where $\hat{g}^*(x) = \hat{\acute{g}}(x) - \hat{\tau}(x)\hat{\acute{f}}(x)$.

Theorem 5.13 *Let the mixing rate* α *satisfy* $\sum_{k=1}^{\infty} \alpha(k) < \infty$ *and suppose that the density functions* f_X *and* $\acute{f}_X$ *are uniformly bounded,* $\acute{f} \in \mathbb{L}_2(\mathbb{R})$ *and* $j_0 \to \infty$ *as* $n \to \infty$,

$$\text{Bias } \hat{\mathrm{g}}^*(\mathrm{x}) \;=\; O\left(\left(\frac{2^{3j_0}\ln(n)}{n}\right)^{1/2}\right) + O(2^{-j_0(1-1/p)}) + O(2^{-j_0 m}) + O\left(\frac{2^{j_0}}{n}\right)$$

and

$$\mathrm{Var}(\hat{\mathrm{g}}^*(\mathrm{x})) = \mathrm{O}\left(\frac{2^{3\mathrm{j}_0}}{\mathrm{n}}\right).$$

Proof. We write

$$\begin{aligned}
\mathbb{E}\hat{g}^*(x) - g^*(x) &= [\mathbb{E}\hat{\acute{g}}(x) - g(x)] - \mathbb{E}\hat{\tau}(x)[\hat{\acute{f}}(x) - \acute{f}(x)] - \acute{f}(x)[\mathbb{E}\hat{\tau}(x) - \tau(x)] \\
&\leq |\text{Bias } \hat{\acute{\mathrm{g}}}(\mathrm{x})| + |\text{ Bias } \hat{\acute{\mathrm{f}}}(\mathrm{x})| + |\acute{\mathrm{f}}(\mathrm{x})||\text{ Bias } \hat{\tau}(\mathrm{x})| = \mathrm{T}_1 + \mathrm{T}_2 + \mathrm{T}_3.
\end{aligned}$$

If $\acute{g}(x) \in F_{s,p,q}$, $0 < p < r$, $1 \geq p, q < \infty$, with $s > 1/p$ and multiresolution analysis is r-regular, then it follows by arguments given in Kerkyacharian and Picard [24] that

$$T_1 = O(2^{-j_0(1-1/p)}). \tag{5.6}$$

By Prakasa Rao [26],

$$T_2 = O\left(\left(\frac{2^{3j_0}\log n}{n}\right)^{1/2}\right) + O(2^{-j_0(1-1/p)}). \tag{5.7}$$

By equation (4.9),

$$T_3 = O(2^{-j_0 m}) + O\left(\frac{2^{j_0}}{n}\right). \tag{5.8}$$

Using (5.5), (5.6) and (5.7) the first assertion of theorem is proved. For, the proof of the second assertion of theorem, note that,

$$\hat{g}^*(x) = \frac{-2^{2j_0}}{n}\sum_{i=1}^{n}\{X_{i+1} - \hat{\tau}(x)\}E(2^{j_0}x, 2^{j_0}X_i).$$

Hence,

$$Var(\hat{g}^*(x)) = \frac{2^{4j_0}}{n^2}\left\{\sum_{i=1}^{n} Var(\{X_{i+1}-\hat{\tau}(x)\}E(2^{j_0}x, 2^{j_0}X_i))\right.$$
$$\left.+2\sum_{1\le i<t\le n} Cov(\{X_{i+1}-\hat{\tau}(x)\}E(2^{j_0}x,2^{j_0}X_i),\{X_{t+1}-\hat{\tau}(x)\}E(2^{j_0}x,2^{j_0}X_t))\right\}$$
$$= \frac{2^{4j_0}}{n^2}\{T_1+T_2\}$$

Now, we have

$$\begin{aligned} T_1 = &\sum_{i=1}^{n}\{\mathbb{E}(\{X_{i+1}-\hat{\tau}(x)\}E(2^{j_0}x,2^{j_0}X_i))^2-(\mathbb{E}\{X_{i+1}-\hat{\tau}(x)\}E(2^{j_0}x,2^{j_0}X_i))^2\}\\ \le\ & n\int E^2(2^{j_0}x,2^{j_0}y)f(y)dy + n\left(\int E(2^{j_0}x,2^{j_0}y)f(y)dy\right)^2\\ =\ & n2^{j_0}\int E^2(2^{j_0}x,y)f(2^{-j_0}y)dy+\left(2^{-j_0}\int|E(2^{j_0}x,y)|f(2^{-j_0}y)dy\right)^2. \end{aligned}$$

Using equation (5.3), we have

$$T_1\le Kn2^{-j_0}(1+o(1)). \tag{5.9}$$

On the other hand, we have

$$\begin{aligned} &Cov(\{X_{i+1}-\hat{\tau}(x)\}E(2^{j_0}x,2^{j_0}X_i),\{X_{t+1}-\hat{\tau}(x)\}E(2^{j_0}x,2^{j_0}X_t))\\ &\quad=\mathbb{E}\{X_{i+1}-\hat{\tau}(x)\}E(2^{j_0}x,2^{j_0}X_i)\{X_{t+1}-\hat{\tau}(x)\}E(2^{j_0}x,2^{j_0}X_t)\\ &\quad-(\mathbb{E}\{X_{i+1}-\hat{\tau}(x)\}E(2^{j_0}x,2^{j_0}X_i))^2\\ &\quad\le\int\int|E(2^{j_0}x,2^{j_0}y)||E(2^{j_0}x,2^{j_0}u)|f_{X_i,X_t}(y,u)dydu\\ &\quad+\left(\int|E(2^{j_0}x,2^{j_0}y)|f(y)\right)^2\le K2^{-2j_0}\alpha(t-1)(1+o(1)). \end{aligned}$$

Hence, $T_2\le Kn2^{-2j_0}\sum_{1\le i<t\le n}\alpha(t-i)\le Kn2^{-2j_0}\sum_{k=1}^{n}\alpha(k)$, and thus,

$$T_2\le Kn2^{-2j_0}. \tag{5.10}$$

Finally, using (5.7) and (5.9) the second assertion is proved. ■

The above results allow us to control the convergence rate of estimators $\hat{\tau}=\frac{\hat{g}^*}{\hat{f}}$. Using expansion (2.6) of Rosenblatt ([28], P13), we have

$$\begin{aligned}\hat{\tau}(x) \ =\ & \frac{\mathbb{E}\hat{g}^*}{\mathbb{E}\hat{f}}+\frac{\hat{g}^*(x)-\mathbb{E}\hat{g}^*(x)}{\mathbb{E}\hat{f}(x)}-\frac{\hat{f}(x)-\mathbb{E}\hat{f}(x)}{[\mathbb{E}\hat{f}(x)]^2}\\ &+O_p([\hat{g}^*(x)-\mathbb{E}\hat{g}^*(x)]^2)+O_p([\hat{f}(x)-\mathbb{E}\hat{f}(x)]^2).\end{aligned}$$

Then, it follows that

$$\mathbb{E}\hat{\acute{\tau}}(x) = \frac{\mathbb{E}\hat{g}^*}{\mathbb{E}\hat{f}} + O(Var\hat{g}^*(x)) + O(Var\hat{f}(x)) \leq \frac{\mathbb{E}\hat{g}^*}{\mathbb{E}\hat{f}} + O\left(\frac{2^{j_0}}{n}\right) + O\left(\frac{2^{3j_0}}{n}\right),$$

by Theorems 2.1, 5.1 and Lemma 3.4. Now, by equation (2.7) of Rosenblatt [28],

$$\begin{aligned}\frac{\mathbb{E}\hat{g}^*}{\mathbb{E}\hat{f}} &= \acute{\tau}(x) + \frac{\mathbb{E}\hat{g}^*(x) - g^*(x)}{f_X(x)} - \frac{\mathbb{E}\hat{f}(x) - f_X(x)}{f_X(x)}\acute{\tau}(x) \\ &\quad + O_p([g^*(x) - \mathbb{E}\hat{g}^*(x)]^2) + O_p([f_X(x) - \mathbb{E}\hat{f}(x)]^2).\end{aligned}$$

By Lemma 3.4, Theorems 2.1 and 5.1, it follows that

$$\frac{\mathbb{E}\hat{g}^*}{\mathbb{E}\hat{f}} \leq \acute{\tau}(x) + O(2^{-j_0 m}) + O\left(\left(\frac{2^{3j_0}\log n}{n}\right)^{1/2}\right) + O(2^{-j_0(1-1/p)}).$$

Therefore, the bias of the estimators $\hat{\acute{\tau}}$, considering (3.9), is

$$\begin{aligned}\mathrm{bias}(\hat{\acute{\tau}}(\mathrm{x})) &= O(2^{-j_0 m}) + O\left(\left(\frac{2^{3j_0}\log n}{n}\right)^{1/2}\right) + O(2^{-j_0(1-1/p)}) \\ &\quad + O\left(\frac{2^{j_0}}{n}\right) + O\left(\frac{2^{j_0}}{n}\right) + O\left(\frac{2^{3j_0}}{n}\right).\end{aligned}$$

For variance of $\hat{\acute{\tau}}$ we have,

$$\begin{aligned}Var(\hat{\acute{\tau}}(x)) &\leq \frac{Var\hat{g}^*(x)}{[\mathbb{E}\hat{f}(x)]^2} + \frac{[\mathbb{E}\hat{g}^*(x)]^2}{[\mathbb{E}\hat{f}(x)]^4}Var\hat{f}(x) \\ &\quad + O_p(\mathbb{E}[g^*(x) - \mathbb{E}\hat{g}^*(x)]^4) + O_p(\mathbb{E}[f_X(x) - \mathbb{E}\hat{f}(x)]^4).\end{aligned}$$

Assuming $f(x) > 0$ for all x, and given the asymptotic biases and variance of $\hat{g}^*(x)$ and $\hat{f}(x)$, using Theorem 2.2, Lemma 3.4 and Theorem 5.1, we easily obtain

$$Var(\hat{\acute{\tau}}(x)) \leq O\left(\frac{2^{3j_0}}{n}\right). \tag{5.11}$$

6. Application in dynamical systems

In this section, we apply our wavelet estimators 4.6 and 5.5 in dynamical systems. We estimate chaotic dynamical system from noisy data and estimate metric entropy of the chaotic dynamical system. In many physical systems what is observed is only data in the form of points $\{x_1, x_2, ..., x_{n+1}\}$ on a set X. The nature of dynamical system producing the data is unknown. Estimating a point transformation $\tau : X \to X$ such that the dynamical system $x_{n+1} = \tau(x_n)$ has f as its invariant probability density function is an important problem in dynamical systems. Estimation of τ has important application in estimating metric entropy of the observed data. Metric entropy is an important measure of chaos in a dynamical system. When dealing with a system modeled by a discrete time, nonlinear difference equation, $x_{n+1} = \tau(x_n)$ the method described by Abarbanel in [1] and implemented by Short [30] provides an algorithm for computing metric entropy. When the system is contaminated by noise , as in $x_{n+1} = \tau(x_n) + \epsilon_n$, a statistical method is described by Babu et al. in [2] for estimating the transformation τ and filtering the metric entropy of τ from the observed data $X^{(n)}_{\text{data}} = \{x_1, x_2, ..., x_{n+1}\}$ of the noisy system.

In the following numerical example, we show the performance of our wavelet method. We assume that the transformation τ admits an absolutely continuous invariant measure. We can extract from $X^{(n)}_{\text{data}}$ the $\tau-$invariant density f_τ. Using Pesin's formula

$$h(\tau) = \int \log|\tau'(x)| f_\tau(x) dx$$

we can estimate the metric entropy $h(\tau)$ of τ.

Now, we present an example of a dynamical system and verify the performance of our wavelet estimators.

Example 6.14 Consider the skew tent map $\tau : [0,1] \to [0,1]$ defined by

$$\tau(x) = \begin{cases} 3x, & 0 \le x < \dfrac{1}{3}, \\ \dfrac{3}{2} - \dfrac{3}{2}x, & \dfrac{1}{3} \le x \le 1. \end{cases} \tag{6.1}$$

By perturbing τ with $\epsilon-$neighborhood noise with zero mean and using Maple 9.5 we produce the data set $X^{(n)}_{data}$ for $n = 64$ and $\epsilon = .04$. Figure 1 is the graph of the chaotic dynamical system (transformation) τ, Figure 2 is the graph of noisy data and Figure 3 is the graph of the estimated transformation $\hat{\tau}$.

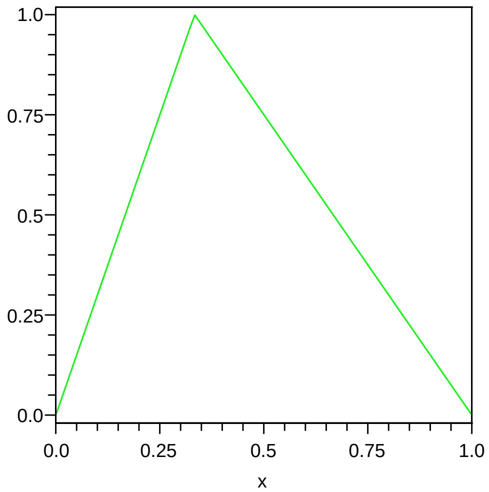

Figure 1: Graph of the transformation τ.

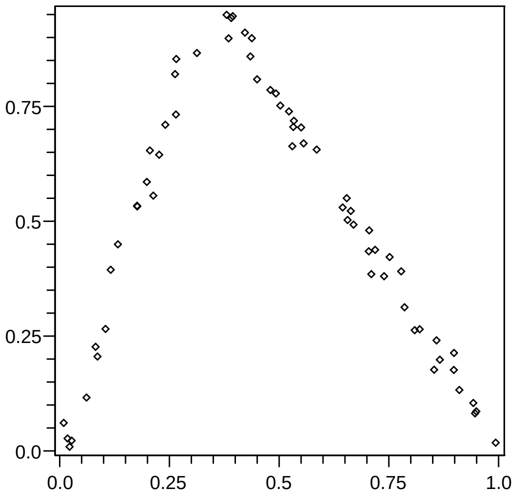

Figure 2: Graph of the noisy data.

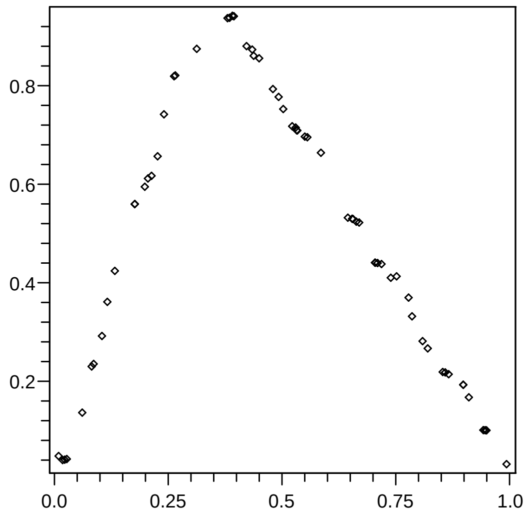

Figure 3: Graph of estimated transformation $\hat{\tau}$.

In this numerical example, we have considered the following scale function

$$\phi(x) = \begin{cases} \dfrac{e^{-\frac{1}{2}(x-\frac{1}{2})^2}}{.382\sqrt{2\pi}} & , \ 0 \le x < 1, \\ 0 & , \ \text{otherwise.} \end{cases} \tag{6.2}$$

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ON PERIODIC SOLUTIONS FOR NESTED POLYGON PLANAR $2N+1$-BODY PROBLEMS WITH ARBITRARY MASSES[1]

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Abstract. In this paper we study some necessary conditions and sufficient conditions for the nested periodic polygon solutions of planar $2N+1$-body problem, in which N-body lie at the vertex of one regular polygon, other N-body lie at the vertex of another regular polygon with a running angle, and $2N+1$th body lies at their geometrical center (origin) of $2N$-body.

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1. Main results

This paper uses the same notations as the paper [6]. For $n \geq 2$, the equations of motion of the planar n-body problem ([1], [2], [3], [5], [6]) can be written in the form

$$\ddot{z}_k = -\sum_{\substack{j=1 \\ j\neq k}}^{n} m_j \frac{z_k - z_j}{|z_k - z_j|^3}, \tag{1.1}$$

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where z_k is the complex coordinate of the kth mass m_k in an inertial coordinate system.

Let ρ_k denote the N complex kth roots of unity; i.e.,

$$\rho_k = \exp(2\pi I k/N), \tag{1.2}$$

hereafter $I = \sqrt{-1}$. This equation will also serve to define ρ_k for any number k. We assume that the mass $m_k (k = 1, ..., N)$ locates at the vertex ρ_k of a regular polygon inscribed on the unit circle, and $\tilde{m}_k \ (k = 1, ..., N)$ locates at

$$\tilde{\rho}_k = a\rho_k \tag{1.3}$$

where $a > 0,\ 0 \le \theta \le 2\pi$, and $a \ne 1$ when $\theta = 0$ or 2π, and m_0 locates at the geometrical center(which is taken as the coordinate origin) of ρ_k and $\tilde{\rho}_k$ $(k = 1, ..., N)$. Then the center of masses $m_1, ..., m_N; \tilde{m}_1, ..., \tilde{m}_N, m_0$ is

$$z_0 = \frac{\sum_j (m_j \rho_j + \tilde{m}_j \tilde{\rho}_j)}{M} \tag{1.4}$$

where $M = \sum_j (m_j + \tilde{m}_j) + m_0$. In (1.4) and throughout this paper, unless specially restricted, all indices and summations will range from 1 to N. The functions describing their rotation about z_0 with angular velocity ω are then given by

$$\begin{aligned} z_k(t) &= (\rho_k - z_0)\exp(I\omega t), \qquad k = 1, ..., N & (1.5)\\ \tilde{z}_k(t) &= (a\rho_k e^{I\theta} - z_0)\exp(I\omega t), \quad k = 1, ..., N & (1.6)\\ \tilde{z}_0(t) &= (0 - z_0)\exp(I\omega t). & (1.7) \end{aligned}$$

Then the equations of motion of the planar $2N$-body problem can be written as the following form,

$$\ddot{z}_k = \sum_{j \ne k} m_j \frac{z_j - z_k}{|z_j - z_k|^3} + \sum_j \tilde{m}_j \frac{\tilde{z}_j - z_k}{|\tilde{z}_j - z_k|^3}, \tag{1.8}$$

$$\ddot{\tilde{z}}_k = \sum_j m_j \frac{z_j - \tilde{z}_k}{|z_j - \tilde{z}_k|^3} + \sum_{j \ne k} \tilde{m}_j \frac{\tilde{z}_j - \tilde{z}_k}{|\tilde{z}_j - \tilde{z}_k|^3}, \tag{1.9}$$

and

$$\ddot{\tilde{z}}_0 = \sum_j m_j \frac{z_j - \tilde{z}_0}{|z_j - \tilde{z}_0|^3} + \sum_j \tilde{m}_j \frac{\tilde{z}_j - \tilde{z}_0}{|\tilde{z}_j - \tilde{z}_0|^3}. \tag{1.10}$$

R. Moeckel and C. Simo ([5]) proved the following result:

Theorem (Moeckel-Simo). *If* $\theta{=}0, m_0{=}0$ *and* $m_1{=}\cdots{=}m_N,\ \tilde{m}_1{=}\cdots{=}\tilde{m}_N,$ *then for every mass ratio* $b = \dfrac{\tilde{m}_1}{m_1} \ne 1$, *there are exactly two planar central configurations consisting of two nested regular* N*-gons. For one of these, the ratio of the sizes of the two polygons is less than* 1, *and for the other it is greater than* 1.

Zhang and Zhou also discussed periodic solutions for planar $2N$-body in [8], [9]. In this paper, we continue to study the inverse problem of the theorem (Moeckel-Simo) for $2N+1$-body problem and the following results are established.

Theorem 1. *For $N \geq 2$, $m_k, \tilde{m}_k, m_0 > 0$, the functions $z_k(t), \tilde{z}_k(t)$ and $\tilde{z}_0(t)$ with ω given by* (1.5)–(1.7) *are solutions of the $2N+1$-body problem* (1.8)–(1.10), *then*

$$\text{(i)}\quad \left(\sum_k m_k\right)\sum_{j\neq N}\left(\frac{1}{|1-\rho_j|^3}-\frac{\omega^2}{M}\right)(1-\rho_j) + \left(\sum_k \tilde{m}_k\right)\sum_j\left(\frac{1}{|1-a\rho_j e^{I\theta}|^3}-\frac{\omega^2}{M}\right)(1-a\rho_j e^{I\theta}) + m_0\cdot N\left(1-\frac{\omega^2}{M}\right)=0 \tag{1.11}$$

$$\left(\sum_k m_k\right)\sum_j\left(\frac{1}{|ae^{I\theta}-\rho_j|^3}-\frac{\omega^2}{M}\right)(ae^{I\theta}-\rho_j) + \left(\sum_k \tilde{m}_k\right)\sum_{j\neq N}\left(\frac{1}{|a-a\rho_j|^3}-\frac{\omega^2}{M}\right)(a-a\rho_j)e^{I\theta} + m_0\cdot N\left(\frac{1}{a^3}-\frac{\omega^2}{M}\right)ae^{I\theta}=0, \tag{1.12}$$

$$\text{(ii)}\quad m_1=m_2=\cdots=m_N \text{ and } \tilde{m}_1=\tilde{m}_2=\cdots=\tilde{m}_N. \tag{1.13}$$

Theorem 2. *For $N \geq 2$, the functions $z_k(t)$, $\tilde{z}_k(t)$ and $\tilde{z}_0(t)$ with ω given by* (1.5)–(1.7) *are solutions of the $2N+1$-body problem* (1.8)–(1.10), *if and only if the followings hold*

$$\text{(i)}\quad m_1=m_2=\cdots=m_N := m \text{ and } \tilde{m}_1=\tilde{m}_2=\cdots=\tilde{m}_N := \tilde{m}, \tag{1.14}$$

$$\text{(ii)}\quad \gamma := \frac{\omega^2}{M} = \frac{1}{N+bN+c}\left[\sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3} + b\sum_j\frac{1-a\rho_j e^{I\theta}}{|1-a\rho_j e^{i\theta}|^3}+c\right] \tag{1.15}$$

$$b = \frac{a^3 e^{I\theta}\sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3} - a^2\sum_j\frac{ae^{I\theta}-\rho_j}{|ae^{I\theta}-\rho_j|^3} + (ca^3-c)e^{I\theta}}{e^{I\theta}\left(\sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3} - a^3\sum_j\frac{1-a\rho_j e^{I\theta}}{|1-a\rho_j e^{I\theta}|^3}\right)}, \tag{1.16}$$

where $b=\tilde{m}/m, c=m_0/m$.

Theorem 3. *When $\theta = 0$, for the given mass ratio: $b=\tilde{m}/m \neq 1$, and the arbitrary mass ratio: $c = m_0/m$, there exists two unique solutions in* (1.8)–(1.10) *satisfying* (1.14) *and one such that $0 < a < 1$, the other one such that $a > 1$.*

When $\theta = \pi/N$, for $b > 1$ there exists a unique solution in (1.8)–(1.10) *satisfying* (1.14) *and $0 < a < 1$, and for $1 > b > 0$ there exists a unique solution such that $a > 1$.*

Remark. It seems that only $\theta = 0$ or $\dfrac{\pi}{N}$, ω^2 and a are positive real numbers, but the proof seems very difficult.

Corollary 1. *For $N \geq 2$, $\theta = \pi/N, a = 1$, if the functions $z_k(t)$, $\tilde{z}_k(t)$ and $\tilde{z}_0(t)$ with ω given by* (1.5)–(1.7) *are solutions of the $2N$+1-body problem* (1.8)–(1.10), *then*

$$\text{(i)} \quad b = 1, \ \textit{i.e.}, \ m_1 = m_2 = \cdots = m_N = \tilde{m}_1 = \tilde{m}_2 = \cdots = \tilde{m}_N, \tag{1.17}$$

$$\text{(ii)} \quad \gamma := \frac{\omega^2}{M} = \frac{1}{2N + c}\left[\frac{1}{4}\sum_{j \neq N} \csc\left(\frac{\pi j}{2N}\right) + c\right]. \tag{1.18}$$

It is the extension of Theorem 1 (Perko-Walter) [6].

Corollary 2. *Under the above assumptions,*

(i) *if $N = 2, \theta = 0, a > 1$, $z_k(t), \tilde{z}_k(t), \tilde{z}_0(t)$ with ω given by* (1.5)–(1.7) *are solutions of the $2 + 2 + 1$-body problems* (1.8)–(1.10), *if and only If*

$$m_1 = m_2, \tilde{m}_1 = \tilde{m}_2,$$

$$\frac{\omega^2}{M} = \frac{1}{2(1+b)+c}\left[\frac{1}{4} - \frac{4ab}{(a^2-1)^2} + c\right] \tag{1.19}$$

and

$$b = \frac{4c(a^3-1)(a^4-2a^2+1) + (a^7 - 2a^5 - 8a^4 + a^3 - 8a^2)}{17a^4 - 2a^2 + 1}, \tag{1.20}$$

hereafter $b = \tilde{m}_1/m_1, c = m_0/m_1$.

(ii) *If $N = 2, \theta = 0, 0 < a < 1, z_k(t), \tilde{z}_k(t), \tilde{z}_0(t)$ with ω given by* (1.5)–(1.7) *are solutions of the $2 + 2 + 1$-body problems* (1.8)–(1.10), *if and only if*

$$m_1 = m_2, \tilde{m}_1 = \tilde{m}_2,$$

$$\frac{\omega^2}{M} = \frac{1}{2(1+b)+c}\left[\frac{1}{4} + \frac{2b(a^2+1)}{(a^2-1)^2} + c\right] \tag{1.21}$$

and

$$b = \frac{(a^7 - 2a^5 + 17a^3) + 4c(a^3-1)(a^4-2a^2+1)}{-8a^5 + a^4 - 8a^3 - 2a^2 + 1}. \tag{1.22}$$

(iii) *For $N = 2$ and $\theta = \dfrac{\pi}{2}$, b, c and a has the following relationship*

$$b = \frac{2^{-2} - 2(a^2+1)^{-3/2} + c - ca^{-3}}{2^{-2}a^{-3} - 2(a^2+1)^{-3/2}}. \tag{1.23}$$

Remark. When $m_0 = 0$, Corollary 2 is conclusions of MacMillan-Bartky [4] in some sense.

Corollary 3. *Under the above assumptions,*

(i) *If* $N = 3, \theta = 0, a > 1, z_k(t), \tilde{z}_k(t), \tilde{z}_0(t)$ *with* ω *given by* (1.5)–(1.7) *are solutions of the* $3+3+1$*-body problems* (1.8)–(1.10), *if and only if*

$$m_1 = m_2 = m_3, \tilde{m}_1 = \tilde{m}_2 = \tilde{m}_3,$$

$$\frac{\omega^2}{M} = \frac{1}{3(1+b)+c}\left[\frac{\sqrt{3}}{3} + \frac{b(2+a)}{(1+a+a^2)^{\frac{3}{2}}} - \frac{b}{(a-1)^2} + c\right] \tag{1.24}$$

and

$$b = \frac{\left[a^2\left(\frac{2a+1}{(1+a+a^2)^{\frac{3}{2}}} + \frac{1}{(a-1)^2}\right) - a^3\frac{\sqrt{3}}{3} - ca^3 + c\right]}{\left[a^3\left(\frac{2a+1}{(1+a+a^2)^{\frac{3}{2}}} - \frac{1}{(a-1)^2}\right) - \frac{\sqrt{3}}{3}\right]}. \tag{1.25}$$

(ii) *If* $N = 3, \theta = 0, 0 < a < 1, z_k(t), \tilde{z}_k(t), \tilde{z}_0(t)$ *with* ω *given by* (1.5)–(1.7) *are solutions of the* $3+3+1$*-body problems* (1.8)–(1.10), *if and only if*

$$m_1 = m_2 = m_3, \tilde{m}_1 = \tilde{m}_2 = \tilde{m}_3,$$

$$\frac{\omega^2}{M} = \frac{1}{3(1+b)+c}\left[\frac{\sqrt{3}}{3} + \frac{b(2+a)}{(1+a+a^2)^{\frac{3}{2}}} + \frac{b}{(a-1)^2} + c\right] \tag{1.26}$$

and

$$b = \frac{\left[a^2\left(\frac{2a+1}{(1+a+a^2)^{\frac{3}{2}}} - \frac{1}{(a-1)^2}\right) - a^3\frac{\sqrt{3}}{3} - ca^3 + c\right]}{\left[a^3\left(\frac{2a+1}{(1+a+a^2)^{\frac{3}{2}}} + \frac{1}{(a-1)^2}\right) - \frac{\sqrt{3}}{3}\right]}. \tag{1.27}$$

Corollary 4. *Under the above assumptions,*

(i) *If* $N = 4, \theta = 0, a > 1, z_k(t), \tilde{z}_k(t), \tilde{z}_0(t)$ *with* ω *given by* (1.5)–(1.7) *are solutions of the* $4+4+1$*-body problems* (1.8)–(1.10), *if and only if*

$$m_1 = \cdots = m_4, \tilde{m}_1 = \cdots = \tilde{m}_4,$$

$$\frac{\omega^2}{M} = \frac{1}{4(1+b)+c}\left[\frac{1}{4} + \frac{\sqrt{2}}{2} + b\left(\frac{2a}{(1+a^2)^{\frac{3}{2}}} - \frac{4a}{(a^2-1)^2}\right) + c\right] \tag{1.28}$$

and

$$b = \frac{\left[a^3\left(\frac{1}{4} + \frac{\sqrt{2}}{2}\right) - a^2\left(\frac{2a}{(1+a^2)^{\frac{3}{2}}} + \frac{2(a^2+1)}{(a^2-1)^2}\right) + ca^3 - c\right]}{\left[\left(\frac{1}{4} + \frac{\sqrt{2}}{2}\right) - a^3\left(\frac{2}{(1+a^2)^{\frac{3}{2}}} - \frac{4a}{(a^2-1)^2}\right)\right]}. \tag{1.29}$$

(ii) *If* $N = 4, \theta = 0, 0 < a < 1, z_k(t), \tilde{z}_k(t), \tilde{z}_0(t)$ *with* ω *given by* (1.5)–(1.7) *are solutions of the* $4+4+1$*-body problems* (1.8)–(1.10), *if and only if*

$$m_1 = \cdots = m_4, \tilde{m}_1 = \cdots = \tilde{m}_4,$$

$$\frac{\omega^2}{M} = \frac{1}{4(1+b)+c}\left[\frac{1}{4} + \frac{\sqrt{2}}{2} + b\left(\frac{2a}{(1+a^2)^{\frac{3}{2}}} - \frac{2(a^2+1)}{(a^2-1)^2}\right) + c\right] \tag{1.30}$$

and

$$b = \frac{\left[a^3\left(\frac{1}{4} + \frac{\sqrt{2}}{2}\right) - a^2\left(\frac{2a}{(1+a^2)^{\frac{3}{2}}} - \frac{4a}{(a^2-1)^2}\right) + ca^3 - c\right]}{\left[\left(\frac{1}{4} + \frac{\sqrt{2}}{2}\right) - a^3\left(\frac{2}{(1+a^2)^{\frac{3}{2}}} + \frac{2(a^2+1))}{(a^2-1)^2}\right)\right]}. \tag{1.31}$$

2. Some lemmas

Definition 2.1. ([3]) If $N \times N$ matrix $A = (a_{i,j})$ satisfies

$$a_{i,j} = a_{i-1,j-1}, \quad 1 \le i, \quad j \le N, \tag{2.1}$$

where we assume $a_{i,0} = a_{i,N}$ and $a_{0,j} = a_{N,j}$, then we call A is a circular matrix.

Lemma 2.1. ([3])

(i) *If* A *and* B *are* $N \times N$ *circular matrices, for any numbers* α *and* β, *then* $A+B, A-B, AB, \alpha A + \beta B$ *are also circular matrices, and* $AB = BA$.

(ii) *Let* $A = (a_{i,j})$ *be a* $N \times N$ *circular matrix, then the eigenvalues* λ_k *and the eigenvectors* $\vec{v}_k$ *of* A *are*

$$\lambda_k(A) = \sum_j a_{1,j}\rho_{k-1}^{j-1}, \tag{2.2}$$

$$\vec{v}_k = (1, \rho_{k-1}, \rho_{k-1}^2, ..., \rho_{k-1}^{N-1})^T. \tag{2.3}$$

(iii) *Let* A, B *be circular matrices,* $\lambda_k(A)$, $\lambda_k(B)$ *are eigenvalues of* A, B. *Then the eigenvalues of* $A+B, A-B, A \cdot B$ *are*

$$\lambda_k(A) + \lambda_k(B), \lambda_k(A) - \lambda_k(B), \lambda_k(A) \cdot \lambda_k(B).$$

It is clear that

Lemma 2.2. *If* $A = (a_{i,j})$ *is a* $N \times N$ *circular matrices, and* $AX = 0$, *where* $X = (x_1, ..., x_n)^T, \quad x_i > 0 \quad (i = 1, ..., N)$, *then*

$$\begin{aligned} a_{1,j} + \cdots + a_{N,j} = 0, \quad 1 \le j \le N, \\ a_{i,1} + \cdots + a_{i,N} = 0, \quad 1 \le i \le N. \end{aligned} \tag{2.4}$$

Lemma 2.3. *Let* A, B *be* $N \times N$ *Hermite circular matrices, then* $A+B$, $A-B$, AB, $\alpha A + \beta B$ $(\alpha, \beta \in R)$ *are also Hermite circular matrices.*

Lemma 2.4. *Let* A *is a Hermite circular matrix, then the eigenvalues of* A *are real number.*

(i) *When* $n = 2m + 1 (m \geq 1)$, A *can be denoted with*

$$A = A_{2m+1} = cir(a, b_1, b_2, ..., b_m, \bar{b}_m, ..., \bar{b}_2, \bar{b}_1),$$

where $a \in R$ *and* $\bar{b}_l$ *is a conjugate complex number of* b_l. *It has eigenvalues*

$$\lambda_0 = a + 2\sum_{l=1}^{m} Reb_l \tag{2.5}$$

$$\lambda_k = a + 2\sum_{l=1}^{m}\left[Reb_l cos\frac{2k\pi l}{2m+1} - Imb_l sin\frac{2k\pi l}{2m+1}\right] \quad 1 \leq k \leq 2m. \tag{2.6}$$

(ii) *When* $n = 2m (m \geq 1)$, A *can be denoted with*

$$A = A_{2m} = cir(a, b_1, b_2, ..., b_{m-1}, b_m, \bar{b}_{m-1}, ..., \bar{b}_2, \bar{b}_1,).$$

It has eigenvalues

$$\lambda_0 = a + 2\sum_{l=1}^{m-1} Reb_l + b_m, \tag{2.7}$$

$$\lambda_m = a + 2\sum_{l=1}^{m-1}(-1)^l Reb_l + (-1)^m b_m, \tag{2.8}$$

$$\lambda_k = a + 2\sum_{l=1}^{m-1}\left[Reb_l cos\frac{2k\pi l}{2m} - Imb_l sin\frac{2k\pi l}{2m}\right] + (-1)^k b_m \quad 1 \leq k \leq 2m-1, \ k \neq m. \tag{2.9}$$

Proof. This lemma can be simply proved by the properties of the circular matrix and the Hermite matrix.

Lemma 2.5. *The complex subspace* L *of* C^N *generated by* $X_1 = (1, 1, ..., 1)$, $X_2 = (1, \rho, ..., \rho^{N-1})$, *where* $N = 2k > 2$ $\left(\rho = \exp\frac{2\pi I}{N}\right)$, *and the complex subspace* $\tilde{L}$ *generated by* X_1, X_2 *and* $X_3 = (1, \rho^{(k+1)}, ..., \rho^{(N-1)(k+1)})$ *where* $N = 2k + 1 > 3$, *are all contains no real vectors other than the multiples of* $(1, 1, ..., 1)$.

Proof. After some algebraic computation, it can be also simply proved.

Lemma 2.6. ([5]) *Let* $A = \frac{1}{4}\sum_{j \neq N} csc(\pi j/N)$, *then* $A(N)$ *has the following asymptotic expansion for* N *large:*

$$A(N) \sim \frac{N}{2\pi}\left(\gamma + log\frac{2N}{\pi}\right) + \sum_{k \geq 0}\frac{(-1)^k(2^{2k-1}-1)B_{2k}^2\pi^{2k-1}}{(2k)(2k)!}\frac{1}{N^{2k-1}}, \tag{2.10}$$

where γ *stands for the Euler-Mascheroni constant and* B_{2k} *stands for the Bernoulli numbers.*

Lemma 2.7. *Let* $\Phi_\lambda(x) = \sum_j \frac{1}{d_j^\lambda}$*, where* $\lambda > 0, d_j = 1 + x^2 - 2x cos\left(\frac{2\pi j}{N} - \frac{\pi}{N}\right)$*, then, for* $0 < x < 1$*,* $\Phi_\lambda(x)$ *and all of its any order derivatives are positive. Moreover, the same is thus for* $\Psi_\lambda(x) = \sum_j \frac{cos\left(\frac{2\pi j}{N} - \frac{\pi}{N}\right)}{d_j^\lambda}$.

Proof. The conclusion and proof are similar to [5].

3. The proof of the main results

For two nested regular polygons, we define

$$\rho_k = \exp(2\pi Ik/N), \tag{3.1}$$

$$\tilde{\rho}_k = a\exp(2\pi Ik/N)e^{I\theta}, \tag{3.2}$$

$$z_0 = \sum_j (m_j\rho_j + \tilde{m}_j\tilde{\rho}_j)/M, \tag{3.3}$$

where

$$M = \sum_j (m_j + \tilde{m}_j) + m_0, \tag{3.4}$$

$$z_k(t) = (\rho_k - z_0)\exp(I\omega t), \qquad k = 1, ..., N, \tag{3.5}$$

$$\tilde{z}_k(t) = (a\rho_k e^{I\theta} - z_0)\exp(I\omega t), \qquad k = 1, ..., N, \tag{3.6}$$

and

$$\tilde{z}_0(t) = (0 - z_0)\exp(I\omega t). \tag{3.7}$$

Proof of Theorem 1. (3.1)–(3.7) imply that the $z_k(t)$, $\tilde{z}_k(t)$ and $\tilde{z}_0(t)$ are the solutions of (1.8) to (1.10) if and only if

$$\begin{aligned}&(\rho_k - z_0)\omega^2 \exp(I\omega t)\\ &\quad = \left(\sum_{j\neq k} m_j \frac{\rho_k - \rho_j}{|\rho_k - \rho_j|^3} + \sum_j \tilde{m}_j \frac{\rho_k - \tilde{\rho}_j}{|\rho_k - \tilde{\rho}_j|^3} + m_0 \frac{\rho_k - 0}{|\rho_k|^3}\right)\exp(I\omega t),\end{aligned} \tag{3.8}$$

$$\begin{aligned}&(\tilde{\rho}_k - z_0)\omega^2 \exp(I\omega t)\\ &\quad = \left(\sum_j m_j \frac{\tilde{\rho}_k - \rho_j}{|\tilde{\rho}_k - \rho_j|^3} + \sum_{j\neq k} \tilde{m}_j \frac{\tilde{\rho}_k - \tilde{\rho}_j}{|\tilde{\rho}_k - \tilde{\rho}_j|^3} + m_0 \frac{\tilde{\rho}_k - 0}{|\tilde{\rho}_k|^3}\right)\exp(I\omega t)\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}&(0 - z_0)\omega^2 \exp(I\omega t)\\ &\quad = \left(\sum_{j\neq k} m_j \frac{0 - \rho_j}{|0 - \rho_j|^3} + \sum_j \tilde{m}_j \frac{0 - \tilde{\rho}_j}{|0 - \tilde{\rho}_j|^3}\right)\exp(I\omega t),\end{aligned} \tag{3.10}$$

or if and only if

$$\begin{aligned}\sum_{j\neq k} m_j\left(\frac{1}{|\rho_k-\rho_j|^3}-\frac{\omega^2}{M}\right)(\rho_k-\rho_j)\\ +\sum_j \tilde{m}_j\left(\frac{1}{|\rho_k-\tilde{\rho}_j|^3}-\frac{\omega^2}{M}\right)(\rho_k-\tilde{\rho}_j)+m_0\left(1-\frac{\omega^2}{M}\right)\rho_k=0,\end{aligned} \tag{3.11}$$

$$\begin{aligned}\sum_{j} m_j\left(\frac{1}{|\tilde{\rho}_k-\rho_j|^3}-\frac{\omega^2}{M}\right)(\tilde{\rho}_k-\rho_j)\\ +\sum_{j\neq k} \tilde{m}_j\left(\frac{1}{|\tilde{\rho}_k-\tilde{\rho}_j|^3}-\frac{\omega^2}{M}\right)(\tilde{\rho}_k-\tilde{\rho}_j)+m_0\left(\frac{1}{a^3}-\frac{\omega^2}{M}\right)a\rho_k=0\end{aligned} \tag{3.12}$$

and

$$z_0\omega^2=\left(\sum_{j\neq k} m_j\frac{\rho_j}{|\rho_j|^3}+\sum_j \tilde{m}_j\frac{\tilde{\rho}_j}{|\tilde{\rho}_j|^3}\right). \tag{3.13}$$

Multiplying both sides by ρ_{N-k} in (3.11), (3.12), noting that $|\rho_k-\rho_j| = |\rho_k||1-\rho_{j-k}| = |1-\rho_{j-k}|$ and using $\tilde{\rho}_k = a\rho_k e^{I\theta}$, we have

$$\begin{aligned}\sum_{j\neq k} m_j\left(\frac{1}{|1-\rho_{j-k}|^3}-\frac{\omega^2}{M}\right)(1-\rho_{j-k})\\ +\sum_j \tilde{m}_j\left(\frac{1}{|1-a\rho_{j-k}e^{I\theta}|^3}-\frac{\omega^2}{M}\right)(1-a\rho_{j-k}e^{I\theta})+m_0(1-\frac{\omega^2}{M})=0,\end{aligned} \tag{3.14}$$

$$\begin{aligned}\sum_{j} m_j\left(\frac{1}{|ae^{I\theta}-\rho_{j-k}|^3}-\frac{\omega^2}{M}\right)(ae^{I\theta}-\rho_{j-k})\\ +\sum_{j\neq k} \tilde{m}_j\left(\frac{1}{|a-a\rho_{j-k}|^3}-\frac{\omega^2}{M}\right)(a-a\rho_{j-k})e^{I\theta}+m_0(\frac{1}{a^3}-\frac{\omega^2}{M})ae^{I\theta}=0\end{aligned} \tag{3.15}$$

and

$$z_0\omega^2=\left(\sum_{j\neq k} m_j\frac{\rho_j}{|\rho_j|^3}+\sum_j \tilde{m}_j\frac{\tilde{\rho}_j}{|\tilde{\rho}_j|^3}\right). \tag{3.16}$$

Notice that every step from (3.8) to (3.16) can be conversed respectively, firstly we discuss (3.14)–(3.16). Now define the $N\times N$ circular matrices $A=[a_{k,j}]$, $B=[b_{k,j}]$, $C=[c_{k,j}]$, $D=[d_{k,j}]$ as follows:

$$a_{k,j}=0, \qquad \text{for} \quad k=j,$$

$$a_{k,j} = \left(\frac{1}{|1-\rho_{j-k}|^3}-\frac{\omega^2}{M}\right)(1-\rho_{j-k}), \qquad \text{for} \quad k\neq j, \tag{3.17}$$

$$b_{k,j} = \left(\frac{1}{|ae^{I\theta}-\rho_{j-k}|^3}-\frac{\omega^2}{M}\right)(ae^{I\theta}-\rho_{j-k}), \tag{3.18}$$

$$c_{k,j} = \left(\frac{1}{|1-a\rho_{j-k}e^{I\theta}|^3}-\frac{\omega^2}{M}\right)(1-a\rho_{j-k}e^{I\theta}), \tag{3.19}$$

$$d_{k,j} = 0, \qquad \text{for} \quad k = j,$$

$$d_{k,j} = \left(\frac{1}{|a - a\rho_{j-k}|^3} - \frac{\omega^2}{M}\right)(a - a\rho_{j-k})e^{I\theta} \quad \text{for} \quad k \neq j, \tag{3.20}$$

$$\vec{1} = (1, ..., 1)^T, \tag{3.21}$$

$$E = (1 - \frac{\omega^2}{M}) \cdot \vec{1}, \tag{3.22}$$

$$F = ae^{I\theta}(\frac{1}{a^3} - \frac{\omega^2}{M}) \cdot \vec{1}. \tag{3.23}$$

Then (3.14) and (3.15) hold if and only if the matrix equation

$$\begin{pmatrix} A & C & E \\ B & D & F \end{pmatrix} \begin{pmatrix} m_1 \\ \vdots \\ m_N \\ \tilde{m}_1 \\ \vdots \\ \tilde{m}_N \\ m_0 \end{pmatrix} = 0 \tag{3.24}$$

has a positive solution.

Let

$$\vec{m} = (m_1, ..., m_N)^T, \vec{\tilde{m}} = (\tilde{m}_1, ..., \tilde{m}_N)^T, \tag{3.25}$$

then (3.24) is equivalent to

$$A\vec{m} + C\vec{\tilde{m}} + Em_0 = \vec{0}, \tag{3.26}$$

$$B\vec{m} + D\vec{\tilde{m}} + Fm_0 = 0. \tag{3.27}$$

Notice that A, B, C, D are $N \times N$ circular matrices, with the properties of circular matrix we know they must have positive real eigenvector $\vec{1}$. Each of (3.26), (3.27) left multiplies $\vec{1}^T = (1, 1, ..., 1)$, there are

$$\begin{aligned} &\left(\sum_k m_k\right) \sum_{j \neq N} \left(\frac{1}{|1 - \rho_j|^3} - \frac{\omega^2}{M}\right)(1 - \rho_j) \\ &\quad + \left(\sum_k \tilde{m}_k\right) \sum_j \left(\frac{1}{|1 - a\rho_j e^{I\theta}|^3} - \frac{\omega^2}{M}\right)(1 - a\rho_j e^{I\theta}) \\ &\quad + m_0 \cdot N \left(1 - \frac{\omega^2}{M}\right) = 0, \end{aligned} \tag{3.28}$$

and

$$\begin{aligned} &\left(\sum_k m_k\right) \sum_j \left(\frac{1}{|ae^{I\theta} - \rho_j|^3} - \frac{\omega^2}{M}\right)(ae^{I\theta} - \rho_j) \\ &\quad + \left(\sum_k \tilde{m}_k\right) \sum_{j \neq N} \left(\frac{1}{|a - a\rho_j|^3} - \frac{\omega^2}{M}\right)(a - a\rho_j)e^{I\theta} \\ &\quad + m_0 \cdot N \left(\frac{1}{a^3} - \frac{\omega^2}{M}\right) ae^{I\theta} = 0. \end{aligned} \tag{3.29}$$

The conclusion (i) of Theorem 1 is proved.

(ii) By (3.26) and (3.27) we have

$$(AD-CB)\vec{m}+m_0\left[\left(1-\frac{\omega^2}{M}\right)D-ae^{I\theta}\left(\frac{1}{a^3}-\frac{\omega^2}{M}\right)C\right]\vec{1}=\vec{0}, \tag{3.30}$$

$$(CB-AD)\vec{\tilde{m}}+m_0\left[\left(1-\frac{\omega^2}{M}\right)B-ae^{I\theta}\left(\frac{1}{a^3}-\frac{\omega^2}{M}\right)A\right]\vec{1}=\vec{0}. \tag{3.31}$$

From Lemma 2.1 we see that

$$\left(1-\frac{\omega^2}{M}\right)D-ae^{I\theta}\left(\frac{1}{a^3}-\frac{\omega^2}{M}\right)C,\ \left(1-\frac{\omega^2}{M}\right)B-ae^{I\theta}\left(\frac{1}{a^3}-\frac{\omega^2}{M}\right)A$$

and $AD-CB$ are circular matrices, we know they must have positive real eigenvector $\vec{1}$. Using the properties of circular matrix, (3.30), (3.31) can be written as

$$(AD-CB)\cdot\vec{m}+\alpha_1\cdot\vec{1}=\vec{0}, \tag{3.32}$$

$$(CB-AD)\cdot\vec{\tilde{m}}+\alpha_2\cdot\vec{1}=\vec{0}, \tag{3.33}$$

where

$$\alpha_1\cdot\vec{1}=m_0\left[\left(1-\frac{\omega^2}{M}\right)D-ae^{I\theta}\left(\frac{1}{a^3}-\frac{\omega^2}{M}\right)C\right]\cdot\vec{1}, \tag{3.34}$$

$$\alpha_2\cdot\vec{1}=m_0\left[\left(1-\frac{\omega^2}{M}\right)B-ae^{I\theta}\left(\frac{1}{a^3}-\frac{\omega^2}{M}\right)A\right]\cdot\vec{1}, \tag{3.35}$$

$$\begin{aligned}\alpha_1 &= m_0\Bigg[\left(1-\frac{\omega^2}{M}\right)\sum_{j\neq N}\left(\frac{1}{|a-a\rho_j|^3}-\frac{\omega^2}{M}\right)(a-a\rho_j)e^{I\theta} \\ &\quad -ae^{I\theta}\left(\frac{1}{a^3}-\frac{\omega^2}{M}\right)\sum_{j}\left(\frac{1}{|1-a\rho_je^{I\theta}|^3}-\frac{\omega^2}{M}\right)(1-a\rho_je^{I\theta})\Bigg],\end{aligned} \tag{3.36}$$

$$\begin{aligned}\alpha_2 &= m_0\Bigg[\left(1-\frac{\omega^2}{M}\right)\sum_{j}\left(\frac{1}{|ae^{i\theta}-\rho_j|^3}-\frac{\omega^2}{M}\right)(ae^{I\theta}-\rho_j) \\ &\quad -ae^{I\theta}\left(\frac{1}{a^3}-\frac{\omega^2}{M}\right)\sum_{j\neq N}\left(\frac{1}{|1-\rho_j|^3}-\frac{\omega^2}{M}\right)(1-\rho_j)\Bigg].\end{aligned} \tag{3.37}$$

From (3.28), (3.29), we have

$$\alpha_1\sum_k\tilde{m}_k+\alpha_2\sum_k m_k=0. \tag{3.38}$$

a. If $\alpha_1=0$, then $\alpha_2=0$ and

$$(AD-CB)\cdot\vec{m}=\vec{0}, \tag{3.39}$$

$$(AD - CB) \cdot \vec{\tilde{m}} = \vec{0}. \tag{3.40}$$

(3.39), (3.40) must have positive real solutions, i.e., the kernel K of circular matrix $AD - CB$ has positive vector(s).

By Lemmas 2.1, 2.2 we have the eigenvalue

$$\lambda_k(AD - CB) = \lambda_k(AD) - \lambda_k(CB) = \lambda_k(A)\lambda_k(D) - \lambda_k(C)\lambda_k(B). \tag{3.41}$$

Hence,

$$\lambda_k(AD - CB) = 0 \tag{3.42}$$

for some $1 \leq k \leq N$ if and only if

$$\lambda_k(A)\lambda_k(D) = \lambda_k(B)\lambda_k(C) \tag{3.43}$$

Since

$$\begin{aligned}
&\sum_{j\neq N}\left(\frac{1}{|1-\rho_j|^3} - \frac{\omega^2}{M}\right)(1-\rho_j)\sum_{j\neq N}\left(\frac{1}{|a-a\rho_j|^3} - \frac{\omega^2}{M}\right)(a-a\rho_j)e^{I\theta} \\
&= \sum_j\left(\frac{1}{|1-a\rho_j e^{i\theta}|^3} - \frac{\omega^2}{M}\right)(1-a\rho_j e^{I\theta})\sum_j\left(\frac{1}{|ae^{I\theta}-\rho_j|^3} - \frac{\omega^2}{M}\right)(ae^{I\theta}-\rho_j)
\end{aligned} \tag{3.44}$$

by (3.36), (3.37), we see $\lambda_1(AD - CB) = 0$ and $\vec{v}_1 = (1, 1, ..., 1)^T \in K$. We know A, B, C, D and $AD - CB$ are all Hermite matrices when $\theta = 0 (a \neq 1), \frac{\pi}{N}$. In this case, by Lemma 2.4 , 2.5 and [7], after many complex calculations it implies that the kernel $K \subseteq L$ or $\subseteq \tilde{L}$ only contains such positive vectors as multiples of $v_1 = (1, 1, ..., 1)^T$. In a general way we shall obtain similar conclusion. Hence,

$$m_1 = m_2 = \cdots = m_N := m \qquad m > 0, \tag{3.45}$$

$$\tilde{m}_1 = \tilde{m}_2 = \cdots = \tilde{m}_N := \tilde{m} \qquad \tilde{m} > 0. \tag{3.46}$$

b. If $\alpha_1 \neq 0$ then $\alpha_2 \neq 0$. From (3.32), (3.33), (3.38), we get

$$(CB - AD)\left[\left(\sum_k m_k\right)\vec{\tilde{m}} - \left(\sum_k \tilde{m}_k\right)\vec{m}\right] = \vec{0}. \tag{3.47}$$

If

$$\left(\sum_k m_k\right)\vec{\tilde{m}} - \left(\sum_k \tilde{m}_k\right)\vec{m} = \vec{0}, \tag{3.48}$$

then $\tilde{m}_j = b m_j$, where $b = \sum_k \tilde{m}_k / \sum_k m_k$. Substitute it into (3.26) and (3.27). Similar to the proof in **a**, we also have (3.45) and (3.46).

If

$$\left(\sum_k m_k\right)\vec{\tilde{m}} - \left(\sum_k \tilde{m}_k\right)\vec{m} \neq \vec{0}, \tag{3.49}$$

let $G = CB - AD = (g_{ij})$, which is nonzero circular matrix, by Lemma 2.1 and 2.2, we have $\sum_j g_{ij} = \sum_i g_{ij} = 0$ and G has eigenvalue 0. Using the properties of circular matrix, we have $(AD-CB)\vec{1}=\vec{0}$ or $\vec{1}^T(AD-CB)=\vec{0}^T$. Let $\vec{1}^T$ left multiplies (3.32) and (3.33) respectively, we get $\alpha_1=\alpha_2=0$, which contradicts the supposition. So (3.48) holds. Hence, Theorem 1 is accomplished.

Proof of Theorem 2.

Proof of the Necessary. From Theorem 1, (1.14) holds. We only prove (1.15), (1.16).

Let $\tilde{m}=bm, m_0=cm$, from (3.28), (3.29), there are

$$\begin{aligned}&\left[\sum_{j\neq N}\left(\frac{1}{|1-\rho_j|^3}-\frac{\omega^2}{M}\right)(1-\rho_j)\right.\\&\quad\left.+b\sum_j\left(\frac{1}{|1-a\rho_je^{I\theta}|^3}-\frac{\omega^2}{M}\right)(1-a\rho_je^{I\theta})\right]+c\left(1-\frac{\omega^2}{M}\right)=0\end{aligned}\tag{3.50}$$

$$\begin{aligned}&\left[\sum_j\left(\frac{1}{|ae^{I\theta}-\rho_j|^3}-\frac{\omega^2}{M}\right)(ae^{I\theta}-\rho_j)\right.\\&\quad\left.+b\sum_{j\neq N}\left(\frac{1}{|a-a\rho_j|^3}-\frac{\omega^2}{M}\right)(a-a\rho_j)e^{I\theta}\right]+c\left(\frac{1}{a^3}-\frac{\omega^2}{M}\right)ae^{I\theta}=0.\end{aligned}\tag{3.51}$$

We know

$$\sum_j(1-\rho_j)=N,\tag{3.52}$$

$$\sum_j b(1-a\rho_je^{I\theta})=bN,\tag{3.53}$$

$$\sum_j(ae^{I\theta}-\rho_j)=aNe^{I\theta},\tag{3.54}$$

$$\sum_j b(ae^{I\theta}-a\rho_j)=abNe^{I\theta}.\tag{3.55}$$

By (3.50) and (3.51) there are

$$\begin{aligned}&\frac{\omega^2}{M}\left[\sum_j(1-\rho_j)+b\sum_j(1-a\rho_je^{I\theta})+c\right]\\&\quad=\sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3}+b\sum_j\frac{1-a\rho_je^{I\theta}}{|1-a\rho_je^{I\theta}|^3}+c,\end{aligned}\tag{3.56}$$

$$\begin{aligned}&\frac{\omega^2}{M}\left[\sum_j(ae^{I\theta}-\rho_j)+b\sum_j(a-a\rho_j)e^{I\theta}+cae^{I\theta}\right]\\&\quad=\sum_j\frac{ae^{I\theta}-\rho_j}{|ae^{I\theta}-\rho_j|^3}+b\sum_{j\neq N}\frac{a-a\rho_j}{|a-a\rho_j|^3}e^{I\theta}+\frac{c}{a^2}e^{I\theta},\end{aligned}\tag{3.57}$$

and we have

$$\gamma := \frac{\omega^2}{M} = \frac{1}{N + bN + c} \left[\sum_{j \neq N} \frac{1 - \rho_j}{|1 - \rho_j|^3} + b \sum_j \frac{1 - a\rho_j e^{I\theta}}{|1 - a\rho_j e^{I\theta}|^3} + c \right], \tag{3.58}$$

$$\gamma := \frac{\omega^2}{M} = \frac{1}{ae^{I\theta}(N+bN+c)} \left[\sum_j \frac{ae^{I\theta} - \rho_j}{|ae^{I\theta} - \rho_j|^3} + b \sum_{j \neq N} \frac{a - a\rho_j}{|a - a\rho_j|^3} e^{I\theta} + \frac{c}{a^2} e^{I\theta} \right]. \tag{3.59}$$

Then

$$\begin{aligned} &ae^{I\theta} \left(\sum_{j \neq N} \frac{1 - \rho_j}{|1 - \rho_j|^3} + b \sum_j \frac{1 - a\rho_j e^{I\theta}}{|1 - a\rho_j e^{I\theta}|^3} + c \right) \\ &\qquad = \sum_j \frac{ae^{I\theta} - \rho_j}{|ae^{I\theta} - \rho_j|^3} + b \sum_{j \neq N} \frac{a - a\rho_j}{|a - a\rho_j|^3} e^{I\theta} + \frac{c}{a^2} e^{I\theta}, \end{aligned} \tag{3.60}$$

and that

$$b = \frac{ae^{I\theta} \sum_{j \neq N} \frac{1-\rho_j}{|1-\rho_j|^3} - \sum_j \frac{ae^{I\theta}-\rho_j}{|ae^{I\theta}-\rho_j|^3} + (ca - \frac{c}{a^2}) e^{I\theta}}{e^{I\theta} (\sum_{j \neq N} \frac{a-a\rho_j}{|a-a\rho_j|^3} - a \sum_j \frac{1-a\rho_j e^{I\theta}}{|1-a\rho_j e^{I\theta}|^3})}, \tag{3.61}$$

i.e.

$$b = \frac{a^3 e^{I\theta} \sum_{j \neq N} \frac{1-\rho_j}{|1-\rho_j|^3} - a^2 \sum_j \frac{ae^{I\theta}-\rho_j}{|ae^{I\theta}-\rho_j|^3} + (ca^3 - c) e^{I\theta}}{e^{I\theta} (\sum_{j \neq N} \frac{1-\rho_j}{|1-\rho_j|^3} - a^3 \sum_j \frac{1-a\rho_j e^{I\theta}}{|1-a\rho_j e^{I\theta}|^3})}. \tag{3.62}$$

Namely (1.15) and (1.16) hold.

The proof of the Sufficiency. For $N \geq 2$, the functions $z_k(t)$, $\tilde{z}_k(t)$ and $\tilde{z}_0(t)$ with ω given by (1.5) to (1.7) are solutions of the $2N$+1-body problem (1.8)–(1.10), if and only if (3.8) to (3.10), or (3.11) to (3.13) have positive solutions. Let

$$m_1 = m_2 = \cdots = m_N := m \qquad m > 0, \tag{3.63}$$

$$\tilde{m}_1 = \tilde{m}_2 = \cdots = \tilde{m}_N := \tilde{m} \quad \tilde{m} > 0, \tag{3.64}$$

where $\tilde{m} = bm, m_0 = cm$, then (3.13) holds, (3.11)–(3.12) or (3.14)–(3.15) are equivalence to (3.56) and (3.57). From the process of the necessary proof for theorem 2, we know that (3.56) and (3.57) are equivalence to (3.58) and (3.62), i.e. equivalence to (1.15)–(1.16). Hence the proof of the sufficiency is finished.

Proof of Theorem 3. Under the assumption of the Theorem 3, we know that the uniqueness of the periodic solution (1.5) to (1.7) or (1.8) to (1.10) is equivalence to that (1.15)–(1.16) or (3.58), (3.59) have a unique positive solution for $0 < a < 1$ or $a > 1$. We only prove the case of $\theta = \pi/N$. Obviously, when $\theta = \pi/N$ the right sides of (3.58)–(3.59) are positive. So the problem is that the following equation has a positive solution "a" ($0 < a < 1$ or $a \geq 1$) for given positive numbers b, c.

$$\begin{aligned} &a \left(\sum_{j \neq N} \frac{1 - \rho_j}{|1 - \rho_j|^3} + b \sum_j \frac{1 - a\rho_j e^{I\frac{\pi}{N}}}{|1 - a\rho_j e^{I\frac{\pi}{N}}|^3} + c \right) \\ &\qquad - = \sum_j \frac{a - \rho_j e^{-I\frac{\pi}{N}}}{|a - \rho_j e^{-I\frac{\pi}{N}}|^3} + b \sum_{j \neq N} \frac{a - a\rho_j}{|a - a\rho_j|^3} + \frac{c}{a^2} \cdot \end{aligned} \tag{3.65}$$

Let $a = x$,

$$\begin{aligned} f(x) = x\left(\sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3} + b\sum_j \frac{1-x\rho_j e^{I\frac{\pi}{N}}}{|1-x\rho_j e^{I\frac{\pi}{N}}|^3} + c\right) \\ -\sum_j \frac{x-\rho_j e^{-I\frac{\pi}{N}}}{|x-\rho_j e^{-I\frac{\pi}{N}}|^3} - b\sum_{j\neq N}\frac{x-x\rho_j}{|x-x\rho_j|^3} - \frac{c}{x^2}. \end{aligned} \tag{3.66}$$

We need to prove $f(x)$ has a unique zero for $0 < x < 1$ or for $x \geq 1$. Let

$$d_j^2 = 1 + x^2 - 2xcos\frac{\pi}{N}(2j-1), \tag{3.67}$$

$$\alpha(x) = \sum_j \frac{1}{d_j^3}, \tag{3.68}$$

$$\beta(x) = \sum_j \frac{cos\frac{\pi}{N}(2j-1)}{d_j^3}, \tag{3.69}$$

$$\xi = \sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3} = \begin{cases} \frac{1}{4}(2\sum\limits_{j=1}^{\frac{N}{2}-1} csc\Big(\frac{\pi j}{N}) + 1\Big), & \text{when } N \text{ is even} \\ \frac{1}{2}\sum\limits_{j=1}^{\frac{N-1}{2}} csc\left(\frac{\pi j}{N}\right), & \text{when } N \text{ is odd.} \end{cases} \tag{3.70}$$

Then

$$\lim_{x\to 0^+} f(x) = -\infty \tag{3.71}$$

$$\begin{aligned} \lim_{x\to 1^-} f(x) &= \sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3} + b\sum_j \frac{1-\rho_j e^{I\frac{\pi}{N}}}{|1-\rho_j e^{I\frac{\pi}{N}}|^3} \\ &\quad -\sum_j \frac{1-\rho_j e^{-I\frac{\pi}{N}}}{|1-\rho_j e^{-I\frac{\pi}{N}}|^3} - b\sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3} \\ &= (b-1)\left[\sum_j \frac{1-\rho_j e^{-I\frac{\pi}{N}}}{|1-\rho_j e^{-i\frac{\pi}{N}}|^3} - \sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3}\right]. \end{aligned} \tag{3.72}$$

But

$$\eta = \sum_{j\neq N}\frac{1-\rho_j e^{-I\frac{\pi}{N}}}{|1-\rho_j e^{-I\frac{\pi}{N}}|^3} = \begin{cases} \frac{1}{4}\left(2\sum\limits_{j=1}^{\frac{N}{2}-1} csc\left(\frac{\pi j}{N} - \frac{\pi}{2N}\right) + 1\right), & \text{when } N \text{ is even} \\ \frac{1}{2}\sum\limits_{j=1}^{\frac{N-1}{2}} csc\left(\frac{\pi j}{N} - \frac{\pi}{2N}\right), & \text{when } N \text{ is odd.} \end{cases} \tag{3.73}$$

Since, when $N \geq 2$ is even

$$\begin{aligned} &\sum_j \frac{1-\rho_j e^{-I\frac{\pi}{N}}}{|1-\rho_j e^{-I\frac{\pi}{N}}|^3} - \sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3} \\ &\quad = \frac{1}{2}\sum_{j=1}^{\frac{N}{2}-1}\left[csc\left(\frac{\pi j}{N} - \frac{\pi}{2N}\right) - csc\left(\frac{\pi j}{N}\right)\right] + \frac{1}{2}\left(csc\frac{\pi}{2N} + 1\right) > 0, \end{aligned} \tag{3.74}$$

and when $N \geq 2$ is odd, also

$$\begin{aligned}\sum_j \frac{1-\rho_j e^{-I\frac{\pi}{N}}}{|1-\rho_j e^{-I\frac{\pi}{N}}|^3} - \sum_{j\neq N} \frac{1-\rho_j}{|1-\rho_j|^3} \\ = \frac{1}{2}\sum_{j=1}^{\frac{N-1}{2}} \left[csc\left(\frac{\pi j}{N} - \frac{\pi}{2N}\right) - csc\left(\frac{\pi j}{N}\right)\right] + \frac{1}{4} > 0.\end{aligned} \tag{3.75}$$

Hence, when $b > 1$, we have

$$\lim_{x\to 1^-} f(x) > 0. \tag{3.76}$$

Obviously $f(x)$ is a continue function for $0 < x < 1$. Thus, to prove the existence of unique zero of $f(x)$, it suffices to show that $f(x)$ is increasing. Now $f(x)$ can be written as follow:

$$f(x) = \left(x - \frac{b}{x^2}\right)\xi + bx(\alpha(x) - x\beta(x)) + (\beta(x) - x\alpha(x)) + c\left(x - \frac{1}{x^2}\right). \tag{3.77}$$

Let

$$\Phi(x) = \sum_j \frac{1}{d_j}. \tag{3.78}$$

It follows from the definitions that

$$\Phi(x) = (1+x^2)\alpha(x) - 2x\beta(x), \tag{3.79}$$

and it implies

$$\alpha(x) - x\beta(x) = \alpha(x) + x(\beta(x) - x\alpha(x)). \tag{3.80}$$

Since

$$\frac{d\Phi}{dx} = \alpha(x) - x\beta(x), \tag{3.81}$$

then, also $f(x)$ can be written as

$$f(x) = \left(x - \frac{b}{x^2}\right)\xi + bx\Phi(x) + (1+bx^2)\frac{d\Phi}{dx} + c\left(x - \frac{1}{x^2}\right). \tag{3.82}$$

From Lemma 2.3, the first and the final terms are clearly increasing. Using Lemma 2.4 and its proof, we know $\Phi(x)$, $\frac{d\Phi}{dx}$ themselves and their derivatives are positive for $0 < x < 1$. But

$$\left[bx\Phi(x) + (1+bx^2)\frac{d\Phi}{dx}\right]' = b\Phi(x) + 3bx\frac{d\Phi}{dx} + (1+bx^2)\frac{d^2\Phi}{dx^2}, \tag{3.83}$$

so

$$\left[bx\Phi(x) + (1+bx^2)\frac{d\Phi}{dx}\right]' > 0, \qquad \text{for } 0 < x < 1. \tag{3.84}$$

Hence, the other two terms are increasing too for $0 < x < 1$, i.e. $f(x) = 0$ has a unique solution for $0 < x < 1$ when $b > 1$.

Let the solution of $f(x)$=0 for $0 < x < 1$ is $x_{problem}(b) := x_p(b), 0 < x_p(b) < 1$. For the case of $x > 1$: we don't directly discuss the equation $f(x) = 0$. Similarly we have a corresponding equation. By the symmetry of the problem, we have the solution $\tilde{x}_{problem}(b) := \tilde{x}_p(b)$ s.t. $\tilde{x}_p(b) = \dfrac{1}{x_p\left(\dfrac{1}{b}\right)} > 1$.

Remark. It implies for $b > 1$, then $0 < a < 1$; for $0 < b < 1$, then $a > 1$; and for $b = 1, \theta = \pi/N$, then $a = 1$.

Proof of Corollary 1. By Theorem 2, we easily prove Corollary 1.

Proof of Corollary 2. We only prove (iii), when $N = 2$, $\theta = \pi/2$,

$$\begin{aligned} & a^3e^{I\theta}\sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3} - a^2\sum_j\frac{ae^{I\theta}-\rho_j}{|ae^{I\theta}-\rho_j|^3} + (ca^3-c)e^{I\theta} \\ &= a^2(\frac{2aI}{8} - \frac{aI-1}{(a^2+1)^{3/2}} - \frac{aI+1}{(a^2+1)^{3/2}}) + (ca^3-c)I \\ &= a^2(\frac{2aI}{8} - \frac{2aI}{(a^2+1)^{3/2}}) + (ca^3-c)I \\ &= 2a^3I\left(\frac{1}{8} - \frac{1}{(a^2+1)^{3/2}}\right) + (ca^3-c)I \end{aligned} \tag{3.85}$$

and

$$\begin{aligned} & e^{I\theta}\left(\sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3} - a^3\sum_j\frac{1-a\rho_je^{I\theta}}{|1-a\rho_je^{I\theta}|^3}\right) \\ &= I\left(\frac{1+1}{2^3} - a^3\left(\frac{1-aI}{|1-aI|^3} + \frac{1-a(-1)I}{|1-a(-1)I|^3}\right)\right) \\ &= 2a^3I\left(\frac{1}{8a^3} - \frac{1}{(a^2+1)^{3/2}}\right). \end{aligned} \tag{3.86}$$

So

$$\begin{aligned} b &= \frac{a^3e^{I\theta}\sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3} - a^2\sum_j\frac{ae^{I\theta}-\rho_j}{|ae^{I\theta}-\rho_j|^3} + (ca^3-c)e^{I\theta}}{e^{I\theta}(\sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3} - a^3\sum_j\frac{1-a\rho_je^{I\theta}}{|1-a\rho_je^{I\theta}|^3})} \\ &= \frac{2^{-2} - 2(a^2+1)^{-3/2} + c - ca^{-3}}{2^{-2}a^{-3} - 2(a^2+1)^{-3/2}}. \end{aligned} \tag{3.87}$$

We omit other proofs.

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ON GENERALIZED PRE-CLOSURE SPACES AND SEPARATION FOR SOME SPECIAL TYPES OF FUNCTIONS

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Abstract. In this paper, we show that a pointwise symmetric pre-isotonic pre-closure functions is uniquely determined by the pairs of sets it separates. We then show that when the pre-closure function of the domain is pre-isotonic and the pre-closure function of the codomain is pre-isotonic and pointwise-pre-symmetric, functions which separate only those pairs of sets which are already separated are pre-continuous.

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1. Introduction

Throughout the paper, (X, τ) (or simply X) will always denote a topological space. For a subset A of X, the closure, interior and complement of A in X are denoted by $Cl(A), Int(A)$ and $X - A$, respectively. By $PO(X, \tau)$ and $PC(X, \tau)$ we denote the collection of all preopen sets and the collection of all preclosed sets of (X, τ), respectively. Let A be a subset of a topological space (X, τ). A is preopen [4] or locally dense [1] if $A \subset Int(Cl(A))$. A is preclosed [4] if $X - A$ is preopen or equivalently if $Cl(Int(A)) \subset A$. The intersection of all preclosed sets containing A is called the preclosure of A [2] and is denoted by $pCl(A)$.

Definition 1.

(1) A generalized pre-closure space is a pair (X, pCl) consisting of a set X and a pre-closure function pCl, a function from the power set of X to itself.

(2) The pre-closure of a subset A of X, denoted pCl, is the image of A under pCl.

(3) The pre-exterior of A is $pExt(A) = X \backslash pCl(A)$, and the pre-Interior of A is $pInt(A) = X \backslash pCl(X \backslash A)$.

(4) We say that A is pre-closed if $A = pCl(A)$, A is pre-open if $A = pInt(A)$ and N is a pre-neighborhood of x if $x \in pInt(N)$.

Definition 2. We say that a pre-closure function pCl defined on X is:

(1) pre-grounded if $pCl(\emptyset) = \emptyset$.

(2) pre-isotonic if $pCl(A) \subseteq pCl(B)$ whenever $A \subseteq B$.

(3) pre-enlarging if $A \subseteq pCl(A)$ for each subset A of X.

(4) pre-idempotent if $pCl(A) = pCl(pCl(A))$ for each subset A of X.

(5) pre-sub-linear if $pCl(A \cup B) \subseteq pCl(A) \cup pCl(B)$ for all $A, B \subseteq X$.

(6) pre-additive if $\cup_{i \in I} pCl(A_i) = pCl(\cup_{i \in I} A_i)$ for $A_i \subseteq X$.

Throughout this paper, we will assume that pCl is pre-enlarging.

Definition 3.

(1) Subsets A and B of X are said to be pre-closure-separated in a generalized pre-closure space (X, pCl) (or simply, pCl-separated) if $A \cap pCl(B) = \emptyset$ and $pCl(A) \cap B = \emptyset$, or equivalently, if $A \subseteq pExt(B)$ and $B \subseteq pExt(A)$.

(2) *pExterior* points are said to be pre-closure-separated in a generalized pre-closure space (X, pCl) if for each $A \subseteq X$ and for each $x \in pExt(A)$, $\{x\}$ and A are pCl-separated.

Theorem 1.1. *Let (X, pCl) be a generalized pre-closure space in which pExterior points are pCl-separated and let S be the pairs of pCl-separated sets in X. Then, for each subset A of X, the pre-closure of A is $pCl(A) = \{x \in X : \{\{x\}, A\} \notin S\}$.*

Proof. In any generalized pre-closure space $pCl(A)) \subseteq \{x \in X : \{\{x\}, A\} \notin S\}$. Really suppose that $y \notin \{x \in X : \{\{x\}, A\} \notin S\}$, that is, $\{\{y\}, A\} \in S$. Then $\{y\} \cap pCl(A) = \emptyset$, and so $y \notin pCl(A)$.

Suppose now that $y \notin pCl(A)$. By hypothesis, $\{\{y\}, A\} \in S$, and hence,

$$y \notin \{x \in X : \{\{x\}, A\} \notin S\}.$$

2. Some fundamental properties

Definition 4. A pre-closure function pCl defined on a set X is said to be pointwise pre-symmetric when, for all $x, y \in X$, if $x \in pCl(\{y\})$, then $y \in pCl(\{x\})$.

A generalized pre-closure space (X, pCl) is said to be pre-R_0 when, for all $x, y \in X$, if x is in each pre-neighborhood of y, then y is in each pre-neighborhood of x.

Corollary 2.1. *Let* (X, pCl) *a generalized pre-closure space in which pExterior points are* pCl*-separated. Then* pCl *is pointwise pre-symmetric and* (X, pCl) *is pre-*R_0*.*

Proof. Suppose that *pExterior* points are pCl-separated in (X, pCl).

If $x \in pCl(\{y\})$, then $\{x\}$ and $\{y\}$ are not pCl-separated and hence, $y \in pCl(\{x\})$. Hence, pCl is pointwise pre-symmetric.

Suppose that x belongs to every pre-neighborhood of y, that is, $x \in M$ whenever $y \in pInt(M)$. Letting $A = X \backslash M$ and rewriting contrapositively, $y \in pCl(A)$ whenever $x \in A$.

Suppose $x \in pInt(N)$. $x \notin pCl(X \backslash N)$, so x is pCl-separated from $X \backslash N$. Hence $pCl(\{x\}) \subseteq N$. $x \in \{x\}$, so $y \in pCl(\{x\}) \subseteq N$. Hence (X, pCl) is pre-R_0.

While these three axioms are not equivalent in general, they are equivalent when the pre-closure function is pre-isotonic:

Theorem 2.2. *Let* (X, pCl) *be a generalized pre-closure space with* pCl *pre-isotonic. Then the following are equivalent:*

(1) *pExterior points are* pCl*-separated.*

(2) pCl *is pointwise pre-symmetric.*

(3) (X, pCl) *is pre-*R_0*.*

Proof. Suppose that (2) is true. Let $A \subseteq X$, and suppose $x \in pExt(A)$. Then, as pCl is pre-isotonic, for each $y \in A$, $x \notin pCl(\{y\})$, and hence, $y \notin pCl(\{x\})$. Hence $A \cap pCl(\{x\}) = \emptyset$. Hence (2) implies (1), and by the previous corollary, (1) implies (2).

Suppose now that (2) is true and let $x, y \in X$ such that x is in every pre-neighborhood of y, that is, $x \in N$ whenever $y \in pInt(N)$. Then $y \in pCl(A)$ whenever $x \in A$, and in particular, since $x \in \{x\}$, $y \in pCl(\{x\})$. Hence $x \in pCl(\{y\})$. Thus if $y \in B$, then $x \in pCl(\{y\}) \subseteq pCl(B)$, as pCl is pre-isotonic. Hence, if $x \in pInt(C)$, then $y \in C$, that is, y is in every pre-neighborhood of x. Hence, (2) implies (3).

Finally, suppose that (X, pCl) is pre-R_0 and suppose that $x \in pCl(\{y\})$. Since pCl is pre-isotonic, $x \in pCl(B)$ whenever $y \in B$, or, equivalently, y is in every pre-neighborhood of x. Since (X, pCl) is pre-R_0, $x \in N$ whenever $y \in pInt(N)$. Hence, $y \in pCl(A)$ whenever $x \in A$, and in particular, since $x \in \{x\}$, $y \in pCl(\{x\})$. Hence (3) implies (2).

Theorem 2.3. *Let S be a set of unordered pairs of subsets of a set X such that, for all $A, B, C \subseteq X$,*

1. *if $A \subseteq B$ and $\{B, C\} \in S$, then $\{A, C\} \in S$ and*
2. *if $\{\{x\}, B\} \in S$ for each $x \in A$ and $\{\{y\}, A\} \in S$ for each $y \in B$, then $\{A, B\} \in S$.*

Then the pre-closure function pCl on X, defined by $pCl(A)=\{x \in X : \{\{x\}, A\} \notin S\}$ for every $A \subseteq X$, is pointwise pre-symmetric pre-isotonic and also, pre-closure-separates the elements of S.

Proof. Define pCl by $pCl(A) = \{x \in X : \{\{x\}, A\} \notin S\}$ for every $A \subseteq X$. If $A \subseteq B \subseteq X$ and $x \in pCl(A)$, then $\{\{x\}, A\} \notin S$. Hence, $\{\{x\}, B\} \notin S$, that is, $x \in pCl(B)$. Hence pCl is pre-isotonic. Also, $x \in pCl(\{y\})$ if and only if $\{\{x\}, \{y\}\} \notin S$ if and only if $y \in pCl(\{x\})$, and thus pCl is pointwise pre-symmetric.

Suppose that $\{A, B\} \in S$. Then $A \cap pCl(B) = A \cap \{x \in X : \{\{x\}, B\} \notin S\} = \{x \in A : \{\{x\}, A\} \notin S\} = \emptyset$. Similarly, $pCl(A) \cap B = \emptyset$. Hence, if $\{A, B\} \in S$, then A and B are pCl-separated.

Now suppose that A and B are pCl-separated.

Then $\{x \in A : \{\{x\}, B\} \notin S\} = A \cap pCl(B) = \emptyset$ and $\{x \in B : \{\{x\}, A\} \notin S\} = pCl(A) \cap B = \emptyset$. Hence, $\{\{x\}, B\} \in S$ for each $x \in A$ and $\{\{y\}, A\} \in S$ for each $y \in B$, and thus, $\{A, B\} \in S$.

Furthermore, many properties of pre-closure functions can be expressed in terms of the sets they separate:

Theorem 2.4. *Let S be the pairs of pCl-separated sets of a generalized pre-closure space (X, pCl) in which pExterior points are pre-closure-separates. Then pCl is*

1. *pre-grounded if and only if for all $x \in X$ $\{\{x\}, \emptyset\} \in S$.*
2. *pre-enlarging if and only if for all $\{A, B\} \in S$, A and B are disjoint.*
3. *pre-sub-linear if and only if $\{A, B \cup C\} \in S$ whenever $\{A, B\} \in S$ and $\{A, C\} \in S$.*

Moreover, if pCl is pre-enlarging and for all $A, B \subseteq X$, $\{\{x\}, A\} \notin S$ whenever $\{\{x\}, B\} \notin S$ and $\{\{y\}, A\} \notin S$ for each $y \in B$, then pCl is pre-idempotent. Also, if pCl is pre-isotonic and pre-idempotent, then $\{\{x\}, A\} \notin S$ whenever $\{\{x\}, B\} \notin S$ and $\{\{y\}, A\} \notin S$ for each $y \in B$.

Proof. Recall that, by Theorem 1.1, $pCl(A) = \{x \in X : \{\{x\}, A\} \notin S\}$ for every $A \subseteq X$. Suppose that for all $x \in X$, $\{\{x\}, \emptyset\} \in S$. Then $pCl(\emptyset) = \{x \in X : \{\{x\}, \emptyset\} \notin S\} = \emptyset$. Hence pCl is pre-grounded.

Conversely, if $\emptyset = pCl(\emptyset) = \{x \in X : \{\{x\}, \emptyset\} \notin S\}$, then $\{\{x\}, \emptyset\} \in S$, for all $x \in X$.

Suppose that for all $\{A,B\}\in S$, A and B are disjoint. Since $\{\{a\},A\}\notin S$ if $a\in A$, $A\subseteq pCl(A)$ for each $A\subseteq X$. Hence, pCl is pre-enlarging. Conversely, suppose that pCl is pre-enlarging and $\{A,B\}\in S$. Then $A\cap B\subseteq pCl(A)\cap B=\emptyset$.

Suppose that $\{A,B\cup C\}\in S$ whenever $\{A,B\}\in S$ and $\{A,C\}\in S$. Let $x\in X$ and $B,C\subseteq X$ such that $\{\{x\},B\cup C\}\notin S$. Then $\{\{x\},B\}\notin S$ or $\{\{x\},C\}\notin S$. Hence $pCl(B\cup C)\subseteq pCl(B)\cup pCl(C)$, and therefore, pCl is pre-sub-linear. Conversely, suppose that pCl is pre-sub-linear, and let $\{A,B\},\{A,C\}\in S$. Then $pCl(B\cup C)\cap A\subseteq(pCl(B)\cup pCl(C))\cap A=(pCl(B)\cap A)\cup(pCl(C))\cap A)=\emptyset$ and $(B\cup C)\cap pCl(A)=(B\cap pCl(A))\cup(C\cap pCl(A))=\emptyset$. Suppose that pCl is pre-enlarging and suppose that $\{\{x\},A\}\notin S$ whenever $\{\{x\},B\}\notin S$ and $\{\{y\},A\}\notin S$ for each $y\in B$. Then $pCl(pCl(A))\subseteq pCl(A)$: If $x\in pCl(pCl(A))$, then $\{\{x\},pCl(A)\}\notin S$. $\{\{y\},A\}\notin S$, for each $y\in pCl(A)$; hence $\{\{x\},A\}\notin S$. And since pCl is pre-enlarging, $pCl(A)\subseteq pCl(pCl(A))$. Thus $pCl(pCl(A))=pCl(A)$, for each $A\subseteq X$.

Finally, suppose that pCl is pre-isotonic and pre-idempotent. Let $x\in X$ and $A,B\subseteq X$ such that $\{\{x\},B\}\notin S$ and, for each $y\in B$, $\{\{y\},A\}\notin S$. Then $x\in pCl(B)$ and for each $y\in B$, $y\in pCl(A)$, that is, $B\subseteq pCl(A)$. Hence, $x\in pCl(B)\subseteq pCl(pCl(A))=pCl(A)$.

Definition 5. If (X,pCl_X) and (Y,pCl_Y) are generalized pre-closure spaces, then a function $f:X\to Y$ is said to be

(1) pre-closure-preserving if $f(pCl_X(A))\subseteq pCl_Y(f(A))$ for each $A\subseteq X$.

(2) pre-continuous if $pCl_X(f^{-1}(B))\subseteq f^{-1}(pCl_Y(B))$ for each $B\subseteq Y$.

In general, neither condition implies the other. However, we easily obtain the following result:

Theorem 2.5. *Let (X,pCl_X) and (Y,pCl_Y) be generalized pre-closure spaces and let $f:X\to Y$.*

(1) *If f is pre-closure-preserving and pCl_Y is pre-isotonic, then f is pre-continuous.*

(2) *If f is pre-continuous and pCl_X is pre-isotonic, then f is pre-closure-preserving.*

Proof. Suppose that f is pre-closure-preserving and pCl_Y is pre-isotonic.

Let $B\subseteq Y$. $f(pCl_X(f^{-1}(B)))\subseteq pCl_Y(f(f^{-1}(B)))\subseteq pCl_Y(B)$ and hence,

$$pCl_X(f^{-1}(B))\subseteq f^{-1}(f(pCl_X(f^{-1}(B))))\subseteq f^{-1}(pCl_Y(B)).$$

Suppose that f is pre-continuous and pCl_X is pre-isotonic.

Let $A\subseteq X$. $pCl_X(A)\subseteq pCl_X(f^{-1}(f(A)))\subseteq f^{-1}(pCl_Y(f(A)))$, and hence

$$f(pCl_X(A))\subseteq f(f^{-1}(pCl_Y(f(A))))\subseteq pCl_Y(f(A)).$$

Definition 6. Let (X, pCl_X) and (Y, pCl_Y) be generalized pre-closure spaces and let $f : X \to Y$ be a function. If for all $A, B \subseteq X$, $f(A)$ and $f(B)$ are not pCl_Y-separated whenever A and B are not pCl_X-separated, then we say that f is non-preseparating.

Note that f is non-preseparating if and only if A and B are pCl_X-separated whenever $f(A)$ and $f(B)$ are pCl_Y-separated.

Theorem 2.6. *Let (X, pCl_X) and (Y, pCl_Y) be generalized pre-closure spaces, and let $f : X \to Y$.*

1. *If pCl_Y is pre-isotonic, and f is non-preseparating, then $f^{-1}(C)$ and $f^{-1}(D)$ are pCl_X-separated whenever C and D are pCl_Y-separated.*
2. *If pCl_X is pre-isotonic, and $f^{-1}(C)$ and $f^{-1}(D)$ are pCl_X-separated whenever C and D are pCl_Y-separated, then f is non-preseparating.*

Proof. Let C and D be pCl_Y-separated subsets, where pCl_Y is pre-isotonic. Let $A = f^{-1}(C)$ and let $B = f^{-1}(D)$. $f(A) \subseteq C$ and $f(B) \subseteq D$, and since pCl_Y is pre-isotonic, $f(A)$ and $f(B)$ are also pCl_Y-separated. Hence, A and B are pCl_X-separated in X.

Suppose that pCl_X is pre-isotonic, and let $A, B \subseteq X$ such that $C = f(A)$ and $D = f(B)$ are pCl_X-separated. Then $f^{-1}(C)$ and $f^{-1}(D)$ are pCl_X-separated and since pCl_X is pre-isotonic, $A \subseteq f^{-1}(f(A)) = f^{-1}(C)$ and $B \subseteq f^{-1}(f(B)) = f^{-1}(D)$ are pCl_X-separated as well.

Theorem 2.7. *Let (X, pCl_X) and (Y, pCl_Y) be generalized pre-closure spaces and let $f : X \to Y$ be a function. If f is pre-closure-preserving, then f is non-preseparating.*

Proof. Suppose that f is pre-closure-preserving and $A, B \subseteq X$ are not pCl_X-separated. Suppose that $pCl_X(A) \cap B \neq \emptyset$. Then $\emptyset \neq f(pCl_X(A) \cap B) \subseteq f(pCl_X(A)) \cap f(B) \subseteq pCl_Y(f(A)) \cap f(B)$. Similarly, if $A \cap pCl_X(B) \neq \emptyset$, then $f(A) \cap pCl_Y(f(B)) \neq \emptyset$. Hence $f(A)$ and $f(B)$ are not pCl_Y-separated.

Corollary 2.8. *Let (X, pCl_X) and (Y, pCl_Y) be generalized pre-closure spaces with pCl_X pre-isotonic and let $f : X \to Y$. If f is pre-continuous, then f is non-preseparating.*

Proof. If f is pre-continuous and pCl_X pre-isotonic, then, by Theorem 2.5 (2), f is pre-closure-preserving. Hence, by Theorem 2.7, f is non-preseparating.

Theorem 2.9. *Let (X, pCl_X) and (Y, pCl_Y) be generalized pre-closure spaces which pExterior points pCl_Y-separated in Y and let $f : X \to Y$ be a function. Then f is pre-closure-preserving if and only if f non-preseparating.*

Proof. By Theorem 2.7, if f is pre-closure-preserving, then f is non-preseparating. Suppose that f is non-preseparating and let $A \subseteq X$. If $pCl_X = \emptyset$, then

$$f(pCl_X(A)) = \emptyset \subseteq pCl_Y(f(A).$$

Suppose $pCl_X(A) \neq \emptyset$. Let S_X and S_Y denote the pairs of pCl_X-separated subsets of X and the pairs of pCl_Y-separated subsets of Y, respectively. Let $y \in f(pCl_X(A))$, and let $x \in pCl_X(A) \cap f^{-1}(\{y\})$. Since $x \in pCl_X(A)$, $\{\{x\}, A\} \notin S_X$, and since f non-preseparating, $\{\{y\}, f(A)\} \notin S_Y$. Since *pExterior* points are pCl_Y-separated, $y \in pCl_Y(f(A))$. Thus $f(pCl_X(A)) \subseteq pCl_Y(f(A))$, for each $A \subseteq X$.

Corollary 2.10. *Let (X, pCl_X) and (Y, pCl_Y) be generalized pre-closure spaces with pre-isotonic closure functions and with pCl_Y-pointwise-presymmetric and let $f : X \to Y$. Then f is pre-continuous if and only if f non-preseparating.*

Proof. Since pCl_Y is pre-isotonic and pointwise-presymmetric, *pExterior* points are pre-closure separated in (Y, pCl_Y) (Theorem 2.2 (1)). Since both pre-closure functions are pre-isotonic, f is pre-closure-preserving (Theorem 2.5) if and only if f is pre-continuous. Hence, we can apply the Theorem 2.9.

3. Preconnected generalized pre-closure spaces

Definition 7. Let (X, pCl) be a generalized pre-closure space. X is said to be preconnected if X is not a union of disjoint nontrivial pre-closure-separated pair of sets.

Theorem 3.1. *Let (X, pCl) be a generalized pre-closure space with pre-grounded pre-isotonic pre-enlarging pCl. Then, the following are equivalent:*

1. *(X, pCl) is preconnected,*
2. *X can not be a union of nonempty disjoint preopen sets.*

Proof. (1)⇒(2): Let X be a union of nonempty disjoint preopen sets A and B. Then, $X = A \cup B$ and this implies that $B = X \backslash A$ and A is a preopen set. Thus, B is preclosed and hence $A \cap pCl(B) = A \cap B = \varnothing$. By using similar way, we obtain $pCl(A) \cap B = \varnothing$. Hence, A and B are pre-closure-separated and hence X is not preconnected. This is a contradiction.

(2)⇒(1): Suppose that X is not preconnected. Then $X = A \cup B$, where A, B are disjoint pre-closure-separated sets, i.e $A \cap pCl(B) = pCl(A) \cap B = \varnothing$. We have $pCl(B) \subset X \backslash A \subset B$. Since pCl is pre-enlarging, we obtain $pCl(B) = B$ and hence, B is preclosed. By using $pCl(A) \cap B = \varnothing$ and similar way, it is obvious that A is preclosed. This is a contradiction.

Definition 8. Let (X, pCl) be a generalized pre-closure space with pre-grounded pre-isotonic pCl. Then, (X, pCl) is called a T_1-pre-grounded pre-isotonic space if $pCl(\{x\}) \subset \{x\}$ for all $x \in X$.

Theorem 3.2. *Let (X, pCl) be a generalized pre-closure space with pre-grounded pre-isotonic pCl. Then, the following are equivalent:*

(1) *(X, pCl) is preconnected,*

(2) *Any precontinuous function $f : X \to Y$ is constant for all T_1-pre-grounded pre-isotonic spaces $Y = \{0, 1\}$.*

Proof. (1)⇒(2): Let X be preconnected. Suppose that $f : X \to Y$ is precontinuous and it is not constant. Then there exists a set $U \subset X$ such that $U = f^{-1}(\{0\})$ and $X \backslash U = f^{-1}(\{1\})$. Since f is precontinuous and Y is T_1-pre-grounded pre-isotonic space, then we have $pCl(U) = pCl(f^{-1}(\{0\})) \subset f^{-1}(pCl\{0\}) \subset f^{-1}(\{0\}) = U$ and hence $pCl(U) \cap (X \backslash U) = \varnothing$. By using similar way we have $U \cap pCl(X \backslash U) = \varnothing$. This is a contradiction. Thus, f is constant.

(2)⇒(1): Suppose that X is not preconnected. Then there exist pre-closure-separated sets U and V such that $U \cup V = X$. We have $pCl(U) \subset U$ and $pCl(V) \subset V$ and $X \backslash U \subset V$. Since pCl is pre-isotonic and U and V are pre-closure-separated, then $pCl(X \backslash U) \subset pCl(V) \subset X \backslash U$. If we consider the space (Y, pCl) by $Y = \{0, 1\}$, $pCl(\varnothing) = \varnothing$, $pCl(\{0\}) = \{0\}$, $pCl(\{1\}) = \{1\}$ and $pCl(Y) = Y$, then the space (Y, pCl) is a T_1-pre-grounded pre-isotonic space. We define the function $f : X \to Y$ as $f(U) = \{0\}$ and $f(X \backslash U) = \{1\}$. Let $A \neq \varnothing$ and $A \subset Y$. If $A = Y$, then $f^{-1}(A) = X$ and hence $pCl(X) = pCl(f^{-1}(A)) \subset X = f^{-1}(A) = f^{-1}(pCl(A))$. If $A = \{0\}$, then $f^{-1}(A) = U$ and hence $pCl(U) = pCl(f^{-1}(A)) \subset U = f^{-1}(A) = f^{-1}(pCl(A))$. If $A = \{1\}$, then $f^{-1}(A) = X \backslash U$ and hence $pCl(X \backslash U) = pCl(f^{-1}(A)) \subset X \backslash U = f^{-1}(A) = f^{-1}(pCl(A))$. Hence, f is precontinuous. Since f is not constant, this is a contradiction.

Theorem 3.3. *Let $f : (X, pCl) \to (Y, pCl)$ and $g : (Y, pCl) \to (Z, pCl)$ be precontinuous functions. Then, $gof : X \to Z$ is precontinuous.*

Proof. Suppose that f and g are precontinuous. For all $A \subset Z$ we have $pCl(gof)^{-1}(A) = pCl(f^{-1}(g^{-1}(A))) \subset f^{-1}(pCl(g^{-1}(A))) \subset f^{-1}(g^{-1}(pCl(A))) = (gof)^{-1}(pCl(A))$. Hence, $gof : X \to Z$ is precontinuous.

Theorem 3.4. *Let (X, pCl) and (Y, pCl) be generalized pre-closure spaces with pre-grounded pre-isotonic pCl and $f : (X, pCl) \to (Y, pCl)$ be a precontinuous function onto Y. If X is preconnected, then Y is preconnected.*

Proof. Suppose that $\{0, 1\}$ is a generalized pre-closure spaces with pre-grounded pre-isotonic pCl and $g : Y \to \{0, 1\}$ is a precontinuous function. Since f is precontinuous, by Theorem 3.3, $gof : X \to \{0, 1\}$ is precontinuous. Since X is preconnected, gof is constant and hence g is constant. By Theorem 3.2, Y is preconnected.

Definition 9. Let (Y, pCl) be a generalized pre-closure space with pre-grounded pre-isotonic pCl and more than one element. A generalized pre-closure space (X, pCl) with pre-grounded pre-isotonic pCl is called Y-preconnected if any precontinuous function $f : X \to Y$ is constant.

Theorem 3.5. *Let (Y, pCl) be a generalized pre-closure space with pre-grounded pre-isotonic pre-enlarging pCl and more than one element. Then every Y-preconnected generalized pre-closure space with pre-grounded pre-isotonic is preconnected.*

Proof. Let (X, pCl) be a Y-preconnected generalized pre-closure space with pre-grounded pre-isotonic pCl. Suppose that $f : X \to \{0, 1\}$ is a precontinuous function, where $\{0, 1\}$ is a T_1-pre-grounded pre-isotonic space. Since Y is a generalized pre-closure space with pre-grounded pre-isotonic pre-enlarging pCl and more than one element, then there exists a precontinuous injection $g : \{0, 1\} \to Y$. By Theorem 3.3, $gof : X \to Y$ is precontinuous. Since X is Y-preconnected, then gof is constant. Thus, f is constant and hence, by Theorem 3.2, X is preconnected.

Theorem 3.6. *Let (X, pCl) and (Y, pCl) be generalized pre-closure spaces with pre-grounded pre-isotonic pCl and $f : (X, pCl) \to (Y, pCl)$ be a precontinuous function onto Y. If X is Z-preconnected, then Y is Z-preconnected.*

Proof. Suppose that $g : Y \to Z$ is a precontinuous function. Then $gof : X \to Z$ is precontinuous. Since X is Z-preconnected, then gof is constant. This implies that g is constant. Thus, Y is Z-preconnected.

Definition 10. A generalized pre-closure space (X, pCl) is strongly preconnected if there is no countable collection of pairwise pre-closure-separated sets $\{A_n\}$ such that $X = \cup A_n$.

Theorem 3.7. *Every strongly preconnected generalized pre-closure space with pre-grounded pre-isotonic pCl is preconnected.*

Theorem 3.8. *Let (X, pCl) and (Y, pCl) be generalized pre-closure spaces with pre-grounded pre-isotonic pCl and $f : (X, pCl) \to (Y, pCl)$ be a precontinuous function onto Y. If X is strongly preconnected, then Y is strongly preconnected.*

Proof. Suppose that Y is not strongly preconnected. Then, there exists a countable collection of pairwise pre-closure-separated sets $\{A_n\}$ such that $Y = \cup A_n$. Since $f^{-1}(A_n) \cap pCl(f^{-1}(A_m)) \subset f^{-1}(A_n) \cap f^{-1}(pCl(A_m)) = \varnothing$ for all $n \neq m$, then the collection $\{f^{-1}(A_n)\}$ is pairwise pre-closure-separated. This is a contradiction. Hence, Y is strongly preconnected.

Theorem 3.9. *Let (X, pCl_X) and (Y, pCl_Y) be generalized pre-closure spaces. Then, the following are equivalent for a function $f : X \to Y$*

(1) *f is pre-continuous,*

(2) *$f^{-1}(pInt(B)) \subseteq pInt(f^{-1}(B))$ for each $B \subseteq Y$.*

Theorem 3.10. *Let (X, pCl) be a generalized pre-closure space with pre-grounded pre-isotonic pre-additive pCl. Then (X, pCl) is strongly preconnected if and only if (X, pCl) Y-preconnected for any countable T_1-pre-grounded pre-isotonic space (Y, pCl).*

Proof. ($\Rightarrow$): Let (X, pCl) be strongly connected. Suppose that (X, pCl) is not Y-preconnected for some countable T_1-pre-grounded pre-isotonic space (Y, pCl). There exists a precontinuous function $f : X \to Y$ which is not constant and hence $K = f(X)$ is a countable set with more than one element. For each $y_n \in K$, there exists $U_n \subset X$ such that $U_n = f^{-1}(\{y_n\})$ and hence $Y = \cup U_n$.

Since f is precontinuous and Y is pre-grounded, then for each $n \neq m$, $U_n \cap pCl(U_m) = f^{-1}(\{y_n\}) \cap pCl(f^{-1}(\{y_m\})) \subset f^{-1}(\{y_n\}) \cap f^{-1}(pCl(\{y_m\})) \subset f^{-1}(\{y_n\}) \cap f^{-1}(\{y_m\}) = \varnothing$. This contradict with the strong preconnectedness of X. Thus, X is Y-preconnected.

($\Leftarrow$): Let X be Y-preconnected for any countable T_1-pre-grounded pre-isotonic space (Y, pCl). Suppose that X is not strongly preconnected. There exists a countable collection of pairwise pre-closure-separated sets $\{U_n\}$ such that $X = \cup U_n$. We take the space (Z, pCl), where Z is the set of integers and $pCl : P(Z) \to P(Z)$ is defined by $pCl(K) = K$ for each $K \subset Z$. Clearly (Z, pCl) is a countable T_1-pre-grounded pre-isotonic space. Put $U_k \in \{U_n\}$. We define a function $f : X \to Z$ by $f(U_k) = \{x\}$ and $f(X \backslash U_k) = \{y\}$ where $x, y \in Z$ and $x \neq y$. Since $pCl(U_k) \cap U_n = \varnothing$ for all $n \neq k$, then $pCl(U_k) \cap \cup_{n \neq k} U_n = \varnothing$ and hence $pCl(U_k) \subset U_k$. Let $\varnothing \neq K \subset Z$. If $x, y \in K$ then $f^{-1}(K) = X$ and $pCl(f^{-1}(K)) = pCl(X) \subset X = f^{-1}(K) = f^{-1}(pCl(K))$. If $x \in K$ and $y \notin K$, then $f^{-1}(K) = U_k$ and $pCl(f^{-1}(K)) = pCl(U_k) \subset U_k = f^{-1}(K) = f^{-1}(pCl(K))$. If $y \in K$ and $x \notin K$ then $f^{-1}(K) = X \backslash U_k$. On the other hand, for all $n \neq k$, $U_k \cap pCl(U_n) = \varnothing$ and hence $U_k \cap \cup_{n \neq k} pCl(U_n) = \varnothing$. This implies that $U_k \cap pCl(\cup_{n \neq k} U_n) = \varnothing$. Thus, $pCl(X \backslash U_k) \subset X \backslash U_k$. Since $pCl(K) = K$ for each $K \subset Z$, we have $pCl(f^{-1}(K)) = pCl(X \backslash U_k) \subset X \backslash U_k = f^{-1}(K) = f^{-1}(pCl(K))$. Hence we obtain that f is precontinuous. Since f is not constant, this is a contradiction with the Z-preconnectedness of X. Hence, X is strongly preconnected.

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CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS INVOLVING SĂLĂGEAN OPERATOR

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Abstract. The familiar Sălăgean operator is used here to define a new subclass of analytic and univalent functions in the open unit disk $\mathbb{U}$. In this note we obtain some sufficient conditions for functions belonging to this class and mention few important consequences of our main results.

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1. Introduction, definitions and key lemmas

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : \ z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Also, the classes $\mathcal{S}^*(\alpha)$, $\mathcal{K}(\alpha)$ and $\mathcal{P}(\alpha)$ defined in the open unit $\mathbb{U}$ are the well known subclasses of the class $\mathcal{A}$ of order $\alpha (0 \leqq \alpha < 1)$ in $\mathbb{U}$ which have been studied quite extensively in the *Geometric Function Theory*, and one may refer to MacGregor [6] and Srivastava and Owa ([11], [12]) for their various details.

Let $\mathcal{S}^*(\alpha_1, \alpha_2)$ be the subclass of $\mathcal{A}$ which satisfies

$$-\frac{\pi\alpha_1}{2} < \arg\left(\frac{zf'(z)}{f(z)}\right) < \frac{\pi\alpha_2}{2} \qquad (z \in \mathbb{U};\ 0 < \alpha_1;\ \alpha_2 \leqq 1) \tag{1.2}$$

and let $\mathcal{K}(\alpha_1, \alpha_2)$ be the subclass of $\mathcal{A}$ which satisfies

$$-\frac{\pi\alpha_1}{2} < \arg\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{\pi\alpha_2}{2} \qquad (z \in \mathbb{U};\ 0 < \alpha_1;\ \alpha_2 \leqq 1), \tag{1.3}$$

where $\mathcal{S}^*(\alpha_1, \alpha_2)$ and $\mathcal{K}(\alpha_1, \alpha_2)$ are the subclasses of $\mathcal{A}$ introduced and studied by Takahashi and Nunokawa [13].

We observe that

$$\mathcal{S}^*(\alpha, \alpha) = \mathcal{S}_s^*(\alpha) \qquad \text{and} \qquad \mathcal{K}(\alpha, \alpha) = \mathcal{K}_c(\alpha),$$

where $\mathcal{S}_s^*(\alpha)$ and $\mathcal{K}_c(\alpha)$, are respectively, the familiar subclasses of $\mathcal{A}$ consisting of functions which are strongly starlike of order α $(0 < \alpha \leq 1)$ in $\mathbb{U}$ and strongly convex of order α $(0 < \alpha \leq 1)$ in $\mathbb{U}$. Also, we note that $\mathcal{S}_s^*(0) = \mathcal{S}^*$ and $\mathcal{K}_c(0) = \mathcal{K}$ (see, for details [11] and [12]).

For $f(z) \in \mathcal{A}$, Sălăgean [10] introduced a derivative operator $\mathcal{D}^n$ of order n which we define here by

$$\mathcal{D}^n f(z) = z + \sum_{k=2}^{\infty} k^n\, a_k z^k \qquad (f \in \mathcal{A}; n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{1.4}$$

In terms of the Sălăgean operator $\mathcal{D}^\lambda$ $(\lambda \in \mathbb{N}_0)$ defined by (1.4) above, we introduce a new subclass of $\mathcal{A}$ denoted by $\mathcal{B}(\lambda, \mu, \alpha)$ consisting of functions of the form (1.1) which satisfy the following inequality:

$$\left| \frac{z^{\mu-2}\, \mathcal{D}^{\lambda+1} f(z)}{(\mathcal{D}^\lambda f(z))^{\mu-1}} - 1 \right| < 1 - \alpha. \tag{1.5}$$

We observe that on specializing the arbitrary parameters λ and μ, the above class $\mathcal{B}(\lambda, \mu, \alpha)$ yields the following:

$$\mathcal{B}(0,2,\alpha)=\mathcal{S}^*(\alpha);\ \mathcal{B}(1,2,\alpha)=\mathcal{K}(\alpha);\ \mathcal{B}(0,1,\alpha)=\mathcal{P}(\alpha) \text{ and } \mathcal{B}(0,3,\alpha)=\mathcal{B}(\alpha), \tag{1.6}$$

where $\mathcal{B}(\alpha)$ is a subclass of $\mathcal{A}$ which was earlier studied by Frasin and Darus [2] (see also [1]).

In order to derive our main results, we recall the following known lemmas.

Lemma 1. (Jack's Lemma [4]) *Let the nonconstant function $w(z)$ be analytic in $\mathbb{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathbb{U}$, then*

$$z_0 w'(z_0) = \gamma w(z_0). \tag{1.7}$$

where γ is real and $\gamma \geqq 1$.

Lemma 2. ([7]) *Let Ω be a set in the complex plane $\mathbb{C}$ and suppose that $\phi(z)$ is a mapping from $\mathbb{C}^2 \times \mathbb{U}$ to $\mathbb{C}$ which satisfies $\Phi(ix, y; z) \notin \Omega$ for $z \in \mathbb{U}$, and for all real x, y such that $y \leqq -(1+x^2)/2$. If the function $q(z) = 1 + q_1 z + q_2 z^2 + \cdots$ is analytic in $\mathbb{U}$ such that $\phi(q(z), zq'(z); z) \in \Omega$ for all $z \in \mathbb{U}$, then $\Re(q(z)) > 0$.*

Lemma 3. ([8]) *Let a function* $q(z)$ *be analytic in* $\mathbb{U}$ *with* $q(0)=1$ *and* $q(z)\neq 0$ $(z\in\mathbb{U})$. *If there exists two points* $z_1,\ z_2\in\mathbb{U}$ *such that*

$$-\frac{\pi\alpha_1}{2}=\arg\big(q(z_1)\big)<\arg\big(q(z)\big)<\arg\big(q(z_2)\big)=\frac{\pi\alpha_2}{2} \tag{1.8}$$

for $\alpha_1>0,\ \alpha_2>0$ *and for all* $|z|<|z_1|=|z_2|$, *then*

$$\frac{z_1q'(z_1)}{q(z_1)}=-i\left(\frac{\alpha_1+\alpha_2}{2}\right)m \qquad and \qquad \frac{z_2q'(z_2)}{q(z_2)}=i\left(\frac{\alpha_1+\alpha_2}{2}\right)m, \tag{1.9}$$

where

$$m\geqq\frac{1-|a|}{1+|a|} \qquad and \qquad a=i\ tan\,\frac{\pi}{2}\left(\frac{\alpha_2-\alpha_1}{\alpha_2+\alpha_1}\right). \tag{1.10}$$

In this note we investigate sufficient conditions for functions in the class $\mathcal{A}$ to be members of the class $\mathcal{B}(\lambda,\mu,\alpha)$ (which is defined by involving the familiar Sălăgean operator). Some corollaries are deduced exhibiting the usefulness of the main results.

2. A set of sufficient conditions

Making use of Lemma 1, we first prove

Theorem 1. *If* $f(z)\in\mathcal{A}$ *satisfies the following inequality:*

$$\left|\frac{\mathcal{D}^{\lambda+2}f(z)}{\mathcal{D}^{\lambda+1}f(z)}-(\mu-1)\left(\frac{\mathcal{D}^{\lambda+1}f(z)}{\mathcal{D}^{\lambda}f(z)}-1\right)-1\right|<\frac{1-\alpha}{2-\alpha} \tag{2.1}$$

$$(z\in\mathbb{U};\ 0\leqq\alpha<1;\ \lambda\in\mathbb{N}_0;\ \mu\geqq 0),$$

then $f(z)\in\mathcal{B}(\lambda,\mu,\alpha)$.

Proof. Let $f(z)\in\mathcal{A}$. Define a function $w(z)$ by

$$\frac{z^{\mu-2}\mathcal{D}^{\lambda+1}f(z)}{(\mathcal{D}^{\lambda}f(z))^{\mu-1}}=1+(1-\alpha)w(z), \tag{2.2}$$

then $w(z)$ is analytic in $\mathbb{U}$ and $w(0)=0$. If follows from (2.2) that

$$\frac{\mathcal{D}^{\lambda+2}f(z)}{\mathcal{D}^{\lambda+1}f(z)}-(\mu-1)\left(\frac{\mathcal{D}^{\lambda+1}f(z)}{\mathcal{D}^{\lambda}f(z)}-1\right)-1=\frac{(1-\alpha)zw'(z)}{1+(1-\alpha)w(z)}. \tag{2.3}$$

Suppose that there exists a point $z_0\in\mathbb{U}$ such that

$$\max_{|z|<|z_0|}|w(z)|=|w(z_0)|=1,$$

then (2.3) in view of (1.7) of Lemma 1 and $w(z_0) = e^{i\theta} (0 \leq \theta < 2\pi)$ yields

$$\begin{aligned} &\left| \frac{\mathcal{D}^{\lambda+2} f(z_0)}{\mathcal{D}^{\lambda+1} f(z_0)} - (\mu - 1) \left(\frac{\mathcal{D}^{\lambda+1} f(z_0)}{\mathcal{D}^{\lambda} f(z_0)} - 1 \right) - 1 \right| \\ &\qquad = \left| \frac{(1-\alpha) z_0 w'(z_0)}{1 + (1-\alpha) w(z_0)} \right| = \frac{|(1-\alpha)\gamma e^{i\theta}|}{|1 + (1-\alpha) e^{i\theta}|} \geqq \frac{1-\alpha}{2-\alpha} \end{aligned} \tag{2.4}$$

which contradicts (2.1).

Therefore $|w(z)| < 1$ holds true for all $z \in \mathbb{U}$ and consequently (2.2) gives

$$\left| \frac{z^{\mu-2} \mathcal{D}^{\lambda+1} f(z)}{(\mathcal{D}^{\lambda} f(z))^{\mu-1}} - 1 \right| = |(1-\alpha) w(z)| < 1 - \alpha \qquad (z \in \mathbb{U}) \tag{2.5}$$

which implies that $f(z) \in \mathcal{B}(\lambda, \mu, \alpha)$, completing the proof of Theorem 1.

Remark 1. In view of the relationships that $\mathcal{B}(0, 2, \alpha) = \mathcal{S}^*(\alpha)$, $\mathcal{B}(1, 2, \alpha) = \mathcal{K}(\alpha)$ and $\mathcal{B}(0, 1, \alpha) = \mathcal{P}(\alpha)$, Theorem 1 would easily lead to the results giving sufficient conditions for the function $f(z)$ defined by (1.1) to belong, respectively, to the subclasses $\mathcal{S}^*(\alpha)$, $\mathcal{K}(\alpha)$ and $\mathcal{P}(\alpha)$. The special cases corresponding to the subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ are also identifiable with the results due to Irmak *et al.* [3, p. 364]. Also, we note that by setting $\lambda = 0$ and $\mu = 3$, Theorem 1 corresponds to the result of Frasin and Darus [2, p. 307, Theorem 2.4].

Next we prove

Theorem 2. *If* $f(z) \in \mathcal{A}$ *satisfies the following inequality:*

$$\begin{aligned} &\Re \left\{ \frac{z^{\mu-2} \mathcal{D}^{\lambda+1} f(z)}{(\mathcal{D}^{\lambda} f(z))^{\mu-1}} \left[\alpha \frac{z^{\mu-2} \mathcal{D}^{\lambda+1} f(z)}{(\mathcal{D}^{\lambda} f(z))^{\mu-1}} + \alpha \frac{\mathcal{D}^{\lambda+2} f(z)}{\mathcal{D}^{\lambda+1} f(z)} \right.\right. \\ &\left.\left. -\alpha(\mu-1) \left(\frac{\mathcal{D}^{\lambda+1} f(z)}{\mathcal{D}^{\lambda} f(z)} - 1 \right) + (1 - 2\alpha) \right] \right\} > \alpha\beta \left(\beta - \frac{1}{2} \right) - \frac{\alpha}{2} + \beta \\ &\qquad\qquad (z \in \mathbb{U};\ \mu \geqq 0;\ \lambda \in \mathbb{N}_0;\ \alpha \geqq 0;\ 0 \leqq \beta < 1), \end{aligned} \tag{2.6}$$

then $f(z) \in \mathcal{B}(\lambda, \mu, \beta)$.

Proof. Define a function $q(z)$ by

$$\frac{z^{\mu-2} D^{\lambda+1} f(z)}{(D^{\lambda} f(z))^{\mu-1}} = \beta + (1-\beta) q(z), \tag{2.7}$$

then $q(z)$ is of the form $q(z) = 1 + q_1 z + q_2 z^2 + ...$ and is analytic in $\mathbb{U}$. Differentiating both sides of (2.7) with respect to z, we get

$$\frac{\mathcal{D}^{\lambda+2} f(z)}{\mathcal{D}^{\lambda+1} f(z)} - (\mu - 1) \left(\frac{\mathcal{D}^{\lambda+1} f(z)}{\mathcal{D}^{\lambda} f(z)} - 1 \right) - 1 = \frac{(1-\beta) z q'(z)}{\beta + (1-\beta) q(z)}. \tag{2.8}$$

Using (2.7) and (2.8), we obtain

$$\begin{aligned}
&\frac{z^{\mu-2}\mathcal{D}^{\lambda+1}f(z)}{(\mathcal{D}^{\lambda}f(z))^{\mu-1}}\left[\alpha\frac{z^{\mu-2}\mathcal{D}^{\lambda+1}f(z)}{(\mathcal{D}^{\lambda}f(z))^{\mu-1}}+\alpha\frac{\mathcal{D}^{\lambda+2}f(z)}{\mathcal{D}^{\lambda+1}f(z)}-\alpha(\mu-1)\left(\frac{\mathcal{D}^{\lambda+1}f(z)}{\mathcal{D}^{\lambda}f(z)}-1\right)+(1-2\alpha)\right]\\
&=\alpha(1-\beta)zq'(z)+\alpha[\beta+(1-\beta)q(z)]^2+(1-\alpha)[\beta+(1-\beta)q(z)]\\
&=\alpha(1-\beta)zq'(z)+\alpha(1-\beta)^2q^2(z)+(1-\beta)(1+2\alpha\beta-\alpha)q(z)+\beta(\alpha\beta+1-\alpha)\\
&=\phi(q(z),zq'(z);z),
\end{aligned}$$

where

$$\phi(r,s;t)=\alpha(1-\beta)s+\alpha(1-\beta)^2r^2+(1-\beta)(1+2\alpha\beta-\alpha)r+\beta(\alpha\beta+1-\alpha). \tag{2.9}$$

For all real values of x and y satisfying $y \leqq -(1+x^2)/2$, we infer that

$$\begin{aligned}
\Re(\phi(ix,y;z)) &= \alpha(1-\beta)y-\alpha(1-\beta)^2x^2+\beta(\alpha\beta+1-\alpha)\\
&\leqq -\frac{\alpha}{2}(1-\beta)-\left[\frac{\alpha}{2}(1-\beta)+\alpha(1-\beta)^2\right]x^2+\beta(\alpha\beta+1-\alpha)\\
&\leqq \beta(\alpha\beta+1-\alpha)-\frac{\alpha}{2}(1-\beta).
\end{aligned}$$

Let $\Omega=\left\{w:\ \Re(w)>\alpha\beta\left(\beta-\frac{1}{2}\right)-\frac{\alpha}{2}+\beta\right\}$, then $\phi(q(z),zq'(z);z)\in\Omega$ and $\phi(ix,y;z)\notin\Omega$ for all real x and $y\leqq-(1+x^2)/2$, $z\in\mathbb{U}$. Applying Lemma 2 we conclude that $\Re(q(z))>0$, which in view of (1.5) and (2.7) implies that $f(z)\in\mathcal{B}(\lambda,\mu,\beta)$.

If we set $\mu=3$ and $\lambda=0$, then Theorem 2 gives the following result.

Corollary 1. *If $f(z)\in\mathcal{A}$ satisfies the inequality*

$$\begin{aligned}
&\Re\left\{\frac{z^2f(z)}{(f(z))^2}\left[\alpha\left(\frac{z^2f(z)}{(f(z))^2}+\frac{zf''(z)}{f'(z)}-\frac{2zf'(z)}{f(z)}+1\right)+1\right]\right\}\\
&\qquad>\alpha\beta\left(\beta-\frac{1}{2}\right)-\frac{\alpha}{2}+\beta \qquad (z\in\mathbb{U};\ \alpha\geqq0;\ 0\leqq\beta<1),
\end{aligned} \tag{2.10}$$

then $f(z)\in\mathcal{B}(\beta)$.

Remark 2. Upon making similar parametric substitutions as pointed out in Remark 1 above, interesting sufficient conditions can be obtained for the subclasses $\mathcal{S}^*(\beta)$, $\mathcal{K}(\beta)$ and $\mathcal{P}(\beta)$ which would evidently include a known result involving the subclass $\mathcal{S}^*$ of starlike functions due to Li and Owa [5] (see also [9, p.3, Corollary 2.2]). Derivation of these special cases being straightforward, we skip mentioning of these results.

3. Argument properties

Making use of Lemma 3, we prove

Theorem 3. *Let*

$$\frac{z^{\mu-2}\mathcal{D}^{\lambda+1}f(z)}{(D^{\lambda}f(z))^{\mu-1}} \neq \beta \qquad (z \in \mathbb{U};\ \mu \geqq 0;\ \lambda \in \mathbb{N}_0;\ 0 \leqq \beta < 1).$$

If $f(z) \in \mathcal{A}$ *satisfies the following inequality:*

$$\begin{aligned} &-\frac{\pi\alpha_1}{2} - tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(1-\beta)}{2\gamma}\right) \\ (3.1) \quad &< arg\left\{\frac{z^{\mu-2}D^{\lambda+1}f(z)}{(D^{\lambda}f(z))^{\mu-1}}\left[\frac{D^{\lambda+2}f(z)}{D^{\lambda+1}f(z)} - (\mu-1)\left(\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)} - 1\right) + \frac{\gamma+\beta-1}{1-\beta}\right] - \frac{\gamma\beta}{1-\beta}\right\} \\ &< \frac{\pi\alpha_2}{2} + tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(1-\beta)}{2\gamma}\right) \qquad (0 < \alpha_1;\ \alpha_2 \leqq 1;\ \gamma > 0), \end{aligned}$$

then

$$(3.2) \qquad -\frac{\pi\alpha_1}{2} < arg\left\{\frac{z^{\mu-2}D^{\lambda+1}f(z)}{(D^{\lambda}f(z))^{\mu-1}} - \beta\right\} < \frac{\pi\alpha_2}{2}.$$

Proof. Let $q(z)$ be the same function as defined in (2.7), then since $q(z)$ is analytic in the open unit disk $\mathbb{U}$ with $q(0) = 1$, it follows from the hypothesis of Theorem 3 that $q(z) \neq 0$. Following (2.8), we obtain

$$(3.3) \qquad \begin{aligned} &\frac{z^{\mu-2}D^{\lambda+1}f(z)}{(D^{\lambda}f(z))^{\mu-1}}\left[\frac{D^{\lambda+2}f(z)}{D^{\lambda+1}f(z)} - (\mu-1)\left(\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)} - 1\right)\right. \\ &\left. + \frac{\gamma+\beta-1}{1-\beta}\right] - \frac{\gamma\beta}{1-\beta} = (1-\beta)zq'(z) + \gamma q(z). \end{aligned}$$

Suppose now that there exists two points $z_1,\ z_2 \in \mathbb{U}$ such that the conditions (1.8) are satisfied. Applying (1.9) and (1.10) of Lemma 3, we get

$$\begin{aligned} &\arg\big(\gamma q(z_1) + (1-\beta)z_1q'(z)\big) = \arg(q(z_1)) + \arg\left(\gamma + (1-\beta)\frac{z_1q'(z_1)}{q(z_1)}\right) \\ &= -\frac{\pi\alpha_1}{2} + \arg\left(\gamma - i\frac{(\alpha_1+\alpha_2)(1-\beta)}{2}\,m\right) = -\frac{\pi\alpha_1}{2} - \tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-\beta)}{2\gamma}\,m\right) \\ &\leqq -\frac{\pi\alpha_1}{2} - \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(1-\beta)}{2\gamma}\,m\right), \end{aligned}$$

which, by virtue of (3.3), contradicts the assumption stated in (3.1). Similarly, we can show that

$$\arg\big(\gamma q(z_2) + (1-\beta)z_2q'(z_2)\big) \geqq \frac{\pi\alpha_1}{2} + \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(1-\beta)}{2\gamma}\,m\right),$$

which again contradicts the assumption mentioned in (3.1). Hence the function $q(z)$ defined by (2.7) satisfies the inequality

$$-\frac{\pi\alpha_1}{2} < \arg(q(z)) < \frac{\pi\alpha_2}{2},$$

which implies that

$$-\frac{\pi\alpha_1}{2} < \arg\left\{\frac{z^{\mu-2}D^{\lambda+1}f(z)}{(D^{\lambda}f(z))^{\mu-1}} - \beta\right\} < \frac{\pi\alpha_2}{2} \qquad (z \in \mathbb{U}).$$

This completes the proof of Theorem 3.

If we set $\alpha_1 = \alpha_2 = \alpha, \mu = 2$ and $\lambda = \beta = 0$ in Theorem 3, we get Corollary 2 below.

Corollary 2. *Let*

$$\frac{zf'(z)}{f(z)} \neq 0 \qquad (z \in \mathbb{U}).$$

If $f(z) \in \mathcal{A}$ satisfies the inequality

$$\left|arg\left\{\frac{zf'(z)}{f(z)}\left(1+\gamma+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right\}\right| < \frac{\pi\alpha}{2} + tan^{-1}\frac{\alpha}{\gamma} \tag{3.4}$$

$$(0 < \alpha \leqq 1;\ \gamma > 0),$$

then $f(z) \in S_s^*(\alpha)$.

Also, if we put $\alpha_1 = \alpha_2 = \alpha, \mu = 2, \lambda = 1$ and $\beta = 0$ in Theorem 3, we get

Corollary 3. *Let*

$$\left(1+\frac{zf''(z)}{f'(z)}\right) \neq 0 \qquad (z \in \mathbb{U}).$$

If $f(z) \in \mathcal{A}$ satisfies the inequality

$$\begin{aligned}\left|arg\left\{\left(1+\frac{zf''(z)}{f'(z)}\right)\left(\frac{z(zf'''(z)+2f''(z))}{zf''(z)+f'(z)}-\frac{zf''(z)}{f'(z)}+\gamma\right)\right\}\right| \\ < \frac{\pi\alpha}{2} + tan^{-1}\frac{\alpha}{\gamma} \qquad (0 < \alpha \leqq 1;\ \gamma > 0),\end{aligned} \tag{3.5}$$

then $f(z) \in \mathcal{K}_c^*(\alpha)$.

Lastly, by choosing $\alpha_1 = \alpha_2 = 1, \mu = 1$ and $\lambda = 0$ in Theorem 3, we obtain

Corollary 4. *Let*

$$f'(z) \neq \beta \qquad (z \in \mathbb{U};\ 0 \leqq \beta < 1).$$

If $f(z) \in \mathcal{A}$ satisfies the inequality

$$\left|arg\left\{f'(z)\left(\frac{zf''(z)}{f'(z)}+\frac{\gamma}{1-\beta}\right)-\frac{\gamma\beta}{1-\beta}\right\}\right| < \frac{\pi}{2} + tan^{-1}\left(\frac{1-\beta}{\gamma}\right) \quad (\gamma > 0), \tag{3.6}$$

then $f(z) \in \mathcal{P}(\beta)$.

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FUZZY STABILITY OF QUARTIC MAPPINGS[1]

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Abstract. We establish some stability results concerning the quartic functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y)$$

in the setting of fuzzy normed spaces that in turn generalize a Hyers–Ulam stability result in the framework of classical normed spaces.

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1. Introduction and preliminaries

In 1984, Katrasas [11] defined a fuzzy norm on a linear space at the same year Wu and Fang [27] introduced fuzzy normed space and gave the generalization of the Kolmogoroff normalized theorem for fuzzy topological linear spaces. Later, some mathematicians have defined fuzzy norms on a linear space from various points of view [7], [14], [26]. In 1994, Cheng and Mordeson introduced a definition of fuzzy norm on a linear space in such a manner that the corresponding induced fuzzy metric is of Kramosil and Michalek type [13]. In 2003, Bag and Samanta [2] modified the definition of Cheng and Mordeson [5] by removing a regular condition. They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy norms (see [3]). Following [2], we give the following notion of a fuzzy norm.

Let X be a real linear space. A function $N\colon X\times\mathbb{R}\to[0,1]$ (the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x,y\in X$ and all $s,t\in\mathbb{R}$,

(N1) $N(x,c)=0$ for $c\le 0$;

(N2) $x=0$ if and only if $N(x,c)=1$ for all $c>0$;

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(N3) $N(cx,t) = N\left(x, \dfrac{t}{|c|}\right)$ if $c \neq 0$;

(N4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;

(N5) $N(x,.)$ is a non-decreasing function on $\mathbb{R}$ and $\lim\limits_{t\to\infty} N(x,t) = 1$.

(N6) For $x \neq 0$, $N(x,.)$ is (upper semi)continuous on $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed linear space. One may regard $N(x,t)$ as the truth value of the statement 'the norm of x is less than or equal to the real number t'.

Example 1.1. Let $(X, \|.\|)$ be a normed linear space. Then

$$N(x,t) = \begin{cases} \dfrac{t}{t+\|x\|} & t > 0, x \in X \\ 0 & t \leq 0, x \in X \end{cases}$$

is a fuzzy norm on X.

Example 1.2. Let $(X, \|.\|)$ be a normed linear space. Then

$$N(x,t) = \begin{cases} 0 & t \leq 0 \\ \dfrac{t}{\|x\|} & 0 < t \leq \|x\| \\ 1 & t > \|x\| \end{cases}$$

is a fuzzy norm on X.

A sequence $\{x_n\}$ in a fuzzy normed linear space (X, N) is said to be convergent if there exists $x \in X$ such that $\lim\limits_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the fuzzy limit of the sequence $\{x_n\}$ and we denote it by $\mathrm{N} - \lim x_n = x$. A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$. It is known that every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be fuzzy complete and the fuzzy normed space is called a fuzzy Banach space.

The concept of stability of a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. In 1940 S.M. Ulam [25] posed the first stability problem. In the next year, D.H. Hyers [8] gave an affirmative answer to the question of Ulam. Hyers' theorem was generalized by T. Aoki [1] for additive mappings and by Th.M. Rassias [23] for linear mappings by considering an unbounded Cauchy difference. The concept of the Hyers–Ulam–Rassias stability was originated from Th.M. Rassias' paper [23] for the stability of the linear mappings and its importance in the proof of further results in functional equations. During the last decades several stability problems for various functional equations have been investigated by many mathematicians; we refer the reader to [6], [9], [10], [24] and references therein.

The functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y) \tag{1.1}$$

is called the *quartic functional equation*, since the function $f(x)=x^4$ is a solution of the functional equation. Note that f is called quartic because of the identity

$$(2x+y)^4+(2x-y)^4=4(x+y)^4+4(x-y)^4+24x^4-6y^4.$$

Every solution of the quartic functional equation is said to be a *quartic mapping*. In [15] it is proved that a function $f : X \to Y$ between real normed spaces is quartic if and only if there exists a symmetric biquadratic function $F : X \times X \to Y$ such that $f(x) = F(x,x)$ for all $x \in X$. The first result on the stability of the quartic functional equation was obtained by J.M. Rassias [22]. Also L. Cădariu [4], H.-M. Kim [12], S.H. Lee, S.M. Im and I.S. Hwang [15], Najati [20] and C. Park [21] investigated the stability of quartic functional equation.

The first result on fuzzy stability of functional equations was given by the present author and M.S. Moslehian [17]. Later, several various fuzzy version of stability concerning Jensen, cubic and quadratic functional equations were investigated [16], [18], [19]. In the next section we prove the Hyers–Ulam–Rassias stability of the quartic functional equation (1.1) in the setting of fuzzy normed spaces that in turn generalize Hyers–Ulam stability results ([15, Theorem 3.1] and [22]) in the framework of classical normed spaces.

2. Stability of quartic mappings in the fuzzy setting

Let

$$Df(x,y)=f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y) \tag{1.1}$$

The central theorem is a fuzzy generalized Hyers–Ulam–Rassias type theorem for the quartic functional equation (1.1).

Theorem 1.3. *Let X be a linear space and let (Z,N') be a fuzzy normed space and $\varphi : X \times X \to Z$ be a function. Let (Y,N) be a fuzzy Banach space and let $f : X \to Y$ be a φ-approximately quartic mapping in the sense that*

$$N(Df(x,y),t) \geq N'(\varphi(x,y),t). \tag{1.2}$$

If for some $\alpha < 16$,

$$N'(\varphi(2x,0),t) \geq N'(\alpha\varphi(x,0),t) \tag{1.3}$$

$f(0)=0$ and $\lim_{n\to\infty} N'(2^{-4n}\varphi(2^n x, 2^n y),t) = 1$ for all x,y in X and $t>0$, then there exists a unique quartic mapping $Q : X \to Y$ such that

$$N\left(Q(x)-f(x),t\right) \geq N'\left(\varphi(x,0),2(16-\alpha)t\right) \tag{1.4}$$

Proof. Let $y = 0$ in (1.2), then we have

$$N(2f(2x) - 2^5 f(x), t) \geq N'(\varphi(x,0), t). \tag{1.5}$$

Replacing x by $2^{k-1}x$ in (1.5) and using (1.3), we obtain

$$\begin{aligned} N\left(\frac{f(2^k x)}{2^{4k}} - \frac{f(2^{k-1}x)}{2^{4(k-1)}}, t\right) &\geq N'\left(\frac{\varphi(2^{k-1}x, 0)}{2^{4k+1}}, t\right) \\ &\geq N'\left(\frac{\alpha^{k-1}}{2^{4k+1}}\varphi(x,0), t\right) \\ &= N'(\varphi(x,0), 2^{4k+1}t/\alpha^{k-1}). \end{aligned} \tag{1.6}$$

Substituting t by $\frac{\alpha^{k-1}t}{2^{4k+1}}$ in (1.6), we get

$$N\left(\frac{f(2^k x)}{2^{4k}} - \frac{f(2^{k-1}x)}{2^{4(k-1)}}, \frac{\alpha^{k-1}t}{2^{4k+1}}\right) \geq N'(\varphi(x,0), t).$$

This together with

$$2^{-4n}f(2^n x) - 2^{-4m}f(2^m x) = \sum_{k=m+1}^{n} \left(2^{-4k}f(2^k x) - 2^{-4(k-1)}f(2^{k-1}x)\right) \qquad (n > m)$$

yields

$$N\left(2^{-4n}f(2^n x) - 2^{-4m}f(2^m x), \sum_{k=m+1}^{n} \frac{\alpha^{k-1}t}{2^{4k+1}}\right) \geq N'(\varphi(x,0), t) \tag{1.7}$$

By replacing t by $t/\left(\sum_{k=m+1}^{n} \frac{\alpha^{k-1}}{2^{4k+1}}\right)$ in (1.7), we observe that

$$\begin{aligned} N(2^{-4n}f(2^n x) - 2^{-4m}f(2^m x), t) &\geq N'\left(\varphi(x,0), t/\Big(\sum_{k=m+1}^{n} \frac{\alpha^{k-1}}{2^{4k+1}}\Big)\right) \\ &= N'\left(\varphi(x,0), 32t/\Big(\sum_{k=m+1}^{n} (\frac{\alpha}{16})^{k-1}\Big)\right) \end{aligned} \tag{1.8}$$

Now the Cauchy criterion for convergence and (N5) show that $\{f(2^n x)/2^{4n}\}$ is a Cauchy sequence in (Y, N), since $\sum_{n=1}^{\infty} \left(\frac{\alpha}{16}\right)^n < \infty$. Due to the assumption that (Y, N) is a fuzzy Banach space, the above sequence converges to some point of Y. Put

$$Q(x) := \mathrm{N} - \lim_{n\to\infty} f(2^n x)/2^{4n} \qquad (x \in X).$$

Set $m = 0$ in (1.8) and use the notion of fuzzy limit to obtain

$$N(2^{-4n}f(2^n x) - f(x), t) \geq N'(\varphi(x,0), 2(16-\alpha)t).$$

Therefore for each $\varepsilon > 0$ and large enough n

$$\begin{aligned} N\left(Q(x)-f(x), t+\varepsilon\right) &\geq \min\{N\left(Q(x)-f(2^n x)/2^{4n}, \varepsilon\right), N\left(f(2^n x)/2^{4n}-f(x), t\right)\} \\ &\geq N'\left(\varphi(x,0), 2(16-\alpha)t\right). \end{aligned}$$

By (N6),

$$N\left(Q(x)-f(x), t\right) \geq N'(\varphi(x,0), 2(16-\alpha)t).$$

Replace x, y by $2^n x, 2^n y$ respectively in (1.2) to get

$$N(Df(2^n x, 2^n y)/2^{4n}, t) \geq N'(2^{-4n}\varphi(2^x, 2^n y), t).$$

By our assumption $\lim_{n\to\infty} N'(2^{-4n}\varphi(2^n x, 2^n y), t) = 1$, it follows that Q satisfies formula (1.1).

To prove the uniqueness, let us assume that there exists a quartic function $S : X \to Y$ which satisfies (1.4). Fix $x \in X$. Clearly $Q(2^n x) = 2^{4n}Q(x)$ and $S(2^n x) = 2^{4n}S(x)$ for all $n \in \mathbb{N}$. We have

$$\begin{aligned} N(Q(x)-S(x), t) &= N(2^{-4n}Q(2^n x) - 2^{-4n}S(2^n x), t) \\ &\geq \min\Big\{N\Big(2^{-4n}(Q(2^n x) - f(2^n x)), t/2\Big), \\ &\qquad N\Big(2^{-4n}(S(2^n x) - f(2^n x)), t/2\Big)\Big\} \\ &\geq N'(\varphi(2^n x, 0), 2^{4n}t(16-\alpha)) \\ &\geq N'\left(\varphi(x,0), \frac{t(16-\alpha)16^n}{\alpha^n}\right) \end{aligned}$$

Since $\lim_{n\to\infty} \frac{t(16-\alpha)16^n}{\alpha^n} = \infty$, the last term in the above inequality tends to 1 as $n \to \infty$. Hence $S = Q$. ∎

3. Applications

This section includes three applications of our main result. These are indeed generalizations of known results in [15] to the framework of fuzzy normed spaces.

Corollary 1.4. *Let X be a normed space, (Y, N) be a fuzzy Banach space, (Z, N') be a fuzzy normed space, p, q be nonnegative real numbers and let $z_0 \in Z$. Suppose that $f : X \to Y$ is a (p, q, z_0)-approximately quartic mapping in the sense that*

$$N(Df(x,y), t) \geq N'((\|x\|^p + \|y\|^q)z_0, t) \qquad (x, y \in X).$$

If $f(0) = 0$ and $p, q < 4$, then there exists a unique quartic mapping $Q : X \to Y$ such that

$$N(Q(x) - f(x), t) \geq N'(\|x\|^p z_0, 2t(16 - 2^p)),$$

for all $x \in X$ and all $t > 0$.

Proof. Let $\varphi : X \times X \to Z$ be defined by $\varphi(x,y) = (\|x\|^p + \|y\|^q)z_0$. Then the Corollary is followed from Theorem 1.3 by $\alpha = 2^p$. ■

Remark 1.5. A similar result to Corollary 1.4, where $p, q > 4$ can be formulated. For this one needs to state a similar result to Theorem 1.3, in which one deals with the sequence $\{2^{4n} f(2^{-n}x)\}$ and appropriate conditions on the control function φ.

Corollary 1.6. *Let X be a linear space, (Y, N) be a fuzzy Banach space, (Z, N') be a fuzzy normed space, $z_0 \in Z$ and $\varepsilon > 0$. Suppose that $f : X \to Y$ is an (ε, z_0)-approximately quartic mapping in the sense that*

$$N(Df(x,y),t) \geq N'(\varepsilon z_0, t)$$

for all $x, y \in X$. Then there exists a unique quartic mapping $Q : X \to Y$ such that

$$N(Q(x) - f(x), t) \geq N'(\varepsilon z_0, 30t),$$

for all $x \in X$ and all $t > 0$.

Proof. The result is deduced from Theorem 1.3, by considering $\varphi : X \times X \to Z$ to be $\varphi(x,y) = \varepsilon z_0$. ■

Corollary 1.7. ([15]) *Let X be a linear space and Y be a Banach space. If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies*

$$||Df(x,y)|| \leq \delta \qquad (x, y \in X),$$

then there exists a unique quartic function $Q : X \to Y$ such that

$$||f(x) - Q(x)|| \leq \delta/30.$$

Proof. Consider the induced fuzzy norms N and N' on Y and $\mathbb{R}$, respectively, defined as in Example 1.1. Now apply Theorem 1.3 for $\varphi(x,y) = \delta$ and $\alpha = 1$. ■

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SOME RESULTS ON NON-COMMUTING GRAPH OF A FINITE GROUP

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Abstract. Let G be a finite non-abelian group. We define a graph Γ_G, called the non-commuting graph of G, with vertex set $G - Z(G)$ such that two vertices x and y adjacent if and only if $xy \neq yx$. In this paper some results on the number of edges of Γ_G and also its chromatic number are obtained in general. For some special group G we will prove that if H is a group such that $\Gamma_G \cong \Gamma_H$, then $|G| = |H|$ and in some cases $G \cong H$.

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1. Introduction

Let G be a group. There are several ways to associate a graph to G. The one we will consider in this paper is denoted by Γ_G and is called the non-commuting graph of G. The vertex set of Γ_G is $V(\Gamma_G) = G - Z(G)$, where $Z(G)$ is the center of G and the edge set $E(\Gamma_G)$ contains (x, y) as an edge if and only if $xy \neq yx$. Since we consider simple graphs, hence (x, y) and (y, x) are the same edge and there is no edge of the shape (x, x) in $E(\Gamma_G)$. It is clear that if G is abelian, then Γ_G is the null graph, hence in what follows we will assume that G is a non-abelian group.

According to [8] the non-commuting graph of a finite group G was first introduced by Paul Erdös in connection with the following problem: let G be a group whose non-commuting graph Γ_G has no infinite complete subgraphs. Is it true that there is a finite bound on the cardinalities of complete subgraphs of Γ_G? By [8] the answer to this question is positive and this was the origin of many similar questions and research.

In [1], relation between some graph theoretical properties of Γ_G and the group theory properties of the group G are studied. In particular the following two conjectures are raised:

Conjecture 1 *Let G be a finite non-abelian group. If there is a group such that $\Gamma_G \cong \Gamma_H$, then $|G| = |H|$.*

Conjecture 2 *Let S be a finite non-abelian simple group. Let G be a group such that $\Gamma_G \cong \Gamma_S$, then $G \cong S$.*

Our aim in this paper is to verify the above conjectures for some classes of finite groups. We also obtain some results about the number of edges of the graph Γ_G. Our notation for graphs is standard and [2] is a general reference.

2. Some results on the number of edges

Let G be a finite non-abelian group. The number of conjugacy classes of G is denoted by $k(G)$. By Lemma 3.27 in [1], the number of edges in Γ_G is

$$|E(\Gamma_G)| = \frac{1}{2}\,|G|\,(|G| - k(G)).$$

Proposition 1 *Let G be a non-abelian finite group and let Γ_G be its non-commuting graph. Then $|E(\Gamma_G)| \geqslant \frac{3}{10}k(G)\,|G|$. Moreover equality holds if and only if G is the direct product of an abelian group and a 2-group H such that $|H| \geqslant 8$ and H is indecomposable with $\frac{k(H)}{|H|} = \frac{5}{8}$.*

Proof. On the contrary assume that $|E(\Gamma_G)| < \frac{3}{10}k(G)\,|G|$. Then by substituting $|E(\Gamma_G)| = \frac{1}{2}\,|G|\,(|G| - k(G))$ we will obtain $|G| < \frac{8}{5}k(G)$. Therefore $\frac{k(G)}{|G|} > \frac{5}{8}$. Now, by [4], the probability of commuting two randomly chosen elements of a finite group G is equal to $\frac{k(G)}{|G|}$, and $\frac{k(G)}{|G|} > \frac{5}{8}$ implies that G is abelian, a contradiction. Therefore, $|E(\Gamma_G)| \geqslant \frac{3}{10}k(G)\,|G|$. Equality holds if and only if $\frac{k(G)}{|G|} = \frac{5}{8}$, which by [4] the conclusion follows. ∎

Proposition 2 *For any finite non-abelian group G we have $|E(\Gamma_G)| \neq 2\,|G|$.*

Proof. Assume $|E(\Gamma_G)| = 2\,|G|$. Then, by substituting the value of $E(\Gamma_G)$ we will obtain $|G| = k(G) + 4$. But by Proposition 1 we have $|E(\Gamma_G)| \geqslant \frac{3}{10}k(G)\,|G|$, hence $k(G) \leqslant \frac{20}{3} < 7$. Therefore, $k(G) = 1, 2, \ldots, 6$. Now, from $|G| = k(G) + 4$, we obtain $(k(G), |G|) = (1, 5), (2, 6), (3, 7), (4, 8), (5, 9), (6, 10)$. Next a routine examination of the group orders eliminates all the above possibilities. ■

Lemma 1 *Let G be a finite non-abelian group and n be a natural number. If $|G| > \frac{16n}{3}$, then $|E(\Gamma_G)| > n\,|G|$.*

Proof. By Proposition 1 we have $|E(\Gamma_G)| \geqslant \frac{3}{10}k(G)\,|G|$. If $|E(\Gamma_G)| \leqslant n\,|G|$, then $k(G) \leqslant \frac{10n}{3}$. But $|E(\Gamma_G)| = \frac{|G|\,(|G| - k(G))}{2} \leqslant n\,|G|$ implies $|G| \leqslant 2n + k(G)$. Hence $|G| \leqslant 2n + k(G) \leqslant 2n + \frac{10n}{3} = \frac{16n}{3}$ a contradiction. ■

Corollary 1 *Let G be a non-abelian finite group. If $|E(\Gamma_G)| \leqslant 31\,|G|$ and G is simple, then $G \cong \mathbb{A}_5$.*

Proof. If $|E(\Gamma_G)| \leqslant 31\,|G|$, then, by Lemma 1, we obtain $|G| \leqslant 166$. Since G is assumed to be simple, hence $G \cong \mathbb{A}_5$. ■

Theorem 1 *Let G be a finite non-abelian group. Then $|E(\Gamma_G)| \geqslant \frac{3}{2}\,|G|$. Moreover equality holds if and only if $G \cong S_3, D_8$ or Q_8.*

Proof. Assume $|E(\Gamma_G)| < \frac{3}{2}\,|G|$. Substituting the value of $E(\Gamma_G)$ we will obtain $|G| - k(G) < 3$. But $|G| - k(G) \geqslant 1$ since G is non-abelian, hence$|G| - k(G) = 1$ or 2. By [4] for any non-abelian finite group we have $\frac{k(G)}{|G|} \leqslant \frac{5}{8}$. Now combining $|G| - k(G) = 1$ or 2 with the last inequality we will obtain $|G| \leqslant 2$ or 5 respectively; and in both cases G will be non-abelian, a contradiction.

If $|E(\Gamma_G)| = \frac{3}{2}G$, then $|G| - k(G) = 3$. Using $\frac{k(G)}{|G|} \leqslant \frac{5}{8}$ we obtain $|G| \leqslant 8$. Now, examination of non-abelian groups of order less than or equal to 8 yields $G \cong S_3, D_8$ or Q_8. ■

3. Partition of non-commuting graph

Let $r \geq 2$ be an integer. A graph $G = (V, E)$ is called r-partite if V admits a partition into r classes that every edge has its ends in different classes and vertices in the same partition class must not be adjacent. A 2-partite graph is said be bipartite. Any graph $G(V, E)$ is a $|V(G)|$-partite. In this position every vertex is a class of partition. this partition is said to be the trivial partition.

In the following by a finite group G we mean a non-abelian finite group.

Proposition 3 *Let G be a group, then Γ_G is not bipartite.*

Proof. Assume that Γ_G is bipartite and V_1 and V_2 are classes of this partition. There are $x_1 \in V_1$ and $x_2 \in V_2$ such that $x_1x_2 \neq x_2x_1$, so we have $G \neq C_G(x_1) \cup C_G(x_2)$. Then there is a $y \in G - Z(G)$ such that $y \notin C_G(x_1) \cup C_G(x_2)$. So y is adjacent to both x_1 and x_2. So $y \notin V_1 \cup V_2 = V(\Gamma_G)$. Therefore $y \in Z(G)$ and it is a contradiction.Thus G is not bipartite. ∎

Proposition 4 *Let G be a group. Then Γ_G is not a complete graph.*

Proof. Assume that Γ_G is a complete graph. In this case first we will prove that any non-central element of G has order 2. If $x \in G - Z(G)$, then $d(x) = |G| - |C_G(x)|$ and by the above discussion we have $d(x) = |G| - |Z(G)| - 1$. So $|G| - |C_G(x)| = |G| - |Z(G)| - 1$ and then $|C_G(x)| = |Z(G)| + 1$. But we know $|Z(G)|\Big|\,|C_G(x)|$, here $|Z(G)|\Big|\,|Z(G)| + 1$ and then $|Z(G)| = 1$. Therefore $|C_G(x)| = 2$ and the order of x is 2. If $y \in G - Z(G)$ and $y \neq x$, then $yx \in G - Z(G) = G - \{1_G\}$ and by above discussion y is adjacent to x. So $o(yx) = 2$ and we have $yxyx = 1$. This implies that $yx = xy$ and this means that $y \in C_G(x)$ and it contradicts the fact that x is adjacent to y. So Γ_G is not a complete graph and the proof is complete. ∎

4. Verification of the conjectures for certain groups

For certain non-abelian finite group G we want to show that if H is a group such that $\Gamma_G \cong \Gamma_H$, then $|G| = |H|$, and in the case that G is a non-abelian simple group, then $G \cong H$. Note that $\Gamma_G \cong \Gamma_H$ is a graph isomorphism, that is a one-to-one correspondence $\varphi : G - Z(G) \longrightarrow H - Z(H)$ such that φ preserves edges, i.e. if $x, y \in G - Z(G)$, and $xy \neq yx$, then $\varphi(x)\varphi(y) \neq \varphi(y)\varphi(x)$. Equivalently if we consider the complimentary graph of Γ_G we have the following condition: $x, y \in G - Z(G), xy = yx \Longrightarrow \varphi(x)\varphi(y) = \varphi(y)\varphi(x)$

The isomorphism $\Gamma_G \cong \Gamma_H$ implies that $|G - Z(G)| = |H - Z(H)|$. Since G is assumed to be non-abelian, hence $0 \neq |G - Z(G)| = |H - Z(H)|$, implying that H is non-abelian. We also have $|Z(H)| \leqslant |H - Z(H)|$, so $Z(H)$ is a finite group. Therefore H is a finite non-abelian group.

The degree of a vertex v in graph Γ is defined to be the number of edges adjacent to v, and is denoted by $d(v)$. Now it is easy to see that the degree of a vertex g in the graph Γ_G is equal to $d(g) = |G| - |C_G(g)|$. Our first result concerns the degree properties of the commuting graph of G.

Proposition 5 *Let G be a finite non-abelian group such that there is an element $g \in G - Z(G)$ with $d(g) = p^n$, $n \in \mathbb{N}$, p a prime number. If H is a group and $\Gamma_G \cong \Gamma_H$, then $|G| = |H|$.*

Proof. We know $d(g) = |G| - |C_G(g)| = |C_G(g)|\,([G : C_G(g)] - 1) = p^n$. Therefore $|C_G(g)| = p^m$ for some m, $1 \leqslant m \leqslant n$. Hence we obtain $|G| = p^n + p^m$. Let

$g' \in H - Z(H)$ be the corresponding vertex in Γ_H under the isomorphism $\Gamma_G \cong \Gamma_H$. Therefore $d(g') = p^n$ and similarly we will obtain $|C_H(g)| = p^{m'}$, $1 \leqslant m' \leqslant n$, and finally $|H| = p^n + p^{m'}$.

From $Z(G) \gneqq C_G(g)$ we can put $|Z(G)| = p^l$ where $1 \leqslant l < m$. Similarly since $Z(H) \gneqq C_G(g')$ we put $|Z(H)| = p^{l'}$, $1 \leqslant l' < m'$. Now using the equality $|G| - |Z(G)| = |H| - |Z(H)|$ we obtain $p^m - p^l = p^{m'} - p^{l'}$, therefore $p^m(p^{m-l} - 1) = p^{m'}(p^{m'-l'} - 1)$. Since $p^{m-l} - 1$ and $p^{m'-l'} - 1$ are non-zero and relatively prime to p we deduce $m = m'$, whence $|G| = |H|$ and the Proposition is proved. ■

Proposition 6 *Let G be a finite non-abelian group such that there is an element $g \in G - Z(G)$ with $d(g) = pq$, where p and q are prime numbers. If H is a group and $\Gamma_G \cong \Gamma_H$, then $|G| = |H|$.*

Proof. From $pq = |C_G(g)|\,([G : C_G(g)] - 1)$ we deduced that $|C_G(g)| = p, q$ or pq, hence $|G| = pq + p$, $pq + q$ or $2pq$. Since the corresponding element $g' \in H - Z(H)$ has also degree pq we will obtain $|H| = pq + p$, $pq + q$ or $2pq$. Therefore to prove $|G| = |H|$ it is enough to prove, for example, $|G| = pq + p$ and $|H| = 2pq$ are impossible. From $|G| = pq + p$ we obtain $|C_G(g)| = p$, hence $|Z(G)| = 1$. Therefore using the equality $|G| - |Z(G)| = |H| - |Z(H)|$ we will obtain $|Z(H)| = pq - p + 1$. we must have $|Z(H)| = pq - p + 1 \mid |H| = 2pq$ and an easy calculation shows that this is impossible. ■

Proposition 7 *Let G be a non-abelian group and Γ_G be the non-commuting graph of G and let $g \in G - Z(G)$ be an element such that $d(g) = pqr$ where p, q and r are distinct primes where $p < q < r$ and $q \nmid r - 1$ and $p \nmid r - 1$. If H is a group such that $\Gamma_G \cong \Gamma_H$, then $|G| = |H|$.*

Proof. From $d(g) = |G| - |C_G(g)|$ it follows that

$$pqr = d(g) = |C_G(g)| \left(\left| \frac{G}{C_G(g)} \right| - 1 \right)$$

and so $|C_G(g)| = p, q, r, pq, pr, qr$ or pqr. Therefore by $|G| = d(g) + |C_G(g)|$ we will obtain $|G| = pqr + p, pqr + q, pqr + r, pqr + pq, pqr + pr, pqr + qr$ or $pqr + pqr$. Because of $\Gamma_G \cong \Gamma_H$, if $h \in H - Z(H)$ is the corresponding element with g, then all of cases will happen for $|C_H(h)|$ and $|H|$. Now we consider different cases and show that $|G| = |H|$.

Case 1. $|G| = pqr + p$ and $|H| = pqr + r$. In this case $|C_G(g)| = p$ and $|C_H(h)| = r$ and then $|Z(G)| = 1 = |Z(H)|$. Now by $|G| - |Z(G)| = |H| - |Z(H)|$ we obtain $|G| = |H|$. Therefore in all cases that $|C_G(g)|$ and $|C_H(h)|$ are prime we can show $|G| = |H|$.

Case 2. $|G| = pqr + p$ and $|H| = pqr + pq$. In this case $|C_G(g)| = p$ and $|C_H(h)| = pq$ and so $|Z(G)| = 1$ and $|Z(H)| = p, q$ or 1. By $|G| - |Z(G)| = |H| - |Z(H)|$ we obtain $pqr + p - 1 = pqr + pq - |Z(H)|$, hence $|Z(H)| = pq - p + 1 = p(q - 1) + 1$. If $|Z(H)| = p$, then $p = p(q - 1) + 1$ which is a contradiction. If $|Z(H)| = q$, then $q = p(q-1)+1$. Thus $p(q-1) = q-1$ that implies $p = 1$ and this is a contradiction. So we will obtain that $|Z(H)| = 1$ and then $|Z(G)| = 1 = |Z(H|$ implying that $|G| = |H|$.

Case 3. $|G| = pqr + +p$ and $|H| = pqr + qr$. In this case $|C_G(g)| = p$ and so $|Z(G)| = 1$ and from $|G| - |Z(G)| = |H| - |Z(H)|$ we will obtain $p - 1 = qr - |Z(H)|$. Now, by $|C_H(h)| = qr$, we have $|Z(H)| = q, r$ or 1. If $|Z(H)| = q$, then $p-1 = qr-q = q(r-1)$ which contradicts $p < q$. If $|Z(H)| = r$, then by the same argument as in the case of $|Z(H)| = q$ we obtain a contradiction. Therefore $|Z(H)| = 1$ and $|G| = |H|$.

Case 4. $|G| = pqr + q$ and $|H| = pqr + pr$. Then with the same argument as in the case 3 we have $|C_G(g)| = q$ and so $|Z(G)| = 1$. Also from $|C_H(h)| = pr$ we have $|Z(H)| = p, r$ or 1. If $|Z(H)| = p$, then $q-1 = pr-p = p(r-1)$ and this contradicts $q < r$. If $|Z(H)| = r$, then $q-1 = pr-r = p(r-1)$ which contradicts $q < r$. So $|Z(H)| = 1$ hence $|G| = |H|$.

Case 5. $|G| = pqr + r$ and $|H| = pqr + pq$. In this case $|C_G(g)| = r$ and so $|Z(G)| = 1$. From $|C_H(h)| = pq$ we will obtain $|Z(H)| = p, q$ or 1. If $|Z(H)| = p$, then $r - 1 = pq - p = p(q-1)$ that implies $p|r-1$ and this is a contradiction. If $|Z(H)| = q$, then $r-1 = pq-q = q(p-1)$ that implies $q|r-1$ and this is a contradiction. Thus $|Z(H)| = 1$ and from $|Z(H)| = 1 = |Z(G)|$ we deduce that $|G| = |H|$.

Case 6. $|G| = pqr+p$ and $|H| = pqr+pqr$. In this case $|C_G(g)| = p$ and $|C_H(h)| = pqr$. So $|Z(G)| = 1$ and by $|G| - |Z(G)| = |H| - |Z(H)|$ we have $|Z(H)| = pqr-(p-1)$. Now, by Lemma 3.1 in [1], we know $|Z(H)|\Big||C_G(g)|-|Z(G)|$, hence $pqr-(p-1)|p-1$ and this is a contradiction because of $pqr-(p-1) > p-1$.

The cases $|G| = pqr + q, |H| = pqr + pqr$ or $|G| = pqr + r, |H| = pqr + pqr$ are same as the case 6 and are omitted.

Case 7. $|G| = pqr + pq$ and $|H| = pqr + pqr$. In this case $|C_G(g)| = pq$ and $|C_H(h)| = pqr$ and so $|Z(G)| = p, q$ or 1. If $|Z(G)| = p$, then from $|G| - |Z(G)| = |H| - |Z(H)|$ we obtain $|Z(H)| = pqr - pq + p$. On the other hand we know $|Z(H)|\Big||C_G(g)| - |Z(G)|$ which implies that $|Z(H)|\Big|p-1$. So $p(qr-q+1)|p-1$ and this is a contradiction. If $|Z(G)| = q$, by the same argument we obtain a contradiction and therefore $|Z(G)| = 1$. Hence $|Z(H)| = pqr - pq + 1 = pq(r-1)+1$. But from $|Z(H)|\Big||C_G(g)|-|Z(G)|$ we get $pq(r-1)+1|pq-1$ which contradicts $pq(r-1)+1 > pq-1$. So this case is impossible. Thus in all of the cases we have $|G| = |H|$ and the proof is completed. ∎

Proposition 8 *Let G be a finite non-abelian group such that there is a $g \in G - Z(G)$ in the non-commuting graph of G with degree $d(g) = p^n q$ where p, q are primes and $p > q$. If H is a group such that $\Gamma_G \cong \Gamma_H$, then $|G| = |H|$.*

Proof. We have $d(G) = p^n q = |G|-|C_G(g)| = |C_G(g)|(|\frac{G}{C_G(g)}|-1)$. So $|C_G(g)| = p^{n''}, q$ or $p^{n'}q$ such that n', n'' are non-negative integers and $n' \leq n$ and $n'' \leq n$.

Hence $|G| = p^nq + p^{n''}, p^nq + q$ or $p^nq + p^{n'}q$. If the corresponding element with g in $\Gamma_G \cong \Gamma_H$ is $h \in H - Z(H)$, then h has also degree p^nq and we deduce that $|H| = p^nq + p^{m''}, p^nq + q$ or $p^nq + p^{m'}q$ such that m', m'' are non-negative integers such that $m'' \leq m$ and $m' \leq m$. Now we consider different cases.

Case 1. $|G| = p^nq + p^{n''}, n'' \leq n$ and $|H| = p^nq + q$. In this case we have $|C_G(g)| = p^{n''}$ and $|C_H(h)| = q$, so $|Z(G)| = p^{n_1}$ such that $n_1 < n''$ and $|Z(H)| = 1$. From $|G| - |Z(G)| = |H| - |Z(H)|$ we get $p^nq + p^{n''} - p^{n_1} = p^nq + q - 1$. So $p^{n''} - p^{n_1} = q - 1$ and then $p^{n_1}(p^{n''-n_1} - 1) = q - 1$. Because of assumption $p > q$, we must have $n_1 = 0$ and then $p^{n''} - 1 = q - 1$. Thus $p^{n''} = q$ which is a contradiction. So this case is impossible.

Case 2. $|G| = p^nq+p^{n''}, n'' \leq n$ and $|H| = p^nq+p^{m''}, m'' \leq n$. Then $|C_G(g)| = p^{n''}$ and $|C_H(h)| = p^{m''}$. Now from $|Z(G)| \Big| |C_G(g)|$ we obtain $|Z(G)| = p^{n_1}$ where $n_1 < n''$. Thus from $|G| - |Z(G)| = |H| - |Z(H)|$ we obtain $p^nq + p^{n''} - p^{n_1} = p^nq + p^{m''} - |Z(H)|$ and so $|Z(H)| = p^{m''} + p^{n_1} - p^{n''}$ such that $p^{m''} > p^{n_1}$(if $p^{m''} < p^{n_1}$, then $p^{m''}+p^{n_1} < 2p^{n_1}$ and so $|Z(H)| = p^{m''}+p^{n_1}-p^{n''} \leq 2p^{n_1}-p^{n''} \leq 0$ which is a contradiction). Therefore $|Z(H)| = p^{n_1}(p^{m''-n_1} - p^{n''-n_1} + 1)$ and $m'' - n_1 \neq 0$ and $n'' - n_1 \neq 0$. On the other hand we know $Z(H) \leq C_H(h)$, so $|Z(H)| = p^{m_2}$ such that $m_2 < m''$ and then $p^{m_2} = p^{n_1}(p^{m''-n_1} - p^{n''-n_1} + 1)$. Hence $p^{m''-n_1} - p^{n''-n_1} + 1 = pk+1$ where k is a non-negative integer. But $gcd(pk+1, p)=1$ and by $p^{m_2} = p^{n_1}(p^{m''-n_1} - p^{n''-n_1} + 1)$ we have $pk = p^{m'-n_1} - p^{n'-n_1} = 0$. So we have $m'' = n''$ and in this case $|G| = |H|$.

Case 3. $|G| = p^nq + q$ and $|H| = p^nq + p^{m'}q$ where $m' \leq n$. Then $|C_G(g)| = q$ and $|C_G(h)| = p^{m'}q$ and so $|Z(G)| = 1$. Now, by Lemma 3.1 in [1], $|Z(H)| \big| q - 1$ and thus $gcd(|Z(H)|, q) = 1$ and from $|C_H(h)| = p^{m'}q$ we obtain $|Z(H)| = p^{m_1}$ where $m_1 \leq m'$. Therefore from $|G| - |Z(G)| = |H| - |Z(H)|$ we will obtain $q - 1 = p^{m'}q - p^{m_1}$ and so $p^{m_1}(p^{m'-m_1}q - 1) = q - 1$. From this equality we must have $m_1 = 0$ and $m' - m_1 = 0$, so $|Z(H)| = 1$ and then $|Z(G)| = |Z(H)| = 1$ completes this case.

Case 4. $|G| = p^nq + p^{n''}$ and $|H| = p^nq + p^{m'}q$ where $n'' \leq n$ and $m' \leq n$. Then $|C_G(g)| = p^{n''}$ and $|C_H(h)| = p^{m'}q$ and so $|Z(G)| = p^{n_1}$ where $n_1 < n''$. By $|G| - |Z(G)| = |H| - |Z(H)|$ we have $p^{n''} - p^{n_1} = p^{m'}q - |Z(H)|$, hence $|Z(H)| = p^{m'}q + p^{n_1} - p^{n''}$. From lemma 3.1 in [1] we obtain $|Z(H)| \Big| p^{n''} - p^{n_1}$ and so $|Z(H)| \Big| p^{n_1}(p^{n''-n_1} - 1)$. But we know that $|Z(H)| \Big| |C_H(h)|$ and so $|Z(H)| = p^{s'}q$ or p^s where $s' < m'$ and $s \leq m'$. Now, by $|Z(G)| \Big| |C_H(h)| - |Z(H)|$ we will obtain $p^{n_1} \Big| p^{s'}q(p^{m'-s'} - 1)$ or $p^s(p^{m'-s}q - 1)$. Therefore from $gcd(p^{m'-s'}q - 1, p) = 1$ and $p > q$ we obtain $p^{n_1} \Big| p^{s'}$ or $p^{n_1} \Big| p^s$, so $n_1 \leq s'$ or $n_1 \leq s$. By considering $|Z(H)| \Big| |C_G(g)| - |Z(G)|$ we obtain $p^{s'} \Big| p^{n_1}(p^{n'-n_1} - 1)$ or $p^s \big| p^{n_1}(p^{n''-n_1} - 1)$ and so $s' \leq n_1$ or $s \leq n_1$. Hence $n_1 = s'$ or $n_1 = s$ and so $|Z(H)| = p^{n_1}q$ or p^{n_1}.

If $|Z(H)| = p^{n_1}$, then $p^{m'}q + p^{n_1} - p^{n''} = p^{n_1}$ and so $p^{m'}q = p^{n''}$ which is a contradiction.

If $|Z(H)| = p^{n_1}q$, then $p^{m'}q + p^{n_1} - p^{n''} = p^{n_1}q$ and so $p^{n_1}(p^{m'-n_1}q - q) = p^{n_1}(p^{n''-n_1} - 1)$ which implies that $p^{m'-n_1}q - q = p^{n'-n_1} - 1$ and hence $q = \frac{p^{n''-n_1} - 1}{p^{m'-n_1} - 1}$. Now, by $1 < q = \frac{p^{n''-n_1} - 1}{p^{m'-n_1} - 1}$ and $p \leq \frac{p^{n''-n_1} - 1}{p^{m'-n1} - 1}$ we will obtain $p \leq q$ that is impossible. Thus this case is impossible.

Case 5. $|G| = p^nq + p^{n'}q$ and $|H| = p^nq + p^{m'}q$ where $n' \leq n$ and $m' \leq n$. Then $|C_G(g)| = p^{n'}q$ and $|C_H(h)| = p^{m'}q$, so $|Z(G)| = p^{n_1}q$ or p^{n_2} where $n_1 < n'$ and $n_2 \leq n'$. By $|G| - |Z(G)| = |H| - |Z(H)|$ we obtain $p^{n'}q - |Z(G)| = p^{m'}q - |Z(H)|$. If $|Z(G)| = p^{n_1}q$, then $|Z(H)| = p^{m'}q + p^{n_1}q - p^{n'}q = qp^{n_1}(p^{m'-n_1} - p^{n'-n_1} + 1)$. (we have $m' > n_1$ because of if $m' \leq n_1$, then $|Z(H)| = p^{m_1} + p^{n_1} - p^{n'} < 2p^{n_1} - p^{n'} \leq 0$ which is contradiction). Thus $|Z(H)| = qp^{n_1}(p^{m'-n_1} - p^{n'-n_1} + 1)$ where $m' - n_1 > 0, n' - n_1 > 0$. So we have $p^{m'-n_1} - p^{n'-n_1} + 1 = pk + 1$ where k is a non-negative integer. If $k = 0$ then, $p^{m'-n_1} - p^{n'-n_1} = 0$ and so $m' = n'$ which implies $|G| = |H|$. If $k > 0$, then from $|Z(H)|\Big||C_H(h)|$ we will obtain $qp^{n_1}(pk + 1)|qp^{m'}$ and so $pk + 1|p^{m'-n_1}$ which contradicts $gcd(p, pk + 1) = 1$.

If $|Z(G)| = p^{n_2}$ where $n_2 \leq n$, then $|Z(H)| = p^{m'}q + p^{n_2} - p^{n'}q$ and by the same argument as above we have $m' > n_2$. Also by $|C_H(h)| = p^{m'}q$ we get $|Z(H)| = p^{m_1}$ or p^{m_2} where $m_1 < m'$ and $m_2 \leq m'$. But from $|Z(H)|$ in this case we obtain q does not divide $|Z(H)|$ and so $|Z(H)| = p^{m_2}$ where $m_2 \leq m'$. Thus from different possibilities for the order of $Z(H)$ we get

$$p^{m_2} = p^{m'}q + p^{n_2} - p^{n'}q = p^{n_2}(p^{m'-n_2}q - p^{n'-n_2}q + 1) = p^{n_2}(qpk + 1),$$

where k is a non-negative integer. If $k = 0$, then $m' - n_2 = n' - n_2$ and so $m' = n'$, which implies $|G| = |H|$. If $k > 0$, then $p^{m_2} = p^{n_2}(qpk + 1)$ which contradicts $gcd(qpk{+}1, p) = 1$ and this case can not happen. This last contradiction completes the proof. ∎

Next, we turn to the groups $PSL_3(q)$ and $PSU_3(q^2)$. The conjugacy classes and the character tables of these groups are calculated in [10] and since we need the size of the centralizer orders for elements of these groups hence we state the following Lemma using [10].

Lemma 2 *The size of centralizer orders for elements of* $PSL_3(q)$ *and* $PSU_3(q^2)$ *are one of the following numbers:* $q^3r'rst$, q^2r', q^2, $qr'rs$, qr', r^2, $r'r$, $r's$, t'. *We have* $r = q - \delta$, $s = q + \delta$, $t = q^2 + \delta q + 1$, $r' = \frac{r}{d}$, $t' = \frac{t}{d}$ *and* $d = (3, r)$, *where* $\delta = 1$ *for* $PSL_3(q)$ *and* $\delta = -1$ *for* $PSU_3(q^2)$.

Proposition 9 *Let* $G{=}PSL_3(q)$ *or* $PSU_3(q^2)$. *If* H *is a group such that* $\Gamma_G{\cong}\Gamma_H$, *then* $|G| = |H|$.

Proof. We consider 2 cases.

Case a. $G = PSL_3(q)$. First suppose $3 \mid q - 1$. Then, by Lemma 2, there are elements $x, y, z \in G$ such that $|C_G(x)| = (q - 1)^2$, $|C_G(y)| = \frac{1}{3}(q^2 + q + 1)$ and $|C_G(g)| = q^2$. Since G is a simple group, we have $Z(G) = 1$.

Let $\varphi : \Gamma_H \to \Gamma_G$ be the given isomorphism of graphs. For each $h \in H - H(Z)$ clearly $|Z(H)|$ divides $|C_H(h)|$ and we have

$$|C_H(h)| - |Z(H)| = |C_G(\varphi(h))| - |Z(G)|.$$

Therefore, for all $\alpha \in G - Z(G)$, we have $|Z(H)| \mid |C_G(\alpha)| - |Z(G)|$.

Using the above divisibility condition, $|Z(H)|$ divides the numbers $|C_G(x)| - 1 = q^2 - 2q$, $|C_G(y)| - 1 = \frac{1}{3}(q^2 + q - 2)$, $|C_G(z)| - 1 = q^2 - 1$. From these we will obtain $|Z(H)| = 1$, and consequently $|G| = |H|$.

Secondly we assume $3 \nmid q - 1$. In this case by Lemma 2 there are elements x, y in G such that $|C_G(x)| = q^2$ and $|C_G(y)| = q^2 - 1$. Since $|Z(H)|$ must divide $|C_G(x)| - 1$ and $|C_G(y)| - 1$ we deduce $|Z(H)| = 1$ and hence $|G| = |H|$.

Case b. $G = PSU_3(q^2)$. First suppose $3 \mid q + 1$. By Lemma 2 there are elements x, y, z and t in G such that $|C_G(x)| = \frac{1}{3}(q^2 + q + 1)$, $|C_G(y)| = \frac{1}{3}(q^2 + q)$, $|C_G(z)| = q^2$ and $|C_G(t)| = (q + 1)^2$. Since $|Z(H)|$ divides $|C_G(\alpha)| - 1$ for all $\alpha \in G - 1$, we will obtain $|Z(H)| \mid \gcd(q^2 - q - 2, q^2 + q - 3, q^2 - 1, q^2 - 2q)$. But it is easy to verify that the greatest common divisor (gcd) written above is 1, hence $|Z(H)| = 1$, implying $|G| = |H|$.

Next assume $3 \nmid q+1$. In this case we consider elements x and y in G such that $|C_G(x)| = q^2 - 1$ and $|C_G(y)| = q^2$. In this case $|Z(H)| \mid \gcd(q^2 - 2, q^2 - 1) = 1$. Therefore $|Z(H)| = 1$, consequently $|G| = |H|$ and the proposition is proved. ■

Lemma 3 *Let G and H be finite centerless groups. If $\Gamma_G \cong \Gamma_H$, then $k(G) = k(H)$.*

Proof. Since $|Z(G)| = |Z(H)| = 1$, we will obtain $|G| = |H|$. From $|E(G)| = \frac{1}{2}|G|(|G| - k(G)) = |E(H)| = \frac{1}{2}|H|(|H| - k(H))$ we deduce that $k(G) = k(H)$. ■

Proposition 10 *If $\Gamma_{\mathbb{A}_5} \cong \Gamma_G$, then $G \cong \mathbb{A}_5$.*

Proof. Let G be a finite group such that $\Gamma_{\mathbb{A}_5} \cong \Gamma_G$. By [1] we obtain $|G| = |\mathbb{A}_5|$ and therefore $Z(G) = 1$. By Lemma 3 we have $k(G) = k(\mathbb{A}_5)$. The group $\mathbb{A}_5$ has five conjugacy classes whose representatives may be taken as $x_1 = 1$, x_2, x_3, x_4 and x_5 with centralizer orders 60, 4, 3, 5, 5 respectively. Since $\Gamma_G \cong \mathbb{A}_5$, hence for each $x_i \neq 1$ there is $g_i \in G$ such that $d(x_i) = d(g_i)$, where d denotes the degree of an element as a graph vertex. But $d(x_i) = |\mathbb{A}_5| - |C_{\mathbb{A}_5}(x_i)| = d(g_i) = |G| - |C_G(g_i)|$ which implies $|C_G(g_i)| = |C_{\mathbb{A}_5}(x_i)|$. Therefore we obtain elements g_2, g_3, g_4 in G such that $|C_G(g_2)| = 4$, $|C_G(g_3)| = 3$, $|C_G(g_4)| = 5$. Let g_1 be the identity element of G. Comparing the centralizer orders we deduce that no pair of the elements g_1, g_2, g_3, g_4 are conjugate in G. Since $k(G) = k(\mathbb{A}_5) = 5$, hence there is exactly one other class representative of G, which we denote by g_5 and $|C_G(g_5)| = 5$. Therefore G has class representatives $g_1, \ldots, g_5$ with class sizes: 1, 15, 20, 12, 12. Now if N is a normal subgroup of G, then $|N|$ must be a sum of the above numbers including 1 as a summand. But an easy calculation shows that $|N| = 1$ or 60. Hence G is a simple group. But it is well known that a simple group of order 60 must be isomorphic to $\mathbb{A}_5$. Therefore $G \cong \mathbb{A}_5$ and the proposition is proved. ■

Proposition 11 *If* $\Gamma_{\mathbb{A}_6} \cong \Gamma_G$*, then* $G \cong \mathbb{A}_6$*.*

Proof. Similar to Proposition 6 we will obtain $|G| = |\mathbb{A}_6|$ and $k(G) = k(\mathbb{A}_6) = 7$. By [3], representatives of the conjugacy classes of $\mathbb{A}_6$ may be taken as $x_1 = 1A$, $x_2 = 2A$, $x_3 = 3A$, $x_4 = 3B$, $x_5 = 4A$, $x_6 = 5A$, $x_7 = 5B$. Considering the degree of vertices in G corresponding to x_i we will obtain elements $g_1 = 1$, g_2, g_3, g_4, g_5, g_6, g_7 of G with centralizer orders 360, 8, 9, 9, 4, 5, and 5 respectively. Certainly the elements g_1, g_2, g_3, g_5 and g_6 are not conjugate in G because they have different centralizer orders. Let g and h be the other two representatives of the conjugacy classes of G with centralizer orders α and β respectively. From the class equation we will obtain $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{14}{45}$. Considering the degrees of g and h as vertices of the graph Γ_G we will see that α as well as β must be one of the centralizer orders in $\mathbb{A}_6$. Therefore we will obtain $\alpha = 9$ and $\beta = 5$. Therefore G has conjugacy classes whose sizes are the same as the size of conjugacy classes in $\mathbb{A}_6$. Now with the same reasoning as at the end of proof of Proposition 6 we can proves that G is a simple group. But any simple group of order 360 is isomorphic to $\mathbb{A}_6$, hence $G \cong \mathbb{A}_6$ and the Proposition is proved. ■

Proposition 12 *Let* G *be a finite p-group such that* $\frac{|G|}{|Z(G)|} = p^2$ *and let* A *be a finite abelian group. If* $\Gamma_{A\times G} \cong \Gamma_H$ *for some group* H*, then* $|A \times G| = |H|$ *and* $H = Q \times B$*, where* B *is an abelian group and* Q *is a non-abelian p-group.*

Proof. By Lemma 3.1 in [1], H is a finite group. if $|G| = p^n$, then $|Z(G)| = p^{n-2}$ and the center of $A \times G$ is of order $p^{n-2}|A|$ and the centralizer of every non-central element of $A \times G$ is of order $p^{n-1}|A|$. It follows that $\Gamma_{A\times G}$ is a regular graph. So Γ_H is a regular graph and since $d(x) = |H| - |C_H(x)|$ for any vertex x, then $|C_H(x)| = |C_H(y)|$ for any non-central element $x, y \in H$. It follows that the conjugacy classes of H have only two sizes. Now, Theorem 1 in [6] implies that H is a nilpotent group and is isomorphic to a direct product of a non-abelian q-subgroup Q (q is a prime) with an abelian group B. Let $|Q|=q^s$, $|Z(Q)|=q^t$, $|B| = b$ and $|A| = a$. Note that $|C_Q(x_1)| = |C_Q(x_2)| = q^r$, for all non-central element $x_1, x_2 \in Q$ and $s > r > t > 0$. Now, using the hypothesis $\Gamma_{A\times G} \cong \Gamma_H$, we obtain

$$\text{(i)} \qquad p^{n-1}a - p^{n-2}a = q^r b - q^t b$$

and $p^n a - p^{n-2}a = q^s b - q^t b$. Thus, $q^s b - q^t b = (p^{n-1}a - p^{n-2}a)(p + 1) = (q^r b - q^t b)(p + 1)$ and so, $(p + 1)(q^r b - q^t b) = q^s - q^t$. It follows that

$$\text{(ii)} \qquad p(q^r - q^t) = q^s - q^r$$

and so, $p(q^{r-t} - 1) = q^{s-t} - q^{r-t} = q^{r-t}(q^{s-r} - 1)$. Thus, $q^{r-t}|p(q^{r-t} - 1)$ and $gcd(q^{r-t}, q^{r-t} - 1) = 1$ and so $q^{r-t}|p$, hence, $q = p$ and $r - t = 1$. Now, (ii) implies that $p(p^{t+1} - p^t) = p^s - p^{t+1}$, so $p^{t+2} = p^s$. Therefore, $s = t + 2$ and now, (i) yields that $p^{n-1}a - p^{n-2}a = p^{t+1}b - p^t b$, so $p^{n-2}a = p^t b$. In this situation we consider three cases.

Case 1. If $t = n - 2$, then $a = b$ and $|H| = bp^s = ap^{t+2} = ap^n = |A \times G|$.

Case 2. if $n - 2 > t$, then $b = ap^{n-2-t}$ and $|H| = bp^s = bp^{t+2} = ap^{n-2-t}p^{t+2} = ap^n = |A \times G|$.

Case 3. If $n - 2 < t$, then $a = p^{t-n+2}$ and $|A \times G| = ap^n = p^{t-n+2}bp^n = bp^{t+2} = bp^s = |H|$.

Now, in every case we have $|A \times G| = |H|$ and $q = p$, so Q is a non-abelian p-group. This completes the proof. ■

5. Chromatic number of the non-commuting graph

First, we recall some terminology from graph theory. Let $\Gamma = (V, E)$ be a simple graph. A subset X of V is called an independent set if the induced subgraph on X is the null graph. Let k be a natural number. A k-vertex coloring of Γ is an assignment of k colors to the vertices of Γ such that no two adjacent vertex has the same color. The chromatic number of the graph Γ is the minimum natural number k for which Γ has a k-vertex coloring, this number is denoted by $\chi(\Gamma)$.

Lemma 4 *Let G be a finite non-abelian group and $N \trianglelefteq G$. Then $\chi(\Gamma_{\frac{G}{N}}) \leqslant \chi(\Gamma_G)$.*

Proof. By [1] $\chi(\Gamma)$ is equal to the minimum number of the abelian subgroups of G which cover G. Therefore if $\chi(\Gamma_G) = n$, then $G = \bigcup_{i=1}^{n} H_i$, where H_i is an abelian subgroup of G and G can not be covered by less than n abelian subgroup of G. Since N is a normal subgroup of G, hence $\frac{G}{N} = \bigcup_{i=1}^{n} \frac{NH_i}{N}$. But $\frac{NH_i}{N} \cong \frac{H_i}{N \cap H_i}$ is abelian from which we deduce $\chi(\Gamma_{\frac{G}{N}}) \leqslant n = \chi(\Gamma_G)$. ■

Proposition 13 *Let G be a finite non-abelian group. Then $\chi(\Gamma_G) \leqslant \frac{|V(\Gamma_G)|}{|Z(G)|}$. If equality holds then $G \cong P \times A$, where P is a p-group, A is an abelian group and G is a nilpotent group of class at most 3.*

Proof. Suppose $[G : Z(G)] = n$ and $\{x_1 = 1, x_2, \ldots, x_n\}$ be a set of left transversals of $Z = Z(G)$ in G. It is clear that x_iZ is a commutative subset of G, hence x_iZ is an independent subset in Γ_G for each i. Since $G - Z(G) = \bigcup_{i=2}^{n} x_iZ(G)$ we deduce that $\chi(\Gamma_G) \leqslant n - 1$. But $n - 1 = \frac{|V(\Gamma_G)|}{|Z(G)|}$ and the inequality stated in the proposition is proved.

Now, assume equality holds, i.e., $\chi(\Gamma_G) = \left|\frac{G}{Z(G)}\right| - 1$. If x_i and x_j commutes, then clearly elements of x_iZ commute with elements of x_jZ. Therefore the above equality implies that $\{x_2, \ldots, x_n\}$ forms the vertices of a complete subgraph of Γ_G. Now, for any t in G, there is an i such that $t = x_iz$, where $z \in Z(G)$. Now, it is easy to verify that $C_G(t) = C_G(x_i) = Z \cup x_i Z$. Therefore, if $t \in Z(G)$, we have $|C_G(t)| = |Z|$ and if $t \notin Z\,|G|$, then $|C_G(t)| = 2\,|Z|$. Hence, Γ_G is a regular graph and the result follows by Proposition 2.6 in [1]. ■

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HYPER K-ALGEBRAS INDUCED BY A DETERMINISTIC FINITE AUTOMATON

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Abstract. In this note first we define a hyper K*-algebra* S on the states of a deterministic finite automaton. Then we obtain some commutative hyper K-ideals of types 3, 4, 5, 6 and 9 and also positive implicative hyper K-ideals of types 1, 2, 3, 4, 5, 6, 7, 8 and 9 of S. Also we prove some theorems and obtain some results, to show that some properties of this hyper K-algebra. Then we define another hyper K*-algebra* on the states of a deterministic finite automaton which is simple and normal. Finally, we introduce a hyper K-algebra on the set of all equivalence classes of an equivalence relation on states.

Keywords: deterministic finite automaton, hyper K-algebra, (commutative, positive implicative) hyper K-ideal.

1. Introduction

The hyper algebraic structure theory was introduced by F. Marty [7] in 1934. Imai and Iseki [6] in 1966 introduced the notion of BCK-algebra. Borzooei, Jun and Zahedi et.al. [1], [2], [13] applied the hyper structure to BCK-algebra and introduced the concept of hyper K-algebra which is a generalization of BCK-algebra. Roodbari and Zahedi [12] introduced 27 different types of positive implicative hyper K-ideals, also they introduced 9 different types of commutative hyper K-ideals. Corsini and Leoreanu [4] found some connections between a deterministic finite automaton and the hyper algebraic structure theory. Now, in this note we define two hyper K-algebras on the states of a deterministic finite automaton. Then we obtain some properties of these hyper K-algebras. Finally, we define a hyper K-algebra on the set of all equivalence classes of an equivalence relation on states.

2. Preliminaries

Let H be a nonempty set and o be a hyper operation on H, that is o is a function from $H \times H$ to $\mathcal{P}^*(H) = \mathcal{P}(H) \backslash \phi$.

Definition 2.1 [2] We say that H is a hyper K-algebra if it contains a constant 0 and satisfies the following axioms:

(HK1) $(xoz)o(yoz) < xoy$,
(HK2) $(xoy)oz = (xoz)oy$,
(HK3) $x < x$,
(HK4) $x < y, \ y < x \Rightarrow x = y$,
(HK5) $0 < x$.

For all $x, y, z \in H$, where $x < y$ is defined by $0 \in xoy$ and for every $A, B \subseteq H$, $A < B$ is defined by $\exists a \in A, \exists b \in B$ such that $a < b$. Note that if $A, B \subseteq H$, then by AoB we mean the subset $\bigcup\limits_{a \in A, b \in B} aob$ of H.

Definition 2.2 [10] Let $(H, o, 0)$ be a hyper K-algebra. Then H is called:

(i) A weak implicative, if for all $x, y \in H$, $x < xo(yox)$,

(ii) An implicative, if for all $x, y \in H$, $x \in xo(yox)$,

(iii) A strong implicative, if for all $x, y \in H$, $xo0 \subseteq xo(yox)$.

Definition 2.3 [9] Let $(H, o, 0)$ be a hyper K-algebra and I be a subset of H and $\phi \neq S \subseteq H$. Then we say that I is an S-absorbing set, whenever $x \in I$ and $y \in S$ imply that $xoy \subseteq I$.

Definition 2.4 [10] Let I be a nonempty subset of a hyper K-algebra H. Then we say that I is closed, whenever $x < y$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$.

Definition 2.5 [2], [10] Let I be a nonempty subset of a hyper K-algebra H and $0 \in I$. Then,

(i) I is called a weak hyper K-ideal of H if $xoy \subseteq I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$.

(ii) I is called a hyper K-ideal of H if $xoy < I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$.

(iii) I is called a strong hyper K-ideal of H if $(xoy) \cap I \neq \phi$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$.

(iv) I is called an implicative hyper K-ideal, if for all $x, y, z \in H$, $(xoz)o(yox) < I$ and $z \in I$ imply that $x \in I$.

(v) I is called a weak implicative hyper K-ideal, if for all $x, y, z \in H$, $(xoz)o(yox) \subseteq I$ and $z \in I$ imply that $x \in I$.

Theorem 2.6 [2] *Any strong hyper K-ideal of a hyper K-algebra H is a hyper K-ideal and a weak hyper K-ideal. Also any hyper K-ideal of a hyper K-algebra H is a weak hyper K-ideal.*

Definition 2.7 [12] Let I be a nonempty subset of a hyper K-algebra H and $0 \in I$. Then I is called a commutative hyper K-ideal of

(i) type 1, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \phi$ and $z \in I$ imply that $(xo(yo(yox))) \subseteq I$,

(ii) type 2, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \phi$ and $z \in I$ imply that $(xo(yo(yox))) \cap I \neq \phi$,

(iii) type 3, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \phi$ and $z \in I$ imply that $(xo(yo(yox))) < I$,

(iv) type 4, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $z \in I$ imply that $(xo(yo(yox))) \subseteq I$,

(v) type 5, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $z \in I$ imply that $(xo(yo(yox))) \cap I \neq \phi$,

(vi) type 6, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $z \in I$ imply that $(xo(yo(yox))) < I$,

(vii) type 7, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $z \in I$ imply that $(xo(yo(yox))) \subseteq I$,

(viii) type 8, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $z \in I$ imply that $(xo(yo(yox))) \cap I \neq \phi$,

(ix) type 9, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $z \in I$ imply that $(xo(yo(yox))) < I$,

Definition 2.8 [10] Let I be a nonempty subset of a hyper K-algebra H and $0 \in I$. Then the following statements hold:

(i) If I is a commutative hyper K-ideal of type 4, then I is a commutative hyper K-ideal of type 6,

(ii) If I is a commutative hyper K-ideal of type 6, then I is a commutative hyper K-ideal of type 9,

(iii) If I is a commutative hyper K-ideal of type 5, then I is a commutative hyper K-ideal of type 6,

(iv) If I is a commutative hyper K-ideal of type 9, then I is a commutative hyper K-ideal of type 3.

Definition 2.9 [12] Let I be a nonempty subset of H such that $0 \in I$. Then I is called a positive implicative hyper K-ideal of

(i) type 1, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $(yoz) \subseteq I$ imply that $(xoz) \subseteq I$,

(ii) type 2, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $(yoz) \subseteq I$ imply that $(xoz) \cap I \neq \phi$,

(iii) type 3, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $(yoz) \subseteq I$ imply that $(xoz) < I$,

(iv) type 4, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $(yoz) \cap I \neq \phi$ imply that $(xoz) \subseteq I$,

(v) type 5, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $(yoz) \cap I \neq \phi$ imply that $(xoz) \cap I \neq \phi$,

(vi) type 6, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $(yoz) \cap I \neq \phi$ imply that $(xoz) < I$,

(vii) type 7, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $(yoz) < I$ imply that $(xoz) < I$,

(viii) type 8, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $(yoz) < I$ imply that $(xoz) \cap I \neq \phi$,

(ix) type 9, if for all $x, y, z \in H$, $((xoy)oz) \subseteq I$ and $(yoz) < I$ imply that $(xoz) \subseteq I$,

(x) type 10, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \phi$ and $(yoz) \subseteq I$ imply that $(xoz) \cap I \neq \phi$,

(xi) type 11, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \phi$ and $(yoz) \subseteq I$ imply that $(xoz) \subseteq I$,

(xii) type 12, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \phi$ and $(yoz) \subseteq I$ imply that $(xoz) < I$,

(xiii) type 13, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \phi$ and $(yoz) \cap I \neq \phi$ imply that $(xoz) \subseteq I$,

(xiv) type 14, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \phi$ and $(yoz) \cap I \neq \phi$ imply that $(xoz) \cap I \neq \phi$,

(xv) type 15, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \phi$ and $(yoz) \cap I \neq \phi$ imply that $(xoz) < I$,

(xvi) type 16, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \phi$ and $(yoz) < I$ imply that $(xoz) < I$,

(xvii) type 17, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \phi$ and $(yoz) < I$ imply that $(xoz) \cap I \neq \phi$,

(xviii) type 18, if for all $x, y, z \in H$, $((xoy)oz) \cap I \neq \phi$ and $(yoz) < I$ imply that $(xoz) \subseteq I$,

(xix) type 19, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $(yoz) \cap I \neq \phi$ imply that $(xoz) < I$,

(xx) type 20, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $(yoz) \cap I \neq \phi$ imply that $(xoz) \subseteq I$,

(xxi) type 21, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $(yoz) \cap I \neq \phi$ imply that $(xoz) \cap I \neq \phi$,

(xxii) type 22, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $(yoz) \subseteq I$ imply that $(xoz) \subseteq I$,

(xxiii) type 23, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $(yoz) \subseteq I$ imply that $(xoz) < I$,

(xxiv) type 24, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $(yoz) \subseteq I$ imply that $(xoz) \cap I \neq \phi$,

(xxv) type 25, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $(yoz) < I$ imply that $(xoz) < I$,

(xxvi) type 26, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $(yoz) < I$ imply that $(xoz) \cap I \neq \phi$,

(xxvii) type 27, if for all $x, y, z \in H$, $((xoy)oz) < I$ and $(yoz) < I$ imply that $(xoz) \subseteq I$,

Definition 2.10 [10] A hyper K-algebra $(H, o, 0)$ is called simple if for all distinct elements $a, b \in H - \{0\}$, $a \not< b$ and $b \not< a$.

Definition 2.11 [11] Let H be a hyper K-algebra and S be a nonempty subset of H. Then the sets

$$\begin{array}{lcl} {}_{l1}S = \{x \in H | a < (aox), \forall a \in S\}, & & {}_{l2}S = \{x \in H | a \in (aox), \forall a \in S\}, \\ S_{r1} = \{x \in H | x < (xoa), \forall a \in S\} & \text{and} & S_{r2} = \{x \in H | x \in (xoa), \forall a \in S\} \end{array}$$

are called left hyper K-stabilizer of type 1 of S, left hyper K-stabilizer of type 2 of S, right hyper K-stabilizer of type 1 of S and right hyper K-stabilizer of type 2 of S.

Definition 2.12 [11] A hyper K-algebra $(H, o, 0)$ is called left (right) hyper normal of type 1(2) if ${}_{li}a(a_{ri})$ of any element $a \in H$ is a hyper K-ideal of H for $i = 1$ or 2. Also if H is both left and right hyper normal of type 1 (2), then H is called hyper normal K-algebra of type 1 (2).

Definition 2.13 [5] A deterministic finite automaton consists of:

(i) A finite set of states, often denoted by S.

(ii) A finite set of input symbols, often denoted by M.

(iii) A transition function that takes as arguments a state and an input symbol and returns a state. The transition function will commonly be denoted by t, and in fact $t : S \times M \to S$ is a function.

(iv) A start state, one of the states in S such as s_0.

(v) A set of final or accepting states F. The set F is a subset of S.

For simplicity of notation, we write (S, M, s_0, F, t) for a deterministic finite automaton.

Remark 2.14 [5] Let (S, M, s_0, F, t) be a deterministic finite automaton. A word of M is the product of a finite sequence of elements in M, λ is empty word and M^* is the set of all words on M. We define recursively the extended transition function, $t^* : S \times M^* \to S$, as follows:

$$\forall s \in S, \forall a \in M, t^*(s, a) = t(s, a),$$
$$\forall s \in S, t^*(s, \lambda) = s,$$
$$\forall s \in S, \forall x \in M^*, \forall a \in M, t^*(s, ax) = t^*(t(s, a), x).$$

Note that the length $\ell(x)$ of a word $x \in M^*$ is the number of its letters; so $\ell(\lambda) = 0$ and $\ell(a_1a_2) = 2$, where $a_1, a_2 \in M$.

Definition 2.15 [4] The state s of $S - s_0$ will be called connected to the state s_0 of S if there exists $x \in M^*$, such that $s = t^*(s_0, x)$.

3. Hyper K-algebras induced by a deterministic finite automaton

In this paragraph, we present some relationships between hyper K-algebras and deterministic finite automata.

Definition 3.1 Let (S, M, s_0, F, t) be a deterministic finite automaton.

If $s \in S - \{s_0\}$ is connected to s_0, then the order of a state s is the natural number $l + 1$, where $l = \min\{\ell(x) | t^*(s_0, x) = s, x \in M^*\}$, and if $s \in S - \{s_0\}$ is not connected to s_0 we suppose that the order of s is 1. Also we suppose that the order of s_0 is 0.

We denote the order of a state s by ord s.

Now, we define the relation $\sim$ on the set of states S, as follows:

$$s_1 \sim s_2 \Leftrightarrow ords_1 = ords_2.$$

It is obvious that this relation is an equivalence relation on S.

Note that we denote the equivalence class of s by $\bar{s}$. Also we denote the set of all these classes by $\bar{S}$.

Theorem 3.2 *Let (S, M, s_0, F, t) be a deterministic finite automaton. We define the following hyper operation on S:*

$$\forall (s_1, s_2) \in S^2, s_1 o s_2 = \begin{cases} \bigcup\limits_{ords \leq ords_2} \bar{s} & , \textit{if } ords_1 < ords_2,\ s_1, s_2 \neq s_0,\ s_1 \neq s_2 \\ \bigcup\limits_{s_0 \neq s, ords \leq ords_1} \bar{s} & , \textit{if } ords_1 \geq ords_2,\ s_1, s_2 \neq s_0,\ s_1 \neq s_2 \\ \bigcup\limits_{ords \leq ords_1} \bar{s} & , \textit{if } s_1 = s_2 \\ s_0 & , \textit{if } s_1 = s_0,\ s_2 \neq s_0 \\ s_1 & , \textit{if } s_2 = s_0,\ s_1 \neq s_0. \end{cases}$$

Then (S, o, s_0) is a hyper K-algebra and s_0 is the zero element of S.

Proof. It is easy to see that (S, o, s_0) satisfies (HK3). Since $tot = \bigcup\limits_{ords \leq ordt} \bar{s}$, we conclude that $s_0 \in tot$. So

$$t < t, \forall t \in S \tag{1}$$

By the definition of the hyper operation o, we know that $s_1 \in s_1 o s_2$, and so, $s_1 o s_2 \neq \phi$ for any $s_1, s_2 \in S$.

Since $s_1 \in s_1 o s_3$ and $s_2 o s_3 \neq \phi$, we obtain that $s_1 \in (s_1 o s_3) o (s_2 o s_3)$.

So, by (1) we get that

$$(s_1 o s_3) o (s_2 o s_3) < s_1 o s_2$$

That is (HK1) holds.

Now, we have to consider the following situations to prove (HK2).

(i) Let $s_1, s_2, s_3 \neq s_0$ and $ords_1 < ords_2 < ords_3$. Then

$$(s_1 o s_2) o s_3 = \left(\bigcup_{ords \leq ords_2} \bar{s} \right) o s_3 = \bigcup_{ords \leq ords_3} \bar{s},$$

and

$$(s_1 o s_3) o s_2 = \left(\bigcup_{ords \leq ords_3} \bar{s} \right) o s_2 = \bigcup_{ords \leq ords_3} \bar{s}.$$

So, in this case (HK2) holds.

(ii) Let $s_1, s_2, s_3 \neq s_0$ and $ords_2 < ords_1 < ords_3$. Then

$$(s_1 o s_2) o s_3 = \left(\bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s} \right) o s_3 = \bigcup_{ords \leq ords_3} \bar{s},$$

and

$$(s_1 o s_3) o s_2 = \left(\bigcup_{ords \leq ords_3} \bar{s} \right) o s_2 = \bigcup_{ords \leq ords_3} \bar{s},$$

since $s_2 \in \bigcup_{ords \leq ords_3} \bar{s}$.

Hence, in this case (HK2) holds.

(iii) Let $s_1, s_2, s_3 \neq s_0$ and $ords_2 < ords_3 < ords_1$. Then

$$(s_1 o s_2) o s_3 = \left(\bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s} \right) o s_3 = \bigcup_{ords \leq ords_1} \bar{s},$$

since $s_3 \in \bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s}$, and

$$(s_1 o s_3) o s_2 = \left(\bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s} \right) o s_2 = \bigcup_{ords \leq ords_1} \bar{s},$$

since $s_2 \in \bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s}$. Thus, in this case (HK2) holds.

The proofs of the following three situations are the same as (i), (ii) and (iii) respectively.

(iv) $s_1, s_2, s_3 \neq s_0$ and $ords_1 < ords_3 < ords_2$,

(v) $s_1, s_2, s_3 \neq s_0$ and $ords_3 < ords_1 < ords_2$,

(vi) $s_1, s_2, s_3 \neq s_0$ and $ords_3 < ords_2 < ords_1$.

(vii) Let $s_1, s_2, s_3 \neq s_0$, $ords_1 = ords_2 < ords_3$ and $s_1 \neq s_2$. Then

$$(s_1 o s_2) o s_3 = \left(\bigcup_{s_0 \neq s, ords \leq ords_2} \bar{s} \right) o s_3 = \bigcup_{ords \leq ords_3} \bar{s},$$

and

$$(s_1 o s_3) o s_2 = \left(\bigcup_{ords \leq ords_3} \bar{s} \right) o s_2 = \bigcup_{ords \leq ords_3} \bar{s}.$$

Therefore, in this case (HK2) holds.

(viii) Let $s_1, s_2, s_3 \neq s_0$,$ords_1 = ords_2 > ords_3$ and $s_1 \neq s_2$. Then

$$(s_1 o s_2) o s_3 = \left(\bigcup_{s_0 \neq s, ords \leq ords_2} \bar{s} \right) o s_3 = \bigcup_{ords \leq ords_2} \bar{s},$$

since $s_3 \in \bigcup_{s_0 \neq s, ords \leq ords_2} \bar{s}$, and

$$(s_1 o s_3) o s_2 = \left(\bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s} \right) o s_2 = \bigcup_{ords \leq ords_2} \bar{s},$$

since $s_2 \in \bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s}$. So, in this case (HK2) holds.

The proofs of the following two situations are the same as (vii) and (viii), respectively.

(ix) $s_1, s_2, s_3 \neq s_0$, $ords_1 = ords_3 < ords_2$ and $s_1 \neq s_3$,

(x) $s_1, s_2, s_3 \neq s_0$, $ords_1 = ords_3 > ords_2$ and $s_1 \neq s_3$.

(xi) Let $s_1, s_2, s_3 \neq s_0$, $ords_2 = ords_3 > ords_1$ and $s_2 \neq s_3$. Then

$$(s_1 o s_2) o s_3 = \left(\bigcup_{ords \leq ords_2} \bar{s} \right) o s_3 = \bigcup_{ords \leq ords_2} \bar{s},$$

and

$$(s_1 o s_3) o s_2 = \left(\bigcup_{ords \leq ords_3} \bar{s} \right) o s_2 = \bigcup_{ords \leq ords_2} \bar{s}.$$

Hence, in this case (HK2) holds.

(xii) Let $s_1, s_2, s_3 \neq s_0$, $ords_2 = ords_3 < ords_1$ and $s_2 \neq s_3$. Then

$$(s_1 o s_2) o s_3 = \left(\bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s} \right) o s_3 = \bigcup_{ords \leq ords_1} \bar{s},$$

and

$$(s_1 o s_3) o s_2 = \left(\bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s} \right) o s_2 = \bigcup_{ords \leq ords_1} \bar{s}.$$

Thus, in this case (HK2) holds.

(xiii) Let $s_1, s_2, s_3 \neq s_0$, $ords_1 = ords_2 = ords_3$ and $s_1 \neq s_2 \neq s_3 \neq s_1$. Then

$$(s_1 o s_2) o s_3 = \left(\bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s} \right) o s_3 = \bigcup_{ords \leq ords_1} \bar{s},$$

since $s_3 \in \bigcup\limits_{s_0 \neq s, ords \leq ords_1} \bar{s}$, and

$$(s_1os_3)os_2 = \left(\bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s} \right) os_2 = \bigcup_{ords \leq ords_1} \bar{s},$$

since $s_2 \in \bigcup\limits_{s_0 \neq s, ords \leq ords_1} \bar{s}$. Therefore, in this case (HK2) holds.

(xiv) Let $s_1, s_2, s_3 \neq s_0$, $ords_1 = ords_3$, $s_1 \neq s_3$ and $s_1 = s_2$. Then

$$(s_1os_2)os_3 = \left(\bigcup_{ords \leq ords_2} \bar{s} \right) os_3 = \bigcup_{ords \leq ords_2} \bar{s},$$

and

$$(s_1os_3)os_2 = \left(\bigcup_{s_0 \neq s, ords \leq ords_3} \bar{s} \right) os_2 = \bigcup_{ords \leq ords_2} \bar{s}.$$

So, in this case (HK2) holds.

The proof of the following situation is the same as (xiv).

(xv) $s_1, s_2, s_3 \neq s_0$, $ords_1 = ords_2$, $s_1 \neq s_2$ and $s_1 = s_3$.

(xvi) Let $s_1, s_2, s_3 \neq s_0$, $ords_1 = ords_2$, $s_1 \neq s_2$ and $s_2 = s_3$. Then

$$(s_1os_2)os_3 = \left(\bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s} \right) os_3 = \bigcup_{ords \leq ords_1} \bar{s},$$

and

$$(s_1os_3)os_2 = \left(\bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s} \right) os_2 = \bigcup_{ords \leq ords_1} \bar{s}.$$

Hence, in this case (HK2) holds.

(xvii) Let $s_1, s_2, s_3 \neq s_0$, $ords_1 < ords_3$ and $s_1 = s_2$. Then

$$(s_1os_2)os_3 = \left(\bigcup_{ords \leq ords_2} \bar{s} \right) os_3 = \bigcup_{ords \leq ords_3} \bar{s},$$

and

$$(s_1os_3)os_2 = \left(\bigcup_{ords \leq ords_3} \bar{s} \right) os_2 = \bigcup_{ords \leq ords_3} \bar{s}.$$

Thus, in this case (HK2) holds.

(xviii) Let $s_1, s_2, s_3 \neq s_0$, $ords_1 > ords_3$ and $s_1 = s_2$. Then

$$(s_1 o s_2) o s_3 = \left(\bigcup_{ords \leq ords_2} \bar{s} \right) o s_3 = \bigcup_{ords \leq ords_2} \bar{s},$$

and

$$(s_1 o s_3) o s_2 = \left(\bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s} \right) o s_2 = \bigcup_{ords \leq ords_2} \bar{s},$$

since $s_2 \in \left(\bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s} \right)$. Therefore, in this case (HK2) holds.

The proofs of the following two situations are the same as (xvii) and (xviii) respectively.

(xix) $s_1, s_2, s_3 \neq s_0$, $ords_1 < ords_2$ and $s_1 = s_3$,

(xx) $s_1, s_2, s_3 \neq s_0$, $ords_1 > ords_2$ and $s_1 = s_3$.

(xxi) Let $s_1, s_2, s_3 \neq s_0$, $ords_1 < ords_2$ and $s_2 = s_3$. Then

$$(s_1 o s_2) o s_3 = \left(\bigcup_{ords \leq ords_2} \bar{s} \right) o s_3 = \bigcup_{ords \leq ords_3} \bar{s},$$

and

$$(s_1 o s_3) o s_2 = \left(\bigcup_{ords \leq ords_3} \bar{s} \right) o s_2 = \bigcup_{ords \leq ords_3} \bar{s}.$$

So, in this case (HK2) holds.

(xxii) Let $s_1, s_2, s_3 \neq s_0$, $ords_1 > ords_2$ and $s_2 = s_3$. Then

$$(s_1 o s_2) o s_3 = \left(\bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s} \right) o s_3 = \bigcup_{ords \leq ords_1} \bar{s},$$

and

$$(s_1 o s_3) o s_2 = \left(\bigcup_{s_0 \neq s, ords \leq ords_1} \bar{s} \right) o s_2 = \bigcup_{ords \neq ords_1} \bar{s}.$$

Hence, in this case (HK2) holds.

(xxiii) Let $s_1 = s_2 = s_3$. Then

$$(s_1 o s_2) o s_3 = (s_1 o s_1) o s_1 = (s_1 o s_3) o s_2.$$

Thus, in this case (HK2) holds.

(xxiv) Let $s_1 = s_0$ and $s_2, s_3 \neq s_0$. Then $(s_1 o s_2) o s_3 = (s_0 o s_2) o s_3 = (s_0 o s_3) = s_0$ and $(s_1 o s_3) o s_2 = (s_0 o s_3) o s_2 = (s_0 o s_2) = s_0$. Therefore, in this case (HK2) holds.

(xxv) Let $s_2 = s_0$ and $s_1, s_3 \neq s_0$. Then $(s_1 o s_2) o s_3 = (s_1 o s_0) o s_3 = s_1 o s_3$ and $(s_1 o s_3) o s_2 = (s_1 o s_3) o s_0 = s_1 o s_3$. So, in this case (HK2) holds.

The proof of the following situation is the same as (xxv).

(xxvi) $s_3 = s_0$ and $s_1, s_2 \neq s_0$.

(xxvii) Let $s_1 \neq s_0$ and $s_2 = s_3 = s_0$. Then $(s_1 o s_2) o s_3 = (s_1 o s_0) o s_0 = (s_1 o s_0) = s_1$ and $(s_1 o s_3) o s_2 = (s_1 o s_0) o s_0 = (s_1 o s_0) = s_1$. Hence, in this case (HK2) holds.

(xxviii) Let $s_3 \neq s_0$ and $s_1 = s_2 = s_0$. Then $(s_1 o s_2) o s_3 = (s_0 o s_0) o s_3 = s_0 o s_3 = s_0$ and $(s_1 o s_3) o s_2 = (s_0 o s_3) o s_0 = s_0 o s_0 = s_0$. Thus, in this case (HK2) holds.

The proof of the following situation is the same as (xxviii).

(xxix) $s_2 \neq s_0$ and $s_1 = s_3 = s_0$. So, we obtain that (S, o, s_0) satisfies (HK2).

To prove (HK4), Let $s_1 < s_2$ and $s_2 < s_1$. If $s_1 = s_2$, then we are done. Otherwise, since $s_1 < s_2$, there exist two cases:

(i) $ords_1 < ords_2$. Then $s_2 o s_1 = \bigcup_{s_0 \neq s, ords \leq ords_2} \bar{s}$. Therefore, $s_2 \not< s_1$, which is a contradiction.

(ii) $s_1 = s_0$, $s_2 \neq s_0$. Then, $s_2 o s_1 = s_2 o s_0 = s_2$. Thus, $s_2 \not< s_1$, which is a contradiction.

Now, to complete the proof, we should prove that (S, o, s_0) satisfies (HK5).

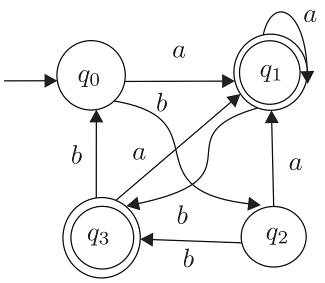
By the definition of the hyper operation o, we know that for any s_1 in S, $s_0 o s_1 = s_0$. Hence, $s_0 < s_1$.

Example 3.3 Let $A = (S, M, s_0, F, t)$ be a deterministic finite automaton such that $S = \{q_0, q_1, q_2, q_3\}$, $M = \{a, b\}$, $s_0 = q_0$, $F = \{q_1, q_3\}$ and t is defined by

$$t(q_0, a) = q_1, \quad t(q_0, b) = q_2, \quad t(q_1, a) = q_1, \quad t(q_1, b) = q_3$$

$$t(q_2, a) = q_1, \quad t(q_2, b) = q_3, \quad t(q_3, a) = q_1, \quad t(q_3, b) = q_0$$

It is clearly that $ordq_1 = ordq_2 = 2$, $ordq_3 = 3$ and $ordq_0 = 0$



According to the definition of the hyper operation "o", which is defined in Theorem 3.2, we have the following table.

o	q_0	q_1	q_2	q_3
q_0	q_0	q_0	q_0	q_0
q_1	q_1	$\{q_0,q_1,q_2\}$	$\{q_1,q_2\}$	$\{q_0,q_1,q_2,q_3\}$
q_2	q_2	$\{q_1,q_2\}$	$\{q_0,q_1,q_2\}$	$\{q_0,q_1,q_2,q_3\}$
q_3	q_3	$\{q_1,q_2,q_3\}$	$\{q_1,q_2,q_3\}$	$\{q_0,q_1,q_2,q_3\}$

Thus, (S,o,s_0) is a hyper K-algebra.

From now on, we let (S,o,s_0) be the hyper K-algebra, which is defined in Theorem 3.2.

Theorem 3.4 *(S,o,s_0) is a (weak, strong) implicative hyper K-algebra.*

Proof. By the definition of the hyper operation "o", we know that $s_1\in s_1os_2$ and $s_1os_2\neq\phi$ for all s_1, s_2 in S. So $s_1\in s_1o(s_2os_1)$, which implies that (S,o,s_0) is implicative.

Also, we show that $s_0\in s_1os_1$ and $s_1\in s_1o(s_2os_1)$ for any s_1,s_2 in S. So, $s_1<s_1o(s_2os_1)$ and we obtain that (S,o,s_0) is weak implicative.

On the other hand, by the definition of the hyper operation "o", we have $s_1o\ 0=s_1$ and $s_1\in s_1o(s_2os_1)$. Thus, $s_1o\ 0\subseteq s_1o(s_2os_1)$, which implies that (S,o,s_0) is strong implicative.

Remark 3.5 In (S,o,s_0), let F be a nonempty subset of $\bar{S}$, $s_0=\overline{s_0}\in F$, $I=\bigcup\limits_{\bar{t}\in F}\bar{t}$ and C be a nonempty subset of S. Then, I may not be a C-absorbing set. Because $s_1,s_2\neq s_0$, $ords_1<ords_2$, $F=s_0\cup\overline{s_1}$ and $s_2\in C$, then $s_1\in I$ and $s_1os_2=\bigcup\limits_{ords\leq ords_2}\bar{s}$. So, $s_2\in s_1os_2$ but $s_2\notin I$.

Theorem 3.6 *In (S,o,s_0), any nonempty subset of S is an $\{s_0\}$-absorbing set.*

Proof. By definition of hyper operation "o" we know that for any s_1 in S, $s_1os_0=s_1$. So it is clearly that for any nonempty subset I of S we have:
If $x\in I$ and $y=s_0\Rightarrow xoy\subseteq I$.

Notation. We denote the class of all states which their order is n by $\overline{s_n}$.

Theorem 3.7 *For any $n\in N$, let $I_n=\{s\in S|s\in\bigcup_{i=0}^{n}\overline{s_i}\}$. Then I_n is:*

(i) *closed,*
(ii) *weak hyper K-ideal,*
(iii) *weak implicative hyper K-ideal for all $n\geq 1$.*

Proof.

(i) Suppose that $s_1<s_2$ and $s_2\in I_n$. Then $s_0\in s_1os_2$. We have three cases:

1. $s_1, s_2 \neq s_0$ and $ords_1 < ords_2$.
   By definition of I_n, we can easily see that $s_1 \in I_n$.
2. $s_1 = s_2$.
   It is clear.
3. $s_1 = s_0$.
   By definition of I_n, it is obvious that $s_1 \in I_n$.

(ii) Assume that $s_1 o s_2 \subseteq I_n$ and $s_2 \in I_n$, then we have to consider the following situations:

1. $s_1 \neq s_2, s_2 \neq s_0$ and $ords_2 < ords_1$.
   Since $s_1 o s_2 = \bigcup\limits_{s_0 \neq s, ords \leq ords_1} \bar{s} \subseteq I_n$, we obtain that $ords_1 \leq n$. Hence $s_1 \in I_n$.
2. $s_1 \neq s_2, s_2 \neq s_0$ and $ords_2 = ords_1$.
   By definition of I_n and the hyper operation "o", it is obvious that $s_1 \in I_n$.
3. $s_1 \neq s_2, s_2 \neq s_0$ and $ords_1 < ords_2$.
   By definition of I_n and the hyper operation "o", it is easy to see that $s_1 \in I_n$.
4. $s_1 = s_2$.
   It is clear.
5. $s_2 = s_0$.
   Since $s_1 o s_2 = s_1 o s_0 = s_1$ and $s_1 o s_2 \subseteq I_n$, we obtain $s_1 \in I_n$.
6. $s_1 = s_0$.
   By definition of I_n, it is obvious that $s_1 \in I_n$.

(iii) Let $(s_1 o s_3) o (s_2 o s_1) \subseteq I_n$ and $s_3 \in I_n$. Since $s_1 \in s_1 o s_3$ and $s_2 o s_1 \neq \phi$ for any s_1, s_2, s_3 in S, we obtain that $s_1 \in (s_1 o s_3) o (s_2 o s_1)$. Therefore $s_1 \in I_n$.

Theorem 3.8 *Let I_n be a set, which is defined in Theorem 3.7. Then, I_n is a commutative hyper K-ideal of types 3,4, 5, 6 and 9.*

Proof. Let $(s_1 o s_2) o s_3 \subseteq I_n$ and $s_3 \in I_n$. Then, we should consider the following situations to prove that I_n is a commutative hyper K-ideal of type 4.

1. $s_1 \neq s_2$, $s_1, s_2 \neq s_0$ and $ords_1 < ords_2$.
   Since I_n is a weak hyper K-ideal, $(s_1 o s_2) o s_3 \subseteq I_n$ and $s_3 \in I_n$, we obtain that $s_1 o s_2 = \bigcup\limits_{ords \leq ords_2} \bar{s} \subseteq I_n$. Also we have: $s_2 o s_1 = \bigcup\limits_{s_0 \neq s, ords \leq ords_2} \bar{s}$, so

$$s_2 o (s_2 o s_1) = s_2 o \left(\bigcup_{s_0 \neq s, ords \leq ords_2} \bar{s} \right) = \bigcup_{ords \leq ords_2} \bar{s}$$

and

$$s_1o(s_2o(s_2os_1)) = s_1o\left(\bigcup_{ords\leq ords_2}\bar{s}\right) = \bigcup_{ords\leq ords_2}\bar{s}.$$

It follows that $s_1o(s_2o(s_2os_1)) \subseteq I_n$.

2. $s_1 \neq s_2$, $s_1, s_2 \neq s_0$ and $ords_1 > ords_2$.
Since $s_0 \in I_n$ and $s_1os_2 = \bigcup_{s_0\neq s, ords\leq ords_1}\bar{s} \subseteq I_n$, we obtain that

$$\bigcup_{ords\leq ords_1}\bar{s} \subseteq I_n.$$

Also we have:

$$s_2os_1 = \bigcup_{ords\leq ords_1}\bar{s},\ s_2o(s_2os_1) = s_2o\left(\bigcup_{ords\leq ords_1}\bar{s}\right) = \bigcup_{ords\leq ords_1}\bar{s}$$

and

$$s_1o(s_2o(s_2os_1)) = \bigcup_{ords\leq ords_1}\bar{s}.$$

Hence, $s_1o(s_2o(s_2os_1)) \subseteq I_n$.

3. $s_1 \neq s_2$, $s_1, s_2 \neq s_0$ and $ords_1 = ords_2$.
Since $s_0 \in I_n$ and $s_1os_2 = \bigcup_{s_0\neq s, ords\leq ords_1}\bar{s} \subseteq I_n$, we get that

$$\bigcup_{ords\leq ords_1}\bar{s} \subseteq I_n.$$

Also we have:

$$s_2os_1 = \bigcup_{s_0\neq s, ords\leq ords_1}\bar{s},\ s_2o(s_2os_1) = s_2o\left(\bigcup_{s_0\neq s, ords\leq ords_1}\bar{s}\right) = \bigcup\nolimits_{ords\leq ords_1}\bar{s}$$

and

$$s_1o(s_2o(s_2os_1)) = s_1o\left(\bigcup_{ords\leq ords_1}\bar{s}\right) = \bigcup_{ords\leq ords_1}\bar{s}.$$

It follows that $s_1o(s_2o(s_2os_1)) \subseteq I_n$.

4. $s_1 = s_2$.
We know that $s_1os_2 = \bigcup_{ords\leq ords_1}\bar{s} \subseteq I_n$ and we have:

$$s_2os_1 = \bigcup_{ords\leq ords_1}\bar{s},\ s_2o(s_2os_1) = s_2o\left(\bigcup_{ords\leq ords_1}\bar{s}\right) = \bigcup_{ords\leq ords_1}\bar{s}$$

and

$$s_1o(s_2o(s_2os_1)) = \bigcup_{ords \leq ords_1} \bar{s}.$$

Hence, $s_1o(s_2o(s_2os_1)) \subseteq I_n$.

5. $s_1 = s_0$, $s_2 \neq s_0$.
   Since $s_0 \in I_n$,

   $$s_2os_1 = s_2os_0 = s_2,\ s_2o(s_2os_1) = s_2o(s_2os_0) = s_2os_2 = \bigcup_{ords \leq ords_2} \bar{s}$$

   and

   $$s_1o(s_2o(s_2os_1)) = s_0o\left(\bigcup_{ords \leq ords_2} \bar{s}\right) = s_0,$$

   we obtain that $s_1o(s_2o(s_2os_1)) \subseteq I_n$.

6. $s_1 \neq s_0$, $s_2 = s_0$.
   Since $s_1os_2 = s_1os_0 = s_1 \subseteq I_n$, we get that $s_1 \in I_n$. On the other hand we have: $(s_2os_1) = s_0os_1 = s_0$, $s_2o(s_2os_1) = s_0o(s_0os_1) = s_0os_0 = s_0$ and $s_1o(s_2o(s_2os_1)) = s_1os_0 = s_1$. It follows that $s_1o(s_2o(s_2os_1)) \subseteq I_n$. So we obtain that I_n is a commutative hyper K-ideal of type 4.

Since $s_1o(s_2o(s_2os_1)) \neq \phi$ and I_n is a commutative hyper K-ideal of type 4, we get that I_n is a commutative hyper K-ideal of type 5.

On the other hand, by Theorem 2.8 we can easily see that I_n is a commutative hyper K-ideal of types 6, 9 and 3.

Theorem 3.9 *Let I_n be a set, which is defined in Theorem 3.7. Then I_n is a positive implicative hyper K-ideal of types 1, 2, 3, 4, 5, 6, 7, 8 and 9.*

Proof. Let for all s_1, s_2, s_3 in S, $(s_1os_2)os_3 \subseteq I_n$. By definition of hyper K-algebra we know that $(s_1os_2)os_3 = (s_1os_3)os_2$. So $(s_1os_3)os_2 \subseteq I_n$, and also we have for any s_1, s_2, s_3 in S, if $s_1os_2 \subseteq I_n$, then $s_1 \in I_n$, because $s_1 \in s_1os_2$. So if $(s_1os_3)os_2 \subseteq I_n$, then $s_1os_3 \subseteq I_n$. Therefore I_n is a positive implicative hyper K-ideal of types 1, 4 and 8. Also since for any s_1, s_3 in S, $s_1os_3 \neq \phi$ and $s_1os_3 \subseteq I_n$, we obtain that $s_1os_3 \cap I_n \neq \phi$ and $s_1os_3 < I_n$. So I_n is a positive implicative hyper K-ideal of types 2, 3, 5, 6, 7 and 9.

Remark 3.10 In (S, o, s_0), let $\exists n, m \in N$ such that $\overline{s_n} \neq \phi$, $\overline{s_m} \neq \phi$ and $m > n$. Here we give a subset I_n of S, which is not a positive implicative hyper K-ideal of types 10, 11, 12, , 26 and 27 but it is similar to the set I_n which is defined in Theorem 3.7.

Let $I_n = \bigcup_{i=0}^{n} \overline{s_i}$, $y \in \overline{s_n}$, $x \in \overline{s_m}$ and $z = s_0$, then we have:

1. Since $((xoy)oz) = (xoy)os_0 = xoy = \bigcup_{s_0\neq s, ords\leq ordx} \bar{s}$, then $y \in \bigcup_{s_0\neq s, ords\leq ordx} \bar{s}$. On the other hand we know that $y < y$. Thus $((xoy)oz) \cap I_n \neq \phi$ and $(xoy)oz < I_n$.
2. Since $yoz = yos_0 = y$, we obtain that $yoz \subseteq I_n$, $yoz < I_n$ and $yoz \cap I_n \neq \phi$.
3. Since $xoz = xos_0 = x$ and $x \notin I_n$, we get that $xoz \nsubseteq I_n$, $xoz \cap I_n = \phi$ and $x \nless I_n$.

By (1), (2), (3) and definition of the positive implicative hyper K-ideals of types 10, 11, 12,..., 26 and 27, we conclude that I_n is not a positive implicative hyper K-ideal of types 10,..., 27.

Theorem 3.11 *(S, o, s_0) is a hyper normal K-algebra of types 1 and 2 but it may not be simple.*

Proof. Since $a \in a$ and $a < a$, for any a, t in S, we have:

$$\begin{array}{lll} {}_{l1}a = \{t \in S | a < aot\} = S, & & {}_{l2}a = \{t \in S | a \in aot\} = S, \ \ \forall a \in S, \\ a_{r1} = \{t \in S | t < toa\} = S & \text{and} & a_{r2} = \{t \in S | t \in toa\} = S, \ \ \forall a \in S. \end{array}$$

On the other hand, it is clear that S is a hyper K-ideal. So, (S, o, s_0) is a hyper normal K-algebra of types 1 and 2.

But, in Example 3.3, we saw that $q_0 \in q_1oq_3$ and $q_0 \notin q_3oq_1$. So $q_1 < q_3$ and $q_3 \nless q_1$. Hence, (S, o, s_0) may not be simple.

Theorem 3.12 *Let (S', M, s'_0, F, t) be a deterministic finite automaton. We define the following hyper operation on S':*

$$\forall (s'_1, s'_2) \in S'^2, s'_1os'_2 = \begin{cases} s'_1 & , if \ \ s'_1 \neq s'_2, \ \ s'_1, s'_2 \neq s_0 \\ s'_1 & , if \ \ s'_2 = s'_0, \ \ s'_1 \neq s'_0 \\ s'_0 & , if \ \ s'_1 = s'_0, \ \ s'_2 \neq s'_0 \\ s'_0 \cup \overline{s'_1} & , if \ \ s'_1 = s'_2. \end{cases}$$

Then (S', o, s'_0) is a hyper K-algebra and s'_0 is the zero element of S'.

Proof. It is easy to see that (S', o, s'_0) satisfies (HK3), (HK4) and (HK5). Also the proof of (HK1) is similar to the proof of (HK1) in Theorem 3.2 by some suitable modifications. Now we consider the following situations to show that (S', o, s'_0) satisfies (HK2).

(i) Let $s'_1, s'_2, s'_3 \neq s'_0$ and $s'_2 \neq s'_3 \neq s'_1 \neq s'_2$. Then $(s'_1os'_2)os'_3 = s'_1os'_3 = s'_1$ and $(s'_1os'_3)os'_2 = s'_1os'_2 = s'_1$. So, in this case (HK2) holds.

(ii) Let $s'_1, s'_2, s'_3 \neq s'_0$ and $s'_3 \neq s'_1 = s'_2$. Then $(s'_1os'_2)os'_3 = (s'_0 \cup \overline{s'_1})os'_3 = s'_0 \cup \overline{s'_1}$ and $(s'_1os'_3)os'_2 = s'_1os'_2 = s'_0 \cup \overline{s'_1}$. Hence, in this case (HK2) holds.

The proof of the following case is the same as (ii).

(iii) $s'_1, s'_2, s'_3 \neq s'_0$ and $s'_2 \neq s'_1 = s'_3$.

(iv) Let $s'_1, s'_2, s'_3 \neq s'_0$ and $s'_1 \neq s'_2 = s'_3$. Then, $(s'_1 o s'_2) o s'_3 = (s'_1 o s'_2) o s'_2$ and $(s'_1 o s'_3) o s'_2 = (s'_1 o s'_2) o s'_2$. Thus, in this case (HK2) holds.

(v) Let $s'_1 = s'_2 = s'_3$. Then $(s'_1 o s'_2) o s'_3 = (s'_1 o s'_1) o s'_1 = (s'_1 o s'_3) o s'_2$.
Therefore, in this case (HK2) holds.

(vi) Let $s'_1 = s'_0$. Then $(s'_1 o s'_2) o s'_3 = (s'_0 o s'_2) o s'_3 = s'_0 o s'_3 = s'_0$ and $(s'_1 o s'_3) o s'_2 = (s'_0 o s'_3) o s'_2 = s'_0 o s'_2 = s'_0$. So in this case (HK2) holds.

(vii) Let $s'_2 = s'_0$. Then $(s'_1 o s'_2) o s'_3 = (s'_1 o s'_0) o s'_3 = s'_1 o s'_3$ and $(s'_1 o s'_3) o s'_2 = (s'_1 o s'_3) o s'_0 = s'_1 o s'_3$. Hence, in this case (HK2) holds.

The proof of the following case is the same as (vii).

(viii) $s'_3 = s'_0$.

(ix) Let $s'_1 = s'_2 = s'_0$. Then $(s'_1 o s'_2) o s'_3 = (s'_0 o s'_0) o s'_3 = s'_0 o s'_3 = s'_0$ and $(s'_1 o s'_3) o s'_2 = (s'_0 o s'_3) o s'_0 = s'_0 o s'_0 = s'_0$. Thus, in this case (HK2) holds.

The proof of the following case is the same as (ix).

(x) $s'_1 = s'_3 = s'_0$.

(xi) Let $s'_2 = s'_3 = s'_0$. Then $(s'_1 o s'_2) o s'_3 = (s'_1 o s'_0) o s'_0 = s'_1 o s'_0 = s'_1$ and $(s'_1 o s'_3) o s'_2 = (s'_1 o s'_0) o s'_0 = s'_1 o s'_0 = s'_1$. So, in this case (HK2) holds.

Finally, we conclude that (S', o, s'_0) is a hyper K-algebra.

Example 3.13 Consider the deterministic finite automaton $A = (S, M, s_0, F, t)$ in Example 3.3. Then the structure of the hyper K-algebra (S, o, s_0) induced on the states of this automaton according to Theorem 3.12 is as follows:

o	q_0	q_1	q_2	q_3
q_0	q_0	q_0	q_0	q_0
q_1	q_1	$\{q_0, q_1, q_2\}$	q_1	q_1
q_2	q_2	q_2	$\{q_0, q_1, q_2\}$	q_2
q_3	q_3	q_3	q_3	$\{q_0, q_3\}$

Note that, if we compare the above table with the table of the Example 3.3, we see that the induced structures of the hyper K-algebras are different. So, the two methods give two different structures.

Theorem 3.14 *Let (S', o, s'_0) be the hyper K-algebra, which is defined in Theorem 3.12, F be a nonempty subset of $\overline{S'}$ and $s'_0 = \overline{s'_0} \in F$. Also, suppose that $I = \bigcup_{\bar{t} \in F} \bar{t}$ and C be a nonempty subset of S'. Then I is a C-absorbing set.*

Proof. Let $s' \in I$ and $t \in C$. then $s'ot = s'$ or $s'ot = s'_0 \cup \overline{s'}$. Since $s' \in I$, by definition of I we know that $\overline{s'} \subseteq I$ and $s'_0 \in I$. Hence $s'ot \subseteq I$.

Theorem 3.15 *Let* (S', o, s'_0) *be the hyper* K*-algebra, which is defined in Theorem* 3.12*. Then* (S', o, s'_0) *is*

1. *a hyper normal* K*-algebra of types* 1 *and* 2,
2. *a simple hyper* K*-algebra*

Proof.

1. Since $a \in aot$ and $a < a$, for any a, t in S', we have:

$$\begin{aligned} {}_{l1}a &= \{t \in S' | a < (aot)\} = S', & {}_{l2}a &= \{t \in S' | a \in (aot)\} = S', \quad \forall a \in S', \\ a_{r1} &= \{t \in S' | t < (toa)\} = S' \quad \text{and} & a_{r2} &= \{t \in S | t \in (toa)\} = S', \quad \forall a \in S'. \end{aligned}$$

   We know that S' is a hyper K-ideal. So, (S', o, s'_0) is a hyper normal K-algebra of types 1 and 2.

2. Let $s'_1 \neq s'_2$ and $s'_1, s'_2 \neq s'_0$, then $s'_1 o s'_2 = s'_1$ and $s'_2 o s'_1 = s'_2$. Hence, $s'_1 \not< s'_2$ and $s'_2 \not< s'_1$. So (S', o, s'_0) is a simple hyper K-algebra.

Theorem 3.16 *Let* (S, M, s_0, F, t) *be a deterministic finite automaton. We define the following hyper operation on* $\bar{S}$ *:*

$$\forall (\overline{s_1}, \overline{s_2}) \in \bar{S}^2, \overline{s_1} o \overline{s_2} = \begin{cases} \{\overline{s_1}, \overline{s_2}\}, & \textit{if} \quad \overline{s_1} \neq \overline{s_2}, \quad \overline{s_1} \neq \overline{s_0} \neq \overline{s_2} \\ \{\overline{s_1}, \overline{s_0}\}, & \textit{if} \quad \overline{s_1} = \overline{s_2} \\ \{\overline{s_2}, \overline{s_0}\}, & \textit{if} \quad \overline{s_1} = \overline{s_0}, \quad \overline{s_2} \neq \overline{s_0} \\ \overline{s_1}, & \textit{if} \quad \overline{s_1} \neq \overline{s_0}, \quad \overline{s_2} = \overline{s_0}. \end{cases}$$

Then, $(\bar{S}, o, \overline{s_0})$ *is a hyper* K*-algebra and* $\overline{s_0}$ *is the zero element of* $\bar{S}$*.*

Proof. It is easy to see that $(\bar{S}, o, \overline{s_0})$ satisfies (HK3), (HK4) and (HK5). Also the proof of (HK1) is similar to the proof of (HK1) in Theorem 3.2 by some suitable modifications. Now we consider the following situations to show that $(\bar{S}, o, \overline{s_0})$ satisfies (HK2).

(i) Let $\overline{s_1}, \overline{s_2}, \overline{s_3} \neq \overline{s_0}$ and $\overline{s_1} \neq \overline{s_2} \neq \overline{s_3} \neq \overline{s_1}$. Then
$(\overline{s_1} o \overline{s_2}) o \overline{s_3} = \{\overline{s_1}, \overline{s_2}\} o \overline{s_3} = \{\overline{s_1}, \overline{s_2}, \overline{s_3}\}$ and
$(\overline{s_1} o \overline{s_3}) o \overline{s_2} = \{\overline{s_1}, \overline{s_3}\} o \overline{s_2} = \{\overline{s_1}, \overline{s_2}, \overline{s_3}\}$.
So, in this case (HK2) holds.

(ii) Let $\overline{s_1}, \overline{s_2}, \overline{s_3} \neq \overline{s_0}$ and $\overline{s_1} = \overline{s_2} \neq \overline{s_3}$. Then
$(\overline{s_1} o \overline{s_2}) o \overline{s_3} = \{\overline{s_1}, \overline{s_0}\} o \overline{s_3} = \{\overline{s_1}, \overline{s_2}, \overline{s_0}\}$ and
$(\overline{s_1} o \overline{s_3}) o \overline{s_2} = \{\overline{s_1}, \overline{s_3}\} o \overline{s_1} = \{\overline{s_1}, \overline{s_3}, \overline{s_0}\}$.
Hence, in this case (HK2) holds.

(iii) Let $\overline{s_1}, \overline{s_2}, \overline{s_3} \neq \overline{s_0}$ and $\overline{s_1} = \overline{s_3} \neq \overline{s_2}$. Then
$(\overline{s_1} o \overline{s_2}) o \overline{s_3} = \{\overline{s_1}, \overline{s_2}\} o \overline{s_1} = \{\overline{s_1}, \overline{s_2}, \overline{s_0}\}$ and
$(\overline{s_1} o \overline{s_3}) o \overline{s_2} = \{\overline{s_1}, \overline{s_0}\} o \overline{s_2} = \{\overline{s_1}, \overline{s_2}, \overline{s_0}\}$.
Thus, in this case (HK2) holds.

(iv) Let $\overline{s_1}, \overline{s_2}, \overline{s_3} \neq \overline{s_0}$ and $\overline{s_2} = \overline{s_3} \neq \overline{s_1}$. Then
$(\overline{s_1} o \overline{s_2}) o \overline{s_3} = \{\overline{s_1}, \overline{s_2}\} o \overline{s_2} = \{\overline{s_1}, \overline{s_2}, \overline{s_0}\}$ and
$(\overline{s_1} o \overline{s_3}) o \overline{s_2} = \{\overline{s_1}, \overline{s_2}\} o \overline{s_2} = \{\overline{s_1}, \overline{s_2}, \overline{s_0}\}$.
Therefore, in this case (HK2) holds.

(v) Let $\overline{s_1}, \overline{s_3} \neq \overline{s_0}$, $\overline{s_1} = \overline{s_0}$ and $\overline{s_2} \neq \overline{s_3}$. Then
$(\overline{s_1} o \overline{s_2}) o \overline{s_3} = \{\overline{s_0}, \overline{s_2}\} o \overline{s_3} = \{\overline{s_2}, \overline{s_3}, \overline{s_0}\}$ and
$(\overline{s_1} o \overline{s_3}) o \overline{s_2} = \{\overline{s_0}, \overline{s_3}\} o \overline{s_2} = \{\overline{s_2}, \overline{s_3}, \overline{s_0}\}$.
So, in this case (HK2) holds.

(vi) Let $\overline{s_1}, \overline{s_3} \neq \overline{s_0}$, $\overline{s_2} = \overline{s_0}$ and $\overline{s_1} \neq \overline{s_3}$. Then,
$(\overline{s_1} o \overline{s_2}) o \overline{s_3} = \overline{s_1} o \overline{s_3} = \{\overline{s_1}, \overline{s_3}\}$ and $(\overline{s_1} o \overline{s_3}) o \overline{s_2} = \{\overline{s_1}, \overline{s_3}\} o \overline{s_0} = \{\overline{s_1}, \overline{s_3}\}$.
Hence, in this case (HK2) holds.

The proof of the following case is the same as (vi).

(vii) $\overline{s_1}, \overline{s_2} \neq \overline{s_0}, \overline{s_3} = \overline{s_0}$ and $\overline{s_1} \neq \overline{s_2}$.

(viii) Let $\overline{s_2}, \overline{s_3} \neq \overline{s_0}, \overline{s_1} = \overline{s_0}$ and $\overline{s_2} = \overline{s_3}$ Then
$(\overline{s_1} o \overline{s_2}) o \overline{s_3} = \{\overline{s_0}, \overline{s_2}\} o \overline{s_2} = \{\overline{s_0}, \overline{s_2}\}$ and $(\overline{s_1} o \overline{s_3}) o \overline{s_2} = \{\overline{s_0}, \overline{s_2}\} o \overline{s_2} = \{\overline{s_0}, \overline{s_2}\}$.
Thus in this case (HK2) holds.

(ix) Let $\overline{s_1}, \overline{s_3} \neq \overline{s_0}$, $\overline{s_2} = \overline{s_0}$ and $\overline{s_1} = \overline{s_3}$. Then
$(\overline{s_1} o \overline{s_2}) o \overline{s_3} = \overline{s_1} o \overline{s_1} = \{\overline{s_0}, \overline{s_1}\}$ and $(\overline{s_1} o \overline{s_3}) o \overline{s_2} = \{\overline{s_0}, \overline{s_1}\} o \overline{s_0} = \{\overline{s_0}, \overline{s_1}\}$.
So in this case (HK2) holds.

The proof of the following case is the same as (ix).

(x) $\overline{s_1}, \overline{s_2} \neq \overline{s_0}, \overline{s_3} = \overline{s_0}$ and $\overline{s_1} = \overline{s_2}$.

(xi) Let $\overline{s_1} = \overline{s_2} = \overline{s_0}$ and $\overline{s_3} \neq \overline{s_0}$ Then
$(\overline{s_1} o \overline{s_2}) o \overline{s_3} = \overline{s_0} o \overline{s_3} = \{\overline{s_0}, \overline{s_3}\}$ and $(\overline{s_1} o \overline{s_3}) o \overline{s_2} = \{\overline{s_0}, \overline{s_3}\} o \overline{s_0} = \{\overline{s_0}, \overline{s_3}\}$.
Hence, in this case (HK2) holds.

(xii) Let $\overline{s_1} = \overline{s_3} = \overline{s_0}$ and $\overline{s_2} \neq \overline{s_0}$ Then
$(\overline{s_1} o \overline{s_2}) o \overline{s_3} = \{\overline{s_0}, \overline{s_2}\} o \overline{s_0} = \{\overline{s_0}, \overline{s_2}\}$ and $(\overline{s_1} o \overline{s_3}) o \overline{s_2} = \overline{s_0} o \overline{s_2} = \{\overline{s_0}, \overline{s_2}\}$.
Thus, in this case (HK2) holds.

(xiii) Let $\overline{s_2} = \overline{s_3} = \overline{s_0}$ and $\overline{s_1} \neq \overline{s_0}$ Then
$(\overline{s_1} o \overline{s_2}) o \overline{s_3} = \overline{s_1} o \overline{s_0} = \overline{s_1}$ and $(\overline{s_1} o \overline{s_3}) o \overline{s_2} = \overline{s_1} o \overline{s_0} = \overline{s_1}$.
Hence, in this case (HK2) holds.

(xiv) Let $\overline{s_1} = \overline{s_2} = \overline{s_3}$. Then
$(\overline{s_1} o \overline{s_2}) o \overline{s_3} = (\overline{s_1} o \overline{s_1}) o \overline{s_1} = (\overline{s_1} o \overline{s_3}) o \overline{s_2}$.
Hence, in this case (HK2) holds.

Finally, we obtain that $(\bar{S}, o, \overline{s_0})$ is a hyper K-algebra.

Theorem 3.17 *Consider the deterministic finite automaton* $A = (S, M, s_0, F, t)$ *in Example* 3.3. *Then the structure of the hyper* K*-algebra* $(\bar{S}, o, \overline{s_0})$ *induced on* $\bar{S}$ *according to Theorem* 3.16 *is as follows:*

o	$\overline{q_0}$	$\overline{q_1}$	$\overline{q_3}$
$\overline{q_0}$	$\overline{q_0}$	$\{\overline{q_0}, \overline{q_1}\}$	$\{\overline{q_0}, \overline{q_3}\}$
$\overline{q_1}$	$\overline{q_1}$	$\{\overline{q_0}, \overline{q_1}\}$	$\{\overline{q_1}, \overline{q_3}\}$
$\overline{q_3}$	$\overline{q_3}$	$\{\overline{q_3}, \overline{q_1}\}$	$\{\overline{q_0}, \overline{q_3}\}$

Theorem 3.18 *Let* $(\bar{S}, o, \overline{s_0})$ *be the hyper* K*-algebra, which is defined in Theorem* 3.16. *Then* $(\bar{S}, o, \overline{s_0})$ *is*

(1) *a hyper normal* K*-algebra of types* 1 *and* 2,

(2) *a simple hyper* K*-algebra.*

Proof.

1. Since $\bar{a} \in \bar{a}o\bar{t}$ and $\bar{a} < \bar{a}$, for any $\bar{a}, \bar{t}$ in $\bar{S}$, we have:

$$\begin{aligned} {}_{l1}a &= \{\bar{t} \in \bar{S} | \bar{a} < \bar{a}o\bar{t}\} = \bar{S}, \qquad & {}_{l2}a &= \{\bar{t} \in \bar{S} | \bar{a} \in \bar{a}o\bar{t}\} = \bar{S}, \quad \forall \bar{a} \in \bar{S}, \\ \bar{a}_{r1} &= \{\bar{t} \in \bar{S} | \bar{t} < \bar{t}o\bar{a}\} = \bar{S} \quad \text{and} & a_{r2} &= \{\bar{t} \in \bar{S} | \bar{t} \in \bar{t}o\bar{a}\} = \bar{S}, \quad \forall \bar{a} \in \bar{S}. \end{aligned}$$

   It is easy to see that $\bar{S}$ is a hyper K-ideal. So $(\bar{S}, o, \overline{s_0})$ is a hyper normal K-algebra of types 1 and 2.

2. Let $\overline{s_1} \neq \overline{s_2}$ and $\overline{s_1}, \overline{s_2} \neq \overline{s_0}$, then $\overline{s_1} o \overline{s_2} = \{\overline{s_1}, \overline{s_2}\}$ and $\overline{s_2} o \overline{s_1} = \{\overline{s_1}, \overline{s_2}\}$. Hence, $\overline{s_1} \not< \overline{s_2}$ and $\overline{s_2} \not< \overline{s_1}$. So $(\bar{S}, o, \overline{s_0})$ is a simple hyper K-algebra.

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COMMON FIXED POINT FOR LIPSCHITZIAN MAPPING SATISFYING RATIONAL CONTRACTIVE CONDITIONS

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Abstract. Common fixed point theorems for a class of mappings called occasionally weakly compatible in a symmetric space (X, d) under Lipschitzian type rational contractive conditions are obtained.

Keywords: weakly occasionally compatible, common fixed point, symmetric space.
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1. Introduction and preliminaries

In 1968, Kannan [14] proved a fixed point theorem for a map satisfying a contractive condition that did not require continuity at each point. This paper was a genesis for a multitude of fixed point papers over the next two decades (see, for example, [11] for a listing and comparison of many of these definitions). A number of these papers dealt with fixed points for more than one map. In some cases commutativity between the maps was required in order to obtain a common fixed point. Sessa [13] coined the term weakly commuting. Jungck [8] generalized the notion of weak commutativity by introducing the concept of compatible maps and then weakly compatible maps [9]. There are examples that show that each of these generalizations of commutativity is a proper extension of the previous definition. Also, during this time a number of authors established fixed point theorems for pair of maps (see, for example, [5], [12]). Recently, Thagafi and Shahzad [4] gave the definition which is a proper generalization of nontrivial weakly compatible maps which do have coincidence points (see, also, [2] and [10]). The aim of this paper is to obtain some fixed points theorem involving occasionally weakly compatible maps in the setting of symmetric space satisfying a rational contractive condition. Our results complement, extend and unify several well known comparable results.

Definition 1.1. Let f and g be self maps of a set X. If $w = fx = gx$ for some x in X, then x is called a *coincidence point* of f and g, and w is called a *point of coincidence* of f and g.

The following concept is a proper generalization of nontrivial weakly compatible maps which a coincidence point.

Definition 1.2. Two selfmaps f and g of a set X are said to be occasionally weakly compatible (owc) iff there is a point x in X which is a coincidence point of f and g at which f and g commute.

Our theorems are proved in symmetric spaces which are more general than metric spaces.

Definition 1.3. Let X be a set. A symmetric on X is a mapping $d : X \times X \to (0, \infty)$ such that

$$d(x, y) = 0 \quad \text{if and only if} \quad x = y,$$

and

$$d(x, y) = d(y, x) \quad \text{for} \quad x, y \in X.$$

We shall also need the following Proposition from [10] (see also [1]).

Proposition 1.4. *Let f and g be occasionally weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

2. Common fixed point theorems

The following result generalizes Theorem 4 of [7].

Theorem 2.1. *Let A, B, S and T be self mappings of a symmetric space X with symmetric d, and*

$$d(Ax, By) \leq a\left[\frac{(d(Ax, Sx))^2 + (d(By, Ty))^2}{d(Ax, Sx) + d(By, Ty)}\right] + bd(Sx, Ty) \tag{2.1}$$

if $d(Ax, Sx) + d(By, Ty) \neq 0$ or $d(Ax, By) = 0$, if $d(Ax, Sx) + d(By, Ty) = 0$, where $a, b > 0$. Then A, B, S and T have a unique common fixed point if the pairs $\{A, S\}$ and $\{B, T\}$ are occasionally weakly compatible.

Proof. Since the pairs $\{A, S\}$ and $\{B, T\}$ are each owc, there exist points $x, y \in X$ such that $Ax = Sx$ and $By = Ty$. From (2.1), we have $Ax = By$. Therefore $Ax = Sx = By = Ty$. Moreover, if there is another point z such that $Az = Sz$, then, using (2.1) it follows that $Az = Sz = By = Ty$, or $Ax = Az$ and $w = Ax = Sx$ is the unique point of coincidence of A and S. By Proposition 1.4, w is the only common fixed point of A and S. By symmetry there is a unique point $z \in X$ such that $z = Bz = Tz$. From (2.1), we obtain, $w = z$ and w is a common fixed point. By the preceding argument it is clear that w is unique.

Theorem 1 of Ahmad and Imdad [3] is a special case of Theorem 2.1.

Theorem 2.2. *Let A, B, S and T be self mappings of a symmetric space X with symmetric d, and*

$$d(Ax, By) \leq a \left[\frac{d(Ax, Sx)d(Sx, By) + d(By, Ty)d(Ty, Ax)}{d(Sx, By) + d(Ty, Ax)} \right] + bd(Sx, Ty) \tag{2.2}$$

if $d(Sx, By) + d(Ty, Ax) \neq 0$ or $d(Ax, By) = 0$, if $d(Sx, By) + d(Ty, Ax) = 0$, where $a, b > 0$ and $b < 1$. Then A, B, S and T have a unique common fixed point if the pairs $\{A, S\}$ and $\{B, T\}$ are occasionally weakly compatible.

Proof. Since the pairs $\{A, S\}$ and $\{B, T\}$ are each owc, there exist points $x, y \in X$ such that $Ax = Sx$ and $By = Ty$. Now we claim that $Ax = By$, if not, then $d(Sx, By) + d(Ty, Ax) \neq 0$. From (2.2),

$$d(Ax, By) \leq bd(Ax, By),$$

a contradiction. Thus, we have $Ax = By$, and $Ax = Sx = By = Ty$. Moreover, if there is another point z such that $Az = Sz$, then, $Az = By$. If not, then $d(Sz, By) + d(Ty, Az) \neq 0$. Using (2.2),

$$d(Az, By) \leq bd(Az, By),$$

a contradiction and hence, it follows that $Az = Sz = By = Ty$, or $Ax = Az$ and $w = Ax = Sx$ is the unique point of coincidence of A and S. By Proposition 1.4, w is the only common fixed point of A and S. By symmetry there is a unique point $z \in X$ such that $z = Bz = Tz$. From (2.2), we obtain, $w = z$ and w is a common fixed point. By the preceding argument it is clear that w is unique.

Theorem 5 of [7] and Theorem 2 of Ahmad and Imdad [6] are a special cases of Theorem 2.2.

Theorem 2.3. *Let A, B, S and T be self mappings of a symmetric space X with symmetric d, and*

$$d(Ax, By) \leq \frac{ad(Ax, Sx)d(Ty, By) + bd(Sx, By)d(Ty, Ax)}{d(Sx, Ax) + d(By, Ty)}] + cd(Sx, Ty) \tag{2.3}$$

if $d(Sx, Ax) + d(By, Ty) \neq 0$ or $d(Ax, By) = 0$, if $d(Sx, Ax) + d(By, Ty) = 0$, where $a, b, c > 0$. Then A, B, S and T have a unique common fixed point if the pairs $\{A, S\}$ and $\{B, T\}$ are occasionally weakly compatible.

Proof. The proof is similar to that of Theorem 2.1, and will therefore be omitted.

Theorem 2.4. *Let* A, B*,and* S *be self mappings of a symmetric space* X *with symmetric* d*, and*

$$\begin{aligned} d(Ax, By) &\leq \left[\frac{ad(Ax, Sx)d(Sy, By) + bd(Sx, By)d(Sy, Ax)}{d(Sx, Ax) + d(By, Sy)}\right] \\ &+ c\left[\frac{d(Ax, Sx)d(Sy, Ax) + d(By, Sy)d(Sx, By)}{d(Sx, Ax) + d(By, Sy)}\right] \end{aligned} \tag{2.4}$$

if $d(Sx, Ax) + d(By, Sy) \neq 0$ *or* $d(Ax, By) = 0$*, if* $d(Sx, Ax) + d(By, Sy) = 0$*, where* $a, b, c > 0$ *and* $c < 1$*. Then* A, B *and* T *have a unique common fixed point if one of the pairs* $\{A, B\}$ *or* $\{A, S\}$ *is occasionally weakly compatible.*

Proof. Suppose that $\{A, S\}$ is owc. Then there exists a point $x \in X$ such that $Ax = Sx$. Now we claim that $Sx = Bx$. If not, then

$$d(Ax, Bx) \leq c\,\frac{(d(Bx, Sx))^2}{d(Bx, Sx)} = cd(Ax, Bx),$$

a contradiction. Thus, $Ax = Bx = Sx$. Now, $AAx = ASx = SAx = SSx$. Since $Bx = Sx$ and $SSx = ASx$, therefore, $d(SSx, ASx) + d(Bx, Sx) = 0$, and $d(ASx, Bx) = 0$. Hence, $ASx = Bx$, which shows Sx is a fixed point of A. Also, Sx is a fixed point of S. Suppose that $Sx \neq BSx$. From (2.4), we have

$$d(Ax, BSx) \leq c\,\frac{(d(BSx, Sx))^2}{d(BSx, Sx)} = cd(BSx, Sx)$$

a contradiction. Therefore, Sx is a common fixed point of A, B and S. Let w and z be two common fixed point of A, B and S. Since

$$d(Sw, Aw) + d(Bz, Sz) = 0,$$

therefore $d(Aw, Bz) = d(w, z) = 0$, $w = z$. A similar argument applies if the pair $\{A, B\}$ is owc.

Theorem 2.5. *Let* A, B*, and* S *be self mappings of a symmetric space* X *with symmetric* d*, and*

$$\begin{aligned} d(Ax, By) &\leq a\,\frac{d(Sx, By)d(Sx, Sy)}{d(Sx, Sy) + d(Sy, By)}] + b[d(Sx, Ax) + d(Sy, By)] \\ & c[d(Sx, By) + d(Sy, Ax)] + dd(Sx, Sy) \end{aligned} \tag{2.5}$$

if $d(Sx, Sy) + d(Sy, By) \neq 0$ *or* $d(Ax, By) = 0$*, if* $d(Sx, Sy) + d(Sy, By) = 0$*, where* $a, b, c > 0$ *and* $a + b + 2c + d < 1$*. Then* A, B *and* T *have a unique common fixed point if one of the pairs* $\{A, B\}$ *or* $\{A, S\}$ *is occasionally weakly compatible.*

Proof. Suppose that $\{A, S\}$ is owc. Then there exist a point $x \in X$ such that $Ax = Sx$. Now we claim that $Sx = Bx$. If not,

$$d(Ax, Bx) \leq bd(Sx, Bx) + cd(Sx, Bx)$$

a contradiction. Thus, $Ax = Bx = Sx$. Now, $AAx = ASx = SAx = SSx$. Since $Bx = Sx$ and $SSx = ASx$, therefore, $d(SSx, ASx) + d(Bx, Sx) = 0$, and $d(ASx, Bx) = 0$. Hence, $ASx = Bx$, which shows Sx is a fixed point of A. Also, Sx is a fixed point of S. Suppose that $Sx \neq BSx$. From (2.5), we have

$$\begin{aligned} d(Ax, BSx) &\leq a\,\frac{d(Sx, BSx)d(Sx, SSx)}{d(SSx, BSx)} \\ &= a\,\frac{d(Sx, BSx)d(Sx, Sx)}{d(Sx, BSx)} = 0 \end{aligned}$$

a contradiction. Therefore, Sx is a common fixed point of A, B and S. Let w and z be two common fixed point of A, B and S. If, $w \neq z$, then,

$$\begin{aligned} d(w, z) &= d(Aw, Bz) \\ &\leq ad(Sw, Bz) + c[d(Sw, Bz) + d(Sz, Aw)] + dd(Sw, Sz) \\ &= (a + 2c + d)d(w, z) \end{aligned}$$

a contradiction, therefore $w = z$. The proof of the result assuming $\{A, B\}$ is owc is similar.

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FUZZY MINIMAL STRUCTURES AND FUZZY MINIMAL SUBSPACES

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Abstract. At the present paper, the notions of induced fuzzy minimal structures, fuzzy minimal subspaces and relatively fuzzy minimal continuous functions are introduced and studied.

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Keywords: fuzzy topological space, induced fuzzy topology, fuzzy continuous function.

1. Introduction

After the discovery of the fuzzy sets by Zadeh [17], many attempts have been made to extend various branches of mathematics to the fuzzy setting. Fuzzy topological spaces as a very natural generalization of topological spaces were first put forward in the literature by Chang [7] in 1968. He studied a number of the basic concepts including interior and closure of a fuzzy set, fuzzy continuous mapping and fuzzy compactness. Many authors used Chang's definition in many direction to obtain some results which are compatible with results in general topology. In 1976, Lowen [9] suggested an alternative and more natural definition for achieving more results which are compatible to the general case in topology. For example with Chang's definition, constant functions between fuzzy topological spaces are not necessarily fuzzy continuous but in Lowen's sense all of the constant functions are fuzzy continuous. In 1985, Sostak [16] introduced the smooth fuzzy topology as an extension of Chang's fuzzy topology.

The concept of minimal structure and minimal spaces, as a generalization of topology and topological spaces were introduced in [11]. Further results about minimal spaces can be found in [2], [5], [10] and [15]. Recently, Alimohammady and Roohi [3], [4] introduced and studied the notions of fuzzy minimal structures and fuzzy minimal spaces.

This paper is organized as follows. In Section 2, some preparatory definitions and results about fuzzy sets which are used in other sections are given. Section 3 is devoted to reviewing some basic definitions and results on fuzzy minimal structures and fuzzy minimal spaces. Also, some new results and an example are investigated. Finally, in Section 4, the concepts of induced fuzzy minimal structures, fuzzy minimal subspaces and relatively fuzzy minimal continuous functions are introduced and studied.

2. Preliminaries

To ease understanding of the material incorporated in this paper, we recall some basic definitions and results. For details on the following notions, we refer to [1], [3], [4], [12], [13] and references therein.

A *fuzzy set* in(on) a universe set X is a function with domain X and values in $I = [0, 1]$. The class of all fuzzy sets on X will be denoted by I^X and symbols A,B,... is used for fuzzy sets on X. For two fuzzy sets A and B in X, we say that A is *contained in* B provided $A(x) \leq B(x)$ for all $x \in X$. The *complement* of A is denoted by A^c and is defined by $A^c(x) = 1 - A(x)$. 01_X is called *empty fuzzy set* where 1_X is the characteristic function on X. A family τ of fuzzy sets in X is called a *fuzzy topology* for X if

(a) $\alpha 1_X \in \tau$ for each $\alpha \in I$,

(b) $A \wedge B \in \tau$, where $A, B \in \tau$ and

(c) $\bigvee_{\alpha \in \mathcal{A}} A_\alpha \in \tau$ whenever, $A_\alpha \in \tau$ for all α in $\mathcal{A}$. The pair (X, τ) is called a *fuzzy topological space* [9]. Every member of τ is called *fuzzy open set* and its complement is called *fuzzy closed sets* [9]. In a fuzzy topological space X, the *interior* and the *closure* of a fuzzy set A (denoted by $Int(A)$ and $Cl(A)$ respectively) are defined by

$$Int(A) = \bigvee \{U \ : \ U \leq A, \ U \text{ is fuzzy open set}\} \text{ and}$$

$$Cl(A) = \bigwedge \{F \ : \ A \leq F, \ F \text{ is fuzzy closed set}\}.$$

Let f be a function from X to Y. It is a fuzzy function defined by

$$f(A)(y) = \begin{cases} \bigvee\limits_{x \in f^{-1}(\{y\})} A(x) & f^{-1}(\{y\}) \neq \emptyset \\ 0 & f^{-1}(\{y\}) = \emptyset, \end{cases}$$

for all y in Y, where A is an arbitrary fuzzy set in X and also $f^{-1}(B) = Bof$ for any fuzzy set B in Y [17].

A fuzzy set in X is called a *fuzzy point* if it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at x is λ $(0 < \lambda \leq 1)$, we denote this fuzzy point by x_λ, where the point x is called its *support* [12], [13].

3. Fuzzy minimal spaces

Definition 3.1. A family $\mathcal{M}$ of fuzzy sets in X is said to be a

(a) *fuzzy minimal structure in Lowen sense* on X if $\lambda 1_X \in \mathcal{M}$ for any $\lambda \in I$, where $I = [0,1]$ ([4]).

(b) *fuzzy minimal structure in Chang sense* on X if $\lambda 1_X \in \mathcal{M}$ for any $\lambda \in \{0,1\}$ ([3]).

In these cases, $(X, \mathcal{M})$ is called a *fuzzy minimal space in Lowen sense* (resp. Chang sense).

In the rest of this paper, fuzzy minimal structure is used for fuzzy minimal structure in Lowen sense.

A fuzzy set $A \in I^X$ is said to be *fuzzy m-open* if $A \in \mathcal{M}$ and also $B \in I^X$ is called a *fuzzy m-closed* set if $B^c \in \mathcal{M}$. Let

$$m - Int(A) = \bigvee\{U : U \leq A, U \in \mathcal{M}\} \quad \text{and} \tag{3.1}$$

$$m - Cl(A) = \bigwedge\{F : A \leq F, F^c \in \mathcal{M}\}. \tag{3.2}$$

Proposition 3.2. [4], [6] *For any two fuzzy sets A and B,*

(a) *m-$Int(A) \leq A$ and m-$Int(A) = A$ if A is a fuzzy m-open set.*

(b) *$A \leq m$-$Cl(A)$ and $A = m$-$Cl(A)$ if A is a fuzzy m-closed set.*

(c) *m-$Int(A) \leq m$-$Int(B)$ and m-$Cl(A) \leq m$-$Cl(B)$ if $A \leq B$.*

(d) *m-$Int(A \wedge B) = (m$-$Int(A)) \wedge (m$-$Int(B))$ and $(m$-$Int(A)) \vee (m$-$Int(B)) \leq m$-$Int(A \vee B)$.*

(e) *m-$Cl(A \vee B) = (m$-$Cl(A)) \vee (m$-$Cl(B))$ and m-$Cl(A \wedge B) \leq (m$-$Cl(A)) \wedge (m$-$Cl(B))$.*

(f) *m-$Int(m$-$Int(A)) = m$-$Int(A)$ and m-$Cl(m$-$Cl(B)) = m$-$Cl(B)$.*

(g) *$(m$-$Cl(A))^c = m$-$Int(A^c)$ and $(m$-$Int(A))^c = m$-$Cl(A^c)$.*

Definition 3.3. [4] A fuzzy minimal space $(X, \mathcal{M})$ enjoys the *property U* if arbitrary union of fuzzy m-open sets is fuzzy m-open. Also, we say that $(X, \mathcal{M})$ has the *property I*, if any finite intersection of fuzzy m-open sets is fuzzy m-open.

Proposition 3.4. [1] *For a fuzzy minimal structure $\mathcal{M}$ on a set X, the following are equivalent.*

(a) *$(X, \mathcal{M})$ has the property U.*

(b) *If m-$Int(A) = A$, then $A \in \mathcal{M}$.*

(c) *If m-$Cl(B) = B$, then $B^c \in \mathcal{M}$.*

Definition 3.5. [4] Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be two fuzzy minimal spaces. We say that a fuzzy function $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is *fuzzy minimal continuous* (briefly fuzzy m-continuous) if $f^{-1}(B) \in \mathcal{M}$, for any $B \in \mathcal{N}$.

Theorem 3.6. [3], [4] *Suppose $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are fuzzy minimal spaces. Then*

(a) *the identity map $id_X : (X, \mathcal{M}) \to (X, \mathcal{M})$ is fuzzy m-continuous,*

(b) *$id_X : (X, \mathcal{M}) \to (X, \mathcal{N})$ is fuzzy m-continuous if and only if $\mathcal{N} \subseteq \mathcal{M}$,*

(c) *Any constant function $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is fuzzy m-continuous.*

Theorem 3.7. [3], [4] *Consider the following properties for a fuzzy function $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ between two fuzzy minimal spaces.*

(a) *f is a fuzzy m-continuous function.*

(b) *$f^{-1}(B)$ is a fuzzy m-closed set for each fuzzy m-closed set $B \in I^Y$.*

(c) *m-$Cl(f^{-1}(B)) \leq f^{-1}(m$-$Cl(B))$ for each $B \in I^Y$.*

(d) *$f(m$-$Cl(A)) \leq m$-$Cl(f(A))$ for any $A \in I^X$.*

(e) *$f^{-1}(m$-$Int(B)) \leq m$-$Int(f^{-1}(B))$ for each $B \in I^Y$.*

Then (a)⇔(b)⇒(c)⇔(d)⇔(e). *Moreover, if $(X, \mathcal{M})$ satisfies in property U then all above statements are equivalent.*

Example 3.8. Let $X=\{x, y\}$, $\mathcal{M}=\{\lambda 1_X : \lambda \in I\} \cup \{x_1\}$ and $\mathcal{N}=\{\lambda 1_X : \lambda \in I\} \cup \{x_{\frac{1}{2}}\}$. It follows from part (b) of Theorem 3.6 that $Id_X : (X, \mathcal{M}) \longrightarrow (X, \mathcal{N})$ is not fuzzy m-continuous. Let m_1-Cl and m_2-Cl are denoted for fuzzy minimal closure in $(X, \mathcal{M})$ and $(X, \mathcal{N})$ respectively. Then, for any fuzzy set B in X with $B(x) = s$ and $B(y) = t$, it follows from (3.2) that m_1-Cl(f^{-1}(B)) $\leq$ f^{-1}(m$_2$-Cl(B)) for each $B \in I^X$.

In [4], for a family of fuzzy functions, authors achieved a weakest fuzzy minimal structure for which all members of it are fuzzy m-continuous. As a consequence, *fuzzy product minimal structure* for an arbitrary family $\{(X_\alpha, \mathcal{M}_\alpha) : \alpha \in \mathcal{A}\}$ of fuzzy minimal spaces are introduced. In fact, fuzzy product minimal structure on $X = \prod_{\alpha \in A} X_\alpha$ is the weakest fuzzy minimal structure on X (denoted by $\mathcal{M} = \prod_{\alpha \in \mathcal{A}} \mathcal{M}_\alpha$) such that for each $\alpha \in \mathcal{A}$ the canonical projection $\pi_\beta : X \longrightarrow X_\beta$ is fuzzy m-continuous. It should be noticed that fuzzy product minimal structure for two fuzzy minimal spaces $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ is the family of fuzzy sets

$$\mathcal{M} \times \mathcal{N} = \{1_X \times V : V \in \mathcal{N}\} \cup \{U \times 1_Y : U \in \mathcal{M}\}.$$

Similarly, one can verify that fuzzy product minimal structure of $\{(X_j, \mathcal{M}_j) : j = 1, 2, ..., n\}$ is

$$\prod_{j=1}^{n} \mathcal{M}_j = \bigcup_{j=1}^{n} \left\{ \prod_{l=1}^{n} F_l : F_l = \left\{ \begin{array}{ll} 1_{X_l} & l \neq j \\ U_j & l = j, \text{ where, } U_j \in \mathcal{M}_j \end{array} \right. \right\}. \tag{3.3}$$

We use $\mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_n$ instead of $\prod\limits_{j=1}^{n} \mathcal{M}_i$ and specially $\mathcal{M}_1 \times \mathcal{M}_2$ instead of $\prod\limits_{j=1}^{2} \mathcal{M}_i$.

Theorem 3.9. [4] *Suppose $\{(X_\alpha, M_\alpha) : \alpha \in \mathcal{A}\}$ is a family of fuzzy minimal spaces. Equip X by the fuzzy product minimal structure $\mathcal{M}$ generated by $\{\pi_\alpha : \alpha \in \mathcal{A}\}$. Then f is fuzzy m-continuous function if and only if $\pi_\alpha o f$ is fuzzy m-continuous for all $\alpha \in \mathcal{A}$, where $f : (Y, \mathcal{N}) \to (X, \mathcal{M})$ is a mapping.*

Theorem 3.10. *Suppose $(X, \mathcal{M})$ is a fuzzy minimal space, $\{(Y_\alpha, \mathcal{M}_\alpha) : \alpha \in \mathcal{A}\}$ is a family of fuzzy minimal spaces and also suppose that $(Y, \mathcal{N})$ is the fuzzy product minimal space of this family. Then for all $\alpha \in \mathcal{A}$, $f_\alpha : (X, \mathcal{M}) \longrightarrow (Y_\alpha, \mathcal{M}_\alpha)$ is fuzzy m-continuous if and only if $f : (X, \mathcal{M}) \longrightarrow (Y, \mathcal{N})$, defined by $f(x) = (f_\alpha(x))_\alpha$, is fuzzy m-continuous.*

Proof. Clearly, $\pi_\alpha o f = f_\alpha$ and hence $\pi_\alpha o f$ is fuzzy m-continuous for all $\alpha \in \mathcal{A}$. That f is fuzzy m-continuous follows from Theorem 3.9.

Theorem 3.11. *Suppose $(X, \mathcal{M})$, $(Y, \mathcal{N})$ are fuzzy minimal spaces. Then for each $y_0 \in Y$ the mapping $i_{y_o} : (X, \mathcal{M}) \longrightarrow (X \times Y, \mathcal{M} \times \mathcal{N})$ defined by $i_{y_0}(x) = (x, y_0)$ is fuzzy m-continuous.*

Proof. By part (c) of Theorem 3.6, the mapping $C_{y_0} : (X, \mathcal{M}) \longrightarrow (Y, \mathcal{N})$ defined by $C_{y_0}(x) = y_0$, for all $x \in X$, is fuzzy m-continuous also part (a) of Theorem 3.6 implies that the identity mapping $id_X : (X, \mathcal{M}) \longrightarrow (X, \mathcal{M})$ is fuzzy m-continuous too. That i_{y_o} is fuzzy m-continuous follows from Theorem 3.10.

Similarly, one can deduce the following result.

Theorem 3.12. *Suppose* $(X,\mathcal{M})$, $(Y,\mathcal{N})$ *are fuzzy minimal spaces. Then for each* $x_0 \in X$ *the mapping* $i_{x_o} : (Y,\mathcal{N}) \longrightarrow (X \times Y, \mathcal{M} \times \mathcal{N})$ *defined by* $i_{x_0}(y) = (x_0, y)$ *is fuzzy* m*-continuous.*

4. Fuzzy minimal subspaces

Definition 4.1. Let A be a fuzzy set in X and $\mathcal{M}$ be a fuzzy minimal space on X. Then $\mathcal{M}_A = \{U \wedge A : U \in \mathcal{M}\}$ is called an *induced fuzzy minimal structure on* A and $(A, \mathcal{M}_A)$ is called *fuzzy minimal subspace* of $(X,\mathcal{M})$.

Proposition 4.2. *Suppose* $(A,\mathcal{M}_A)$ *is a fuzzy minimal subspace of fuzzy minimal space. If*

(a) $(X,\mathcal{M})$ *has the property* U*, then* $(A,\mathcal{M}_A)$ *has this property,*

(b) $(X,\mathcal{M})$ *has the property* I*, then* $(A,\mathcal{M}_A)$ *has this property too.*

Proof. Consider a family $\{V_\alpha : \alpha \in \mathcal{A}\}$ of fuzzy sets in $\mathcal{M}_A$, then there exists a family $\{U_\alpha : \alpha \in \mathcal{A}\}$ of fuzzy m-open sets in $(X,\mathcal{M})$ such that $V_\alpha = U_\alpha \wedge A$ for all $\alpha \in \mathcal{A}$. Therefore,

$$\bigvee_{\alpha\in\mathcal{A}} V_\alpha = \bigvee_{\alpha\in\mathcal{A}} (U_\alpha \wedge A) = \left(\bigvee_{\alpha\in\mathcal{A}} U_\alpha\right) \wedge A.$$

$\bigvee_{\alpha\in\mathcal{A}} V_\alpha \in \mathcal{M}_A$ follows from the fact that $(X,\mathcal{M})$ has the property U, which proves (a). The proof of (b) is similarly.

Definition 4.3. Suppose $(A,\mathcal{M}_A)$ and $(B,\mathcal{N}_A)$ are fuzzy minimal subspaces of fuzzy minimal spaces $(X,\mathcal{M})$ and $(Y,\mathcal{N})$ respectively. Also, suppose that $f : (X,\mathcal{M}) \longrightarrow (Y,\mathcal{N})$ is a mapping. We say that f is a mapping from $(A,\mathcal{M}_A)$ into $(B,\mathcal{N}_A)$ if $f(A) \leq B$.

Definition 4.4. Suppose $(A,\mathcal{M}_A)$ and $(B,\mathcal{N}_A)$ are fuzzy minimal subspaces of fuzzy minimal spaces $(X,\mathcal{M})$ and $(Y,\mathcal{N})$ respectively. The mapping f from $(A,\mathcal{M}_A)$ into $(B,\mathcal{N}_B)$ is said to be

(a) *relatively fuzzy minimal continuous* (briefly, (rfm)-continuous),
if $f^{-1}(W) \wedge A \in \mathcal{M}_A$ for every fuzzy set W in $\mathcal{N}_B$,

(b) *relatively fuzzy minimal open* (briefly, (rfm)-open),
if $f(V) \in \mathcal{N}_B$ for every fuzzy set V in $\mathcal{M}_A$.

Theorem 4.5. *Suppose* $(A,\mathcal{M}_A)$ *and* $(B,\mathcal{N}_B)$ *are fuzzy minimal subspaces of fuzzy minimal spaces* $(X,\mathcal{M})$ *and* $(Y,\mathcal{N})$ *respectively. If* $f : (X,\mathcal{M}) \longrightarrow (Y,\mathcal{N})$ *is fuzzy* m*-continuous with* $f(A) \leq B$*, then* $f : (A,\mathcal{M}_A) \longrightarrow (B,\mathcal{N}_B)$ *is* (rfm)*-continuous.*

Proof. For any given fuzzy m-open set $W \in \mathcal{N}_B$, there exists $\mu \in \mathcal{N}$ for which $W = \mu \wedge B$. Since $f(A) \leq B$, then $A \leq f^{-1}(f(A)) \leq f^{-1}(B)$ and hence

$$\begin{aligned} f^{-1}(W) \wedge A &= f^{-1}(\mu \wedge B) \wedge A \\ &= f^{-1}(\mu) \wedge f^{-1}(B) \wedge A \\ &= f^{-1}(\mu) \wedge A. \end{aligned}$$

Since f is fuzzy m-continuous, so $f^{-1}(\mu) \in \mathcal{M}$, which implies that $f^{-1}(W) \wedge A \in \mathcal{M}_A$. Therefore, $f : (A, \mathcal{M}_A) \longrightarrow (B, \mathcal{N}_B)$ is (rfm)-continuous.

The following example shows that the converse of Theorem 4.5 does not hold.

Example 4.6. Suppose $X = \{a, b\}$, $\mathcal{M} = \{\alpha 1_X : \alpha \in I\}$ and $\mathcal{N} = \{\alpha 1_X : \alpha \in I\} \cup b_1$. Let $A = B = a_1$. Since $\mathcal{N} \not\subseteq \mathcal{M}$, so it follows from part (b) of Theorem 3.6 that the identity map $id_X : (X, \mathcal{M}) \longrightarrow (Y, \mathcal{N})$ is not fuzzy m-continuous. Clearly, $\mathcal{M}_A = \mathcal{N}_B = \{a_\alpha : \alpha \in [0, 1]\}$ where $a_0 = 01_X$. Also, $id_X^{-1}(a_\alpha) \wedge a_1 = a_\alpha \wedge a_1 = a_\alpha \in \mathcal{M}_A$ and so $id_X : (A, \mathcal{M}_A) \longrightarrow (B, \mathcal{N}_B)$ is (rfm)-continuous.

Theorem 4.7. *The composition of two (rfm)-continuous functions is (rfm)-continuous too.*

Proof. Let $(A, \mathcal{M}_A)$, $(B, \mathcal{N}_B)$ and $(C, \mathcal{Q}_C)$ be fuzzy minimal subspaces of fuzzy minimal spaces $(X, \mathcal{M})$, $(Y, \mathcal{N})$ and $(Z, \mathcal{Q})$ respectively. Suppose $f : (A, \mathcal{M}_A) \longrightarrow (B, \mathcal{N}_B)$ and $g : (B, \mathcal{N}_B) \longrightarrow (C, \mathcal{Q}_C)$ are (rfm)-continuous. We must prove that $gof : (A, \mathcal{M}_A) \longrightarrow (C, \mathcal{Q}_C)$ is (rfm)-continuous. To see this, suppose W is an arbitrary element of $\mathcal{Q}_C$. Then $g^{-1}(W) \wedge B \in \mathcal{N}_B$ and so $f^{-1}(g^{-1}(W) \wedge B)) \wedge A \in \mathcal{M}_A$, i.e., $(gof)^{-1}(W) \wedge A \in \mathcal{M}_A$. Therefore, $gof : (A, \mathcal{M}_A) \longrightarrow (C, \mathcal{Q}_C)$ is (rfm)-continuous.

Similarly, one can deduce the following result.

Theorem 4.8. *Let $(A, \mathcal{M}_A)$, $(B, \mathcal{N}_B)$ and $(C, \mathcal{Q}_C)$ be fuzzy minimal subspaces of fuzzy minimal spaces $(X, \mathcal{M})$, $(Y, \mathcal{N})$ and $(Z, \mathcal{Q})$ respectively. Suppose $f : (A, \mathcal{M}_A) \longrightarrow (B, \mathcal{N}_B)$ and $g : (B, \mathcal{N}_B) \longrightarrow (C, \mathcal{Q}_C)$ are (rfm)-open. Then $gof : (A, \mathcal{M}_A) \longrightarrow (C, \mathcal{Q}_C)$ is (rfm)-open.*

Definition 4.9. [14] For each $j \in \{1, 2, ..., n\}$, let A_j be a fuzzy set in X_j. The *fuzzy product* $A = \prod_{j=1}^{n} A_j$ as a fuzzy set of $X = \prod_{j=1}^{n} X_i$ is defined by

$$A(x_1, x_2, ..., x_n) = \min\{A_1(x_1), A_2(x_2), ..., A_n(x_n)\}.$$

We use $A_1 \times A_2 \times \cdots \times A_n$ instead of $\prod_{j=1}^{n} A_j$ and, especially, $A_1 \times A_2$ instead of $\prod_{j=1}^{2} A_j$.

Lemma 4.10. *Suppose A_j is a fuzzy set in X_j for each $j \in \{1, 2, ..., n\}$ and A is the corresponding fuzzy product. Then $\pi_j(A) \leq A_j$ for all $j \in \{1, 2, ..., n\}$.*

Theorem 4.11. *Suppose* $\{(X_j,\mathcal{M}_j) : j \in \{1,...,n\}\}$ *is a family of fuzzy minimal spaces,* $(X,\mathcal{M})$ *is the corresponding fuzzy product minimal space,* A_i *is a fuzzy set in* X_j *for each* $j \in \{1,...,n\}$ *and* $A = \prod_{j=1}^{n} A_j$. *Let* $(B,\mathcal{N}_B)$ *be a fuzzy minimal subspace of the fuzzy minimal space* $(Y,\mathcal{N})$. *Then* $f : (B,\mathcal{N}_B) \longrightarrow (A,\mathcal{M}_A)$ *is* (rfm)*-continuous if and only if* $\pi_j of : (B,\mathcal{N}_B) \longrightarrow (A_j,\mathcal{M}_{j_{A_j}})$ *is* (rfm)*-continuous for all* $j \in \{1,...,n\}$.

Proof. One direction is an immediate consequence of Theorem 4.5 and Theorem 4.7. For the converse, on the contrary suppose $\pi_i of$ is (rfm)-continuous for each $i \in \{1,...,n\}$ and f is not (rfm)-continuous. Hence, there exists $V \in \mathcal{M}_A$ such that $f^{-1}(V) \wedge B \notin \mathcal{N}_B$ and now by Definition 4.1 there exists $U \in \mathcal{M}$ such that $f^{-1}(U \wedge A) \wedge B \notin \mathcal{N}_B$. According to (3.3) there exist $l \in \{1,...,n\}$ and $U_l \in \mathcal{M}_l$ for which

$$f^{-1}(1_{X_1} \times \cdots \times 1_{X_{l-1}} \times U_l \times 1_{X_{l+1}} \times \cdots \times 1_{X_n}) \wedge f^{-1}(A) \wedge B \notin \mathcal{N}_B.$$

Then, Lemma 4.10 and the fact that $B \leq f^{-1}(A) \leq (\pi_l of)^{-1}(A_l)$ imply

$$(\pi_l of)^{-1}(U_l \wedge A_l) \wedge B \notin \mathcal{N}_B;$$

i.e., $\pi_l of$ is not (rfm)-continuous, which is a contradiction.

Corollary 4.12. *Suppose* $(X,\mathcal{M})$ *is a minimal space and* $\{(Y_j,\mathcal{N}_j) : j \in \{1,...,n\}\}$ *is a finite family of fuzzy minimal spaces and* $(Y,\mathcal{N})$ *is their corresponding fuzzy product minimal spaces. Also, suppose* A *and* B_j *are respectively fuzzy sets in* X *and* Y_j *for each* $j \in \{1,...,n\}$ *and* $B = \prod_{i=1}^{n} B_i$. *Let* f_j *be a mapping of* $(A,\mathcal{M}_A)$ *to* $(B_j,\mathcal{N}_{j_{B_j}})$. *Then* $f : (A,\mathcal{M}_A) \longrightarrow (B,\mathcal{N}_B)$ *defined by* $f(x) = (f_1(x)...f_n(x))$ *is* (rfm)*-continuous if and only if* $f_j : (A,\mathcal{M}_A) \longrightarrow (B_j,\mathcal{N}_{j_{B_j}})$ *is* (rfm)*-continuous for each* $j = 1,...,n$.

Proof. It follows from Theorem 4.11 and the fact that $\pi_j of = f_j$.

Theorem 4.13. *Suppose* $(X,\mathcal{M})$, $(Y,\mathcal{N})$ *are fuzzy minimal spaces,* $C = A \times B$, $\mathcal{Q} = \mathcal{M} \times \mathcal{N}$ *and also* A *and* B *are fuzzy sets in* X *and* Y *respectively. Then for each* $y_0 \in Y$ *with* $B(y_0) \geq A(x)$ *for all* $x \in X$, *the mapping* $i_{y_o} : (A,\mathcal{M}_A) \longrightarrow (C,\mathcal{Q}_C)$ *defined by* $i_{y_0}(x) = (x,y_0)$ *is* (rfm)*-continuous.*

Proof. First, we show that $i_{y_0}(A) \leq C$. It is easy to see that

$$i_{y_0}(A)(x,y) = \begin{cases} A(x) & y = y_0 \\ 0 & \text{otherwise} . \end{cases}$$

Since for each $y_0 \in Y$ with $B(y_0) \geq A(x)$ for all $x \in X$, so one can deduce that $i_{y_0}(A) \leq C$. That i_{y_0} is (rfm)-continuous follows from Theorem 3.11 and Theorem 4.5.

Similarly, using Theorem 3.12 and Theorem 4.5 one can deduce the following result.

Theorem 4.14. *Suppose* $(X, \mathcal{M})$, $(Y, \mathcal{N})$ *are fuzzy minimal spaces,* $C = A \times B$, $\mathcal{Q} = \mathcal{M} \times \mathcal{N}$ *and also* A *and* B *are fuzzy sets in* X *and* Y *respectively. Then for each* $x_0 \in X$ *with* $A(x_0) \geq B(y)$ *for all* $y \in Y$*, the mapping* $i_{x_o} : (B, \mathcal{N}_B) \longrightarrow (C, \mathcal{Q}_C)$ *defined by* $i_{x_0}(y) = (x_0, y)$ *is* (rfm)*-continuous.*

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INTERVAL-VALUED INTUITIONISTIC FUZZY SUBSEMIMODULES WITH (S,T)-NORMS

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Abstract. On the basis of the concept of the interval valued intuitionistic fuzzy sets introduced by K. Atanassov, the notion of interval valued intuitionistic fuzzy subsemimodule of a semimodule with respect to t-norm T and s-norm S is given and the characteristic properties are described. The homomorphic image and inverse image are investigated. In particular, by the help of the congruence relations on semimodules, new interval valued intuitionistic (S,T)-fuzzy subsemimodules are constructed.

Keywords: semimodule, subsemimodule, interval valued intuitionistic (S,T)-fuzzy subsemimodule.

1. Introduction

After the introduction of fuzzy sets by Zadeh [14], there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [1] is one among them. For more details on intuitionistic fuzzy sets, we refer the reader to [1], [2], [3]. In 1975, Zadeh [15] introduced the concept of interval valued fuzzy subsets, where the values of the membership functions are intervals of numbers instead of the numbers. Such fuzzy sets have some applications in the technological scheme of the functioning of a silo-farm with pneumatic transportation, in a plastic products company and in medicine (see the book [3]).

The fuzzy algebraic structures play a prominent role in mathematics with wide applications in many other branches such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces, logic, set theory, group theory, groupoids, real analysis, measure theory etc. Also the notion of fuzzy submodules in modules and semimodules (in different views) have seriously studied by many mathematicians ([11], [12]). Recently, some researchers are trying to present new views of fuzzy algebraic structures as intuitionistic fuzzy algebraic structures ([10], [16]). In algebra, we notice that the subsemimodules of semimodules play a crucial role in the structure theory, but they do not in general coincide with the usual submodules, for this reason, their usage is somewhat limited when we try to obtain some analogous module theorems for semimodules. Indeed, many results in modules apparently have no analogous in semimodules by using only submodules. In this paper we introduce the notion of interval valued intuitionistic fuzzy subsemimodules of a semimodule with respect to t-norm T and s-norm S. Then we characterize all of them base on special kind of levels $\mathfrak{U}(\widetilde{M_{\mathcal{A}}};[t,s])$ and $\mathfrak{L}(\widetilde{N}_{\mathcal{A}};[t,s])$, which is a generalization of classic level subsets. At the following the behaviour of this structure under homomorphisms is investigated. In particular, by the help of the congruence relations on semimodules, we construct new interval valued intuitionistic (S,T)-fuzzy subsemimodules on semimodule of quotient.

2. Preliminaries and notations

Let $\mathcal{SR}$ be a semiring. A *left* $\mathcal{SR}$*-semimodule* is a commutative semigroup $\mathcal{SM}$ which we have a function $\mathcal{SR}\times\mathcal{SM}\longrightarrow\mathcal{SM}$, denote by $(r,m)\mapsto rm$ and called *scalar multiplication*, which satisfies the following conditions for all $r,r'\in\mathcal{SR}$ and $m,m'\in\mathcal{SM}$:

(1) $(rr')m=r(r'm)$;

(2) $r(m+m')=rm+rm'$;

(3) $(r+r')m=rm+rm'$.

Right semimodules over $\mathcal{SR}$ are defined in an analogous manner. A *semimodule* is both left and right semimodule (see [6]).

A non-empty subset $\mathcal{SN}$ of a left $\mathcal{SR}$-semimodule $\mathcal{SM}$ is a *subsemimodule* of $\mathcal{SM}$ if and only if $\mathcal{SN}$ is closed under addition and scalar multiplication.

An equivalence relation ρ on a semigroup $(\mathcal{SM},.)$ is said to be a *congruence relation*, if for all $x,y,z\in\mathcal{SM}$, $x\rho y$ implies $(xz)\rho(yz)$, where by $x\rho y$ we mean $(x,y)\in\rho$. Also by $\mathcal{SM}/\rho$ we mean the set of all equivalence classes with respect to ρ, or $\mathcal{SM}/\rho=\{\rho(x):x\in\mathcal{SM}\}$ (see [6]). Also an equivalence relation θ on a semiring $(\mathcal{SR},+,.)$ is said to be a congruence relation, if for all $x,y,z\in\mathcal{SR}$, $x\theta y$ implies $(x+z)\theta(y+z)$ and $(xz)\theta(yz)$ (see [6]).

By an *interval number* $\widetilde{a}$ we mean ([15]) an interval $[a^-,a^+]$, where $0\leq a^-\leq a^+\leq 1$. The set of all interval number is denoted by $D[0,1]$. The interval $[a,a]$ is identified whit the number $a\in[0,1]$.

For interval numbers $\widetilde{a}_i=[a_i^-,a_i^+]\in D[0,1], i\in I$, we define (see [3] and [15])

$$\inf\widetilde{a}_i=\left[\bigwedge_{i\in I}a_i^-,\bigwedge_{i\in I}a_i^+\right],\quad \sup\widetilde{a}_i=\left[\bigvee_{i\in I}a_i^-,\bigvee_{i\in I}a_i^+\right]$$

and put

(1) $\widetilde{a_1} \le \widetilde{a_2} \Longleftrightarrow a_1^- \le a_2^-$ and $a_1^+ \le a_2^+$,

(2) $\widetilde{a_1} = \widetilde{a_2} \Longleftrightarrow a_1^- = a_2^-$ and $a_1^+ = a_2^+$,

(3) $\widetilde{a_1} < \widetilde{a_2} \Longleftrightarrow \widetilde{a_1} \le \widetilde{a_2}$ and $\widetilde{a_1} \ne \widetilde{a_2}$,

(4) $k\tilde{a} = [ka^-, ka^+]$, whenever $0 \le k \le 1$.

It is clear that $(D[0,1], \le, \bigvee, \bigwedge)$ is a complete lattice whit $0 = [0,0]$ as the least element and $1 = [1,1]$ as the greatest element.

By an *interval number fuzzy set* F on X we mean ([15]) the set

$$F = \{(x, [\mu_F^-(x), \mu_F^+(x)]) : x \in X\},$$

where μ_F^- and μ_F^+ are two fuzzy subset of X such that $\mu_F^-(x) \le \mu_F^+(x)$ for all $x \in X$. Putting $\mu_F(x) = [\mu_F^-(x), \mu_F^+(x)]$, we see that $F = \{(x, \mu_F(x)) : x \in X\}$, where $\mu_F : X \longrightarrow D[0,1]$.

As it is well-known, the function $\delta : [0,1] \times [0,1] \longrightarrow [0,1]$ is called a *t-norm* (resp. *s-norm*) if δ satisfied the conditions:

(i) $\delta(x,1) = x$ (resp. $\delta(x,0) = x$),

(ii) $\delta(x,y) = \delta(y,x)$,

(iii) $\delta(\delta(x,y),z) = \delta(x,\delta(y,z))$,

(iv) $\delta(x,u) \le \delta(x,w)$, for all $x, y, z, u, w \in [0,1]$, where $u \le w$.

A t-norm (resp. s-norm) δ is called an *idempotent t-norm* if $\delta(x,x) = x$, for all $x \in [0,1]$, (see [17]).

If δ is an idempotent t-norm (s-norm), then the mapping

$$\Delta : D[0,1] \times D[0,1] \longrightarrow D[0,1]$$

defined by

$$\Delta(\widetilde{a}_1, \widetilde{a}_2) = [\delta(a_1^-, a_2^-), \delta(a_1^+, a_2^+)]$$

is, as it is not difficult to verify, an idempotent t-norm (s-norm, respectively) and is called an *idempotent interval t-norm (s-norm, respectively).*

According to Atanassov ([1], [2], [3]), *an interval valued intuitionistic fuzzy* set on X is defined as an object of the form

$$\mathcal{A} = \{(x, \widetilde{M}_{\mathcal{A}}(x), \widetilde{N}_{\mathcal{A}}(x)) : x \in X\},$$

where $\widetilde{M}_{\mathcal{A}}(x)$ and $\widetilde{N}_{\mathcal{A}}(x)$ are interval valued fuzzy sets on X such that

$$0 \le \sup \widetilde{M}_{\mathcal{A}}(x) + \sup \widetilde{N}_{\mathcal{A}}(x) \le 1 \ \text{ for all } x \in X.$$

For the sake of simplicity, in the following such interval valued intuitionistic fuzzy sets will be denoted by $\mathcal{A} = (\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$.

3. Interval valued intuitionistic (s, t)-fuzzy subsemimodules of semimodules

In what follows, let $\mathcal{SM}$ denote a $\mathcal{SR}$-semimodule unless otherwise specified.

Definition 3.1. An interval valued intuitionistic fuzzy set $\mathcal{A} = (\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ of $\mathcal{SM}$ is called an interval valued intuitionistic (S, T)-fuzzy left subsemimodule of $\mathcal{SM}$ if for all $x, y \in \mathcal{SM}$ and $r \in \mathcal{SR}$ we have

(1) $\widetilde{M}_{\mathcal{A}}(x+y) \geq T(\widetilde{M}_{\mathcal{A}}(x), \widetilde{M}_{\mathcal{A}}(y)),\ \widetilde{N}_{\mathcal{A}}(x+y) \leq S(\widetilde{N}_{\mathcal{A}}(x), \widetilde{N}_{\mathcal{A}}(y))$,

(2) $\widetilde{M}_{\mathcal{A}}(rx) \geq \widetilde{M}_{\mathcal{A}}(x),\ \widetilde{N}_{\mathcal{A}}(rx) \leq \widetilde{N}_{\mathcal{A}}(x)$.

Similarly, we define an interval valued intuitionistic (S, T)-fuzzy right subsemimodule. An interval valued intuitionistic (S, T)-fuzzy subsemimodule is both interval valued intuitionistic (S, T)-fuzzy left and right subsemimodule.

Example. A commutative semigroup $(\mathcal{SM}, +)$ is a $\mathbb{N}$-semimodule with the function $\mathbb{N} \times \mathcal{SM} \longrightarrow \mathcal{SM}$ defined by $(i, m) \mapsto im = m$. Let $\mathcal{SN}$ be a subsemimodule of $\mathcal{SM}$ and let

$$\widetilde{M}_{\mathcal{A}}(x) = \begin{cases} [0.8, 0.9], & \text{if } x \in \mathcal{SN} \\ [0.1, 0.2], & \text{if } x \notin \mathcal{SN} \end{cases}$$

$$\widetilde{N}_{\mathcal{A}}(x) = \begin{cases} [0.2, 0.3], & \text{if } x \in \mathcal{SN} \\ [0.7, 0.8], & \text{if } x \notin \mathcal{SN} \end{cases}$$

it can easily to be checked that $\mathcal{A} = (\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ is an interval valued intuitionistic (S, T)-fuzzy subsemimodule of $\mathcal{SM}$.

Example. $\mathbb{Z}^- = \{0, -1, -2, -3, ...\}$ with the rule $\mathbb{N} \times \mathbb{Z}^- \longrightarrow \mathbb{Z}^-$ defined by $(n, a) \mapsto na$ is a $\mathbb{N}$-semimodule. Let

$$\widetilde{M}_{\mathcal{A}}(x) = \begin{cases} [0.9, 1], & \text{if } x = 0, -2, -4, -6, ... \\ [0, 0.1], & \text{if } x = -1, -3, -5, ... \end{cases}$$

$$\widetilde{N}_{\mathcal{A}}(x) = \begin{cases} [0, 0.1], & \text{if } x = 0, -2, -4, -6, ... \\ [0.9, 1], & \text{if } x = -1, -3, -5, ... \end{cases}$$

it is easy to calculate that $\mathcal{A} = (\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ is an interval valued intuitionistic (S, T)-fuzzy subsemimodule of $\mathbb{Z}^-$. Now let

$$\widetilde{M}_{\mathcal{A}}(x) = \begin{cases} [0.1, 0.2], & \text{if } x = 0, -2, -4, -6, ... \\ [0.8, 0.9], & \text{if } x = -1, -3, -5, ... \end{cases}$$

$$\widetilde{N}_{\mathcal{A}}(x) = \begin{cases} [0.8, 0.9], & \text{if } x = 0, -2, -4, -6, ... \\ [0.1, 0.2], & \text{if } x = -1, -3, -5, ... \end{cases}$$

since $\widetilde{M}_{\mathcal{A}}(2 \times (-3)) = \widetilde{M}_{\mathcal{A}}(-6) = [0.1, 0.2]$ and $\widetilde{M}_{\mathcal{A}}(-3) = [0.8, 0.9]$ and so, $\widetilde{M}_{\mathcal{A}}(2 \times (-3)) \ngeq \widetilde{M}_{\mathcal{A}}(-3)$, therefore $\mathcal{A} = (\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ is not an interval valued intuitionistic (S, T)-fuzzy subsemimodule of $\mathbb{Z}^-$.

With any interval valued intuitionistic fuzzy set $\mathcal{A} = (\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ of $\mathcal{SM}$ there are connected two levels:

$$\mathfrak{U}(\widetilde{M}_{\mathcal{A}}; [t, s]) = \{x \in \mathcal{SM} : \widetilde{M}_{\mathcal{A}}(x) \geq [t, s]\},$$

$$\mathfrak{L}(\widetilde{N}_{\mathcal{A}}; [t, s]) = \{x \in \mathcal{SM} : \widetilde{N}_{\mathcal{A}}(x) \leq [t, s]\}.$$

Theorem 3.2. *Let T and S be idempotent intervals t-norm and s-norm respectively. Then $\mathcal{A} = (\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ is an interval valued intuitionistic (S, T)-fuzzy submodule if and only if for all $t, s \in [0, 1], t \leq s, \mathfrak{U}(\widetilde{M}_{\mathcal{A}}; [t, s])$ and $\mathfrak{L}(\widetilde{N}_{\mathcal{A}}; [t, s])$ are subsemimodules of $\mathcal{SM}$.*

Proof. Let $\mathcal{A} = (\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ be an interval valued intuitionistic (S, T)-fuzzy subsemimodle of $\mathcal{SM}$. Then for every $x, y \in \mathfrak{U}(\widetilde{M}_{\mathcal{A}}; [t, s])$ we have $\widetilde{M}_{\mathcal{A}}(x) \geq [t, s]$ and $\widetilde{M}_{\mathcal{A}}(y) \geq [t, s]$. Hence $T(\widetilde{M}_{\mathcal{A}}(x), \widetilde{M}_{\mathcal{A}}(y)) \geq T([t, s], [t, s]) = [t, s]$, and so $\widetilde{M}_{\mathcal{A}}(x+y) \geq [t, s]$. Therefore $x+y \in \mathfrak{U}(\widetilde{M}_{\mathcal{A}}; [t, s])$. If $r \in \mathcal{SR}$ and $x \in \mathfrak{U}(\widetilde{M}_{\mathcal{A}}; [t, s])$, then $\widetilde{M}_{\mathcal{A}}(x) \geq [t, s]$. On the other hand $\widetilde{M}_{\mathcal{A}}(rx) \geq \widetilde{M}_{\mathcal{A}}(x) \geq [t, s]$. Therefore

$$rx \in \mathfrak{U}(\widetilde{M}_{\mathcal{A}}; [t, s]).$$

This proves that $\mathfrak{U}(\widetilde{M}_{\mathcal{A}}; [t, s])$ is a subsemimodule of $\mathcal{SM}$.

Conversely, assume that for every $[t, s] \in D[0, 1]$ any non-empty $\mathfrak{U}(\widetilde{M}_{\mathcal{A}}; [t, s])$ is a subsemimodule of $\mathcal{SM}$. If $[t_0, s_0] = T(\widetilde{M}_{\mathcal{A}}(x), \widetilde{M}_{\mathcal{A}}(y))$ for some $x, y \in \mathcal{SM}$, then $x, y \in \mathfrak{U}(\widetilde{M}_{\mathcal{A}}; [t_0, s_0])$ and so $x + y \in \mathfrak{U}(\widetilde{M}_{\mathcal{A}}; [t_0, s_0])$. Therefore

$$\widetilde{M}_{\mathcal{A}}(x + y) \geq [t_0, s_0] = T(\widetilde{M}_{\mathcal{A}}(x), \widetilde{M}_{\mathcal{A}}(y)).$$

Also if $[t_1, s_1] = \widetilde{M}_{\mathcal{A}}(x)$, for some $x \in \mathcal{SM}$, then $x \in \mathfrak{U}(\widetilde{M}_{\mathcal{A}}; [t_1, s_1])$, and so $rx \in \mathfrak{U}(\widetilde{M}_{\mathcal{A}}; [t_1, s_1])$, for every $r \in \mathcal{SR}$, hence $\widetilde{M}_{\mathcal{A}}(rx) \geq [t_1, s_1] = \widetilde{M}_{\mathcal{A}}(x)$. This proves that $\widetilde{M}_{\mathcal{A}}$ is an interval valued intuitionistic left T-fuzzy subsemimodule of $\mathcal{SM}$. The proof of $\widetilde{M}_{\mathcal{A}}$ is an interval valued intuitionistic right T-fuzzy subsemimodule of $\mathcal{SM}$ is similar. Analogously, we can show that $\widetilde{N}_{\mathcal{A}}$ is an interval valued intuitionistic S-fuzzy ideal of $\mathcal{SM}$. Therefore $\mathcal{A} = (\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ is an interval valued intuitionistic (S, T)-fuzzy subsemimodule. ∎

Let $\mathcal{A} = (\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ be an interval valued intuitionistic fuzzy set of $\mathcal{SR}$ and let $t, s, t', s' \in [0,1]$ such that $t \leq s$ and $t' \leq s'$. Put

$$\mathcal{M}^{[t,s]}_{[t',s']} = \{x \in \mathcal{SM} : \widetilde{M}_{\mathcal{A}}(x) \geq [t,s],\ \widetilde{N}_{\mathcal{A}}(x) \leq [t',s']\}.$$

Clearly,

$$\mathcal{M}^{[t,s]}_{[t',s']} = \mathfrak{U}(\widetilde{M}_{\mathcal{A}}; [t,s]) \bigcap \mathfrak{L}(\widetilde{M}_{\mathcal{A}}; [t',s']).$$

Corollary 3.3. *Let T and S be idempotent intervals t-norm and s-norm respectively. Then $\mathcal{A} = (\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ is an interval valued intuitionistic (S,T)-fuzzy subsemimodule of $\mathcal{SM}$ if and only if for all $t, s, t', s' \in [0,1], t \leq s,\ t' \leq s',\ \mathcal{M}^{[t,s]}_{[t',s']}$ is a subsemimodule of $\mathcal{SM}$.*

Proof. It is immediately followed by Theorem 3.2. ∎

Definition 3.4. Let $f : X \to Y$ be a mapping and $\mathcal{A}=(\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ and $\mathcal{B}=(\widetilde{M}_{\mathcal{B}}, \widetilde{N}_{\mathcal{B}})$ interval valued intuitionistic sets X and Y, respectively. Then the image of $f[\mathcal{A}] = (f(\widetilde{M}_{\mathcal{A}}), f(\widetilde{N}_{\mathcal{A}}))$ of $\mathcal{A}$ is the interval valued intuitionistic fuzzy set of Y defined by

$$f(\widetilde{M}_{\mathcal{A}})(y) = \begin{cases} \sup\limits_{z \in f^{-1}(y)} \widetilde{M}_{\mathcal{A}}(z) & \text{if } f^{-1}(y) \neq \emptyset, \\ & \\ [0,0] & \text{otherwise} \end{cases}$$

$$f(\widetilde{N}_{\mathcal{A}})(y) = \begin{cases} \inf\limits_{z \in f^{-1}(y)} \widetilde{N}_{\mathcal{A}}(z) & \text{if } f^{-1}(y) \neq \emptyset, \\ & \\ [1,1] & \text{otherwise} \end{cases}$$

for all $y \in Y$.

The inverse image $f^{-1}(\mathcal{B})$ of $\mathcal{B}$ is an interval valued intuitionistic fuzzy set defined by

$$f^{-1}(\widetilde{M}_{\mathcal{B}})(x) = \widetilde{M}_{f^{-1}(\mathcal{B})}(x) = \widetilde{M}_{\mathcal{B}}(f(x)),$$

$$f^{-1}(\widetilde{N}_{\mathcal{B}})(x) = \widetilde{N}_{f^{-1}(\mathcal{B})}(x) = \widetilde{N}_{\mathcal{B}}(f(x))$$

for all $x \in X$.

Definition 3.5. Let $\mathcal{SM}$ and $\mathcal{SN}$ be two semimodules over a semiring $\mathcal{SR}$. A mapping $f : \mathcal{SM} \longrightarrow \mathcal{SN}$ is called a homomorphism if for all $x, y \in \mathcal{SM}$ and $r \in \mathcal{SR}$ we have $f(x+y) = f(x) + f(y)$ and $f(r.x) = r.f(x)$.

Lemma 3.6. *Let $\mathcal{SM}_1$ and $\mathcal{SM}_2$ be two semimodules over a semiring $\mathcal{SR}$ and $f : \mathcal{SM}_1 \longrightarrow \mathcal{SM}_2$ an epimorphism.*

- (i) *If $\mathcal{SN}_1$ is a subsemimodule of $\mathcal{SM}_1$, then $f(\mathcal{SN}_1)$ is a subsemimodule of $\mathcal{SM}_2$.*
- (ii) *If $\mathcal{SN}_2$ is a subsemimodule of $\mathcal{SM}_2$, then $f^{-1}(\mathcal{SN}_2)$ is a subsemimodule of $\mathcal{SM}_1$.*

Proof. Straightforward. ∎

Theorem 3.7. *Let $\mathcal{SM}_1$ and $\mathcal{SM}_2$ be two subsemimodules, and $f : \mathcal{SM}_1 \to \mathcal{SM}_2$ an epimorphism and T and S idempotent intervals t-norm and s-norm respectively.*

(i) *If $\mathcal{A} = (\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ is an interval valued intuitionistic (S,T)-fuzzy subsemimodule of $\mathcal{SM}_1$, then the image $f[\mathcal{A}]$ of $\mathcal{A}$ is an interval valued intuitionistic (S,T)-fuzzy subsemimodule of $\mathcal{SM}_2$.*

(ii) *If $\mathcal{B} = (\widetilde{M}_{\mathcal{B}}, \widetilde{N}_{\mathcal{B}})$ is an interval valued intuitionistic (S,T)-fuzzy subsemimodule of $\mathcal{SM}_2$, then the inverse image $f^{-1}(\mathcal{B}) = (f^{-1}(\widetilde{M}_{\mathcal{B}}), f^{-1}(\widetilde{N}_{\mathcal{B}}))$ of $\mathcal{B}$ is an interval valued intuitionistic (S,T)-fuzzy subsemimodule of $\mathcal{SM}_1$.*

Proof. (i) Let $\mathcal{A}=(\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ be an interval valued intuitionistic (S,T)-fuzzy subsemimodule of $\mathcal{SM}_1$. By Theorem 3.2, $\mathfrak{U}(\widetilde{M}_{\mathcal{A}};[t,s])$ and $\mathfrak{L}(\widetilde{N}_{\mathcal{A}};[t,s])$ are subsemimodules of $\mathcal{SM}_1$ for every $[t,s]\in D[0,1]$. Therefore, by Lemma 3.6, $f(\mathfrak{U}(\widetilde{M}_{\mathcal{A}};[t,s]))$ and $f(\mathfrak{L}(\widetilde{N}_{\mathcal{A}};[t,s]))$ are subsemimodules of $\mathcal{SM}_2$. But

$$\mathfrak{U}(f(\widetilde{M}_{\mathcal{A}});[t,s]) = f(\mathfrak{U}(\widetilde{M}_{\mathcal{A}};[t,s])) \text{ and } \mathfrak{L}(f(\widetilde{N}_{\mathcal{A}});[t,s]) = f(\mathfrak{L}(\widetilde{N}_{\mathcal{A}};[t,s])),$$

so, $\mathfrak{U}(f(\widetilde{M}_{\mathcal{A}});[t,s])$ and $\mathfrak{L}(f(\widetilde{N}_{\mathcal{A}});[t,s])$ are subsemimodules of $\mathcal{SM}_2$. Therefore $f[A]$ is an interval valued intuitionistic (S,T)-fuzzy subsemimodule of $\mathcal{SM}_2$.

(ii) For any $x, y \in \mathcal{SM}_1$, we have

$$\begin{aligned} \widetilde{M}_{f^{-1}(\mathcal{B})}(x+y) &= \widetilde{M}_{\mathcal{B}}(f(x+y)) \geq T(\widetilde{M}_{\mathcal{B}}(f(x)), \widetilde{M}_{\mathcal{B}}(f(y))) \\ &= T(\widetilde{M}_{f^{-1}(\mathcal{B})}(x), \widetilde{M}_{f^{-1}(\mathcal{B})}(y)). \end{aligned}$$

Also, if $x \in \mathcal{SM}_1$ and $r \in SR$, we have

$$\widetilde{M}_{f^{-1}(\mathcal{B})}(r.x) = \widetilde{M}_{\mathcal{B}}(f(r.x)) = \widetilde{M}_{\mathcal{B}}(r.f(x)) \geq \widetilde{M}_{\mathcal{B}}(f(x)) = \widetilde{M}_{f^{-1}(\mathcal{B})}(x).$$

This completes the proof that $\widetilde{M}_{f^{-1}(\mathcal{B})}$ is an interval valued T-fuzzy subsemimodule of $\mathcal{SR}_1$. Similarly we can prove $\widetilde{N}_{f^{-1}(\mathcal{B})}$ is an interval valued S-fuzzy subsemimodule of $\mathcal{SR}_1$. Similarly $f^{-1}(\mathcal{B}) = (f^{-1}(\widetilde{M}_{f^{-1}(\mathcal{B})}), f^{-1}(\widetilde{N}_{f^{-1}(\mathcal{B})}))$ is an interval valued intuitionistic (S,T)-fuzzy subsemimodule of $\mathcal{SR}_1$. ∎

Let γ be a congruence relation on $\mathcal{SM}$ and θ a congruence relation on $\mathcal{SR}$. Then it is easy to verify that $\mathcal{SM}/\gamma$ is a semimodule over semiring $\mathcal{SR}/\theta$, by the rule $\odot : \mathcal{SM}/\gamma \times \mathcal{SR}/\theta \longrightarrow \mathcal{SM}/\gamma$ define by $\gamma(x) \odot \theta(r) = \gamma(x.r)$.

Definition 3.8. Let $\mathcal{A} = (\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ be an interval valued intuitionistic fuzzy set. The intuitionistic fuzzy set $\mathcal{A}/\gamma = (\widetilde{M}_{\mathcal{A}/\gamma}, \widetilde{N}_{\mathcal{A}/\gamma})$ is defined as a pair of maps

$$\begin{cases} \widetilde{M}_{\mathcal{A}/\gamma} : \mathcal{SR}/\gamma \longrightarrow D[0,1] \\ \\ \widetilde{N}_{\mathcal{A}/\gamma} : \mathcal{SR}/\gamma \longrightarrow D[0,1] \end{cases}$$

Such that $\widetilde{M}_{\mathcal{A}/\gamma}(\gamma(x)) = \sup\limits_{a\in\gamma(x)} \widetilde{M}_{\mathcal{A}}(a)$ and $\widetilde{N}_{\mathcal{A}/\gamma}(\gamma(x)) = \inf\limits_{a\in\gamma(x)} \widetilde{N}_{\mathcal{A}}(a)$.

Theorem 3.9. *Let $\mathcal{SM}$ be a semimodule over $\mathcal{SR}$. If $\mathcal{A} = (\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ is an interval valued intuitionistic (S,T)-fuzzy subsemimodule of $\mathcal{SM}$, then $\mathcal{A}/\gamma = (\widetilde{M}_{\mathcal{A}/\gamma}, \widetilde{N}_{\mathcal{A}/\gamma})$ is an interval valued intuitionistic (S,T)-fuzzy subsemimodule of semimodule $\mathcal{SM}/\gamma$ over $\mathcal{SR}/\theta$.*

Proof. Let $\gamma(x), \gamma(y) \in \mathcal{SM}/\gamma$, we have

$$\begin{aligned}
T(\widetilde{M}_{\mathcal{A}/\gamma}(\gamma(x)), \widetilde{M}_{\mathcal{A}/\gamma}(\gamma(y))) &= T\left(\sup_{a\in\gamma(x)} \widetilde{M}_{\mathcal{A}}(a), \sup_{b\in\gamma(y)} \widetilde{M}_{\mathcal{A}}(b)\right)\\
&= \sup_{a\in\gamma(x),\ b\in\gamma(y)} T(\widetilde{M}_{\mathcal{A}}(a), \widetilde{M}_{\mathcal{A}}(b))\\
&\leq \sup_{a\in\gamma(x),\ b\in\gamma(y)} \widetilde{M}_{\mathcal{A}}(a+b)\\
&\leq \sup_{a\in\gamma(x),\ b\in\gamma(y)} \left(\sup_{t\in\gamma(a+b)} \widetilde{M}_{\mathcal{A}}(t)\right)\\
&= \sup_{a\in\gamma(x),\ b\in\gamma(y)} \widetilde{M}_{\mathcal{A}/\gamma}(\gamma(a+b))\\
&= \widetilde{M}_{\mathcal{A}/\gamma}(\gamma(a+b)),
\end{aligned}$$

for all $a \in \gamma(x), b \in \gamma(y)$. On the other hand, we have

$$\widetilde{M}_{\mathcal{A}/\gamma}(\gamma(a+b)) = \widetilde{M}_{\mathcal{A}/\gamma}(\gamma(a) \oplus \gamma(b)) = \widetilde{M}_{\mathcal{A}/\gamma}(\gamma(x) \oplus \gamma(y)) = \widetilde{M}_{\mathcal{A}/\gamma}(\gamma(x+y)).$$

So,

$$T(\widetilde{M}_{\mathcal{A}/\gamma}(\gamma(x)), \widetilde{M}_{\mathcal{A}/\gamma}(\gamma(y))) \leq \widetilde{M}_{\mathcal{A}/\gamma}(\gamma(x) \oplus \gamma(y)).$$

The proof of the inequality

$$S(\widetilde{N}_{\mathcal{A}/\gamma}(\gamma(x)), \widetilde{N}_{\mathcal{A}/\gamma}(\gamma(y))) \geq \widetilde{N}_{\mathcal{A}/\gamma}(\gamma(x) \oplus \gamma(y)),$$

is similar.

To prove the second condition, let $\gamma(x) \in \mathcal{SM}/\gamma$ and $\theta(r) \in \mathcal{SR}/\theta$, then for every $b \in \gamma(x)$ we have

$$\widetilde{M}_{\mathcal{A}/\gamma}(\theta(r) \odot \gamma(x)) = \widetilde{M}_{\mathcal{A}/\gamma}(\theta(r) \odot \gamma(b)) = \widetilde{M}_{\mathcal{A}/\gamma}(\gamma(r.b)).$$

On the other hand

$$\widetilde{M}_{\mathcal{A}/\gamma}(\gamma(r.b)) = \sup_{t\in\gamma(r.b)} \widetilde{M}_{\mathcal{A}}(t) \geq \widetilde{M}_{\mathcal{A}}(r.b) \geq \widetilde{M}_{\mathcal{A}}(b),$$

and so for every $b \in \gamma(x)$, we have $\widetilde{M}_{\mathcal{A}/\gamma}(\theta(r) \odot \gamma(x)) \geq \widetilde{M}_{\mathcal{A}}(b)$. Hence

$$\widetilde{M}_{\mathcal{A}/\gamma}(\theta(r) \odot \gamma(x)) \geq \sup_{b\in\gamma(x)} \widetilde{M}_{\mathcal{A}}(b) = \widetilde{M}_{\mathcal{A}/\gamma}(\gamma(x)).$$

Similarly, we can obtain

$$\widetilde{N}_{\mathcal{A}/\gamma}(\theta(r) \odot \gamma(x)) \leq \widetilde{N}_{\mathcal{A}/\gamma}(\gamma(x)).$$

This completes the proof.

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α-GENERALIZED-CONVERGENCE THEORY OF L-FUZZY NETS AND ITS APPLICATIONS

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Abstract. The convergence theory not only is an significantly basic theory of fuzzy topology and fuzzy analysis but also has wide applications in fuzzy inference and some other aspects. In this paper, we introduce the concept of α-generalized-remote-neighborhood of fuzzy points and establish the Moore-Smith α-generalized-convergence theory of L-fuzzy nets. Then, we introduce and study the concept of L-fuzzy α-generalized-irresolute mappings and L-fuzzy α-generalized compactness. Also we discuss the applications of α-generalized-convergence of L-fuzzy nets.

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1. Introduction

The usual notion of a set was generalized with the introduction of fuzzy sets by Zadeh in the classical paper [15] of 1965. Since then many authors have expansively developed the theory of fuzzy sets and its applications to several sectors of both pure and applied sciences, such as [6], [10]-[14]. As it is known now that the traditional neighborhood method is not effective any longer in fuzzy topology, in order to overcome this deficiency Pu and Liu introduced the concepts of the fuzzy point and the Q-neighborhood and established a systematic Moore-Smith convergence theory of fuzzy nets [10]. It paved a new way for the study of the fuzzy topology. Later on, Wang introduced the concept of remote-neighborhood systems [13], this concept is an abstraction of the concept of neighborhood in point set topology and the concept of Q-neighborhood in fuzzy topology. Q-neighborhood and remote-neighborhood can be used in wide aspect [3], [8]-[13].

In this paper, we introduce the concept of α-generalized-remote-neighborhood of fuzzy points with the concept of remote-neighborhood. In Section 3, using the concepts α-generalized fuzzy closed sets and L-fuzzy α-generalized-remote-neighborhood, we establish the Moore-Smith α-generalized-convergence theory of L-fuzzy nets. In Section 4, we discuss the applications of α-generalized-convergence of L-fuzzy nets.

2. Preliminaries

Throughout this paper, $L = L(\leq, \vee, \wedge, ^{'})$ will denote a fuzzy lattice, i.e., a completely distributive lattice with a smallest element 0 and largest element 1 ($0 \neq 1$) and with an order reversing involution $a \to a^{'} (a \in L)$. Let X be a nonempty crisp set, and we shall denote by L^X the lattice of all L-subsets of X and if $A \subseteq X$ by χ_A the characteristic function of A. An element p of L is called prime iff $p \neq 1$ and whenever $a, b \in L$ with $a \wedge b \leq p$ then $a \leq p$ or $b \leq p$ [14]. The set of all prime elements of L will be denoted by $Pr(L)$. An element α of L is called union-irreducible or coprime iff whenever $a, b \in L$ with $\alpha \leq a \vee b$ then $\alpha \leq a$ or $\alpha \leq b$ [14]. The set of all non-zero union-irreducible elements of L will be denoted by $M(L)$. It is obvious that $p \in pr(L)$ iff $p^{'} \in M(L)$. We denote $M^*(L^X) = \{x_\alpha : x \in X \text{ and } \alpha \in M(L)\}$.

For the definition of a fuzzy point x_α we follow Pu and Liu [10]. When the support and value of a fuzzy point are trivial, we use briefly the symbols e to denote fuzzy point. A fuzzy point $x_\alpha \in A$, where A is an L-fuzzy set in X, iff $\alpha \leq A(x)$. The constant L-fuzzy sets taking on the values 0 and 1 on X are designated by 0_X and 1_X, respectively. An L-fuzzy net $S = \{S(n), n \in D\}$ is a function $S : D \to \xi$ where D is a directed set with order relation $\geq$ and ξ the collection of all the fuzzy points in X [14]. A net S is called an $\alpha-$net ($\alpha \in M(L)$) if for each $\lambda \in \beta^{'}(\alpha)$ (where $\beta^{'}(\alpha)$ denotes the union of all minimal sets relative to α), there is $n_0 \in D$ such that $V(S(n)) \geq \lambda$ whenever $n \geq n_0$, where $V(S(n))$ is the height of point $S(n)$.

Definition 2.1. Let L be a fuzzy lattice, X be a nonempty crisp set and $\delta \subseteq L^X$. An L-fuzzy topology is a family δ of L-subsets of X which satisfies the following conditions:

(a) $0, 1 \in \delta$,
(b) If $A, B \in \delta$, then $A \wedge B \in \delta$,
(c) If $A_i \in \delta$ for each $i \in I$, then $\vee_{i \in I} A_i \in \delta$.

δ is called an L-fuzzy topology on X, and the pair (L^X, δ) is an L-fuzzy topological space, or L-fts for short. Every member of δ is called L-fuzzy open set.

Example 2.2. Let $L = \{0, 1\}$, then $L^X \cong 2^X$ and (L^X, δ) is just the general topological space.

Definition 2.3. Let $L = [0, 1]$, then (L^X, δ) is called fuzzy topological space.

Remark 2.4. From the definitions above, we know that L-fuzzy topological space is the generalization of both general topological space and fuzzy topological space.

Example 2.5. Let $X = [0, 1]$ and define fuzzy sets on X as:

$$\mu_1(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq 1/2; \\ 2x - 1, & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

$$\mu_2(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1/4; \\ 2 - 4x, & \text{if } 1/4 \leq x \leq 1/2; \\ 0, & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

$$\mu_3(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq 1/4; \\ (4x - 1)/3, & \text{if } 1/4 \leq x \leq 1. \end{cases}$$

Put $\tau = \{0, 1, \mu_3\}$ and $\sigma = \{0, 1, \mu_1, \mu_2, \mu_1 \vee \mu_2\}$. Then (X, τ) and (X, σ) are both fuzzy topological spaces and hence L-fuzzy topological spaces.

Let (L^X, δ) be an L-fuzzy topological space (briefly, L-fts), e be a fuzzy point and P an L-fuzzy closed set in (L^X, δ). Then P is called a remote-neighborhood of e, if $e \notin P$. The set of all remote-neighborhoods of e will be denoted by $\eta(e)$. A^o, A^- and $A^{'}$ will denote the interior, closure and complement of the L-fuzzy set A in X, respectively. For definitions and results not explained in this paper, the reader is referred to [10], [13] assuming them to be well known.

Example 2.6. Let $x_1, x_2, ...$ be a sequence in a set X. Then it is a net with an index set $D = \{1, 2, ...\}$. So the concept of a net is a generalization of the concept of a sequence.

Definition 2.7. Let L_1 and L_2 be fuzzy lattices. A mapping $f : L_1 \rightarrow L_2$ is called an order-homomorphism (briefly, OH) if the following conditions hold:

(1) $f(0) = 0$.

(2) $f(\vee A_i) = \vee f(A_i)$ for $\{A_i\} \subset L_1$.

(3) $f^{-1}(B^{'}) = (f^{-1}(B))'$ for each $B \in L_2$.

In general topological spaces, generalized closed sets were introduced by Norman Levine [5]. G. Balasubramanian and P. Sundaram extended this definition to L-topological spaces ($L = [0, 1]$) [2].

Definition 2.8. (G. Balasubramanian and P. Sundaram [2]) Let (L^X, δ) be an L-fts and $f \in L^X$. Then f is called generalized fuzzy closed (in short gfc) iff $cl(f) \leq \mu$ whenever $f \leq \mu$ and μ is L-fuzzy open. An L-set λ is called generalized fuzzy open (in short gfo) iff $1-\lambda$ is gfc. It can be proved that λ is gfo iff $\mu \leq Int(\lambda)$ whenever $\mu \leq \lambda$ and μ is L-fuzzy closed. And the union of any two gf-closed sets is also a gf-closed set.

And in general topological spaces, α-generalized closed sets were introduced by H. Maki, R. Devi, and K. Balachandran in [7]. M.E. El-Shafei and A. Zakari extended this definition to L-topological spaces and studied its basic properties in [3].

Definition 2.9. (M.E. El-Shafei and A. Zakari [3]) Let (L^X, δ) be an L-fts and $f \in L^X$. Then f is called α-generalized fuzzy closed (in short α-g-closed) iff $cl_\alpha(f) \leq \mu$ whenever $f \leq \mu$ and μ is L-fuzzy open. It's easy to see that a finite union of α-generalized fuzzy closed sets is always α-generalized fuzzy closed set. And the complement of a α-g-closed fuzzy set is called α-g-open.

Proposition 2.10. [3] *Every generalized fuzzy closed set is α-g-closed.*

Let (L^X, δ) be an L-fts, and A an L-set of (L^X, δ). Then

$$A_\triangle = \cup\{B : B \in \alpha GO(L^X),\ B \leq A\},\ \ A_\sim = \cap\{B : B \in \alpha GC(L^X),\ A \leq B\}$$

are called the α-generalized-interior and α-generalized-closure of A, respectively. $\alpha GO(L^X)$ and $\alpha GC(L^X)$ will always denote the family of α-g-open sets and the family of α-g-closed sets of an L-fts (L^X, δ), respectively.

α-Generalized-convergence of L-fuzzy nets

Definition 3.1. Let (L^X, δ) be an L-fts, x_α be a fuzzy point and $P \in \alpha GC(L^X)$. P is called an L-fuzzy α-generalized-remote-neighborhood, or briefly, αGC-RN of x_α, if $x_\alpha \notin P$. The set of all αGC-RNs of x_α will be denoted by ζ_{x_α}.

Definition 3.2. Let A an L-set of an L-fts (L^X, δ). Then a fuzzy point x_α is called a α-g-adherence point of A if $A \not\leq P$ for each $P \in \zeta_{x_\alpha}$. If x_α is a α-g-adherence point of A and $x_\alpha \notin A$, or $x_\alpha \in A$ and for each fuzzy point x_μ satisfying $x_\alpha \leq x_\mu \in A$ we have $A \not\leq x_\mu \vee P$, then x_α is called a α-g-accumulation point of A. The union of all α-g-accumulation points of A will be called the α-G-derived set of A and denoted $A^{d(\alpha-G)}$.

Remark 3.3. In general topological space (or in mathematical analysis), a point x is an adherence point of a subset A iff every neighborhood of x intersects A; A point x is an accumulation point of a subset A iff every neighborhood of x contains points of A other than x.

In general topological space, P is a closed remote-neighborhood of a point x iff P' is an open neighborhood of x. Then every closed remote-neighborhood P of x does not contain A equivalent to every open neighborhood P' of x intersected A. So the concept of adherence point in L-fts is a generalization of the adherence point in general topological space (or in analysis). Similarly, the concept of accumulation point in L-fts is a generalization of the accumulation point in general topological space (or in analysis).

Definition 3.4. Let (L^X, δ) be an L-fts, $e \in M^*(L^X)$ and $S = \{S(n) : n \in D\}$ an L-fuzzy net in L^X. Then

(1) e is said to be a α-g-limit point of S (or S α-g-converges to e; in symbols, $S \to e(\alpha)$), if for each $P \in \zeta(e)$, $S(n) \notin P$ is eventually true (i.e. if there exists $n_0 \in D$ such that for every $n \in D$, $n \geq n_0$, always possess $S(n) \notin P$).

(2) e is said to be a α-g-cluster point of S (or S α-g-accumulates to e; in symbols, $S \infty e(\alpha)$), if for each $P \in \zeta(e)$, $S(n) \notin P$ is frequently true (i.e. if for every $n_0 \in D$, there always exist $n \in D$, $n \geq n_0$, such that $S(n) \notin P$).

The union of all α-g-limit points and all α-g-cluster points of S will be denoted by α-g-lim S and α-g-ad S, respectively. Obviously, α-g-lim $S \leq \alpha$-g-ad S. One can readily check the following proposition.

Proposition 3.5. *Let (L^X, δ) be an L-fts, $e \in M^*(L^X)$ and $S = \{S(n) : n \in D\}$ an L-fuzzy net in L^X. Then the following statements are valid:*

(1) *If $S = \{S(n) : n \in D\} \to e(\alpha)$, $T = \{T(n) : n \in D\}$ is an L-fuzzy net with the same domain as S and for each $n \in D$, $T(n) \geq S(n)$ holds. Then $T = \{T(n) : n \in D\} \to e(\alpha)$.*

(2) *If $S = \{S(n) : n \in D\} \infty e(\alpha)$, $T = \{T(n) : n \in D\}$ is an L-fuzzy net with the same domain as S and for each $n \in D$, $T(n) \geq S(n)$ holds. Then $T = \{T(n) : n \in D\} \infty e(\alpha)$.*

(3) *If $S=\{S(n) : n \in D\} \to e(\alpha)$ and $d \leq e$. Then $S = \{S(n) : n \in D\} \to d(\alpha)$.*

(4) *If $S = \{S(n) : n \in D\} \infty e(\alpha)$ and $d \leq e$. Then $S = \{S(n) : n \in D\} \infty d(\alpha)$.*

Theorem 3.6. *Let (L^X, δ) be an L-fts, $e \in M^*(L^X)$ and $S = \{S(n) : n \in D\}$ an L-fuzzy net in L^X. Then:*

(1) *$S \to e(\alpha)$ iff $e \in \alpha$-g-lim S.*

(2) *$S \infty e(\alpha)$ iff $e \in \alpha$-g-ad S.*

Proof. (1) $\Rightarrow$ Suppose that $S \to e(\alpha)$, then by the Definition 3.4, e is said to be a α-g-limit point of S. And α-g-lim S is the union of all α-g-limit points of S, then we have $e \in \alpha$-g-lim S.

$\Leftarrow$ Suppose that $e \in \alpha$-g-lim S and $P \in \zeta(e)$. Then $e \notin P$, and so α-g-lim $S \nleq P$. By the definition of α-g-lim S, there must exist a α-g-limit point d of S such that $d \notin P$, i.e., $P \in \zeta(d)$. Hence, S is eventually not in P, i.e., $S \to e(\alpha)$.

(2) $\Rightarrow$ Suppose that $S \infty e(\alpha)$, then by Definition 3.4, e is said to be a α-g-cluster point of S. And α-g-ad S is the union of all α-g-cluster points of S, then we have $e \in \alpha$-g-ad S.

$\Leftarrow$ Suppose that $e \in \alpha$-g-ad S and $P \in \zeta(e)$. Then $e \notin P$, and so α-g-ad $S \nleq P$. By the definition of α-g-ad S, there must exist a α-g-cluster point d of S such that $d \notin P$, i.e., $P \in \zeta(d)$. Hence, $S \notin P$ is frequently true, i.e., $S \infty e(\alpha)$.

Theorem 3.7. *Let* (L^X, δ) *be an L-fts,* $e \in M^*(L^X)$ *and* $S = \{S(n) : n \in D\}$ *an L-fuzzy net in* L^X*. Then* α*-g-lim* S *and* α*-g-ad* S *are* α*-g-closed.*

Proof. Let $e \in (\alpha\text{-g-lim } S)_\sim$. Then $\alpha\text{-g-lim } S \not\leq P$ for each $P \in \zeta(e)$. Hence there exists $d \in M^*(L^X)$ such that $d \in \alpha\text{-g-lim } S$ and $d \notin P$. Then $P \in \zeta(d)$. By Theorem 3.6 (1), $S \to d(\alpha)$, i.e., $S(n) \notin P$ is eventually true. Thus, $e \in \alpha\text{-g-lim } S$. This implies that α-g-lim S is α-g-closed. Similarly, α-g-ad S is α-g-closed.

Theorem 3.8. *Let* (L^X, δ) *be an L-fts,* $e \in M^*(L^X)$ *and* $A \in L^X$*.*

(1) *If there exists in* A *an L-fuzzy net* $S = \{S(n) : n \in D\}$ *such that* $S \infty e(\alpha)$*, then* e *is a* α*-g-adherence point of* A*.*

(2) *If* e *is a* α*-g-adherence point of* A*, then there exists in* A *an L-fuzzy net* $S = \{S(n) : n \in D\}$ *such that* $S \to e(\alpha)$*.*

Proof. (1) Let $S \infty e(\alpha)$ and $S(n) \in A$ for each $n \in D$. Then for each $P \in \zeta(e)$. $A \not\leq P$ because of the fact that $S(n) \notin P$ is frequently true. Hence, e is a α-g-adherence point of A.

(2) If e is a α-g-adherence point of A, then for each $P \in \zeta(e)$ there exists a point $S(P)$ such that $S(P) \leq A$ and $S(P) \not\leq P$. Define $S = \{S(P), P \in \zeta(e)\}$, then S is an L-fuzzy net in A because of the fact that $\zeta(e)$ is a directed set in which the order is defined by inclusion. Clearly, $S \to e(\alpha)$.

Definition 3.9. Let $S = \{S(n) : n \in D\}$ and $T = \{T(m) : m \in E\}$ be two nets in L^X. Call T the subnet of S, if there exists a mapping $N : E \to D$ such that

(1) $T = SN$;

(2) For every $n_0 \in D$, there exists $m_0 \in E$ such that $N(m) \geq n_0$ for $m \geq m_0$.

Theorem 3.10. *Let* (L^X, δ) *be an L-fts,* $e \in M^*(L^X)$ *and* $S = \{S(n) : n \in D\}$ *an L-fuzzy net in* L^X*. Then* S *has a subnet* T *such that* $T \to e(\alpha)$ *iff* $S \infty e(\alpha)$*.*

Proof. Suppose that $T = \{T(m) : m \in E\}$ is a subnet of S, $T \to e(\alpha)$, $P \in \zeta(e)$ and $n_0 \in D$. By the definition of subnet, there exists a mapping $N : E \to D$ and $m_0 \in E$ such that $N(m) \geq n_0 (N(m) \in D)$ when $m \geq m_0 (m \in E)$. Since T α-g-converges to e, there is $m_1 \in E$. When $m \geq m_1 (m \in E)$, $T(m) \notin P$. Because E is a directed set, there exists $m_2 \in E$ such that $m_2 \geq m_0$ and $m_2 \geq m_1$. Hence, $T(m_2) \notin P$ and $N(m_2) \geq n_0$. Let $n = N(m_2)$. Then $S(n) = S(N(m_2)) = T(m_2) \notin P$ and $n \geq n_0$. This means that $S(n) \notin P$ is frequently true. Thus $S \infty e(\alpha)$.

Conversely, suppose that $S \infty e(\alpha)$. Then for each $P \in \zeta(e)$ and each $n \in D$, there exists $N(P, n) \in D$ such that $N(P, n) \geq n$ and $S(N(P, n)) \notin P$. Let $E = \{(N(P, n), P) : P \in \zeta(e), n \in D\}$, and define $(N(P_1, n_1), P_1) \leq (N(P_2, n_2), P_2)$ iff $n_1 \leq n_2$ and $P_1 \leq P_2$. Thus E is a directed set because:

(a) For each $(N(P,n),P)$, since $n\in D$ and D is a directed set, we have $n\le n$. Also, since $P\in\zeta(e)$ and $\zeta(e)$ is a directed set, we have $P\le P$. Hence we have $n\le n$ and $P\le P$ which equivalent that $(N(P,n),P)\le(N(P,n),P)$. Thus the relation $\le$ is reflexive on E.

(b) Let $(N(P_1,n_1),P_1)$, $(N(P_2,n_2),P_2)$ and $(N(P_3,n_3),P_3)$ belong to E with $(N(P_1,n_1),P_1)\le(N(P_2,n_2),P_2)$ and $(N(P_2,n_2),P_2)\le(N(P_3,n_3),P_3)$. Thus we have $n_1\le n_2$, $P_1\le P_2$ and $n_2\le n_3$ and $P_2\le P_3$. Since D and $\zeta(e)$ are directed sets, we get $n_1\le n_3$ and $P_1\le P_3$ which equivalent that $(N(P_1,n_1),P_1)\le(N(P_3,n_3),P_3)$. Thus the relation $\le$ is transitive on E.

(c) Let $(N(P_1,n_1),P_1)$ and $(N(P_2,n_2),P_2)$ belong to E. Since $n_1,n_2\in D$ and D is a directed set, there is $n\in D$ such that $n_1\le n$ and $n_2\le n$. Also, since $P_1,P_2\in\zeta(e)$, we have $P=P_1\vee P_2\in\zeta(e)$ and $P_1\le P$, $P_2\le P$. Hence there exists $(N(P,n),P)\in E$ with $(N(P_1,n_1),P_1)\le(N(P,n),P)$ and $(N(P_2,n_2),P_2)\le(N(P,n),P)$.

Hence $(E,\le)$ is a direct set. Let $T(N(P,n),P)=S(N(P,n))$. Then T is a subnet of S and $T\to e(\alpha)$.

Theorem 3.11. *Let (L^X,δ) be an L-fts, $e\in M^*(L^X)$ and $S=\{S(n):n\in D\}$ an L-fuzzy net in L^X. If T is a subnet of S, then:*

1. *If $S\to e(\alpha)$, then $T\to e(\alpha)$.*
2. *If $T\infty e(\alpha)$, then $S\infty e(\alpha)$.*
3. *α-g-lim $S\le\alpha$-g-lim T.*
4. *α-g-ad $T\le\alpha$-g-ad S.*

Proof. (1) Suppose $T=\{T(m):m\in E\}$ is a subnet of S, $S\to e(\alpha)$ and $P\in\zeta(e)$, then $S(n)\notin P$ is eventually true. From the definition of subnet, there exists a mapping $N:E\to D$ and for every $m\in E$, there exists $n\in D$ such that $T(m)=S(N(m))=S(n)$. That is to say, every element of the net T is actually the element of the net S. So $T(m)\notin P$ is eventually true. Thus we have $T\to e(\alpha)$.

(2) Suppose that $T=\{T(m):m\in E\}$ is a subnet of S, $T\infty e(\alpha)$, $P\in\zeta(e)$ and $n_0\in D$. By the definition of subnet, there exists a mapping $N:E\to D$ and $m_0\in E$ such that $N(m)\ge n_0(N(m)\in D)$ when $m\ge m_0(m\in E)$. Since T α-g-accumulates to e, for $m_0\in E$ there is $m_1\in E$. When $m_1\ge m_0(m_1\in E)$, $T(m_1)\notin P$. Let $n=N(m_1)$. Then $S(n)=S(N(m_1))=T(m_1)\notin P$ and $n\ge n_0$. This means that $S(n)\notin P$ is frequently true. Thus $S\infty e(\alpha)$.

(3) By Theorem 3.6, $S\to e(\alpha)$ means $e\in\alpha$-g-lim S and $T\to e(\alpha)$ means $e\in\alpha$-g-lim T. Thus by (1), we have α-g-lim $S\le\alpha$-g-lim T.

(4) By Theorem 3.6, $S\infty e(\alpha)$ means $e\in\alpha$-g-ad S and $T\infty e(\alpha)$ means $e\in\alpha$-g-ad T. Thus by (2), we have α-g-ad $T\le\alpha$-g-ad S.

4. Applications

Definition 4.1. An OH $f : (L_1^X, \delta) \to (L_2^Y, \tau)$ is said to be α-g-irresolute if $f^{-1}(B) \in \alpha GO(L_1^X)$ for each $B \in \alpha GO(L_2^Y)$.

Theorem 4.2. *For an OH $f : (L_1^X, \delta) \to (L_2^Y, \tau)$ the following are equivalent:*

(1) *f is α-g-irresolute.*

(2) *$f^{-1}(B) \in \alpha GC(L_1^X)$ for each $B \in \alpha GC(L_2^Y)$.*

(3) *$(f^{-1}(B))_\sim \leq f^{-1}(B_\sim)$ for each $B \in L_2^Y$.*

Proof. (1)⇒(2): f is α-g-irresolute if $f^{-1}(A) \in \alpha GO(L_1^X)$ for each $A \in \alpha GO(L_2^Y)$. For each $B \in \alpha GC(L_2^Y)$, $B' \in \alpha GO(L_2^Y)$. So we have $(f^{-1}(B))' = f^{-1}(B') \in \alpha GO(L_1^X)$. This shows $f^{-1}(B) \in \alpha GC(L_1^X)$.

(2)⇒(1): For each $A \in \alpha GO(L_2^Y)$, $A' \in \alpha GC(L_2^Y)$. Then by (2) we have $(f^{-1}(A))' = f^{-1}(A') \in \alpha GC(L_1^X)$. This shows $f^{-1}(A) \in \alpha GO(L_1^X)$. Hence by Definition 4.1, f is α-g-irresolute.

(2)⇒(3): For each $B \in L_2^Y$, $B_\sim \in \alpha GC(L_2^Y)$. Then by (2) we have $f^{-1}(B_\sim) \in \alpha GC(L_1^X)$. And $B \leq B_\sim$ implies $f^{-1}(B) \leq f^{-1}(B_\sim)$. From the definition of α-generalized-closure we have $(f^{-1}(B))_\sim \leq f^{-1}(B_\sim)$.

(3)⇒(1): Let $B \in \alpha GC(L_2^Y)$, then $B = B_\sim$. By (3) we have $f^{-1}(B) \leq (f^{-1}(B))_\sim \leq f^{-1}(B_\sim) = f^{-1}(B)$, i.e., $f^{-1}(B) = (f^{-1}(B))_\sim$. Hence $f^{-1}(B) \in \alpha GC(L_2^X)$ and consequently, f is α-g-irresolute.

Definition 4.3. An OH $f : (L_1^X, \delta) \to (L_2^Y, \tau)$ is said to be α-g-irresolute at a point $e \in M^*(L_1^X)$ if $(f^{-1}(P))_\sim \in \zeta_1(e)$ for each $P \in \zeta_2(f(e))$, where $\zeta_1(e)$ and $\zeta_2(f(e))$ denote the set of all αGC-RNs of e and $f(e)$, respectively.

Theorem 4.4. *An OH $f : (L_1^X, \delta) \to (L_2^Y, \tau)$ is α-g-irresolute iff f is α-g-irresolute for each point $e \in M^*(L_1^X)$.*

Proof. Suppose that f is α-g-irresolute and $e \in M^*(L_1^X)$. Then $f^{-1}(P)$ is α-g-closed for each $P \in \zeta_2(f(e))$. Clearly, $e \notin f^{-1}(P)$. Hence $f^{-1}(P) = (f^{-1}(P))_\sim \in \zeta_1(e)$ and so f is α-g-irresolute at e.

Conversely, suppose that f is α-g-irresolute for each $e \in M^*(L_1^X)$ and $P \in \alpha GC(L_2^Y)$. We may assume that $f^{-1}(P) \neq 1_X$ and suppose that $e \notin f^{-1}(P)$. Then $f(e) \notin P$ and so $P \in \zeta_2(f(e))$. Hence, $(f^{-1}(P))_\sim \in \zeta_1(e)$, i.e., $e \notin f^{-1}(P)$ implies that $e \notin (f^{-1}(P))_\sim$ or $(f^{-1}(P))_\sim \leq f^{-1}(P)$. Thus, $f^{-1}(P)$ is α-g-closed in (L_1^X, δ), i.e., f is α-g-irresolute.

Now we discuss the applications of α-g-convergence of L-fuzzy nets.

Theorem 4.5. *Let $f : (L_1^X, \delta) \to (L_2^Y, \tau)$ be α-g-irresolute at $e \in M^*(L_1^X)$ and S an L-fuzzy net in L_1^X. If $S \to e(\alpha)$ we have $f(S)$ α-g-converges to $f(e)$ where $f(S) = \{f(S(n)),\ n \in D\}$ is an L-fuzzy net in L_2^Y.*

Proof. Suppose that f is α-g-irresolute at $e \in M^*(L_1^X)$ and $S \to e(\alpha)$. Let $P \in \zeta_2(f(e))$. Then S is eventually not in $(f^{-1}(P))_\sim \in \zeta_1(e)$, and hence $f(S)$ is eventually not in P, i.e., $f(S) \to f(e)(\alpha)$.

Theorem 4.6. *Let* f *:*$(L_1^X, \delta) \to (L_2^Y, \tau)$ *be* α*-g-irresolute. Then for each* L*-fuzzy net* S *in* L_1^X *we have* $f(\alpha\text{-g-lim } S) \leq \alpha\text{-g-lim } f(S)$.

Proof. Suppose that $e \in M^*(L_1^X)$, S is an L-fuzzy net in L_1^X and $f(e) \in f(\alpha$-g-lim $S)$. Then $e \in \alpha$-g-lim S. By Theorem 3.6 we have $S \to e(\alpha)$. Since f is α-g-irresolute, we have $f(S) \to f(e)(\alpha)$ based on Theorems 4.4 and 4.5. And by Theorem 3.6 we have $f(e) \in \alpha$-g-lim $f(S)$. Thus, $f(\alpha\text{-g-lim } S) \leq \alpha\text{-g-lim } f(S)$.

Theorem 4.7. *Let* $f : (L_1^X, \delta) \to (L_2^Y, \tau)$ *be* α*-g-irresolute. Then for each* L*-fuzzy net* T *in* L_2^Y *we have* $\alpha\text{-g-lim } f^{-1}(T) \leq f^{-1}(\alpha\text{-g-lim } T)$.

Proof. Let $T = \{T(n) : n \in D\}$ be an L-fuzzy net in L_2^Y. Then $f^{-1}(T) = \{f^{-1}(T(n)) : n \in D\}$ an L-fuzzy net in L_1^X. Since f is α-g-irresolute, according to Theorem 4.6 we have $f(\alpha\text{-g-lim } f^{-1}(T)) \leq \alpha\text{-g-lim } f(f^{-1}(T)) \leq \alpha\text{-g-lim } T$. Hence, $\alpha\text{-g-lim } f^{-1}(T) \leq f^{-1}(\alpha\text{-g-lim } T)$.

Definition 4.8. (Aygün [1]) Let (L^X, δ) be an L-fts and $g \in L^X$, $r \in L$.

(1) A collection $\mu = \{f_i\}_{i \in J}$ of L-subsets is called an r-level cover of g iff $(\vee_{i \in J} f_i)(x) \not\leq r$ for all $x \in X$ with $g(x) \geq r'$. If each f_i is open then μ is called an r-level open cover of g. If g is the whole space 1_X, then μ is called an r-level cover of 1_X iff $(\vee_{i \in J} f_i)(x) \not\leq r$ for all $x \in X$.

(2) An r-level cover $\mu = \{f_i\}_{i \in J}$ of g is said to have a finite r-level subcover if there exists a finite subset F of J such that $(\vee_{i \in F} f_i)(x) \not\leq r$ for all $x \in X$ with $g(x) \geq r'$.

Definition 4.9. (Kudri [4]) Let (L^X, δ) be an L-fts and $g \in L^X$. The L-fuzzy subset g is said to be compact iff for every prime $p \in L$ and every collection $\{f_i\}_{i \in J}$ of open L-subsets with $(\vee_{i \in J} f_i)(x) \not\leq p$ for all $x \in X$ with $g(x) \geq p'$, there exists a finite subset F of J such that $(\vee_{i \in F} f_i)(x) \not\leq p$ for all $x \in X$ with $g(x) \geq p'$, i.e. every p-level open cover of g has a finite p-level subcover, where $p \in pr(L)$. If g is the whole space, then the L-tfs (L^X, δ) is called compact.

Definition 4.10. Let (L^X, δ) be an L-fts and $g \in L^X$. The L-fuzzy subset g is called α-g-compact iff every p-level cover of g consisting of α-g-open L-subsets has a finite p-level subcover, where $p \in pr(L)$. If g is the whole space, then we say that the L-fts (X, δ) is α-g-compact.

Theorem 4.11. *Let* (L^X, δ) *be an* L*-fts and* $g \in L^X$*. The* L*-fuzzy subset* g *is said to be* α*-g-compact if and only if for every* $\alpha \in M(L)$ *and every collection* $(f_i)_{i \in J}$ *of* α*-g-closed* L*-fuzzy sets with* $(\wedge_{i \in J} f_i)(x) \not\geq \alpha$ *for all* $x \in X$ *with* $g(x) \geq \alpha$*, there exists a finite subset* F *of* J *with* $(\wedge_{i \in F} f_i)(x) \not\geq \alpha$ *for all* $x \in X$ *with* $g(x) \geq \alpha$*, i.e.,* L*-fuzzy points* $x_\alpha \in M(L^X)$ *such that* $x_\alpha \leq g$.

Proof. This follows immediately from Definition 4.10 and the duality of p and α.

Definition 4.12. Let (L^X,δ) be an L-fts , $x_\alpha \in M^*(L^X)$ and $S = (S_m)_{m\in D}$ be a net. x_α is called a α-g-cluster of S iff for each α-g-closed L-subset f with $f(x) \not\geq \alpha$ and for all $n \in D$, there is $m \in D$ such that $m \geq n$ and $S_m \not\leq f$, i.e., $h(S_m) \not\leq f(SuppS_m)$.

Theorem 4.13. *Let (L^X,δ) be an L-fts and $g \in L^X$. The L-fuzzy subset g is said to be α-g-compact if and only if for every constant $\alpha-$net $(S_m)_{m\in D}$ contained in $g(S_m \leq g$ for every $m \in D)$ has a α-g-cluster point with height $\alpha, x_\alpha \in M^*(L^X)$, contained in $g(x_\alpha \leq g$ for each $\alpha \in M(L))$.*

Proof. *Necessity:* Let $\alpha \in M(L)$ and $S = (S_m)_{m\in D}$ be a constant α-net in g without any α-g-cluster point with height α in g. Then for each $x \in X$ with $g(x) \geq \alpha$, x_α is not a α-g-cluster point of S, i.e., there are $n_x \in D$ and a α-g-closed L-subset f_x with $f_x(x) \not\geq \alpha$ and $S_m \leq f_x$ for each $m \geq n_x$. Let $x^1, ..., x^k$ be elements of X with $g(x^i) \geq \alpha$ for each $i \in \{1, ..., k\}$. Then there are $n_{x1}, ..., n_{xk} \in D$ and α-g-closed L-subset f_{xi} with $f_{xi}(x^i) \not\geq \alpha$ and $S_m \leq f_{xi}$ for each $m \geq n_{xi}$ and for each $i \in \{1, ..., k\}$. Since D is a directed set, there is $n_o \in D$ such that $n_o \geq n_{xi}$ for each $i \in \{1, ..., k\}$ and $S_m \leq f_{xi}$ for $i \in \{1, ..., k\}$ and each $m \geq n_o$. Now, consider the family $\mu = \{f_x\}_{x\in X}$ with $g(x) \geq \alpha$. Then $(\wedge_{f_x\in\mu} f_x)(y) \not\geq \alpha$ for all $y \in X$ with $g(y) \geq \alpha$ because $f_y(y) \not\geq \alpha$. We also have that for any finite subfamily $\nu = \{f_{x1}, ..., f_{xk}\}$ of μ, there is $y \in X$ with $g(y) \geq \alpha$ and $(\wedge_{i=1}^k f_{xi})(y) \geq \alpha$ since $S_m \leq \wedge_{i=1}^k f_{xi}$ for each $m \geq n_o$ because $S_m \leq f_{xi}$ for each $i \in \{1, ..., k\}$ and for each $m \geq n_o$. Hence, by Theorem 4.11, g is not α-g-compact.

Sufficiency: Suppose that g is not α-g-compact. Then, by Theorem 4.11, there exist $\alpha \in M(L)$ and a collection $\mu = \{f_i\}_{i\in J}$ of α-g-closed L-subsets with $(\wedge_{i\in J} f_i)(x) \not\geq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$, but for any finite subfamily ν of μ there is $x \in X$ with $g(x) \geq \alpha$ and $(\wedge_{f\in\nu} f)(x) \geq \alpha$. Consider the family of all finite subsets of μ, $2^{(\mu)}$, with the order $\nu_1 \leq \nu_2$ iff $\nu_1 \subseteq \nu_2$. Then $2^{(\mu)}$ is a directed set. So, writing x_α as S_ν for every $\nu \in 2^{(\mu)}$, $(S_\nu)_{\nu\in 2^{(\mu)}}$ is a constant α-net in g because the height of S_ν for all $\nu \in 2^{(\mu)}$ is α and $S_\nu \leq g$ for all $\nu \in 2^{(\mu)}$, i.e., $g(x) \geq \alpha$. $(S_\nu)_{\nu\in 2^{(\mu)}}$ also satisfies the condition that for each α-g-closed L-subset $f_i \in \nu$ we have $x_\alpha = S_\nu \leq f_i$. Let $y \in X$ with $g(y) \geq \alpha$. Then $(\wedge_{i\in J} f_i)(y) \not\geq \alpha$, i.e., there exists $j \in J$ with $f_j(y) \not\geq \alpha$. Let $\nu_0 = \{f_j\}$. So, for any $\nu \geq \nu_0$, $S_\nu \leq \wedge_{f_i\in\nu} f_i \leq \wedge_{f_i\in\nu_0} f_i = f_j$. Thus, we get a α-g-closed L-subset f_j with $f_j(y) \not\geq \alpha$ and $\nu_0 \in 2^{(\mu)}$ such that for any $\nu \geq \nu_0$, $S_\nu \leq f_j$. That means that $y_\alpha \in M^*(L^X)$ is not a α-g-cluster point of $(S_\nu)_{\nu\in 2^{(\mu)}}$ for all $y \in X$ with $g(y) \geq \alpha$. Hence, the constant α-net $(S_\nu)_{\nu\in 2^{(\mu)}}$ has no α-g-cluster point in g with height α.

Corollary 4.14. *An L-fts is α-g-compact iff every constant α-net in (L^X,δ) has a α-g-cluster point with height α, where $\alpha \in M(L)$.*

5. Conclusion and future research

The theory of fuzzy lattices is one of the most important branches in fuzzy systems. The theory of α-g-closed sets, α-g-convergence of L-fuzzy nets and α-g-irresolute functions and α-g-compact which presented in this paper by using molecules and remoted neighborhoods are very significant tools to studying the theory of L-fuzzy topological spaces. There is still a lot of results for future investigations, for example the consideration of this theory on topological molecular lattices [14] will lead to some interesting research from the view point of fuzzy mathematics.

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CERTAIN TRANSFORMATION AND SUMMATION FORMULAE FOR q-SERIES

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Abstract. In this paper, making use of certain summation formulae, an attempt has been made to establish certain new and interesting transformation and summation formulae for q-series.

Keywords: summation, transformation, q-series.
AMS Subject Classification: 33A30.

1. Introduction

Bailey [1] established a simple but very useful identity:

If

$$\beta_n = \sum_{r=0}^{n} u_{n-r}\, v_{n+r}\, \alpha_r \tag{1.1}$$

and

$$\gamma_n = \sum_{r=n}^{\infty} u_{r-n}\, v_{r+n}\, \delta_r \tag{1.2}$$

where α_r, δ_r, u_r and v_r are any functions of r only such that series γ_n exists, then subject to the convergence of the series.

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \ \delta_n. \tag{1.3}$$

Making use of (1.3), Slater [3] gave a long list of Rogers-Ramanujan type identities. Later on, a number of mathematicians, notably, Verma [7], Verma and Jain [9], Singh [5], Denis [2], Singh [6] and others made use of Bailey's identity (1.3) and established a number of transformation formulae and also Rogers-Ramanujan type identities of different moduli. In this paper, making use of certain known summation formulae due to Verma and Jain [9] and identity (1.3), an attempt has been made to establish certain very interesting transformation formulae for q-hypergeometric series.

In the last section of this paper, making use of the following identity due to Verma [8], viz.,

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-z)^n q^{n(n-1)/2}}{[q;q]_n\,[\gamma q^n;q]_n} \sum_{k=0}^{\infty} \frac{[\alpha,\beta;q]_{n+k} B_{n+k}\, z^k}{[q;q]_k\,[\gamma q^{2n+1};q]_k} \sum_{j=0}^{n} \frac{[q^{-n}.\gamma q^n;q]_j\, A_j (wq)^j}{[q;q]_j\,[\alpha,\beta;q]_j} \\ = \sum_{n=0}^{\infty} A_n B_n \,\frac{(zw)^n}{[q;q]_n}\end{aligned} \tag{1.4}$$

and summation formulae due to Verma and Jain [9], an attempt has been made to establish certain new transformation and summation formulae for basic hypergeometric series.

2. Definitions and notations

A basic (q-) hypergeometric series is generally defined to be a series of the type $\sum\limits_{n=0}^{\infty} a_n z^n$, where a_{n+1}/a_n is a rational function of q^n, q being a fixed complex-parameter, called the base of the series, usually with modules less than one. An explicit representation of such series is given by

$${}_r\Phi_s\left[\begin{array}{ll} a_1, a_2, \cdots, a_r & ; q; z \\ b_1, b_2, \cdots, b_s & ; q^i \end{array}\right] = \sum_{n=0}^{\infty} q^{in(n-1)/2}\, \frac{[a_1, a_2, \cdots a_r; q]_n\, z^n}{[q, b_1, b_2, \cdots, b_s; q]_n} \tag{2.1}$$

and

$$[a_1, a_2, \cdots, a_r; q]_n = [a_1;q]_n\,[a_2;q]_n \cdots [a_r;q]_n$$

with the q-shifted factorial defined by

$$[a;q]_n = \begin{cases} 1, & \text{if } n=0 \\ (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}), & \text{if } \ n=1,2,\cdots \end{cases} \tag{2.2}$$

For convergence of the series (2.1) we need $|q| < 1$ and $|z| < \infty$, when $i > 0$, or $\max\{|q|, |z|\} < 1$, when $i = 0$, provided no zeros appear in the denominator. Following summations are needed in our analysis,

$$ {}_2\Phi_1\left[\begin{matrix} a,\, b & ;q & ;c/ab \\ c & & \end{matrix}\right] = \frac{[c/a, c/b; q]_\infty}{[c, c/ab; q]_\infty}. \tag{2.3} $$

(Slater [2; App.IV(IV.2)])

$$ {}_2\Phi_1\left[\begin{matrix} a,\, b & ;q & ;c/ab \\ cq & & \end{matrix}\right] = \frac{[cq/a,\, cq/b; q]_\infty}{[cq,\, cq/ab; q]_\infty}\left\{\frac{ab(1+c) - c(a+b)}{ab - c}\right\}. \tag{2.4} $$

(Verma [7; (1.4)])

$$ {}_4\Phi_3\left[\begin{matrix} q^{-n},\, x^2y^2q^{n+1},\, x,\, -xq & ;q & ;q \\ xyq,\, -xyq,\, x^2q & & \end{matrix}\right] = \frac{x^n[q;q]_n\,[x^2q^2;q^2]_m\,[y^2q^2;q^2]_m}{[x^2q;q]_n\,[x^2y^2q^2;q^2]_m\,[q^2;q^2]_m}, \tag{2.5} $$

where m is greatest integer $\leq n/2$.

$$ {}_4\Phi_3\left[\begin{matrix} q^{-2n},\, b^2x^4y^2q^{2n+2},\, x^2,\, x^2q & ;q^2 & ;q^2 \\ bx^2q,\, bx^2q^2,\, x^4q^2 & & \end{matrix}\right] = \frac{x^{2n}[-q;q]_n\,[bq;q]_n}{[-x^2q;q]_n\,[bx^2q;q]_n}, \tag{2.6} $$

[Verma and Jain [9; (2.32)])

$$ \begin{aligned} &{}_4\Phi_3\left[\begin{matrix} q^{-n},\, bx^2q^{n+2},\, x,\, -xq & ;q & ;q \\ xq\sqrt{b},\, -xq\sqrt{b},\, x^2q^2 & & \end{matrix}\right] \\ &\qquad = \frac{x^n[q;q]_n\,[bxq^2;q]_n\,[bx^2q^3;q^2]_m\,[bq^2;q^2]_m\,[xq^2;q]_{2m}}{[xq;q]_n\,[bx^2q^2;q]_n\,[q^2;q^2]_m\,[x^2q^3;q^2]_m\,[bxq^2;q]_{2m}}, \end{aligned} \tag{2.7} $$

[Verma and Jain [9; (3.2)])

where m is greatest integer $< n/2$.

$$ {}_5\Phi_4\left[\begin{matrix} a,\, aq,\, aq^2,\, a^3q^{3n+3},\, q^{-3n} & ;q^3 & ;q^3 \\ (aq)^{3/2},\, -(aq)^{3/2},\, a^{3/2}q^3,\, -a^{3/2}q^3 & & \end{matrix}\right] = \frac{a^n[q^3;q^3]_n\,[aq;q]_n}{[a^3q^3;q^3]_n\,[q;q]_n}, \tag{2.8} $$

[Verma and Jain [9; (4.2)])

$$ \begin{aligned} &{}_5\Phi_4\left[\begin{matrix} x,\, \omega xq,\, \omega^2xq,\, x^3q^{n+4},\, q^{-n} & ;q & ;q \\ (xq)^{3/2},\, -(xq)^{3/2},\, x^{3/2}q^2,\, -x^{3/2}q^2 & & \end{matrix}\right] \\ &\qquad = \frac{x^n[x^2q^4;q]_n\,[q;q]_n\,[x^3q^6;q^3]_n\,[xq^3;q]_{3m}}{[x^3q^4;q]_n\,[xq;q]_n\,[q^3;q^3]_m\,[x^2q^4;q]_{3m}}, \end{aligned} \tag{2.9} $$

[Verma and Jain [9; (4.4)])

where m is greatest integer $\leq n/3$ and $\omega = e^{2\pi i/3}$.

$$(2.10)\quad {}_5\Phi_4\left[\begin{matrix} a^{1/3},\ \omega a^{1/3},\ \omega^2 a^{1/3},\ aq^{n+1},\ q^{-n} & ;q & ;q \\ q\sqrt{a},\ -q\sqrt{a},\ \sqrt{aq},\ -\sqrt{aq} & & \end{matrix}\right] = \frac{(\sqrt{a})^{n-m}\,[q;q]_n\,[aq^3;q^3]_m}{[aq;q]_n\,[q^3;q^3]_m},$$

[Verma and Jain [9; (4.5)])

where m is greatest integer $\leq n/3$ and $\omega = e^{2\pi i/3}$.

$$(2.11)\quad \begin{aligned} &{}_6\Phi_5\left[\begin{matrix} a^{1/3},\ \omega a^{1/3},\ \omega^2 a^{1/3},\ q\sqrt{a},\ aq^{n+1},\ q^{-n} & ;q & ;q \\ \sqrt{a},\ -\sqrt{a},\ \sqrt{aq},\ -\sqrt{aq},\ q^2\sqrt{a} & & \end{matrix}\right] \\ &\qquad = \frac{[q;q]_n[\sqrt{a};q]_n[aq^3;q^3]_m[q^6\sqrt{a};q^3]_m(\sqrt{a})^{n-m}}{[aq;q]_n\,[q^2\sqrt{a};q]_n[q^3;q^3]_m[\sqrt{a};q^3]_m}, \end{aligned}$$

[Verma and Jain [9; (4.8)])

where m is greatest integer $\leq n/3$.

3. Main Results

In this section, we shall establish the transformation formulae by making use of (1.3).

Taking $u_r = \dfrac{1}{[q;q]_r}$, $v_r = \dfrac{[aq;q]_r}{q^{r^2/2}}$ in (1.1) and (1.2), we get:

If

$$(3.1)\qquad \beta_n = \frac{[aq;q]_n}{[q;q]_n\,q^{n^2/2}}\sum_{r=0}^{n}\frac{(-1)^r q^{r/2}[q^{-n};q]_r\,[aq^{n+1};q]_r\,\alpha_r}{q^{r^2}} \qquad \text{infty}$$

and

$$(3.2)\qquad \gamma_n = \frac{[aq;q]_{2n}}{q^{2n^2}}\sum_{r=0}^{\infty}\frac{[aq^{2n+1};q]_r\,\delta_{r+n}}{[q;q]_r\,q^{r^2/2+2nr}},$$

then

$$(3.3)\qquad \sum_{n=0}^{\infty}\alpha_n\gamma_n = \sum_{n=0}^{\infty}\beta_n\ \delta_n,$$

provided the series involving are convergent. We shall now use (3.1), (3.2) and (3.3) in order to establish the required transformations.

(i) Replacing a by x^2y^2 in (3.1) and (3.2), and then taking

$$\alpha_r = \frac{[x,-xq;q]_r\,q^{r^2+r/2}(-)^r}{[q,xyq,-xyq,x^2q;q]_r}$$

in (3.1) and making use of (2.5), we have:

$$\beta_n = \frac{[x^2y^2q;q]_n\, x^n [x^2q^2;q^2]_m\, [y^2q^2;q^2]_m}{q^{n^2/2}\, [x^2q;q]_n\, [x^2y^2q^2;q^2]_m\, [q^2;q^2]_m} \tag{3.4}$$

where m is the greatest integer $< n/2$.

Again taking $\delta_r = z^r q^{r^2/2}$ in (3.2), we get after some simplification:

$$\gamma_n = \frac{[x^2y^2zq;q]_\infty\, [x^2y^2q;q^2]_{2n}\, (-)^n\, q^{n/2}}{[z;q]_\infty\, q^{n^2}\, [x^2q^2zq;q]_n\, [q/z;q]_n}. \tag{3.5}$$

Now, putting these values of α_n, β_n, γ_n and δ_n in (3.3) we get:

$$\begin{aligned} &\frac{[x^2y^2zq;q]_\infty}{[z;q]_\infty}\, {}_4\Phi_3\left[\begin{matrix} x,\ -xq,\ xyq^{1/2},\ -xyq^{1/2} & ;q\ ;\ q \\ x^2q,\ x^2y^2zq,\ q/z & \end{matrix}\right] \\ &\qquad = {}_2\Phi_1\left[\begin{matrix} x^2y^2q,\ y^2q^2 & ;q^2\ ;\ x^2z^2 \\ x^2q & \end{matrix}\right] \\ &\qquad + \frac{xz(1-x^2y^2q)}{(1-x^2q)}\, {}_2\Phi_1\left[\begin{matrix} x^2y^2q^3,\ y^2q^2 & ;q^2\ ;\ x^2z^2 \\ x^2q^3 & \end{matrix}\right], \qquad |xz| < 1. \end{aligned} \tag{3.6}$$

(ii) Next, replacing a by b^2x^4 and q by q^2 in (3.1) and (3.2) and then taking $\alpha_r = \dfrac{[x^2, x^2q; q^2]_r\, q^{2r^2+r}(-)^r}{[bx^2q, bx^2q^2, x^4q^2; q^2]_r\, [q^2;q^2]_r}$ in (3.1) and making use of (2.6) we have:

$$\beta_n = \frac{[b^2x^4q^2;q^2]_n\, x^{2n}\, [-q;q]_n\, [bq;q]_n}{[q^2;q^2]_n\, [-x^2q;q]_n\, [bx^2q;q]_n\, q^{n^2}}. \tag{3.7}$$

Again,setting $\delta_r = z^r q^{r^2}$ in (3.2) we have:

$$\gamma_n = \frac{[b^2x^4zq^2;q^2]_\infty\, [b^2x^4q^2;q^2]_{2n}\, (-)^n\, q^n}{[z;q^2]_\infty\, q^{2n^2}\, [b^2x^4zq^2;q^2]_n\, [q^2/z;q^2]_n}. \tag{3.8}$$

Putting these values of α_n, β_n, γ_n and δ_n in (3.3) we get the following transformation:

$$\begin{aligned} &\frac{[b^2x^4zq^2;q^2]_\infty}{[z;q^2]_\infty}\, {}_4\Phi_3\left[\begin{matrix} x^2,\ x^2q,\ -bx^2q,\ -bx^2q^2 & ;q^2\ ;\ q^2 \\ x^4q^2,\ b^2x^4zq^2,\ q^2/z & \end{matrix}\right] \\ &\qquad = {}_2\Phi_1\left[\begin{matrix} -bx^2q,\ bq & ;q\ ;\ x^2z \\ -x^2q & \end{matrix}\right], \qquad |x^2z| < 1. \end{aligned} \tag{3.9}$$

(iii) Again, putting $a = bx^2q$ in (3.1) and (3.2) and then taking $\alpha_r = \dfrac{[x, -xq; q]_r\, (-)^r\, q^{r^2+r/2}}{[x^2q^2, xq\sqrt{b}, -xq\sqrt{b}; q]_r\, [q;q]_r}$ in (3.1) and making use of (2.7) we get:

$$\beta_n = \frac{x^n [bxq^2;q]_n\,[bx^2q^3;q^2]_m\,[bq^2;q^2]_m\,[xq^2;q]_{2m}}{q^{n^2/2}[xq;q]_n\,[q^2;q^2]_m\,[x^2q^3;q^2]_m\,[bxq^2;q]_{2m}}, \tag{3.10}$$

where m is the greatest integer $\leq n/2$.

Again, taking $\delta_r = z^r q^{r^2/2}$ in (3.2) we get:

$$\gamma_n = \frac{[bx^2q^2z;q]_\infty\,[bx^2q^2;q]_{2n}\,(-)^n\,q^{n/2}}{[z;q]_\infty\,q^{n^2}\,[bx^2q^2z;q]_n\,[q/z;q]_n}. \tag{3.11}$$

Now, putting these values in (3.3) we get the following transformation:

$$\begin{aligned} &\frac{[bzx^2q^2;q]_\infty}{[z;q]_\infty}\,{}_4\Phi_3\left[\begin{matrix} x,\,-xq,\,xq\sqrt{bq},\,-xq\sqrt{bq} & ;q\,;\,q \\ bzx^2q^2,\,x^2q^2,\,q/z & \end{matrix}\right] \\ &= {}_3\Phi_2\left[\begin{matrix} bx^2q^3,\,bq^2,\,xq^3 & ;q^2\,;\,x^2z^2 \\ xq,\,x^2q^3 & \end{matrix}\right] \\ &+\frac{xz(1-bxq^2)}{(1-xq)}\,{}_3\Phi_2\left[\begin{matrix} bxq^4,\,bx^2q^3,\,bq^2 & ;q^2\,;\,x^2z^2 \\ x^2q^3,\,bxq^2 & \end{matrix}\right], \quad |x^2z^2|<1. \end{aligned} \tag{3.12}$$

(iv) Next, replacing a by a^3 and q by q^3 in (3.1) and (3.2) and then taking $\alpha_r = \dfrac{[a,aq,aq^2;q^3]_r\,q^{3r+3r/2}\,(-)^r}{[q^3,(aq)^{3/2},-(aq)^{3/2},\,a^{3/2}q^3,\,-a^{3/2}q^3;q^3]_r}$ in (3.1) and making use of (2.8) we have:

$$\beta_n = \frac{[aq;q]_n\,a^n}{q^{3n^2/2}\,[q;q]_n}. \tag{3.13}$$

Again, taking $\delta_r = z^r q^{3r^2/2}$ in (3.2) we get after some simplifications,

$$\gamma_n = \frac{[a^3q^3;q^3]_{2n}\,[a^3q^3z;q^3]_\infty\,(-)^n\,q^{3n/2}}{[z;q^3]_\infty\,[a^3q^3z;q^3]_n\,[q^3/z;q^3]_n}. \tag{3.14}$$

Substituting these values of α_n, β_n, γ_n and δ_n in (3.3) we get the following summation,

$${}_3\Phi_2\left[\begin{matrix} a,\,aq,\,aq^2 & ;q^3\,;\,q^3 \\ a^3q^3z,\,q^3/z & \end{matrix}\right] = \frac{[z;q^3]_\infty\,[a^2zq;q]_\infty}{[a^3q^3z;q^3]_\infty\,[az;q]_\infty}. \tag{3.15}$$

(v) Next, replaying a by x^3q^3 in (3.1) and (3.2) and then taking $\alpha_r = \dfrac{[x,\omega xq,\omega^2xq;q]_r q^{r^2+r/2}(-)^r}{[q,(xq)^{3/2},-(xq)^{3/2},x^{3/2}q^2,-x^{3/2}q^2;q]_r}$ in (3.1) and making use of (2.9) we get:

$$\beta_n = \frac{x^n [x^2q^4;q]_n\,[x^3q^6;q^3]_m\,[xq^3;q]_{3m}}{q^{n^2/2}\,[xq;q]_n\,[q^3;q^3]_m\,[x^2q^4;q]_{3m}}, \tag{3.16}$$

where m is the greatest integer $\leq n/3$ and $\omega = e^{2\pi i/3}$.

Again taking $\delta_r = z^r q^{r^2/2}$ in (3.2) we get:

$$\gamma_n = \frac{[x^3zq^4;q]_\infty\,[x^3q^4;q]_{2n}\,(-)^n\,q^{n/2}}{[z;q]_\infty\,q^{n^2}\,[x^3zq^4;q]_n\,[q/z;q]_n}. \tag{3.17}$$

Now, putting these values of α_n, β_n, γ_n and δ_n in (3.3) we have:

$$\begin{aligned}
&\frac{[x^3zq^4;q]_\infty}{[z;q]_\infty}\,{}_5\Phi_4\left[\begin{matrix} x,\,\omega xq,\,\omega^2xq,\,x^{3/2}q^{5/2},\,-x^{3/2}q^{5/2} & ;q\,;\,q \\ (xq)^{3/2},\,-(xq)^{3/2},\,x^3zq^4,\,q/z & \end{matrix}\right] \\
&\quad = {}_3\Phi_2\left[\begin{matrix} x^2q^5, x^2q^6, x^3q^6\,;\,q^3\,;\,x^3z^3 \\ xq,\,xq^2 \end{matrix}\right] \\
&\quad + \frac{xz(1-x^2q^4)}{(1-xq)}\,{}_4\Phi_3\left[\begin{matrix} x^2q^5, x^2q^6, x^2q^7, x^3q^6\,;q^3\,;\,x^3z^3 \\ xq^2,\,xq^4,\,x^2q^4 \end{matrix}\right] \\
&\quad + \frac{x^2z^2(1-x^2q^4)(1-x^2q^5)}{(1-xq)(1-xq^2)}\,{}_4\Phi_3\left[\begin{matrix} x^2q^6, x^2q^7, x^2q^8, x^3q^6\,;q^3\,;\,x^3z^3 \\ xq^4,\,xq^5,\,x^2q^4 \end{matrix}\right],
\end{aligned} \tag{3.18}$$

$$|x^3z^3| < 1.$$

(vi) Taking $\alpha_r = \dfrac{[a^{1/3},\omega a^{1/3},\omega^2a^{1/3};q]_r q^{r^2+r/2}(-)^r}{[q,q\sqrt{a},-\sqrt{a},\sqrt{aq},-\sqrt{aq};q]_r}$ in (3.1) and making use of (2.10) we get:

$$\beta_n = \frac{(a)^{n-m/2}[aq^3;q^3]_m}{q^{n^2/2}\,[q^3;q^3]_m}, \tag{3.19}$$

where m is the greatest integer $\leq n/3$ and $\omega = e^{2\pi i/3}$.

Again taking $\delta_r = z^r q^{r^2/2}$ in (3.2), we get:

$$\gamma_n = \frac{[azq;q]_\infty\,[aq;q]_{2n}\,(-)^n\,q^{n/2}}{[z;q]_\infty\,q^{n^2}\,[azq;q]_n\,[q/z;q]_n}. \tag{3.20}$$

Now, putting these values of α_n, β_n, γ_n and δ_n in (3.3) we get:

$$\begin{aligned}
&{}_4\Phi_3\left[\begin{matrix} a^{1/3},\,\omega a^{1/3},\,\omega^2a^{1/3},\,-q\sqrt{a}\,;q\,;\,q \\ azq,\,q/z,\,-\sqrt{a} \end{matrix}\right] \\
&\qquad = \frac{[z;q]_\infty[a^2z^3q^3;q^3]_\infty}{[azq;q]_\infty[az^3;q^3]_\infty}\,\{1+a^{1/2}z+az^2\}.
\end{aligned} \tag{3.21}$$

(vii) Lastly, taking $\alpha_r = \dfrac{[a^{1/3}, \omega a^{1/3}, \omega^2 a^{1/3}, q\sqrt{a}; q]_r (-)^r q^{r^2+r/2}}{[q, q\sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, q^2\sqrt{a}; q]_r}$ in (3.1) and making use of (2.11), we get:

$$\beta_n = \frac{[\sqrt{a}; q]_n \, [aq^3; q^3]_m \, [q^6\sqrt{a}; q^3]_m \, (\sqrt{a})^{n-m}}{q^{n^2/2} \, [q^2\sqrt{a}; q]_n \, [q^3; q^3]_m \, [\sqrt{a}; q^3]_m}, \tag{3.22}$$

where m is the greatest integer $\leq n/3$ and $\omega = e^{2\pi i/3}$.

Again, taking $\delta_r = z^r q^{r^2/2}$ in (3.2), we get:

$$\gamma_n = \frac{[azq; q]_\infty \, [aq; q]_{2n} \, (-)^n \, q^{n/2}}{[z; q]_\infty \, q^{n^2} \, [azq, q/z; q]_n}. \tag{3.23}$$

Now, putting these values in (3.3), we get:

$$\begin{aligned}
&\frac{[azq; q]_\infty}{[z; q]_\infty} \, {}_6\Phi_5 \left[\begin{matrix} a^{1/3}, \, \omega a^{1/3}, \, \omega^2 a^{1/3}, \, q\sqrt{a}, \, -q\sqrt{a}, \, q\sqrt{a} \;\; ; q \; ; q \\ azq, \, q/z, \, \sqrt{a}, \, -\sqrt{a}, \, q^2\sqrt{a} \end{matrix} \right] \\
&\quad = {}_3\Phi_2 \left[\begin{matrix} q\sqrt{a}, \, aq^3, \, q^6\sqrt{a} \, ; \, q^3 \, ; \, az^3 \\ q^3\sqrt{a}, \, q^4\sqrt{a} \end{matrix} \right] \\
&\quad + \frac{z a^{1/2}(1-\sqrt{a})}{(1-q^2\sqrt{a})} \, {}_3\Phi_2 \left[\begin{matrix} q\sqrt{a}, \, aq^3, \, q^6\sqrt{a} \, ; \, q^3 \, ; \, az^3 \\ q^4\sqrt{a}, \, \sqrt{a} \end{matrix} \right] \\
&\quad + \frac{az^2(1-\sqrt{a})(1-q\sqrt{a})}{(1-q^2\sqrt{a})(1-q^3\sqrt{a})} \, {}_3\Phi_2 \left[\begin{matrix} q^2\sqrt{a}, \, q^3\sqrt{a}, \, aq^3 \, ; \, q^3 \, ; \, az^3 \\ q^5\sqrt{a}, \, \sqrt{a} \end{matrix} \right], \\
&\qquad |az^3| < 1.
\end{aligned} \tag{3.24}$$

4. Certain transformations and summations

In this section, we shall make use of (1.4) and summation formulae (2.3)–(2.11) to establish certain transformation and summation formulae for q-series.

If we take $B_n = 1$, $z = \gamma q/\alpha\beta$ in (1.4) and make use of (2.3) to sum of inner ${}_2\Phi_1$ series, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{[\gamma, \alpha, \beta; q]_n \, (1-\gamma q^{2n}) \, (-\gamma q/\alpha\beta)^n \, q^{n(n-1)/2}}{[q, \gamma q\alpha, \gamma q/\beta]_n (1-\gamma)} \sum_{j=0}^{n} \frac{[q^{-n}, \, \gamma q^n; q]_j \, A_j (wq)^j}{[q, \, \alpha, \, \beta; q]_j} \\
&\quad = \frac{[\gamma q, \, \gamma q/\alpha\beta; q]_\infty}{[\gamma q/\alpha, \, \gamma q/\beta; q]_\infty} \sum_{n=0}^{\infty} A_n \frac{(w\gamma q/\alpha\beta)^n}{[q; q]_n}.
\end{aligned} \tag{4.1}$$

Again taking $B_n = 1$, $z = \gamma/\alpha\beta$ in (1.4) and making use of (2.4) in order to sum the inner ${}_2\Phi_1$ series, we get:

$$\sum_{n=0}^{\infty} \frac{[\gamma, \alpha, \beta; q]_n \, (1 - \gamma q^{2n}) \, (-\gamma q/\alpha\beta)^n \, q^{n(n-1)/2}}{[q, \gamma q/\alpha, \gamma q/\beta]_n (1-\gamma)}$$

$$\times \left\{ \frac{\alpha\beta(1+\gamma q^{2n}) - \gamma q^n(\alpha+\beta)}{\alpha\beta - \gamma} \right\} \sum_{j=0}^{n} \frac{[q^{-n}, \, \gamma q^n; q]_j \, A_j (wq)^j)}{[q, \, \alpha, \, \beta; q]_j} \tag{4.2}$$

$$= \sum_{n=0}^{\infty} A_n \frac{(w\gamma/\alpha\beta)^n}{[q;q]_n}.$$

We shall make use of (4.1) and (4.2) in order to establish our main results.

(i) Replacing q by q^2 and then taking $\gamma = b^2x^4q^2$, $\alpha = bx^2q$, $\beta = bx^2q^2$, $w = 1$ and $A_j = \dfrac{[x^2, x^2q; q^2]_j}{[x^4q^2; q^2]_j}$ in (4.1), we get:

$$\sum_{n=0}^{\infty} \frac{[b^2x^4q^2, \, bx^2q, \, x^2q^2; q^2]_n (1 - b^2x^4 \, q^{4n+2} \, (-)^n \, q^{n(n-1)/2} \, q^n}{[q^2, \, bx^2q^3, \, b^2x^2q^2; q^2]_n \, (1 - b^2x^4q^2)}$$

$$\times \, {}_4\Phi_3 \left[\begin{matrix} q^{-2n}, b^2x^4q^{2n+2}, x^2, x^2q \, ; q^2 \, ; q^2 \\ bx^2q, \, bx^2q^2, \, x^4q^2 \end{matrix} \right] \tag{4.3}$$

$$= \frac{[b^2x^4q^4, \, q; q^2]_\infty}{[bx^2q^3, bx^2q^2; q^2]_\infty} \, {}_2\Phi_1 \left[\begin{matrix} x^2, x^2q \, ; q^2 \, ; \, q \\ x^4q^2 \end{matrix} \right].$$

Now, summing the inner ${}_4\Phi_3$-series on the left hand side and ${}_2\Phi_1$ on the right hand side of (4.3) with the help of (2.6) and (2.3), respectively, we get:

$${}_2\Phi_1 \left[\begin{matrix} bq, \, -bx^2q^3 & ; q \, ; \, -x^2q \\ - \, x^2q & ; q^2 \end{matrix} \right] = \frac{[bx^2q; q]_\infty}{[-x^2q; q]_\infty}. \tag{4.4}$$

(ii) Taking $\gamma = bx^2q^2$, $\alpha = xq\sqrt{b}$, $\beta = -xq\sqrt{b}$, $A_j = \dfrac{[x, -xq; q]_j}{[x^2q^2; q]_j}$ and $w = 1$ in (4.1), we get:

$$\sum_{n=0}^{\infty} \frac{[bx^2q^2, \, xq\sqrt{b}, \, -xq\sqrt{b}; q]_n (1 - bx^2 \, q^{2n+2}) \, q^{n(n-1)/2} \, q^n}{[q, \, xq^2\sqrt{b}, \, -xq^2\sqrt{b}^{2}; q]_n \, (1 - bx^2q^2)}$$

$$\times \, {}_4\Phi_3 \left[\begin{matrix} q^{-n}, bx^2q^{n+2}, x, -xq \, ; q \, ; q \\ xq\sqrt{b}, \, -xq\sqrt{b}, \, x^2q^2 \end{matrix} \right] \tag{4.5}$$

$$= \frac{[bx^2q^3, \, -q; q]_\infty}{[xq^2\sqrt{b}, \, -xq^2\sqrt{b}; q^2]_\infty} \, {}_2\Phi_1 \left[\begin{matrix} x, -xq \, ; q \, ; \, -q \\ x^2q^2 \end{matrix} \right].$$

Now, summing the inner ${}_4\Phi_3$-series on the left hand side and ${}_2\Phi_1$ series on the right hand side of (4.5), with the help of (2.7) and (2.3) respectively, we get the following summation formula:

$$\begin{aligned}
&{}_3\Phi_2\left[\begin{matrix} bx^2q^3,\, bq^2,\, xq^3 & ;q^2\ ;\, x^2q^3 \\ xq,\, x^2q^3 & ;q^4 \end{matrix}\right] \\
&\qquad +\frac{xq}{(1-xq)}\,{}_3\Phi_2\left[\begin{matrix} bx^2q^3,\, bq^2,\, bxq^4 & ;q^2\ ;\, x^2q^5 \\ x^2q^3,\, bxq^2 & ;q^4 \end{matrix}\right] \\
&\qquad = \frac{[bx^2q^3,\, -xq,\, xq^2;q]_\infty}{[x^2q^2,\, xq^2\sqrt{b},\, -xq^2\sqrt{b};q]_\infty}.
\end{aligned}\tag{4.6}$$

(iii) Replacing q by q^3 and then taking $\gamma = a^3q^3$, $\alpha = (aq)^{3/2}$, $\beta = -(aq)^{3/2}$, $A_j = \dfrac{[a, aq, aq^2; q^3]_j}{[a^{3/2}q^3, -a^{3/2}q^3; q^3]_j}$ and $w = 1$ in (4.1), we get:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{[a^3q^3,\, (aq)^{3/2},\, -(aq)^{3/2}; q^3]_n(1-a^3q^{6n+3})\, q^{3n}\, q^{3n(n-1)/2}}{[q^3,\, (aq)^{3/2}q^3,\, -(aq)^{3/2}q^3; q^3]_n\,(1-a^3q^3)} \\
&\qquad \times {}_5\Phi_4\left[\begin{matrix} q^{-3n}, a^3q^{3n+3}, a, aq, aq^2 & ;q^3\ ;q^3 \\ (aq)^{3/2},\, -(aq)^{3/2},\, a^{3/2}q^3,\, -a^{3/2}q^3 & \end{matrix}\right] \\
&\qquad = \frac{[a^3q^6,\, -q^3;\, q^3]_\infty}{[(aq)^{3/2}q^3,\, -(aq)^{3/2}q^3; q^3]_\infty}\,{}_3\Phi_2\left[\begin{matrix} a,\, aq,\, aq^2 & ;q^3\ ;\, -q^3 \\ a^{3/2}q^3,\, -a^{3/2}q^3 & \end{matrix}\right].
\end{aligned}\tag{4.7}$$

Now, summing the inner ${}_5\Phi_4$-series on the right hand side of (4.7) with the help of (2.8), we get:

$$\begin{aligned}
&{}_1\Phi_0\left[\begin{matrix} aq & ;q\ ;\, aq^3 \\ - & ;q^3 \end{matrix}\right] \\
&\qquad = \frac{[-q^3,\, a^3q^6;\, q^3]_\infty}{[(aq)^{3/2}q^3,\, -(aq)^{3/2}q^3; q^3]_\infty}\,{}_3\Phi_2\left[\begin{matrix} a,\ \ aq,\ \ aq^2; q^3\ ;\, -q^3 \\ a^{3/2}q^3, -a^{3/2}q^3 \end{matrix}\right].
\end{aligned}\tag{4.8}$$

(iv) Taking

$$\gamma = x^3q^4,\ \alpha = x^{3/2}q^2,\ \beta = -x^{3/2}q^2,\ w = 1 \text{ and } A_j = \frac{[x,\, \omega xq,\, \omega^2xq; q]_j}{[(xq)^{3/2},\, -(xq)^{3/2}; q]_j}$$

in (4.1), we get:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{[x^3q^4;q]_n\, q^{n+n(n-1)/2}}{[q;q]_n}\,{}_5\Phi_4\left[\begin{matrix} q^{-n}, x^3q^{n+4}, x, \omega xq,\, \omega^2xq\ ;q\ ;q \\ x^{3/2}q^2,\, -x^{3/2}q^2,\, (xq)^{3/2},\, -(xq)^{3/2} \end{matrix}\right] \\
&\qquad = \frac{[x^3q^5,\, -q;q]_\infty}{[x^{3/2}q^3,\, -x^{3/2}q^3; q]_\infty}\,{}_3\Phi_2\left[\begin{matrix} x,\, \omega xq,\, \omega^2xq\ ;q\ ;\, -q \\ (xq)^{3/2},\, -(xq)^{3/2} \end{matrix}\right],
\end{aligned}\tag{4.9}$$

where $\omega = e^{2\pi i/3}$.

Now, summing the inner ${}_5\Phi_4$-series on the right hand side of (4.9) with the help of (2.9), we get:

$$\begin{aligned}
&{}_3\Phi_2\left[\begin{matrix} x^3q^6,\, xq^4,\, xq^5 & ;q^3\,;\,x^3q^6 \\ xq,\, xq^2 & ;q^9 \end{matrix}\right] \\
&+\frac{xq(1-x^2q^4)}{(1-xq)}\,{}_3\Phi_2\left[\begin{matrix} x^3q^6,\, x^2q^7,\, xq^5 & ;q^3\,;\,x^3q^9 \\ xq^2,\, x^2q^4 & ;q^9 \end{matrix}\right] \\
&+\frac{x^2q^3(1-x^2q^4)(1-x^2q^5)}{(1-xq)(1-xq^2)}\,{}_3\Phi_2\left[\begin{matrix} x^3q^6,\, x^2q^7,\, x^2q^8 & ;q^3\,;\,x^3q^{12} \\ x^2q^4,\, x^2q^5 & ;q^9 \end{matrix}\right] \\
&=\frac{[x^3q^5,\, -q;q]_\infty}{[x^{3/2}q^3,\, -x^{3/2}q^3;q]_\infty}\,{}_3\Phi_2\left[\begin{matrix} x,\, \omega xq,\, \omega^2xq & ;q\,;\,-q \\ (xq)^{3/2},\, -(xq)^{3/2} & \end{matrix}\right],
\end{aligned}\tag{4.10}$$

where $\omega = e^{2\pi i/3}$ and $|x| < 1$, $|q| < 1$.

(v) Lastly, taking

$$\gamma = aq,\quad \alpha = \sqrt{aq},\quad \beta = -\sqrt{aq},\quad w = 1 \quad\text{and}$$

$$A_j = \frac{[a^{1/3},\, \omega a^{1/3},\, \omega^2 a^{1/3}, q\sqrt{a}; q]_j}{[\sqrt{a},\, -\sqrt{a},\, q^2\sqrt{a}; q]_j}$$

in (4.1), we get:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{[aq;q]_n\, q^{n(n-1)/2}}{[q;q]_n}\,{}_6\Phi_5\left[\begin{matrix} q^{-n}, aq^{n+1}, a^{1/3}, \omega a^{1/3},\, \omega^2a^{1/3},\, q\sqrt{a} & ;q\,;q \\ \sqrt{aq},\, -\sqrt{aq},\, \sqrt{a},\, -\sqrt{a},\, q^2\sqrt{a} & \end{matrix}\right] \\
&\quad=\frac{[aq^2,\, -q;q]_\infty}{[q\sqrt{aq},\, -q\sqrt{aq};q]_\infty}\,{}_4\Phi_3\left[\begin{matrix} a^{1/3}, \omega a^{1/3},\, \omega^2a^{1/3},\, q\sqrt{a} & ;q\,;-q \\ \sqrt{a},\, -\sqrt{a},\, q^2\sqrt{a} & \end{matrix}\right],
\end{aligned}\tag{4.11}$$

Now, summing the inner ${}_6\Phi_5$-series on the left hand side of (4.11) with the help of (2.11), we get:

$$\begin{aligned}
&{}_3\Phi_2\left[\begin{matrix} aq^3,\, q\sqrt{a},\, q^6\sqrt{a} & ;q^3\,;\,aq^6 \\ q^3\sqrt{a},\, q^4\sqrt{a} & ;q^9 \end{matrix}\right] \\
&+\frac{q\sqrt{a}(1-\sqrt{a})}{(1-q^2\sqrt{a})}\,{}_4\Phi_3\left[\begin{matrix} q\sqrt{a},\, q^2\sqrt{a},\, aq^3,\, q^6\sqrt{a} & ;q^3\,;\,aq^9 \\ q^4\sqrt{a},\, q^5\sqrt{a},\, \sqrt{a} & ;q^9 \end{matrix}\right] \\
&+\frac{aq^3(1-\sqrt{a})(1-q\sqrt{a})}{(1-q^2\sqrt{a})(1-q^3\sqrt{a})}\,{}_3\Phi_2\left[\begin{matrix} q^2\sqrt{a},\, q^3\sqrt{a},\, aq^3 & ;q^3\,;\,aq^{12} \\ q^5\sqrt{a},\, \sqrt{a} & ;q^9 \end{matrix}\right] \\
&=\frac{[-q,\, aq^2;q]_\infty}{[q\sqrt{aq},\, -q\sqrt{aq};q]_\infty}\,{}_4\Phi_3\left[\begin{matrix} a^{1/3},\, \omega a^{1/3},\, \omega^2a^{1/3}, q\sqrt{a} & ;q\,;\,-q \\ \sqrt{a},\, -\sqrt{a},\, q^2\sqrt{a} & \end{matrix}\right],
\end{aligned}\tag{4.12}$$

where $\omega = e^{2\pi i/3}$, $|a| < 1$ and $|q| < 1$.

Proceeding in the same way one can also establish certain summation and transformation formulae for q-series by making use of the summation (2.5)–(2.11) and identity (4.2).

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A NOTE ON CONTINUED FRACTIONS AND ${}_3\psi_3$ SERIES

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Abstract. The present paper concerns with the continued fraction representation for ${}_3\psi_3$ basic bilateral hypergeometric series. Several special cases are also discussed.

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1. Introduction

Continued fraction has been centre of attraction for applied mathematicians as well as pure mathematicians of previous centuries. In previous centuries there are so many results, which are established in terms of continued fraction. It is also a tool which act as a bridge between pure and applied mathematicians. So, the attraction of continued fraction for today's mathematicians has also amplified. R.P. Agrawal [1] has given the continued fraction representation for the basic hypergeometric series ${}_2\phi_1$. R.Y. Denis [4] has also developed a continued fraction representation for the ratio of two basic bilateral hypergeometric series ${}_2\psi_2$. There are also a number of researchers like R.P. Agarwal [2], G.E. Andrews and D. Bowman, [3], R.Y. Denis and S.N. Singh [5], P. Rai [7], S.N. Singh [8], etc., who have established a number of interesting results for hypergeometric function, basic hypergeometric function and basic bilateral hypergeometric function in terms of continued fraction.In this paper we are developing results for ${}_3\psi_3$ ratio's in terms of continued fraction and we also deduce some interesting special cases with the help of these results. These results may be helpful for further research in this area to developed more results in terms of continued fraction.

S.N. Singh [8] has derived an interesting transformation formula which transforms a basic bilateral hypergeometric function into basic hypergeometric function. In this paper, we shall use this transformation formula for the development of our main results. We shall use the following results in order to establish main results.

$$
\begin{aligned}
&{}_{r+3}\psi_{r+3}\left(\begin{matrix} \frac{a}{b}, & cq, & dq, & \frac{b_1}{b}q^{m_1}, & ...., & \frac{b_r}{b}q^{m_r} \\ & & & & & & ;q, \quad z \\ \frac{q}{b}, & c, & d, & \frac{b_1}{b}, & ...., & \frac{b_r}{b} \end{matrix}\right) \\
&= \frac{\left(q, \frac{bq}{az}, \frac{az}{b}, \frac{q}{a}; q\right)_\infty}{\left(\frac{q}{b}, \frac{q}{az}, az, \frac{bq}{a}; q\right)_\infty} \frac{(1-bc)(1-bd)(b_1;q)_{m_1}...(b_r;q)_{m_r}}{(1-c)(1-d)\left(\frac{b_1}{b};q\right)_{m_1} ... \left(\frac{b_r}{b};q\right)_{m_r}} \\
&\times {}_{r+3}\phi_{r+2}\left(\begin{matrix} a, & bcq, & bdq, & b_1q^{m_1}, & ...., & b_rq^{m_r} \\ & & & & & & ;q, \quad z \\ & bc, & bd, & b_1, & ...., & b_r \end{matrix}\right) [6;(11)]
\end{aligned} \tag{1.1}
$$

$$
\frac{{}_3\phi_2\left(\begin{matrix} aq, & b, & c \\ & & & ;q, \quad \frac{de}{abcq} \\ & d, & e \end{matrix}\right)}{{}_3\phi_2\left(\begin{matrix} a, & b, & c \\ & & & ;q, \quad \frac{de}{abc} \\ & d, & e \end{matrix}\right)}
= 1 + \frac{A_0}{(1-a)\left(1-\frac{de}{abcq}\right)+} \, \frac{B_0}{1+} \cdots \frac{A_n}{(1-a)\left(1-\frac{de}{abcq}\right)+} \, \frac{B_n}{1+} \cdots [1;(2)], \tag{1.2}
$$

where

$$
\begin{aligned}
A_n &= (de/abcq)(1-bq^n)(1-cq^n), \quad n = 0,1,2,... \\
B_n &= a[1-(dq^n/a)][1-(eq^n/a)], \quad n = 0,1,2,...
\end{aligned}
$$

$$
\begin{aligned}
&\frac{{}_3\phi_2\left(\begin{matrix} \frac{xq}{a_1a_2}, & a_3, & a_4 \\ & & & ;q, \quad \frac{xq}{a_3a_4} \\ & \frac{xq}{a_1}, & \frac{xq}{a_2} \end{matrix}\right)}{{}_3\phi_2\left(\begin{matrix} \frac{xq^2}{a_1a_2}, & a_3, & a_4 \\ & & & ;q, \quad \frac{xq}{a_3a_4} \\ & \frac{xq^2}{a_1}, & \frac{xq^2}{a_2} \end{matrix}\right)} \\
&= \frac{T(x)P(x)}{S(x)Q(x)} + \cfrac{R(x)/S(x)Q(x)}{\frac{T(xq)P(xq)}{S(xq)Q(xq)} + \cfrac{R(xq)/S(xq)Q(xq)}{\frac{T(xq^2)P(xq^2)}{S(xq^2)Q(xq^2)} + ...}} \; [2;(3.15)],
\end{aligned} \tag{1.3}
$$

where

$$\begin{aligned}
T(x) &= (1-xq^2/a_1)(1-xq^2/a_2)(1-xq^2/a_3/a_4),\\
S(x) &= (xq/a_1)_2(xq/a_2)_2(xq/a_3a_4)_2,\\
P(x) &= 1-xqS_1+\{(S_3-S_4)(1+q)+S_4q^2\}(x^2q^2+x^4q^5S_4)\\
&\quad +x^5q^7S_4^2S_1-x^6q^9S_4^3,
\end{aligned}$$

$$\begin{aligned}
S_M(x) &= (1-xq^{1+M}/a_1)(xq^{1+M}/a_2)_2(xq/a_3a_4)_2(xq/a_1a_2)_2,\\
Q(x) &= 1-x^2q^4S_4,\\
R(x) &= xq(1-x^2q^2S_4)(1-xq^2/a_1a_2)(1-xq^2/a_1a_3).
\end{aligned}$$

2. Definition and notations

A continued fraction is a ratio of the type:

$$a_1+\frac{a_2}{a_3+}\,\frac{a_4}{a_5+}\,\frac{a_6}{a_7+}\,\frac{a_8}{a_9+}\,\frac{a_{10}}{a_{11}+}...,$$

where $a_1, a_2, a_3, a_4, ...$ are real or complex numbers.

A Basic hypergeometric series is denoted and defined as:

$${}_A\phi_B\left(\begin{matrix}(a)\\ \\ (b)\end{matrix}\;;q,\quad z\right)=\sum_{n=0}^{\infty}\frac{[(a);q]_n}{[(b);q]_n}\frac{z^n}{(q;q)_n},\quad (|z|<1,|q|<1),$$

where (a) represents sequence of A parameters, (b) represents sequence of B parameters.

$$(a;q)_n=\begin{cases}(1-a)(1-aq)(1-aq^2)......(1-aq^{n-1}), & \text{when } n\neq 0;\\ 1, & \text{n=0.}\end{cases}$$

A Basic bilateral hypergeometric series is denoted and defined as:

$${}_r\psi_r\left(\begin{matrix}(a)\\ \\ (b)\end{matrix}\;;q,\quad z\right)=\sum_{n=-\infty}^{\infty}\frac{[(a);q]_n}{[(b);q]_n}z^n,\quad (|z|<1,|q|<1),$$

where (a) and (b) represent sequences of r parameters.

All the parameters and variable may be real or complex numbers. The other notations appearing in this paper carry their usual meaning.

3. Main results

We shall establish the following results.

$$
\frac{{}_3\psi_3\left(\begin{matrix} \frac{aq}{b}, & cq, & dq & & \\ & & & ;q, & \frac{1}{aq^3} \\ \frac{q}{b}, & c, & d & & \end{matrix}\right)}{{}_3\psi_3\left(\begin{matrix} \frac{a}{b}, & cq, & dq & & \\ & & & ;q, & \frac{1}{aq^2} \\ \frac{q}{b}, & c, & d & & \end{matrix}\right)} \tag{3.1}
$$

$$
= \frac{(a{-}1)}{(a{-}b)}\left[1 + \frac{A_0}{(1{-}a)\left(1{-}\frac{1}{aq^3}\right)+}\frac{B_0}{1+}\cdots\cdots\frac{A_n}{(1{-}a)\left(1{-}\frac{1}{aq^3}\right)+}\frac{B_n}{1+}\cdots\right],
$$

where

$$
\begin{aligned}
A_n &= \frac{1}{aq^3}(1 - bcq^{n+1})(1 - bdq^{n+1}), \quad n = 0, 1, 2, ... \\
B_n &= a\left(1 - \frac{bcq^n}{a}\right)\left(1 - \frac{bdq^n}{a}\right), \quad n = 0, 1, 2, ...
\end{aligned}
$$

and

$$
\frac{{}_3\psi_3\left(\begin{matrix} \frac{azq}{a_1a_2b}, & \frac{azq^2}{a_1b}, & \frac{azq^2}{a_2b} & & \\ & & & ;q, & \frac{a_1a_2}{azq^3} \\ \frac{q}{b}, & \frac{azq}{a_1b}, & \frac{azq}{a_2b} & & \end{matrix}\right)}{{}_3\psi_3\left(\begin{matrix} \frac{azq^2}{a_1a_2b}, & \frac{azq^3}{a_1b}, & \frac{azq^3}{a_2b} & & \\ & & & ;q, & \frac{a_1a_2}{azq^5} \\ \frac{q}{b}, & \frac{azq^2}{a_1b}, & \frac{azq^2}{a_2b} & & \end{matrix}\right)} \tag{3.2}
$$

$$
= \frac{b(azq{-}ba_1a_2)}{(azq{-}a_1a_2)} \times \frac{(1{-}azq/a_1)(1{-}azq/a_2)(1{-}azq^2/a_1b)}{(1{-}azq/a_1b)(1{-}azq/a_2b)(1{-}azq^2/a_1)} \times \frac{(1{-}azq^2/a_2b)}{(1{-}azq^2/a_2)}
$$

$$
\times \left[\frac{T(az)P(az)}{S(az)Q(az)} + \cfrac{R(az)/S(az)Q(az)}{\frac{T(azq)P(azq)}{S(azq)Q(azq)} + \cfrac{R(azq)/S(azq)Q(azq)}{\frac{T(azq^2)P(azq^2)}{S(azq^2)Q(azq^2)} + \cdots}}\right],
$$

where

$$\begin{aligned}
T(az) &= (1-azq^2/a_1)(1-azq^2/a_2)(1-azq^2/a_1/a_2),\\
S(az) &= (azq/a_1)_2(azq/a_2)_2(azqa_1a_2/a^2z^2q^4)_2,\\
P(az) &= 1-azqS_1+\{(S_3-S_4)(1+q)+S_4q^2\}(a^2z^2q^2+a^4z^4q^5S_4)\\
&\quad +a^5z^5q^7S_4^2S_1-a^6z^6q^9S_4^3,\\
S_M(az) &= (1-azq^{1+M}/a_1)(azq^{1+M}/a_2)_2(azqa_1a_2/a^2z^2q^4)_2(azq/a_1a_2)_2,\\
Q(az) &= 1-a^2z^2q^4S_4,\\
R(az) &= azq(1-a^2z^2q^2S_4)(1-azq^2/a_1a_2)(1-azq^2a_2/a_1azq^2).
\end{aligned}$$

4. Proof of main results

Proof of (3.1). Taking $r=0$ in (1.1), we get

$$\begin{aligned}
{}_3\psi_3\left(\begin{matrix} a/b, & cq, & dq & & \\ & & & ;q, & z \\ q/b, & c, & d & & \end{matrix}\right) &= \frac{(q,bq/az,az/b,q/a;q)_\infty}{(q/b,q/az,az,bq/a;q)_\infty}\\
&\times\frac{(1-bc)(1-bd)}{(1-c)(1-d)}\times {}_3\phi_2\left(\begin{matrix} a, & bcq, & bdq & & \\ & & & ;q, & z \\ & bc, & bd & & \end{matrix}\right)
\end{aligned} \tag{4.1}$$

Replacing a by aq in (4.1), we get

$$\begin{aligned}
{}_3\psi_3\left(\begin{matrix} aq/b, & cq, & dq & & \\ & & & ;q, & z \\ q/b, & c, & d & & \end{matrix}\right) &= \frac{(q,b/az,aqz/b,1/a;q)_\infty}{(q/b,1/az,aqz,b/a;q)_\infty}\\
&\times\frac{(1-bc)(1-bd)}{(1-c)(1-d)}\times {}_3\phi_2\left(\begin{matrix} aq, & bcq, & bdq & & \\ & & & ;q, & z \\ & bc, & bd & & \end{matrix}\right)
\end{aligned} \tag{4.2}$$

Taking $z=1/aq^2$ in (4.1) and $z=1/aq^3$ in (4.2) and then taking ratio of (4.1) and (4.2) and using result (1.2), we get

$$\frac{{}_3\psi_3\left(\begin{matrix} \frac{aq}{b}, & cq, & dq & & \\ & & & ;q, & \frac{1}{aq^3} \\ \frac{q}{b}, & c, & d & & \end{matrix}\right)}{{}_3\psi_3\left(\begin{matrix} \frac{a}{b}, & cq, & dq & & \\ & & & ;q, & \frac{1}{aq^2} \\ \frac{q}{b}, & c, & d & & \end{matrix}\right)}$$

$$= \frac{(a-1)}{(a-b)}\left[1+\frac{A_0}{(1-a)\left(1-\frac{1}{aq^3}\right)+}\,\frac{B_0}{1+}\cdots\frac{A_n}{(1-a)\left(1-\frac{1}{aq^3}\right)+}\,\frac{B_n}{1+}\cdots\right].$$

Proof of (3.2). Now, replacing a by azq/a_1a_2, bc by azq/a_1, bd by azq/a_2, c by azq/a_1b, d by azq/a_2b, z by a_1a_2/azq^3 in (4.1), we get

$$\begin{aligned}
&{}_3\psi_3\left(\begin{matrix} azq/a_1a_2b, & azq^2/a_1b, & azq^2/a_2b \\ q/b, & azq/a_1b, & azq/a_2b \end{matrix} ;q,\quad a_1a_2/azq^3\right) \\
&= \frac{(q, bq^3, 1/bq^2, a_1a_2/az;q)_\infty}{(q/b, q^3, 1/q^2, ba_1a_2/az;q)_\infty} \times \frac{(1-azq/a_1)(1-azq/a_2)}{(1-azq/a_1b)(1-azq/a_2b)} \\
&\times {}_3\phi_2\left(\begin{matrix} azq/a_1a_2, & azq^2/a_1, & azq^2/a_2 \\ & azq/a_1, & azq/a_2 \end{matrix} ;q,\quad a_1a_2/azq^3\right).
\end{aligned} \tag{4.3}$$

Replacing a by azq^2/a_1a_2, bc by azq^2/a_1, bd by azq^2/a_2, c by azq^2/a_1b, d by azq^2/a_2b, z by a_1a_2/azq^5 in (4.1), we get

$$\begin{aligned}
&{}_3\psi_3\left(\begin{matrix} azq^2/a_1a_2b, & azq^3/a_1b, & azq^3/a_2b \\ q/b, & azq^2/a_1b, & azq^2/a_2b \end{matrix} ;q,\quad a_1a_2/azq^5\right) \\
&= \frac{(q, bq^4, 1/bq^3, a_1a_2/azq;q)_\infty}{(q/b, q^4, 1/q^3, ba_1a_2/azq;q)_\infty} \times \frac{(1-azq^2/a_1)(1-azq^2/a_2)}{(1-azq^2/a_1b)(1-azq^2/a_2b)} \\
&\times {}_3\phi_2\left(\begin{matrix} azq^2/a_1a_2, & azq^3/a_1, & azq^3/a_2 \\ & azq^2/a_1, & azq^2/a_2 \end{matrix} ;q,\quad a_1a_2/azq^5\right).
\end{aligned} \tag{4.4}$$

Now, taking ratio of (4.3) and (4.4), then simplifying and using the result (1.3), we get

$$\frac{{}_3\psi_3\left(\begin{matrix} \frac{azq}{a_1a_2b}, & \frac{azq^2}{a_1b}, & \frac{azq^2}{a_2b} \\ \frac{q}{b}, & \frac{azq}{a_1b}, & \frac{azq}{a_2b} \end{matrix} ;q,\quad \frac{a_1a_2}{azq^3}\right)}{{}_3\psi_3\left(\begin{matrix} \frac{azq^2}{a_1a_2b}, & \frac{azq^3}{a_1b}, & \frac{azq^3}{a_2b} \\ \frac{q}{b}, & \frac{azq^2}{a_1b}, & \frac{azq^2}{a_2b} \end{matrix} ;q,\quad \frac{a_1a_2}{azq^5}\right)}$$

$$= \frac{b(azq - ba_1a_2)}{(azq - a_1a_2)} \times \frac{(1 - azq/a_1)(1 - azq/a_2)}{(1 - azq/a_1b)(1 - azq/a_2b)} \times \frac{(1 - azq^2/a_1b)(1 - azq^2/a_2b)}{(1 - azq^2/a_1)(1 - azq^2/a_2)}$$

$$\times \left[\frac{T(az)P(az)}{S(az)Q(az)} + \cfrac{R(az)/S(az)Q(az)}{\cfrac{T(azq)P(azq)}{S(azq)Q(azq)} + \cfrac{R(azq)/S(azq)Q(azq)}{\cfrac{T(azq^2)P(azq^2)}{S(azq^2)Q(azq^2)} + \ldots}} \right]$$

5. Special cases

Here, we shall deduce certain interesting cases of our main results.

Putting $c = q$ in (3.1), we get

$$\frac{{}_3\phi_2\left(\begin{matrix} aq/b, & q^2, & dq \\ & q/b, & d \end{matrix} ; q, \; 1/aq^3\right)}{{}_3\phi_2\left(\begin{matrix} a/b, & q^2, & dq \\ & q/b, & d \end{matrix} ; q, \; 1/aq^2\right)} \tag{5.1}$$

$$= \frac{(a-1)}{(a-b)} \left[1 + \frac{P_0}{(1-a)\left(1-\frac{1}{aq^3}\right)+} \frac{Q_0}{1+} \cdots \frac{P_n}{(1-a)\left(1-\frac{1}{aq^3}\right)+} \frac{Q_n}{1+} \cdots \right]$$

where

$$P_n = \frac{1}{aq^3}(1 - bq^{n+2})(1 - bdq^{n+1}), \quad n = 0, 1, 2, \ldots$$

$$Q_n = a\left(1 - \frac{bq^{n+1}}{a}\right)\left(1 - \frac{bdq^n}{a}\right), \quad n = 0, 1, 2, \ldots$$

Putting $d = 1/b$ in (5.1), we get

$$\frac{{}_2\phi_1\left(\begin{matrix} aq/b, & q^2 \\ & 1/b \end{matrix} ; q, \; 1/aq^3\right)}{{}_2\phi_1\left(\begin{matrix} a/b, & q^2 \\ & 1/b \end{matrix} ; q, \; 1/aq^2\right)} \tag{5.2}$$

$$= \frac{(a-1)}{(a-b)} \left[1 + \frac{L_0}{(1-a)\left(1-\frac{1}{aq^3}\right)+} \frac{M_0}{1+} \cdots \frac{L_n}{(1-a)\left(1-\frac{1}{aq^3}\right)+} \frac{M_n}{1+} \cdots \right]$$

where

$$L_n = \frac{1}{aq^3}(1-bq^{n+2})(1-q^{n+1}), \quad n=0,1,2,...$$

$$M_n = a\left(1-\frac{bq^{n+1}}{a}\right)\left(1-\frac{q^n}{a}\right), \quad n=0,1,2,...$$

Replacing $1/b$ by q in (5.2), we get

$$\frac{{}_2\phi_1\left(\begin{matrix} aq^2, & q^2 \\ & q \end{matrix}; q, \ 1/aq^3\right)}{{}_2\phi_1\left(\begin{matrix} aq, & q^2 \\ & q \end{matrix}; q, \ 1/aq^2\right)} \tag{5.3}$$

$$= \frac{(a-1)q}{(aq-1)}\left[1+\frac{R_0}{(1-a)\left(1-\frac{1}{aq^3}\right)+}\,\frac{S_0}{1+}\cdots\frac{R_n}{(1-a)\left(1-\frac{1}{aq^3}\right)+}\,\frac{S_n}{1+}\cdots\right]$$

where

$$R_n = \frac{1}{aq^3}(1-q^{n+1})^2, \quad n=0,1,2,...$$

$$S_n = a\left(1-\frac{q^n}{a}\right)^2, \quad n=0,1,2,...$$

Putting $b=1$ in (3.2), we get

$$\frac{{}_3\phi_2\left(\begin{matrix} azq/a_1a_2, & azq^2/a_1, & azq^2/a_2 \\ & azq/a_1, & azq/a_2 \end{matrix}; q, \ a_1a_2/azq^3\right)}{{}_3\phi_2\left(\begin{matrix} azq^2/a_1a_2, & azq^3/a_1, & azq^3/a_2 \\ & azq^2/a_1, & azq^2/a_2 \end{matrix}; q, \ a_1a_2/azq^5\right)} \tag{5.4}$$

$$= \frac{T(az)P(az)}{S(az)Q(az)} + \cfrac{R(az)/S(az)Q(az)}{\cfrac{T(azq)P(azq)}{S(azq)Q(azq)} + \cfrac{R(azq)/S(azq)Q(azq)}{\cfrac{T(azq^2)P(azq^2)}{S(azq^2)Q(azq^2)} + ...}}$$

Replacing az by z in (5.4), we get

$$\frac{{}_3\phi_2\left(\begin{matrix} zq/a_1a_2, & zq^2/a_1, & zq^2/a_2 \\ & zq/a_1, & zq/a_2 \end{matrix}\; ;q,\quad a_1a_2/zq^3\right)}{{}_3\phi_2\left(\begin{matrix} zq^2/a_1a_2, & zq^3/a_1, & zq^3/a_2 \\ & zq^2/a_1, & zq^2/a_2 \end{matrix}\; ;q,\quad a_1a_2/zq^5\right)} \tag{5.5}$$

$$= \frac{T(z)P(z)}{S(z)Q(z)} + \cfrac{R(z)/S(z)Q(z)}{\frac{T(zq)P(zq)}{S(zq)Q(zq)} + \cfrac{R(zq)/S(zq)Q(zq)}{\frac{T(zq^2)P(zq^2)}{S(zq^2)Q(zq^2)} + \ldots}}$$

Similarly, some other interesting special cases could be deduced.

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A CONNECTION BETWEEN CATEGORIES OF (FUZZY) MULTIALGEBRAS AND (FUZZY) ALGEBRAS

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Abstract. The purpose of this paper is study the relationship between the categories of fuzzy multialgebras and crisp algebras. In this regards first we briefly study the categories of multialgebras and fuzzy multialgebras and then by using the fundamental relation of multialgebras we construct a functor from category of fuzzy multialgebras to the category of fuzzy algebras and hence. Hence it can be derived a fuzzy algebra from every fuzzy multialgebras through the fundamental relation.

Keywords: universal algebra, multialgebra, fuzzy multialgebra, fundamental relation, fuzzy set, homomorphism.

1. Introduction

Several aspects of homomorphisms, subalgebras and subdirect decompositions of multialgebras also called hyperalgebra were developed for special cases in [13], [14] by Picket and in [9] by Hansoul. In [17], D. Schweigert studied the congruences of multialgebras and the exponentiations of universal hyperalgebras. Ameri and Zahedi introduced the notion of hyperalgebraic systems in [2]. Ameri et.al. in [3] introduced congruence of multialgebras. The notion of direct product, identities and fundamental relation of multialgebras in [10], [11] and [12] introduced and studied by Pelea.

In this paper, we follow [2] and [3] to study the relationship between the category of fuzzy multialgebras and category of fuzzy algebras. The paper is organized in four sections. In Section 2, we gather the definitions and basic properties of multialgebras and fuzzy algebras that will be used in the next sections. In Section 3, the category of multialgebras are briefly discussed. In Section 4, first the category of fuzzy multialgebras are investigated and then, we use the fundamental

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relation to assigning to every fuzzy multialgebra a fuzzy algebra. Finally, we construct a functor, which is called the fundamental functor from category of fuzzy multialgebras to the category of fuzzy algebras.

2. Preliminaries

In this section, we present some definitions and simple properties of multialgebras which will be used in the next section. In the sequel, H is a fixed nonvoid set, $P^*(H)$ is the family of all nonvoid subsets of H, and for a positive integer n we denote for H^n the set of n-tuples over H (for more see [6] and [7]).

For a positive integer n, an n-ary *hyperoperation* β on H is a function $\beta : H^n \to P^*(H)$. We say that n is the *arity* of β. A subset S of H is *closed* under the n-ary hyperoperation β if $(x_1, \dots, x_n) \in S^n$ implies that $\beta(x_1, \dots, x_n) \subseteq S$. A nullary hyperoperation on H is just an element of $P^*(H)$; i.e., a nonvoid subset of H.

A *hyperalgebraic system* or a *multialgebra* $\langle H, (\beta_i, \mid i \in I)\rangle$ is the set H with together a collection $(\beta_i, \mid i \in I)$ of hyperoperations on H.

A subset S of a multialgebra $\mathbb{H} = \langle H, (\beta_i, \mid i \in I)\rangle$ is a *submultialgebra* of $\mathbb{H}$ if S is closed under each hyperoperation β_i, for all $i \in I$, that is $\beta_i(a_1, ..., a_n) \subseteq S$, whenever $(a_1, ..., a_n) \in S^n$. The *type* of $\mathbb{H}$ is the map from I into the set $\mathbb{N}^*$ of nonnegative integers assigning to each $i \in I$ the arity of β_i. Two multialgebras of the same type are similar.

For $n > 0$, we extend an n-ary hyperoperation β on H to an n-ary operation $\overline{\beta}$ on $P^*(H)$ by setting for all $A_1, ..., A_n \in P^*(H)$

$$\overline{\beta}(A_1, ..., A_n) = \bigcup\{\beta(a_1, ..., a_n) | a_i \in A_i (i = 1, ..., n)\}$$

It is easy to see that $\overline{\mathbb{H}}\langle P^*(H), (\overline{\beta}_i, \mid i \in I)\rangle$ is an algebra of the same type of $\mathbb{H}$. Whenever possible we write a instead of the the singleton $\{a\}$; e.g. for a binary hyperoperation $\circ$ and $a, b, c \in H$ we write $a \circ (b \circ c)$ for $\{a\} \circ (\{b\} \circ \{c\}) = \bigcup\{a \circ u | u \in b \circ c\}$.

Example 2.1.

(i) A *hypergroupoid* is a multialgebra of type (2), that is a set H together with a (binary) hyperoperation $\circ$. A hypergroupoid $(H, \circ)$, which is associative, that is $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in H$ is called a *semihypergroup.*

(ii) A *hypergroup* is a semihypergroup such that for all $x \in H$ we have $x \circ H = H = H \circ x$ (called the *reproduction axiom*).

An element e in a hypergroup $\mathbb{H} = (H, \circ)$ is called an *identity* of $\mathbb{H}$ if, for all $x \in H$, $x \in (e \circ x) \cap (x \circ e)$.

(iii) A *polygroup* (or multigroup) is a semihypergroup $\mathbb{H} = (H, \circ)$ with $e \in H$ such that for all $x, y \in H$.

(iv) $e \circ x = x = x \circ e$.

(v) there exists a unique element, $x^{-1} \in H$ such that

$$e \in (x \circ x^{-1}) \cap (x^{-1} \circ x), x \in \bigcap_{z \in x \circ y} (z \circ y^{-1}), y \in \bigcap_{z \in x \circ y} (x^{-1} \circ z).$$

In fact, a polygroup is a multialgebra of type $(2, 1, 0)$.

Definition 2.2. Let $\mathbb{H} = \langle H, (\beta_i, \mid i \in I)\rangle$ and $\overline{\mathbb{H}} = \langle \overline{H}, (\overline{\beta}_i, \mid i \in I)\rangle$ be two similar multialgebras. A map h from $\mathbb{H}$ into $\overline{\mathbb{H}}$ is called a

(i) A *homomorphism* if for every $i \in I$ and all $(a_1, ..., a_{n_i}) \in H^{n_i}$ we have that $h(\beta_i((a_1, ..., a_{n_i})) \subseteq \overline{\beta}_i(h(a_1), ..., h(a_{n_i}))$;

(ii) a *good homomorphism* if for every $i \in I$ and all $(a_1, ..., a_{n_i}) \in H^{n_i}$ we have that $h(\beta_i((a_1, ..., a_{n_i})) = \overline{\beta}_i(h(a_1), ..., h(a_{n_i}))$.

Definition 2.3. A *universal algebra* or *algebra* whose hyperoperations are singleton valued (i.e. $|\beta(a_1, ..., a_n)| = 1$ for all $a_1, ..., a_n \in H$) viewed as maps from H^n into H and called operation.

Definition 2.4. Let X be a nonempty set. A fuzzy subset μ of X is a function

$$\mu : X \to [0, 1].$$

Let μ and υ be two fuzzy subset of X, we say that μ is contained in υ, if

$$\mu(x) \leq \upsilon(x), \forall x \in X.$$

If μ_i be a collection of fuzzy subsets of X, then we define the fuzzy subset $\bigcap_{i \in I} \mu_i$ by:

$$\left(\bigcap_{i \in I} \mu_i\right)(x) = \inf_{i \in I}\{\mu_i(x)\}, \quad \forall x \in X.$$

Definition 2.5. Let μ be a fuzzy subset of X and $t \in [0, 1]$. The set $A_t = \{x \in X \mid A(x) \geq t\}$ is called a level subset of X.

3. The category of multialgebras

Definition 3.1. Let $\mathbb{H}$ be a multialgebra. Define inductively a relation ϵ^* on $\mathbb{H}$ as follows:

Set $\epsilon_0 = \{(x, x) \mid x \in H\}$. If $n \geq 0$ and ϵ_n has been defined set

$$\epsilon_{n+1} = \epsilon_n \cup \{\beta_i(a_1, ..., a_{n_i}) \times \beta_i(b_1, ..., b_{n_i}) : i \in I \text{ and } (a_j, b_j) \in \epsilon_n \text{ for all } j = 1, ..., n_i\}$$

and set $\epsilon = \bigcup_{n=0}^{\propto} \epsilon_n$. Finally, set ϵ^* be the transitive closure of ϵ (here, as usual the transitive closure of ϵ is the set of all $(a,b) \in H^2$ such that there exists $k \geq 0$ and $a = a_0, a_1, ..., a_k = b$ with $(a_j, a_{j+1}) \in \epsilon$ for all $j = 0, ..., k-1$). A direct induction shows that ϵ_n is an equivalence relation on $\mathbb{H}$ for all $n \geq 0$ and hence ϵ^* is an equivalence relation on $\mathbb{H}$. Denote by H/ϵ^* the set of blocks (also called equivalence classes) of ϵ^*. Let $i \in I$ and $(a_j, b_j) \in \epsilon^*$ for all $j = 1, ..., n_i$. Then there exist $m_j \geq 0$ such that $(a_j, b_j) \in \epsilon_{m_j}$ for all $j = 1, ..., n_i$. Let $m = max(m_1, ..., m_{n_i})$. By the definition of ϵ^* clearly $(\beta_i(a_1, ..., a_{n_i}), \beta_i(b_1, ..., b_{n_i})) \in \epsilon_{m+1} \subseteq \epsilon^*$. This shows that for arbitrary blocks $B_1, ..., B_{n_i}$ of ϵ^* the set $\beta_i(B_1, ..., B_{n_i})$ is included in a block B of ϵ^*. It follows that $\mathbb{H}/\epsilon^* = (H/\epsilon^*, (\beta_i : i \in I))$ is a universal algebra. It can be verified that ϵ^* is the least equivalence relation such that $\mathbb{H}/\epsilon^*$ is an universal algebra.

Remark 3.2. Consider a hyperalgebra $\mathbb{H}$. The smallest equivalence relation such that the factor algebra $\mathbb{H}/\epsilon^*$ is an algebra is called the fundamental relation of $\mathbb{H}$ (for more see [11]). In [11], Pelea introduced and studied the fundamental relation of a multialgebra based on term functions. In this paper we present a different approach of [11] to introduce the fundamental relation of a hyperalgebra.

Theorem 3.3. *The relation ϵ^* is the fundamental relation on A.*

Definition 3.4. Let $\mathbb{H}_j = \langle H_j, (\beta_{ji} : i \in I)\rangle, (j \in J)$ be a nonvoid family of similar multialgebras. Set $H = \bigcup_{j \in J} H_j$ and denote by $X = \prod_{j \in J} H_j$ the set of maps $f : J \to H$ such that $f(j) \in H_j$ for all $j \in J$. For $i \in I$ and $g_1, ..., g_{n_i} \in X$ define $g = \beta'_i(g_1, ..., g_{n_i})$ by setting $g(j) = \beta_{ji}(g_1(j), ..., g_{n_i}(j))$ for all $j \in J$. Clearly g is a nonvoid subset of X and so β'_i is a hyperoperation on X. Therefore, $\prod_{j \in J} \mathbb{H}_j = \mathbb{X} = \langle X, (\beta'_i : i \in I)\rangle$ is a multialgebra.

In this section, we briefly introduce the category of multialgebras.

Definition 3.5. The category $\mathcal{MA}$ of multialgebras, is defined as follows:

(i) The objects of $\mathcal{MA}$ are the multialgebras;

(ii) For objects A, B of $\mathcal{MA}$, of the same type, the set $Hom(A, B)$ of morphism from A to B, is the set all homomorphism from A to B;

(iii) The composition gf of morphisms $f : A \to B$ and $g : B \to C$ is defined by setting $(gf)(x) = g(f(x))$ for all $x \in A$.

(iv) For any object A, the morphism $1_A : A \to A$ $(x \mapsto x)$, is the identity morphism.

Note that in the part (ii) of above if we replace $Hom(A, B)$ with $Hom_g(A, B)$, the set of all good homomorphisms, we will obtain a new category, which it is denoted by $\mathcal{MA}_g$. In fact, $\mathcal{MA}_g$ is a subcategory of $\mathcal{MA}$, which is not full.

We denote by $\mathcal{A}$ the category of (universal) algebras .

Lemma 3.6. *Let* $\mathbb{M} = \langle M, (\beta_i : i \in I)\rangle$ *and* $\mathbb{N} = \langle N, (\beta'_i : i \in I)\rangle$ *be two similar multialgebras and* f *be a good homomorphism from* $\mathbb{M}$ *into* $\mathbb{N}$. *Let* ϵ^* *and* ϵ'^* *be the fundamental relations of* $\mathbb{M}$ *and* $\mathbb{N}$ *respectively. Then*

(i) $f([x]_{\epsilon^*}) \subseteq [f(x)]_{\epsilon'^*}$; *for all* $x \in M$;

(ii) *the map* $f^* : M/\epsilon^* \to N/\epsilon'^*$ *by setting* $f^*([x]_{\epsilon^*}) = [f(x)]_{\epsilon'^*}$ *for all* $x \in M$ *is a homomorphism of the algebra* M/ϵ^* *into the algebra* N/ϵ'^* *(where* $[x]_{\epsilon^*}$ *and* $[f(x)]_{\epsilon'^*}$ *are the blocks of* M/ϵ^* *and* N/ϵ'^* *containing* x *and* $f(x)$*).*

Proof. (i) We proof by induction on $n \geq 0$ that $f(\epsilon_n) \subseteq \epsilon^*_n$. Let $n = 0$. Then, the blocks of ϵ_0 are the singletons of M and their image consists of certain singletons of N. Suppose $n \geq 0$ and $f(\epsilon_n) \subseteq \epsilon'_n$, we show that $f(\epsilon_{n+1}) \subseteq \epsilon'_{n+1}$ (where ϵ'_{n+1} denotes the corresponding relation on N). Let $(a, b) \in \epsilon_{n+1} \setminus \epsilon_n$. Then, there exist $i \in I$ and $(a_j, b_j) \in \epsilon_n (j = 1, ..., n_i)$ such that $a = \beta_i(a_1, ..., a_{n_i})$ and $b = \beta_i(b_1, ..., b_{n_i})$. Clearly, $f(a) = \beta'_i(f(a_1), ..., f(a_{n_i}))$, $f(b) = \beta'_i(f(b_1), ..., f(b_{n_i}))$ and so $(f(a), f(b)) \in \epsilon'_{n+1}$. This proves that $f(\epsilon_{n+1}) \subseteq \epsilon'_{n+1}$. By the definition of transitive closure, we obtain that $f(\epsilon^*) \subseteq \epsilon'^*$, which proves the theorem.

(ii) Straightforward.

Theorem 3.7. *The mapping* $F : \mathcal{MA} \to \mathcal{A}$ *defined by* $F(M) = M/\epsilon^*$ *and* $F(f) = f^*$ *is a functor.*

Proof. It is an immediate consequence of Lemma 3.6.

Theorem 3.8. *Let* $\mathbb{H}_j (j \in J)$ *be a family of similar multialgebras* $\mathbb{H}_j$ *with fundamental relations* ϵ_j*. *Then the fundamental relation* ϵ^* *on* $\prod_{j \in J} \mathbb{H}_j$ *consists of* (f, g) *with* $f, g \in \prod_{j \in J} \mathbb{H}_j$ *satisfying* $(f(j), g(j)) \in \epsilon_j*$ *for all* $j \in J$.

Proof. By induction on $n \geq 0$, one proves the result for ϵ_n.

Remark 3.9. The theorem guarantees the existence of the fundamental relation for special multialgebras such as, semihypergoups, hypergroups, hyperrings, hypermodules, hypervector spaces.

Corollary 3.10. *Let* $f : \mathbb{M} \to \mathbb{N}$ *be a morphism in* $\mathcal{MA}$ *and let* φ_M *and* φ_N *denote the canonical projections of* $\mathbb{M}$ *and* $\mathbb{N}$ *into* $\mathbb{M}/\epsilon^*$ *and* $\mathbb{N}/\epsilon'^*$, *respectively. Then the following diagram is commutative:*

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\varphi_M"'] & N \arrow[d, "\varphi_N"] \\
M/\varepsilon^* \arrow[r, "f^*"] & N/\varepsilon^*
\end{tikzcd}$$

4. Category of fuzzy multialgebras

Definition 4.1. Let $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ be a multialgebra and μ be a fuzzy subset of H. We say that μ is a fuzzy multialgebra in symbol $\mu <_{FMA} \mathbb{H}$ if

(i) for every $i \in I$, such that the arity n_i of β_i is positive, for all $a_1, ..., a_{n_i} \in H$ and every $z \in \beta_i(a_1, ..., a_{n_i})$ we have $\mu(z) \geq \mu(a_1) \wedge ... \wedge \mu(a_{n_i})$;

(ii) if there exist a nullary β_i then: the image in μ has a greatest element m and $\mu(z) = m$ for every $z \in \beta_j$ with $n_j = 0$. Denote by $FMA(\mathbb{H})$, the set of fuzzy multialgebras of $\mathbb{H}$.

For a (universal) algebra in the above definition, the condition in (i) reduces to

$$\mu(\beta_i(a_1, ..., a_{n_i})) \geq \mu(a_1) \wedge ... \wedge \mu(a_{n_i}),$$

and the condition (ii) became $\mu(\beta) = m$.

Denote by $FA(\mathbb{H})$, the set of all fuzzy algebras of $\mathbb{H}$.

Definition 4.2. The category of fuzzy multialgebras of a given type τ denoted by $\mathcal{FMA}_\tau$, is defined as follows:

(i) The objects of $\mathcal{FMA}$ are the fuzzy multialgebras;

(ii) For the objects $\mu <_{FMA} \mathbb{M}$ and $\mu' <_{FMA} \mathbb{N}$ a morphism is a homomorphism f, from $\mathbb{M}$ to $\mathbb{N}$, such that $\mu(z) \geq t \Rightarrow \mu'(f(z)) \geq t$, for all $z \in M$ and $t \in L$;

(iii) the composition of morphisms is the composition of homomorphisms and the identity morphism is the identity selfmap.

Clearly, fuzzy algebras (considered as singleton valued multialgebras) form a subcategory of $\mathcal{FMA}_\tau$, we denote this subcategory by $\mathcal{FA}$.

Definition 4.3. Let $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ be an algebra and μ be a fuzzy subalgebra of H. Define the following hyperoperations on H by

$$\beta_i^* : H \times ... \times H \longrightarrow P^*(H)$$

$$\beta_i^*(x_1, ...x_{n_i}) = \{t \mid \mu(t) = \mu(\beta_i(x_1, ..., x_{n_i}))\}, \text{ for every } i \in I$$

then $\mathbb{H} =< H, (\beta_i^* : i \in I) >$ is a multialgebra and μ is a fuzzy multialgebra.

Proof. We must prove that for every $i \in I$ and all $z \in \beta_i^*(a_1, ..., a_{n_i})$

$$\mu(z) \geq \mu(a_1) \wedge ... \wedge \mu(a_{n_i}).$$

But if $z \in \beta_i^*(a_1, ..., a_{n_i})$ then $\mu(z) = \mu(\beta_i(a_1, ..., a_{n_i})) \geq \mu(a_1) \wedge ... \wedge (a_{n_i})$ (since μ is a fuzzy subalgebra). ■

Theorem 4.4. *Let* $\mathbb{H}$ *be a multialgebra and* μ *be a fuzzy subset of* H. *Then,* μ *is a fuzzy submultialgebra if and only if for every* t *in* $[0, 1]$, *the level subset* μ_t *be a submultialgebra of* $\mathbb{H}$.

Proof. ($\Rightarrow$) Let μ be a fuzzy multialgebra of $\mathbb{H}$. Let $t \in [0,1]$, $i \in I$ and $a_1, ..., a_{n_i} \in \mu_t$. To prove that $\beta_i(a_1, ..., a_{n_i}) \subseteq \mu_t$, let $z \in \beta_i(a_1, ..., a_{n_i})$ be arbitrary . Then $\mu(a_1) \wedge ... \wedge \mu(a_{n_i}) \leq \mu(z)$, and hence we have $\mu(a_1) \geq t, ..., \mu(a_{n_i}) \geq t$, thus $\mu(z) \geq \mu(a_1) \wedge ... \wedge \mu(a_{n_i}) \geq t$ proving $z \in \mu_t$.

($\Leftarrow$) Suppose that μ_t, for every $0 \leq t \leq 1$, is a submultialgebra of $\mathbb{H}$. Let $i \in I$ and $a_1, ..., a_{n_i} \in H$. Set $t = \mu(a_1) \wedge ... \wedge \mu(a_{n_i})$. Now, clearly, $\mu(a_i) \geq t$, hence, $a_i \in \mu_t$ for all $i = 1, ..., n_i$. As μ_t is a submultialgebra of $\mathbb{H}$, clearly $\beta_i(a_1, ..., a_{n_i}) \subseteq \mu_t$ proving $\mu(z) \geq t = \mu(a_1) \wedge ... \wedge \mu(a_{n_i})$ for all $z \in \beta_i(a_1, ..., a_{n_i})$. ■

Lemma 4.5. *Let* $\mathbb{H}=\langle H, (\beta_i : i \in I)\rangle$ *be a multialgebra and* μ *be a fuzzy submultialgebra of* H. *If* $0 \leq t_1 < t_2 \leq 1$, *then* $\mu_{t_1} = \mu_{t_2}$ *if and only if there is no* x *in* H *such that* $t_1 \leq \mu(x) \leq t_2$.

Proof. Straightforward. ■

Let μ be a fuzzy multialgebra on $\mathbb{H}$ and α^* be the fundamental relation on $\mathbb{H}$. Define μ^* on H/α^*, such that $\mu^* = w(\mu)$, where $w : \mathbb{H} \to \mathbb{H}/\alpha^*$ is the natural homomorphism.

Lemma 4.6. *Let* $\mu <_{FMA} \mathbb{H}$ *and let* ϵ^* *be the fundamental relation on* $\mathbb{H}$. *Define* $\mu^* : H/\epsilon^* \to [0,1]$ *by setting* $\mu^*(B) = \vee\{\mu(b) : b \in B\}$, *then* μ^* *is a fuzzy algebra on* $\mathbb{H}/\alpha^*$.

Proof. Let $i \in I$ and $B_1, ..., B_{n_i}$ be blocks of ϵ^*. Then $\beta_i(B_1, ..., B_{n_i}) \subseteq B$ for some block B of ϵ^*. Now,

$$\begin{aligned}\mu^*(B_1) \wedge ... \wedge \mu^*(B_{n_i}) &= \bigwedge_{i=1}^{n_i} \left(\bigvee_{t_j \in B_i} \mu(t_j) \right) \\ &= \bigvee\{\mu(t_1) \wedge ... \wedge \mu(t_{n_i}) : t_1 \in B_1, ..., t_{n_i} \in B_{n_i}\} \\ &\leq \bigvee\{\mu(z) : z \in \beta_i(t_1, ..., t_{n_i}), t_1 \in B_1, ..., t_{n_i} \in B_{n_i}\} \leq \mu^*(B). \quad \blacksquare\end{aligned}$$

Lemma 4.7. *Let* f *be a homomorphism of* $\mu <_{FMA} \mathbb{H}$ *into* $\mu' <_{FMA} \mathbb{H}'$ *and let* ϵ^* *and* ϵ'^* *be the fundamental relations of* $\mathbb{H}$ *and* $\mathbb{H}'$. *Define* $f^* : H/\epsilon^* \to H'/\epsilon'^*$ *by setting* $f^*(B) = B'$ *where* B' *is the block of* ϵ'^* *containing* $f(B)$. *Then* f^* *is a homomorphism from the fuzzy algebra* $\mu^* <_{FA} \mathbb{H}/\epsilon^*$ *to* $\mu'^* <_{FA} \mathbb{H}'/\epsilon'^*$.

Proof. It follows from Lemma 3.2 and the definitions. ■

The next result follows immediately from Lemmas 4.6 and 4.7.

Theorem 4.8. *The map F:* $\mathcal{FMA} \longrightarrow \mathcal{FA}$ *defined by* $F(\mu) = \mu^*$ *and* $F(f) = f^*$ *is a functor.*

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LIPSCHITZ ESTIMATES FOR MULTILINEAR COMMUTATOR OF LITTLEWOOD-PALEY OPERATOR

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Abstract. In this paper, we will study the continuity of multilinear commutator generated by Littlewood-Paley operator and the functions b_j on Triebel-Lizorkin space, Hardy space and Herz-Hardy space, where the functions b_j belong to Lipschiz space.

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1. Introduction

We know, the commutator $[b,T](f)(x) = b(x)T(f)(x) - T(bf)(x)$ is bounded on $L^p(R^n)$ for $1 < p < \infty$ when T is the Calderón-Zygmund operator and $b \in BMO(R^n)$. Janson and Paluszynski study the commutator for the Triebel-Lizorkin space and the case $b \in Lip_\beta(R^n)$, where $Lip_\beta(R^n)$ is the homogeneous Lipschitz space. Chanillo (see [2]) proves a similar result when T is replaced by the fractional operators. The main purpose of this paper is to discuss the boundedness of Littlewood-Paley multilinear commutator generated by Littlewood-Paley operator and Lipschiz functions on Triebel-Lizorkin space, Hardy space and Herz-Hardy space.

2. Preliminaries and Definitions

Throughout this paper, $M(f)$ will denote the Hardy-Littlewood maximal function of f, and write $M_p(f) = (M(f^p))^{1/p}$ for $0 < p < \infty$. Q will denote a cube of R^n with side parallel to the axes.

Let $f_Q = |Q|^{-1}\int_Q f(x)dx$ and $f^{\#}(x) = \sup_{x\in Q}|Q|^{-1}\int_Q |f(y) - f_Q|dy$ denote the Hardy spaces by $H^p(R^n)$. It is well known that $H^p(R^n)$ $(0 < p \le 1)$ has the atomic decomposition characterization (see [11], [16], [17]). For $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta,\infty}(R^n)$ be the homogeneous Tribel-Lizorkin space.

The Lipschitz space $Lip_\beta(R^n)$ is the space of functions f such that

$$||f||_{Lip_\beta} = \sup_{\substack{x,y\in R^n \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\beta} < \infty.$$

Lemma 1. (see [15]) *For* $0<\beta<1,\ 1<p<\infty,$ *we have*

$$\begin{aligned} ||f||_{\dot{F}_p^{\beta,\infty}} &\approx \left\| \sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q |f(x)-f_Q|dx \right\|_{L^p} \\ &\approx \left\| \sup_{\cdot\in Q} \inf_c \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q |f(x)-c|dx \right\|_{L^p}. \end{aligned}$$

Lemma 2. (see [15]) *For* $0<\beta<1,\ 1\le p\le\infty,$ *we have*

$$\begin{aligned} ||f||_{Lip_\beta} &\approx \sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q |f(x)-f_Q|dx \\ &\approx \sup_Q \frac{1}{|Q|^{\frac{\beta}{n}}} \left(\frac{1}{|Q|} \int_Q |f(x)-f_Q|^p dx \right)^{1/p}. \end{aligned}$$

Lemma 3. (see [2]) *For* $1\le r<\infty$ *and* $\beta>0,$ *let*

$$M_{\beta,r}(f)(x) = \sup_{x\in Q} \left(\frac{1}{|Q|^{1-\frac{\beta r}{n}}} \int_Q |f(y)|^r dy \right)^{1/r},$$

suppose that $r<p<n/\beta,$ *and* $1/q = 1/p - \beta/n,$ *then*

$$||M_{\beta,r}(f)||_{L^q} \le C||f||_{L^p}.$$

Lemma 4. (see [5]) *Let* $Q_1 \subset Q_2,$ *then*

$$|f_{Q_1} - f_{Q_2}| \le C||f||_{\dot{\Lambda}_\beta} |Q_2|^{\beta/n}.$$

Definition 1. Let $0<p,\ q<\infty$, $\alpha\in R$, $B_k = \{x\in R^n, |x|\le 2^k\}$, $A_k = B_k\backslash B_{k-1}$ and $\chi_k = \chi_{Ak}$ for $k \in \mathbf{Z}$.

1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,p}(R^n) = \{f\in L^q_{Loc}(R^n\backslash\{0\}) : ||f||_{\dot{K}_q^{\alpha,p}} < \infty\},$$

where

$$||f||_{\dot{K}_q^{\alpha,p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} ||f\chi_k||_{L^q}^p \right]^{1/p};$$

2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(R^n) = \{f \in L^q_{Loc}(R^n) : ||f||_{K_q^{\alpha,p}} < \infty\},$$

where

$$||f||_{K_q^{\alpha,p}} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p}||f\chi_k||^p_{L^q} + ||f\chi_{B_0}||^p_{L^q}\right]^{1/p}.$$

Definition 2. Let $\alpha \in R$, $0 < p, q < \infty$.

(1) The homogeneous Herz type Hardy space is defined by

$$H\dot{K}_q^{\alpha,p}(R^n) = \{f \in S'(R^n) : G(f) \in \dot{K}_q^{\alpha,p}(R^n)\},$$

and

$$||f||_{H\dot{K}_q^{\alpha,p}} = ||G(f)||_{\dot{K}_q^{\alpha,p}};$$

(2) The nonhomogeneous Herz type Hardy space is defined by

$$HK_q^{\alpha,p}(R^n) = \{f \in S'(R^n) : G(f) \in K_q^{\alpha,p}(R^n)\},$$

and

$$||f||_{HK_q^{\alpha,p}} = ||G(f)||_{K_q^{\alpha,p}};$$

where $G(f)$ is the grand maximal function of f.

The Herz type Hardy spaces have the atomic decomposition characterization.

Definition 3. Let $\alpha \in R$, $1 < q < \infty$. A function $a(x)$ on R^n is called a central (α, q)-atom (or a central (a, q)-atom of restrict type), if

1) supp$a \subset B(0, r)$ for some $r > 0$ (or for some $r \geq 1$),

2) $||a||_{L^q} \leq |B(0, r)|^{-\alpha/n}$,

3) $\int_{R^n} a(x)x^\eta dx = 0$ for $|\eta| \leq [\alpha - n(1 - 1/q)]$.

Lemma 5. (see [6], [14]) *Let* $0 < p < \infty$, $1 < q < \infty$ *and* $\alpha \geq n(1 - 1/q)$. *A temperate distribution* f *belongs to* $H\dot{K}_q^{\alpha,p}(R^n)$ (*or* $HK_q^{\alpha,p}(R^n)$) *if and only if there exist central* (α, q)*-atoms* (*or central* (α, q)*-atoms of restrict type*) a_j *supported on* $B_j = B(0, 2^j)$ *and constants* λ_j, $\sum_j |\lambda_j|^p < \infty$ *such that* $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (*or* $f = \sum_{j=0}^{\infty} \lambda_j a_j$) *in the* $S'(R^n)$ *sense, and*

$$||f||_{H\dot{K}_q^{\alpha,p}} (or\ ||f||_{HK_q^{\alpha,p}}) \sim \left(\sum_j |\lambda_j|^p\right)^{1/p}.$$

Definition 4. Let $0 < \delta < n$, $\varepsilon > 0$ and ψ be a fixed function which satisfies the following properties:

1) $\int_{R^n} \psi(x)dx = 0$,
2) $|\psi(x)| \le C(1+|x|)^{-(n+1-\delta)}$,
3) $|\psi(x+y) - \psi(x)| \le C|y|^{\varepsilon}(1+|x|)^{-(n+1+\varepsilon-\delta)}$ when $2|y| < |x|$.

Let m be a positive integer and $b_j (1 \le j \le m)$ be the locally integrable function, set $\vec{b} = (b_1, \cdots, b_m)$. The multilinear commutator of Littlewood-Paley operator is defined by

$$g^{\vec{b}}_{\psi,\delta}(f)(x) = \left(\int_0^\infty |F^{\vec{b}}_t(x)|^2 \frac{dt}{t}\right)^{1/2},$$

where

$$F^{\vec{b}}_t(f)(x) = \int_{R^n} \prod_{j=1}^m (b_j(x) - b_j(y))\psi_t(x-y)f(y)dy,$$

and

$$\psi_t(x) = t^{-n+\delta}\psi(x/t)$$

for $t > 0$. Set $F_t(f) = \psi_t * f$. We also define that

$$g_{\psi,\delta}(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

which is the Littlewood-Paley g function (see [17]).

Let H be the space

$$H(R^n) = \{h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t\right)^{1/2} < \infty\},$$

then, for each fixed $x \in R^n$ $F_t(f)(x)$ may be viewed as a mapping from $[0, +\infty)$ to H, and it is clear that

$$g_{\psi,\delta}(f)(x) = ||F_t(f)(x)|| \text{ and } g^{\vec{b}}_{\psi,\delta}(f)(x) = ||F^{\vec{b}}_t(f)(x)||.$$

Note that when $b_1 = \cdots = b_m$, $g^{\vec{b}}_{\psi,\delta}$ is just the m order commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1-4], [7-10], [12], [15]). Our main purpose is to establish the boundedness of the multilinear commutator on Triebel-Lizorkin space, Hardy space and Herz-Hardy space.

Given a positive integer m and $1 \le j \le m$, we set

$$||\vec{b}||_{Lip_\beta} = \prod_{j=1}^m ||b_j||_{Lip_\beta}$$

and denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{Lip_\beta} = ||b_{\sigma(1)}||_{Lip_\beta} \cdots ||b_{\sigma(j)}||_{Lip_\beta}$.

Lemma 6. (see [10]) *Let* $0 < \beta \leq 1, 0 < \delta < n$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ *and* $b \in Lip_\beta(R^n)$. *Then* $g^b_{\psi,\delta}$ *is bounded from* $L^p(R^n)$ *to* $L^q(R^n)$.

3. Theorems and proofs

Theorem 1. *Let* $0 < \delta < n$, $0 < \beta < min(1, \varepsilon/m)$, $1 < p < \infty$, $\vec{b} = (b_1, \cdots, b_m)$ *with* $b_j \in Lip_\beta(R^n)$ *for* $1 \leq j \leq m$ *and* $g^{\vec{b}}_{\psi,\delta}$ *be the multilinear commutator of Littlewood-Paley operator as in Definition* 4. *Then*

a) $g^{\vec{b}}_{\psi,\delta}$ *is bounded from* $L^p(R^n)$ *to* $\dot{F}_p^{m\beta,\infty}(R^n)$ *for* $1<p<n/\delta$ *and* $1/p-1/q=\delta/n$.

b) $g^{\vec{b}}_{\psi,\delta}$ *is bounded from* $L^p(R^n)$ *to* $L^q(R^n)$ *for* $1/p - 1/q = m\beta + \delta/n$ *and* $1/p > m\beta + \delta/n$.

Proof. (a). Fixed a cube $Q = (x_0, l)$ and $\tilde{x} \in Q$, see [10] when $m = 1$.

Consider now the case $m \geq 2$. Set

$$\vec{b}_Q = ((b_1)_Q, \cdots, (b_m)_Q),$$

where

$$(b_j)_Q = |Q|^{-1} \int_Q b_j(y)dy, \ 1 \leq j \leq m.$$

Writing $f = f_1 + f_2$, where $f_1 = f\chi_{2Q}, f_2 = f\chi_{R^n \setminus 2Q}$, we have

$$\begin{aligned} F_t^{\vec{b}}(f)(x) &= \int_{R^n} (b_1(x) - b_1(y)) \cdots (b_m(x) - b_m(y))\psi_t(x-y)f(y)dy \\ &= (b_1(x) - (b_1)_Q) \cdots (b_m(x) - (b_m)_Q)F_t(f)(x) \\ &\quad +(-1)^m F_t((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q)f)(x) \\ &\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j}(b(x) - \vec{b}_Q)_\sigma \int_{R^n} (b(y) - \vec{b}_Q)_{\sigma^c} \psi_t(x-y)f(y)dy \\ &= (b_1(x) - (b_1)_Q) \cdots (b_m(x) - (b_m)_Q)F_t(f)(x) \\ &\quad +(-1)^m F_t((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q)f_1)(x) \\ &\quad +(-1)^m F_t((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q)f_2)(x) \\ &\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j}(b(x) - \vec{b}_Q)_\sigma F_t((b - \vec{b}_Q)_{\sigma^c} f)(x), \end{aligned}$$

then

$$
\begin{aligned}
&|g_{\psi,\delta}^{\vec{b}}(f)(x)-g_{\psi,\delta}(((b_1)_Q-b_1)\cdots((b_m)_Q-b_m)f_2)(x_0)|\\
&\le ||F_t^{\vec{b}}(f)(x)-F_t(((b_1)_Q-b_1)\cdots((b_m)_Q-b_m)f_2)(x_0)||\\
&\le ||(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)F_t(f)(x)||\\
&+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||(b(x)-\vec{b}_Q)_\sigma F_t((b-\vec{b}_Q)_{\sigma^c}f)(x)||\\
&+||F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(x)||\\
&+||F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(x)-F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(x_0)||\\
&=I_1(x)+I_2(x)+I_3(x)+I_4(x),
\end{aligned}
$$

so,

$$
\begin{aligned}
&\frac{1}{|Q|^{1+\frac{m\beta}{n}}}\int_Q|g_{\psi,\delta}^{\vec{b}}(f)(x)-g_{\psi,\delta}((b_1)_Q-b_1)\cdots((b_m)_Q-b_m)f_2)(x_0)|dx\\
&\quad\le\frac{1}{|Q|^{1+\frac{m\beta}{n}}}\int_Q I_1(x)dx+\frac{1}{|Q|^{1+\frac{m\beta}{n}}}\int_Q I_2(x)dx\\
&\quad+\frac{1}{|Q|^{1+\frac{m\beta}{n}}}\int_Q I_3(x)dx+\frac{1}{|Q|^{1+\frac{m\beta}{n}}}\int_Q I_4(x)dx\\
&\quad=I+II+III+IV.
\end{aligned}
$$

For I, by using Lemma 2, we have

$$
\begin{aligned}
I &\le \frac{1}{|Q|^{1+\frac{m\beta}{n}}}\sup_{x\in Q}|b_1(x)-(b_1)_Q|\cdots|b_m(x)-(b_m)_Q|\int_Q|g_{\psi,\delta}(f)(x)|dx\\
&\le C||\vec{b}||_{Lip_\beta}\frac{1}{|Q|^{1+\frac{m\beta}{n}}}|Q|^{\frac{m\beta}{n}}\int_Q|g_{\psi,\delta}(f)(x)|dx\\
&\le C||\vec{b}||_{Lip_\beta}M(g_{\psi,\delta}(f))(\tilde{x}).
\end{aligned}
$$

For II, taking $1<r<p<q<n/\delta$, $1/q'+1/q=1, 1/s'+1/s=1, 1/q=1/p-\delta/n$, $ps=r$ by using the Hölder's inequality and the boundedness of $g_{\psi,\delta}$ from $L^p(R^n)$ to $L^q(R^n)$ and Lemma 2, we get

$$
\begin{aligned}
II &\le \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{1+m\beta/n}}\int_Q|(\vec{b}(x)-\vec{b}_Q)_\sigma||g_{\psi,\delta}((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(x)|dx\\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{m\beta/n}}\left(\frac{1}{|Q|}\int_Q|(\vec{b}(x)-\vec{b}_Q)_\sigma|^{q'}dx\right)^{1/q'}\\
&\qquad\times\left(\frac{1}{|Q|}\int_{R^n}|g_{\psi,\delta}((\vec{b}-\vec{b}_Q)_{\sigma^c}f\chi_Q)(x)|^q dx\right)^{1/q}
\end{aligned}
$$

$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{m\beta/n}}||\vec{b}_\sigma||_{Lip_\beta}|Q|^{|\sigma|\beta/m}\frac{1}{|Q|^{1/q}}\left(\int_{R^n}|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}f\chi_Q|^p dx\right)^{1/p}$$

$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{m\beta/n}}||\vec{b}_\sigma||_{Lip_\beta}|Q|^{(-1/q)+(1/ps')+(1-\delta ps/n)/ps}$$

$$\times\left(\frac{1}{|Q|}\int_Q|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}|^{ps'}\right)^{1/ps'}\left(\frac{1}{|Q|^{1-\frac{\delta ps}{n}}}\int_Q|f(x)|^{ps}dx\right)^{1/ps}$$

$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{m\beta/n}}||\vec{b}_\sigma||_{Lip_\beta}|Q|^{|\sigma|\beta/n}||\vec{b}_{\sigma^c}||_{Lip_\beta}|Q|^{|\sigma^c|\beta/n}M_{r,\delta}(f)(\tilde{x})$$

$$\leq C||\vec{b}||_{Lip_\beta}M_{r,\delta}(f)(\tilde{x}).$$

For III, we choose $1<r<p<q<n/\delta$, $0<\delta<n$, $1/q=1/p-\delta/n$, $r=ps$, by the boundness of $g_{\psi,\delta}$ from $L^p(R^n)$ to $L^q(R^n)$ and Hölder's inequality with $1/s+1/s'=1$, we get

$$\begin{aligned}
III &= \frac{1}{|Q|^{1+m\beta/n}}\int_Q|g_{\psi,\delta}((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(x)|dx\\
&\leq C\frac{1}{|Q|^{m\beta/n}}\left(\frac{1}{|Q|}\int_{R^n}|g_{\psi,\delta}(\prod_{j=1}^m(b_j(y)-(b_j)_Q)f\chi_Q)(x)|^q dx\right)^{1/q}\\
&\leq C\frac{1}{|Q|^{m\beta/n}}\frac{1}{|Q|^{1/q}}\left(\int_{R^n}|\prod_{j=1}^m(b_j(y)-(b_j)_Q)|^p|f\chi_Q|^p dx\right)^{1/p}\\
&\leq C\frac{1}{|Q|^{m\beta/n}}|Q|^{(-1/q)+1/ps'-(1-(\delta ps/n)/ps)}\left(\frac{1}{|Q|}\int_Q|\prod_{j=1}^m(b_j(y)-(b_j)_Q)|^{ps'}dx\right)^{1/ps'}\\
&\quad\times\left(\frac{1}{|Q|^{1-\delta ps/n}}\int_Q|f(x)|^{ps}dx\right)^{1/ps}\\
&\leq C||\vec{b}||_{Lip_\beta}M_{r,\delta}(f)(\tilde{x}).
\end{aligned}$$

For IV, since $|x_0-y|\approx|x-y|$ for $y\in(2Q)^c$, by Lemma 4 and the condition on ψ, we have

$$||F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(x)-F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(x_0)||$$

$$\leq\left[\int_0^\infty\left(\int_{(2Q)^c}|\psi_t(x-y)-\psi_t(x_0-y)||f(y)|\prod_{j=1}^m|b_j(y)-(b_j)_Q|dy\right)^2\frac{dt}{t}\right]^{1/2}$$

$$\leq C\left[\int_0^\infty \left(\int_{(2Q)^c} \frac{t|x-x_0|^\varepsilon}{(t+|x_0-y|)^{n+1+\varepsilon-\delta}}|f(y)|\prod_{j=1}^m |b_j(y)-(b_j)_Q|dy\right)^2 \frac{dt}{t}\right]^{1/2}$$

$$\leq C\int_{(2Q)^c} |x_0-x|^\varepsilon |x_0-y|^{-(n+\varepsilon-\delta)}|f(y)|\prod_{j=1}^m |b_j(y)-(b_j)_Q|dy$$

$$\leq C\sum_{k=1}^\infty \int_{2^{k+1}Q\setminus 2^kQ} |x_0-x|^\varepsilon |x_0-y|^{-(n+\varepsilon-\delta)}|f(y)|\prod_{j=1}^m |b_j(y)-(b_j)_Q|dy$$

$$\leq C\sum_{k=1}^\infty 2^{-k\varepsilon}|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}|f(y)|\prod_{j=1}^m (|b_j(y)-(b_j)_{2^{k+1}Q}|+|(b_j)_{2^{k+1}}-(b_j)_Q|)dy$$

$$\leq C\sum_{k=1}^\infty 2^{-k\varepsilon}|2^{k+1}Q|^{\delta/n}\frac{1}{|2^{k+1}Q|^{1-\delta/n}}||\vec{b}||_{Lip_\beta}M_{r,\delta}(f)(\tilde{x})$$

$$\leq C||\vec{b}||_{Lip_\beta}|Q|^{\frac{m\beta}{n}}M_{r,\delta}(f)(\tilde{x}),$$

thus

$$IV \leq C||\vec{b}||_{Lip_\beta}M_{r,\delta}(f)(\tilde{x}).$$

We put these estimates together, by using Lemma 1 and taking the supremum over all Q such that $x \in Q$, we obtain

$$||g_{\psi,\delta}^{\vec{b}}(f)||_{\dot{F}_p^{m\beta,\infty}} \leq C||\vec{b}||_{Lip_\beta}||f||_{L^p}.$$

This complete the proof of (a).

(b) By some argument as in proof of (a), we have

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |g_{\psi,\delta}^{\vec{b}}(f)(x) - g_{\psi,\delta}(((b_1)_Q-b_1)\cdots((b_m)_Q-b_m)f_2)(x_0)|dx\\
&\leq \frac{1}{|Q|}\int_Q I_1(x)dx + \frac{1}{|Q|}\int_Q I_2(x)dx + \frac{1}{|Q|}\int_Q I_3(x)dx + \frac{1}{|Q|}\int_Q I_4(x)dx\\
&\leq C||\vec{b}||_{Lip_\beta}(M_{m\beta,1}(g_{\psi,\delta}(f)(\tilde{x})) + M_{m\beta+\delta,r}(f)(\tilde{x})),
\end{aligned}$$

thus

$$(g_{\psi,\delta}^{\vec{b}}(f))^\# \leq C||\vec{b}||_{Lip_\beta}(M_{m\beta,1}(g_{\psi,\delta}(f)(\tilde{x})) + M_{m\beta+\delta,r}(f)(\tilde{x})).$$

By using Lemma 3 and the boundedness of $g_{\psi,\delta}$, we have

$$\begin{aligned}
||g_{\psi,\delta}^{\vec{b}}(f)||_{L^q} &\leq C||(g_{\psi,\delta}^{\vec{b}}(f))^\#||_{L^q}\\
&\leq C||\vec{b}||_{Lip_\beta}(||M_{m\beta,1}(g_{\psi,\delta}(f)(\tilde{x})) + M_{m\beta+\delta,r}(f)(\tilde{x}))||_{L^q}\\
&\leq C||\vec{b}||_{Lip_\beta}||f||_{L^p}.
\end{aligned}$$

This complete the proof of (b) and the theorem.

Theorem 2. *Let* $0 < \delta < n$, $0 < \beta+\delta/m < \min(\gamma/m, 1/2m)$, $n/(n+\beta+\delta/m) < p \leq 1$, $1/q = 1/p-(m\beta+\delta)/n$, $\vec{b} = (b_1, \cdots, b_m)$ *with* $b_j \in Lip_\beta(R^n)$ *for* $1 \leq j \leq m$. *Then* $g^{\vec{b}}_{\psi,\delta}$ *is bounded from* $H^p(R^n)$ *to* $L^q(R^n)$.

Proof. It suffices to show that there exists a constant $C > 0$ such that for every H^p-atom a,

$$||g^{\vec{b}}_{\psi,\delta}(a)||_{L^q} \leq C.$$

Let a be a H^p-atom, that is that a supported on a cube $Q = Q(x_0, r)$, $||a||_{L^\infty} \leq |Q|^{-1/p}$ and $\int_{R^n} a(x)x^\gamma dx = 0$ for $|\gamma| \leq [n(1/p-1)]$.

When $m = 1$, see [10]. Now consider the case $m \geq 2$. Write

$$\begin{aligned}||g^{\vec{b}}_{\psi,\delta}(a)(x)||_{L^q} &\leq \left(\int_{|x-x_0|\leq 2r} |g^{\vec{b}}_{\psi,\delta}(a)(x)|^q dx\right)^{1/q} + \left(\int_{|x-x_0|>2r} |g^{\vec{b}}_{\psi,\delta}(a)(x)|^q dx\right)^{1/q}\\ &= I + II.\end{aligned}$$

For I, choose $1 < p_1 < n/(m\beta+\delta)$ and q_1 such that $1/q_1 = 1/p_1 - m\beta + \delta/n$. By the boundednss of $g^{\vec{b}}_{\psi,\delta}$ from $L^{p_1}(R^n)$ to $L^{q_1}(R^n)$ (see Theorem 1), we get

$$\begin{aligned} I &\leq C||g^{\vec{b}}_{\psi,\delta}(a)||^q_{L^{q_1}}|Q(x_0,2r)|^{1-q/q_1} \leq C||a||^q_{L^{p_1}}||\vec{b}||_{Lip_\beta}|Q|^{1-q/q_1}\\ &\leq C||\vec{b}||_{Lip_\beta}|Q|^{-q/p+q/p_1+1-q/q_1} \leq C||\vec{b}||_{Lip_\beta}.\end{aligned}$$

For II, let $\tau, \tau' \in N$ such that $\tau+\tau' = m$, and $\tau' \neq 0$. We get

$$|F^{\vec{b}}_t(a)(x)| \leq |(b_1(x)-b_1(x_0))\cdots(b_m(x)-b_m(x_0))\int_B(\psi_t(x-y)-\psi_t(x-x_0))a(y)dy|$$

$$+\sum_{j=1}^{m}\sum_{\sigma\in C^m_j}|(b(x)-b(x_0))_{\sigma^c}\int_B(b(y)-b(x_0))_\sigma\psi_t(x-y)a(y)dy|$$

$$\leq C||\vec{b}||_{Lip_\beta}|x-x_0|^{m\beta}\cdot\int_B|\psi_t(x-y)-\psi_t(x-x_0)||a(y)|dy$$

$$+C||\vec{b}||_{Lip_\beta}\sum_{\tau+\tau'=m}|x-x_0|^{\tau\beta}\int_B|y-x_0|^{\tau'\beta}|\psi_t(x-y)||a(y)|dy$$

$$\leq C||\vec{b}||_{Lip_\beta}\frac{|x-x_0|^{m\beta}t}{(t+|x-x_0|)^{n+1+\varepsilon-\delta}}\int_B|x_0-y|^\varepsilon|a(y)|dy$$

$$+C||\vec{b}||_{Lip_\beta}\sum_{\tau+\tau'=m}|x-x_0|^{\tau\beta}\frac{t}{(t+|x-x_0|)^{n+1-\delta}}\int_B|y-x_0|^{\tau'\beta}|a(y)|dy$$

$$\leq C||\vec{b}||_{Lip_\beta}\frac{t}{(t+|x-x_0|)^{n+1+\varepsilon-\delta}}\cdot r^{m\beta+\varepsilon+n(1-\frac{1}{p})}$$

$$+C||\vec{b}||_{Lip_\beta}\frac{t}{(t+|x-x_0|)^{n+1-\delta}}\cdot r^{m\beta+n(1-\frac{1}{p})},$$

thus

$$\begin{aligned}|g_{\psi,\delta}^{\vec{b}}(a)(x)| &\leq C||\vec{b}||_{Lip_\beta}\left(\int_0^\infty\left(\frac{t}{(t+|x-x_0|)^{n+1+\varepsilon-\delta}}\right)^2\frac{dt}{t}\right)^{1/2}\cdot r^{m\beta+\varepsilon+n(1-\frac{1}{p})}\\ &+ C||\vec{b}||_{Lip_\beta}\left(\int_0^\infty\left(\frac{t}{(t+|x-x_0|)^{n+1-\delta}}\right)^2\frac{dt}{t}\right)^{1/2}\cdot r^{m\beta+n(1-\frac{1}{p})}\\ &\leq C||\vec{b}||_{Lip_\beta}|x-x_0|^{-n+\delta}\cdot r^{m\beta+n(1-\frac{1}{p})},\end{aligned}$$

so,

$$II\leq C||\vec{b}||_{Lip_\beta}\cdot r^{m\beta+n(1-\frac{1}{p})}\left(\int_{|x-x_0|>2r}|x-x_0|^{-nq+q\delta}dx\right)^{1/q}\leq C||\vec{b}||_{Lip_\beta}.$$

This complete the proof of Theorem 2.

Theorem 3. *Let* $0<\beta\leq 1$, $0<\delta<n$, $0<p<\infty$, $1<q_1,q_2<\infty$, $1/q_1-1/q_2=m\beta+\delta/n$, $n(1-1/q_1)\leq\alpha<n(1-1/q_1)+\beta+\delta/m$, $\vec{b}=(b_1,\cdots,b_m)$ *with* $b_j\in Lip_\beta(R^n)$ *for* $1\leq j\leq m$. *Then* $g_{\psi,\delta}^{\vec{b}}$ *is bounded from* $H\dot{K}_{q_1}^{\alpha,p}(R^n)$ *to* $\dot{K}_{q_2}^{\alpha,p}(R^n)$.

Proof. By Lemma 5, let $f\in H\dot{K}_{q_1}^{\alpha,p}(R^n)$ and $f=\sum_{j=-\infty}^{\infty}\lambda_j a_j$, $supp\, a_j\subset B_j=B(0,2^j)$, a_j be a central $(\alpha,q)-$atom, and $\sum_{j=-\infty}^{\infty}|\lambda_j|^p<\infty$. We have

$$\begin{aligned}||g_{\psi,\delta}^{\vec{b}}(f)||_{\dot{K}_{q_2}^{\alpha,p}}^p &\leq C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|||g_{\psi,\delta}^{\vec{b}}(a_j)\chi_k||_{L^{q^2}}\right)^p\\ &+ C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-1}^{\infty}|\lambda_j|||g_{\psi,\delta}^{\vec{b}}(a_j)\chi_k||_{L^{q_2}}\right)^p\\ &= I+II.\end{aligned}$$

For II, by the boundedness of $g_{\psi,\delta}^{\vec{b}}$ on (L^{q_1},L^{q_2}), we have

$$\begin{aligned}II &\leq C||\vec{b}||_{Lip_\beta}^p\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-1}^{\infty}|\lambda_j|||a_j||_{L^{q_1}}\right)^p\\ &\leq C||\vec{b}||_{Lip_\beta}^p\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-1}^{\infty}|\lambda_j|\cdot 2^{-j\alpha}\right)^p\\ &\leq C||\vec{b}||_{Lip_\beta}^p\begin{cases}\displaystyle\sum_{k=-\infty}^{\infty}\sum_{j=k-1}^{\infty}|\lambda_j|^p\cdot 2^{(k-j)\alpha p},\ 0<p\leq 1\\ \displaystyle\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-1}^{\infty}|\lambda_j|^p\cdot 2^{-j\alpha p/2}\right)\left(\sum_{j=k-1}^{\infty}2^{-j\alpha p'/2}\right)^{p/p'},\ 1<p<\infty\end{cases}\end{aligned}$$

$$\leq C||\vec{b}||^p_{Lip_\beta}\sum_{j=-\infty}^{\infty}|\lambda_j|^p.$$

For I, when $m=1$, we have

$$\begin{aligned}
|F_t^{b_1}(a_j)(x)| &\leq \left|(b_1(x)-b_1(0))\int_{B_j}(\psi_t(x-y)-\psi_t(x))a_j(y)dy\right|\\
&+ \left|\int_{B_j}\psi_t(b_1(y)-b_1(0))a_j(y)dy\right|\\
&\leq C||b_1||_{Lip_\beta}\left[\int_{B_j}\frac{|x|^\beta|y|^\varepsilon t}{(t+|x|)^{n+1+\varepsilon-\delta}}\cdot|a_j(y)|dy\right.\\
&+ \left.\int_{B_j}\frac{t|y|^\beta}{(t+|x-y|)^{n+1-\delta}}\cdot|a_j(y)|dy\right]\\
&\leq C||b_1||_{Lip_\beta}\left[\frac{|x|^\beta t}{(t+|x|)^{n+1+\varepsilon-\delta}}\int_{B_j}|y|^\varepsilon|a_j(y)|dy\right.\\
&+ \left.\frac{t}{(t+|x|)^{n+1-\delta}}\int_{B_j}|y|^\varepsilon|a_j(y)|dy\right]\\
&\leq C||b_1||_{Lip_\beta}\left[\frac{|x|^\beta t}{(t+|x|)^{n+1+\varepsilon-\delta}}\cdot 2^{j(\varepsilon+n(1-\frac{1}{q_1})-\alpha)}\right.\\
&+ \left.\frac{t}{(t+|x|)^{n+1-\delta}}\cdot 2^{j(\beta+n(1-\frac{1}{q_1})-\alpha)}\right],
\end{aligned}$$

thus

$$\begin{aligned}
g^{b_1}_{\psi,\delta}(a_j)(x) &\leq C||b_1||_{Lip_\beta}\left[\left(\int_0^\infty\left(\frac{t}{(t+|x|)^{n+1+\varepsilon-\delta}}\right)^2\right)^{1/2}\cdot|x|^\beta\cdot 2^{j(\varepsilon+n(1-\frac{1}{q_1})-\alpha)}\right.\\
&+ \left.\left(\int_0^\infty\left(\frac{t}{(t+|x|)^{n+1-\delta}}\right)^2\frac{dt}{t}\right)^{1/2}\cdot 2^{j(\beta+n(1-\frac{1}{q_1})-\alpha)}\right]\\
&\leq C||b_1||_{Lip_\beta}\left[|x|^{-(n+\varepsilon-\delta)}\cdot|x|^\beta\cdot 2^{j(\varepsilon+n(1-\frac{1}{q_1})-\alpha)}|x|^{-n+\delta}\cdot 2^{j(\beta+n(1-\frac{1}{q_1})-\alpha)}\right]\\
&\leq C||b_1||_{Lip_\beta}|x|^{-n+\delta}\cdot 2^{j(\beta+n(1-\frac{1}{q_1})-\alpha)},
\end{aligned}$$

from that we have

$$\begin{aligned}
||g^{b_1}_{\psi,\delta}(a_j)\chi_k||_{L^{q_2}} &\leq C||b_1||_{Lip_\beta}\cdot 2^{j(\beta+n(1-\frac{1}{q_1})-\alpha)}\left(\int B_k|x|^{-nq_2+q_2\delta}dx\right)^{1/q_2}\\
&\leq C||b_1||_{Lip_\beta}\cdot 2^{j(\beta+n(1-\frac{1}{q_1})-\alpha)}\cdot 2^{-kn(1-\frac{1}{q_2})+k\delta}\\
&\leq C||b_1||_{Lip_\beta}\cdot 2^{[j(\beta+n(1-\frac{1}{q_1})-\alpha)-k(\beta+n(1-\frac{1}{q_1}))]},
\end{aligned}$$

so,

$$I \leq C||b_1||^p_{Lip_\beta}\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{\infty}|\lambda_j|\cdot 2^{[j(\beta+n(1-\frac{1}{q_1})-\alpha)-k(\beta+n(1-\frac{1}{q_1}))]}\right)^p$$

$$
\begin{aligned}
&\leq C||b_1||^p_{Lip_\beta}\begin{cases} \displaystyle\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-2}|\lambda_j|^p\cdot 2^{(j-k)(\beta+n(1-\frac{1}{q_1})-\alpha)p}, \quad 0<p\leq 1\\ \displaystyle\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|^p\cdot 2^{\frac{p}{2}[j(\beta+n(1-\frac{1}{q_1})-\alpha)-k(\beta+n(1-\frac{1}{q_1}))]}\right)\\ \qquad\times\left(\displaystyle\sum_{j=-\infty}^{k-2}2^{\frac{p'}{2}[j(\beta+n(1-\frac{1}{q_1})-\alpha)-k(\beta+n(1-\frac{1}{q_1}))]}\right)^{p/p'}, \quad 1<p<\infty \end{cases}\\
&\leq C||b_1||^p_{Lip_\beta}\begin{cases} \displaystyle\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+2}^{\infty}2^{(j-k)(\beta+n(1-\frac{1}{q_1})-\alpha)p}, \quad 0<p\leq 1\\ \displaystyle\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+2}^{\infty}2^{\frac{p}{2}[(j-k)(\beta+n(1-\frac{1}{q_1})-\alpha)]}, \quad 1<p<\infty \end{cases}\\
&\leq C||b_1||^p_{Lip_\beta}\sum_{j=-\infty}^{\infty}|\lambda_j|^p.
\end{aligned}
$$

Then

$$||g^{b_1}_{\psi,\delta}(f)||_{\dot{K}^{\alpha,p}_{q_2}}\leq C||b_1||_{Lip_\beta}(\sum_{j=-\infty}^{\infty}|\lambda_j|^p)^{1/p}\leq C||f||_{H\dot{K}^{\alpha,p}_{q_1}}.$$

When $m>1$, we have

$$
\begin{aligned}
|F^{\vec{b}}_t(a_j))(x)| &\leq |(b_1(x)-b_1(0))\cdots(b_m(x)-b_m(0))\int_{B_j}(\psi_t(x-y)-\psi_t(x))a_j(y)dy|\\
&\quad+\sum_{j=1}^{\infty}\sum_{\sigma\in C^m_j}|(b(x)-b(0))_{\sigma^c}\int_{B_j}(b(y)-b(0))_\sigma\psi_t(x-y)a_j(y)dy|\\
&\leq C||\vec{b}||_{Lip_\beta}|x|^{m\beta}\int_{B_j}|\psi_t(x-y)-\psi_t(x)||a_j(y)|dy\\
&\quad+C||\vec{b}||_{Lip_\beta}\sum_{\tau+\tau'=m}|x|^{\tau\beta}\int_{B_j}|y|^{\tau'\beta}|\psi_t(x-y)||a_j(y)|dy\\
&\leq C||\vec{b}||_{Lip_\beta}\frac{|x|^{m\beta}t}{(t+|x|)^{n+1+\varepsilon-\delta}}\int_{B_j}|y|^{\varepsilon}|a_j(y)|dy\\
&\quad+C||\vec{b}||_{Lip_\beta}\sum_{\tau+\tau'=m}\frac{|x|^{\tau\beta}t}{(t+|x|)^{n+1-\delta}}\int_{B_j}|y|^{\tau'\beta}|a_j(y)|dy\\
&\leq C||\vec{b}||_{Lip_\beta}\frac{|x|^{m\beta}t}{(t+|x|)^{n+1+\varepsilon-\delta}}\cdot 2^{j(\varepsilon+n(1-\frac{1}{q_1})-\alpha)}\\
&\quad+C||\vec{b}||_{Lip_\beta}\sum_{\tau+\tau'=m}\frac{|x|^{\tau\beta}t}{(t+|x|)^{n+1-\delta}}\cdot 2^{j(\tau'\beta+n(1-\frac{1}{q_1})-\alpha)},
\end{aligned}
$$

thus

$$
\begin{aligned}
g^{\vec{b}}_{\psi,\delta}(a_j)(x) &= \left(\int_0^\infty|F^{\vec{b}}_t(a_j)(x)|^2\frac{dt}{t}\right)^{1/2}\\
&\leq C||\vec{b}||_{Lip_\beta}|x|^{m\beta}\cdot 2^{j(\varepsilon+n(1-\frac{1}{q_1})-\alpha)}\cdot\left(\int_0^\infty\left(\frac{t}{(t+|x|)^{n+1+\varepsilon-\delta}}\right)^2\frac{dt}{t}\right)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&+C||\vec{b}||_{Lip_\beta}\sum_{\tau+\tau'=m}|x|^{\tau\beta}\cdot 2^{j(\tau'\beta+n(1-\frac{1}{q_1})-\alpha)}\cdot\left(\int_0^\infty\left(\frac{t}{(t+|x|)^{n+1-\delta}}\right)^2\frac{dt}{t}\right)^{1/2}\\
&\leq C||\vec{b}||_{Lip_\beta}|x|^{m\beta}|x|^{-(n+\varepsilon-\delta)}\cdot 2^{j(\varepsilon+n(1-\frac{1}{q_1})-\alpha)}\\
&+C||\vec{b}||_{Lip_\beta}\sum_{\tau+\tau'=m}|x|^{\tau\beta}|x|^{-n+\delta}\cdot 2^{j(\tau'\beta+n(1-\frac{1}{q_1})-\alpha)}\\
&\leq C||\vec{b}||_{Lip_\beta}|x|^{-n+\delta}\cdot 2^{j(m\beta+n(1-\frac{1}{q_1})-\alpha)},
\end{aligned}
$$

then

$$
\begin{aligned}
||g^{\vec{b}}_{\psi,\delta}(a_j)\chi_k||_{L^{q_2}} &\leq C||\vec{b}||_{Lip_\beta}\cdot 2^{j(m\beta+n(1-\frac{1}{q_1})-\alpha)}\cdot\left(\int_{B_j}|x|^{-nq_2+q_2\delta}dx\right)^{1/q_2}\\
&\leq C||\vec{b}||_{Lip_\beta}\cdot 2^{[j(m\beta+n(1-\frac{1}{q_1})-\alpha)-k(m\beta+n(1-\frac{1}{q_1}))]},
\end{aligned}
$$

so,

$$
\begin{aligned}
I &\leq C||\vec{b}||^p_{Lip_\beta}\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|\cdot 2^{[j(m\beta+n(1-\frac{1}{q_1})-\alpha)-k(m\beta+n(1-\frac{1}{q_1}))]}\right)^p\\
&\leq C||\vec{b}||^p_{Lip_\beta}\begin{cases}\displaystyle\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-2}|\lambda_j|^p\cdot 2^{(j-k)(m\beta+n(1-\frac{1}{q_1})-\alpha)p}, & 0<p\leq 1\\[2ex]
\displaystyle\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|^p\cdot 2^{\frac{p}{2}\left[j\left(m\beta+n\left(1-\frac{1}{q_1}\right)-\alpha\right)-k\left(m\beta+n\left(1-\frac{1}{q_1}\right)\right)\right]}\right)\\
\displaystyle\quad\times\left(\sum_{j=-\infty}^{k-2}2^{\frac{p'}{2}[j(m\beta+n(1-\frac{1}{q_1})-\alpha)-k(m\beta+n(1-\frac{1}{q_1}))]}\right)^{p/p'}, & 1<p<\infty\end{cases}\\
&\leq C||\vec{b}||^p_{Lip_\beta}\sum_{j=-\infty}^{\infty}|\lambda_j|^p.
\end{aligned}
$$

From I and II, we have

$$
||g^{\vec{b}}_{\psi,\delta}(f)||\leq C||\vec{b}||_{Lip_\beta}\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p}\leq C||f||_{H\dot{K}^{\alpha,p}_{q_1}}.
$$

This completes the proof of Theorem 3.

Theorem 4. *Let* $0<\beta<\min(\gamma/m,1/2m)$, $0<p\leq 1$, $1<q_1,q_2<\infty$, $0<\delta<n$, $1/q_2=1/q_1-(m\beta+\delta)/n$, $\vec{b}=(b_1,\cdots,b_m)$ *with* $b_j\in Lip_\beta(R^n)$ *for* $1\leq j\leq m$. *Then* $g^{\vec{b}}_{\psi,\delta}$ *maps* $H\dot{K}^{n(1-1/q_1)+\beta+\delta/m,p}_{q_1}(R^n)$ *continuously into* $W\dot{K}^{n(1-1/q_1)+\beta+\delta/m,p}_{q_2}(R^n)$.

Proof. We write

$$
f=\sum_{k=-\infty}^{\infty}\lambda_k a_k,
$$

where each a_k is a central $(n(1-1/q_1)+\beta+\delta/m, q_1)$ atom supported on B_k and $\sum\limits_{k=-\infty}^{\infty}|\lambda_k|^p<\infty$. Write

$$\begin{aligned}
&||g_{\psi,\delta}^{\vec{b}}||_{W\dot{K}_{q_2}^{n(1-1/q_1)+\beta+\delta/m,p}}\\
&\leq \sup_{\lambda>0}\lambda\left\{\sum_{l=-\infty}^{\infty}2^{l(n(1-1/q_1)+\beta+\delta/m)p}\left|\left\{x\in E_l:\left|g_{\psi,\delta}^{\vec{b}}\left(\sum_{k=l-3}^{\infty}\lambda_k a_k\right)(x)\right|>\lambda/2\right\}\right|^{p/q_2}\right\}^{1/p}\\
&+\sup_{\lambda>0}\lambda\left\{\sum_{l=-\infty}^{\infty}2^{l(n(1-1/q_1)+\beta+\delta/m)p}\left|\left\{x\in E_l:\left|g_{\psi,\delta}^{\vec{b}}\left(\sum_{k=-\infty}^{l-4}\lambda_k a_k\right)(x)\right|>\lambda/2\right\}\right|^{p/q_2}\right\}^{1/p}\\
&=G_1+G_2.
\end{aligned}$$

By the (L^{q_1},L^{q_2}) boundedness of $g_{\psi,\delta}^{\vec{b}}$ and an estimate similar to that for I_1 in Theorem 3, we get

$$G_1^p\leq C\sum_{l=-\infty}^{\infty}2^{lp(n(1-1/q_1)+\beta+\delta/m)}\left||g_{\psi,\delta}^{\vec{b}}\left(\sum_{l-3}^{\infty}\lambda_k a_k\right)(x)\chi_l\right||_{q_2}^p\leq C||\vec{b}||_{Lip_\beta}^p\sum_{k=-\infty}^{\infty}|\lambda_k|^p.$$

To estimate G_2, let us now use the estimate

$$|g_{\psi,\delta}^{\vec{b}}(a_k)|\ \leq\ C||\vec{b}||_{Lip_\beta}|x|^{\delta-n}(2^k)^{m\beta+n(1-1/q_1)-\alpha},$$

which we get in the proof of Theorem 3.

Note that when $x\in E_l$, $\alpha=n(1-1/q_1)+\beta+\delta/m$,

$$\begin{aligned}
\lambda<\sum_{k=-\infty}^{l-4}|\lambda_k||g_{\psi,\delta}^{\vec{b}}(a_k)| &\leq C||\vec{b}||_{Lip_\beta}\sum_{k=-\infty}^{l-4}|\lambda_k||x|^{\delta-n}(2^k)^{m\beta+n(1-1/q_1)-\alpha}\\
&\leq C||\vec{b}||_{Lip_\beta}\sum_{k=-\infty}^{l-4}|\lambda_k||2^l|^{\delta-n}\sum_{k=-\infty}^{l-4}(2^k)^{m\beta+n(1-1/q_1)-\alpha}\\
&\leq C||\vec{b}||_{Lip_\beta}\sum_{k=-\infty}^{l-4}|\lambda_k|(2^l)^{((m-1)\beta+\delta-n-\delta/m)}\\
&\leq C||\vec{b}||_{Lip_\beta}2^{l((m-1)\beta+\delta-n-\delta/m)}\left(\sum_{k=-\infty}^{\infty}|\lambda_k|^p\right)^{1/p},
\end{aligned}$$

for $\lambda>0$, let l_λ be the maximal positive integer satisfying

$$2^{l_\lambda(n+\delta/m-(m-1)\beta-\delta)}\leq C||\vec{b}||_{Lip_\beta}\lambda^{-1}\left(\sum_{k=-\infty}^{\infty}|\lambda_k|^p\right)^{1/p},$$

then if $l>l_\lambda$, we have

$$|\{x\in E_l:|g_{\psi,\delta}^{\vec{b}}\left(\sum_{k=-\infty}^{l-4}\lambda_k a_k\right)|>\lambda/2\}|=0.$$

So, we obtain

$$\begin{aligned} G_2 &\leq \sup_{\lambda>0} \lambda \left\{ \sum_{l=-\infty}^{l_\lambda} 2^{l(n(1-1/q_1)+\beta+\delta/m)p} (2^l)^{np/q_2} \right\}^{1/p} \\ &\leq \sup_{\lambda>0} \lambda \left\{ \sum_{l=-\infty}^{l_\lambda} (2^l)^{(n-(m-1)\beta-\delta)} \right\}^{1/p} \\ &\leq \sup_{\lambda>0} \lambda 2^{l_\lambda(n-(m-1)\beta-\delta)} \leq C||\vec{b}||_{Lip_\beta} \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^p \right)^{1/p} . \end{aligned}$$

Now, combining the above estimates for G_1 and G_2, we obtain

$$||g^{\vec{b}}_{\psi,\delta}(f)||_{W\dot{K}_{q_2}^{n(1-1/q_1)+\beta+\delta/m,p}} \leq C||\vec{b}||_{Lip_\beta} \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^p \right)^{1/p} .$$

Theorem 4 follows by taking the infimum over all central atomic decompositions.

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A FUNCTIONAL ASSOCIATED WITH TWO BOUNDED LINEAR OPERATORS IN HILBERT SPACES AND RELATED INEQUALITIES

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Abstract. In this paper, several inequalities for the functional

$$\mu(A,B) := \sup_{\|x\|=1} \{\|Ax\| \, \|Bx\|\}$$

under various assumptions for the operators involved, including operators satisfying the uniform (α,β)-property and operators for which the transform $C_{\alpha,\beta}(\cdot,\cdot)$ is accretive, are given.

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1. Introduction

Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [9, p. 1]:

$$W(T) = \{\langle Tx, x\rangle, \ x \in H, \ \|x\| = 1\}.$$

The *numerical radius* $w(T)$ of an operator T on H is given by [9, p. 8]:

$$w(T) = \sup\{|\lambda|, \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{1.1}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T : H \to H$. This norm is equivalent to the operator norm. In fact, the following more precise result holds [9, p. 9]:

$$w(T) \le \|T\| \le 2w(T) \tag{1.2}$$

for any $T \in B(H)$

For other results on numerical radii, see [10], Chapter 11. For some recent and interesting results concerning inequalities for the numerical radius, see [11], [12].

If A, B are two bounded linear operators on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$, then

$$w(AB) \leq 4w(A)\,w(B). \tag{1.3}$$

In the case that $AB = BA$, then

$$w(AB) \leq 2w(A)\,w(B). \tag{1.4}$$

The following results are also well known [9, p. 38]:

If A is a unitary operator that commutes with another operator B, then

$$w(AB) \leq w(B). \tag{1.5}$$

If A is an isometry and $AB = BA$, then (1.5) also holds true.

We say that A and B *double commute* if $AB = BA$ and $AB^* = B^*A$. If the operators A and B double commute, then [9, p. 38]

$$w(AB) \leq w(B)\,\|A\|. \tag{1.6}$$

As a consequence of the above, we have [9, p. 39]:

If A is a normal operator commuting with B, then

$$w(AB) \leq w(A)\,w(B). \tag{1.7}$$

For other results and historical comments on the above see [9, pp. 39–41].

For two bounded linear operators A, B in the Hilbert space $(H, \langle\cdot,\cdot\rangle)$, we define the functional

$$\mu(A,B) := \sup_{\|x\|=1}\{\|Ax\|\,\|Bx\|\}\;(\geq 0). \tag{1.8}$$

It is obvious that μ is symmetric and sub-additive in each variable, $\mu(A,A)=\|A\|^2$, $\mu(A,I) = \|A\|$, where I is the identity operator, $\mu(\alpha A, \beta B) = |\alpha\beta|\,\mu(A,B)$ and $\mu(A,B) \leq \|A\|\,\|B\|$. We also have the following inequalities

$$\mu(A,B) \geq w(B^*A) \tag{1.9}$$

and

$$\mu(A,B)\,\|A\|\,\|B\| \geq \mu(AB, BA). \tag{1.10}$$

Inequality (1.9) follows by the Schwarz inequality $\|Ax\|\|Bx\| \geq |\langle Ax, Bx\rangle|$, $x \in H$, while (1.10) can be obtained by multiplying the inequalities $\|ABx\| \leq \|A\|\,\|Bx\|$ and $\|BAx\| \leq \|B\|\,\|Ax\|$.

From (1.9) we also get

$$\|A\|^2 \geq \mu(A, A^*) \geq w\left(A^2\right) \quad \text{for any } A. \tag{1.11}$$

Motivated by the above results we establish in this paper several inequalities for the functional $\mu(\cdot,\cdot)$ under various assumptions for the operators involved, including operators satisfying the uniform (α,β) −property and operators for which the transform $C_{\alpha,\beta}(\cdot,\cdot)$ is accretive.

2. General inequalities

The following result concerning some general power operator inequalities may be stated:

Theorem 2.1 *For any* $A,B\in B(H)$ *and* $r\geq 1$ *we have the inequality*

$$\mu^{r}(A,B)\leq\frac{1}{2}\left\|(A^{*}A)^{r}+(B^{*}B)^{r}\right\|. \tag{2.1}$$

The constant $\frac{1}{2}$ *is best possible.*

Proof. Using the arithmetic mean - geometric mean inequality and the convexity of the function $f(t)=t^{r}$ for $r\geq 1$ and $t\geq 0$ we have successively

$$\begin{aligned}\|Ax\|\,\|Bx\| &\leq \frac{1}{2}\left[\langle A^{*}Ax,x\rangle+\langle B^{*}Bx,x\rangle\right]\\ &\leq\left[\frac{\langle A^{*}Ax,x\rangle^{r}+\langle B^{*}Bx,x\rangle^{r}}{2}\right]^{\frac{1}{r}}\end{aligned} \tag{2.2}$$

for any $x\in H$.

It is well known that if P is a positive operator, then for any $r\geq 1$ and $x\in H$ with $\|x\|=1$ we have the inequality (see for instance [13])

$$\langle Px,x\rangle^{r}\leq\langle P^{r}x,x\rangle . \tag{2.3}$$

Applying this inequality to the positive operators $A^{*}A$ and $B^{*}B$ we deduce that

$$\left[\frac{\langle A^{*}Ax,x\rangle^{r}+\langle B^{*}Bx,x\rangle^{r}}{2}\right]^{\frac{1}{r}}\leq\left\langle\frac{\left[(A^{*}A)^{r}+(B^{*}B)^{r}\right]x}{2},x\right\rangle^{\frac{1}{r}} \tag{2.4}$$

for any $x\in H$ with $\|x\|=1$.

Now, on making use of the inequalities (2.2) and (2.4) we get

$$\|Ax\|\,\|Bx\|\leq\left\langle\frac{\left[(A^{*}A)^{r}+(B^{*}B)^{r}\right]x}{2},x\right\rangle^{\frac{1}{r}} \tag{2.5}$$

for any $x\in H$ with $\|x\|=1$. Taking the supremum over $x\in H$ with $\|x\|=1$ we obtain the desired result (2.1).

For $r=1$ and $B=A$ we get in both sides of (2.1) the same quantity $\|A\|^{2}$ which shows that the constant $\frac{1}{2}$ is best possible in general in the inequality (2.1). ∎

Corollary 2.1 *For any $A \in B(H)$ and $r \geq 1$ we have the inequality*

$$\mu^{r}(A, A^{*}) \leq \frac{1}{2}\left\|(A^{*}A)^{r} + (AA^{*})^{r}\right\| \tag{2.6}$$

and the inequality

$$\|A\|^{r} \leq \frac{1}{2}\left\|(A^{*}A)^{r} + I\right\|, \tag{2.7}$$

respectively.

The following similar result for powers of operators can be stated as well:

Theorem 2.2 *For any $A, B \in B(H)$, any $\alpha \in (0, 1t)$ and $r \geq 1$ we have the inequality*

$$\mu^{2r}(A, B) \leq \left\|\alpha \cdot (A^{*}A)^{r/\alpha} + (1-\alpha) \cdot (B^{*}B)^{r/(1-\alpha)}\right\|. \tag{2.8}$$

The inequality is sharp.

Proof. Observe that, for any $\alpha \in (0, 1)$ we have

$$\begin{aligned} \|Ax\|^{2}\|Bx\|^{2} &= \langle (A^{*}A)x, x\rangle \langle (B^{*}B)x, x\rangle \\ &= \left\langle \left[(A^{*}A)^{1/\alpha}\right]^{\alpha} x, x\right\rangle \left\langle \left[(B^{*}B)^{1/(1-\alpha)}\right]^{1-\alpha} x, x\right\rangle, \end{aligned} \tag{2.9}$$

where $x \in H$.

It is well known that (see for instance [13]), if P is a positive operator and $q \in (0, 1)$, then

$$\langle P^{q}x, x\rangle \leq \langle Px, x\rangle^{q}. \tag{2.10}$$

Applying this property to the positive operators $(A^{*}A)^{1/\alpha}$ and $(B^{*}B)^{1/(1-\alpha)}$, where $\alpha \in (0, 1)$, we have

$$\begin{aligned} &\left\langle \left[(A^{*}A)^{1/\alpha}\right]^{\alpha} x, x\right\rangle \left\langle \left[(B^{*}B)^{1/(1-\alpha)}\right]^{1-\alpha} x, x\right\rangle \\ &\qquad \leq \left\langle (A^{*}A)^{1/\alpha} x, x\right\rangle^{\alpha} \left\langle (B^{*}B)^{1/(1-\alpha)} x, x\right\rangle^{1-\alpha} \end{aligned} \tag{2.11}$$

for any $x \in H$ with $\|x\| = 1$.

Now, by using the weighted arithmetic mean-geometric mean inequality, i.e.,

$$a^{\alpha}b^{1-\alpha} \leq \alpha a + (1-\alpha)b, \quad \text{where } \alpha \in (0, 1) \text{ and } a, b \geq 0,$$

we get

$$\begin{aligned}\left\langle (A^*A)^{1/\alpha} x, x \right\rangle^{\alpha} \left\langle (B^*B)^{1/(1-\alpha)} x, x \right\rangle^{1-\alpha} \\ \leq \alpha \cdot \left\langle (A^*A)^{1/\alpha} x, x \right\rangle + (1-\alpha) \cdot \left\langle (B^*B)^{1/(1-\alpha)} x, x \right\rangle\end{aligned} \tag{2.12}$$

for any $x \in H$ with $\|x\| = 1$.

Moreover, by the elementary inequality

$$\alpha a + (1-\alpha) b \leq (\alpha a^r + (1-\alpha) b^r)^{1/r}, \quad \text{where } \alpha \in (0,1) \text{ and } a, b \geq 0;$$

we have successively

$$\begin{aligned}&\alpha \cdot \left\langle (A^*A)^{1/\alpha} x, x \right\rangle + (1-\alpha) \cdot \left\langle (B^*B)^{1/(1-\alpha)} x, x \right\rangle \\ &\leq \left[\alpha \cdot \left\langle (A^*A)^{1/\alpha} x, x \right\rangle^r + (1-\alpha) \cdot \left\langle (B^*B)^{1/(1-\alpha)} x, x \right\rangle^r\right]^{\frac{1}{r}} \\ &\leq \left[\alpha \cdot \left\langle (A^*A)^{r/\alpha} x, x \right\rangle + (1-\alpha) \cdot \left\langle (B^*B)^{r/(1-\alpha)} x, x \right\rangle\right]^{\frac{1}{r}},\end{aligned} \tag{2.13}$$

for any $x \in H$ with $\|x\| = 1$, where for the last inequality we have used the property (2.3) for the positive operators $(A^*A)^{1/\alpha}$ and $(B^*B)^{1/(1-\alpha)}$.

Now, by making use of the identity (2.9) and the inequalities (2.11)-(2.13), we get

$$\|Ax\|^2 \|Bx\|^2 \leq \left[\left\langle \left[\alpha \cdot (A^*A)^{r/\alpha} + (1-\alpha) \cdot (B^*B)^{r/(1-\alpha)}\right] x, x \right\rangle\right]^{\frac{1}{r}}$$

for any $x \in H$ with $\|x\| = 1$. Taking the supremum over $x \in H$ with $\|x\| = 1$ we deduce the desired result (2.8).

Notice that the inequality is sharp since for $r = 1$ and $B = A$ we get in both sides of (2.8) the same quantity $\|A\|^4$. ∎

Corollary 2.2 *For any $A \in B(H)$, any $\alpha \in (0,1)$ and $r \geq 1$, we have the inequalities*

$$\mu^{2r}(A, A^*) \leq \left\|\alpha \cdot (A^*A)^{r/\alpha} + (1-\alpha) \cdot (AA^*)^{r/(1-\alpha)}\right\|,$$

$$\|A\|^{2r} \leq \left\|\alpha \cdot (A^*A)^{r/\alpha} + (1-\alpha) \cdot I\right\|$$

and

$$\|A\|^{4r} \leq \left\|\alpha \cdot (A^*A)^{r/\alpha} + (1-\alpha) \cdot (A^*A)^{r/(1-\alpha)}\right\|,$$

respectively.

The following reverse of inequality (1.9) may be stated as well:

Theorem 2.3 *For any* $A, B \in B(H)$ *we have the inequality*

$$(0 \leq)\, \mu(A,B) - w(B^*A) \leq \frac{1}{2}\|A-B\|^2 \tag{2.14}$$

and the inequality

$$\mu\left(\frac{A+B}{2}, \frac{A-B}{2}\right) \leq \frac{1}{2} w(B^*A) + \frac{1}{4}\|A-B\|^2, \tag{2.15}$$

respectively.

Proof. We have

$$\begin{aligned} \|Ax-Bx\|^2 &= \|Ax\|^2 + \|Bx\|^2 - 2\operatorname{Re}\langle B^*Ax, x\rangle \\ &\geq 2\|Ax\|\|Bx\| - 2|\langle B^*Ax, x\rangle|, \end{aligned} \tag{2.16}$$

for any $x \in H$, $\|x\| = 1$, which gives the inequality

$$\|Ax\|\|Bx\| \leq |\langle B^*Ax, x\rangle| + \frac{1}{2}\|Ax-Bx\|^2,$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $\|x\| = 1$ we deduce the desired result (2.14).

By the parallelogram identity in the Hilbert space H, we also have

$$\begin{aligned} \|Ax\|^2 + \|Bx\|^2 &= \frac{1}{2}\left(\|Ax+Bx\|^2 + \|Ax-Bx\|^2\right) \\ &\geq \|Ax+Bx\|\|Ax-Bx\|, \end{aligned}$$

for any $x \in H$.

Combining this inequality with the first part of (2.16), we get

$$\|Ax+Bx\|\|Ax-Bx\| \leq \|Ax-Bx\|^2 + 2|\langle B^*Ax, x\rangle|,$$

for any $x \in H$. Taking the supremum in this inequality over $\|x\| = 1$ we deduce the desired result (2.15). ∎

Corollary 2.3 *Let* $A \in B(H)$. *If*

$$\operatorname{Re}(A) := \frac{A+A^*}{2} \text{ and } \operatorname{Im}(A) := \frac{A-A^*}{2i}$$

are the real and imaginary parts of A, *then we have the inequality*

$$(0 \leq)\, \mu(A, A^*) - w\left(A^2\right) \leq 2 \cdot \|\operatorname{Im}(A)\|^2$$

and

$$\mu(\operatorname{Re}(A), \operatorname{Im}(A)) \leq \frac{1}{2} w\left(A^2\right) + \|\operatorname{Im}(A)\|^2,$$

respectively.

Moreover, we have

$$(0 \leq)\, \mu(\operatorname{Re}(A), \operatorname{Im}(A)) - w(\operatorname{Re}(A)\operatorname{Im}(A)) \leq \frac{1}{2}\|A\|^2.$$

Corollary 2.4 *For any* $A \in B(H)$ *and* $\lambda \in \mathbb{C}$ *with* $\lambda \neq 0$ *we have the inequality* (see also [6])

$$(0 \leq) \|A\| - w(A) \leq \frac{1}{2|\lambda|} \|A - \lambda I\|^2 . \tag{2.17}$$

For a bounded linear operator T consider the quantity

$$\ell(T) := \inf_{\|x\|=1} \|Tx\| .$$

We can state the following result as well.

Theorem 2.4 *For any* $A, B \in B(H)$ *with* $A \neq B$ *and such that* $\ell(B) \geq \|A - B\|$ *we have*

$$(0 \leq) \mu^2(A, B) - w^2(B^*A) \leq \|A\|^2 \|A - B\|^2 . \tag{2.18}$$

Proof. Denote $r := \|A - B\| > 0$. Then for any $x \in H$ with $\|x\| = 1$ we have $\|Bx\| \geq r$ and by the first part of (2.16) we can write that

$$\|Ax\|^2 + \left(\sqrt{\|Bx\|^2 - r^2}\right)^2 \leq 2|\langle B^*Ax, x\rangle| \tag{2.19}$$

for any $x \in H$ with $\|x\| = 1$.

On the other hand, we have

$$\|Ax\|^2 + \left(\sqrt{\|Bx\|^2 - r^2}\right)^2 \geq 2 \cdot \|Ax\| \sqrt{\|Bx\|^2 - r^2} \tag{2.20}$$

for any $x \in H$ with $\|x\| = 1$.

Combining (2.19) with (2.20), we deduce

$$\|Ax\| \sqrt{\|Bx\|^2 - r^2} \leq |\langle B^*Ax, x\rangle|$$

which is clearly equivalent to

$$\|Ax\|^2 \|Bx\|^2 \leq |\langle B^*Ax, x\rangle|^2 + \|Ax\|^2 \|A - B\|^2 \tag{2.21}$$

for any $x \in H$ with $\|x\| = 1$. Taking the supremum in (2.21) over $x \in H$ with $\|x\| = 1$, we deduce the desired inequality (2.18). ∎

Corollary 2.5 *For any* $A \in B(H)$ *a non-self-adjoint operator in* $B(H)$ *and such that* $\ell(A^*) \geq \|\mathrm{Im}(A)\|$ *we have*

$$(0 \leq) \mu^2(A, A^*) - w^2\left(A^2\right) \leq 4 \cdot \|A\|^2 \|\mathrm{Im}(A)\|^2 . \tag{2.22}$$

Corollary 2.6 *For any* $A \in B(H)$ *and* $\lambda \in \mathbb{C}$ *with* $\lambda \neq 0$ *and* $|\lambda| \geq \|A - \lambda I\|$ *we have the inequality* (see also [6])

$$(0 \leq) \|A\|^2 - w^2(A) \leq \frac{1}{|\lambda|^2} \cdot \|A\|^2 \|A - \lambda I\|^2$$

or, equivalently,

$$(0 \leq) \sqrt{1 - \frac{\|A - \lambda I\|^2}{|\lambda|^2}} \leq \frac{w(A)}{\|A\|} (\leq 1).$$

3. Inequalities for operators satisfying the uniform (α, β)-property

The following result that may be of interest in itself holds:

Lemma 3.1 *Let* $T \in B(H)$ *and* $\alpha, \beta \in \mathbb{C}$ *with* $\alpha \neq \beta$. *The following statements are equivalent:*

(i) *We have*

$$\operatorname{Re}\langle \beta y - Tx, Tx - \alpha y\rangle \geq 0 \tag{3.1}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$;

(ii) *We have*

$$\left\| Tx - \frac{\alpha + \beta}{2} \cdot y \right\| \leq \frac{1}{2} |\alpha - \beta| \tag{3.2}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$.

Proof. This follows by the following identity

$$\operatorname{Re}\langle \beta y - Tx, Tx - \alpha y\rangle = \frac{1}{4} |\alpha - \beta|^2 - \left\| Tx - \frac{\alpha + \beta}{2} \cdot y \right\|^2,$$

that holds for any $x, y \in H$ with $\|x\| = \|y\| = 1$. ■

Remark 3.1 *For any operator* $T \in B(H)$ *if we choose* $\alpha = a\|T\|(1 + 2i)$ *and* $\beta = a\|T\|(1 - 2i)$ *with* $a \geq 1$, *then*

$$\frac{\alpha + \beta}{2} = a\|T\| \quad \textit{and} \quad \frac{|\alpha - \beta|}{2} = 2a\|T\|$$

showing that

$$\left\| Tx - \frac{\alpha + \beta}{2} \cdot y \right\| \leq \|Tx\| + \left| \frac{\alpha + \beta}{2} \right| \leq \|T\| + a\|T\|$$
$$\leq 2a\|T\| = \frac{1}{2} \cdot |\alpha - \beta|,$$

that holds for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$, *i.e.,* T *satisfies condition* (3.1) *with the scalars* α *and* β *given above.*

Definition 3.1 For given $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq \beta$ and $y \in H$ with $\|y\| = 1$, we say that the operator $T \in B(H)$ has the (α, β, y)-property if either (3.1) or, equivalently, (3.2) holds true for any $x \in H$ with $\|x\| = 1$. Moreover, if T has the (α, β, y)-property for any $y \in H$ with $\|y\| = 1$, then we say that this operator has the uniform (α, β)-property.

Remark 3.2 The above Remark 3.1 shows that any bounded linear operator has the uniform (α, β)-property for infinitely many (α, β) appropriately chosen. For a given operator satisfying an (α, β)-property, it is an open problem to find the possibly nonzero lower bound for the quantity $|\alpha - \beta|$.

The following results may be stated:

Theorem 3.1 *Let $A, B \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ with $\alpha \neq \beta$ and $\gamma \neq \delta$. For $y \in H$ with $\|y\| = 1$ assume that A^* has the (α, β, y)-property while B^* has the (γ, δ, y)-property. Then*

$$\left|\|Ay\| \|By\| - \|BA^*\|\right| \leq \frac{1}{4} |\beta - \alpha| |\gamma - \delta| . \tag{3.3}$$

Moreover, if A^* has the uniform (α, β)-property and B^* has the uniform (γ, δ)-property, then

$$|\mu(A, B) - \|BA^*\|| \leq \frac{1}{4} |\beta - \alpha| |\gamma - \delta| . \tag{3.4}$$

Proof. A^* has the (α, β, y)-property while B^* has the (γ, δ, y)-property, then on making use of Lemma 3.1 we have that

$$\left\|A^*x - \frac{\alpha + \beta}{2} \cdot y\right\| \leq \frac{1}{2} |\beta - \alpha| \quad \text{and} \quad \left\|B^*z - \frac{\gamma + \delta}{2} \cdot y\right\| \leq \frac{1}{2} |\gamma - \delta|$$

for any $x, z \in H$ with $\|x\| = \|z\| = 1$.

Now, we make use of the following Grüss type inequality for vectors in inner product spaces obtained by the author in [1] (see also [2] or [7, p. 43]:

Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $u, v, e \in H$, $\|e\| = 1$, and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ such that

$$\operatorname{Re} \langle \beta e - u, u - \alpha e \rangle \geq 0, \qquad \operatorname{Re} \langle \delta e - v, v - \gamma e \rangle \geq 0 \tag{3.5}$$

or, equivalently,

$$\left\|u - \frac{\alpha + \beta}{2} e\right\| \leq \frac{1}{2} |\beta - \alpha|, \qquad \left\|v - \frac{\gamma + \delta}{2} e\right\| \leq \frac{1}{2} |\delta - \gamma| . \tag{3.6}$$

Then

$$|\langle u, v \rangle - \langle u, e \rangle \langle e, v \rangle| \leq \frac{1}{4} |\beta - \alpha| |\delta - \gamma| . \tag{3.7}$$

Applying (3.7) for $u = A^*x$, $v = B^*z$ and $e = y$ we deduce

$$|\langle BA^*x, z\rangle - \langle x, Ay\rangle \langle By, z\rangle| \leq \frac{1}{4} |\beta - \alpha| |\delta - \gamma|, \tag{3.8}$$

for any $x, z \in H$, $\|x\| = \|z\| = 1$, which is an inequality of interest in itself.

Observing that

$$||\langle BA^*x, z\rangle| - |\langle x, Ay\rangle \langle z, By\rangle|| \leq |\langle BA^*x, z\rangle - \langle x, Ay\rangle \langle By, z\rangle|,$$

then by (3.8) we deduce the inequality

$$||\langle BA^*x, z\rangle| - |\langle x, Ay\rangle \langle z, By\rangle|| \leq \frac{1}{4} |\beta - \alpha| |\delta - \gamma|$$

for any $x, z \in H$, $\|x\| = \|z\| = 1$. This is equivalent with the following two inequalities

$$|\langle BA^*x, z\rangle| \leq |\langle x, Ay\rangle \langle z, By\rangle| + \frac{1}{4} |\beta - \alpha| |\delta - \gamma| \tag{3.9}$$

and

$$|\langle x, Ay\rangle \langle z, By\rangle| \leq |\langle BA^*x, z\rangle| + \frac{1}{4} |\beta - \alpha| |\delta - \gamma| \tag{3.10}$$

for any $x, z \in H, \|x\| = \|z\| = 1$.

Taking the supremum over $x, z \in H$, $\|x\| = \|z\| = 1$, in (3.9) and (3.10) we get the inequalities

$$\|BA^*\| \leq \|Ay\| \|By\| + \frac{1}{4} |\beta - \alpha| |\delta - \gamma| \tag{3.11}$$

and

$$\|Ay\| \|By\| \leq \|BA^*\| + \frac{1}{4} |\beta - \alpha| |\delta - \gamma|, \tag{3.12}$$

which are clearly equivalent to (3.3).

Now, if A^* has the uniform (α, β)-property and B^* has the uniform (γ, δ)-property, then the inequalities (3.11) and (3.12) hold for any $y \in H$ with $\|y\| = 1$. Taking the supremum over $y \in H$ with $\|y\| = 1$ in these inequalities we deduce

$$\|BA^*\| \leq \mu(A, B) + \frac{1}{4} |\beta - \alpha| |\delta - \gamma|$$

and

$$\mu(A, B) \leq \|BA^*\| + \frac{1}{4} |\beta - \alpha| |\delta - \gamma|,$$

which are equivalent to (3.4). ■

Corollary 3.7 *Let $A \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ with $\alpha \neq \beta$ and $\gamma \neq \delta$. For $y \in H$ with $\|y\| = 1$ assume that A has the (α, β, y)-property while A^* has the (γ, δ, y)-property. Then*

$$\left| \|A^* y\| \, \|Ay\| - \left\|A^2\right\| \right| \leq \frac{1}{4} |\beta - \alpha| \, |\gamma - \delta| \, .$$

Moreover, if A has the uniform (α, β)-property and A^* has the uniform (γ, δ)-property, then

$$\left| \mu(A, A^*) - \left\|A^2\right\| \right| \leq \frac{1}{4} |\beta - \alpha| \, |\gamma - \delta| \, .$$

The following results may be stated as well:

Theorem 3.2 *Let $A, B \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ with $\alpha + \beta \neq 0$ and $\gamma + \delta \neq 0$. For $y \in H$ with $\|y\| = 1$ assume that A^* has the (α, β, y)-property while B^* has the (γ, δ, y)-property. Then*

$$(3.13) \qquad |\|Ay\|\|By\| - \|BA^*\|| \leq \frac{1}{4} \cdot \frac{|\beta - \alpha||\delta - \gamma|}{\sqrt{|\beta + \alpha||\delta + \gamma|}} \sqrt{(\|A\| + \|Ay\|)(\|B\| + \|By\|)}.$$

Moreover, if A^* has the uniform (α, β)-property and B^* has the uniform (γ, δ)-property, then

$$(3.14) \qquad |\mu(A, B) - \|BA^*\|| \leq \frac{1}{2} \cdot \frac{|\beta - \alpha| \, |\delta - \gamma|}{\sqrt{|\beta + \alpha| \, |\delta + \gamma|}} \sqrt{\|A\| \, \|B\|}.$$

Proof. We make use of the following inequality obtained by the author in [5] (see also [7, p. 65]):

Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $u, v, e \in H$, $\|e\| = 1$, and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ with $\alpha + \beta \neq 0$ and $\gamma + \delta \neq 0$ and such that

$$\operatorname{Re} \langle \beta e - u, u - \alpha e \rangle \geq 0, \qquad \operatorname{Re} \langle \delta e - v, v - \gamma e \rangle \geq 0$$

or, equivalently,

$$\left\| u - \frac{\alpha + \beta}{2} e \right\| \leq \frac{1}{2} |\beta - \alpha| \, , \left\| v - \frac{\gamma + \delta}{2} e \right\| \leq \frac{1}{2} |\delta - \gamma| \, .$$

Then

$$(3.15) \quad |\langle u, v \rangle - \langle u, e \rangle \langle e, v \rangle| \leq \frac{1}{4} \cdot \frac{|\beta - \alpha| \, |\delta - \gamma|}{\sqrt{|\beta + \alpha| \, |\delta + \gamma|}} \sqrt{(\|u\| + |\langle u, e \rangle|)(\|v\| + |\langle v, e \rangle|)}.$$

Applying (3.15) for $u = A^* x$, $v = B^* z$ and $e = y$ we deduce

$$|\langle BA^* x, z \rangle - \langle x, Ay \rangle \langle By, z \rangle| \leq \frac{1}{4} \cdot \frac{|\beta - \alpha| \, |\delta - \gamma|}{\sqrt{|\beta + \alpha| \, |\delta + \gamma|}} \sqrt{(\|A^* x\| + |\langle x, Ay \rangle|)(\|B^* z\| + |\langle z, By \rangle|)}$$

for any $x, y, z \in H$, $\|x\| = \|y\| = \|z\| = 1$.

Now, on making use of a similar argument to the one from the proof of Theorem 3.1, we deduce the desired results (3.13) and (3.14). The details are omitted. ■

Corollary 3.8 *Let* $A \in B(H)$ *and* $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ *with* $\alpha + \beta \neq 0$ *and* $\gamma + \delta \neq 0$. *For* $y \in H$ *with* $\|y\| = 1$ *assume that* A *has* (α, β, y)*-property while* A^* *has the* (γ, δ, y)*-property. Then*

$$\left| \|A^* y\| \, \|Ay\| - \|A^2\| \right| \leq \frac{1}{4} \cdot \frac{|\beta - \alpha| \, |\delta - \gamma|}{\sqrt{|\beta + \alpha| \, |\delta + \gamma|}} \sqrt{(\|A\| + \|A^* y\|)(\|A\| + \|Ay\|)}.$$

Moreover, if A *has the uniform* (α, β)*-property and* A^* *has the uniform* (γ, δ)*-property, then*

$$\left| \mu(A, A^*) - \|A^2\| \right| \leq \frac{1}{2} \cdot \frac{|\beta - \alpha| \, |\delta - \gamma|}{\sqrt{|\beta + \alpha| \, |\delta + \gamma|}} \|A\|.$$

4. The transform $C_{\alpha,\beta}(\cdot,\cdot)$ and other inequalities

For two given operators $T, U \in B(H)$ and two given scalars $\alpha, \beta \in \mathbb{C}$ consider the transform

$$C_{\alpha,\beta}(T, U) = (T^* - \bar{\alpha} U^*)(\beta U - T).$$

This transform generalizes the transform

$$C_{\alpha,\beta}(T) := (T^* - \bar{\alpha} I)(\beta I - T) = C_{\alpha,\beta}(T, I),$$

where I is the identity operator, which has been introduced in [8] in order to provide some generalizations of the well known Kantorovich inequality for operators in Hilbert spaces.

We recall that a bounded linear operator T on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ is called *accretive* if $\operatorname{Re} \langle Ty, y \rangle \geq 0$ for any $y \in H$.

Using the following identity

$$\begin{aligned} \operatorname{Re} \langle C_{\alpha,\beta}(T, U) x, x \rangle &= \operatorname{Re} \langle C_{\beta,\alpha}(T, U) x, x \rangle \\ &= \frac{1}{4} |\beta - \alpha|^2 \|Ux\|^2 - \left\| Tx - \frac{\alpha + \beta}{2} \cdot Ux \right\|^2, \end{aligned} \tag{4.1}$$

that holds for any scalars α, β and any vector $x \in H$, we can give a simple characterization result that is useful in the following:

Lemma 4.2 *For* $\alpha, \beta \in \mathbb{C}$ *and* $T, U \in B(H)$ *the following statements are equivalent:*

(i) *The transform* $C_{\alpha,\beta}(T, U)$ *(or, equivalently,* $C_{\beta,\alpha}(T, U)$*) is accretive;*

(ii) *We have the norm inequality*

$$\left\| Tx - \frac{\alpha+\beta}{2} \cdot Ux \right\| \leq \frac{1}{2} \left|\beta - \alpha\right| \left\|Ux\right\| \tag{4.2}$$

for any $x \in H$.

As a consequence of the above lemma, we can state

Corollary 4.9 *Let $\alpha, \beta \in \mathbb{C}$ and $T, U \in B(H)$. If $C_{\alpha,\beta}(T,U)$ is accretive, then*

$$\left\| T - \frac{\alpha+\beta}{2} \cdot U \right\| \leq \frac{1}{2} \left|\beta - \alpha\right| \left\|U\right\| . \tag{4.3}$$

Remark 4.3 *In order to give examples of linear operators $T, U \in B(H)$ and numbers $\alpha, \beta \in \mathbb{C}$ such that the transform $C_{\alpha,\beta}(T,U)$ is accretive, it suffices to select two bounded linear operator S and V and the complex numbers z, w $(w \neq 0)$ with the property that $\left\|Sx - zVx\right\| \leq |w| \left\|Vx\right\|$ for any $x \in H$, and, by choosing $T = S$, $U = V$, $\alpha = \frac{1}{2}(z+w)$ and $\beta = \frac{1}{2}(z-w)$, we observe that T and U satisfy (4.2), i.e., $C_{\alpha,\beta}(T,U)$ is accretive.*

We are able now to give the following result concerning other reverse inequalities for the case when the involved operators satisfy the accretivity property described above.

Theorem 4.1 *Let $\alpha, \beta \in \mathbb{C}$ and $A, B \in B(H)$. If $C_{\alpha,\beta}(A,B)$ is accretive, then*

$$(0 \leq)\, \mu^2(A,B) - w^2(B^*A) \leq \frac{1}{4} \cdot \left|\beta - \alpha\right|^2 \left\|B\right\|^4 . \tag{4.4}$$

Moreover, if $\alpha + \beta \neq 0$, then

$$(0 \leq)\, \mu(A,B) - w(B^*A) \leq \frac{1}{4} \cdot \frac{\left|\beta - \alpha\right|^2}{\left|\beta + \alpha\right|} \left\|B\right\|^2 . \tag{4.5}$$

In addition, if $\operatorname{Re}\left(\alpha\bar{\beta}\right) > 0$ and $B^*A \neq 0$, then also

$$(1 \leq)\, \frac{\mu(A,B)}{w(B^*A)} \leq \frac{1}{2} \cdot \frac{\left|\beta + \alpha\right|}{\sqrt{\operatorname{Re}\left(\alpha\bar{\beta}\right)}} \tag{4.6}$$

and

$$(0 \leq)\, \mu^2(A,B) - w^2(B^*A) \leq \left(\left|\beta + \alpha\right| - 2 \cdot \sqrt{\operatorname{Re}\left(\alpha\bar{\beta}\right)}\right) w(B^*A) \left\|B\right\|^2 , \tag{4.7}$$

respectively.

Proof. By Lemma 4.2, since $C_{\alpha,\beta}(A,B)$ is accretive, then

$$\left\| Ax - \frac{\alpha+\beta}{2} \cdot Bx \right\| \leq \frac{1}{2} |\beta - \alpha| \|Bx\| \tag{4.8}$$

for any $x \in H$.

We use the following reverse of the Schwarz inequality in inner product spaces obtained by the author in [3] (see also [7, p. 4]):

If $\gamma, \Gamma \in \mathbb{K}$ ($\mathbb{K} = \mathbb{C}, \mathbb{R}$) and $u, v \in H$ are such that

$$\operatorname{Re} \langle \Gamma v - u, u - \gamma v \rangle \geq 0 \tag{4.9}$$

or, equivalently,

$$\left\| u - \frac{\gamma+\Gamma}{2} \cdot v \right\| \leq \frac{1}{2} |\Gamma - \gamma| \|v\|, \tag{4.10}$$

then

$$0 \leq \|u\|^2 \|v\|^2 - |\langle u, v \rangle|^2 \leq \frac{1}{4} |\Gamma - \gamma|^2 \|v\|^4. \tag{4.11}$$

Now, by making use of (4.11) for $u = Ax$, $v = Bx$, $x \in H$, $\|x\| = 1$ and $\gamma = \alpha, \Gamma = \beta$, we can write the inequality

$$\|Ax\|^2 \|Bx\|^2 \leq |\langle B^* Ax, x \rangle|^2 + \frac{1}{4} |\beta - \alpha|^2 \|Bx\|^4,$$

for any $x \in H$, $\|x\| = 1$. Taking the supremum over $\|x\| = 1$ in this inequality produces the desired result (4.4).

Now, by using the result from [5] (see also [7, p. 29]) namely:

If $\gamma, \Gamma \in \mathbb{K}$ with $\gamma + \Gamma \neq 0$ and $u, v \in H$ are such that either (4.9) or, equivalently, (4.10) holds true, then

$$0 \leq \|u\| \|v\| - |\langle u, v \rangle| \leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \|v\|^2. \tag{4.12}$$

Now, by making use of (4.12) for $u = Ax$, $v = Bx$, $x \in H$, $\|x\| = 1$ and $\gamma = \alpha, \Gamma = \beta$ and using the same procedure outlined above, we deduce the second inequality (4.5).

The inequality (4.6) follows from the result presented below obtained in [4] (see also [7, p. 21]):

If $\gamma, \Gamma \in \mathbb{K}$ with $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$ and $u, v \in H$ are such that either (4.9) or, equivalently, (4.10) holds true, then

$$\|u\| \|v\| \leq \frac{1}{2} \cdot \frac{|\Gamma + \gamma|}{\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}} |\langle u, v \rangle|, \tag{4.13}$$

by choosing $u = Ax$, $v = Bx$, $x \in H$, $\|x\| = 1$ and $\gamma = \alpha, \Gamma = \beta$ and taking the supremum over $\|x\| = 1$.

Finally, by making use of the inequality (see [6])

$$\|u\|^2 \|v\|^2 - |\langle u, v\rangle|^2 \leq \left(|\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\right) |\langle u, v\rangle| \, \|v\|^2 \tag{4.14}$$

that is valid provided $\gamma, \Gamma \in \mathbb{K}$ with $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$ and $u, v \in H$ are such that either (4.9) or, equivalently, (4.10) holds true, we obtain the last inequality (4.7). The details are omitted. ∎

Remark 4.4 *Let $M, m > 0$ and $A, B \in B(H)$. If $C_{m,M}(A, B)$ is accretive, then*

$$\begin{aligned}
(0 \leq)\, \mu^2(A, B) - w^2(B^*A) &\leq \frac{1}{4} \cdot (M - m)^2 \|B\|^4, \\
(0 \leq)\, \mu(A, B) - w(B^*A) &\leq \frac{1}{4} \cdot \frac{(M - m)^2}{m + M} \|B\|^2, \\
(1 \leq)\, \frac{\mu(A, B)}{w(B^*A)} &\leq \frac{1}{2} \cdot \frac{m + M}{\sqrt{mM}} \\
(0 \leq)\, \mu^2(A, B) - w^2(B^*A) &\leq \left(\sqrt{M} - \sqrt{m}\right)^2 w(B^*A) \|B\|^2,
\end{aligned}$$

respectively.

Corollary 4.10 *Let $\alpha, \beta \in \mathbb{C}$ and $A \in B(H)$. If $C_{\alpha,\beta}(A, A^*)$ is accretive, then*

$$(0 \leq)\, \mu^2(A, A^*) - w^2\left(A^2\right) \leq \frac{1}{4} \cdot |\beta - \alpha|^2 \|A\|^4.$$

Moreover, if $\alpha + \beta \neq 0$, then

$$(0 \leq)\, \mu(A, A^*) - w\left(A^2\right) \leq \frac{1}{4} \cdot \frac{|\beta - \alpha|^2}{|\beta + \alpha|} \|A\|^2.$$

In addition, if $\operatorname{Re}\left(\alpha\bar{\beta}\right) > 0$ and $A^2 \neq 0$, then also

$$(1 \leq)\, \frac{\mu(A, A^*)}{w(A^2)} \leq \frac{1}{2} \cdot \frac{|\beta + \alpha|}{\sqrt{\operatorname{Re}\left(\alpha\bar{\beta}\right)}}$$

and

$$(0 \leq)\, \mu^2(A, A^*) - w^2\left(A^2\right) \leq \left(|\beta + \alpha| - 2 \cdot \sqrt{\operatorname{Re}\left(\alpha\bar{\beta}\right)}\right) w\left(A^2\right) \|A\|^2,$$

respectively.

Remark 4.5 *In a similar manner, if $N, n > 0$, $A \in B(H)$ and $C_{n,N}(A, A^*)$ is accretive, then*

$$\begin{aligned}
(0 \leq)\, \mu^2(A, A^*) - w^2\left(A^2\right) &\leq \frac{1}{4} \cdot (N - n)^2 \|A\|^4, \\
(0 \leq)\, \mu(A, A^*) - w\left(A^2\right) &\leq \frac{1}{4} \cdot \frac{(N - n)^2}{n + N} \|A\|^2, \\
(1 \leq)\, \frac{\mu(A, A^*)}{w(A^2)} &\leq \frac{1}{2} \cdot \frac{n + N}{\sqrt{nN}} \ \textit{(for } A^2 \neq 0\textit{)}
\end{aligned}$$

and

$$(0 \leq) \, \mu^2 (A, A^*) - w^2 \left(A^2\right) \leq \left(\sqrt{N} - \sqrt{n}\right)^2 w \left(A^2\right) \|A\|^2 ,$$

respectively.

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GENERALIZATION OF GOLDBACH'S CONJECTURE AND SOME SPECIAL CASES

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Abstract. Concerned with Goldbach's conjecture, we accomplished a generalization that we called *generalized Goldbach's conjecture* and proved their equivalency. However, the generalized Goldbach's conjecture reveals a new direction for a potential generalized proof. In this paper we prove both claims for certain cases.

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1. Introduction

The Goldbach's conjecture [4],

> *every even positive integer number* (*i.e., every positive integer multiple of* 2) *besides* 2*, is analyzed* (*not necessarily uniquely*) *as the sum of two positive prime numbers*[1],

gives rise to the question whether a similar conjecture can be stated for the positive multiples of every positive integer number. In particular whether

> *every positive multiple of every positive integer a, except itself, is analyzed* (*not necessarily uniquely*) *as the sum of a prime numbers.*

By considering simple examples, it is confirmed that the statement holds. Since it is unproved, it remains a simple conjecture and because it generalizes Goldbach's conjecture we characterized it as *generalized Goldbach's conjecture.* However, considering their elaborations we have concluded that Goldbach's conjecture implies the generalized Goldbach's conjecture and vice versa. It is worth

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[1] The statement concerns the binary conjecture also known as the *strong Goldbach's conjecture*, in contrast to Goldbach's original ternary conjecture which states that every odd integer number greater or equal to seven is the sum of three primes.

noting that Goldbach's conjecture has received a lot of attention; see, for example, [1], [2], [3], [5], [6], [7].

For proving the equivalency of both conjectures we will proceed inductively. From the beginning, let us assume the division

$$am/(a-1) \tag{1}$$

of the positive multiple am of a by $a-1$, that is the relationship

$$am = (a-1)n + r$$

where n and r are the quotient and the remainder, respectively. Thus we have $0 \leq r < a-1$. Next we assume that $a-1$ is analyzed as the sum of $a-1$ prime numbers and we will prove the correspondence for a.

2. Base cases $a = 1, 2, 3, 4, 5$ for the induction

The case for $a = 1$ is obviously excluded, unless m prime, for $a = 2$ we have Goldbach's conjecture that we accept as a proved statement or as an axiom. Thus we have $a \geq 3$ and for the induction we will consider the following cases ($a = 3, 4, 5$).

2.1. Case $a = 3$

From the division $3m/2$ we have the cases $3m = 2n + 1$ and $3m = 2n$.

(3.*i*) $3m = 2n + 1 = p_1 + q_1 + 1$, where $p_1 + q_1$ by the analysis of Goldbach of $2n$ as the sum of two prime numbers (for $n > 1$ and, thus, for $m > 1$) and because we exclude $p_1 = q_1 = 2$ (for otherwise we would have $3m = 5$, not a multiple of 3) one of p_1, q_1 will be odd number. Let such a number be q_1. Then $q_1 + 1$ is even and by Goldbach $q_1 + 1 = p_2 + p_3$ sum of two primes. Hence we conclude that $3m = p_1 + p_2 + p_3$ sum of three prime numbers.

Example 2.1.

$$3 \cdot 7 = 2 \cdot 10 + 1 = 21 = 2 + 2 + 17 = 3 + 5 + 13 = 5 + 5 + 11 = 7 + 7 + 7.$$

(3.*ii*) $3m = 2n = 2(n-1) + 2 = p_1 + p_2 + 2$, where $p_1 + p_2$ one analysis of $2(n-1)$ as the sum of two prime numbers, since $n - 1 > 1$. This means for $n > 2$ we have $m > 1$.
In such a case $3m$ is obviously an even multiple of 3. (Moreover we have $3m = 2n = 6m'$ because 2 as a divisor of $3m$ and prime number with respect to 3 divides m and thus $m = 2m'$.)

Example 2.2.

$$3 \cdot 6 = 3(2 \cdot 3) = 2 \cdot 9 = 18 = 2 + 3 + 13 = 2 + 5 + 11.$$

Remark 2.1.

a) In every analysis that concerns (as it is non-unique) every even multiple of 3 into a sum of three prime numbers, one of them is 2.

b) The above analysis is unique only if $p_1 = p_2 = p_3 = 2$, in which case we have $2 + 2 + 2 = 3 \cdot 2$. Thus it holds for $m = 2$ which is true for $a \in \mathbb{N}$: $a \cdot 2 = 2 + 2 + \cdots + 2$, sum of a numbers.

c) Moreover it follows that for every $a \in \mathbb{N}$ there is no additive analysis for the multiple of am for $m = 1$, i.e., for a itself, as the sum of a prime numbers. Thus we will assume in general that $m \geq 2$.

2.2. Case $a = 4$

We have (from the division $4m/3$) the case $4m = 3n + 1$, $4m = 3n + 2$, and $4m = 3n$.

(4.*i*) $4m = 3n + 1 = p_1 + p_2 + p'_3 + 1$, where p_1, p_2, p'_3 are prime numbers of an additive analysis of $3n$ and, as before, the case for which $p_1 = p_2 = p'_3 = 2$, is excluded. At least one of them (more precisely two) will be odd number. Let such a number be p'_3. Then $p'_3 + 1$ is even and thus $p'_3 + 1 = p_3 + p_4$ is sum of two prime numbers. Hence $4m = p_1 + p_2 + p_3 + p_4$ is an additive analysis of $4m$ into four additive prime numbers.

Example 2.3.

$$\begin{aligned} 4 \cdot 4 &= 16 = 3 \cdot 5 + 1 = (2 + 2 + 11) + 1 = 2 + 2 + 5 + 7 \\ &= 3 + 3 + 3 + 7 = 3 + 3 + 5 + 5. \end{aligned}$$

$$\begin{aligned} 4 \cdot 7 &= 28 = 3 \cdot 9 + 1 = 2 + 2 + 5 + 19 = 3 + 3 + 3 + 19 \\ &= 3 + 3 + 5 + 17 = 5 + 5 + 5 + 13 = 7 + 7 + 7 + 7. \end{aligned}$$

(4.*ii*) $4m = 3n + 2 = p_1 + p_2 + p_3 + 2$ is the sum of four primes one of which is 2. We have then that (excluding the case $p_1 = p_2 = p_3 = 2$ that holds for $4m = 4 \cdot 2$, i.e., $m = 2$) since the sum $p_1 + p_2 + p_3$ is even with three additives, one of them must be 2. Let such a number be $p_3 = 2$. Finally we have $4m = p_1 + p_2 + 2 + 2$ being the sum of four prime numbers, not only in this form (where two additives being 2), as we have the following examples.

Example 2.4.

$$\begin{aligned} 4 \cdot 5 &= 20 = 3 \cdot 6 + 2 = 2 + 2 + 3 + 13 \\ &= 2 + 2 + 5 + 11 = 3 + 3 + 7 + 7. \end{aligned}$$

$$\begin{aligned} 4 \cdot 8 &= 32 = 3 \cdot 10 + 2 = 2 + 2 + 5 + 23 = 2 + 2 + 11 + 17 \\ &= 3 + 3 + 7 + 19 = 3 + 3 + 3 + 23 = 5 + 5 + 5 + 17 = 7 + 7 + 7 + 11. \end{aligned}$$

(4.iii) $4m = 3n = 3(n-1)+3 = p_1+p_2+p_3+3$ is the additive analysis of $4m$ as sum of four primes one of which is 3. Moreover, we have

$$4m = 3n = 12m' = (12m'-3)+3 = 3(4m'-1)+3 = p_1+p_2+p_3+3.$$

However, there are analysis of $4m$ without necessarily one of the four additives being 3. Indeed, we have

$$\begin{aligned} 4m = 3n &= 12m' = (12m'-2)+2 = 2(6m'-1)+2 = p_1'+p_2'+2 \\ &= (p_1'+1)+(p_2'+1) = p_1+p_2+p_3+p_4, \end{aligned}$$

excluding the case for which $p_1' = p_2' = 2$. The numbers p_1', p_2' are odd primes and thus the sums $p_1'+1, p_2'+1$ are even numbers.

Example 2.5.

$$\begin{aligned} 4\cdot 6 &= 3\cdot 8 = 12\cdot 2 = (12\cdot 2-2)+2 = 2(6\cdot 2-1)+2 = (19+3)+2 \\ &= (17+5)+2 = (11+11)+2 = (19+1)+(3+1) \\ &= (17+1)+(5+1) = (11+1)+(11+1) = 20+4 = 18+6 \\ &= 12+12 = (17+3)+(2+2) = (13+7)+(2+2) \\ &= (13+5)+(3+3) = (11+7)+(3+3) = (5+7)+(5+7) \\ &= 2+2+3+17 = 2+2+7+13 = 3+3+5+13 \\ &= 3+3+7+11 = 5+5+7+7. \end{aligned}$$

Furthermore, we have $3+3+5+13 = 3+5+5+11$ and $2+2+3+17 = 3+7+7+7$. Finally, we obtain

$$\begin{aligned} 4\cdot 6 &= 2+2+3+17 = 2+2+7+13 \\ &= 3+3+5+13 = 3+3+7+11 \\ &= 5+5+7+7 = 3+5+5+11 = 3+7+7+7. \end{aligned}$$

2.3. Case $a = 5$

We distinguish the following cases according to the remainders of the division $5m/4$:

$$5m = 4n+1,\ 5m = 4n+2,\ 5m = 4n+3,\ 5m = 4n = 20m'.$$

We assume an additive analysis of $4n$ as the sum of four primes. Excluding the case $4n = 2+2+2+2$, we have

(5.i) $5m = 4n+1 = (p_1+p_2+p_3+p_4')+1 = p_1+p_2+p_3+(p_4'+1) = p_1+p_2+p_3+p_4+p_5$, $p_4'+1 = p_4+p_5$ sum of two primes as an even number (p_4' is an odd number).

(5.*ii*) $5m = 4n + 2 = (p_1 + p_2 + p_3 + p_4) + 2$ is the sum of five prime numbers one of which being 2.

(5.*iii*) $5m = 4n + 3 = (p_1 + p_2 + p_3 + p_4) + 3$ is the sum of five prime numbers one of which being 3 (or and another analysis as in the case (5.*i*)).

(5.*iv*) $5m = 4n = 20m' = (20m' - 4) + 4 = p'_1 + p'_2 + 4 = (p'_1 + 1) + (p'_2 + 1) + 2 = p_1 + p_2 + p_3 + p_4 + 2$ as in the case (5.*ii*) (where $20m' - 4 = p'_1 + p'_2$, sum of two primes being an even number).

Remark 2.2.

By considering the case $a = 5$ (and the previous) it is obvious that instead of examining separately each of the cases 0, 1, 2, 3, 4 of the remainder from the division $5m/4$, it is enough to consider the cases of even $2k$ and odd $2k + 1$ remainders. That is, for the cases

$$5m = 4n + 2k, \text{ and } 5m = 4n + 2k + 1.$$

3. Goldbach's conjecture implies generalized Goldbach's conjecture

Next, we proceed inductively in order to generalize the additive analysis of every positive multiple of am as the sum of a prime numbers for $a \in \mathbb{N} \setminus \{2\}$ according to the previous facts and assuming the validity of Goldbach's conjecture. We assume the division (1) and distinguish its remainders for the multipliers $m \in \mathbb{N}$ with respect to even $2k$ and prime $2k + 1$ numbers. We have two cases, a being an even number and a being an odd number.

I. ***a*** **even.** We have the cases

$$\begin{aligned} am &= (a-1)n + 2k \text{ and} \\ am &= (a-1)n + 2k + 1. \end{aligned}$$

i. Let $am = (a-1)n + 2k$. It is obvious that am and $2k$ are even numbers and thus $(a-1)n$ is even. According to the induction hypothesis we have $(a-1)n = p_1 + \cdots + p_{a-1}$ is a sum of additives prime numbers where their sum $(a-1)n$ is even and thus

$$am = (p_1 + \cdots + p_{a-1}) + 2k.$$

Because the number of the additives is odd and their sum is even, it follows that at least one of $p_1, \ldots, p_{a-1}$, is even, and more precisely being 2. Let such a number be p_{a-1}. Then,

$$am = (p_1 + \cdots + p_{a-2}) + 2 + 2k.$$

The sum $2+2k$ is even and according to Goldbach's conjecture, $2+2k = p+q$ is the sum of two prime numbers. Hence

$$\begin{aligned} am &= (p_1 + \cdots + p_{a-2}) + p + q \\ &= p_1 + \cdots + p_{a-2} + p_{a-1} + p_a, \end{aligned}$$

sum of a additive primes for each of its multiplier.

Remark 3.1.

a) We observe that in the two assumed additive analysis of $(a-\ 1)n$ and am as sum of primes, $a-2$ additives are common. These two additive analysis where the one of am follows from that of $(a-1)n$ with the above procedure, are called *corresponding.*

b) In the above case of the additive analysis

$$(a-1)n = p_1 + \cdots + p_{a-1}$$

we can have not only one but an odd number of the additives being equal to 2, depending of course by the multiplier m. Because $a-1$ is odd, in the special case where each of the additives is 2, we have $(a-1)n = (a-1)2$, and thus $n = 2$. By $am = 2(a-1)+2k$ assumed as division $am/2$, we have $2k = 2$, and thus $am = 2(a-1)+2 = 2a$ and $m = 2$.

c) In the case for which we have $am = (a-1)n + 2k$ and $k = 0$ then (likewise in the special cases for $a = 3, 4, 5$) we have

$$am = (a-1)n = a(a-1)n'$$

(because $a-1$ is prime with respect to a and divides m so that $m = (a-1)n'$). Hence,

$$\begin{aligned} am &= (a-2)an' + an' \\ &= p_1 + \cdots + p_{a-2} + an'. \end{aligned}$$

Since an' is even, we have $an' = p+q$, and thus again

$$am = p_1 + \cdots + p_{a-2} + p + q,$$

is sum of a prime additives.

ii. Let $am = (a-1)n + 2k + 1$. According to the induction hypothesis we have $(a-1)n = p_1 + \cdots + p_{a-1}$ and thus

$$am = p_1 + \cdots + p_{a-1} + 2k + 1,$$

where we have an odd number of additives (since am is even for every m). Then at least one of the additives is odd (or an odd number of them). Let such a number be p_{a-1}. Then the sum $p_{a-1}+2k+1$ is even and thus $p_{a-1}+2k+1=p+q$. Hence,

$$\begin{aligned} am &= p_1+\cdots+p_{a-2}+p+q \\ &= p_1+\cdots+p_{a-2}+p_{a-1}+p_a, \end{aligned}$$

is the sum of a prime additives.

II. a **odd** We have again the cases

$$\begin{aligned} am &- (a-1)n+2k \quad \text{and} \\ am &= (a-1)n+2k+1. \end{aligned}$$

It is obvious that the first case occurs only for the even multipliers of a and the second one only for the odds (because $a-1$ is an even number). Moreover we have in both cases by the induction hypothesis the additive analysis

$$(a-1)n=p_1+\cdots+p_{a-1}$$

with an even number of prime additives.

i. Let $am=(a-1)n+2k$ with $m=2m'$ (even). We exclude the case for which $p_1=\cdots=p_{a-1}=2$, so that

$$am=(a-1)2+2k$$

and as before we have $k=1$, that is, we exclude the case $am=(a-1)2+2$, so that $m=2$. For every other even multiple of a we have that at least two of the additives $p_1,\ldots,p_{a-1}$ are odd numbers (or an even number of them less than $a-1$). Let them be p_{a-2},p_{a-1}. Then we have,

$$\begin{aligned} am &= (p_1+\cdots+p_{a-3})+p_{a-2}+p_{a-1}+(2k-2)+2 \\ &= p_1+\cdots+p_{a-3}+2+(p_{a-2}+p_{a-1}+2k-2) \\ &= p_1+\cdots+p_{a-3}+2+p+q, \end{aligned}$$

where $p+q$ is an analysis of the even number $p_{a-2}+p_{a-1}+2k-2$ as sum of two primes according to Goldbach's conjecture. Hence am is written as the sum of a prime numbers.

Remark 3.2.

a) As in the corresponding case for even a, in the additive analysis of $(a-1)n$ an even number of additives can be odd numbers. For instance,

$$\begin{aligned} am=7\cdot 10=70 &= 6\cdot 11+2+2 \\ &= 3+5+13+13+17+17+2 \\ &= 2+2+13+13+19+19+2 \\ &= 2+2+2+23+13+13+13+2. \end{aligned}$$

b) In every analysis of an even multiplier of an odd number as the sum of primes, at least one of the additives is 2.

c) If $am = (a-1)n$ (a odd, m even, $k = 0$), then we have

$$am = (a-3)n + 2n = (a-3)n + 2(n-1) + 2.$$

Hence

$$am = p_1 + \cdots + p_{a-3} + p + q + 2,$$

is the sum of a prime numbers ($2(n-1) = p + q$, since $n > 1$).

ii. Let $am = (a-1)n + 2k + 1$ with m being an odd number. Then, by the induction hypothesis,

$$am = p_1 + \cdots + p_{a-1} + 2k + 1$$

where the sum $p_1 + \cdots + p_{a-1}$ is an even number. If $p_1 = \cdots = p_{a-1} = 2$ then

$$\begin{aligned} am &= (a-3)2 + 2 + 2 + 2k + 1 \\ &= (a-3)2 + 3 + (2 + 2k) \\ &= (a-3)2 + 3 + p + q \\ &= (2 + \cdots + 2) + 3 + p + q. \end{aligned}$$

Otherwise, there exists an even number of additives different than 2. If p_{a-1} is one of them, we have

$$\begin{aligned} am &= (p_1 + \cdots + p_{a-2}) + p_{a-1} + 2k + 1 \\ &= p_1 + \cdots + p_{a-2} + p + q, \end{aligned}$$

which is the sum of a prime numbers, because the number $p_{a-1} + 2k + 1$ is even.

From the previous facts we obtain the following fundamental theorem (that we already mentioned in the Introduction).

Theorem 3.1. *The axiomatic acceptance of Goldbach's conjecture implies the validity of the generalized conjecture.*

From the proof of the theorem and as a starting point the division $am/(a-1)$, i.e., the relationship

$$am = (a-1)n + r,$$

and according to Goldbach's conjecture, we devise several properties of the generalized conjecture, some of which are mentioned in their proper positions but others have not been mentioned. For that reason we conclude all of them in the following proposition.

Proposition 3.1. *In every additive analysis of* $(a-1)n$ *as a sum of* $a-1$ *additive primes corresponds an analysis of* am *as the sum of* a *prime numbers from which the* $a-2$ *are common in both those analysis, except the case* a *being odd and* $r=2k$ *for which the common additives are* $a-3$. *In such a case at least two of the even number of prime additives of the analysis of* $a-1$ *are different than* 2 (*possible all of them*)*. In such an additive analysis of* $a-1$ *if* a *is even and* $r=2k$ *then one of the additives is* 2 .

Remark 3.3. Therefore by Goldbach's conjecture we conclude not only the generalized Goldbach's conjecture but we obtain also the properties of Proposition 3.1.

Corollary 3.1. *By the acceptance of Goldbach's conjecture we have that every integer* $a\in\mathbb{N}$ *admits a number of additives analysis into the sum of primes which is equal to the number of its divisibles with the number of additives in each one* (*not necessarily unique*) *as the corresponding divisor.*

Example 3.1. Let $a=12$. Then the divisors are $(1),2,3,4,6,(12)$.

$$\begin{array}{llll} \text{For } a &= 2 & \text{we have} & 12=5+7. \\ \text{For } a &= 3 & \text{we have} & 12=2+3+7=2+5+5. \\ \text{For } a &= 4 & \text{we have} & 12=2+2+3+5=3+3+3+3. \\ \text{For } a &= 6 & \text{we have} & 12=2+2+2+2+2+2. \end{array}$$

4. Generalized Goldbach's conjecture implies Goldbach's conjecture

For the converse, now we examine whether the axiomatic acceptance of the generalized Goldbach's conjecture and Proposition 3.1 for numbers $a\in\mathbb{N}\setminus\{2\}$ imply Goldbach's conjecture.

In order to show the validity of the generalized Goldbach's conjecture we focused in each of the cases of the relationship

$$am=(a-1)n+r,\ 0\leq r<a-1,$$

for a even with $r=2k$ and $r=2k+1$, and for a odd with $r=2k$ and $r=2k+1$. We also proceed in a similar fashion.

Let $2s$ be a positive multiplier of 2 with $s\neq 1$ and let an even integer a such that $a>2s+1$. We consider a proper multiple am of a so that the remainder of the division $am/(a-1)$ is $2(s-1)$ (for instance, this is determined with an indefinite analysis of the equation $ax-(a-1)y=2(s-1)$). If n is the quotient, we have

$$am=(a-1)n+2(s-1).$$

Because the product $(a-1)n$ is an even number (a is even and $2(s-1)$ is even), the additive analysis of $(a-1)n=p_1+\cdots+p_{a-1}$ has an odd number of prime additives one of which is 2. Let such a number be p_{a-1}. Thus we have

$$am=p_1+\cdots+p_{a-2}+2+2(s-1)=p_1+\cdots+p_{a-2}+2s.$$

Since am is analyzed into a sum of a prime additives from which $a-2$ are according to Proposition 3.1 additives of the additive analysis of $(a-1)n$ into $a-1$ primes, the rest two primes p and q of the analysis of am are expressed by the remainder $2s$. This means that

$$am = p_1 + \cdots + p_{a-2} + p + q$$

and thus

$$2s = p + q,$$

which is exactly Goldbach's conjecture. Therefore we prove the following important theorem.

Theorem 4.1. *If for the positive multiples of any two numbers* $a,\ a-1 \in \mathbb{N} \setminus \{2\}$, *the generalized Goldbach's conjecture holds, then so does the Goldbach's conjecture.*

5. Proofs of the conjectures for some cases

The generalized Goldbach's conjecture opens a new direction for proving Goldbach's conjecture. Indeed it remains to prove that the positive multiples of any two consecutive integers a and $a-1$ can be analyzed as sums of a and $a-1$, respectively, additive primes. It is natural to expect that the effort begins with the numbers of the smallest pair $(4, 3)$ of consecutives $(a, a-1)$. The smaller pair $(3, 2)$ is excluded from the effort, since analyzing its numbers implies the proof for Goldbach's conjecture. Thus we have,

5.1 Case $a = 3$

In order to apply induction we let at the beginning multiples of 3 with additive analysis $p_1 + p_2 + p_3$ in which two of the three prime numbers to be the smallest pairs of integers $(2, 2), (2, 3)$.

$$\begin{aligned} 3 \cdot 3 &= 3(2+1) = 9 = 2+2+5 \\ 3 \cdot 4 &= 3(2 \cdot 2) = 12 = 2+3+7 \\ 3 \cdot 5 &= 3(2 \cdot 2+1) = 15 = 2+2+11 \\ 3 \cdot 6 &= 3(2 \cdot 3) = 18 = 2+3+13 \end{aligned}$$

And generalizing by induction:

$$\begin{aligned} 3(2m+1) &= 2+2+p \ (p \text{ prime, thus } m \geq 1) \\ 3(2m) &= 2+3+p \ (p \text{ prime, thus } m \geq 2) \end{aligned}$$

Furthermore, we obtain

a) $3(2m+1) = 2+2+p \Rightarrow p = 6m+3-4 = 6m-1$, true.

Because, as it is known from Number Theory, *every positive prime number different than* 2 *and* 3 *is of the form* mul$6+1$ *or* mul$6-1$. The converse however does not hold[2] (for instance, $4 \cdot 6+1$, $6 \cdot 6-1$ are not prime numbers).

For the converse now, for each positive prime p of the form $6m-1$ corresponds a (positive) odd multiple of 3 with an additive analysis $2+2+p$. Indeed,

$$p = 6m-1 \Rightarrow 6m = p+1 \Rightarrow 6m+3 = p+4$$

and thus

$$3(2m+1) = 2+2+p.$$

Moreover every odd multiple of 3 does not admit the additive analysis $2+2+p$ where p is prime. In order to admit such an analysis, we need the difference

$$3(2m+1)-(2+2) = 6m+3-4 = 6m-1$$

to be prime number. For instance, for $3(2m+1) = 3(2\cdot2+1)$ we have $6\cdot2-1 = 11$ which is a prime number. And thus

$$3(2\cdot2+1) = 15 = 2+2+11.$$

Whereas for $3(2m+1) = 3(2\cdot6+1)$ we have $6\cdot6-1 = 35$ which is a non-prime number and hence the multiple $3(2\cdot6+1)$ of 3 does not admit such an analysis $2+2+p$ where p is prime. Therefore we have the following theorem.

Theorem 5.1. *The class of the odd multiples* $3(2m+1)$ *of* 3 *for which the number* $6m-1 = p$ *is prime, verifies the generalized Goldbach's conjecture with an additive analysis* $2+2+p$ *for each such a multiple.*

Related to the odd multiples of 3, we observe that every such number is odd and for which only one class of numbers according to the theorem admits an additive analysis $2+3+p$ for p prime. Thus we conclude for the odd numbers the following theorem.

Theorem 5.2. *The class of odd numbers of the form* $3(2m+1)$ *for which the number* $6m-1 = p$ *is prime, verifies the Goldbach's conjecture* (*for the odd numbers*) *with an additive analysis* $2+3+p$ *for each such number.*

Indeed

$$3(2m+1) = 6m+3 = (6m-1)+4 = p+2+3.$$

For instance, for $3(2m+1) = 3(2\cdot1+1)$ we have $6\cdot1-1 = 5 = p$ prime number and thus $3(2\cdot1+1) = 9 = 2+2+5$. Whereas for $3(2m+1) = 3(2\cdot6+1)$ we have $6\cdot6-1 = 35$ non-prime and thus the number $3(2\cdot6+1) = 39$ does not

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[2] Because one of the numbers $p-1, p, p+1$ is divisible by 3 we have either $p+1 =$mul3 or $p-1 =$mul3, since p is prime and different than 3. Moreover, each of the $p-1$, $p+1$ is even and if it is also multiple of 3 then it is also multiple of 6. Thus we have $p =$mul6+1 or $p =$mul6$-$1.

admit an additive analysis $2+2+p$, for p prime. It admits however according to Goldbach's conjecture, other kind of analysis; for instance,

$$\begin{aligned} 39 &= 3+5+31 = 3+7+29 = 3+13+23 \\ &= 3+17+19 = 5+5+29 = \\ &= 5+11+23 = 5+17+17 = 7+13+19 \\ &= 17+11+11 = 13+13+13. \end{aligned}$$

b) $3(2m) = 2+3+p \Rightarrow p = 6m-5 = (6m-6)+1 = 6(m-1)+1$ true,

according to the above facts. For the converse, for every positive prime number of the form $p = 6m+1$ there is an even multiple of 3 with an additive analysis $2+3+p$. Indeed,

$$p = 6m+1 \Rightarrow 6m = p-1 \Rightarrow 6m+6 = p+5$$

and thus

$$3\,(2(m+1)) = 2+3+p.$$

In a similar fashion with the case above, every even multiple of 3 does not admit an analysis of the form $2+3+p$. In order to admit such an analysis, we need the difference

$$\begin{aligned} &3(2m)-(2+3) \qquad \text{or} \\ &6m-5 = (6m-6)+1 = 6(m-1)+1 \end{aligned}$$

to be a prime number. For instance, for $3(2m) = 3(2\cdot 3)$ we have $6\cdot 2+1 = 13$ prime number and thus

$$3(2\cdot 3) = 18 = 2+3+13.$$

Whereas for $3(2m) = 3(2\cdot 9)$ we have $6\cdot 8+1 = 49$ non-prime number and thus the multiple $3(2\cdot 9)$ of 3 does not admit the analysis $2+3+p$ where p is a prime number.

Therefore in correspondence with Theorem 5.1, we have the following:

Theorem 5.3. *The class of the even multiples* $3(2m)$ *of* 3 *for which the number* $6(m-1)+1 = p$ *is prime, verifies the generalized Goldbach's conjecture with an additive analysis* $2+3+p$ *for each such a multiple.*

Suppose now the even multiple $3(2m)$ of 3 admits an additive analysis $2+3+p$ where p is prime, i.e., the relationship $3(2m) = 2+3+p$ with $p = 6(m-1)+1$ form the previous theorem. By such an equivalence we have that $2(3m) = 5+p$ which means that we refer to Goldbach's conjecture with the following important theorem:

Theorem 5.4. *The class of the even number of the form* $2(3m)$ *for which the number* $6(m-1)+1 = p$ *is prime, verifies Goldbach's conjecture with an additive analysis* $5+p$ *for each such a number.*

Indeed

$$2(3m) = 6m = (6(m-1)+1) + 5 = 5 + p.$$

For instance, for $2(3m) = 2(3\cdot 3)$ we have $6(3-1)+1 = 13 = p$ prime number and thus $2(3\cdot 3) = 18 = 5+13$. Whereas for $2(3m) = 2(3\cdot 5)$ we have $6(5-1)+1 = 25$ non-prime number and the number $2(3\cdot 5) = 30$ does not admit the analysis $5+p$ but according to Goldbach's conjecture it admits several others (in the considered case we have $30 = 7+23 = 11+19$).

Especially for the even multiples with an additive analysis $2+3+p$ where only one of the added numbers is 2, we observe that this is a general property of the even multiples, since with the acceptance of the generalized Goldbach's conjecture we have for every $m \geq 2$,

$$3(2m) = p + q + u.$$

One of the odd number additive primes must be even number, i.e., equal to 2, since for otherwise their sum would have been odd number and not even $3(2m)$. Let such a number be u. Then

$$3(2m) = 2 + p + q.$$

This implies that for the next odd multiple we have

$$3(2m+1) = 3(2m) + 3 = 5 + p + q$$

so that it admits an analysis of sum of three primes. This particular implies that in order to prove the generalized conjecture for 3, it suffices to show the analysis of every multiple of 3 as the sum of three prime numbers. We deduce that this equals with the Goldbach's conjecture. Therefore it also remains as an open problem waiting for its answer.

5.2. Case $a = 4$

In general, the analysis of the multiples $4m$ of 4 remains as an open problem.

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STRONG COLOURINGS OF HYPERGRAPHS

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Abstract. We define a new method of colouring for a hypergraph, in particular for a graph. Such a method is as usual meant as a partition of a hypergraph, in particular of a graph. However, it is more intrinsically linked to the geometric structure of the hypergraph and therefore enables us to obtain stronger results than in the classical case. For instance, we prove theorems concerning 3-colourings, 4-colourings and 5-colourings, while we have no analogous results in the classical case. Moreover, we prove that there are no semi-hamiltonian regular simple graphs of positive degree admitting a hamiltonian 1-colouring. Finally, we characterize the above graphs admitting a hamiltonian 2-colouring and a hamiltonian 3-colouring.

1. Introduction

A hypergraph [2] is a pair $(\mathcal{S}, \mathcal{B})$ where $\mathcal{S}$ is a non-empty finite set whose elements we call *vertices* and $\mathcal{B}$ is a non-empty family of non-empty subsets of $\mathcal{S}$, whose elements we call *edges*, such that $\mathcal{B}$ is a covering of $\mathcal{S}$. We denote by $\deg P$, degree of P, the number of edges through the vertex P. A hypergraph is also called *geometric space*. In this case, the vertices are called *points* and the edges are called *blocks*.

Let $|\mathcal{S}| = v, |\mathcal{B}| = b$. From now on we adopt the terminology of the geometric spaces, taking into account that it can be immediately translated into the language of the hypergraphs.

Let

$$\begin{aligned} r &= \max_{P \in \mathcal{S}} \deg P, \\ k &= \min_{B \in \mathcal{B}} |B|, \\ k' &= \max_{B \in \mathcal{B}} |B|. \end{aligned}$$

Let $\mathcal{I} = \{1, 2, \ldots, v\}$ and φ be a bijection $\varphi : \mathcal{I} \longrightarrow \mathcal{S}$.

A block B gives rise to the set $\{\varphi^{-1}(P)\}_{P\in B}=\left\{n_1,n_2,...,n_{|B|}\right\}$, with $n_1<n_2<\cdots<n_{|B|}$.

We call *i-th point* of B, i=1, 2, ..., $|B|$, the point $P\in B$ such that $\varphi^{-1}(P)$=n_i.

For every $j=0,1,\ldots,r$ and for every $i=1,2,...,k'$, we get the set

$$I_\varphi(j,i)=\left\{\begin{array}{c} P\in\mathcal{S}:\text{there are } j \text{ blocks through } P\\ \text{such that } P \text{ is their } i\text{-th point}\end{array}\right\}$$

For any i, $1\le i\le k'$, we get the set of indices

$$J_\varphi(i)=\{j,\ 0\le j\le r: I_\varphi(j,i)\neq\emptyset\}\,.$$

Obviously the family $\{I_\varphi(j,i)\}_{j\in J_\varphi(i)}$ is a partition of $\mathcal{S}$.

We call the pair

$$\left(\{I_\varphi(j,i)\}_{j\in J_\varphi(i)}\,,J_\varphi(i)\right)$$

strong colouring of base φ *and index* i *of the geometric space* $(\mathcal{S},\mathcal{B})$ or simply *strong colouring* of ($\mathcal{S}$,$\mathcal{B}$) and we denote it by $c\,(\varphi,i)$. The indices $j\in J_\varphi(i)$ are called the *colours of* $c\,(\varphi,i)$, hence every vertex of $I_\varphi(j,i)$ is said to have the colour j.

Now let $(\mathcal{S},\mathcal{B})$ be a graph $G=(V(G),E(G))$. Then $\mathcal{S}=(V(G)$, $\mathcal{B}=E(G)$, $v=|V(G)|$, $s=|E(G)|$, $k=k'=2$, $i\in\{1,2\}$.

We call strong colouring of a graph G the colouring $c\,(\varphi,i)$ just defined for the geometric space. Thus, every bijection gives rise to two strong colourings, since $i=\{1,2\}$. According to this definition, the colour of a vertex V, that is the number of edges through V admitting V as i-th vertex, is determined by the geometric structure of the graph around V and consequently we get deeper results than in the classical case, where the colour of a vertex is arbitrarily assigned, with the only condition that two vertices have different colours. The following results hold.

- If G is a simple graph, that is a graph without loops and multiedges, every strong colouring of G has the colour $j=0$.
- A simple graph G is strongly 1-colorable, if and only if, G is a null graph (that is $E(G)=\emptyset$).
- A regular simple graph is strongly 2-colorable if, and only if, G is a bipartite graph.
- If G is a regular simple graph, of degree $r=2p$, p a prime, $v=|G|$ even, $v<2r$, strong 3-colourings of G do not exist.

A graph G is called *semi-hamiltonian,* if it contains a path through all the vertices of G, called *semi-hamiltonian path*.

If the path is closed, the graph G is called *hamiltonian.*

Consider the following semi-hamiltonian path $\ell=V_1\to V_2\to\cdots\to V_v$.

We define the bijection $\varphi_\ell: n\in\mathcal{I}=\{1,2,\ldots,v\}\longrightarrow V_n\in V(G)$. For any $i\in\{1,2\}$ we get the strong colouring $c\,(\varphi_\ell,i)$, which we call *strong hamiltonian colouring of index* i *associated with the path* ℓ.

Let G denote a semi-hamiltonian regular simple graph of positive degree. We prove that the only graph G admitting a hamiltonian strong 2-colouring is K_2. The only graphs G admitting a hamiltonian strong 3-colouring, are the circuit-graphs. If G has a hamiltonian strong 4-colouring, then $r \geq 3$ and the colours of $c\left(\varphi_\ell, i\right)$, are $0, 1, r-1, r$. Moreover, the number of vertices of color 1 equals the number of vertices of color $r-1$, which is $\dfrac{v}{2}-1$, hence v is even.

The following theorem holds:

Theorem 1 (cubic simple semi-hamiltonian graphs theorem)**.** *If G is a simple regular semi-hamiltonian graph with* $\deg G = 3$ *and if $c\left(\varphi_\ell, i\right)$ is a hamiltonian strong colouring of G, then $c\left(\varphi_\ell, i\right)$ is a strong* 4*-colouring with colours* $0, 1, 2, 3$. *Moreover the number of vertices of colour* 1 *equals the number of vertices of color* 2*, which is $v/2-1$. Hence v is even.*

Finally if G has a hamiltonian strong 5-colouring, we get $r \geq 4$ and the colours of $c\left(\varphi_\ell, i\right)$ are $0, 1, j, r-1, r$, $1 < j < r-1$.

The number of vertices of colour 1 *and the number of vertices of colour $r-1$ are both less than $\dfrac{v}{2}-1$. If v is even, there are at least two vertices of colour j and, if such vertices are two, we get $j = \dfrac{r}{2}$, hence r is even.*

Moreover, the number of vertices of colour 1 *and the number of vertices of colour $r-1$ are both equal to $\dfrac{v}{2}-2$.*

2. Strong colourings of a geometric space

Let $(\mathcal{S}, \mathcal{B})$ be a finite geometric space and $c(\varphi, i)$ a strong colouring of $(\mathcal{S}, \mathcal{B})$, that is the pair $\left(\{I_\varphi(j,i)\}_{j \in J_\varphi(i)}, J_\varphi(i)\right)$. The indices $j \in J_\varphi(i)$ are the colours of $c(\varphi, i)$. We say that $j \in J_\varphi(i)$ is the colour of $I_\varphi(j,i)$ and that $P \in I_\varphi(j,i)$ has the colour j.

Obviously the number of colours $|J_\varphi(i)|$ satisfies the condition $1 \leq |J_\varphi(i)| \leq r+1$. For any integer k, $1 \leq k \leq r+1$, we say that $(\mathcal{S}, \mathcal{B})$ is strongly k-colourable, if there is a strong colouring $c(\varphi, i)$ of $(\mathcal{S}, \mathcal{B})$ with k colours. Such $c(\varphi, i)$ is called *strong k-colouring* of $(\mathcal{S}, \mathcal{B})$.

Let $t(j,i) = |I_\varphi(j,i)|$. Obviously

$$\sum_{j=0}^{r} t(j,i) = v, \quad i = 1, 2, \ldots, k'. \tag{1}$$

Moreover we get:

$$\sum_{j=0}^{r} j t(j,i) = b, \quad \forall i = 1, 2, \ldots, k. \tag{2}$$

We remark that (1) and (2) hold for any bijection $\varphi : I \to \mathcal{S}$.

3. The strong colourings of a graph

Let us prove the following

Theorem 2. *Let G be a simple graph, then every strong colouring of G has the colour $j = 0$.*

Proof. If G is the null graph, the theorem is obvious. Then assume that G is not the null graph and then it has two distinct vertices. Let $c\,(\varphi, i)$ be a strong colouring of G. Let V_M and V_m be the vertices such that $\varphi^{-1}(V_M) = |G| = v$, $\varphi^{-1}(V_m) = 1$. Such vertices are distinct, since $|G| \geq 2$. If $i = 1$, there is no edge through V_M admitting V_M as first vertex, therefore $V_M \in I_\varphi(0, 1)$ and so $I_\varphi(0, 1) \neq \emptyset$. It follows that $j = 0 \in J_\varphi(1)$. If $i = 2$, there is no edge through V_m admitting V_m as second vertex, therefore $V_m \in I_\varphi(0, 2)$ and so $I_\varphi(0, 2) \neq \emptyset$. It follows $j = 0 \in J_\varphi(2)$. ■

Theorem 3. *A simple graph G is strongly 1-colourable if, and only if, G is the null graph.*

Proof. Obviously, if G is the null graph, it is strongly 1-colourable, with the colour $j = 0$. Conversely, let G be strongly 1-colorable and let $c\,(\varphi, i)$ be a strong 1-colouring of G. Then, by Theorem 2, the colour of $c\,(\varphi, i)$ is $j = 0$. Assume now G is not the null graph. Then in G there is an edge $\{V', V''\}$. In this case either V', or V'' cannot have the colour 0, a contradiction. ■

The following theorem holds.

Theorem 4. *Let G be a non-null simple graph and let $c\,(\varphi, i)$ be a strong colouring of G. Then there is at least a colour $j \neq 0$ of $c\,(\varphi, i)$ such that $j \leq |I_\varphi(0, i)|$.*

Proof. By Theorems 2 and 3 it follows that the strong colouring $c\,(\varphi, i)$ has at least two distinct colours and one of them is $j = 0$. Then, there is a vertex V_1 of colour $j \neq 0$ and so $V_1 \notin I_\varphi(0, i)$. Assume that every colour $j \neq 0$ satisfies the condition $j > |I_\varphi(0, i)|$. Then there is an edge $\{V_1, V_2\}$, with $V_2 \notin I_\varphi(0, i)$, which admits V_1 as i-th vertex. Since V_2 has a colour different from zero, there is an edge $\{V_2, V_3\}$, with $V_3 \notin I_\varphi(0, i)$, which admits V_2 as i-th vertex. Moreover we get $V_3 \neq V_1$, since

$$\begin{aligned} \varphi^{-1}(V_1) &> \varphi^{-1}(V_2) > \varphi^{-1}(V_3), \quad \text{if } i = 2, \\ \varphi^{-1}(V_1) &< \varphi^{-1}(V_2) < \varphi^{-1}(V_3), \quad \text{if } i = 1. \end{aligned}$$

Similarly, since V_3 has a colour different from zero, there is an edge V_3, V_4, with $V_4 \notin I_\varphi(0, i)$, which admits V_3 as i-th vertex and such that $V_4 \neq V_1$, $V_4 \neq V_2$, $V_4 \neq V_3$. This procedure continues indefinitely and so the set $V(G) - I_\varphi(0, i)$ is not finite: a contradiction, since G is finite. The contradiction proves that $j > |I_\varphi(0, i)|$, for every colour $j \neq 0$ of $c\,(\varphi, i)$ is impossible. ■

Now let G be a strongly 2-colorable graph and let $c\,(\varphi, i)$ be a strong 2-colouring of G with colours 0 and j, $j \leq r$. Obviously one of the two vertices of an edge ℓ is the i-th vertex of ℓ. It follows that ℓ cannot have both the vertices in $I_\varphi(0, i)$ and that, if both the vertices of ℓ are in $I_\varphi(j, i)$, there is at least a vertex of $I_\varphi(j, i)$ which is the i-th vertex of ℓ. Let $\ell = \{V', V''\}$ with $V' \in I_\varphi(j, i)$ and $V'' \in I_\varphi(0, i)$. Then V' is the i-th vertex of ℓ, since there is no edge admitting

V'' as i-th vertex. It follows that for any such an edge ℓ of G, there is a vertex $V \in I_\varphi(j,i)$, which is the i-th vertex of ℓ. Then, any edge ℓ of G has a vertex $V \in I_\varphi(j,i)$. Thus it follows that $s = j\,|I_\varphi(j,i)|$, as it can be proved also by (2).

So the following theorem holds

Theorem 5. *Let $c(\varphi, i)$ be a strong 2-colouring of a simple graph G. Then the colours of G are 0 and j, $j > 0$, and the following holds:*

- **a)** *two distinct vertices of $I_\varphi(0,i)$ are not adjacent;*
- **b)** *for any edge ℓ of G, there is a vertex of $I_\varphi(j,i)$ which is i-th vertex of ℓ;*
- **c)** $|I_\varphi(j,i)| = \frac{s}{j}$, *where s is the number of edges of G.*

We provide an example of a strongly 2-colorable graph whose colours are $j_1 = 0$ and $j_2 = 3$.

Example 1. (See Figure 1.)

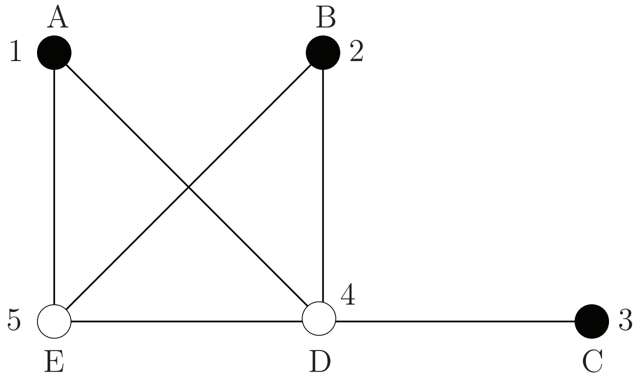

Figure 1:

$$\varphi : (1,2,3,4,5) \to (A,B,C,D,E), \quad I_\varphi(0,2) = \{A,B,C\}, \quad I_\varphi(3,2) = \{D,E\}.$$

We remark that this strong colouring is not classical, since the two adjacent vertices D and E have both the colour 3. Moreover $c(\varphi, 1)$ is a strong 3-colouring of G with colours 0,1,2, since $I_\varphi(0,1) = \{E,D\}$, $I_\varphi(1,1) = \{C\}$, $I_\varphi(2,1) = \{A,B\}$. This confirms that the strong colouring depends on i.

4. Strong colourings of regular simple graphs

A graph is *regular* if all its vertices have the same degree.

Here we consider the strong colourings $c(\varphi, i)$ of a regular simple graph.The following theorem holds

Theorem 6. *A strong colouring $c(\varphi, i)$ of a regular simple graph G of positive degree r has at least the colours $j_1 = 0$ and $j_2 = r$.*

Proof. Let $c(\varphi, i)$ be a strong colouring of a regular simple graph G of degree $r > 0$. Let V_M and V_m be the vertices of G such that $\varphi^{-1}(V_M) = |G| = v$, $\varphi^{-1}(V_m) = 1$. We remark that $V_M \neq V_m$, since $|G| \geq 2$ (we have $|G| \geq 2$, since $r > 0$). Then $I_\varphi(r,1) = I_\varphi(0,2) \neq \emptyset$, since $V_m \in I_\varphi(r,1)$. Moreover $I_\varphi(0,1) = I_\varphi(r,2) \neq \emptyset$, since $V_M \in I_\varphi(0,1)$. It follows that $j_1 = 0$ and $j_2 = r$ are colours of $c(\varphi, i)$. ■

Theorem 7. *Let G be a regular simple graph of positive degree. Then G is strongly 2-colorable if, and only if, G is a bipartite graph $G(\mathcal{V}_1, \mathcal{V}_2)$, with $|\mathcal{V}_1| = |\mathcal{V}_2| = |G|/2$.*

Proof. Let G be strongly 2-colourable and let $c(\varphi, i)$ be a strong 2-colouring of G. By Theorem 6 it follows that the colours of $c(\varphi, i)$ are $j_1 = 0$ and $j_2 = r$. Since the colours are two, we have $I_\varphi(r,1) = I_\varphi(0,2)$, and $I_\varphi(0,1) = I_\varphi(r,2)$. By Theorem 5 it follows that two distinct vertices of $I_\varphi(r,i)$ are not adjacent. By (1) and (2) and since in a regular graph of degree r it is $s = vr/2$, we have

$$t(r,i) = t(0,i) = \frac{v}{2}. \tag{3}$$

Then by the previous arguments, it follows that G is a bipartite graph $G(\mathcal{V}_1, \mathcal{V}_2)$, with $|\mathcal{V}_1| = t(r,i) = |\mathcal{V}_2| = t(0,i) = v/2$.

Conversely, let $G = G(\mathcal{V}_1, \mathcal{V}_2)$ be a bipartite regular simple graph of degree $r > 0$.

Let $\mathcal{V}_1 = \{V_1, V_2, \ldots, V_m\}$, $\mathcal{V}_2 = \{V_{m+1}, V_{m+2}, \ldots, V_v\}$. Let

$$\varphi : n \in \{1, 2, \ldots, v\} \longmapsto V_n \in \mathcal{V}_1 \cup \mathcal{V}_2.$$

The strong colouring $c(\varphi, 1)$ is a strong 2-colouring of G. For, through any vertex $V \in \mathcal{V}_1$ there are r edges admitting V as first vertex (and then all the vertices of $\mathcal{V}_1$ have the colour r) and as a consequence through any vertex $V' \in \mathcal{V}_2$ there is no edge admitting V' as first vertex (and all the vertices of $\mathcal{V}_2$ have the colour 0).

This theorem holds also for the classical colourings of graphs.

An example of a strongly 2-colourable graph of degree 2 (the colours are 0 and 2) is the following.

Example 2. (See Figure 2.)

$$\begin{aligned}
V(G) &= \{A, B, C, D, E, F\}, \\
E(G) &= \{\{A,F\}, \{A,E\}, \{B,F\}, \{B,D\}, \{C,E\}, \{C,D\}\}, \\
\varphi &: (1,2,3,4,5,6) \longrightarrow (A,B,C,D,E,F); \quad i = 2, \\
I_\varphi(0,2) &= \{A, B, C\}, \quad I_\varphi(2,2) = \{D, E, F\}.
\end{aligned}$$

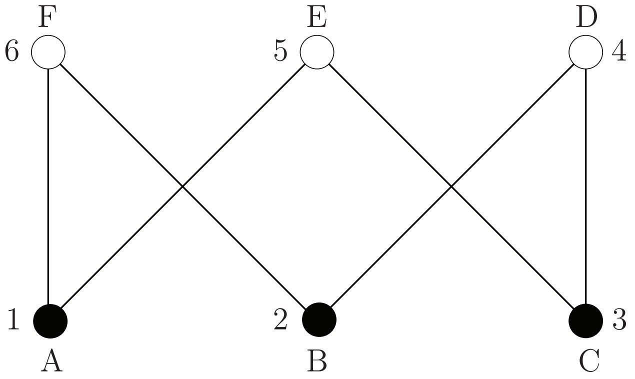

Figure 2:

An example of strongly 2-colorable graph of degree 3 (the colours are 0 and 3) is the following.

Example 3. (See Figure 3.)

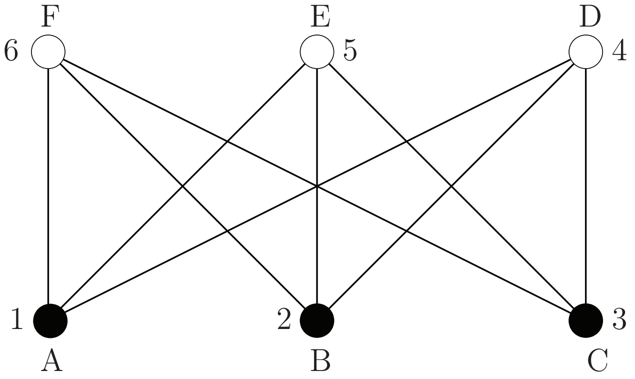

Figure 3:

$$\begin{aligned} V(G) &= \{A,B,C,D,E,F\}, \\ E(G) &= \{\{A,F\},\{A,E\},\{A,D\},\{B,F\},\{B,E\},\{B,D\},\{C,F\},\{C,E\},\{C,D\}\}, \\ \varphi &: (1,2,3,4,5,6) \longrightarrow (A,B,C,D,E,F)\,; \quad i=2 \\ I_\varphi(0,2) &= \{A,B,C\}, \quad I_\varphi(3,2) = \{E,F,D\}. \end{aligned}$$

By the definition of complete graph and by definition of $c\,(\varphi,i)$ the following theorem hold

Theorem 8. *Every strong colouring $c\,(\varphi,i)$ of a complete graph K_n is a strong n-colouring, that is distinct vertices of K_n have different colours.*

This theorem holds also for the classical colourings. Let G be a strongly 3-colorable regular simple graph, of degree $r>0$. Let $c\,(\varphi,i)$ be a strong 3-colouring of G. By Theorem 6, the colours of $c\,(\varphi,i)$ are $0,j,r$ with $0<j<r$. It is

$$c\,(\varphi,i) = \left(\left\{I_\varphi(0,i), I_\varphi(j,i), I_\varphi(r,i)\right\}, \{0,j,r\}\right).$$

Let us prove the following

Theorem 9. *Let G be a regular simple graph of degree $r > 0$. Let $c(\varphi, i)$ be a strong 3-colouring of G. Then the following inequalities hold:*

$$\begin{aligned} r - |I_\varphi(r,i)| &\leq j \leq |I_\varphi(0,i)|, \\ |I_\varphi(0,i)| + |I_\varphi(r,i)| &\geq r, \\ |I_\varphi(j,i)| &\leq v - r. \end{aligned}$$

If in the last inequality the equality holds, then $j = |I_\varphi(j,i)|$.

Proof. Let us prove that $j \leq |I_\varphi(0,i)|$. If $r \leq |I_\varphi(0,i)|$, we get $j < |I_\varphi(0,i)|$. If $r > |I_\varphi(0,i)|$, by Theorem 4 it immediately follows that $j \leq |I_\varphi(0,i)|$. The strong colouring $c(\varphi, i')$ with $i' = \{1,2\} - \{i\}$, has obviously the colours $0, r-j, r$. Therefore $c(\varphi, i') = (\{I_\varphi(0,i'), I_\varphi(r-j,i'), I_\varphi(r,i')\}, \{0, r-j, r\})$, where $I_\varphi(0,i') = I_\varphi(r,i)$, $I_\varphi(r-j,i') = I_\varphi(j,i)$, $I_\varphi(r,i') = I_\varphi(0,i)$. Applying to $c(\varphi, i')$ the arguments of $c(\varphi, i)$, we get

$$r - j \leq |I_\varphi(0,i')| = |I_\varphi(r,i)|. \tag{4}$$

By (4) it follows $j \geq r - |I_\varphi(r,i)|$. Thus

$$r - |I_\varphi(r,i)| \leq j \leq |I_\varphi(0,i)| \tag{5}$$

and so $|I_\varphi(0,i)| + |I_\varphi(r,i)| \geq r$, hence $|I_\varphi(j,i)| \leq v - r$.

If $I_\varphi(j,i) = v - r$ we get $j = |I_\varphi(0,i)|$. ■

Theorem 10. *Let G be a regular simple graph of degree r, r an odd prime, $c(\varphi, i)$ a strong 3-colouring of G, then $|I_\varphi(j,i)| \equiv 0 \bmod r$.*

Proof. By (2) we get:

$$j\,|I_\varphi(j,i)| + r\,|I_\varphi(r,i)| = \frac{vr}{2}. \tag{6}$$

By (6) and since r is odd, it follows

$$j\,|I_\varphi(j,i)| \equiv 0 \bmod r.$$

The integers j and r are coprime, since r is prime and $0 < j < r$. It follows that $|I_\varphi(j,i)| \equiv 0 \bmod r$ and so the theorem is proved. ■

Theorem 11. *Let G be a regular simple graph of degree $r = p^h$, $h \geq 1$, p a prime, $|G| = v$ even, $v \leq 2r$. Let $c(\varphi, i)$ be a strong 3-colouring of G with colours $0, j, r$, $0 < j < r$. We get either:*

i) $v = 2r$, $|I_\varphi(j,i)| = r$, $j = |I_\varphi(0,i)|$,

or

ii) $j = p^{h'}$, $1 \leq h' < h$.

It follows that if $h = 1$, only **i)** *occurs.*

Proof. By (2) it follows

$$j\,|I_\varphi(j,i)| \equiv 0 \bmod r. \tag{7}$$

a) $|I_\varphi(j,i)| = kr$, with k positive integer;

b) $|I_\varphi(j,i)| \neq kr$.

In the case a) we remark that $k = 1$. Assume $k \geq 2$. By (1), since $|I_\varphi(0,i)| \geq 1$, $|I_\varphi(r,i)| \geq 1$, it follows $v \geq 2r+2$, a contradiction, since $v \leq 2r$. Therefore

$$|I_\varphi(j,i)| = r. \tag{8}$$

By (8) and by the third inequality of Theorem 9 we get $r \leq v - r$, that is

$$v \geq 2r, \tag{9}$$

hence

$$v = 2r. \tag{10}$$

By (8) and (10) it follows

$$|I_\varphi(j,i)| = r = v - r. \tag{11}$$

By (11) and Theorem 9 it follows $j = |I_\varphi(0,i)|$.

In the case b), by (7) it follows $\gcd(j,r) \neq 1$, since both the integers j and $r = p^h$ have at least the factor p in common. Then, since $0 < j < r$, it follows $j = p^{h'}$, $1 \leq h' < h$. ■

We provide some examples concerning Theorems 10 and 11.

Example 4. (See Figure 4.)

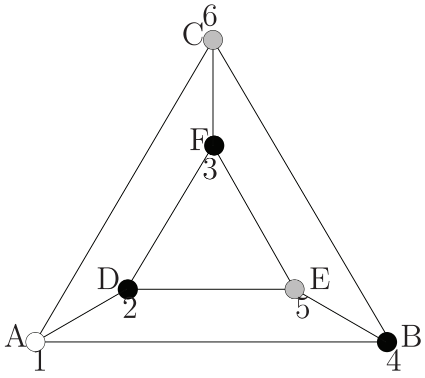

Figure 4:

$$\begin{aligned} V(G) &= \{A,B,C,D,E,F\}, \\ E(G) &= \{\{A,D\},\{A,B\},\{A,C\},\{D,E\},\{D,F\}, \\ &\quad \{F,E\},\{C,F\},\{B,E\},\{B,C\}\}, \\ \varphi &: (1,2,3,4,5,6) \longrightarrow (A,D,F,B,E,C), \quad i=2. \end{aligned}$$

The colouring $c(\varphi,2)$ is a strong 3-colouring of G with colours $0,1,3$. For,

$$I_\varphi(0,2)=\{A\}, \quad I_\varphi(1,2)=\{B,D,F\}, \quad I_\varphi(3,2)=\{C,E\}.$$

This colouring is not classical, since the adjacent vertices D and E have the same colour.

Example 5. (See Figure 5.) This graph G is the complete bipartite graph $K_{3,3}$.

$$\begin{aligned} V(G) &= \{A,B,C,D,E,F\}, \\ E(G) &= \{\{A,D\},\{A,E\},\{A,F\},\{B,D\},\{B,E\}, \\ &\quad \{B,F\},\{C,D\},\{C,E\},\{C,F\}\}, \\ \varphi &: (1,2,3,4,5,6) \longrightarrow (A,B,D,E,F,C), \quad i=1. \end{aligned}$$

The strong colouring $c(\varphi,1)$ is a strong 3-colouring of colours $0,1,3$.

$$I_\varphi(0,1)=\{C\}, \quad I_\varphi(1,1)=\{D,E,F\}, \quad I_\varphi(3,1)=\{A,B\}.$$

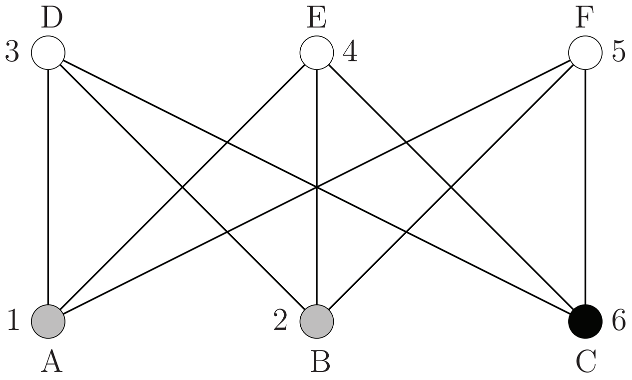

Figure 5:

Example 6. Cubic Petersen Graph. (Figure 6.)

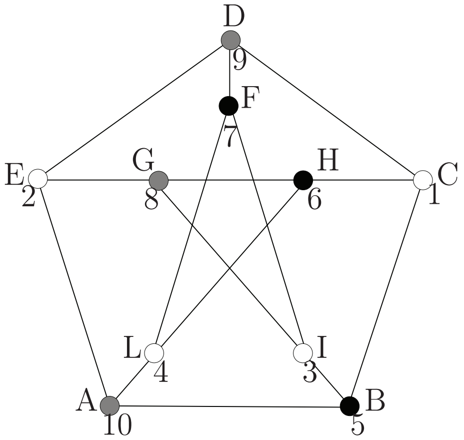

Figure 6:

$$
\begin{aligned}
V(G) &= \{A,B,C,D,E,F,G,H,I,L\},\\
E(G) &= \{\{A,B\},\{B,C\},\{C,D\},\{D,E\},\{E,A\},\{E,G\},\{A,L\},\{B,I\},\\
&\quad \{C,H\},\{D,F\},\{G,I\},\{G,H\},\{F,L\},\{F,I\},\{L,H\}\},\\
\varphi &: (1,2,3,4,5,6,7,8,9,10) \longrightarrow (C,E,I,L,B,H,F,G,D,A)\,; \quad i=2.
\end{aligned}
$$

The strong colouring $c\,(\varphi,2)$ is a strong 3-colouring with colours $0,2,3$, since

$$
\begin{aligned}
I_\varphi(0,2) &= \{C,E,L,I\},\\
I_\varphi(2,2) &= \{B,F,H\},\\
I_\varphi(3,2) &= \{A,D,G\}.
\end{aligned}
$$

Example 7. (Figure 7.)

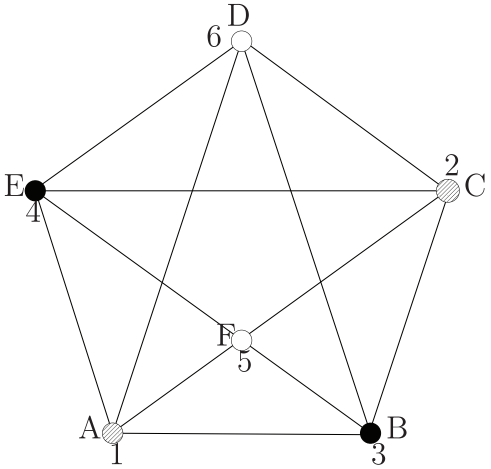

Figure 7: graph with 3-strong colourings of colours $0,2,3$.

$$\begin{aligned} V(G) &= \{A,B,C,D,E,F\},\\ E(G) &= \{\{A,B\},\{B,C\},\{C,D\},\{D,E\},\{A,E\},\{E,F\},\\ &\quad \{C,F\},\{B,F\},\{A,F\},\{A,D\},\{B,D\}\},\\ \varphi &: (1,2,3,4,5,6)\longrightarrow(A,C,B,E,F,D)\,;\quad i=1. \end{aligned}$$

We have a 3-colouring of colours $0,2,3$, with

$$\begin{aligned} I_\varphi(0,1) &= \{D,F\},\\ I_\varphi(2,1) &= \{B,E\},\\ I_\varphi(3,1) &= \{A,C\}. \end{aligned}$$

This example satisfies the hypotheses of Theorem 11 and ii) holds, but not i).

Example 8. (Figure 8.)

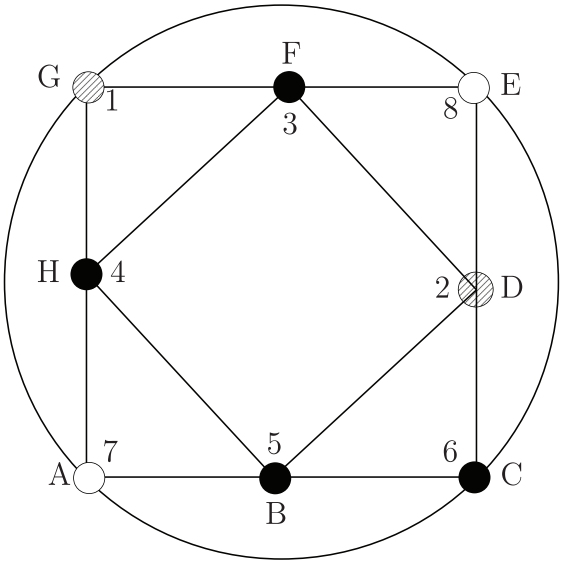

Figure 8:

$$\begin{aligned} V(G) &= \{A,B,C,D,E,F,G,H\},\\ E(G) &= \{\{A,B\},\{B,C\},\{C,D\},\{D,E\},\{E,F\},\{F,G\},\{G,H\},\{H,A\},\\ &\quad \{H,B\},\{B,D\},\{D,F\},\{F,H\},\{A,C\},\{C,E\},\{E,G\},\{G,A\}\},\\ \varphi &: (1,2,3,4,5,6,7,8)\longrightarrow(G,D,F,H,B,C,A,E)\,;\quad i=1,\\ I_\varphi(0,1) &= \{A,E\},\\ I_\varphi(2,1) &= \{B,C,F,H\},\\ I_\varphi(3,1) &= \{D,G\}. \end{aligned}$$

This graph satisfies the hypotheses of Theorem 11 and both i) and ii) hold. Moreover this strong colouring is not classical, since the adjacent vertices B and C have the same colour.

By Theorem 11 it follows

Theorem 12. *Let G be a simple regular graph of degree $r = p$, p a prime, $|G| = v$, $v < 2r$. Then strong 3-colourings of G do not exist.*

We provide an example of a graph satisfying the hypoteses of Theorem 12 and therefore not admitting a strong 3-colouring.

Example 9. (Figure 9.)

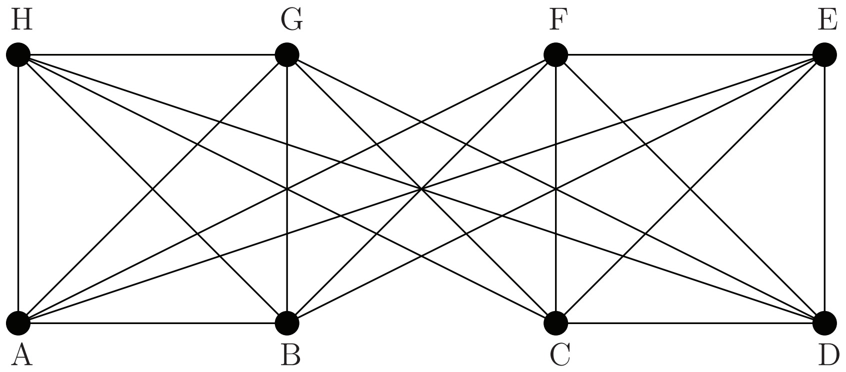

Figure 9:

$$\begin{aligned} V(G) &= \{A, B, C, D, E, F, G, H\}, \\ E(G) &= \{\{A, H\}, \{A, G\}, \{A, F\}, \{A, E\}, \{B, H\}, \{B, G\}, \{B, F\}, \\ &\quad \{B, E\}, \{C, H\}, \{C, G\}, \{C, F\}, \{C, E\}, \{D, H\}, \{D, G\}, \\ &\quad \{D, F\}, \{D, E\}, \{A, B\}, \{C, D\}, \{F, E\}, \{G, H\}\}. \end{aligned}$$

5. Hamiltonian strong colourings of regular simple graphs

A *path* of a graph G is a finite sequence of edges such as $V_1V_2, V_2V_3, \ldots, V_mV_{m+1}$, denoted also $V_1 \to V_2 \to \cdots \to V_m \to V_{m+1}$, where the edges and the vertices are distinct (may be, eventually, $V_1 = V_{m+1}$).

A graph G is called *semi-hamiltonian* if there is a path through every vertex of G. If the path is closed, G is called *hamiltonian.*

Let G be a simple semi-hamiltonian graph and ℓ be a path through every vertex of G.

Let $\mathcal{V}$ be the set of vertices of G and let $v = |\mathcal{V}|$. Let

$$\ell = V_1 \to V_2 \to \cdots \to V_v.$$

The following bijection arises

$$\varphi_\ell : n \in \mathcal{I} = \{1, 2, \ldots, v\} \longmapsto V_n \in \mathcal{V}.$$

For every $i \in \{1, 2\}$, we get the strong colouring $c(\varphi_\ell, i)$ which is called *hamiltonian strong colouring of index i associated with ℓ.*

By Theorem 6 we have that, if G is regular of degree $r > 0$, the strong colouring $c(\varphi_\ell, i)$ has the colours 0 and r. If $c(\varphi_\ell, i)$ is a hamiltonian strong colouring, the following theorem holds

Theorem 13. *Let G be a semi-hamiltonian regular simple graph of positive degree r and let $c(\varphi_\ell, i)$ a hamitonian strong colouring of G, with $\ell = V_1 \to V_2 \to \cdots \to V_v$, $v = |G|$. Then there is a unique vertex of colour 0, which is V_1 and a unique vertex of colour r, which is V_r.*

Proof. Let $i = 1$. Then V_1 has the colour r, and V_v has the colour 0. Any vertex V_n, $1 < n < v$, has a colour which is neither 0, nor r. Therefore, V_1 and V_v are the only vertices with colours r and 0, respectively. The same result holds in the case $i = 2$, but V_1 has the colour 0. and V_v has the colour r. ∎

Theorem 14. *Semi-hamiltonian regular simple graphs of positive degree having a hamiltonian strong 1-colouring do not exists.*

Proof. This results follows by Theorem 3, since a hamiltonian graph cannot be the null graph. ∎

Theorem 15. *The only semi-hamiltonian regular simple graph of positive degree admitting a hamiltonian strong 2-colouring is K_2.*

Proof. Let G be a regular simple graph of positive degree having a hamiltonian strong 2-colouring $c(\varphi_\ell, i)$ and let $\ell = V_1 \to V_2 \to \cdots \to V_v$, where $v = |G|$. By Theorem 13 it follows $\ell = V_1 \to V_2$, then $G = K_2$. Conversely, K_2 is a regular simple graph of degree 1 having a hamiltonian strong 2-colouring with colours 0 and 1. ∎

Theorem 16. *The only semi-hamiltonian regular simple graph of positive degree having a hamiltonian strong 3-colouring are the circuit-graphs.*

Proof. Let G be a semi-hamiltonian regular simple graph of positive degree r having a hamiltonian strong 3-colouring $c(\varphi_\ell, i)$, with $\ell = V_1 \to V_2 \to \cdots \to V_v$, where $v = |G|$. We get $r > 1$, since $r = 1$ implies $\ell = V_1 \to V_2$ and then $v = 2$: a contradiction, since G admits a strong 3-colouring, then $r \geq 2$. By Theorem 13 it follows that $|I_\varphi(r, i)| = |I_\varphi(0, i)| = 1$. By Theorem 9 it follows that $r \leq |I_\varphi(0, i)| + |I_\varphi(r, i)| = 2$. Then $r = 2$ and G is a connected regular simple graph of degree 2 and then a circuit-graph. The converse is obvious, since a simple circuit-graph admits a hamiltonian strong 3-colouring with colours $0, 1, 2$. ∎

We remark that in the case of classical colourings there is no characterization of strongly 3-colorable graphs.

Theorem 17. *Let G, $v = |G|$, be a semi-hamiltonian regular simple graph of positive degree r admitting a hamiltonian strong 4-colouring $c(\varphi_\ell, i)$. Then $r \geq 3$ and the colours of $c(\varphi_\ell, i)$ are $0, 1, r-1, r$. Moreover, the number of vertices with colour 1 equals that of vertices of colour $r-1$. This number is $\frac{v}{2} - 1$, hence v is even.*

Proof. Let G be a semi-hamiltonian regular simple graph of degree $r > 0$ admitting a hamiltonian strong 4-colouring $c(\varphi_\ell, i)$ and let $\ell = V_1 \to V_2 \to \cdots \to V_v$, where $v = |G|$. Obviously $v \geq 4$. Let $0, j_1, j_2, r$ the colours of $c(\varphi_\ell, i)$, $0 < j_1 < j_2 < r$. It is $r \geq 3$. By Theorem 13 it follows $|I_{\varphi_\ell}(0,i)| = 1$. By Theorem 4 it follows the existence of a colour $j \neq 0$ such that $j \leq |I_{\varphi_\ell}(0,i)| = 1$. Therefore $j_1 = 1$. Let us consider the strong colouring $c(\varphi_\ell, i')$, $i' = \{1,2\} - \{i\}$. The colours of $c(\varphi_\ell, i')$ are $0, r - j_2, r - j_1 = r - 1, r$, with $0 < r - j_2 < r - j_1 = r - 1 < r$. By Theorem 13 it follows $|I_{\varphi_\ell}(0,i')| = 1$. By Theorem 4 it follows the existence of a colour $j \neq 0$ such that $j \leq |I_{\varphi_\ell}(0,i')| = 1$. Therefore $r - j_2 = 1$, that is $j_2 = r - 1$. So the colours of $c(\varphi_\ell, i)$ are $0, 1, r-1, r$. By (1) and (2), we get:

$$\begin{aligned} t(0,i) + t(1,i) + t(r-1,i) + t(r,i) &= v, \\ t(1,i) + (r-1)t(r-1,i) + rt(r,i) &= \frac{vr}{2}. \end{aligned} \tag{12}$$

By Theorem 13 it follows

$$t(0,i) = t(r,i) = 1. \tag{13}$$

By (12) and (13) we get:

$$\begin{aligned} t(1,i) + t(r-1,i) &= v - 2, \\ t(1,i) + (r-1)t(r-1,i) &= \frac{vr}{2} - r. \end{aligned} \tag{14}$$

By (14) we get

$$(r-2)t(r-1,i) = \frac{v(r-2)}{2} - (r-2).$$

Since $r - 2 \neq 0$ (it is $r \geq 3$), we get $t(r-1,i) = v/2 - 1$. By previous conditions we get $t(1,i) = v/2 - 1$. Since $t(j,i) = |I_{\varphi_\ell}(j,i)|$, $j = 0, 1, ...r$, the theorem is proved. ∎

By Theorems 14, 15, 16, 17 it follows immediately

Theorem 18 (Theorem of cubic simple graphs)**.** *Let G be a semi-hamiltonian regular simple graph of degree 3 and let $c(\varphi_\ell, i)$ be a hamiltonian strong colouring of G. Then $c(\varphi_\ell, i)$ is a strong 4-colouring with colours $0, 1, 2, 3$. Moreover, the number of vertices of colour 1 equals the number of vertices of colour 2. This number is $\frac{v}{2} - 1$, hence $v = |G|$ is even.*

Example 10. (Figure 10.) This example is an explanation of Theorem 18.

Now let G be a semi-hamiltonian regular simple graph of degree $r > 0$ admitting a hamiltonian strong 5-colouring $c(\varphi_\ell, i)$. Like in Theorem 17, we prove that $r \geq 4$ and that the colours of $c(\varphi_\ell, i)$ are $0, 1, j, r-1, r$, with $1 < j < r - 1$. By (1) and (2), we get:

$$\begin{aligned} t(0,i) + t(1,i) + t(j,i) + t(r-1,i) + t(r,i) &= v, \\ t(1,i) + jt(j,i) + (r-1)t(r-1,i) + rt(r,i) &= \frac{vr}{2}, \end{aligned} \tag{15}$$

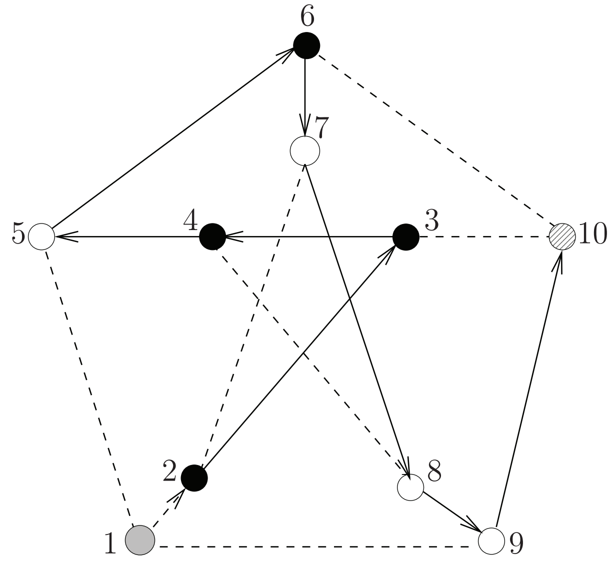

Figure 10:

where $v = |G|$. By Theorem 13 it follows

$$t(0,i) = t(r,i) = 1. \tag{16}$$

By (15) and (16) we have

$$\begin{aligned} t(1,i) + t(j,i) + t(r-1,i) &= v-2, \\ t(1,i) + jt(j,i) + (r-1)t(r-1,i) &= \frac{vr}{2} - r. \end{aligned} \tag{17}$$

By (17), we have:

$$j\,[t(1,i) + t(r-1,i) - (v-2)] = t(1,i) + (r-1)t(r-1,i) - \frac{r}{2}(v-2).$$

Since $t(1,i) + t(r-1,i) \leq v-3$, the integer $t(1,i) + t(r-1,i) - (v-2)$ is negative, then we get

$$j = \frac{t(1,i) + (r-1)t(r-1,i) - \frac{r}{2}(v-2)}{t(1,i) + t(r-1,i) - (v-2)} > 1. \tag{18}$$

By (18), since $r-2 > 0$, we get

$$t(r-1,i) < \frac{v}{2} - 1. \tag{19}$$

We denote by $c(\varphi_\ell, i')$ the hamiltonian strong 5-colouring, with $i' = \{1,2\} - \{i\}$, whose colours are $0, 1, r-j, r-1, r$. Applying (19), to this strong colouring, since $t(r-1,i') = t(1,i)$, we get

$$t(1,i) < \frac{v}{2} - 1. \tag{20}$$

Now assume v even. By (19) and (20) we have

$$\begin{aligned} t(r-1,i) &\leq \frac{v}{2}-2, \\ t(1,i) &\leq \frac{v}{2}-2. \end{aligned} \tag{21}$$

By the first of (17) and by (21) we get

$$v-2-t(j,i)=t(1,i)+t(r-1,i)\leq v-4. \tag{22}$$

Then

$$t(j,i)\geq 2.$$

If $t(j,i)=2$, by (21) and (22) we get

$$t(1,i)=t(r-1,i)=\frac{v}{2}-2. \tag{23}$$

By (18) and (23) it follows

$$j=\frac{r}{2}.$$

Then the following theorem holds

Theorem 19. *Let G be a semi-hamiltonian regular simple graph of positive degree r admitting a hamiltonian strong 5-colouring $c(\varphi_\ell,i)$ and let $v=|G|$. Then $r\geq 4$, the colours of $c(\varphi_\ell,i)$ are $0,1,j,r-1,r$, with $1<j<r-1$. The number of the vertices of colour 1 and that of the vertices of colour $r-1$ are both less than $v/2-1$. If v is even, the number of vertices of colour j is greater than or equal 2 and if this number equals 2, we get $j=r/2$. Therefore r is even and the number of vertices of colour 1 and that of vertices of colour $r-1$ are both equal to $v/2-2$.*

Example 11. (Figure 11.) This example provides a hamiltonian strong 5-colouring with $i=1$ of a regular simple graph of degree 4 with 6 vertices. The colours are $0,1,2,3,4$ and $j=2=r/2$. This strong colouring is not classical, since there are two adjacent vertices having the same colour 2. We remark that in the case of classical colourings, we have no result concerning 4-colourings and 5-colourings.

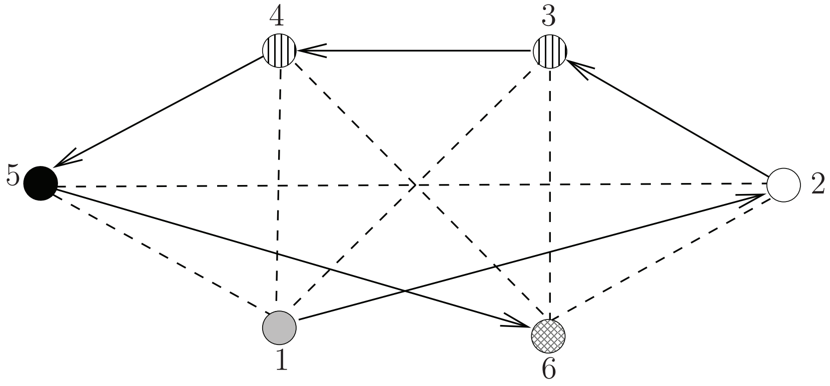

Figure 11:

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p-FUZZY HYPERGROUPS AND p-FUZZY JOIN SPACES OBTAINED FROM p-FUZZY HYPERGRAPHS[1]

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Abstract. We construct a fuzzy hyperoperation from a p-fuzzy hypergraph and then use it to construct a p-fuzzy hypergroup and a p-fuzzy join space. Also, we study generalizations of this fuzzy hyperoperation.

Keywords: fuzzy hypergroupoid; p-fuzzy hypergroup; p-fuzzy hypergraph; p-fuzzy join space.

1. Introduction and preliminaries

The connections between graphs and hypergroups had been looked into by several researchers (see, for instance, [4], [6]). Corsini [5] and Ali [1] studied the connections between hypergraphs and hypergroups. In this paper, we construct a fuzzy hyperoperation from a p-fuzzy hypergraph and then use it to construct a p-fuzzy hypergroup and a p-fuzzy join space. Also, we study generalizations of this fuzzy hyperoperation. This paper can be seen as a fuzzy version of [5].

We recall some notations of fuzzy hyperstructure theory. A fuzzy subset of a nonempty set H is a function $M : H \to [0,1]$; The collection of all fuzzy subsets of H is denoted by $F(H)$. The p-cut of a fuzzy subset M of H is defined by

$$M_p \doteq \{x \in H \mid M(x) \geq p\}.$$

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Given a fuzzy hyperoperation $* : H \times H \to F(H)$, for all $a \in H$, $B \in F(H)$, the fuzzy subset $a * B$ of H is defined by

$$(a * B)(x) \doteq \bigvee_{B(b)>0} (a * b)(x).$$

Given $A, B \in F(H)$, we give the following definitions

$$\begin{aligned}
A \subseteq B &\doteq A(x) \leq B(x), \quad \forall x \in H.\\
A = B &\doteq A(x) = B(x), \quad \forall x \in H.\\
(A \cup B)(x) &\doteq A(x) \vee B(x), \quad \forall x \in H.\\
(A \cap B)(x) &\doteq A(x) \wedge B(x), \quad \forall x \in H.
\end{aligned}$$

Proposition 0.1 ([7]) $\forall A, B, C \in F(H)$, *we have the following properties*

(1) $A \cup A = A, A \cap A = A$;
(2) $A \cup B = B \cup A, A \cap B = B \cap A$;
(3) $(A \cup B) \cup C = A \cup (B \cup C), (A \cap B) \cap C = A \cap (B \cap C)$;
(4) $A \cap (A \cup B) = A, A \cup (A \cap B) = A$;
(5) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C), (A \cap B) \cup C = (A \cup C) \cap (B \cup C)$;
(6) $A \cup \emptyset = A, A \cap \emptyset = \emptyset, A \cup H = H, A \cap H = A$.

A fuzzy hypergroupoid $\langle H; *\rangle$ is a nonempty set H endowed with a fuzzy hyperoperation (i.e., a function $*$ from $H \times H$ to $F(H)$). A p-fuzzy quasi-hypergroup is a fuzzy hypergroupoid such that

$$(x * H)_p = H = (H * x)_p, \quad \forall x \in H.$$

A p-fuzzy hypergroup is a p-fuzzy quasi-hypergroup such that for all $x, y, z \in H$, we have

$$(x * y) * z = x * (y * z).$$

The readers can consult [2], [3], [7] to learn more about hyperstructures and fuzzy sets.

2. Fuzzy Hyperoperation $*$

Definition 2.1 H is a nonempty set, $\{A_i\}_i$ is a family of fuzzy subsets of H, if there exists a $p \in (0, 1]$ such that

$$\bigcup_i (A_i)_p = H,$$

then $\langle H; \{A_i\}_i\rangle$ is called a p-fuzzy hypergraph.

Definition 2.2 Let $\Gamma = \langle H; \{A_i\}_i \rangle$ be a p-fuzzy hypergraph, set

$$E_p(x) = \bigcup_{A_i(x) \geq p} A_i.$$

The fuzzy hypergroupoid $H_\Gamma = \langle H; * \rangle$ where the fuzzy hyperoperation $*$ is defined by

$$x * y = E_p(x) \cup E_p(y), \quad \forall x, y \in H$$

is called a p-fuzzy hypergraph hypergroupoid or a p-f.h.g. hypergroupoid.

Proposition 2.3 *The p-f.h.g. hypergroupoid H_Γ has the following properties for any $x, y \in H$:*

(1) $x * y = x * x \cup y * y$;

(2) $x \in (x * x)_p$;

(3) $y \in (x * x)_p \Leftrightarrow x \in (y * y)_p$;

(4) $\{x, y\} \subseteq (x * y)_p$;

(5) $x * y = y * x$;

(6) $(x * H)_p = H$;

(7) $\langle H; \{x * x\}_{x \in H} \rangle$ *is a p-fuzzy hypergraph;*

(8) $x * x * x = \displaystyle\bigcup_{(x*x)(z)>0} z * z$;

(9) $(x * x) * (x * x) = x * x * x$.

Proof.

(1) $x * y = E_p(x) \cup E_p(y) = (E_p(x) \cup E_p(x)) \cup (E_p(y) \cup E_p(y)) = x * x \cup y * y$.

(2) It is a special case of (4).

(3) Since $\bigcup_i (A_i)_p = H$, then for any $x \in H$ there exists some $A_i \in F(H)$ such that $A_i(x) \geq p$.

We only prove the implication "$\Rightarrow$". Since

$$\begin{aligned}(x * x)(y) &= (E_p(x) \cup E_p(x))(y) = (E_p(x))(y) = \left(\bigcup_{A_i(x) \geq p} A_i \right)(y) \\ &= \bigvee_{A_i(x) \geq p} A_i(y) \geq p,\end{aligned}$$

then there exists $A_i \in F(H)$ such that $A_i(x) \geq p$ and $A_i(y) \geq p$. So,

$$(y * y)(x) = \bigvee_{A_j(y) \geq p} A_j(x) \geq p.$$

Thus $x \in (y * y)_p$.

(4) $(x * y)(x) = (E_p(x) \cup E_p(y))(x) = (E_p(x))(x) \vee (E_p(y))(x) \geq (E_p(x))(x) = \bigvee_{A_i(x) \geq p} A_i(x) \geq p$. So $x \in (x * y)_p$.

Similarly, we can prove $y \in (x * y)_p$.

(5) $x * y = E_p(x) \cup E_p(y) = E_p(y) \cup E_p(x) = y * x$.

(6) For any $y \in H$,

$$\begin{aligned}(x * H)(y) &= \left(\bigcup_{t \in H} x * t\right)(y) = \left(\bigcup_{t \in H} (E_p(x) \cup E_p(t)\right)(y) \\ &\geq \left(\bigcup_{t \in H} E_p(t)\right)(y) \geq (E_p(y))(y) \\ &= \left(\bigcup_{A_i(y) \geq p} A_i\right)(y) = \bigvee_{A_i(y) \geq p} A_i(y) \geq p.\end{aligned}$$

So, $H \subseteq (x * H)_p$ and thus $(x * H)_p = H$.

(7) From $x \in (x * x)_p$ we know

$$\bigcup_{x \in H} (x * x)_p = H.$$

And then $\langle H; \{x * x\}_{x \in H}\rangle$ is a p-fuzzy hypergraph.

(8) $x * x * x = \bigcup_{(x*x)(z)>0} z * x = \bigcup_{(x*x)(z)>0} (z * z) \cup (x * x) = \bigcup_{(x*x)(z)>0} z * z$.

(9) $(x * x) * (x * x) = \bigcup_{(x*x)(a)>0,(x*x)(b)>0} a * b = \bigcup_{(x*x)(a)>0,(x*x)(b)>0} (a * a \cup b * b) = \bigcup_{(x*x)(a)>0} a * a = x * x * x$. ■

Remark 2.4 From (5), (6) of the above Proposition we know that H_Γ is a commutative p-fuzzy quasi-hypergroup.

Theorem 2.5 *A p-fuzzy hypergroupoid $\langle H; *\rangle$ satisfying* (1), (2) *and* (3) *of Proposition* 2.3 *is a p-fuzzy hypergroup if and only if*

$$a * a * a \cup c * c = a * a * a \cup c * c * c, \quad \forall a, c \in H.$$

Proof. First, let's prove the implication "$\Leftarrow$". It is enough to verify the associativity. We have:

$(a*b)*c=(a*a\cup b*b)*c=(a*a)*c\cup(b*b)*c,$
$a*(b*c)=(b*c)*a=(b*b)*a\cup(c*c)*a,\quad \forall a,b,c\in H.$

Moreover,

$$\begin{aligned}(a*a)*c &= \bigcup_{(a*a)(u)>0} u*c=\bigcup_{(a*a)(u)>0}(u*u\cup c*c)\\ &= c*c\cup\left(\bigcup_{(a*a)(u)>0} u*u\right)=c*c\cup a*a*a.\end{aligned}$$

Also, we have

$(b*b)*c=b*b*b\cup c*c.$

Therefore,

$(a*b)*c=a*a*a\cup b*b*b\cup c*c=b*b*b\cup(a*a*a\cup c*c)$ and

$a*(b*c)=a*a\cup b*b*b\cup c*c*c=b*b*b\cup(a*a\cup c*c*c).$

By the hypothesis, we have

$a*a*a\cup c*c=a*a*a\cup c*c*c=a*a\cup c*c*c.$

And so, $(a*b)*c=a*(b*c)$.

Let's now prove the implication "$\Rightarrow$". From the associativity it follows

$(a*a)*c=a*(a*c),\quad \forall a,c\in H.$

From above we have

$(a*a)*c=a*a*a\cup c*c,\ a*(a*c)=a*a\cup a*a*a\cup c*c*c=a*a*a\cup c*c*c.$

So, $a*a*a\cup c*c=a*a*a\cup c*c*c.$ ■

Corollary 2.6 *If a p-fuzzy hypergroupoid $\langle H;*\rangle$ satisfies* (1), (2) *and* (3) *of Proposition* 2.3 *and the condition*

$$x*x*x=x*x,\quad \forall x\in H,$$

then it is a p-fuzzy hypergroup.

Example 2.7 Let $\Gamma=\langle\{a,b\};\{A_1,A_2\}\rangle$, where $A_1=\dfrac{0.5}{a}+\dfrac{0.5}{b}$, $A_2=\dfrac{0.5}{a}+\dfrac{0.5}{b}$. Since $\bigcup_{i=1}^{2}(A_i)_{0.5}=(A_1)_{0.5}\cup(A_2)_{0.5}=\{a,b\}\cup\{a,b\}=\{a,b\}$, then Γ is a 0.5-fuzzy hypergraph. Moreover,

$a*a=b*b=a*b=b*a=a*a*a=b*b*b=\dfrac{0.5}{a}+\dfrac{0.5}{b}.$

So, from above Corollary we know that $\langle\{a,b\};*\rangle$ is a 0.5-fuzzy hypergroup.

Example 2.8 Let $\Gamma = \langle \{a, b\}; \{A_1, A_2\}\rangle$, where $A_1 = \dfrac{0.5}{a} + \dfrac{0.8}{b}$, $A_2 = \dfrac{0.7}{a} + \dfrac{0.2}{b}$. Since $\bigcup\limits_{i=1}^{2}(A_i)_{0.5} = (A_1)_{0.5} \cup (A_2)_{0.5} = \{a, b\} \cup \{a\} = \{a, b\}$, then Γ is a 0.5-fuzzy hypergraph. Moreover,

$$a * a = E_{0.5}(a) = A_1 \cup A_2 = \frac{0.7}{a} + \frac{0.8}{b}.$$

$$b * b = E_{0.5}(b) = A_1 = \frac{0.5}{a} + \frac{0.8}{b}.$$

$$a * a * a = \left(\frac{0.7}{a} + \frac{0.8}{b}\right) * a = a * a \cup a * b = \frac{0.7}{a} + \frac{0.8}{b}.$$

$$b * b * b = \left(\frac{0.5}{a} + \frac{0.8}{b}\right) * b = a * b \cup b * b = \frac{0.7}{a} + \frac{0.8}{b}.$$

We have $b * b * b \neq b * b$.

But $x * x * x \cup y * y = x * x * x \cup y * y * y$, $\forall x, y \in \{a, b\}$.

So, from Theorem 2.5, we know that $\langle \{a, b\}; *\rangle$ is a 0.5-fuzzy hypergroup.

Definition 2.9 An associative p-f.h.g. quasi-hypergroup is called a p-f.h.g. hypergroup.

Definition 2.10 Let $\langle H; *\rangle$ be a commutative p-fuzzy hypergroup, $\langle H; *, /\rangle$ is called a p-fuzzy join space if and only if

$$(x/y \cap z/w)_p \neq \emptyset \Rightarrow (x * w \cap y * z)_p \neq \emptyset$$

where $(x/y)(t) = (t * y)(x)$.

Theorem 2.11 *Let $\langle H; *\rangle$ be a p-fuzzy hypergroup satisfying* (1), (2) *and* (3) *of Proposition* 2.3. *Then $\langle H; *, /\rangle$ is a p-fuzzy join space.*

Proof. We prove the following implication is valid:

$$(x/y \cap z/w)_p \neq \emptyset \Rightarrow (x * w \cap y * z)_p \neq \emptyset \text{ where } (x/y)(t) = (t * y)(x).$$

We have

$$u \in (x/y \cap z/w)_p \Leftrightarrow [x \in (u * y)_p \text{ and } z \in (u * w)_p].$$

Moreover,

$$x \in (u * y)_p \Leftrightarrow x \in (u * u \cup y * y)_p = (u * u)_p \cup (y * y)_p \text{ and}$$

$$z \in (u * w)_p = z \in (u * u \cup w * w)_p = (u * u)_p \cup (w * w)_p.$$

Four cases are possible:

(1) if $x \in (u * u)_p, z \in (u * u)_p$, then $u \in (x * x)_p \cap (z * z)_p = (x * x \cap z * z)_p$ and therefore $u \in (x * w \cap y * z)_p$.

(2) if $x \in (u*u)_p, z \in (w*w)_p$, then $w \in (z*z)_p$ and therefore $w \in (x*w \cap y*z)_p$.

(3) if $x \in (y*y)_p, z \in (u*u)_p$, then $y \in (x*x)_p$ and therefore $y \in (x*w \cap y*z)_p$.

(4) if $x \in (y*y)_p, z \in (w*w)_p$, then $w \in (z*z)_p$ and therefore $w \in (x*w \cap y*z)_p$. ■

3. Generalizations of $*$

We can generalize the fuzzy hyperoperation $*$ in following ways.

Definition 3.1 Let $\Gamma = \langle H; \{A_i\}_i \rangle$ be a p-fuzzy hypergraph, for all $q \in (0, p]$, set

$$E_q(x) = \bigcup_{A_i(x) \geq q} A_i$$

and the fuzzy hyperoperation $*_q$ is defined by

$$x *_q y = E_q(x) \cup E_q(y), \quad \forall x, y \in H.$$

Proposition 3.2 *The fuzzy hyperoperation $*_q$ has the following properties for any $x, y \in H$:*

(1) $x *_q y = x *_q x \cup y *_q y$;

(2) $x \in (x *_q x)_p$;

(3) $y \in (x *_q x)_q \Leftrightarrow x \in (y *_q y)_q$;

(4) $\{x, y\} \subseteq (x *_q y)_p$;

(5) $x *_q y = y *_q x$;

(6) $(x *_q H)_p = H$;

(7) $\langle H; \{x *_q x\}_{x \in H} \rangle$ *is a p-fuzzy hypergraph;*

(8) $x *_q x *_q x = \bigcup_{(x *_q x)(z) > 0} z *_q z$;

(9) $(x *_q x) *_q (x *_q x) = x *_q x *_q x$.

Proof. A straightforward verification. ■

Definition 3.3 Let $\Gamma = \langle H; \{A_i\}_i \rangle$ be a p-fuzzy hypergraph. For all $q \in (0, p]$, set

$$E_q(x) = \bigcup_{A_i(x) \geq q} A_i.$$

For all $s, t \in (0, p]$, the fuzzy hyperoperation $*_s^t$ is defined by

$$x *_s^t y = E_s(x) \cup E_t(y), \quad \forall x, y \in H.$$

Proposition 3.4 *The fuzzy hyperoperation* $*_s^t$ *has the following properties for any* $x, y \in H$:

(1) $x *_s^t y = x *_s^s x \cup y *_t^t y$;

(2) $x \in (x *_s^t x)_p$;

(3) $y \in (x *_s^t x)_{s \vee t} \Leftrightarrow x \in (y *_s^t y)_{s \vee t}$;

(4) $\{x, y\} \subseteq (x *_s^t y)_p$;

(5) $x *_s^t y = y *_t^s x$;

(6) $(x *_s^t H)_p = H$;

(7) $\langle H; \{x *_s^t x\}_{x \in H}\rangle$ *is a p-fuzzy hypergraph;*

Proof. A straightforward verification. ∎

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FUZZY LIE IDEALS OVER A FUZZY FIELD

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Abstract. The concept of fuzzy Lie ideals of a Lie algebra over a fuzzy field is introduced and some fundamental properties of such fuzzy Lie ideals are given. We then characterize the Artinian and Noetherian Lie algebras by considering their fuzzy Lie ideals over a fuzzy field.

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1. Introduction

The concept of Lie groups was first introduced by Sophus Lie in nineteenth century through his studies in geometry and integration methods for differential equations. Lie algebras were also discovered by him when he attempted to classify certain smooth subgroups of a general linear group. The importance of Lie algebras in mathematics and physics has become increasingly evident in recent years. In applied mathematics, Lie theory remains a powerful tool for studying differential equations, special functions and perturbation theory. It is noted that Lie theory has applications not only in mathematics and physics but also in diverse fields such as continuum mechanics, cosmology and life sciences. Lie algebra has nowadays even been applied by electrical engineers in solving problems in mobile robot control.

On the other hand, Zadeh [13] introduced the notion of a fuzzy subset of a set in 1965. By using fuzzy sets, people have established the theory for study uncertainty. Fuzzy mathematics have become a vigorous area of research in different domains such as engineering, medical science, social science, artificial intelligence, signal processing, pattern recognition, computer networks,automata theory and so on. The notions of fuzzy ideals and fuzzy subalgebras of Lie algebras over a field were first introduced by Yehia in [12]. In this paper, the notion of fuzzy Lie ideals of a Lie algebra over a fuzzy field (in short, fuzzy Lie $\mathbb{F}$-ideals) is considered and some properties of fuzzy Lie $\mathbb{F}$-ideal of a Lie algebra are presented. By fuzzy Lie $\mathbb{F}$-ideals, we give characterizations for Artinian and Noetherian Lie algebras.

The definitions and terminologies that we used in this paper are standard. For other notations, terminologies and applications, the readers are referred to [1], [2], [3], [5], [8]-[12].

2. Preliminaries

We first review some elementary aspects which are useful in the sequel. Throughout this paper, L is a Lie algebra and X is a field. It is clear that the multiplication of a Lie algebra is not necessary associative, that is, $[[x, y], z] = [x, [y, z]]$ does not hold in general, however it is *anti- commutative*, that is, $[x, y] = -[y, x]$. Let μ be a *fuzzy set* on L, that is, a map $\mu : L \to [0, 1]$. For any fuzzy set μ in L and any $t \in [0, 1]$, we define set $U(\mu; t) = \{x \in L \mid \mu(x) \geq t\}$, which is called *upper t-level cut* of μ.

Definition 2.1. [14] A mapping $f : L \to L$ is called a *closure* if, for every $x, y \in L$,

(i) $x \geq y \Rightarrow f(x) \geq f(y)$ (monotony)

(ii) $x \leq f(x)$ (inclusion)

(iii) $f(f(x)) = f(x)$ (idempotence).

Definition 2.2. [1] A fuzzy set $\mu : L \to [0, 1]$ is called a *fuzzy Lie ideal* of L over a field X if the following conditions:

(1) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$,

(2) $\mu(\alpha x) \geq \mu(x)$,

(3) $\mu([x, y]) \geq \mu(x)$

hold for all $x, y \in L$ and $\alpha \in X$.

Definition 2.3. [10] A fuzzy set F of X is called a *fuzzy field* if the following conditions are satisfied:

- $(\forall\, m,\, n \in X)(F(m - n) \geq \min\{F(m), F(n)\})$,
- $(\forall m, n \in X, n \neq 0)(F(mn^{-1}) \geq \min\{F(m), F(n)\})$.

Lemma 2.4. [10] *If F is a fuzzy subfield of X, then*

$$\begin{array}{lll} F(0) \geq F(1) \geq F(m) = F(-m) & \textit{for all } \ m \in X & \textit{and} \\ F(-m) = F(m^{-1}) & \textit{for all } m \in X - \{0\}. & \end{array}$$

Lemma 2.5. [10] *Let F be a fuzzy subfield of X. Then for $t \in [0,1]$, the fuzzy-cut $U(F;t)$ is a crisp subfield of X.*

3. Fuzzy Lie ideals over a fuzzy field

Definition 3.1. Let μ be a fuzzy set of L and F a fuzzy field of X. Then μ is called a *fuzzy Lie ideal over a fuzzy field* F(briefly, fuzzy Lie $\mathbb{F}$-ideal) if the following conditions:

(a) $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$,

(b) $\mu(\alpha x) \geq \max\{F(\alpha), \mu(x)\}$,

(c) $\mu([x,y]) \geq \mu(x)$

hold for all $x, y \in L$ and $\alpha \in X$.

From condition (b) above, it follows that $\mu(0) \geq F(0)$.

Example 3.2. Let $\Re^2 = \{(x,y) : x, y \in R\}$ be the set of all 2-dimensional real vectors. Then it is clear that $\Re^2$ endowed with the operation defined by $[x,y] = x \times y$ form a real Lie algebra. Define a fuzzy set $\mu : \Re^2 \to [0,1]$ by

$$\mu(x,y) = \begin{cases} 0 & \text{if } x = y = 0, \\ 1 & \text{otherwise.} \end{cases}$$

and define $F : \mathbb{R} \to [0,1]$ for all $\alpha \in \mathbb{R}$ by

$$F(\alpha) = \begin{cases} 0 & \text{if } \alpha \in \mathbb{Q}, \\ 1 & \text{if } \alpha \in \mathbb{R} - \mathbb{Q}(\sqrt{3}). \end{cases}$$

By routine computations, one can easily check that μ is a fuzzy Lie $\mathbb{F}$-ideal.

Definition 3.3. Let μ be a fuzzy set and $s \in [0,\ 1]$. Define:

(d) the *t-cut* of μ by the non-empty set $U(\mu;t) = \{x \in L \,|\, \mu(x) \geq t\}$.

(e) the strong *t-cut* of μ by the non-empty set ${}^{>}U(\mu;t) = \{x \in L \,|\, \mu(x) > t\}$.

(f) the image of μ by the set $t \in Im(\mu)$.

We now formulate the following theorem of fuzzy Lie $\mathbb{F}$-ideals of L.

Theorem 3.4. *Let μ be a fuzzy Lie $\mathbb{F}$-ideal of L and ν the closure of the image of μ. Then the following conditions are equivalent:*

(g) *μ is a fuzzy Lie $\mathbb{F}$-ideal of L,*

(h) *the non-empty strong level subset ${}^{>}U(\mu;t)$ of μ is a Lie ideal of L, for all $t \in [0,1]$,*

(i) *the non-empty strong level subset ${}^{>}U(\mu;t)$ of μ is a Lie ideal of L, for all $t \in Im(\mu) \setminus \nu$,*

(j) *the nonempty level subset $U(\mu;t)$ of μ is a Lie ideal of L,for all $t \in Im(\mu)$,*

(k) *the nonempty level subset $U(\mu;t)$ of μ is a Lie ideal of L, for all $t \in [0,1]$.*

Proof. *(g)* $\Leftrightarrow$ *(h)*: Let $t \in [0,1]$ be such that the strong t-cut of μ is nonempty, that is, ${}^{>}U(\mu;t) \neq \emptyset$. Then for $x, y \in L$, $\alpha \in X$ satisfying the condition $x \in {}^{>}U(\mu;t)$, $y \in {}^{>}U(\mu;t)$, $\alpha \in {}^{>}U(F;t)$, we have $\mu(x) > t$ and $\mu(y) > t$, $F(\alpha) > t$. From Definition 3.1, it follows that

$$\begin{aligned} \mu(x+y) &\geq \min(\mu(x), \mu(y)) > t, \\ \mu(\alpha x) &\geq \max(F(\alpha), \mu(x)) > t, \\ \mu([x,y]) &\geq \mu(x) > t \end{aligned}$$

and hence $x + y \in {}^{>}U(\mu;t)$, $\alpha x \in {}^{>}U(\mu;t)$ and $[x,y] \in {}^{>}U(\mu;t)$.

(h) $\Leftrightarrow$ (i), (i) $\Leftrightarrow$ (j), (j) $\Leftrightarrow$ (k) are obvious.

(k) $\Leftrightarrow$ (g): suppose that $U(\mu;t) \neq \emptyset$ is a Lie ideal of L for every $t \in [0,1]$. If

$$\mu(x+y) < \min\{\mu(x), \mu(y)\}$$

for some x, $y \in L$, then by taking

$$s_0 := \frac{1}{2}\{\mu(x+y) + \min\{\mu(x) + \mu(y)\}\},$$

we have $\mu(x+y) < s_0 < \min\{\mu(x), \mu(y)\}$. This shows that $x + y \notin U(\mu;t)$, $x \in U(\mu;t)$ and $y \in U(\mu;t)$, however, this is a contradiction. Hence $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$ for all x, $y \in L$. By using the same argumentations we can prove $\mu(\alpha x) \geq \max(F(\alpha), \mu(y))$, $\mu([x,y]) \geq \mu(x)$.

The proofs of the following Propositions are obvious.

Proposition 3.5.

(i) *Let A be a nonempty subset of L. Define a fuzzy set μ by*

$$\mu(x) = \begin{cases} \beta_2 & \text{if } x \in A \\ \beta_1 & \text{otherwise} \end{cases}$$

Clearly, μ with $0 \leq \beta_1 < \beta_2 \leq 1$ is a fuzzy Lie $\mathbb{F}$-ideal of L if and only if A is a Lie ideal of L.

(ii) *If μ and ν are fuzzy Lie $\mathbb{F}$-ideals of L, then $\mu + \nu$ and $\mu \cap \nu$ are clearly fuzzy Lie $\mathbb{F}$-ideals of L.*

(iii) *If* $\{\mu_i : i \in \Lambda\}$ *is a family of fuzzy Lie* $\mathbb{F}$*-ideals of* L*, then* $\bigcap_{i\in\Lambda} \mu_i$ *is also a fuzzy Lie* $\mathbb{F}$*-ideal of* L*.*

Proposition 3.6.

(iv) *Let* $f : L_1 \to L_2$ *be a homomorphism of Lie algebras. If* μ *is a fuzzy Lie* $\mathbb{F}$*-ideal of* L_1*, then* $f(\mu)$ *is a fuzzy Lie* $\mathbb{F}$*-ideal of* L_2*.*

(v) *A Lie algebra homomorphism image of a fuzzy Lie* $\mathbb{F}$*-ideal having the* sup *property is a fuzzy Lie* $\mathbb{F}$*-ideal.*

(vi) *Let* $f : L_1 \to L_2$ *be an onto homomorphism of Lie algebras. If* μ *is a fuzzy Lie* $\mathbb{F}$*-ideal of* L_2*, then* $f^{-1}(\mu)$ *is a fuzzy Lie* $\mathbb{F}$*-ideal of* L_1*.*

(vii) *Let* $f : L_1 \to L_2$ *be an onto homomorphism of Lie algebras. If* μ *is a fuzzy Lie* $\mathbb{F}$*-ideal of* L_2*, then* $f^{-1}(\mu^c) = (f^{-1}(\mu))^c$*.*

Definition 3.7. Let L_1 and L_2 be Lie algebras. If f is a function of a fuzzy set μ in L_1, then the *image* of μ under f is a fuzzy set defined by

$$f(\mu)(y) = \begin{cases} \sup\{\mu(t) \mid t \in L_1, f(t) = y\}, & \text{if } f^{-1}(y) \neq \emptyset\ , \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3.8. Let L_1 and L_2 be any sets and $f : L_1 \to L_2$ any function. Then a fuzzy set μ is called *f-invariant* if and only if for x, $y \in L_1$, $f(x) = f(y)$ implies $\mu(x) = \mu(y)$.

Theorem 3.9. *Let* $f : L_1 \to L_2$ *be an epimorphism of Lie algebras. Then* μ *is an* f*-invariant fuzzy Lie* $\mathbb{F}$*-ideal of* L_1 *if and only if* $f(\mu)$ *is a fuzzy Lie* $\mathbb{F}$*-ideal of* L_2*.*

Proof. Let x, $y \in L_2$ and $\alpha \in X$. Then there exist a, $b \in L_1$ such that $f(a) = x$, $f(b) = y$ and $x + y = f(a + b)$ with $\alpha x = \alpha f(a)$. Since μ is f-invariant, we conclude that

$$\begin{aligned} f(\mu)(x+y) &= \mu(a+b) \geq \min(\mu(a), \mu(b)) \\ &= \min(f(\mu)(x), f(\mu)(y)), \\ f(\mu)(\alpha x) &= \mu(\alpha a) \geq \max(F(\alpha), \mu(a)) \\ &= \max(f(F(\alpha)), f(\mu)(x)), \\ f(\mu)([x,y]) &= \mu([a,b]) = [\mu(a), \mu(b)] \geq \mu(a) = f(\mu)(x). \end{aligned}$$

Hence $f(\mu)$ is a fuzzy Lie $\mathbb{F}$-ideal of L_2.
Conversely, if $f(\mu)$ is a fuzzy Lie $\mathbb{F}$-ideal of L_2, then for any $x \in L_1$ we have

$$\begin{aligned} f^{-1}(f(\mu))(x) &= f(\mu)(f(x)) \\ &= \sup\{\mu(t) \mid t \in L_1, f(t) = f(x)\} \\ &= \sup\{\mu(t) \mid t \in L_1, \mu(t) = \mu(x)\} \\ &= \mu(x). \end{aligned}$$

This shows that $f^{-1}(f(\mu)) = \mu$ is a fuzzy Lie $\mathbb{F}$-ideal of L, by Proposition 3.6 (iv).

Definition 3.10. An ideal A of L is said to be a *characteristic* ideal of L if $f(A) = A$, for all $f \in \mathrm{Aut}(L)$, where $\mathrm{Aut}(L)$ is the set of all automorphisms of L. A fuzzy Lie $\mathbb{F}$-ideal μ of L is said to be *fuzzy characteristic* if $\mu^f(x) = \mu(x)$ for all $x \in L$ and $f \in \mathrm{Aut}(L)$.

Definition 3.11. An ideal A of Lie algebra L is said to be *fully invariant* if $f(C) \subseteq C$ for all $f \in \mathrm{End}(L)$, where $\mathrm{End}(L)$ is the set of all endomorphisms of L. A fuzzy Lie $\mathbb{F}$-ideal μ is said to be *fuzzy fully invariant* if $\mu^f(x) \leq \mu(x)$, for all $x \in L$ and $f \in \mathrm{End}(L)$.

Lemma 3.12. *Let μ be a fuzzy Lie $\mathbb{F}$-ideal of L. Then for any $x \in L$, $\mu(x) = s$ if and only if $x \in U(\mu; s)$ and $x \notin U(\mu; t)$ for all $s < t$.*

We now characterize the characteristic fuzzy Lie $\mathbb{F}$-ideals of L.

Theorem 3.13. *A fuzzy Lie $\mathbb{F}$-ideal μ of L is characteristic if and only if each of its level set is a characteristic Lie ideal of L.*

Proof. Suppose that μ is fuzzy characteristic over a fuzzy field F and $s \in Im(\mu)$, $f \in \mathrm{Aut}(L)$ and $x \in U(\mu; s)$. Then $\mu^f(x) = \mu(x)$ implies $\mu(f(x)) \geq s$, and whence $f(x) \in U(\mu; s)$. Thus $f(U(\mu; s)) \subseteq U(\mu; s)$. On the other hand, if $x \in U(\mu; s)$ and $y \in L$ such that $f(y) = x$, then $\mu(y) = \mu^f(y) = \mu(f(y)) = \mu(x) \geq s \Rightarrow y \in U(\mu; s)$. Consequently, $y \in U(\mu; s)$ and so $x = f(y) \in U(\mu; s)$. This leads to $U(\mu; s) \subseteq f(U(\mu; s))$. Hence, $f(U(\mu; s)) = U(\mu; s)$, that is , $U(\mu; s)$ is a characteristic ideal .

Conversely, if each level Lie ideal of μ is a characteristic ideal of L with $x \in L$, $f \in Aut(L)$ and $\mu(x) = s$, then, by Lemma 3.12 , $x \in U(\mu; s)$ and $x \notin U(\mu; t)$ for all $s < t$. Hence, by our assumption, we have $f(x) \in f(U(\mu; s)) = U(\mu; s)$ and so $\mu^f(x) = \mu(f(x))) \leq s$. Let $t = \mu^f(x)$ and assume that $s < t$. Then $f(x) \in U(\mu; t) = f(U(\mu; t))$. This implies from the injectivity of f that $x \in U(\mu; t)$, a contradiction. This shows that $\mu^f(x) = \mu(f(x)) = s = \mu(x)$ and so μ is fuzzy characteristic over the fuzzy field F.

In view of the above Theorem, we deduce immediately the following theorem.

Theorem 3.14. *If μ is a fully invariant fuzzy Lie $\mathbb{F}$-ideal of L , then it is a characteristic ideal.*

Definition 3.15.

(i) A fuzzy relation on any set L is defined as a fuzzy set $\mu : L \times L \to [0, 1]$.

(ii) If μ is a fuzzy relation on a set L and ν is a fuzzy set in L, then μ is a *fuzzy relation* on ν if $\mu(x, y) \leq \min\{\nu(x), \nu(y)\}$ for all x, $y \in L$.

(iii) Let μ and ν be the fuzzy sets in a set L. The *cartesian product* of μ and ν is defined by $(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\}$ for all x, $y \in L$.

Theorem 3.16. *If μ and ν are two fuzzy Lie $\mathbb{F}$-ideals of L, then $\mu \times \nu$ is a fuzzy Lie $\mathbb{F}$-ideal of $L \times L$.*

Proof. We restrict our proof on condition (b) of μ in Definition 3.1. Let $x = (x_1, x_2) \in L \times L$ and $\alpha \in X$. Then

$$\begin{aligned}(\mu \times \nu)(\alpha x) &= (\mu \times \nu)(\alpha(x_1, x_2)) = (\mu \times \nu)((\alpha x_1, \alpha x_2)) \\ &= \min(\mu(\alpha x_1), \nu(\alpha x_2)) \geq \min(\max(F(\alpha), \mu(x_1)), \max(F(\alpha), \nu(x_2))) \\ &= \min(\max(F(\alpha), F(\alpha)), \max(\mu(x_1), \nu(x_2))) \\ &= \max(F(\alpha), \min(\mu(x_1), \nu(x_2))) = \max(F(\alpha), (\mu \times \nu)(x_1, x_2) \\ &= \max(F(\alpha), (\mu \times \nu)(x)).\end{aligned}$$

The verifications for other conditions are analogous. Hence,$\mu \times \nu$ is a fuzzy Lie $\mathbb{F}$-ideal of $L \times L$.

Definition 3.17. Let ν be a fuzzy set in a set L. Then *the strongest fuzzy relation* on L is the fuzzy relation on ν is μ_ν which is defined by $\mu_\nu(x, y) = \min\{\nu(x), \nu(y)\}$, for all x, $y \in L$.

We now characterize the fuzzy Lie $\mathbb{F}$-ideal of L.

Theorem 3.18. *Let ν be a fuzzy set in L and μ_ν a strongest fuzzy relation on L. Then ν is a fuzzy Lie $\mathbb{F}$-ideal of L if and only if μ_ν is a fuzzy Lie $\mathbb{F}$-ideal of $L \times L$.*

Proof. We restrict our proof on the verification of condition (b) of μ in Definition 3.1. Suppose that ν is a fuzzy Lie $\mathbb{F}$-ideal of L. Then For any $x = (x_1, x_2) \in L \times L$ and $\alpha \in X$,we have

$$\begin{aligned}\mu_\nu(\alpha x) &= \mu_\nu(\alpha(x_1, x_2)) = \mu_\nu(\alpha x_1, \alpha x_2) \\ &= \min\{\nu(\nu(\alpha x_1)), \nu(\alpha x_2)\} \\ &\geq \max\{\min(F(\alpha), \nu(x_1)), \min(F(\alpha), \nu(x_2))\} \\ &= \max\{F(\alpha), \min(\nu(x_1), \nu(x_2))\} \\ &= \max\{F(\alpha), \mu_\nu((x_1, x_2))\} \\ &= \max\{F(\alpha), \mu_\nu(x)\}.\end{aligned}$$

The verifications for other conditions are analogous and we omit the details. Hence μ_λ is a fuzzy Lie $\mathbb{F}$-ideal of $L \times L$. The proof of the converse part is easy.

Definition 3.19. Let μ be a fuzzy Lie $\mathbb{F}$-ideal in L and $\mu_n = [\mu, \mu_{n-1}]$ for $n > 0$, where $\mu_0 = \mu$. If there exists a positive integer n such that $\mu_n = 0$, then a fuzzy Lie $\mathbb{F}$-ideal is called *nilpotent*.

Definition 3.20. Let μ be a fuzzy Lie $\mathbb{F}$-ideal in L. Define a sequence of fuzzy Lie $\mathbb{F}$-ideals in L by $\mu^0 = \mu$, $\mu^n = [\mu^{n-1}, \mu^{n-1}]$ for $n > 0$. If there exists a positive integer n such that $\mu^n = 0$, then a fuzzy Lie $\mathbb{F}$-ideal is called *solvable*.

By using similar method as in the proof of Theorem 3.21 in [1], we obtain the following Proposition.

Proposition 3.21.

(I) *The homomorphic image of a solvable fuzzy Lie* $\mathbb{F}$*-ideal is a solvable fuzzy Lie* $\mathbb{F}$*-ideal.*

(II) *The homomorphic image of a nilpotent fuzzy Lie* $\mathbb{F}$*-ideal is a nilpotent fuzzy Lie* $\mathbb{F}$*-ideal.*

(III) *If* μ *is a nilpotent fuzzy Lie* $\mathbb{F}$*-ideal, then it is solvable.*

4. Artinian and Noetherian Lie algebras

Definition 4.1. An Lie algebra L is said to satisfy the *descending chain condition* for Lie ideals if for any sequence of Lie ideals $I_1, I_2, \cdots, I_i, \cdots$ of L such that

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \supseteq I_i \cdots ,$$

there exists an element $n \in \mathbb{N}$ such that $I_m = I_n$ for each $m \in \mathbb{N}$, $m \leq n$. $\mathbb{N} = \{1, 2, \cdots\}$ always denotes the set of natural numbers.

L is called *Artinian* if it satisfies the descending chain condition on its Lie ideals. Similarly, L is called *Noetherian* if it satisfies the ascending chain condition on its Lie ideals.

The following Lemma is immediate.

Lemma 4.2. *Let* μ *be a fuzzy Lie* $\mathbb{F}$*-ideal of a Lie algebra* L *with* $s, t \in Im(\mu)$*. Then* $U(\mu; s) = U(\mu; t) \iff s = t$.

Theorem 4.3. *Every fuzzy Lie* $\mathbb{F}$*-ideal of a Lie algebra* L *has finite number of values if and only if a Lie algebra* L *is Artinian.*

Proof. Suppose that every fuzzy Lie $\mathbb{F}$-ideal of a Lie algebra L has finite number of values but L is not Artinian. Then there exists a strictly descending chain

$$L = U_0 \supset U_1 \supset U_2 \supset \cdots$$

of ideals of L. Define a fuzzy set μ in L by μ be a fuzzy set in L defined by

$$\mu(x) := \begin{cases} \dfrac{n}{n+1} & \text{if } x \in U_n \setminus U_{n+1}, n = 0, 1, 2, ..., \\ 1 & \text{if } x \in \displaystyle\bigcap_{n=0}^{\infty} U_n. \end{cases}$$

Let $x, y \in L$, $\alpha \in X$. Then $x+y$, αx, $[x, y] \in U_n \backslash U_{n+1}$ for some $n = 0, 1, 2, ...$, and either $x \notin U_{n+1}$ or $y \notin U_{n+1}$. Now, let $y \in U_n \setminus U_{n+1}$ for $k \leq n$. Then by

Definition 3.1, we have

$$\begin{aligned}\mu(x+y) &= \frac{n}{n+1} \geq \frac{k}{k+1} \geq \min(\mu(x),\mu(y)),\\ \mu(\alpha x) &= \frac{n}{n+1} \geq \frac{k}{k+1} \geq \max(F(\alpha),\mu(x)),\\ \mu([x,y]) &= \frac{n}{n+1} \geq \frac{k}{k+1} \geq \mu(x).\end{aligned}$$

Thus, μ is fuzzy Lie $\mathbb{F}$-ideal of L and μ has infinite number of different values. This contradiction proves that L is an Artinian Lie algebra.

Conversely, Suppose that L is an Artinian Lie algebra such that μ is a fuzzy Lie $\mathbb{F}$-ideal of L. If $Im(\mu)$ is infinite, then every subset of $[0,1]$ contains either a strictly increasing or a strictly decreasing sequence.

If $t_1 < t_2 < t_3 < \cdots$ is a strictly increasing sequence in $Im(\mu)$, then the following chain

$$U(\mu;t_1) \supset U(\mu;t_2) \supset U(\mu;t_3) \supset$$

is a strictly descending chain of ideals of L. Since L is Artinian, there exists a natural number i such that $U(\mu;t_i) = U(\mu;t_{i+n})$ for all $n \geq 1$. Since $t_i \in Im(\mu)$ for all i,it follows from Lemma 4.2 that $t_i = t_{i+n}$, for all $n \geq 1$. However, this is a contradiction because t_i are different.

On the other hand, if $t_1 > t_2 > t_3 > \cdots$ is a strictly decreasing sequence in $Im(\mu)$, then

$$U(\mu;t_1) \subset U(\mu;t_2) \subset U(\mu;t_3) \subset$$

is an ascending chain of ideals of L. Since L is Noetherian, there exists a natural number j such that $U(\mu;t_j) = U(\mu;t_{j+n})$ for all $n \geq 1$. Since $t_j \in Im(\mu)$ for all j, by Lemma 4.2, $t_j = t_{j+n}$, for all $n \geq 1$. This is again a contradiction because t_j are distinct. This shows that $Im(\mu)$ is finite.

Theorem 4.4. *Let L be an Artinian Lie algebra and μ a fuzzy Lie $\mathbb{F}$-ideal of L. Then $|U_\mu| = |Im(\mu)|$, where U_μ is a family of all level ideals of L with respect to μ.*

Proof. Since L is Artinian, by Theorem 4.3, $Im(\mu)$ is finite. Let $Im(\mu) = \{t_1, t_2, \cdots, t_n\}$, where $t_1 < t_2 < \cdots < t_n$. Then,it suffices to show that U_μ consists of level ideals of L with respect to μ, for all $t_i \in Im(\mu)$, that is, $U_\mu = \{U(\mu;t_i) \mid 1 \leq i \leq n\}$. It is clear that $U(\mu;t_i) \in U_\mu$ for all $t_i \in Im(\mu)$. Let $0 \leq t \leq \mu(0)$ and $U(\mu;t)$ a level ideal of L with respect to μ. Assume that $t \notin Im(\mu)$. If $t < t_1$, then clearly $U(\mu;t) = U(\mu;t_1)$, and so $t_i < t < t_{i+1}$ for some i. Hence, $U(\mu;t_{i+1}) \subseteq U(\mu;t)$. Let $x \in U(\mu;t)$. Then $\mu(x) > t$ since $t \notin Im(\mu)$, and so $\mu(x) \geq U(\mu;t_{i+1})$. Thus $U(\mu;t) = U(\mu;t_{i+1})$. This shows that U_μ consists of the level ideals of L with respect to μ, for all $t_i \in Im(\mu)$. Hence $|U_\mu| = |Im(\mu)|$.

Theorem 4.5. *Let L be an Artinian Lie algebra. If μ and ν are fuzzy Lie $\mathbb{F}$-ideals of L, then $|U_\mu| = |U_\nu|$ and $Im(\mu)$=$Im(\nu)$ if and only if μ= ν.*

Proof. If $\mu = \nu$, then $U_\mu = U_\nu$ and $Im(\mu) = Im(\nu)$. Suppose that $U_\mu = U_\nu$ and $Im(\mu) = Im(\nu)$. Then, by Theorem 4.3 and 4.4, $Im(\mu) = Im(\nu)$ are finite and $|U_\mu| = |Im(\mu)|$ and $|U_\nu| = |Im(\nu)|$. Let

$$Im(\mu) = \{t_1, t_2, ..., t_n\} \quad \text{and} \quad Im(\nu) = \{s_1, s_2, ..., s_n\},$$

where $t_1 < t_2 < \cdots < t_n$ and $s_1 < s_2 < \cdots < s_n$. Thus, $t_i = s_i$ for all i. We now prove that $U(\mu; t_i) = U(\nu; t_i)$, for all i. Observe that $U(\mu; t_1) = L = U(\nu; t_1)$. Consider $U(\mu; t_2)$, $U(\nu; t_2)$. If $U(\mu; t_2) \neq U(\nu; t_2)$, then $U(\mu; t_2) = U(\nu; t_k)$ for some $k > 2$ and $U(\nu; t_2) = U(\mu; t_j)$ for some $j > 2$. If there exists $x \in L$ such that $\mu(x) = t_2$, then

$$\mu(x) < t_j \text{ for all } j > 2. \tag{1}$$

Since $U(\mu; t_2) = U(\nu; t_k)$, $x \in U(\nu; t_k)$, $\nu(x) \geq t_k > t_2$, $k > 2$. Thus $x \in U(\nu; t_2)$. Now we have $x \in U(\mu; t_j)$ because $U(\nu; t_2) = U(\mu; t_j)$. Thus we deuce that

$$\mu(x) \geq t_j \text{ for some } j > 2. \tag{2}$$

Clearly, (1) and (2) contradict each other, and so $U(\mu; t_2) = U(\nu; t_2)$. Continuing in this manner, we deduce $U(\mu; t_i) = U(\nu; t_i)$ for all i, as required.

Now let $x \in L$. Suppose that $\mu(x) = t_i$ for some i. Then $x \notin U(\mu; t_j)$, for all $i + 1 \leq j \leq n$. This implies that $x \notin U(\nu; t_j)$ for all $i + 1 \leq j \leq n$. But then $\nu(x) < t_j$, for all $i + 1 \leq j \leq n$. Suppose that $\nu(x) = t_m$ for some $i \leq m \leq i$. If $i \neq m$, then $x \in U(\nu; t_i)$. On the other hand, since $\mu(x) = t_i$, $x \in U(\mu; t_i) = U(\nu; t_i)$, and hence we obtain a contradiction. Thus $i = m$ and $\mu(x) = t_i = \nu(x)$, and consequently $\mu = \nu$.

Theorem 4.6. *A Lie algebra L is Noetherian if and only if the set of values of any fuzzy Lie $\mathbb{F}$-ideal of L is a well ordered subset of* $[0, 1]$.

Proof. We first suppose that μ is a fuzzy Lie $\mathbb{F}$-ideal of L whose set of values is not a well ordered subset of $[0, 1]$. Then considering the strictly decreasing sequence $\{\lambda_n\}$ such that $\mu(x_n) = \lambda_n$. Denote by U_n the set $\{x \in L \mid \mu(x) \geq \lambda_n\}$. Then $U_1 \subset U_2 \subset U_3 \cdots$ is a strictly ascending chain of ideals of L, which contradicts that L is Noetherian.

Conversely, assume that the set values of any fuzzy Lie $\mathbb{F}$-ideal of L is a well ordered subset of $[0, 1]$ and L is not Noetherian Lie algebra. Then there exists a strictly ascending chain

$$U_1 \subset U_2 \subset U_3 \cdots \tag{$*$}$$

of ideals of L. Define a fuzzy set μ in L by

$$\mu(x) := \begin{cases} \dfrac{1}{k} & \text{for } x \in U_k \backslash U_{k-1}, \\ 0 & \text{for } x \notin \displaystyle\bigcup_{k=1}^{\infty} U_k, \end{cases}$$

Now,by using similar argument as Theorem 4.4, one can easily show that μ is a fuzzy Lie $\mathbb{F}$-ideal of L. Since the chain $(*)$ is not terminating, μ has a strictly descending sequence of values which leads to a contradiction. Thus, L is Noetherian.

The following propositions follows easily and we omit their proofs.

Proposition 4.7. *Let* $L = \{\lambda_n \in (0,1) \mid n \in \mathbb{N}\} \cup \{0\}$, *where* $\lambda_i > \lambda_j$ *whenever* $i < j$. *If* $\{U_n \mid n \in \mathbb{N}\}$ *is a family of ideals of Lie algebra* L *such that* $U_1 \subset U_2 \subset U_3 \subset \cdots$, *then the fuzzy set* μ *in* L *defined by*

$$\mu(x) := \begin{cases} \lambda_1 & \text{if } x \in U_1, \\ \lambda_n & \text{if } x \in U_n \setminus U_{n-1}, n = 2, 3, ... \\ 0 & \text{if } x \in L \setminus \bigcup\limits_{n=1}^{\infty} U_n, \end{cases}$$

is a fuzzy Lie $\mathbb{F}$*-ideal of* L.

Proposition 4.8. *Let* $L = \{\lambda_1, \lambda_2, ..., \lambda_n, ...\} \cup \{0\}$, *where* $\{\lambda_n\}$ *is a fixed sequence, strictly decreasing to* 0 *and* $0 < \lambda_n < 1$. *Then a Lie algebra* L *is Noetherian if and only if for each fuzzy Lie* $\mathbb{F}$*-ideal* μ *of* L, $Im(\mu) \subset L \Longrightarrow \exists\, n_0 \in \mathbb{N}$ *such that* $Im(\mu) \subset \{\lambda_1, \lambda_2, ..., \lambda_{n_0}\} \cup \{0\}$.

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ON A FINER TOPOLOGICAL SPACE THAN τ_θ AND SOME MAPS

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Abstract. In 1943, Fomin [7] introduced the notion of θ-continuity. In 1966, the notions of θ-open subsets, θ-closed subsets and θ-closure were introduced by Veličko [18] for the purpose of studying the important class of H-closed spaces in terms of arbitrary filterbases. He also showed that the collection of θ-open sets in a topological space (X, τ) forms a topology on X denoted by τ_θ (see also [12]). Dickman and Porter [4], [5], Joseph [11] continued the work of Veličko. Noiri and Jafari [15], Caldas et al. [1] and [2], Steiner [16] and Cao et al [3] have also obtained several new and interesting results related to these sets.

In this paper, we will offer a finer topology on X than τ_θ by utilizing the new notions of ω_θ-open and ω_θ-closed sets. We will also discuss some of the fundamental properties of such sets and some related maps.

Key words and phrases: topological spaces, θ-open sets, θ-closed sets, ω_θ-open sets, ω_θ-closed sets, anti locally countable, ω_θ-continuity.

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1. Introduction

In 1982, Hdeib [8] introduced the notion of ω-closedness by which he introduced and investigated the notion of ω-continuity. In 1943, Fomin [7] introduced the notion of θ-continuity. In 1966, the notions of θ-open subsets, θ-closed subsets and θ-closure were introduced by Veličko [18] for the purpose of studying the important class of H-closed spaces in terms of arbitrary filterbases. He also showed that the collection of θ-open sets in a topological space (X, τ) forms a topology on X denoted by τ_θ (see also [12]). Dickman and Porter [4], [5], Joseph [11] continued the work of Veličko. Noiri and Jafari [15], Caldas et al. [1] and [2], Steiner [16] and Cao et al [3] have also obtained several new and interesting results related to these sets. In this paper, we will offer a finer topology on X than τ_θ by utilizing the new notions of ω_θ-open and ω_θ-closed sets. We will also discuss some of the fundamental properties of such sets and some related maps.

Throughout this paper, by a space we will always mean a topological space. For a subset A of a space X, the closure and the interior of A will be denoted by $cl(A)$ and $int(A)$, respectively. A subset A of a space X is said to be α-open [14] (resp. preopen [13], regular open [17], regular closed [17]) if $A \subset int(cl(int(A)))$ (resp. $A \subset int(cl(A))$, $A = int(cl(A))$, $A = cl(int(A))$).

A point $x \in X$ is said to be in the θ-closure [18] of a subset A of X, denoted by $\theta\text{-}cl(A)$, if $cl(U) \cap A \neq \varnothing$ for each open set U of X containing x. A subset A of a space X is called θ-closed if $A = \theta\text{-}cl(A)$. The complement of a θ-closed set is called θ-open. The θ-interior of a subset A of X is the union of all open sets of X whose closures are contained in A and is denoted by $\theta\text{-}int(A)$. Recall that a point p is a condensation point of A if every open set containing p must contain uncountably many points of A. A subset A of a space X is ω-closed [8] if it contains all of its condensation points. The complement of an ω-closed subset is called ω-open. It was shown that the collection of all ω-open subsets forms a topology that is finer than the original topology on X. The union of all ω-open sets of X contained in a subset A is called the ω-interior of A and is denoted by $\omega\text{-}int(A)$.

The family of all ω-open (resp. θ-open, α-open) subsets of a space (X, τ) is denoted by $\omega O(X)$ (resp, $\tau_\theta = \theta O(X)$, $\alpha O(X)$).

A function $f : X \to Y$ is said to be ω-continuous [9] (resp. θ-continuous [7]) if $f^{-1}(V)$ is ω-open (resp. θ-open) in X for every open subset V of Y. A function $f : X \to Y$ is called weakly ω-continuous [6] if for each $x \in X$ and each open subset V in Y containing $f(x)$, there exists an ω-open subset U in X containing x such that $f(U) \subset cl(V)$.

2. A finer topology than τ_θ

Definition 1 A subset A of a space (X, τ) is called ω_θ-open if for every $x \in A$, there exists an open subset $B \subset X$ containing x such that $B \backslash \theta\text{-}int(A)$ is countable. The complement of an ω_θ-open subset is called ω_θ-closed.

The family of all ω_θ-open subsets of a space (X, τ) is denoted by $\omega_\theta O(X)$.

Theorem 2 *$(X, \omega_\theta O(X))$ is a topological space for a topological space (X, τ).*

Proof. It is obvious that $\varnothing, X \in \omega_\theta O(X)$. Let $A, B \in \omega_\theta O(X)$ and $x \in A \cap B$. There exist the open sets $U, V \subset X$ containing x such that $U \backslash \theta - int(A)$ and $V \backslash \theta - int(B)$ are countable. Then

$$(U \cap V) \backslash \theta - int(A \cap B)$$

$$= (U \cap V) \backslash [\theta - int(A) \cap \theta - int(B)] \subset [(U \backslash \theta - int(A)) \cup (V \backslash \theta - int(B))].$$

Thus, $(U \cap V) \backslash \theta - int(A \cap B)$ is countable and hence $A \cap B \in \omega_\theta O(X)$. Let $\{A_i : i \in I\}$ be a family of ω_θ-open subsets of X and $x \in \cup_{i \in I} A_i$. Then $x \in A_j$ for some $j \in I$. This implies that there exists an open subset B of X containing x such that $B \backslash \theta - int(A_j)$ is countable.

Since $B \backslash \theta - int\left(\bigcup_{i \in I} A_i\right) \subset B \backslash \bigcup_{i \in I} \theta - int(A_i) \subset B \backslash \theta - int(A_j)$, then $B \backslash \theta - int\left(\bigcup_{i \in I} A_i\right)$ is countable. Hence, $\bigcup_{i \in I} A_i \in \omega_\theta O(X)$. ■

Theorem 3 *Let A be a subset of a space (X, τ). Then A is ω_θ-open if and only if for every $x \in A$, there exists an open subset U containing x and a countable subset V such that $U \backslash V \subset \theta$-$int(A)$.*

Proof. Let $A \in \omega_\theta O(X)$ and $x \in A$. Then there exists an open subset U containing x such that $U \backslash \theta - int(A)$ is countable.

Take $V = U \backslash \theta - int(A) = U \cap (X \backslash \theta - int(A))$. Thus, $U \backslash V \subset \theta - int(A)$.

Conversely, let $x \in A$. There exists an open subset U containing x and a countable subset V such that $U \backslash V \subset \theta - int(A)$. Hence, $U \backslash \theta - int(A)$ is countable. ■

Remark 4 The following diagram holds for a subset A of a space X:

$$\begin{tikzcd}
\omega_\theta\text{-open} \arrow[r] & \omega\text{-open} \\
\theta\text{-open} \arrow[r] \arrow[u] & \text{open} \arrow[u]
\end{tikzcd}$$

The following examples show that these implications are not reversible.

Example 5

(1) Let R be the real line with the topology $\tau = \{\varnothing, R, R \backslash (0,1)\}$. Then the set $R \backslash (0,1)$ is open but it is not ω_θ-open.

(2) Let R be the real line with the topology $\tau = \{\varnothing, R, Q'\}$ where Q' is the set of irrational numbers. Then the set $A = Q' \cup \{1\}$ is ω-open but it is not ω_θ-open.

Example 6 Let $X = \{a, b, c, d\}$ and $\tau = \{X, \varnothing, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Then the set $A = \{a, b, d\}$ is ω_θ-open but it is not open.

Theorem 7 *Let A be an ω_θ-closed subset of a space X. Then θ-$cl(A) \subset K \cup V$ for a closed subset K and a countable subset V.*

Proof. Since A is ω_θ-closed, then $X \backslash A$ is ω_θ-open. For every $x \in X \backslash A$ there exists an open set U containing x and a countable set V such that

$$U \backslash V \subset \theta - int(X \backslash A) = X \backslash \theta - cl(A).$$

Hence,

$$\theta - cl\,(A) \subset X \backslash\,(U \backslash V) = X \cap ((X \backslash U) \cup V) = (X \backslash U) \cup V.$$

Take $K = X \backslash U$. Thus, K is closed and $\theta - cl(A) \subset K \cup V$. ■

Definition 8 The intersection of all ω_θ-closed sets of X containing a subset A is called the ω_θ-closure of A and is denoted by ω_θ-$cl(A)$. The union of all ω_θ-open sets of X contained in a subset A is called the ω_θ-interior of A and is denoted by ω_θ-$int(A)$.

Lemma 9 *Let A be a subset of a space X. Then*

(1) ω_θ-$cl(A)$ *is* ω_θ*-closed in* X.

(2) ω_θ-$cl(X \backslash A) = X \backslash \omega_\theta - int(A)$.

(3) $x \in \omega_\theta$-$cl(A)$ *if and only if* $A \cap G \neq \emptyset$ *for each* ω_θ*-open set* G *containing* x.

(4) A *is* ω_θ*-closed in* X *if and only if* $A = \omega_\theta$-$cl(A)$.

Definition 10 A subset A of a topological space (X, τ) is said to be an (ω_θ, ω)-set if ω_θ-$int(A) = \omega - int(A)$.

Definition 11 A subset A of a topological space (X, τ) is said to be an (ω_θ, θ)-set if $\omega_\theta - int(A) = \theta - int(A)$.

Remark 12 Every ω_θ-open set is an (ω_θ, ω)-set and every θ-open set is an (ω_θ, θ)-set but not conversely.

Example 13

(1) Let R be the real line with the topology $\tau = \{\varnothing, R, Q'\}$ where Q' is the set of irrational numbers. Then the natural number set N is an (ω_θ, ω)-set but it is not ω_θ-open.

(2) Let R be the real line with the topology $\tau = \{\varnothing, R, (2, 3)\}$. Then the set $A = (1, \frac{3}{2})$ is an (ω_θ, θ)-set but it is not θ-open.

Theorem 14 *Let A be a subset of a space X. Then A is ω_θ-open if and only if A is ω-open and an (ω_θ, ω)-set.*

Proof. Since every ω_θ-open is ω-open and an (ω_θ, ω)-set, it is obvious.

Conversely, let A be an ω-open and (ω_θ, ω)-set. Then

$$A = \omega - int(A) = \omega_\theta - int(A).$$

Thus, A is ω_θ-open. ■

Theorem 15 *Let A be a subset of a space X. Then A is θ-open if and only if A is ω_θ-open and an (ω_θ, θ)-set.*

Proof. Necessity. It follows from the fact that every θ-open set is ω_θ-open and an (ω_θ, θ)-set.

Sufficiency. Let A be an ω_θ-open and (ω_θ, θ)-set. Then

$$A = \omega_\theta - int(A) = \theta - int(A).$$

Thus, A is θ-open. ■

Recall that a space X is called locally countable if each $x \in X$ has a countable neighborhood.

Theorem 16 *Let (X, τ) be a locally countable space and $A \subset X$.*

(1) *$\omega_\theta O(X)$ is the discrete topology.*

(2) *A is ω_θ-open if and only if A is ω-open.*

Proof. (1) : Let $A \subset X$ and $x \in A$. Then there exists a countable neighborhood B of x and there exists an open set U containing x such that $U \subset B$. We have $U \backslash \theta - int(A) \subset B \backslash$ θ-$int(A) \subset B$. Thus $U \backslash \theta - int(A)$ is countable and A is ω_θ-open. Hence, $\omega_\theta O(X)$ is the discrete topology.

(2) : Necessity. It follows from the fact that every ω_θ-open set is ω-open.

Sufficiency. Let A be an ω-open subset of X. Since X is a locally countable space, then A is ω_θ-open. ■

Corollary 17 *If (X, τ) is a countable space, then $\omega_\theta O(X)$ is the discrete topology.*

A space X is called anti locally countable if nonempty open subsets are uncountable. As an example, observe that in Example 5 (1), the topological space (R, τ) is anti locally countable.

Theorem 18 *Let (X, τ) be a topological space and $A \subset X$. The following hold:*

(1) *If X is an anti locally countable space, then $(X, \omega_\theta O(X))$ is anti locally countable.*

(2) *If X is anti locally countable regular space and A is θ-open, then*

$$\theta - cl(A) = \omega_\theta - cl(A).$$

Proof. (1) : Let $A \in \omega_\theta O(X)$ and $x \in A$. There exists an open subset $U \subset X$ containing x and a countable set V such that $U \backslash V \subset \theta\text{-}int(A)$. Thus, $\theta\text{-}int(A)$ is uncountable and A is uncountable.

(2) : It is obvious that $\omega_\theta\text{-}cl(A) \subset \theta\text{-}cl(A)$.

Let $x \in \theta\text{-}cl(A)$ and B be an ω_θ-open subset containing x. There exists an open subset V containing x and a countable set U such that $V \backslash U \subset \theta\text{-}int(B)$. Then $(V \backslash U) \cap A \subset \theta\text{-}int(B) \cap A$ and $(V \cap A) \backslash U \subset \theta\text{-}int(B) \cap A$. Since X is regular, $x \in V$ and $x \in \theta\text{-}cl(A)$, then $V \cap A \neq \varnothing$. Since X is regular and V and A are ω_θ-open, then $V \cap A$ is ω_θ-open. This implies that $V \cap A$ is uncountable and hence $(V \cap A) \backslash U$ is uncountable. Since $B \cap A$ contains the uncountable set $\theta\text{-}int(B) \cap A$, then $B \cap A$ is uncountable. Thus, $B \cap A \neq \varnothing$ and $x \in \omega_\theta\text{-}cl(A)$. ■

Corollary 19 *Let (X, τ) be an anti locally countable regular space and $A \subset X$. The following hold:*

(1) *If A is θ-closed, then $\theta - int(A) = \omega_\theta - int(A)$.*

(2) *The family of (ω_θ, θ)-sets contains all θ-closed subsets of X.*

Theorem 20 *If X is a Lindelof space, then $A \backslash \theta - int(A)$ is countable for every closed subset $A \in \omega_\theta O(X)$.*

Proof. Let $A \in \omega_\theta O(X)$ be a closed set. For every $x \in A$, there exists an open set U_x containing x such that $U_x \backslash \theta - int(A)$ is countable. Thus, $\{U_x : x \in A\}$ is an open cover for A. Since A is Lindelof, it has a countable subcover $\{U_n : n \in N\}$. Since $A \backslash \theta - int(A) = \bigcup\limits_{n \in N} (U_n \backslash \theta - int(A))$, then $A \backslash \theta - int(A)$ is countable. ■

Theorem 21 *If A is ω_θ-open subset of (X, τ), then $\omega_\theta O(X)|_A \subset \omega_\theta O(A)$.*

Proof. Let $G \in \omega_\theta O(X)|_A$. We have $G = V \cap A$ for some ω_θ-open subset V. Then for every $x \in G$, there exist $U, W \in \tau$ containing x and countable sets K and L such that

$$U \backslash K \subset \theta - int(V) \text{ and } W \backslash L \subset \theta - int(A).$$

We have $x \in A \cap (U \cap W) \in \tau|_A$. Thus, $K \cup L$ is countable and

$$\begin{aligned} A \cap (U \cap W) \backslash (K \cup L) &\subset (U \cap W) \cap (X \backslash K) \cap (X \backslash L) \\ &= (U \backslash K) \cap (W \backslash L) \subset \theta - int(V) \cap \theta - int(A) \cap A \\ &= \theta - int(V \cap A) \cap A \\ &= \theta - int(G) \cap A \subset \theta - int_A(G). \end{aligned}$$

Hence, $G \in \omega_\theta O(A)$. ■

3. Continuities via ω_θ-open sets

Definition 22 A function $f : X \to Y$ is said to be ω_θ-continuous if for every $x \in X$ and every open subset V in Y containing $f(x)$, there exists an ω_θ-open subset U in X containing x such that $f(U) \subset V$.

Theorem 23 *For a function $f : X \to Y$, the following are equivalent:*

(1) *f is ω_θ-continuous.*

(2) *$f^{-1}(A)$ is ω_θ-open in X for every open subset A of Y,*

(3) *$f^{-1}(K)$ is ω_θ-closed in X for every closed subset K of Y.*

Proof. (1) $\Rightarrow$ (2) : Let A be an open subset of Y and $x \in f^{-1}(A)$. By (1), there exists an ω_θ-open set B in X containing x such that $B \subset f^{-1}(A)$. Hence, $f^{-1}(A)$ is ω_θ-open.

(2) $\Rightarrow$ (1) : Let A be an open subset in Y containing $f(x)$. By (2), $f^{-1}(A)$ is ω_θ-open. Take $B = f^{-1}(A)$. Hence, f is ω_θ-continuous.

(2) $\Leftrightarrow$ (3) : Let K be a closed subset of Y. By (2), $f^{-1}(Y \backslash K) = X \backslash f^{-1}(K)$ is ω_θ-open. Hence, $f^{-1}(K)$ is ω_θ-closed. ■

Theorem 24 *The following are equivalent for a function $f : X \to Y$:*

(1) *f is ω_θ-continuous.*

(2) *$f : (X, \omega_\theta O(X)) \to (Y, \sigma)$ is continuous.*

Definition 25 A function $f : X \to Y$ is called weakly ω_θ-continuous at $x \in X$ if for every open subset V in Y containing $f(x)$, there exists an ω_θ-open subset U in X containing x such that $f(U) \subset cl(V)$. If f is weakly ω_θ-continuous at every $x \in X$, it is said to be weakly ω_θ-continuous.

Remark 26 The following diagram holds for a function $f : X \to Y$:

$$\begin{tikzcd}
\text{weakly } \omega_\theta\text{-continuous} \arrow[r] & \text{weakly } \omega\text{-continuous} \\
\omega_\theta\text{-continuous} \arrow[r] \arrow[u] & \omega\text{-continuous} \arrow[u] \\
\theta\text{-continuous} \arrow[r] \arrow[u] & \text{continuous} \arrow[u]
\end{tikzcd}$$

The following examples show that these implications are not reversible.

Example 27 Let R be the real line with the topology $\tau = \{\varnothing, R, (2,3)\}$. Let $Y = \{a, b, c\}$ and $\sigma = \{Y, \varnothing, \{a\}, \{c\}, \{a, c\}\}$.

Define a function $f : (X, \tau) \to (Y, \sigma)$ as follows:

$$f(x) = \begin{cases} a & , \text{if } x \in (0,1) \\ b & , \text{if } x \notin (0,1) \end{cases} .$$

Then f is weakly ω_θ-continuous but it is not ω_θ-continuous.

Example 28 Let R be the real line with the topology $\tau = \{\varnothing, R, Q'\}$ where Q' is the set of irrational numbers. Let $Y = \{a, b, c, d\}$ and $\sigma = \{Y, \varnothing, \{c\}, \{d\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}$. Define a function $f : (R, \tau) \to (Y, \sigma)$ as follows:

$$f(x) = \begin{cases} a & \text{, if } x \in Q' \cup \{1\} \\ b & \text{, if } x \notin Q' \cup \{1\} \end{cases}.$$

Then f is ω-continuous but it is not weakly ω_θ-continuous.

Example 29 Let $X = \{a, b, c, d\}$ and $\tau = \{X, \varnothing, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ as follows: $f(a) = a$, $f(b) = a$, $f(c) = c$, $f(d) = a$. Then f is ω_θ-continuous but it is not θ-continuous.

For the other implications, the contra examples are as shown in [6, 9].

Definition 30 A function $f : X \to Y$ is said to be (ω_θ, ω)-continuous if $f^{-1}(A)$ is an (ω_θ, ω)-set for every open subset A of Y.

Definition 31 A function $f : X \to Y$ is said to be (ω_θ, θ)-continuous if $f^{-1}(A)$ is an (ω_θ, θ)-set for every open subset A of Y.

Remark 32 Every θ-continuous function is (ω_θ, θ)-continuous and every ω_θ-continuous function is (ω_θ, ω)-continuous but not conversely.

Example 33 Let R be the real line with the topology $\tau = \{\varnothing, R, Q'\}$ where Q' is the set of irrational numbers. Define a function $f : (R, \tau) \to (R, \tau)$ as follows:

$$f(x) = \begin{cases} \pi & \text{, if } x \in N \\ 1 & \text{, if } x \notin N \end{cases}.$$

Then f is (ω_θ, ω)-continuous but it is not ω_θ-continuous.

Example 34 Let R be the real line with the topology $\tau = \{\varnothing, R, (2, 3)\}$. Let $A = (1, \frac{3}{2})$ and $\sigma = \{R, \varnothing, A, R \backslash A\}$. Define a function $f : (R, \tau) \to (R, \sigma)$ as follows:

$$f(x) = \begin{cases} \frac{5}{4} & \text{, if } x \in (1, 2) \\ 4 & \text{, if } x \notin (1, 2) \end{cases}.$$

Then f is (ω_θ, θ)-continuous but it is not θ-continuous.

Definition 35 A function $f : X \to Y$ is coweakly ω_θ-continuous if for every open subset A in Y, $f^{-1}(fr(A))$ is ω_θ-closed in X, where $fr(A) = cl(A) \backslash int(A)$.

Theorem 36 *Let $f : X \to Y$ be a function. The following are equivalent:*

(1) *f is ω_θ-continuous,*

(2) *f is ω-continuous and (ω_θ, ω)-continuous,*

(3) *f is weakly ω_θ-continuous and coweakly ω_θ-continuous.*

Proof. (1) $\Leftrightarrow$ (2) : It is an immediate consequence of Theorem 14.

(1) $\Rightarrow$ (3) : Obvious.

(3) $\Rightarrow$ (1) : Let f be weakly ω_θ-continuous and coweakly ω_θ-continuous. Let $x \in X$ and V be an open subset of Y such that $f(x) \in V$. Since f is weakly ω_θ-continuous, then there exists an ω_θ-open subset U of X containing x such that $f(U) \subset cl(V)$. We have $fr(V) = cl(V)\backslash V$ and $f(x) \notin fr(V)$. Since f is coweakly ω_θ-continuous, then $x \in U\backslash f^{-1}(fr(V))$ is ω_θ-open in X. For every $y \in f(U\backslash f^{-1}(fr(V)))$, $y = f(x_1)$ for a point $x_1 \in U\backslash f^{-1}(fr(V))$. We have $f(x_1) = y \in f(U) \subset cl(V)$ and $y \notin fr(V)$. Hence, $f(x_1) = y \notin fr(V)$ and $f(x_1) \in V$. Thus, $f(U\backslash f^{-1}(fr(V))) \subset V$ and f is ω_θ-continuous. ■

Theorem 37 *The following are equivalent for a function $f : X \to Y$:*

(1) *f is θ-continuous,*

(2) *f is ω_θ-continuous and (ω_θ, θ)-continuous.*

Proof. It is an immediate consequence of Theorem 15. ■

Theorem 38 *Let $f : X \to Y$ be a function. The following are equivalent:*

(1) *f is weakly ω_θ-continuous,*

(2) *ω_θ-$cl(f^{-1}(int(cl(K)))) \subset f^{-1}(cl(K))$ for every subset K of Y,*

(3) *ω_θ-$cl(f^{-1}(int(A))) \subset f^{-1}(A)$ for every regular closed set A of Y,*

(4) *ω_θ-$cl(f^{-1}(A)) \subset f^{-1}(cl(A))$ for every open set A of Y,*

(5) *$f^{-1}(A) \subset \omega_\theta$-$int(f^{-1}(cl(A)))$ for every open set A of Y,*

(6) *ω_θ-$cl(f^{-1}(A)) \subset f^{-1}(cl(A))$ for each preopen set A of Y,*

(7) *$f^{-1}(A) \subset \omega_\theta$-$int(f^{-1}(cl(A)))$ for each preopen set A of Y.*

Proof. (1) $\Rightarrow$ (2) : Let $K \subset Y$ and $x \in X\backslash f^{-1}(cl(K))$. Then $f(x) \in Y\backslash cl(K)$. This implies that there exists an open set A containing $f(x)$ such that $A \cap K = \emptyset$. We have, $cl(A) \cap int(cl(K)) = \emptyset$. Since f is weakly ω_θ-continuous, then there exists an ω_θ-open set B containing x such that $f(B) \subset cl(A)$. We have $B \cap f^{-1}(int(cl(K))) = \emptyset$. Thus, $x \in X\backslash\omega_\theta$-$cl(f^{-1}(int(cl(K))))$ and ω_θ-$cl(f^{-1}(int(cl(K)))) \subset f^{-1}(cl(K))$.

(2) $\Rightarrow$ (3) : Let A be any regular closed set in Y. Thus, ω_θ-$cl(f^{-1}(int(A))) = \omega_\theta$-$cl(f^{-1}(int(cl(int(A))))) \subset f^{-1}(cl(int(A))) = f^{-1}(A)$.

(3) $\Rightarrow$ (4) : Let A be an open subset of Y. Since $cl(A)$ is regular closed in Y, ω_θ-$cl(f^{-1}(A)) \subset \omega_\theta$-$cl(f^{-1}(int(cl(A)))) \subset f^{-1}(cl(A))$.

(4) $\Rightarrow$ (5) : Let A be any open set of Y. Since $Y\backslash cl(A)$ is open in Y, then $X\backslash\omega_\theta$-$int(f^{-1}(cl(A))) = \omega_\theta$-$cl(f^{-1}(Y\backslash cl(A))) \subset f^{-1}(cl(Y\backslash cl(A))) \subset X\backslash f^{-1}(A)$. Thus, $f^{-1}(A) \subset \omega_\theta$-$int(f^{-1}(cl(A)))$.

(5) $\Rightarrow$ (1) : Let $x \in X$ and A be any open subset of Y containing $f(x)$. Then $x \in f^{-1}(A) \subset \omega_\theta\text{-}int(f^{-1}(cl(A)))$. Take $B = \omega_\theta\text{-}int(f^{-1}(cl(A)))$. Thus $f(B) \subset cl(A)$ and f is weakly ω_θ-continuous at x in X.

(1) $\Rightarrow$ (6) : Let A be any preopen set of Y and $x \in X\backslash f^{-1}(cl(A))$. Then there exists an open set W containing $f(x)$ such that $W \cap A = \emptyset$. We have $cl(W \cap A) = \emptyset$. Since A is preopen, then $A \cap cl(W) \subset int(cl(A)) \cap cl(W) \subset cl(int(cl(A)) \cap W)$ $\subset cl(int(cl(A) \cap W)) \subset cl(int(cl(A \cap W))) \subset cl(A \cap W) = \emptyset$. Since f is weakly ω_θ-continuous and W is an open set containing $f(x)$, there exists an ω_θ-open set B in X containing x such that $f(B) \subset cl(W)$. We have $f(B) \cap A = \emptyset$ and hence $B \cap f^{-1}(A) = \emptyset$. Thus, $x \in X\backslash\omega_\theta\text{-}cl(f^{-1}(A))$ and $\omega_\theta\text{-}cl(f^{-1}(A)) \subset f^{-1}(cl(A))$.

(6) $\Rightarrow$ (7) : Let A be any preopen set of Y. Since $Y\backslash cl(A)$ is open in Y, then $X \backslash \omega_\theta\text{-}int(f^{-1}(cl(A))) = \omega_\theta\text{-}cl(f^{-1}(Y \backslash cl(A))) \subset f^{-1}(cl(Y \backslash cl(A))) \subset X \backslash f^{-1}(A)$. Hence, $f^{-1}(A) \subset \omega_\theta\text{-}int(f^{-1}(cl(A)))$.

(7) $\Rightarrow$ (1) : Let $x \in X$ and A any open set of Y containing $f(x)$. Then $x \in f^{-1}(A) \subset \omega_\theta\text{-}int(f^{-1}(cl(A)))$. Take $B = \omega_\theta\text{-}int(f^{-1}(cl(A)))$. Then $f(B) \subset cl(A)$. Thus, f is weakly ω_θ-continuous at x in X. ■

Theorem 39 *The following properties are equivalent for a function $f : X \to Y$:*

(1) *$f : X \to Y$ is weakly ω_θ-continuous at $x \in X$.*

(2) *$x \in \omega_\theta\text{-}int(f^{-1}(cl(A)))$ for each neighborhood A of $f(x)$.*

Proof. (1) $\Rightarrow$ (2) : Let A be any neighborhood of $f(x)$. There exists an ω_θ-open set B containing x such that $f(B) \subset cl(A)$. Since $B \subset f^{-1}(cl(A))$ and B is ω_θ-open, then $x \in B \subset \omega_\theta - int(B) \subset \omega_\theta - int(f^{-1}(cl(A)))$.

(2) $\Rightarrow$ (1) : Let $x \in \omega_\theta - int(f^{-1}(cl(A)))$ for each neighborhood A of $f(x)$. Take $U = \omega_\theta - int(f^{-1}(cl(A)))$. Thus, $f(U) \subset cl(A)$ and U is ω_θ-open. Hence, f is weakly ω_θ-continuous at $x \in X$. ■

Theorem 40 *Let $f : X \to Y$ be a function. The following are equivalent:*

(1) *f is weakly ω_θ-continuous,*

(2) *$f(\omega_\theta\text{-}cl(K)) \subset \theta\text{-}cl(f(K))$ for each subset K of X,*

(3) *$\omega_\theta\text{-}cl(f^{-1}(A)) \subset f^{-1}(\theta\text{-}cl(A))$ for each subset A of Y,*

(4) *$\omega_\theta\text{-}cl(f^{-1}(int(\theta\text{-}cl(A)))) \subset f^{-1}(\theta\text{-}cl(A))$ for every subset A of Y.*

Proof. (1) $\Rightarrow$ (2) : Let $K \subset X$ and $x \in \omega_\theta\text{-}cl(K)$. Let U be any open set of Y containing $f(x)$. Then there exists an ω_θ-open set B containing x such that $f(B) \subset cl(U)$. Since $x \in \omega_\theta\text{-}cl(K)$, then $B \cap K \neq \emptyset$. Thus, $\emptyset \neq f(B) \cap f(K) \subset cl(U) \cap f(K)$ and $f(x) \in \theta\text{-}cl(f(K))$. Hence, $f(\omega_\theta\text{-}cl(K)) \subset \theta\text{-}cl(f(K))$.

(2) $\Rightarrow$ (3) : Let $A \subset Y$. Then $f(\omega_\theta\text{-}cl(f^{-1}(A))) \subset \theta\text{-}cl(A)$. Thus, $\omega_\theta\text{-}cl(f^{-1}(A)) \subset f^{-1}(\theta\text{-}cl(A))$.

(3) $\Rightarrow$ (4) : Let $A \subset Y$. Since $\theta\text{-}cl(A)$ is closed in Y, then $\omega_\theta\text{-}cl(f^{-1}(int(\theta\text{-}cl(A)))) \subset f^{-1}(\theta\text{-}cl(int(\theta\text{-}cl(A)))) = f^{-1}(cl(int(\theta\text{-}cl(A)))) \subset f^{-1}(\theta\text{-}cl(A))$.

(4) $\Rightarrow$ (1) : Let U be any open set of Y. Then $U \subset int(cl(U)) = int(\theta\text{-}cl(U))$. Thus, $\omega_\theta\text{-}cl(f^{-1}(U)) \subset \omega_\theta\text{-}cl(f^{-1}(int(\theta\text{-}cl(U)))) \subset f^{-1}(\theta\text{-}cl(U)) = f^{-1}(cl(U))$. By Theorem 38, f is weakly ω_θ-continuous. ■

Recall that a space is rim-compact [10] if it has a basis of open sets with compact boundaries.

Theorem 41 *Let $f : X \to Y$ be a function with the closed graph. Suppose that X is regular and Y is a rim-compact space. Then f is weakly ω_θ-continuous if and only if f is ω_θ-continuous.*

Proof. Let $x \in X$ and A be any open set of Y containing $f(x)$. Since Y is rim-compact, there exists an open set B of Y such that $f(x) \in B \subset A$ and ∂B is compact. For each $y \in \partial B$, $(x, y) \in X \times Y \backslash G(f)$. Since $G(f)$ is closed, there exist open sets $U_y \subset X$ and $V_y \subset Y$ such that $x \in U_y$, $y \in V_y$ and $f(U_y) \cap V_y = \emptyset$. The family $\{V_y\}_{y \in \partial B}$ is an open cover of ∂B. Then there exist a finite number of points of ∂B, say, y_1, y_2, ..., y_n such that $\partial B \subset \cup\{V_{y_i}\}_{i=1}^n$. Take $K = \cap\{U_{y_i}\}_{i=1}^n$ and $L = \cup\{V_{y_i}\}_{i=1}^n$. Then K and L are open sets such that $x \in K$, $\partial B \subset L$ and $f(K) \cap \partial B \subset f(K) \cap L = \emptyset$. Since f is weakly ω_θ-continuous, there exists an ω_θ-open set G containing x such that $f(G) \subset cl(B)$. Take $U = K \cap G$. Then, U is an ω_θ-open set containing x, $f(U) \subset cl(B)$ and $f(U) \cap \partial B = \emptyset$. Hence, $f(U) \subset B \subset A$ and f is ω_θ-continuous.

The converse is obvious. ■

Definition 42 If a space X can not be written as the union of two nonempty disjoint ω_θ-open sets, then X is said to be ω_θ-connected.

Theorem 43 *If $f : X \to Y$ is a weakly ω_θ-continuous surjection and X is ω_θ-connected, then Y is connected.*

Proof. Suppose that Y is not connected. There exist nonempty open sets U and V of Y such that $Y = U \cup V$ and $U \cap V = \emptyset$. This implies that U and V are clopen in Y. By Theorem 38, $f^{-1}(U) \subset \omega_\theta\text{-}int(f^{-1}(cl(U))) = \omega_\theta\text{-}int(f^{-1}(U))$. Hence $f^{-1}(U)$ is ω_θ-open in X. Similarly, $f^{-1}(V)$ is ω_θ-open in X. Hence, $f^{-1}(U) \cap f^{-1}(V) = \emptyset$, $X = f^{-1}(U) \cup f^{-1}(V)$ and $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty. Thus, X is not ω_θ-connected. ■

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HOMOMORPHISMS AND EPIMORPHISMS OF SOME HYPERGROUPS

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Abstract. By a *homomorphism* of a hypergroup $(H, \circ)$ we mean a function $f : H \to H$ satisfying $f(x \circ y) \subseteq f(x) \circ f(y)$ for all $x, y \in H$. A homomorphism f of a hypergroup $(H, \circ)$ is called an *epimorphism* if $f(H) = H$. For a hypergroup $(H, \circ)$, denote by $\mathrm{Hom}(H, \circ)$ and $\mathrm{Epi}(H, \circ)$ the set of all homomorphisms and the set of all epimorphisms of $(H, \circ)$, respectively. For a positive integer n, let $(\mathbb{Z}, \circ_n)$ be the hypergroup where $x \circ_n y = x + y + n\mathbb{Z}$ for all $x, y \in \mathbb{Z}$. In this paper, we characterize the elements of $\mathrm{Hom}(\mathbb{Z}, \circ_n)$ and $\mathrm{Epi}(\mathbb{Z}, \circ_n)$. In addition, we show that $|\mathrm{Hom}(\mathbb{Z}, \circ_n)|=|\mathrm{Epi}(\mathbb{Z}, \circ_n)|=2^{\aleph_0}$.

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1. Introduction

A *hyperoperation* on a nonempty set H is a function $\circ : H \times H \to \mathcal{P}^*(H)$ where $\mathcal{P}(H)$ is the power set of H and $\mathcal{P}^*(H) = \mathcal{P}(H) \smallsetminus \{\emptyset\}$. The value of $(x, y) \in H \times H$ under $\circ$ is denoted by $x \circ y$. The system $(H, \circ)$ is called a *hypergroupoid.* For $A, B \subseteq H$ and $x \in H$, let

$$A \circ B = \bigcup_{\substack{a\in A \\ b\in B}} a \circ b,\ A \circ x = A \circ \{x\} \text{ and } x \circ A = \{x\} \circ A.$$

The hypergroupoid $(H, \circ)$ is called a *semihypergroup* if

$$x \circ (y \circ z) = (x \circ y) \circ z \quad \text{for all } x, y, z \in H.$$

A *hypergroup* is a semihypergroup $(H, \circ)$ satisfying

$$H \circ x = x \circ H = H \quad \text{for all } x \in H.$$

Then hypergroups are a generalization of groups.

By a *homomorphism* of a hypergroup $(H, \circ)$ we mean $f : H \to H$ such that

$$f(x \circ y) \subseteq f(x) \circ f(y) \text{ for all } x, y \in H.$$

If the equality is valid, f is called a *good homomorphism* of $(H, \circ)$. A [good] homomorphism of a hypergroup $(H, \circ)$ is called an [*good*] *epimorphism* of $(H, \circ)$ if $f(H) = H$. For a hypergroup $(H, \circ)$, let $\mathrm{Hom}(H, \circ)$ and $\mathrm{Epi}(H, \circ)$ be the set of all homomorphisms and the set of all epimorphisms of $(H, \circ)$, respectively.

If G is a group, N is a normal subgroup of G and $\circ_N$ is the hyperoperation on G defined by

$$x \circ_N y = xyN \quad \text{for all } x, y \in G,$$

then $(G, \circ_N)$ is a hypergroup ([2], p.11). Observe that if $N=\{e\}$, then $(G, \circ_N)=G$ where e is the identity of G. Let $\mathbb{Z}$ be the set of integers and n a positive integer. Then $(\mathbb{Z}, \circ_n)$ is a hypergroup where

$$x \circ_n y = x + y + n\mathbb{Z} \quad \text{for all } x, y \in \mathbb{Z}.$$

In [3], the authors characterized the good homomorphisms and good epimorphisms of the hypergroup $(\mathbb{Z}, \circ_n)$. Such homomorphisms were also counted in [3]

The cardinality of a set X is denoted by $|X|$.

For $a \in \mathbb{Z}$, let $g_a : \mathbb{Z} \to \mathbb{Z}$ be defined by $g_a(x) = ax$ for all $x \in \mathbb{Z}$. Then

$$\mathrm{Hom}(\mathbb{Z}, +) = \{g_a \mid a \in \mathbb{Z}\} \quad \text{and} \quad \mathrm{Epi}(\mathbb{Z}, +) = \{g_1, g_{-1}\}.$$

Hence $|\mathrm{Hom}(\mathbb{Z}, +)| = \aleph_0$ and $|\mathrm{Epi}(\mathbb{Z}, +)| = 2$.

Our objective is to

(1) characterize the elements of $\mathrm{Hom}(\mathbb{Z}, \circ_n)$ and $\mathrm{Epi}(\mathbb{Z}, \circ_n)$,

(2) show that $|\mathrm{Hom}(\mathbb{Z}, \circ_n)| = |\mathrm{Epi}(\mathbb{Z}, \circ_n)| = 2^{\aleph_0}$.

The following fact of infinite cardinal numbers will be used. If p is an infinite cardinal number, then $p^p = 2^p$ ([6], p.161).

We note here that some results of homomorphisms and good homomorphisms of certain hypergroups can be seen in [1]. Homomorphisms of some multiplicative hyperrings were also studied in [4] and [5].

Let $\mathbb{Z}^+$ stand for the set of positive integers.

2. Homomorphisms of the Hypergroup $(\mathbb{Z}, \circ_n)$

The following lemma is needed to characterize the elements of $\mathrm{Hom}(\mathbb{Z}, \circ_n)$.

Lemma 2.1. *Let G be a group, N a normal subgroup of G and $\circ_N$ the hyperoperation on G. Then the following statements hold for $f \in \mathrm{Hom}(G, \circ_N)$.*

(i) $f(N) \subseteq N$.

(ii) *For all $x \in G$, $f(xN) \subseteq f(x)N$.*

(iii) *For all* $x, y \in G$, $f(xyN) \subseteq f(xy)N = f(x)f(y)N$.
(iv) *For all* $x \in G$, $f(x^{-1}N) \subseteq f(x^{-1})N = f(x)^{-1}N$.
(v) *For all* $x \in G$ *and* $k \in \mathbb{Z}$, $f(x^kN) \subseteq f(x^k)N = f(x)^kN$.

Proof. First, we recall that for all $x, y \in G$, $xN \cap yN \neq \emptyset$ implies $xN = yN$.

(i) We have that

$$f(N) = f(eeN) = f(e \circ_N e) \subseteq f(e) \circ_N f(e) = f(e)f(e)N.$$

Then $f(e) \in f(N) \subseteq f(e)f(e)N$. Since G is cancellative, we have $e \in f(e)N$ which implies that $N = f(e)N$, so $f(N) \subseteq f(e)f(e)N = N$.

(ii) By (i), $f(e) \in N$. If $x \in G$, then

$$f(xN) = f(xeN) = f(x \circ_N e) \subseteq f(x) \circ_N f(e) = f(x)f(e)N = f(x)N.$$

(iii) Let $x, y \in G$. Then by (ii),

$$f(xyN) \subseteq f(xy)N.$$

We also have that

$$f(xyN) = f(x \circ_N y) \subseteq f(x) \circ_N f(y) = f(x)f(y)N.$$

This implies that $f(xy)N = f(x)f(y)N$. Hence (iii) holds.

(iv) If $x \in G$, then

$$f(N) = f(xx^{-1}N) = f(x \circ_N x^{-1}) \subseteq f(x)f(x^{-1})N.$$

But $f(N) \subseteq N$ by (i), so $f(N) \subseteq N \cap f(x)f(x^{-1})N$. Then $N = f(x)f(x^{-1})N$ which implies that $f(x^{-1})N = f(x)^{-1}N$. By (ii), $f(x^{-1}N) \subseteq f(x^{-1})N$. Hence (iv) holds.

(v) Let $x \in G$. Then by (ii), for all $k \in \mathbb{Z}$, $f(x^kN) \subseteq f(x^k)N$. It remains to show that $f(x^k)N = f(x)^kN$ for all $k \in \mathbb{Z}$. This is true for $k = 0$ and 1. Assume that $k \in \mathbb{Z}^+$ and $f(x^k)N = f(x)^kN$. Then

$$\begin{aligned} f(x^{k+1})N &= f(xx^k)N \\ &= f(x)f(x^k)N && \text{from (iii)} \\ &= f(x)(f(x^k)N) \\ &= f(x)(f(x)^kN) && \text{from the assumption} \\ &= f(x)^{k+1}N. \end{aligned}$$

This shows that $f(y^l)N = f(y)^lN$ for all $y \in G$ and $l \in \mathbb{Z}^+$. If $k \in \mathbb{Z}^+$, then

$$\begin{aligned} f(x^{-k})N &= f((x^{-1})^k)N \\ &= f(x^{-1})^kN \\ &= (f(x^{-1})N)\dots(f(x^{-1})N) && (k \text{ brackets}) \\ &= (f(x)^{-1}N)\dots(f(x)^{-1}N) && \text{from (iv)} \\ &= (f(x)^{-1})^kN \\ &= f(x)^{-k}N. \end{aligned}$$

Hence (v) is proved. ∎

Theorem 2.2. *For $f : \mathbb{Z} \to \mathbb{Z}$, the following statements are equivalent.*

(i) $f \in \operatorname{Hom}(\mathbb{Z}, \circ_n)$.

(ii) $f(x + n\mathbb{Z}) \subseteq xf(1) + n\mathbb{Z}$ *for all* $x \in \mathbb{Z}$.

(iii) *There exists an integer a such that* $f(x + n\mathbb{Z}) \subseteq xa + n\mathbb{Z}$ *for all* $x \in \mathbb{Z}$.

Proof. (i)⇒(ii) follows directly from Lemma 2.1(v).

(ii)⇒(iii) is evident.

(iii)⇒(i). Let $x, y \in \mathbb{Z}$. Then $f(x) \in f(x) + n\mathbb{Z}$ and $f(y) \in f(y) + n\mathbb{Z}$. Since $f(x) \in f(x + n\mathbb{Z}) \subseteq xa + n\mathbb{Z}$ and $f(y) \in f(y + n\mathbb{Z}) \subseteq ya + n\mathbb{Z}$, it follows that $f(x) + n\mathbb{Z} = xa + n\mathbb{Z}$ and $f(y) + n\mathbb{Z} = ya + n\mathbb{Z}$. Consequently,

$$\begin{aligned} f(x \circ_n y) &= f(x + y + n\mathbb{Z}) \subseteq (x + y)a + n\mathbb{Z} = xa + n\mathbb{Z} + ya + n\mathbb{Z} \\ &= f(x) + n\mathbb{Z} + f(y) + n\mathbb{Z} = f(x) + f(y) + n\mathbb{Z} = f(x) \circ_n f(y). \end{aligned}$$

Hence $f \in \operatorname{Hom}(\mathbb{Z}, \circ_n)$, as desired. ■

Remark 2.3. For $f : \mathbb{Z} \to \mathbb{Z}$ and $a \in \mathbb{Z}$, if f and a satisfies (iii) of Theorem 2.2, then $a \equiv f(1) \pmod{n}$ since $f(1) \in f(1 + n\mathbb{Z}) \subseteq a + n\mathbb{Z}$.

Recall that for any nonempty sets X and Y,

$$|\{f \mid f : X \to Y\}| = |Y|^{|X|}$$

and in particular, if X is an infinite set, then

$$|\{f \mid f : X \to X\}| = |X|^{|X|} = 2^{|X|}.$$

Lemma 2.4. *Let G be a group and N a normal subgroup of G. For $f \in \operatorname{Hom}(G)$, $f(N) \subseteq N$ if and only if $f \in \operatorname{Hom}(G, \circ_N)$.*

Proof. First, assume that $f(N) \subseteq N$. Then for all $x, y \in G$,

$$f(x \circ_N y) = f(xyN) = f(x)f(y)f(N) \subseteq f(x)f(y)N = f(x) \circ_N f(y).$$

Thus $f \in \operatorname{Hom}(G, \circ_N)$.

For the converse, assume that $f \in \operatorname{Hom}(G, \circ_N)$. Since $f \in \operatorname{Hom}(G)$, $f(e) = e$. Then

$$f(N) = f(eeN) = f(e \circ_N e) \subseteq f(e) \circ_N f(e) = f(e)f(e)N = N.$$ ■

Theorem 2.5. $\operatorname{Hom}(\mathbb{Z}, +) \subseteq \operatorname{Hom}(\mathbb{Z}, \circ_n)$.

Proof. Recall that

$$\operatorname{Hom}(\mathbb{Z}, +) = \{g_a \mid a \in \mathbb{Z}\}$$

where $g_a(x) = ax$ for all $x \in \mathbb{Z}$. Since $g_a(n\mathbb{Z}) = an\mathbb{Z} \subseteq n\mathbb{Z}$ for all $a \in \mathbb{Z}$, by Lemma 2.4, $g_a \in \operatorname{Hom}(\mathbb{Z}, \circ_n)$ for all $a \in \mathbb{Z}$ and the desired result follows. ■

From Theorem 2.5, we have $|\operatorname{Hom}(\mathbb{Z}, \circ_n)| \geq \aleph_0$. In fact, $\operatorname{Hom}(\mathbb{Z}, \circ_n)$ is an uncountable set, as shown by the next theorem.

Lemma 2.6. *If G is a group, then* $\operatorname{Hom}(G, \circ_G) = \{f \mid f : G \to G\}$.

Proof. If $f : G \to G$, then for all $x, y \in G$,

$$f(x \circ_G y) = f(xyG) = f(G) \subseteq G = f(x)f(y)G = f(x) \circ_G f(y),$$

so $f \in \operatorname{Hom}(G, \circ_G)$. Hence the result follows. ■

Theorem 2.7. $|\operatorname{Hom}(\mathbb{Z}, \circ_n)| = 2^{\aleph_0}$.

Proof. By Lemma 2.6, $\operatorname{Hom}(\mathbb{Z}, \circ_1) = \{f \mid f : \mathbb{Z} \to \mathbb{Z}\}$. Then

$$|\operatorname{Hom}(\mathbb{Z}, \circ_1)| = |\{f \mid f : \mathbb{Z} \to \mathbb{Z}\}| = \aleph_0^{\aleph_0} = 2^{\aleph_0}.$$

Next, assume that $n > 1$. Let $K = \{g \mid g : n\mathbb{Z} \to n\mathbb{Z}\}$. Then $|K| = \aleph_0^{\aleph_0} = 2^{\aleph_0}$. Recall that for each $x \in \mathbb{Z}$, there are unique $q_x \in \mathbb{Z}$ and $r_x \in \{0, 1, \ldots, n-1\}$ such that $x = nq_x + r_x$. For each $g \in K$, define $\bar{g} : \mathbb{Z} \to \mathbb{Z}$ by

$$\bar{g}(x) = r_x + g(nq_x) \text{ for all } x \in \mathbb{Z}.$$

Then for every $g \in K$, $\bar{g}_{|n\mathbb{Z}} = g$ and for $x \in \mathbb{Z}$,

$$\begin{aligned} \bar{g}(x + n\mathbb{Z}) &= \bar{g}(r_x + nq_x + n\mathbb{Z}) = \bar{g}(r_x + n\mathbb{Z}) = r_x + g(n\mathbb{Z}) \subseteq r_x + n\mathbb{Z} \\ &= r_x + nq_x + n\mathbb{Z} = x + n\mathbb{Z} \end{aligned}$$

By Theorem 2.2, we have $\bar{g} \in \operatorname{Hom}(\mathbb{Z}, \circ_n)$ for all $g \in K$. It follows that

$$\begin{aligned} 2^{\aleph_0} = |K| &= |\{\bar{g} \mid g \in K\}| \\ &\leq |\operatorname{Hom}(\mathbb{Z}, \circ_n)| \\ &\leq |\{f \mid f : \mathbb{Z} \to \mathbb{Z}\}| = \aleph_0^{\aleph_0} = 2^{\aleph_0} \end{aligned}$$

which implies that $|\operatorname{Hom}(\mathbb{Z}, \circ_n)| = 2^{\aleph_0}$.

Hence the theorem is proved. ■

3. Epimorphisms of the Hypergroup $(\mathbb{Z}, \circ_n)$

First, we provide the following general fact. It is used to characterize the elements of $\operatorname{Epi}(\mathbb{Z}, \circ_n)$.

Lemma 3.1. *Let G be a group and N a normal subgroup of G. If the index $[G : N]$ of N in G is finite and $f \in \operatorname{Epi}(G, \circ_N)$, then $f(xN) = f(x)N$ for all $x \in G$.*

Proof. Let $[G : N] = n$. Then there are $x_1, \ldots, x_n \in G$ such that $G = \bigcup_{i=1}^{n} x_i N$. Then $x_1N, \ldots, x_nN$ are mutually disjoint. By Lemma 2.1(ii), $f(x_iN) \subseteq f(x_i)N$ for all $i \in \{1, \ldots, n\}$. Hence

$$G = f\left(\bigcup_{i=1}^{n} x_i N\right) = \bigcup_{i=1}^{n} f(x_iN) \subseteq \bigcup_{i=1}^{n} f(x_i)N,$$

which implies that

$$G = \bigcup_{i=1}^{n} f(x_iN) = \bigcup_{i=1}^{n} f(x_i)N.$$

Since $[G : N] = n$, it follows that $f(x_1)N, \ldots, f(x_n)N$ are mutually disjoint. But $f(x_iN) \subseteq f(x_i)N$ for all $i \in \{1, \ldots, n\}$, thus we have

$$f(x_iN) = f(x_i)N \text{ for all } i \in \{1, \ldots, n\}.$$

Next, let $x \in G$. Then $xN = x_jN$ for some $j \in \{1, \ldots, n\}$. By Lemma 2.1(ii), $f(xN) \subseteq f(x)N$. Hence

$$f(x_j)N = f(x_jN) = f(xN) \subseteq f(x)N$$

which implies that $f(x)N = f(x_j)N$. Consequently,

$$f(xN) = f(x_jN) = f(x_j)N = f(x)N$$

■

Theorem 3.2. *For $f : \mathbb{Z} \to \mathbb{Z}$, $f \in \operatorname{Epi}(\mathbb{Z}, \circ_n)$ if and only if*

(i) *$f(x + n\mathbb{Z}) = xf(1) + n\mathbb{Z}$ for all $x \in \mathbb{Z}$ and*

(ii) *$f(1)$ and n are relatively prime.*

Proof. First, assume that $f \in \operatorname{Epi}(\mathbb{Z}, \circ_n)$. By Lemma 3.1, $f(x+n\mathbb{Z}) = f(x)+n\mathbb{Z}$ for all $x \in \mathbb{Z}$. But by Lemma 2.1(v), $f(x) + n\mathbb{Z} = xf(1) + n\mathbb{Z}$ for all $x \in \mathbb{Z}$, thus (i) holds. The fact that $f(\mathbb{Z}) = \mathbb{Z}$ and (i) yield

$$\mathbb{Z} = f\big(\bigcup_{x \in \mathbb{Z}} (x + n\mathbb{Z})\big) = \bigcup_{x \in \mathbb{Z}} (xf(1) + n\mathbb{Z}).$$

Then $1 \in yf(1) + n\mathbb{Z}$ for some $y \in \mathbb{Z}$. Thus $1 = yf(1) + tn$ for some $t \in \mathbb{Z}$ which implies that $f(1)$ and n are relatively prime. Therefore (ii) holds.

For the converse, assume that (i) and (ii) hold. Then from (i) and Theorem 2.2, $f \in \operatorname{Hom}(\mathbb{Z}, \circ_n)$. From (ii), $sf(1) + tn = 1$ for some $s, t \in \mathbb{Z}$. But since

$$\begin{aligned}
\text{for every } x \in \mathbb{Z},\ x + n\mathbb{Z} &= x(sf(1) + tn) + n\mathbb{Z} \\
&= xsf(1) + n\mathbb{Z} \\
&= f(xs + n\mathbb{Z}) \qquad\qquad \text{from (i)} \\
&\subseteq f(\mathbb{Z}),
\end{aligned}$$

it follows that $f(\mathbb{Z}) = \mathbb{Z}$. Hence $f \in \mathrm{Epi}(\mathbb{Z}, \circ_n)$. ∎

To show that $|\mathrm{Epi}(\mathbb{Z}, \circ_n)| = 2^{\aleph_0}$, the following lemma is also needed.

Lemma 3.3. *If X is an infinite set, then $|\{f : X \to X \mid f(X) = X\}| = 2^{|X|}$.*

Proof. Since X is an infinite set, there are subsets X_1 and X_2 such that $X = X_1 \cup X_2, X_1 \cap X_2 = \emptyset$ and $|X_1| = |X_2| = |X|$. Let b and c be two distinct fixed points in X. Then $|X_1| = |X \smallsetminus \{b\}|$. Let $\varphi : X_1 \to X \smallsetminus \{b\}$ be a bijection. For each nonempty subset Y of X_2, define $g_Y : X \to X$ by a bracket notation as follows:

$$g_Y = \begin{pmatrix} s & Y & t \\ \varphi(s) & b & c \end{pmatrix}_{\substack{s \in X_1 \\ t \in X_2 \smallsetminus Y}}$$

Then $g_Y(X) = X$ for every nonempty subset Y of X_2. If Y_1 and Y_2 are distinct nonempty subsets of X_2, then $g_{Y_1}^{-1}(b) = Y_1 \neq Y_2 = g_{Y_2}^{-1}(b)$, so $g_{Y_1} \neq g_{Y_2}$. Hence

$$\begin{aligned}
2^{|X|} = |X|^{|X|} = |\{f \mid f : X \to X\}| &\geq |\{f : X \to X \mid f(X) = X\}| \\
&\geq |\{g_Y \mid \emptyset \neq Y \subseteq X_2\}| \\
&= |\{Y \mid \emptyset \neq Y \subseteq X_2\}| \\
&= 2^{|X_2|} = 2^{|X|}
\end{aligned}$$

which implies that $|\{f : X \to X \mid f(X) = X\}| = 2^{|X|}$, as desired. ∎

Theorem 3.4. $|\mathrm{Epi}(\mathbb{Z}, \circ_n)| = 2^{\aleph_0}$.

Proof. By Lemma 2.6, we have that $\mathrm{Epi}(\mathbb{Z}, \circ_1) = \{f : \mathbb{Z} \to \mathbb{Z} \mid f(\mathbb{Z}) = \mathbb{Z}\}$. Then by Lemma 3.3, $|\mathrm{Epi}(\mathbb{Z}, \circ_1)| = 2^{\aleph_0}$.

Assume that $n > 1$. Let $L = \{g : n\mathbb{Z} \to n\mathbb{Z} \mid g(n\mathbb{Z}) = n\mathbb{Z}\}$. Also, by Lemma 3.3, $|L| = 2^{\aleph_0}$. For each $x \in \mathbb{Z}$, let $q_x, r_x \in \mathbb{Z}$ be such that $r_x \in \{0, 1, \ldots, n-1\}$ and $x = nq_x + r_x$. Note that q_x and r_x are unique. For each $g \in L$, define $\bar{g} : \mathbb{Z} \to \mathbb{Z}$ by

$$\bar{g}(x) = r_x + g(nq_x) \text{ for all } x \in \mathbb{Z}.$$

Then for $g \in L, \bar{g}_{|n\mathbb{Z}} = g$ and we can see from the proof of Theorem 2.7 with $g(n\mathbb{Z}) = n\mathbb{Z}$ that

$$\bar{g}(x + n\mathbb{Z}) = x + n\mathbb{Z} \text{ for all } x \in \mathbb{Z}.$$

It follows from Theorem 2.2 that $\bar{g} \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$ for all $g \in L$. We also have that

$$\bar{g}(\mathbb{Z}) = \bar{g}\big(\bigcup_{x\in\mathbb{Z}} (x+n\mathbb{Z})\big) = \bigcup_{x\in\mathbb{Z}} \bar{g}(x+n\mathbb{Z}) = \bigcup_{x\in\mathbb{Z}} (x+n\mathbb{Z}) = \mathbb{Z}.$$

Hence $\bar{g} \in \mathrm{Epi}(\mathbb{Z}, \circ_n)$ for all $g \in L$. Consequently,

$$\begin{aligned} 2^{\aleph_0} = |L| &= |\{\bar{g} \mid g \in L\}| \\ &\le |\mathrm{Epi}(\mathbb{Z}, \circ_n)| \\ &\le |\{f \mid f : \mathbb{Z} \to \mathbb{Z}\}| = \aleph_0^{\aleph_0} = 2^{\aleph_0}, \end{aligned}$$

so the desired result follows. ■

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ON HOMOMORPHISMS OF SOME MULTIPLICATIVE HYPERRINGS

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Abstract. A *homomorphism* of a multiplicative hyperring $(A,+,\circ)$ is a function $f:A\to A$ satisfying the conditions $f(x+y)=f(x)+f(y)$ and $f(x\circ y)\subseteq f(x)\circ f(y)$ for all $x,y\in A$. Denote by $\mathrm{Hom}(A,+,\circ)$ and $\mathrm{Hom}(A,+)$ the set of all homomorphisms of the multiplicative hyperring $(A,+,\circ)$ and the set of all homomorphisms of the group $(A,+)$. Then $\mathrm{Hom}(A,+,\circ)\subseteq\mathrm{Hom}(A,+)$. It is known that if $(R,+,\cdot)$ is a ring and I is an ideal of R, then $(R,+,\circ)$ is a strongly distributive hyperring where $x\circ y=xy+I$ for all $x,y\in R$, and we shall write $(R,+,I)$ for $(R,+,\circ)$. The purpose of this paper is to prove the following results for positive integers m,n : $\mathrm{Hom}(\mathbb{Z},+,m\mathbb{Z})$ is infinite. $\mathrm{Hom}(\mathbb{Z},+,m\mathbb{Z})=\mathrm{Hom}(\mathbb{Z},+)$ if and only if $m\le 2$. If $m>2$, then $\mathrm{Hom}(\mathbb{Z},+)\smallsetminus\mathrm{Hom}(\mathbb{Z},+,m\mathbb{Z})$ is infinite. If $(m,n)>1$, then $|\mathrm{Hom}(\mathbb{Z}_n,+,m\mathbb{Z}_n)|\ge\frac{2n}{(m,n)}$. $\mathrm{Hom}(\mathbb{Z}_n,+,m\mathbb{Z}_n)=\mathrm{Hom}(\mathbb{Z}_n,+)$ if and only if $(m,n)\le 2$. If $(m,n)>2$, then $|\mathrm{Hom}(\mathbb{Z}_n,+)\smallsetminus\mathrm{Hom}(\mathbb{Z}_n,+,m\mathbb{Z}_n)|\ge\frac{n}{(m,n)}$.

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1. Introduction

The cardinality of a set X is denoted by $|X|$. Let $(\mathbb{Z},+,\cdot)$ and $(\mathbb{Z}_n,+,\cdot)$ be the ring of integers and the ring of integers modulo n, respectively, under usual addition and multiplication. The residue class of $x\in\mathbb{Z}$ modulo n will be denoted by $\overline{x}$. Then

$$\mathbb{Z}_n=\{\overline{x}\mid x\in\mathbb{Z}\}=\{\overline{0},\overline{1},...,\overline{n-1}\}\ \text{ and } |\mathbb{Z}_n|=n.$$

For $a\in\mathbb{Z}$, define $g_a:\mathbb{Z}\to\mathbb{Z}$ and $h_{\overline{a}}:\mathbb{Z}_n\to\mathbb{Z}_n$ by

$$g_a(x)=ax \text{ and } h_{\overline{a}}(\overline{x})=\overline{ax} \text{ for all } x\in\mathbb{Z}.$$

For groups G, G', let $\mathrm{Hom}(G, G')$ be the set of all homomorphisms from G into G' and let $\mathrm{Hom}(G)$ stand for Hom (G, G). Then

$$\mathrm{Hom}(\mathbb{Z},+) = \{g_a \mid a \in \mathbb{Z}\} \text{ and } \mathrm{Hom}(\mathbb{Z}_n,+) = \{h_{\overline{a}} \mid a \in \mathbb{Z}\}.$$

Since $g_a \neq g_b$ if $a \neq b$ and $h_{\overline{a}} \neq h_{\overline{b}}$ if $\overline{a} \neq \overline{b}$, it follows that $|\mathrm{Hom}(\mathbb{Z},+)| = \aleph_0$ and $|\mathrm{Hom}(\mathbb{Z}_n,+)| = n$.

We know that for $I \subseteq \mathbb{Z}, I$ is an ideal of the ring $(\mathbb{Z},+,\cdot)$ if and only if $I = m\mathbb{Z}$ for some $m \in \mathbb{Z}$. Since $x \mapsto \overline{x}$ is an epimorphism from the ring $(\mathbb{Z},+,\cdot)$ onto the ring $(\mathbb{Z}_n,+,\cdot)$, it follows that for $I \subseteq \mathbb{Z}_n, I$ is an ideal of the ring $(\mathbb{Z}_n,+,\cdot)$ if and only if $I = m\mathbb{Z}_n$ for some $m \in \mathbb{Z}$ where $m\mathbb{Z}_n = \{m\overline{x} \mid x \in \mathbb{Z}\}(= \{\overline{mx} \mid x \in \mathbb{Z}\})$. It is easy to see that

$$m\mathbb{Z}_n = (m,n)\mathbb{Z}_n = \left\{\,\overline{0}, \overline{(m,n)}, ..., \left(\tfrac{n}{(m,n)} - 1\right)\overline{(m,n)}\,\right\} \text{ and } |m\mathbb{Z}_n| = \tfrac{n}{(m,n)}$$

where (m,n) denotes the g.c.d. of m and n.

A *hyperoperation* on a nonempty set H is a function $\circ : H \times H \to \mathcal{P}(H) \smallsetminus \{\emptyset\}$ where $\mathcal{P}(H)$ is the power set of H. The value of (x,y) under the hyperoperation $\circ$ is denoted by $x \circ y$. The system $(H, \circ)$ is called a *hypergroupoid.* For $A, B \subseteq H$ and $x \in H$, let

$$A \circ B = \bigcup_{\substack{a\in A\\ b\in B}} a \circ b,\ A \circ x = A \circ \{x\} \text{ and } x \circ A = \{x\} \circ A.$$

The hypergroupoid $(H, \circ)$ is called a *semihypergroup* if

$$x \circ (y \circ z) = (x \circ y) \circ z \quad \text{for all} \quad x, y, z \in H.$$

For a hypergroupoid $(H, \circ)$, a function $f : H \to H$ is called a *homomorphism* of $(H, \circ)$ if

$$f(x \circ y) \subseteq f(x) \circ f(y) \quad \text{for all } x, y \in H \quad \text{([1], p.12).}$$

A *multiplicative hyperring* is a system $(A,+,\circ)$ such that

1. $(A,+)$ is an abelian group,
2. $(A,\circ)$ is a semihypergroup,
3. for all $x, y, z \in A, x \circ (y+z) \subseteq x \circ y + x \circ z$ and $(y+z) \circ x \subseteq y \circ x + z \circ x$,
4. for all $x, y \in A,\ x \circ (-y) = (-x) \circ y = -(x \circ y)$.

If in the condition 3, the equalities are valid, then the multiplicative hyperring $(A,+,\circ)$ is called *strongly distributive.* Several results on multiplicative hyperrings were provided by Rota [5] and [6] and Oslon and Ward [3]. In [7], quasi-hyperideals in multiplicative hyperrings were defined and studied. If $(A,+)$ is an abelian group and $\circ$ is the hyperoperation on A defined by

$$x \circ y = \mathbb{Z}x + \mathbb{Z}y \text{ (the subgroup of } (A,+) \text{ generated by } x \text{ and } y) \\ \text{for all } x, y \in A,$$

then $(A, +, \circ)$ is a multiplicative hyperring ([1], p.177) which is not generally strongly distributive. Note that if $f \in \operatorname{Hom}(A, +)$, then for all $x, y \in A$,

$$\begin{aligned} f(x \circ y) = f(\mathbb{Z}x + \mathbb{Z}y) &= \bigcup_{k,\, l \,\in\, \mathbb{Z}} f(kx + ly) = \bigcup_{k,\, l \,\in\, \mathbb{Z}} (kf(x) + lf(y)) \\ &= \mathbb{Z}f(x) + \mathbb{Z}f(y) = f(x) \circ f(y). \end{aligned}$$

We are interested in multiplicative hyperrings defined from rings as follows: Let $(R, +, \cdot)$ be a ring, I an ideal of R and $\circ$ the hyperoperation defined on R by

$$x \circ y = xy + I \quad \text{for all } x, y \in R.$$

Then $(R, +, \circ)$ is a strongly distributive multiplicative hyperring ([1], p.177). For convenience, the multiplicative hyperring $(R, +, \circ)$ will be denoted by $(R, +, I)$.

By a *homomorphism* of a multiplicative hyperring $(A, +, \circ)$ we mean a function $f : A \to A$ such that f is a homomorphism from the group $(A, +)$ into itself and a homomorphism from the semihypergroup $(A, \circ)$ into itself, that is,

$$f(x + y) = f(x) + f(y) \text{ and } f(x \circ y) \subseteq f(x) \circ f(y) \quad \text{for all } x, y \in A.$$

Denote by $\operatorname{Hom}(A, +, \circ)$ the set of all homomorphisms of $(A, +, \circ)$. Notice that $\operatorname{Hom}(A, +, \circ) \subseteq \operatorname{Hom}(A, +)$.

It can be seen from our previous observation that if $(A, +, \circ)$ is the multiplicative hyperring defined from an abelian group $(A, +)$ by $x \circ y = \mathbb{Z}x + \mathbb{Z}y$ for all $x, y \in A$, then $\operatorname{Hom}(A, +, \circ) = \operatorname{Hom}(A, +)$.

In the remainder of this paper, let m and n be positive integers. Notice that $(-m)\mathbb{Z} = m\mathbb{Z}$ and $(-m)\mathbb{Z}_n = m\mathbb{Z}_n$. In [2], the authors characterized the elements of $\operatorname{Hom}(\mathbb{Z}, +, m\mathbb{Z})$ and $\operatorname{Hom}(\mathbb{Z}_n, +, m\mathbb{Z}_n)$ when m is a prime. In this case, $|\operatorname{Hom}(\mathbb{Z}, +, m\mathbb{Z})|$ and $|\operatorname{Hom}(\mathbb{Z}_n, +, m\mathbb{Z}_n)|$ were also determined. Some results on homomorphisms of certain multiplicative hyperrings were given in [4]. The purpose of this paper is to provide the following facts.

1. $\operatorname{Hom}(\mathbb{Z}, +, m\mathbb{Z})$ is infinite. $\operatorname{Hom}(\mathbb{Z}, +, m\mathbb{Z}) = \operatorname{Hom}(\mathbb{Z}, +)$ if and only if $m \leq 2$. If $m > 2$, then $\operatorname{Hom}(\mathbb{Z}, +) \smallsetminus \operatorname{Hom}(\mathbb{Z}, +, m\mathbb{Z})$ is infinite.
2. If $(m, n) > 1$, then $|\operatorname{Hom}(\mathbb{Z}_n, +, m\mathbb{Z}_n)| \geq \frac{2n}{(m,n)}$. $\operatorname{Hom}(\mathbb{Z}_n, +, m\mathbb{Z}_n) = \operatorname{Hom}(\mathbb{Z}_n, +)$ if and only if $(m, n) \leq 2$. If $(m, n) > 2$, then $|\operatorname{Hom}(\mathbb{Z}_n, +) \smallsetminus \operatorname{Hom}(\mathbb{Z}_n, +, m\mathbb{Z}_n)| \geq \frac{n}{(m,n)}$.

2. Main Results

To obtain the main results, the following series of lemmas is needed.

Lemma 2.1. *For $a \in \mathbb{Z}$, $g_a \in \operatorname{Hom}(\mathbb{Z}, +, m\mathbb{Z})$ if and only if $m \,|\, (a^2 - a)$.*

Proof. Assume that $g_a \in \operatorname{Hom}(\mathbb{Z},+,m\mathbb{Z})$. Then $g_a(1\circ 1) \subseteq g_a(1)\circ g_a(1)$, so

$$a+am\mathbb{Z} = a(1+m\mathbb{Z}) = a(1{\cdot}1+m\mathbb{Z}) = g_a(1\circ 1) \subseteq g_a(1)\circ g_a(1) = a\circ a = a^2+m\mathbb{Z}.$$

This implies that $a = a^2 + mt$ for some $t \in \mathbb{Z}$. Thus $m\,|(a^2-a)$.

Conversely, assume that $m\,|(a^2-a)$. Then $a^2 - a = mt$ for some $t \in \mathbb{Z}$, so $a = a^2 - mt$. Thus for all $x, y \in \mathbb{Z}$,

$$\begin{aligned} g_a(x\circ y) &= g_a(xy+m\mathbb{Z}) = a(xy+m\mathbb{Z}) = axy + am\mathbb{Z} \\ &= (a^2-mt)xy + am\mathbb{Z} \subseteq a^2xy + m\mathbb{Z} + am\mathbb{Z} \\ &= a^2xy + m\mathbb{Z} = (ax)(ay) + m\mathbb{Z} = g_a(x)g_a(y) + m\mathbb{Z} = g_a(x)\circ g_a(y). \end{aligned}$$

Hence $g_a \in \operatorname{Hom}(\mathbb{Z},+,m\mathbb{Z})$, as desired. ■

Lemma 2.2. $\{\, g_a \mid a \in m\mathbb{Z} \cup (m\mathbb{Z}+1)\} \subseteq \operatorname{Hom}(\mathbb{Z},+,m\mathbb{Z})$.

Proof. If $a \in m\mathbb{Z}\cup(m\mathbb{Z}+1)$, then $m\,|\,a$ or $m\,|(a-1)$, so $m\,|(a^2-a)$. By Lemma 2.1, the lemma is proved. ■

Lemma 2.3. *If* $m > 2$*, then* $\{\, g_a \mid a \in m\mathbb{Z}+2\} \subseteq \operatorname{Hom}(\mathbb{Z},+) \smallsetminus \operatorname{Hom}(\mathbb{Z},+,m\mathbb{Z})$.

Proof. Assume $m > 2$ and let $a \in m\mathbb{Z}+2$. Then $a = mk+2$ for some $k \in \mathbb{Z}$. But

$$a^2 - a = m^2k^2 + 3mk + 2,$$

so $m \nmid (a^2-a)$. By Lemma 2.1, $g_a \notin \operatorname{Hom}(\mathbb{Z},+,m\mathbb{Z})$. Hence the desired result follows. ■

Lemma 2.4. *For* $a \in \mathbb{Z}$*,* $h_{\bar a} \in \operatorname{Hom}(\mathbb{Z}_n,+,m\mathbb{Z}_n)$ *if and only if* $(m,n)|(a^2-a)$.

Proof. Assume that $h_{\bar a} \in \operatorname{Hom}(\mathbb{Z}_n,+,m\mathbb{Z}_n)$. Then

$$\begin{aligned} \bar a + am\mathbb{Z}_n = \bar a(\bar 1\cdot\bar 1 + m\mathbb{Z}_n) &= \bar a(\bar 1\circ\bar 1) = h_{\bar a}(\bar 1\circ\bar 1) \subseteq h_{\bar a}(\bar 1)\circ h_{\bar a}(\bar 1) \\ &= \bar a\circ\bar a = \bar a^2 + m\mathbb{Z}_n = \bar a^2 + (m,n)\mathbb{Z}_n, \end{aligned}$$

so $\bar a - \bar a^2 = (m,n)\bar s$ for some $s \in \mathbb{Z}$. Hence $a - a^2 - (m,n)s = nt$ for some $t \in \mathbb{Z}$. Thus $a - a^2 = (m,n)s + nt$. But $(m,n)\,|\,((m,n)s+nt)$, so $(m,n)\,|\,(a^2-a)$.

For the converse, assume that $(m,n)\,|\,(a^2-a)$. Then $a^2 - a = (m,n)s$ for some $s \in \mathbb{Z}$, so $a = a^2 - (m,n)s$. If $x, y \in \mathbb{Z}$, then

$$\begin{aligned} h_{\bar a}(\bar x\circ\bar y) &= h_{\bar a}(\overline{xy} + m\mathbb{Z}_n) \\ &= \bar a(\overline{xy} + m\mathbb{Z}_n) \\ &= \overline{axy} + am\mathbb{Z}_n \\ &= \overline{(a^2-(m,n)s)xy} + am\mathbb{Z}_n \\ &= \overline{a^2xy} - \overline{(m,n)sxy} + am\mathbb{Z}_n \\ &\subseteq \overline{a^2xy} + (m,n)\mathbb{Z}_n + am\mathbb{Z}_n \\ &= \overline{a^2xy} + m\mathbb{Z}_n + am\mathbb{Z}_n \\ &= \overline{a^2xy} + m\mathbb{Z}_n \\ &= \overline{ax}\cdot\overline{ay} + m\mathbb{Z}_n \\ &= h_{\bar a}(\bar x)\cdot h_{\bar a}(\bar y) + m\mathbb{Z}_n \\ &= h_{\bar a}(\bar x)\circ h_{\bar a}(\bar y). \end{aligned}$$

Hence $h_{\bar{a}} \in \mathrm{Hom}(\mathbb{Z}_n, +, m\mathbb{Z}_n)$. ■

Lemma 2.5. $\{ h_{\bar{a}} \mid a \in (m,n)\mathbb{Z} \cup ((m,n)\mathbb{Z}+1)\} \subseteq \mathrm{Hom}(\mathbb{Z}_n, +, m\mathbb{Z}_n)$.

Proof. If $a \in (m,n)\mathbb{Z} \cup ((m,n)\mathbb{Z}+1)$, then $(m,n) \mid a$ or $(m,n) \mid (a-1)$, thus $(m,n) \mid (a^2 - a)$. Hence by Lemma 2.3, the result follows. ■

Lemma 2.6. *If* $(m,n) > 2$, *then*

$$\{ h_{\bar{a}} \mid a \in (m,n)\mathbb{Z}+2\} \subseteq \mathrm{Hom}(\mathbb{Z}_n, +) \smallsetminus \mathrm{Hom}(\mathbb{Z}_n, +, m\mathbb{Z}_n).$$

Proof. If $(m,n) > 2$ and $a \in (m,n)\mathbb{Z}+2$, then $a = (m,n)k+2$, so

$$a^2 - a = (m,n)^2k^2 + 3(m,n)k + 2$$

which is not divided by (m,n), so by Lemma 2.4, $h_{\bar{a}} \notin \mathrm{Hom}(\mathbb{Z}_n, +, m\mathbb{Z}_n)$, that is, $h_{\bar{a}} \in \mathrm{Hom}(\mathbb{Z}_n, +) \smallsetminus \mathrm{Hom}(\mathbb{Z}_n, +, m\mathbb{Z}_n)$. ■

Theorem 2.7. *The following statements hold.*

(i) $\mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z})$ *is infinite.*

(ii) $\mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}, +)$ *if and only if* $m \leq 2$.

(iii) *If* $m > 2$, *then* $\mathrm{Hom}(\mathbb{Z}, +) \smallsetminus \mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z})$ *is infinite.*

(iv) *If* m *is a prime power, then* $\mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z}) = \{ g_a \mid a \in m\mathbb{Z} \cup (m\mathbb{Z}+1)\}$.

Proof. (i) Since $g_a \neq g_b$ if $a \neq b$ in $\mathbb{Z}$, (i) follows from Lemma 2.2.

(ii) If $m > 2$, then by Lemma 2.3, $\mathrm{Hom}(\mathbb{Z}, +) \smallsetminus \mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z}) \neq \emptyset$, so $\mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z}) \neq \mathrm{Hom}(\mathbb{Z}, +)$. This shows that if $\mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}, +)$, then $m \leq 2$.

Assume that $m \leq 2$. Since $1\mathbb{Z} \cup (1\mathbb{Z}+1) = \mathbb{Z}$ and $2\mathbb{Z} \cup (2\mathbb{Z}+1) = \mathbb{Z}$, we have $m\mathbb{Z} \cup (m\mathbb{Z}+1) = \mathbb{Z}$. It follows that $\{ g_a \mid a \in m\mathbb{Z} \cup (m\mathbb{Z}+1)\} = \mathrm{Hom}(\mathbb{Z}, +)$. Hence by Lemma 2.2, $\mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}, +)$.

(iii) follows directly from Lemma 2.3.

(iv) Assume that m is a prime power. Let $a \in \mathbb{Z}$ be such that $g_a \in \mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z})$. By Lemma 2.1, $m \mid a^2 - a$. Since $a^2 - a = a(a-1)$ and a and $a-1$ are relatively prime, we have that $m \mid a$ or $m \mid a-1$. Therefore $a \in m\mathbb{Z}$ or $a - 1 \in m\mathbb{Z}$. Hence $a \in m\mathbb{Z} \cup (m\mathbb{Z}+1)$. This shows that $\mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z}) \subseteq \{ g_a \mid a \in m\mathbb{Z} \cup (m\mathbb{Z}+1)\}$. This implies that $\mathrm{Hom}(\mathbb{Z}, +, m\mathbb{Z}) = \{ g_a \mid a \in m\mathbb{Z} \cup (m\mathbb{Z}+1)\}$ by Lemma 2.2. ■

Theorem 2.8. *The following statements hold.*

(i) *If* $(m,n) > 1$, *then* $|\mathrm{Hom}(\mathbb{Z}_n, +, m\mathbb{Z}_n)| \geq \frac{2n}{(m,n)}$.

(ii) $\mathrm{Hom}(\mathbb{Z}_n, +, m\mathbb{Z}_n) = \mathrm{Hom}(\mathbb{Z}_n, +)$ *if and only if* $(m,n) \leq 2$.

(iii) *If* $(m,n) > 2$, *then* $|\mathrm{Hom}(\mathbb{Z}_n, +) \smallsetminus \mathrm{Hom}(\mathbb{Z}_n, +, m\mathbb{Z}_n)| \geq \frac{n}{(m,n)}$.

(iv) *If* (m,n) *is a prime power, then*

$$\mathrm{Hom}(\mathbb{Z}_n,+,m\mathbb{Z}_n) = \{\, h_{\bar a} \mid a \in (m,n)\mathbb{Z} \cup ((m,n)\mathbb{Z}+1)\}$$

and thus $|\mathrm{Hom}(\mathbb{Z}_n,+,m\mathbb{Z}_n)| = \frac{2n}{(m,n)}$.

Proof. (i) Assume that $(m,n) > 1$. Then $|(m,n)\mathbb{Z}_n| = \frac{n}{(m,n)} < n$. This implies that $(m,n)\mathbb{Z}_n \cap ((m,n)\mathbb{Z}_n + 1) = \emptyset$. Since $h_{\bar a} \neq h_{\bar b}$ for all distinct $\bar a, \bar b \in \mathbb{Z}_n$, it follows that

$$\begin{aligned}|\mathrm{Hom}(\mathbb{Z}_n,+,m\mathbb{Z}_n)| &\geq |\{\, h_{\bar a} \mid a \in (m,n)\mathbb{Z} \cup ((m,n)\mathbb{Z}+1)\}| \\ &= |\{\, h_{\bar a} \mid a \in \mathbb{Z} \text{ and } \bar a \in (m,n)\mathbb{Z}_n \cup ((m,n)\mathbb{Z}_n + \bar 1)\}| \\ &= |(m,n)\mathbb{Z}_n| + |(m,n)\mathbb{Z}_n + \bar 1| \\ &= \frac{n}{(m,n)} + \frac{n}{(m,n)} = \frac{2n}{(m,n)}.\end{aligned}$$

(ii) If $(m,n) > 2$, then by Lemma 2.6, $\mathrm{Hom}(\mathbb{Z}_n,+) \smallsetminus \mathrm{Hom}(\mathbb{Z}_n,+,m\mathbb{Z}_n) \neq \emptyset$, so $\mathrm{Hom}(\mathbb{Z}_n,+,m\mathbb{Z}_n) \neq \mathrm{Hom}(\mathbb{Z}_n,+)$. Hence if $\mathrm{Hom}(\mathbb{Z}_n,+,m\mathbb{Z}_n) = \mathrm{Hom}(\mathbb{Z}_n,+)$, then $(m,n) \leq 2$.

Assume that $(m,n) \leq 2$. Then $(m,n)\mathbb{Z} \cup ((m,n)\mathbb{Z}+1) = \mathbb{Z}$. This implies that $\{h_{\bar a} | \, a \in (m,n)\mathbb{Z} \cup ((m,n)\mathbb{Z}+1)\} = \mathrm{Hom}(\mathbb{Z}_n,+)$. Therefore by Lemma 2.5, we have $\mathrm{Hom}(\mathbb{Z}_n,+,m\mathbb{Z}_n) = \mathrm{Hom}(\mathbb{Z}_n,+)$.

(iii) Assume that $(m,n) > 2$. Then

$$\begin{aligned}&|\mathrm{Hom}(\mathbb{Z}_n,+) \smallsetminus \mathrm{Hom}(\mathbb{Z}_n,+,m\mathbb{Z}_n)| \\ &\qquad \geq |\{h_{\bar a} | \, a \in (m,n)\mathbb{Z} + 2\}| \quad \text{from Lemma 2.6} \\ &\qquad = |\{h_{\bar a} | \, a \in \mathbb{Z} \text{ and } \bar a \in (m,n)\mathbb{Z}_n + \bar 2\}| \\ &\qquad = |(m,n)\mathbb{Z}_n + \bar 2| = |(m,n)\mathbb{Z}_n| = \frac{n}{(m,n)}.\end{aligned}$$

(iv) Let (m,n) be a prime power and let $a \in \mathbb{Z}$ be such that $h_{\bar a} \in \mathrm{Hom}(\mathbb{Z}_n,+,m\mathbb{Z}_n)$. By Lemma 2.4, $(m,n) \,|\, a^2 - a$. But $a^2 - a = a(a-1)$ and $(a, a-1) = 1$, so $(m,n) \,|\, a$ or $(m,n) \,|\, a-1$. Thus $a \in (m,n)\mathbb{Z} \cup ((m,n)\mathbb{Z}+1)$. This shows that $\mathrm{Hom}(\mathbb{Z}_n,+,m\mathbb{Z}_n) \subseteq \{\, h_{\bar a} \mid a \in (m,n)\mathbb{Z} \cup ((m,n)\mathbb{Z}+1)\,\}$. Hence by Lemma 2.5, we have $\mathrm{Hom}(\mathbb{Z}_n,+,m\mathbb{Z}_n)=\{\, h_{\bar a} \mid a \in (m,n)\mathbb{Z} \cup ((m,n)\mathbb{Z}+1)\}$. ■

Example 2.9. By Theorem 2.7(iv),

$$\mathrm{Hom}(\mathbb{Z},+,4\mathbb{Z}) = \{\, g_a \mid a \in 4\mathbb{Z} \cup (4\mathbb{Z}+1)\}$$

and hence

$$\mathrm{Hom}(\mathbb{Z},+) \smallsetminus \mathrm{Hom}(\mathbb{Z},+,4\mathbb{Z}) = \{\, g_a \mid a \in (4\mathbb{Z}+2) \cup (4\mathbb{Z}+3)\}.$$

By Theorem 2.8(iv), $|\mathrm{Hom}(\mathbb{Z}_{20},+,4\mathbb{Z}_{20})| = \dfrac{2 \times 20}{(4,20)} = 10$ and

$$\begin{aligned}\mathrm{Hom}(\mathbb{Z}_{20},+,4\mathbb{Z}_{20}) &= \{\, h_{\bar{a}} \mid a \in 4\mathbb{Z} \cup (4\mathbb{Z}+1)\} \\ &= \{\, h_{\bar{a}} \mid a \in \mathbb{Z},\ \bar{a} \in 4\mathbb{Z}_{20} \cup (4\mathbb{Z}_{20}+\bar{1})\} \\ &= \{\, h_{\bar{0}}, h_{\bar{4}}, h_{\bar{8}}, h_{\overline{12}}, h_{\overline{16}}, h_{\bar{1}}, h_{\bar{5}}, h_{\bar{9}}, h_{\overline{13}}, h_{\overline{17}}\}.\end{aligned}$$

Thus

$$\mathrm{Hom}(\mathbb{Z}_{20},+) \smallsetminus \mathrm{Hom}(\mathbb{Z}_{20},+,4\mathbb{Z}_{20}) = \{h_{\bar{2}}, h_{\bar{3}}, h_{\bar{6}}, h_{\bar{7}}, h_{\overline{10}}, h_{\overline{11}}, h_{\overline{14}}, h_{\overline{15}}, h_{\overline{18}}, h_{\overline{19}}\}.$$

It follows from Theorem 2.8(i) and (iii) that

$$|\mathrm{Hom}(\mathbb{Z}_{18},+,6\mathbb{Z}_{18})| \geq \frac{2 \times 18}{(6,18)} = 6$$

and

$$|\mathrm{Hom}(\mathbb{Z}_{18},+) \smallsetminus \mathrm{Hom}(\mathbb{Z}_{18},+,6\mathbb{Z}_{18})| \geq \frac{18}{(6,18)} = 3.$$

From Lemma 2.5 and Lemma 2.6, we have respectively that

$$\begin{aligned}\mathrm{Hom}(\mathbb{Z}_{18},+,6\mathbb{Z}_{18}) &\supseteq \{\, h_{\bar{a}} \mid a \in 6\mathbb{Z} \cup (6\mathbb{Z}+1)\} \\ &= \{\, h_{\bar{a}} \mid a \in \mathbb{Z} \text{ and } \bar{a} \in 6\mathbb{Z}_{18} \cup (6\mathbb{Z}_{18}+\bar{1})\} \\ &= \{\, h_{\bar{0}}, h_{\bar{6}}, h_{\overline{12}}, h_{\bar{1}}, h_{\bar{7}}, h_{\overline{13}}\,\},\end{aligned}$$

$$\begin{aligned}\mathrm{Hom}(\mathbb{Z}_{18},+) \smallsetminus \mathrm{Hom}(\mathbb{Z}_{18},+,6\mathbb{Z}_{18}) &\supseteq \{\, h_{\bar{a}} \mid a \in 6\mathbb{Z}+2\} \\ &= \{\, h_{\bar{a}} \mid a \in \mathbb{Z} \text{ and } \bar{a} \in 6\mathbb{Z}_{18}+\bar{2}\} \\ &= \{\, h_{\bar{2}}, h_{\bar{8}}, h_{\overline{14}}\,\}.\end{aligned}$$

Let us consider $h_{\bar{a}}$ where $a \in (6\mathbb{Z}+3) \cup (6\mathbb{Z}+4) \cup (6\mathbb{Z}+5)$. If $k \in \mathbb{Z}$, then

$$6 \mid (6k+3)^2-(6k+3), 6 \mid (6k+4)^2-(6k+4) \text{ and } 6 \nmid (6k+5)^2-(6k+5),$$

so by Lemma 2.4,

$$\{\, h_{\bar{a}} \mid a \in (6\mathbb{Z}+3) \cup (6\mathbb{Z}+4)\} \subseteq \mathrm{Hom}(\mathbb{Z}_{18},+,6\mathbb{Z}_{18})$$

and

$$\{\, h_{\bar{a}} \mid a \in 6\mathbb{Z}+5\} \subseteq \mathrm{Hom}(\mathbb{Z}_{18},+) \smallsetminus \mathrm{Hom}(\mathbb{Z}_{18},+,6\mathbb{Z}_{18}).$$

Consequently,

$$\begin{aligned}\mathrm{Hom}(\mathbb{Z}_{18},+,6\mathbb{Z}_{18}) &= \{\, h_{\bar{a}} \mid a \in 6\mathbb{Z} \cup (6\mathbb{Z}+1) \cup (6\mathbb{Z}+3) \cup (6\mathbb{Z}+4)\} \\ &= \{h_{\bar{0}}, h_{\bar{6}}, h_{\overline{12}}, h_{\bar{1}}, h_{\bar{7}}, h_{\overline{13}}, h_{\bar{3}}, h_{\bar{9}}, h_{\overline{15}}, h_{\bar{4}}, h_{\overline{10}}, h_{\overline{16}}\}, \\ |\mathrm{Hom}(\mathbb{Z}_{18},+,6\mathbb{Z}_{18})| &= 12, \\ \mathrm{Hom}(\mathbb{Z}_{18},+) \smallsetminus \mathrm{Hom}(\mathbb{Z}_{18},+,6\mathbb{Z}_{18}) &= \{\, h_{\bar{a}} \mid a \in (6\mathbb{Z}+2) \cup (6\mathbb{Z}+5)\} \\ &= \{h_{\bar{2}}, h_{\bar{8}}, h_{\overline{14}}, h_{\bar{5}}, h_{\overline{11}}, h_{\overline{17}}\}, \\ |\mathrm{Hom}(\mathbb{Z}_{18},+) \smallsetminus \mathrm{Hom}(\mathbb{Z}_{18},+,6\mathbb{Z}_{18})| &= 6.\end{aligned}$$

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UPPER TOPOLOGICAL GENERALIZED GROUPS

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Abstract. Here, we introduce the notion of generalized universal covers for topological generalized groups and present a method for constructing new topological generalized groups by using of universal covers. As a result a generalization of notion of fundamental groups which is called the generalized fundamental groups is deduced.

Key words and phrases: universal cover, topological generalized group, fundamental group, semilocally simple connected space, locally compact topological group.

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1. Introduction

A new mathematics, isomathematics, was proposed by Santilli when he was studying the mathematical models for electroweak and gravitational theories [8]. The notion of generalized groups, first was introduced by Molaei [4], [5], has an important role in the construction of a geometric unified theory by use of Santilli's isotheory.

Molaei used generalized groups in order to introduce a new kind of dynamics on top spaces [4]. He showed that each generalized group is isometric to a Rees matrix semigroup, see [4]. Also, he introduced the notion of topological generalized groups and proved that if X and Y are Hausdroff topological spaces, G is a topological group and $s : Y \times X \to G$ is a continuous mapping, then the Rees matrix $P = X \times G \times Y$ is a topological generalized group [4]. Topological generalized groups can also be used for modelizing the set of genetic codes, for more details see [5]. Recently, Farhangdoost and Molaei presented a method for constructing new top spaces by using of universal covering spaces of special Lie subsemigroups of a top space, see [6]. Moreover, they deduced a generalization of the notion of fundamental groups which was a completely simple semigroup. Here, we extend their results for semilocally simply connected topological generalized groups. Moreover, we use a generalized notion of universal cover which is developed by Berestovskii and Plaut in the covering group theory for a category of coverable topological groups which requires any form of local simple connectivity [1], [2]. Then, by using of this universal cover for a locally arcwise connected, locally compact topological generalized group, we construct a new topological generalized group.

2. Preliminaries and main results

We introduce some common notations and preliminaries. First, we recall the definition of a generalized group. A generalized group is a non-empty set G admitting an operation called multiplication with the following properties:

i) $(xy)z = x(yz)$, for all $x, y, z \in G$;

ii) for each x belongs to G, there exists a unique element in G, we denote by $e(x)$, such that $x \cdot e(x) = e(x) \cdot x = x$;

iii) for each $x \in G$, there exists $y \in G$ such that $xy = yx = e(x)$.

One can see that each x in a generalized group G has a unique inverse in G [4], we denote it by x^{-1}.

A topological generalized group is a generalized group G equipped with a Hausdorff topology such that the mappings $m : G \times G \to G$, defined by $(g, h) \mapsto g \cdot h$ and $m' : G \to G$, defined by $g \mapsto g^{-1}$ are continuous.

A topological generalized group G is called a normal topological generalized group if G is a normal generalized group, i.e., $e(xy) = e(x) \cdot e(y)$, for all $x, y \in G$.

Now, let G be a normal topological generalized group and let

$$G_{e(g)} = \{h \in G : e(g) = e(h)\}, \text{ for each } g \in G. \text{ Then, } G = \bigcup_{g \in G} G_{e(g)}.$$

It is easy to see that, for each $g \in G$, $G_{e(g)}$ with subspace topology and product of G is a topological group.

We note that if G and H are two normal topological generalized groups and $f : G \to H$ is an algebraic homomorphism, then $f(e(g)) = e(f(g))$ and $f : G_{e(g)} \to H_{e(f(g))}$ is a group homomorphism, for each $g \in G$.

Let G be a topological space. G is called semilocally simply connected, if for each $x \in G$, there is an open set U of x such that the inclusion of U in G induces the trivial homomorphism on their fundamental groups. Topological space G is called simply connected if G is arcwice connected and $\pi_1(G) \simeq \{e\}$, where $\pi_1(G)$ denotes the (Poincare) fundamental group of G.

If X is a topological space and if $\mathcal{C}$ is a collection of subspaces of X whose union is X, the topology of X is said to be coherent with the collection $\mathcal{C}$, provided a set A is closed in X if and only if $A \cap C$ is closed in C, for each $C \in \mathcal{C}$. It is equivalent to require that U is open in X if and only if $U \cap C$ is open in C, for each $C \in \mathcal{C}$.

Let G and H be two topological groups and $\phi : G \to H$ be an open epimorphism with discrete kernel. Then ϕ is called a (traditional) cover. It is easy to see that, if ϕ is a cover then ϕ is a local homeomorphism. A universal cover for a topological group H is a covering epimorphism $\phi : G \to H$ such that for any cover $\rho : F \to H$ of topological groups, there is a homomorphism $\psi : G \to F$ such that $\phi = \rho\psi$. If G and F are connected, it follows easily that ψ is a cover and unique, see [1].

In [1], Berestovskii and Plaut utilized a generalized notion of cover, namely an open epimorphism between topological groups whose kernel is central and prodiscrete, i.e., the inverse limit of discrete groups. They proved that, for a large category C of topological groups, called coverable topological groups, the following assertions hold:

(1) For every $G \in \mathcal{C}$, there exists a cover $\phi : \widetilde{G} \to G$.

(2) Covers are morphisms in $\mathcal{C}$, (i.e., the composition of covers between elements of $\mathcal{C}$ is a cover).

(3) The cover $\phi : \widetilde{G} \to G$ has the traditional universal property of the universal cover in the category $\mathcal{C}$ with covers as morphisms.

Here, we define the notion of a cover for topological generalized groups. Let G and $\widetilde{G}$ be two normal topological generalized groups and $\phi : \widetilde{G} \to G$ be an algebraic homomorphism. So, the restriction of ϕ to each $\widetilde{G}_{e(\widetilde{g})}$ is a group homomorphism from $\widetilde{G}_{e(\widetilde{g})}$ to $G_{e(\phi(\widetilde{g}))}$. For simplicity, we denote it by $\phi_{\widetilde{g}}$. We say that ϕ is a (traditional) generalized cover of topological generalized groups, if ϕ is an open epimorphism with discrete kernel, where $\ker \phi = \bigcup\limits_{\widetilde{g} \in \widetilde{G}} \ker \phi_{\widetilde{g}}$.

Moreover, by using of notion of covers in the sense of Berestovskii and Plaut, we define another notion of cover for topological generalized groups. We say that an open epimorphism $\phi : \widetilde{G} \to G$ of normal topological generalized groups $\widetilde{G}$ and G is a generalized cover in the sense of Berestovskii and Plaut, if the restriction of ϕ to each $\widetilde{G}_{e(\widetilde{g})}$ is an open epimorphism of topological groups with central and prodiscrete kernel.

Let G be a normal topological generalized group and $\phi : \widetilde{G} \to G$ be a universal generalized cover of G. Then we call the pair $(\widetilde{G}, \phi)$ an upper topological generalized group.

In this article, we present a method for constructing new topological generalized groups by using of two kinds of universal covers, traditional universal covers and the universal covers in the sense of Berestovskii and Plaut. As a result, generalization of th notion of fundamental groups, which are generalized fundamental groups is deduced.

Theorem 2.1. *Let G be a locally acrwise connected and semilocally simply connected normal topological generalized group such that its topology coherent with the collection $\{G_{e(g)} : g \in G\}$. Then there exist a normal topological generalized group $\widetilde{G}$ and a (traditional) generalized universal cover $\phi : \widetilde{G} \to G$ associated to G. Moreover, $\widetilde{G}_{e(\widetilde{g})}$ is a connected and simply connected topological group for each $\widetilde{g} \in \widetilde{G}$.*

We denote the restriction of ϕ to $\widetilde{G}_{e(\widetilde{g})}$ by $\phi_g : \widetilde{G}_{e(\widetilde{g})} \to G_{e(g)}$.

Corollary 2.2. *With the above assumption, $\ker\phi = \bigcup_{g\in G} \ker\phi_g$ is a discrete topological generalized subgroup of $\widetilde{G}$. In particular, $\pi_1(G_{e(g)})$ is abstractly isomorphic to $\ker\phi_g$, where $\pi_1(G_{e(g)})$ is the (Poincaré) fundamental group of $G_{e(g)}$.*

Also, we have the following results for locally compact topological generalized groups. In this case, the notion of (traditional) generalized universal cover substituted by generalized universal cover in the sense of Berestovskii and Plaut.

Theorem 2.3. *Let G be a locally arcwise connected, locally compact normal generalized topological group such that its topology is coherent with the collection $\{G_{e(g)} : g \in G\}$. Then there exist a normal topological generalized group $\widetilde{G}$ and a generalized universal cover (in the sense of Berestovskii and Plaut) $\phi : \widetilde{G} \to G$ associated to G. Moreover, $\widetilde{G}_{e(\widetilde{g})}$ is connected and simply connected topological group for each $\widetilde{g} \in \widetilde{G}$.*

Corollary 2.4. *With the above assumptions, $\ker\phi = \bigcup_{g\in G} \ker\phi_g$ is a prodiscrete topological generalized subgroup of $\widetilde{G}$. In particular, $\pi_1(G_{e(g)})$ is abstractly isomorphic to $\ker\phi_g$.*

3. Proof of Theorem 2.1

In this section, we will prove Theorem 2.1 and Corollary 2.2. Let G be a topological group. If G is connected, locally arcwise connected and semilocally simply connected, then G has a universal cover $\phi : \widetilde{G} \to G$. Moreover, $\widetilde{G}$ is a connected, locally arcwise connected and simply connected topological group. In fact, we have the following results (see [9] and [1]):

Proposition 3.1. *Let G be connected and locally arcwise connected. G admits a universal cover $\phi : \widetilde{G} \to G$ if and only if it is semilocally simply connected. Moreover, ϕ is unique up to isomorphism.*

Also, we need the following proposition of [1], (Proposition 81).

Proposition 3.2. *Let $G, \widetilde{G}$ be topological groups and $\phi : \widetilde{G} \to G$ be a cover. Suppose that X is a connected, locally arcwise connected and simply connected* *topological space. If* $f:(X,p)\to(G,e)$ *is a continuous function, then there is a unique lift* $g:(X,p)\to(\widetilde{G},\widetilde{e})$ *such that* $f=\phi\circ g$.

Proposition 3.3. *Assume that* G *is a locally arcwise connected and semilocally simply connected topological generalized group such that its topology coherent with* $\{G_{e(g)}:g\in G\}$. *Then each* $G_{e(g)}$ *has a universal cover,* $\phi_g:\widetilde{G}_g\to G_{e(g)}$, *which is unique up to isomorphism.*

Proof. Let G be a normal topological generalized group. Then the mapping $g\mapsto e(g)$ is continuous, see [4]. Therefore, $G_{e(g)}$ is a closed subspace of G. If we put $G=\bigcup\limits_{g\in G}G_{e(g)}$, then for each $g,h\in G$, the topological groups $G_{e(g)}$ and $G_{e(h)}$ are either disjoint or identical, i.e., $G_{e(g)}=G_{e(h)}$.

Suppose that the topology of G is coherent with the collection $\{G_{e(g)}:g{\in}G\}$. Then, each $G_{e(g)}$ is also an open subspace of G. Moreover, if G is locally arcwise connected and semilocally simply connected, then each $G_{e(g)}$ is an arcwise connected, locally arcwise connected and semilocally simply connected topological group.

Now, by Proposition 3.1, $G_{e(g)}$ has a universal cover, say $\phi_g:\widetilde{G}_g\to G_{e(g)}$, which is unique up to isomorphism. ∎

Note that, by Proposition 3.1, $\widetilde{G}_g$ is connected and simply connected topological group. Also, $\widetilde{G}_g$ is locally arcwise connected. This implies that $\widetilde{G}_g\times\widetilde{G}_g$ is also connected, locally arcwise connected and simply connected topological space. Therefore, by Proposition 3.2, the mapping $m\circ(\phi_g\times\phi_g):\widetilde{G}_g\times\widetilde{G}_g\to G_{e(g)}$, has a unique lifting $\widetilde{m}:\widetilde{G}_g\times\widetilde{G}_g\to\widetilde{G}_g$ such that $\widetilde{m}(\widetilde{e}_g,\widetilde{e}_g)=\widetilde{e}_g$, where $\widetilde{e}_g$ is the unit element of $\widetilde{G}_g$.

Proposition 3.4. $\widetilde{G}_g$ *with the multiplication defined by* $\widetilde{m}$ *is also a topological group. Moreover, its structure group with product* $\widetilde{m}$ *is the same as to its original structure group up to isomorphism.*

Proof. First, we have

$$\begin{aligned}\phi_g\circ(\widetilde{m}\circ(id_{\widetilde{G}_g}\times\widetilde{m})) &= m\circ(\phi_g\times\phi_g)\circ(id_{\widetilde{G}_g}\times\widetilde{m})\\ &= m\circ(\phi_g\circ\widetilde{m}\times\phi_g)\\ &= m\circ((m\circ(\phi_g\times\phi_g))\times\phi_g)\\ &= m\circ(m\times id_{G_{e(g)}})\circ(\phi_g\times\phi_g\times\phi_g)\end{aligned}$$

and

$$\begin{aligned}\phi_g\circ(\widetilde{m}\circ(\widetilde{m}\times id_{\widetilde{G}_g})) &= m\circ(\phi_g\times\phi_g)\circ(\widetilde{m}\times id_{\widetilde{G}_g})\\ &= m\circ(\phi_g\circ\widetilde{m}\times\phi_g)\\ &= m\circ((m\circ(\phi_g\times\phi_g))\times\phi_g)\\ &= m\circ(m\times id_{G_{e(g)}})\circ(\phi_g\times\phi_g\times\phi_g).\end{aligned}$$

Since the multiplication on G is associative, it follows that $\widetilde{m}\circ(id_{\widetilde{G}_g}\times\widetilde{m})$ and $\widetilde{m}\circ(\widetilde{m}\times id_{\widetilde{G}_g})$ are the lifts of the space map from $\widetilde{G}_g\times\widetilde{G}_g\times\widetilde{G}_g$ into $G_{e(g)}$. Since both maps map $(\widetilde{e}_g,\widetilde{e}_g,\widetilde{e}_g)$ into $\widetilde{e}_g$, it follows that they are identical, i.e., the operation $\widetilde{m}$ is associative.

Also, we have

$$\phi_g(\widetilde{m}(\widetilde{g},\widetilde{e}_g)) = m(\phi_g(\widetilde{g},e(g))) = \phi_g(\widetilde{g}).$$

Therefore, $\widetilde{g}\mapsto\widetilde{m}(\widetilde{g},\widetilde{e}_g)$ is the lifting of $\phi_g:\widetilde{G}_g\to G_{e(g)}$. Since $\widetilde{m}(\widetilde{e}_g,\widetilde{g})=\widetilde{e}_g$, this map is the identity on $\widetilde{G}_g$, i.e., $\widetilde{m}(\widetilde{g},\widetilde{e}_g)=\widetilde{g}$, for all $\widetilde{g}\in\widetilde{G}_g$.

Analogously, we have

$$\phi_g(\widetilde{m}(\widetilde{e}_g,\widetilde{g})) = m(e(g);\phi_g(\widetilde{g})) = \phi_g(\widetilde{g}).$$

Hence, $\widetilde{g}\mapsto\widetilde{m}(\widetilde{e}_g,\widetilde{g})$ is the lifting of $\phi_g:\widetilde{G}_g\to G_{e(g)}$. Since $\widetilde{m}(\widetilde{e}_g,\widetilde{g})=\widetilde{e}_g$, this map is the identity on $\widetilde{G}_g$, i.e., $\widetilde{m}(\widetilde{e}_g,\widetilde{g})=\widetilde{g}$ for all $\widetilde{g}\in\widetilde{G}_g$.

It follows that $\widetilde{e}_g$ is the identity in $\widetilde{G}_g$. Let $\widetilde{m}':\widetilde{G}_g\to\widetilde{G}_g$ be the lifting of the map $m'\circ\phi_g:\widetilde{G}_g\to\widetilde{G}_{e(g)}$ such that $\widetilde{m}'(\widetilde{e}_g)=\widetilde{e}_g$.

Then we have

$$\begin{aligned}\phi_g(\widetilde{m}(\widetilde{g},\widetilde{m}'(\widetilde{g}))) &= m(\phi_g(\widetilde{g});\phi_g(\widetilde{m}'(\widetilde{g})))\\ &= m(\phi_g(\widetilde{g}),(\phi_g(\widetilde{g}))^{-1})\\ &= e(g).\end{aligned}$$

Therefore, $\widetilde{g}\mapsto\widetilde{m}(\widetilde{g},\widetilde{m}'(\widetilde{g}))$ is the lifting of constant map of $\widetilde{G}_g$ into $e(g)$.

Since $\widetilde{m}(\widetilde{e}_g,\widetilde{m}'(\widetilde{e}_g)=\widetilde{e}_g$, we conclude that this map is constant and its value is equal to $\widetilde{e}_g$.

Therefore, we have

$$\widetilde{m}(\widetilde{g},\widetilde{m}'(\widetilde{g}))=\widetilde{e}_g,\ \text{for all } \widetilde{g}\in\widetilde{g}_g.$$

Analogously, we have

$$\begin{aligned}\phi_g(\widetilde{m}'(\widetilde{g}),\widetilde{g}) &= m(\phi_g(\widetilde{m}'(\widetilde{g})),\phi_g(\widetilde{g}))\\ &= m((\phi_g(\widetilde{g})))^{-1}\\ &= e(g).\end{aligned}$$

Therefore, $\widetilde{g}\mapsto\widetilde{m}(\widetilde{m}'(\widetilde{g}),\widetilde{g})$ is the lifting of the constant map of $\widetilde{G}_g$ into $e(g)\in G$. Since $\widetilde{m}(\widetilde{m}'(\widetilde{e}_g),\widetilde{e}_g)=\widetilde{e}_g$, we conclude that this map is constant and its value is equal to $\widetilde{e}_g$.

Therefore, we have $\widetilde{m}(\widetilde{m}'(\widetilde{g}),\widetilde{g})=\widetilde{g}$, for all $\widetilde{g}\in\widetilde{G}_g$. This implies that any element $\widetilde{g}\in\widetilde{G}_g$ has an inverse $\widetilde{m}'(g)=\widetilde{g}'1$. Therefore, $\widetilde{G}_g$ with operation $\widetilde{m}$ is a group. Moreover, since $\widetilde{m}$ and $\widetilde{m}'$ are continuous, $\widetilde{G}_g$ with this operation is a topological group, which we denote it by $\overline{G}_g$. It is easy to see that $\phi_g:\overline{G}_g\to G_{e(g)}$ is a cover and $\overline{G}$ is a connected and simply connected topological group. Then, by

Proposition 3.1, since $G_{e(g)}$ is semilocally simply connected, the topological groups $\widetilde{G}_g$ and $\overline{G}_g$ are the same up to isomorphism (from now on we use the notation $\widetilde{G}_g$ for both of them). ∎

Now, we construct a new normal topological generalized groups $\widetilde{G}$ such as follows:

Let $\widetilde{G}$ be the disjoint union of $\widetilde{G}_g$, where $g \in G$. We consider a topology on $\widetilde{G}$ which is coherent with the collection $\mathcal{C} = \{\widehat{G}_g : g \in G\}$, provided a set U is open in $\widetilde{G}$ if and only if $U \cap \widetilde{G}_g$ is open in $\widetilde{G}_g$, for each $\widetilde{G}_g \in \mathcal{C}$. Clearly, the topology of $\widetilde{G}_g$ as a subspace of $\widetilde{G}$ is equivalent to original topology of $\widetilde{G}_g$, see [6]. So, $\widetilde{G}_g$ is connected, locally path connected and simply connected as a subspace of $\widetilde{G}$. This implies that $\widetilde{G}_g \times \widetilde{G}_h$ is also connected, locally arcwise connected and simply connected. Then, by Proposition 3.2, the mapping $m \circ (\phi_g \times \phi_h) : \widetilde{G}_g \times \widetilde{G}_h \to G_{e(gh)}$ has a unique lifting $\widetilde{m}_{gh} \times \widetilde{G}_h \to \widetilde{G}_{gh}$ such that $\widetilde{m}_{gh}(\widetilde{e}_g, \widetilde{e}_h) = \widetilde{e}_{gh}$. In this way, we can define the product $\widetilde{m}$ on $\widetilde{G} \times \widetilde{G}$ by $\widetilde{m}(\widetilde{g}, \widetilde{h}) = \widetilde{m}_{gh}(\widetilde{g}, \widetilde{h})$.

Proposition 3.5. $(\widetilde{g}, \widetilde{m})$ *is a normal topological generalized group.*

Proof. First, we show that $\widetilde{m}$ is associative. We have

$$\begin{aligned}
\phi_{g(hk)} \circ (\widetilde{m}_{g(hk)} \circ (id_{\widetilde{G}} \times \widetilde{m}_{hk})) &= m \circ (\phi_g \times \phi_{hk}) \circ (id_{\widetilde{G}} \times \widetilde{m}_{hk}) \\
&= m \circ (\phi_{gh} \circ \widetilde{m} \times \phi_k) \\
&= m \circ (m \circ (m \circ \phi_g \times \phi_h) \times \phi_k) \\
&= m \circ (m \times id_G) \circ (\phi_g \times \phi_h \times \phi_k)
\end{aligned}$$

and

$$\begin{aligned}
\phi_{ghk} \circ (\widetilde{m} \circ (\widetilde{m} \times id_{\widetilde{G}})) &= m \circ (\phi_{gh} \times \phi_k) \circ (\widetilde{m} \times id_{\widetilde{G}}) \\
&= m \circ (\phi_{ghk} \circ \widetilde{m} \times \phi_k) \\
&= m \circ ((m \circ (\phi_g \times \phi_h)) \times \phi_k) \\
&= m \circ (m \times id_G) \circ (\phi_g \times \phi_h \times \phi_k).
\end{aligned}$$

Since the multiplication on G is associative, it follows that $\widetilde{m} \circ (id_{\widetilde{G}} \times \widetilde{m})$ and $\widetilde{m} \circ (\widetilde{m} \times id_{\widetilde{G}})$ are the lifts of the same map from $\widetilde{G}_g \times \widetilde{G}_h \times \widetilde{G}_k$ into $G_{e(ghk)}$. Since both maps map $(\widetilde{e}_g, \widetilde{e}_h, \widetilde{e}_k)$ into $\widetilde{e}_{e(ghk)}$, it follows that they are identical, i.e., the operation $\widetilde{m}$ is associative. Also, for each $\widetilde{h} \in \widetilde{G}$, there exist $g \in G$ such that $\widetilde{h} \in \widetilde{G}_g$. Therefore, $\widetilde{m}(\widetilde{h}, \widetilde{e}_g) = \widetilde{m}(\widetilde{e}_g, \widetilde{h}) = \widetilde{h}$ and $\widetilde{h}$ has a unique inverse in $\widetilde{G}_g$. (Note that $\widetilde{G}_g$'s are disjoint.) Clearly, the product $\widetilde{m}$ and the mapping $\widetilde{m}' : \widetilde{G} \to \widetilde{G}$, $\widetilde{g} \mapsto \widetilde{g}^{-1}$ are continuous and this implies that $\widetilde{G}$ with product $\widetilde{m}$ is a normal topological generalized group. ∎

We note that, the mapping $\phi : \widetilde{G} \to G$ defined by $\phi(\widetilde{g}) = \phi_g(\widetilde{g})$, where $\widetilde{g} \in \widetilde{G}$ for some $g \in G$, is a homomorphism of topological generalized groups $\widetilde{G}$ and G.

If we define the kernel of ϕ by $\ker \phi = \bigcup\limits_{g \in G} \ker \phi_g$, then $\ker \phi$ is discrete. Moreover, ϕ is an open mapping. For, let $\widetilde{U} \subset \widetilde{G}$ be open in $\widetilde{G}$. Then, since the topology of $\widetilde{G}$ is coherent with the collection $\{\widetilde{G}_g : g \in G\}$, $\widetilde{U} \cap \widetilde{G}_g$ is open in $\widetilde{G}_g$, for each $g \in G$. On the other hand, ϕ_g is an open mapping. Therefore, $\phi_g(\widetilde{U} \cap \widetilde{G})$ is open in $G_{e(g)}$. Since $\phi(\widetilde{U}) = \bigcup_{g \in G} \phi_g(\widetilde{U} \cap \widetilde{G}_g)$, this implies that $\phi(\widetilde{U})$ is open in G, i.e., ϕ is an open mapping. Hence, ϕ is an open epimorphism with discrete kernel and restriction of ϕ to each $\widetilde{G}_g$ is a universal cover of $\widetilde{G}_g$ to $G_{e(g)}$. Therefore, ϕ is a universal generalized cover between topological generalized groups $\widetilde{G}$ and G. This complete the proof of Theorem 2.1. ∎

In the sequel, we need the following result of [1] (see Corollary 85).

Proposition 3.6. *If G is a topological group, $\phi : \widetilde{G} \to G$ is a universal cover, and $\widetilde{G}$ is arcwise connected and $\pi_1(\widetilde{G}) = e$, then $\pi_1(G)$ is absolutely isomorphic to* $\ker \phi$.

Now, we consider the universal covers $\phi_g : \widetilde{G}_g \to G_{e(g)}$, for each $g \in G$. As we have already seen that, $\widetilde{G}_g$ is connected, locally arcwise connected and simply connected. So, $\widetilde{G}_g$ is also arcwise connected and Proposition 3.2 implies that $\pi_1(G_{e(g)})$ is absolutely isomorphic to $\ker \phi_g$. Then, $\ker \phi = \bigcup_{g \in G} \ker \phi_g \simeq \bigcup_{g \in G} \pi_1(G_{e(g)})$. It is easy to see that $\ker \phi$ is also a topological generalized subgroup of $\widetilde{G}$. Therefore, the assertion of Corollary 2.4 holds.

Proposition 3.7. *Let G be a locally arcwise connected and semilocally simply connected normal topological generalized group such that its topology is coherent with the collection $\{G_{e(g)} : g \in G_g\}$. If $(\widetilde{G}, \phi)$ and $(\widetilde{G}, \psi)$ be two upper topological generalized groups of G, then* $\ker \phi$ *is isomorphic to* $\ker \psi$.

Proof. Since the topology of G is coherent with $\{G_{e(g)} : g \in G_g\}$, then $G_{e(g)}$ is open in G, for each $g \in G$. This implies that $G_{e(g)}$ is semilocally simply connected, connected and locally arcwise connected. Therefore, by Proposition 3.2, the universal cover $(\widetilde{G}_g, \phi_g)$ is unique up to isomorphism. Then, $\ker \phi_g$ is isomorphic to $\ker \psi_g$ and this implies that $\ker \phi$ is also isomorphic to $\ker \psi$. ∎

4. Locally compact topological generalized groups

In this section, we consider the locally compact topological generalized groups and we use a generalization notion of universal cover which is developed by Berestivskii and Plaut, that is $\phi : \widetilde{G} \to G$ is a cover of topological groups G and $\widetilde{G}$ if ϕ is an open epimorphism whose kernel is central and prodiscrete (i.e., the inverse limit of discrete groups), see [2]. They proved that, for any topological group G there is a topological group $\widetilde{G}$ and a natural homomorphism $\phi : \widetilde{G} \to G$. In particular, if G is coverable then $\widetilde{G}$ is coverable and ϕ is a universal cover in the category of coverable groups and covers. In the sequel, we use the following results of [2].

Theorem 4.1. *Let G be a locally compact topological group. Then the following are equivalent:*

1. *G is coverable,*
2. *$\phi : \widetilde{G} \to G$ is a cover,*
3. *$\phi : \widetilde{G} \to G$ is open and G is connected,*
4. *$\phi : \widetilde{G} \to G$ is surjective,*
5. *G is connected and locally arcwise connected.*

Moreover, if G is metrizable, then G is coverable if and only if G is connected and locally connected.

Proof of Theorem 2.3. By Theorem 4.1, if G is connected, locally arcwise connected and locally compact topological group then G is coverable and natural homomorphism $\phi : \widetilde{G} \to G$ is a cover.

Moreover, $\widetilde{G}$ is simply connected, see [1]. Therefore, if G is connected, locally arcwise connected and locally compact topological group, then by Proposition 3.2, the mapping $m \circ (\phi \times \phi) : \widetilde{G} \times \widetilde{G} \to G$ has a unique lifting $\widetilde{m} : \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ such that $\widetilde{m}(\widetilde{e}, \widetilde{e}) = \widetilde{e}$, where $\widetilde{e}$ is the unit element of $\widetilde{G}$. Now, Proposition 3.4 implies that $\widetilde{G}$ with product $\widetilde{m}$ is a topological group. Moreover, its structure group with product $\widetilde{m}$ is the same as its original structure group up to isomorphism (we note that the Proposition 3.2 also holds for covers in the sense of Berestovskii and Plaut).

Now, suppose that G be a locally arcwise connected and locally compact topological generalized group with its topology coherent with the collection $\{G_{e(g)} : g \in G\}$. Then, each $G_{e(g)}$ is connected, locally arcwise connected and locally compact topological group and therefore has a natural cover $\phi_g : \widetilde{G}_g \to G_{e(g)}$.

Now, we construct a new normal topological generalized group $\widetilde{G}$ as follows:

Let $\widetilde{G}$ be the disjoint union of $\widetilde{G}_g$, where $g \in G$. We consider a topology on $\widetilde{G}$ which is coherent with the collection $\mathcal{C} = \{\widetilde{G}_g : g \in G_g\}$. Therefore, the topology of $\widetilde{G}_g$ as a subspace of $\widetilde{G}$ is equivalent to original topology of $\widetilde{G}_g$. Since ϕ_g is surjective and $G_{e(g)}$ is connected, then $\widetilde{G}_g$ is also connected. Moreover, Theorem 3 of [2] implies that $\widetilde{G}_g$ is locally arcwise connected. We have already seen that $\widetilde{G}_g$ is simply connected. Therefore, $\widetilde{G}_g \times \widetilde{G}_h$ is also connected, locally arcwise connected and simply connected. Then, by Proposition 3.2, the mapping $m \circ (\phi_g \times \phi_h) : \widetilde{G}_g \times \widetilde{G}_h \to G_{e(gh)}$ has a unique lifting $\widetilde{m}_{gh} : \widetilde{G}_g \times \widetilde{G}_h \to \widetilde{G}_{gh}$ such that $\widetilde{m}_{gh}(\widetilde{e}_g, \widetilde{e}_h) = \widetilde{e}_{gh}$. In this way, we can define the product $\widetilde{m}$ on $\widetilde{G} \times \widetilde{G}$ by $\widetilde{m}(e_g, e_h) = \widetilde{m}_{gh}(\widetilde{g}, \widetilde{h})$, where $\widetilde{g} \in \widetilde{G}_g$ and $\widetilde{h} \in \widetilde{G}_h$. Now, Proposition 3.5 implies that $(\widetilde{G}, \widetilde{m})$ is a normal topological generalized group. Clearly, the mapping $\phi : \widetilde{G} \to G$ defined by $\phi(\widetilde{g}) = \phi_g(\widetilde{g})$, where $\widetilde{g} \in \widetilde{G}_g$ for some $g \in G$ is an algebraic homomorphism of topological generalized groups $\widetilde{G}$ and G. We define the kernel of ϕ by $\ker\phi = \bigcup_{g\in G} \ker\phi_g$. Moreover, ϕ is an open mapping (for proof, see the argument used in proof of Theorem 2.1). Hence, ϕ is a generalized cover. This complete the proof of Theorem 4.1. ■

Proposition 4.2. *The kernel of* ϕ, $\ker\phi$, *is totally disconnected.*

Proof. By Lemma 32 of [1], prodiscrete topological groups are totally disconnected. So, for each $g \in G$, $\ker\phi_g$ is totally disconnected. Since $\ker\phi = \bigcup_{g\in G} \ker\phi_g$ and topology of $\widetilde{G}$ is coherent with $\{\widetilde{G}_g : g \in G\}$, then $\ker\phi$ is also totally disconnected. ■

Now, we need the following results of [2].

Proposition 4.3. *If* G *is locally compact topological group, then* $\pi_1(G)$ *is abstractly isomorphic to the prodiscrete topological group* $\ker\phi$, *where* $\phi : \widetilde{G} \to G$ *is the natural homomorphism.*

If we consider the universal covers $\phi_g : \widetilde{G}_g \to G_{e(g)}$, for each $g \in G$, as we have already seen, $G_{e(g)}$ is locally compact topological group and then, by Proposition 4.3, $\pi_1(G)$ absolutely isomorphic to the prodiscrete topological group $\ker\phi_g$. Then, $\ker\phi = \cup \ker\phi_g \simeq \cup\pi_1(G_{e(g)})$, which is also a topological generalized subgroup of $\widetilde{G}$. Therefore, Corollary 2.4 holds.

Theorem 4.4. *Let* G *be a normal topological generalized group. Then* $G_{e(g)}$ *and* $G_{e(h)}$ *are homomorphic, for each* $g, h \in G$.

Proof. By Lemma 2.1 of [4], if G is a topological generalized group, then $e(g)G = gG$ for each $g \in G$. Let $g, h \in G$. Then $e(g)h = gg'$, for some $g' \in G$,

$$e(g)e(h) = e(e(g)h) = e(gg') = e(g)e(g') \Longrightarrow e(h) = e(g').$$

So,

$$gg'h^{-1} = e(g)hh^{-1} = e(g)e(h) = e(g)e(g') = e(gg').$$

Therefore, $(gg')^{-1} = h^{-1}$, that is $gg' = h$.

Now, we define $R_{g'} : G_{e(g)} \to G_{e(h)}$, by right translation, $k \mapsto kg'$. Then mapping $R_{g'}$ is well-defined, since

$$e(kg') = e(k)e(g') = e(g)e(g') = e(gg') = e(h).$$

On the other hand, since product on G is continuous, then $R_{g'}$ is continuous.

Also $(R_{g'})^{-1} = R_{g'-1}$. For,

$$\begin{aligned} R_{g'} \circ R_{g'-1}(k) &= R_{g'}(kg'^{-1}) \\ &= (kg'^{-1})g' \\ &= k(g'^{-1}g') \\ &= ke(g') \\ &= ke(h) \\ &= ke(k) \\ &= k. \end{aligned}$$

Similarly, $R_{g'-1} \circ R_{g'} = id_{G_{e(g)}}$. So, $R_{g'}$ is a homeomorphism and $G_{e(g)}$ is homeomorphic to $G_{e(h)}$. ■

Remark 4.5. We note that, if G is a normal topological generalized group that satisfies the assumptions of Theorem 2.1 or 2.3, then, for each $g \in G$, $G_{e(g)}$ is a path component of G. Therefore, by the above theorem, path components of G are homeomorphic. This implies that the fundamental group of G does not depend on the base point.

Remark 4.6. We note that Biss [3] puts a topology on the fundamental groups of topological spaces. Let (X, x) be a pointed space. He equipped the space of continuous based maps $Hom((S^1, 1), (X, x))$ with the compact-open topology. Then by using the surjection $Hom((S^1, 1), (X, x)) \rightarrow \pi_1(X, x)$, he defined a quotient topology on $\pi_1(X, x)$. As we saw here, by using the notion of universal covers for coverable topological groups in the sense of Berestovskii and Plaut, the fundamental groups admit a natural prodiscrete topology as the kernel of their universal covers. The fundamental groups with this topology are always Hausdorff, however with the compact-open topology introduced by Biss, in general, they would not be a Hausdorff topological space. So, in general, these two topology are not the same.

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ANALYSIS OF A TWO-STEP METHOD FOR NUMERICAL SOLUTION OF FUZZY ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. Recently, fuzzy initial value problems or fuzzy differential equations have received considerable amount of attentions ([3], [4] and [5]). In all of them, one-step numerical methods have been considered, but in this paper we have a *two-step method* for solving fuzzy ordinary differential equations. In the first section, we present the necessary and introductory materials to deal with the fuzzy initial value differential equations. In the second section, a modified two-step Simpson method and the corresponding convergence theorem of our method are presented. In the last section, we will present an example of fuzzy differential equations. Our numerical results can compare with the results of the existing methods.

Keywords: fuzzy differential equations; two-step methods; Simpson method; ordinary differential equations.
AMSC: 65L05, 65L06, 03E72.

1. Preliminaries

A general definition of fuzzy numbers may be found in [1]. However, our fuzzy numbers will be almost always triangular or triangular shaped fuzzy numbers. Let T be the set of all triangular or triangular shaped fuzzy numbers and $u \in T$. We define the r-level sets:

$$[u]_r = \{x : u(x) \geq r\} \quad , \quad 0 \leq r \leq 1 \tag{1.1}$$

which are closed bounded intervals and we denote by $[u]_r = [\underline{u}(r), \overline{u}(r)]$. For more details see [1], [2].

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Let S be the set of all closed bounded intervals in R and $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$ be two members of S. The interval metric d_I on S is defined as:

$$d_I(I_1, I_2) = \frac{|a_1 - a_2| + |b_1 - b_2|}{2}. \tag{1.2}$$

Consider the first-order one dimensional fuzzy initial value differential equation given by

$$\begin{cases} y'(t) = f(t, y(t)) & t \in [t_0, T] \\ y(t_0) = y_0 \end{cases} \tag{1.3}$$

where y is a fuzzy function of t, $f(t, y(t))$ is a fuzzy function of the crisp variable t and the fuzzy variable y, y' is the fuzzy derivative of y and $y(t_0) = y_0$ is a triangular or a triangular shaped fuzzy number. Therefore we have a fuzzy Cauchy problem [4]. We denote the fuzzy function y by $y = [\underline{y}, \overline{y}]$. It means that the r-level sets of $y(t)$ for $t \in [t_0, T]$ is $[y(t)]_r = [\underline{y}(t; r), \overline{y}(t; r)]$. Also

$$[y'(t)]_r = [\underline{y}'(t; r), \overline{y}'(t; r)] \quad , \quad [f(t, y(t))]_r = [\underline{f}(t, y(t); r), \overline{f}(t, y(t); r)]. \tag{1.4}$$

We write $f(t, y) = [\underline{f}(t, y), \overline{f}(t, y)]$ such that $\underline{f}(t, y) = F[t, \underline{y}, \overline{y}]$ and $\overline{f}(t, y) = G[t, \underline{y}, \overline{y}]$. Because of $y' = f(t, y)$ we have:

$$\begin{aligned} \underline{y}'(t; r) &= \underline{f}(t, y(t); r) = F[t, \underline{y}(t; r), \overline{y}(t; r)] \\ \overline{y}'(t; r) &= \overline{f}(t, y(t); r) = G[t, \underline{y}(t; r), \overline{y}(t; r)]. \end{aligned} \tag{1.5}$$

Also we write

$$[y(t_0)]_r = [\underline{y}(t_0; r), \overline{y}(t_0; r)] \quad , \quad [y_0]_r = [\underline{y}_0(r), \overline{y}_0(r)] \tag{1.6}$$

where $\underline{y}(t_0; r) = \underline{y}_0(r)$ and $\overline{y}(t_0; r) = \overline{y}_0(r)$. By integration of the system (1.3), from t_{n-1} to t_{n+1} and using the Simpson method for the right hand side of the equation

$$\int_{t_{n-1}}^{t_{n+1}} y'(s)ds = \int_{t_{n-1}}^{t_{n+1}} f(s, y(s))ds, \tag{1.7}$$

we will have

$$\begin{aligned} y(t_{n+1}) &= y(t_{n-1}) + \frac{h}{3} f(t_{n-1}, y(t_{n-1})) + \frac{4h}{3} f(t_n, y(t_n)) \\ &+ \frac{h}{3} f(t_{n+1}, y(t_n) + h f(t_n, y(t_n))) \\ &+ \frac{h^3}{6} f'(\xi_2, y(\xi_2)) f_y(t_{n+1}, \xi_3) - \frac{h^5}{90} f^{(4)}(\xi_1, y(\xi_1)) \end{aligned} \tag{1.8}$$

where $t_{n-1} \le \xi_1 \le t_{n+1}$, $t_n \le \xi_2 \le t_{n+1}$ and ξ_3 is between $y(t_n) + h f(t_n, y(t_n))$ and

$y(t_n) + hf(t_n, y(t_n)) + \frac{h^2}{2}f'(\xi_2, y(\xi_2))$. We have the modified two-step Simpson method

$$y_{n+1} = y_{n-1} + \frac{h}{3}f(t_{n-1}, y_{n-1}) + \frac{4h}{3}f(t_n, y_n) + \frac{h}{3}f(t_{n+1}, y_n + hf(t_n, y_n)) \tag{1.9}$$

for numerical solutions of the fuzzy differential equation (1.3) with initial value $y_0 = y(t_0)$ and $y_1 = y_0 + hf(t_0, y_0) + \frac{h^2}{2}f(t_0, y_0)$.

2. A modified explicit two-step Simpson method of order two

We note that throughout each integration step, the value of r is unchanged. We calculate the exact and Simpson approximation solution at grid points $t_n = t_0 + nh$, $0 \le n \le N$ where $h = \frac{T - t_0}{N}$. Let $y(t_n)$ be the exact solution and y_n be the Simpson approximation solution of the fuzzy initial value problem (1.3) at t_n. We denote the exact and approximation solution at t_n by:

$$[y(t_n)]_r = [\underline{y}(t_n; r), \overline{y}(t_n; r)] \quad , \quad [y_n]_r = [\underline{y}_n(r), \overline{y}_n(r)] \qquad 0 \le n \le N \tag{2.1}$$

respectively. We know that the exact solution satisfies to:

$$\begin{aligned} \underline{y}(t_{n+1}; r) = \underline{y}(t_{n-1}; r) &+ \frac{h}{3}F[t_{n-1}, \underline{y}(t_{n-1}; r), \overline{y}(t_{n-1}; r)] \\ &+ \frac{4h}{3}F[t_n, \underline{y}(t_n; r), \overline{y}(t_n; r)] \\ &+ \frac{h}{3}F[t_{n+1}, \underline{y}(t_n; r) + hF[t_n, \underline{y}(t_n; r), \overline{y}(t_n; r)] \\ &\qquad , \overline{y}(t_n; r) + hG[t_n, \underline{y}(t_n; r), \overline{y}(t_n; r)]] + h^3\underline{A}(r) \end{aligned} \tag{2.2}$$

$$\begin{aligned} \overline{y}(t_{n+1}; r) = \overline{y}(t_{n-1}; r) &+ \frac{h}{3}G[t_{n-1}, \underline{y}(t_{n-1}; r), \overline{y}(t_{n-1}; r)] \\ &+ \frac{4h}{3}G[t_n, \underline{y}(t_n; r), \overline{y}(t_n; r)] \\ &+ \frac{h}{3}G[t_{n+1}, \underline{y}(t_n; r) + hF[t_n, \underline{y}(t_n; r), \overline{y}(t_n; r)] \\ &\qquad , \overline{y}(t_n; r) + hG[t_n, \underline{y}(t_n; r), \overline{y}(t_n; r)]] + h^3\overline{A}(r) \end{aligned} \tag{2.3}$$

where $A = [\underline{A}, \overline{A}]$, $[A]_r = [\underline{A}(r), \overline{A}(r)]$ and:

$$[A]_r = \left[\frac{1}{6}f'(\xi_2, y(\xi_2))f_y(t_{n+1}, \xi_3) - \frac{h^2}{90}f^{(4)}(\xi_1, y(\xi_1))\right]_r. \tag{2.4}$$

In order to approximate the solution of the fuzzy differential equation (1.3), we will use the two-step explicit Simpson method:

$$\begin{aligned} \underline{y}_{n+1}(r) = \underline{y}_{n-1}(r) + \frac{h}{3}F[t_{n-1}, \underline{y}_{n-1}(r), \overline{y}_{n-1}(r)] + \frac{4h}{3}F[t_n, \underline{y}_n(r), \overline{y}_n(r)] \\ +\frac{h}{3}F[t_{n+1}, \underline{y}_n(r) + hF[t_n, \underline{y}_n(r), \overline{y}_n(r)], \overline{y}_n(r) + hG[t_n, \underline{y}_n(r), \overline{y}_n(r)]] \end{aligned} \tag{2.5}$$

$$\begin{aligned}\overline{y}_{n+1}(r) &= \overline{y}_{n-1}(r) + \frac{h}{3}G[t_{n-1}, \underline{y}_{n-1}(r), \overline{y}_{n-1}(r)] + \frac{4h}{3}G[t_n, \underline{y}_n(r), \overline{y}_n(r)] \\ &+\frac{h}{3}G[t_{n+1}, \underline{y}_n(r) + hF[t_n, \underline{y}_n(r), \overline{y}_n(r)], \overline{y}_n(r) + hG[t_n, \underline{y}_n(r), \overline{y}_n(r)]]. \end{aligned} \tag{2.6}$$

The following lemmas will be applied to show the convergence of our method. For more details see [4].

Lemma 2.1. *Suppose a sequence of non negative numbers* $\{W_n\}_{n=0}^N$ *satisfy:*

$$W_n \le AW_{n-1} + B \quad , \quad 1 \le n \le N \tag{2.7}$$

where A *and* B *are two given positive constants. Then, for* $s = 0, 1, 2, ..., n$,

$$W_n \le A^{n-s}W_s + B\frac{A^{n-s}-1}{A-1} \quad , \quad s \le n \le N. \tag{2.8}$$

Lemma 2.2. *Suppose that a sequence of non negative numbers* $\{P_n\}_{n=0}^N$ *satisfy*

$$P_{n+1} \le AP_n + BP_{n-1} + C \quad , \quad 1 \le n \le N-1 \tag{2.9}$$

for some given positive constants A , B *and* C. *Then, for* $\alpha = \dfrac{\sqrt{A^2+4B}+A}{2}$, *we have*

$$P_{n+1} + (\alpha - A)P_n \le \alpha^n[P_1 + (\alpha - A)P_0] + C\frac{\alpha^n - 1}{\alpha - 1}. \tag{2.10}$$

Proof. It is obvious that $A = \dfrac{\sqrt{A^2+4B}+A}{2} - \dfrac{\sqrt{A^2+4B}-A}{2}$. Therefore, we have:

$$\begin{aligned} &P_{n+1} + \frac{\sqrt{A^2+4B}-A}{2}P_n \\ &\qquad \le \frac{\sqrt{A^2+4B}+A}{2}\left(P_n + \frac{\sqrt{A^2+4B}-A}{2}P_{n-1}\right) + C. \end{aligned} \tag{2.11}$$

If we set $T_{n+1} = P_{n+1} + \dfrac{\sqrt{A^2+4B}-A}{2}P_n$ and $\alpha = \dfrac{\sqrt{A^2+4B}+A}{2}$, then

$$T_{n+1} \le \alpha T_n + C \quad , \quad 1 \le n \le N-1. \tag{2.12}$$

By using Lemma 2.1 with $s = 1$, the proof is completed. ■

Let $F[t, u, v]$ and $G[t, u, v]$ be the functions which are given by the equations (1.5) where u and v are constants and $u \le v$. Thus, the domain of F and G are defined as $K = \{(t, u, v) : t_0 \le t \le T, -\infty < u \le v, -\infty < v < +\infty\}$. Now, we will present the convergence theorem.

Theorem 2.1. *Let $F[t,u,v]$ and $G[t,u,v]$ belong to $C^1(K)$ and suppose that the partial derivatives of F and G be bounded on K. Then for arbitrary fixed $0 \le r \le 1$ the Simpson approximations y_N converge to the exact solution $y(T)$ uniformly in t. In other words,*

$$\lim_{h\to 0} d_I([\underline{y}_N(r), \overline{y}_N(r)], [\underline{y}(t_N;r), \overline{y}(t_N;r)]) = 0. \tag{2.13}$$

Proof. Let $W_n = |\underline{y}(t_n;r) - \underline{y}_n(r)|$ and $V_n = |\overline{y}(t_n;r) - \overline{y}_n(r)|$. By using the equations (2.2), (2.3), (2.5) and (2.6) we conclude that [4]:

$$\begin{aligned} W_{n+1} &\le W_{n-1} + \frac{2Lh}{3}\max\{W_{n-1},V_{n-1}\} + \frac{8Lh}{3}\max\{W_n,V_n\} \\ &+ \frac{2Lh}{3}[2Lh\max\{W_n,V_n\} + \max\{W_n,V_n\}] + h^3\underline{M} \end{aligned} \tag{2.14}$$

$$\begin{aligned} V_{n+1} &\le V_{n-1} + \frac{2Lh}{3}\max\{W_{n-1},V_{n-1}\} + \frac{8Lh}{3}\max\{W_n,V_n\} \\ &+ \frac{2Lh}{3}[2Lh\max\{W_n,V_n\} + \max\{W_n,V_n\}] + h^3\overline{M} \end{aligned} \tag{2.15}$$

where $\underline{M}$ and $\overline{M}$ are upper bound for $\underline{A}(r)$ and $\overline{A}(r)$ respectively which

$$[A]_r = \left[\underline{A}(r), \overline{A}(r)\right] = \left[\frac{1}{6}f'(\xi_2, y(\xi_2))f_y(t_{n+1},\xi_3) - \frac{h^2}{90}f^{(4)}(\xi_1, y(\xi_1))\right]_r. \tag{2.16}$$

We see that $\max\{W_i, V_i\} \le W_i + V_i$. Therefore,

$$\begin{aligned} W_{n+1} &\le W_{n-1} + \frac{2Lh}{3}(W_{n-1}+V_{n-1}) + \frac{8Lh}{3}(W_n+V_n) \\ &+ \frac{2Lh}{3}(1+2Lh)(W_n+V_n) + h^3\underline{M} \end{aligned} \tag{2.17}$$

$$\begin{aligned} V_{n+1} &\le V_{n-1} + \frac{2Lh}{3}(W_{n-1}+V_{n-1}) + \frac{8Lh}{3}(W_n+V_n) \\ &+ \frac{2Lh}{3}(1+2Lh)(W_n+V_n) + h^3\overline{M}. \end{aligned} \tag{2.18}$$

By adding above two equations and setting $U_n = W_n + V_n$, we obtain

$$U_{n+1} \le \frac{4Lh}{3}(5+2Lh)U_n + \left(1 + \frac{4Lh}{3}\right)U_{n-1} + 2h^3M \tag{2.19}$$

where $M = \max\{\underline{M}, \overline{M}\}$. By using Lemma 2.2, we have:

$$U_{n+1} + (\alpha - A)U_n \le \alpha^n[U_1 + (\alpha - A)U_0] + C\frac{\alpha^n - 1}{\alpha - 1} \tag{2.20}$$

where $\alpha = \dfrac{\sqrt{A^2+4B}+A}{2}$. Because of $U_0 = 0$, for $n = N-1$ we have:

$$\lim_{h\to 0}\left[\alpha^{N-1}[U_1 + (\alpha - A)U_0] + C\frac{\alpha^{N-1}-1}{\alpha - 1}\right] = 0. \tag{2.21}$$

Therefore, we have $\lim\limits_{h\to 0}[U_N + (\alpha - A)U_{N-1}] = 0$ and consequently $\lim\limits_{h\to 0} U_N = 0$. In other words, $\lim\limits_{h\to 0} W_N = \lim\limits_{h\to 0} V_N = 0$ and the proof is completed. ■

3. Numerical result

In this section, we will present a numerical example. For this example, the theoretical exact solution and the numerical solutions via our method are shown in the figures and tables at the end of this section. As well as the convergence theorem shows, the numerical results also show that for smaller stepsize h we get smaller errors and hence better results. This example has chosen in comparison with the results of other methods [4].

Example 3.1. Consider the fuzzy initial value problem

$$\begin{cases} y'(t) = ty(t), \quad t \in [-1,1] \\ [y(-1)]_r = [0.5\sqrt{r} - 0.3, 0.2\sqrt{1-r} + 0.2]. \end{cases} \tag{3.1}$$

The exact solution is separated between two steps. If $t < 0$ then with $t_0 = -1$ we have:

$$\begin{cases} \underline{y}(t;r) = \dfrac{A+B}{2}\underline{y}_0(r) + \dfrac{A-B}{2}\overline{y}_0(r) \\ \overline{y}(t;r) = \dfrac{A-B}{2}\underline{y}_0(r) + \dfrac{A+B}{2}\overline{y}_0(r), \end{cases} \tag{3.2}$$

where

$$A = \frac{1}{2}e^{\frac{t^2-t_0^2}{2}} \quad , \quad B = \frac{1}{A}, \tag{3.3}$$

and if $t \geq 0$ with $t_0 = 0$ we have:

$$\underline{y}(t;r) = \underline{y}_0(r)e^{\frac{t^2-t_0^2}{2}} \quad , \quad \overline{y}(t;r) = \overline{y}_0(r)e^{\frac{t^2-t_0^2}{2}}. \tag{3.4}$$

By using the fuzzy two-step modified Simpson method approximation and denoting

$$\begin{aligned} \underline{y}_1(r) &= \underline{y}_0(r) + ht_0\overline{y}_0 + \frac{h^2}{2}(1+t_0^2)\underline{y}_0(r) \\ \overline{y}_1(r) &= \overline{y}_0(r) + ht_0\underline{y}_0 + \frac{h^2}{2}(1+t_0^2)\overline{y}_0(r) \end{aligned} \tag{3.5}$$

as the initial values, we have

$$\begin{aligned} \underline{y}_{i+1}(r) &= \underline{y}_{i-1}(r) + \frac{h}{3}t_{i-1}\overline{y}_{i-1}(r) + \frac{4h}{3}t_i\overline{y}_i(r) + \frac{h}{3}t_{i+1}(\overline{y}_i(r) + ht_i\underline{y}_i(r)) \\ \overline{y}_{i+1}(r) &= \overline{y}_{i-1}(r) + \frac{h}{3}t_{i-1}\underline{y}_{i-1}(r) + \frac{4h}{3}t_i\underline{y}_i(r) + \frac{h}{3}t_{i+1}(\underline{y}_i(r) + ht_i\overline{y}_i(r)) \end{aligned} \tag{3.6}$$

where $t_i < 0$ and

$$\begin{aligned} \underline{y}_{i+1}(r) &= \underline{y}_{i-1}(r) + \frac{h}{3}t_{i-1}\overline{y}_{i-1}(r) + \frac{h}{3}t_{i+1}\underline{y}_i(r) \\ \overline{y}_{i+1}(r) &= \overline{y}_{i-1}(r) + \frac{h}{3}t_{i-1}\underline{y}_{i-1}(r) + \frac{h}{3}t_{i+1}\overline{y}_i(r) \end{aligned} \tag{3.7}$$

where $t_i = 0$ and

$$\begin{aligned} \underline{y}_{i+1}(r) &= \underline{y}_{i-1}(r) + \frac{h}{3}t_{i-1}\underline{y}_{i-1}(r) + \frac{4h}{3}t_i\underline{y}_i(r) + \frac{h}{3}t_{i+1}(\underline{y}_i(r) + ht_i\underline{y}_i(r)) \\ \overline{y}_{i+1}(r) &= \overline{y}_{i-1}(r) + \frac{h}{3}t_{i-1}\overline{y}_{i-1}(r) + \frac{4h}{3}t_i\overline{y}_i(r) + \frac{h}{3}t_{i+1}(\overline{y}_i(r) + ht_i\overline{y}_i(r)) \end{aligned} \tag{3.8}$$

where $t_i > 0$. The theoretical exact solution and the numerical solutions via our method with different stepsize h are shown in Figures 3.1, 3.2 and 3.3. Also the r-level sets of the fuzzy modified two-step Simpson approximations for $r = 0.2$ and $r = 0.7$ are given in Tables 3.1 and 3.2, respectively.

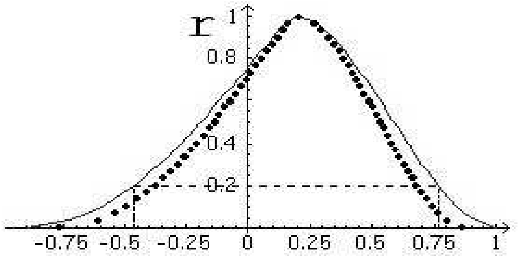

Figure 3.1: $[y(1)]_r$ and $[y_N]_r$ with $h = 2^0$.

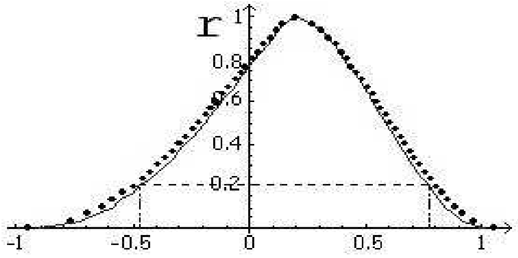

Figure 3.2: $[y(1)]_r$ and $[y_N]_r$ with $h = 2^{-1}$.

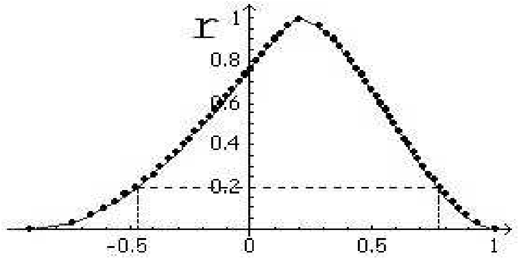

Figure 3.3: $[y(1)]_r$ and $[y_N]_r$ with $h = 2^{-2}$.

h or N ($N = \frac{2}{h}$)	$\underline{y}_N(0.2)$	$\overline{y}_N(0.2)$
$h = 1$ or $N = 2$	-0.37991229588	0.68240453183
$h = 0.5$ or $N = 4$	-0.50158297633	0.79864855758
$h = 0.25$ or $N = 8$	-0.47712052458	0.77860199758
$h = 0.1$ or $N = 20$	-0.46916539153	0.77158097615
$h = 0.01$ or $N = 200$	-0.46755835823	0.77005051182
$h = 0.001$ or $N = 2000$	-0.46754187646	0.77003411232
Exact solution	-0.46754170963	0.77003394558

Table 3.1: 0.2-levelsets

h or N ($N = \frac{2}{h}$)	$\underline{y}_N(0.7)$	$\overline{y}_N(0.7)$
$h = 1$ or $N = 2$	-0.00914631888	0.43702084365
$h = 0.5$ or $N = 4$	-0.06294574081	0.48314427584
$h = 0.25$ or $N = 8$	-0.05047583804	0.47692064109
$h = 0.1$ or $N = 20$	-0.04667023792	0.47443633957
$h = 0.01$ or $N = 200$	-0.04595721883	0.47383162709
$h = 0.001$ or $N = 2000$	-0.04595025566	0.47382478031
Exact solution	-0.04595018555	0.47382471032

Table 3.2: 0.7-levelsets

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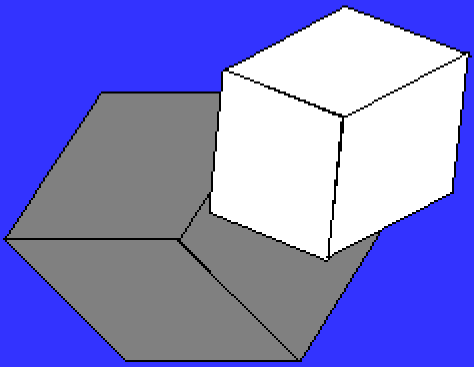

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