Analysis for fractional integro-differential equation with time delay

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Abstract. The main purpose of this paper is to solve some linear and nonlinear Caputo fractional Volterra-Fredholm integro-differential equations with delay. In addition, we investigate the convergence analysis for these equations. A semi-analytical method is described and illustrated with examples.
Keywords: convergence, stability, fractional integro-differential equation.

1. Introduction

Fractional calculus (shortly FC) plays a major role in the modeling of mathematical engineering and physical systems. FC is one of the most important branches of mathematical analysis. The basic advantage of this calculus is in a generalization of the concepts of integer-order differentiation and integration [7]. This calculus used to illuminate physical, bioengineering, aerodynamics, biological control, electro-dynamics and electrochemistry numerous models [5, 12, 14, 15].

A delay fractional integro-differential equation is an integro-differential equation in which the fractional derivative of the unknown function at a certain time [1]. In recent years, the fractional differential equations with delays are developed in [6, 16, 18, 19]. If the behavior of systems includes information on the former state, then these systems are called time delay systems. There are various

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fractional differential operators, as Riemann-Liouville, Caputo and Grunwald-Letnikov. Among these operators, we interest to use Caputo fractional operator where, it allows usual initial conditions when operating on integral transform for instance Laplace transform. In particular, Volterra integro- differential equations without delay have been studied [4]. This study contributes to getting a better progressing and closing the above-mentioned powerful information.

There are much more difficult to study the stability analysis for systems. But indeed, the stability is verified for Caputo fractional differential equations and Volterra integro-differential equations in [8, 9, 10, 16, 20, 22].

Decomposition method for solving non-linear integro-differential equations has been presented by Al Khaled [2]. The modified Laplace Adomain decomposition method (MLADM) for solving Volterra integral equation has been discussed by Wazwaz [21]. Then MLADM applied to solve the integro-differential equation [11].

In this paper, we discuss the analysis of fractional constant time delay integro-differential equations with the Caputo fractional derivative. This article focuses on the convergence and stability analysis for the fractional integro-differential equations with constant time delay. Also, we derive the existence and uniqueness of the solution. Finally, we develop and apply MLADM to fractional integro-differential equations with constant time delay.

The outline of this work as follows. Section 1 presents a brief introduction to fractional calculus, recalls the basic notations that help us to get the desired results. Section 2 offers the suggested equation and then investigates the existed unique solution of the proposed problem in this paper. Section 3 states the stability of fractional integro-differential equations with the Caputo derivative. Section 4 proves the convergence of the proposed equation and gives experimental examples.

1.1 Notations and preliminaries

Here, we state some definitions, theorems and preliminary demanded facts that apply to this paper.

Definition 1.1. For $\alpha \in (0,1)$ and a function $u : [0,T] \to \mathbb{R}$ Caputo fractional derivative

$$D^{\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u^{(1)}(s)ds}{(t-s)^{\alpha}},$$

where $\Gamma(\alpha)$ denotes the gamma function.

Some important mathematical properties of Caputo fractional derivative:

1. The fractional product rule [17]

$$D^{\alpha}[u(t)v(t)] = \sum_{i=1}^{\infty} \binom{\alpha}{i} D^{\alpha-i}u(t)D^{i}v(t), \tag{1.1}$$

where

$$\begin{pmatrix} \alpha \\ i \end{pmatrix} = \frac{\Gamma(\alpha+1)}{\Gamma(i+1)\Gamma(\alpha+1-i)}.$$

2. Linearity of Caputo fractional operater,

$$D^{\alpha}(au(t) \pm bv(t)) = aD^{\alpha}u(t) \pm bD^{\alpha}v(t).$$

3. Caputo fractional operater for composite function,

$$D^{\alpha}u(v(t)) = D^{\alpha}v(t)u^{(1)}v(t). \tag{1.2}$$

4. Caputo fractional differential operator for Riemann fractional integral operator of the same order α is also defined as

$$D^{\alpha}I^{\alpha}u(t) = u(t). \tag{1.3}$$

5. Riemann fractional integral operator of Caputo fractional differential operator for the same order α is also defined as

$$I^{\alpha}D^{\alpha}u(t) = u(t) - \sum_{i=0}^{k-1} u^{(i)}(0)\frac{t^{(i)}}{\Gamma(i+1)},\ k-1 \leq \alpha \leq k \in \mathbb{N}. \tag{1.4}$$

Definition 1.2. The α-order Riemann-Liouville fractional integral operator of u is defined for all $a \leq t \leq b$ by

$$I^{\alpha}u(t) := \begin{cases} u(t), & \alpha = 0, \\ \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t \frac{u(\tau)d\tau}{(t-\tau)^{\alpha-1}}, & \alpha > 0. \end{cases} \tag{1.5}$$

The Riemann fractional integral operator for a negative power α converts into Riemann fractional differential operator, i.e. $\alpha < 0$, $I^{-\alpha} = D^{\alpha}$.

Definition 1.3. $C^1[0,1]$ denotes the space of continuously differentiable functions on [0,1] with the norm

$$\|u(t)\| = \max_{0\leq t\leq 1} |u(t)|. \tag{1.6}$$

Definition 1.4. The functions $G, H \in (C^1[0,1], \mathbb{R})$ satisfy the Lipschitz conditions if there exist real constants ξ, $\mu > 0$, such that

$$|G(u(t)) - G(v(t))| \leq \xi|u-v|,\ |H(u(t)) - H(v(t))| \leq \mu|u-v|, \tag{1.7}$$

where u, $v \in C^1[0,1]$, $0 \leq t \leq 1$.

Definition 1.5. Laplace transform for Caputo fractional derivative and fractional Riemann integral are expressed respectively as follows

$$L(D^{\alpha}u(t)) = s^{\alpha}U(s) - \sum_{j=0}^{n-1} s^{\alpha-j-1}u^{j}(0),\ \alpha \in (n-1,n) \tag{1.8}$$

$$L(I^{\alpha}u(t)) = s^{-\alpha}U(s),\ U(s) = L(u(t)). \tag{1.9}$$

Also, the linearity of Laplace operator

$$L(au(t) + bv(t)) = aL(u(t)) + bL(v(t)). \tag{1.10}$$

Definition 1.6. $L^{\infty}[0,1]$ is the space of all essentially bounded functions with norm

$$\|u\| = \sup_{0\le t\le 1} |u(t)|. \tag{1.11}$$

Theorem 1.1. *Assume that $u, v \in S$, if $(S, \|.\|)$ is a Banach space and $\Phi : S \to S$ satisfying*

$$\|\Phi u - \Phi v\| \le L\|u - \Phi v\| + \chi\|u - v\|, \tag{1.12}$$

where $L \ge 0$, $0 \le \chi < 1$. Suppose that Φ has a fixed point. Then, Φ is a picard $\Phi-$ stable.

2. Statement of the problem

Throughout this paper, we will consider Caputo fractional integro-differential equation of the form:

$$D^{\alpha}u(t) = f\bigg(t, u(t-\tau), \int_0^t G(x,u(x))dx, \int_0^1 H(x,u(x))\ dx\bigg).$$

With another more detailed form:

$$\begin{aligned} D^{\alpha}u(t) = {} & u(t-\tau) + \lambda_1 \int_0^t K(t,x)G(u(x))\ dx + f(t) \\ & + \lambda_2 \int_0^1 M(t,x)H(u(x))\ dx,\ 0 < \alpha \le 1, \end{aligned} \tag{2.1}$$

subject to the initial condition:

$$u(0) = u_0, \tag{2.2}$$

where $D^{\alpha}u$ denotes the Caputo fractional derivative of the unknown function, $u(t) \in S = C([0,1], \mathbb{R}^{+})$, τ is the time delay; $\tau > 0$, $K, M : [0,1] \times \mathbb{R} \to \mathbb{R}$ are the kernels of the equation Eq.(2.1), $f(t)$ is an analytic function, $G, H : \mathbb{R} \to \mathbb{R}$ and λ_1, λ_2 are arbitrary constants.

Lemma 2.1. *If the function $f(t)$ is a continuous function, then a nonlinear fractional integro-differential equation Eq.(2.1) is equivalent to the integral equation,*

$$\begin{aligned} u(t) = \; & u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[u(s-\tau) + f(s)]ds \\ & + \frac{\lambda_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\int_0^s K(t,w)G(u(w))dw \right) ds \\ & + \frac{\lambda_2}{\Gamma(\alpha)} \int_0^1 (t-s)^{\alpha-1} \left(\int_0^1 M(t,w)H(u(w))dw \right) ds, \quad t \in [0,1]. \end{aligned} \tag{2.3}$$

Proof. Suppose that $u(t)$ is a solution of Eq.(2.1), let

$$\begin{aligned} w(t) = \; & u(t-\tau) + \lambda_1 \int_0^t K(t,x)G(u(x)) \; dx + f(t) \\ & + \lambda_2 \int_0^1 M(t,x)H(u(x)) \; dx, \end{aligned}$$

then, $w(t) = D^\alpha u(t)$. Since $D^\alpha u(t) = D^\alpha(u(t) - u_0)$, where D^α is a Caputo fractional derivative of order α,

$$w(t) = D^\alpha(u(t) - u_0) = I^{1-\alpha} \frac{d}{dt}(u(t) - u_0).$$

This gives

$$D^{1-\alpha} w(t) = \frac{d}{dt}(u(t) - u_0),$$

upon applying the integral operator I on both sides of the previous relation

$$ID^{1-\alpha} w(t) = u(t) - u_0 + k,$$

where k is a constant of integration. Since $w(t)$ and $u(t) - u_0$ are both continuous functions, for $t = 0$ we obtain $k = 0$. This gives

$$I^\alpha w(t) = u(t) - u_0.$$

By virtue of the definition of $w(t)$, we obtain Eq.(2.3). Conversely, suppose that $u(t)$ is the solution of the Eq.(2.3). Then, it can be written as

$$u(t) = u_0 + I^\alpha w(t),$$

where

$$w(t) \;\; = \;\; u(t-\tau) + \lambda_1 \textstyle\int_0^t K(t,x)G(u(x)) \; dx + f(t) + \lambda_2 \int_0^1 M(t,x)H(u(x)) \; dx,$$

since $w(t)$ is continuous and u_0 is continuous, operating the Caputo fractional differential operator D^α on both sides of Eq.(2.3) then, we get

$$D^\alpha u(t) = D^\alpha u_0 + D^\alpha I^\alpha w(t) = w(t).$$

This gives

$$D^{\alpha}u(t) = u(t-\tau) + \lambda_1 \int_0^t K(t,x)G(u(x))\,dx + f(t) + \lambda_2 \int_0^1 M(t,x)H(u(x))\,dx,$$

From Eq.(2.3), we have $u(0) = u_0$. This completes the proof. □

2.1 Main results

In this section, we will prove the existence and uniqueness of the solution to Eq.(2.1).

Theorem 2.1. *Assume that G, H are nonlinear Lipschitz continuous functions as in* (1.7) *and $f(t)$ is a continuous function, then the non-linear integro-differential equation* (2.1) *has at least a unique solution $u \in S$.*

Proof. First, we examine the continuity and bounded of Φ. Define the operator $\Phi : S \to S$ as follows

$$\begin{aligned}\Phi u(t) &= u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[u(s-\tau) + f(s)\Big] ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[\int_0^s K(s,x)F(u(x))dx \\ &+ \int_0^1 M(s,x)G(u(x))dx\Big] ds.\end{aligned}$$

The answer divided into two steps.
Step 1. In fact, for $u \in S$,

$$\begin{aligned}|\Phi u(t)| &\leq |u_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[|u(s-\tau)| + |f(s)|\Big] ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[\int_0^s |K(s,x)||F(u(x))|dx \\ &+ \int_0^1 |M(s,x)||G(u(x))|dx\Big] ds.\end{aligned}$$

From definition (1.6) and continuity of u

$$|\Phi u(t)| \leq (\|u\|_\infty + \|f\|_\infty)(\frac{t^\alpha}{\Gamma(\alpha+1)}) + \frac{kt^\alpha}{\Gamma(\alpha+1)} \leq \frac{\sigma t^\alpha}{\Gamma(\alpha+1)}.$$

Consequently,

$$|\Phi u(t)| \leq C,$$

where k, σ and C are constants. This mean that Φ is well-defined.
Step 2. Thus,

$$\begin{aligned}|\Phi u(t)-u_0| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[|u(s-\tau)|+|f(s)|\Big]ds\\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[\int_0^s |K(s,x)||F(u(x))|dx\\ &+ \int_0^1 |M(s,x)||G(u(x))|dx\Big]ds.\end{aligned}$$

From the Lipschitz conditions of the nonlinear functions F, G

$$|\Phi u(t)-u_0| \leq \frac{\omega}{\Gamma(\alpha)}\int_0^t s^{\alpha-1}ds \leq \frac{\omega}{\Gamma(\alpha+1)}t^{\alpha}.$$

Hence, for u bounded, continuous, which leads to Φu is also bounded and continuous.
Note that. Since u is continuous, so Φu is also continuous. Then, Φ is a well defined operator.

Finally, from (2.3), for t_1, $t_2 \in [0,1]$ with $t_2 > t_1$, we gain

$$\begin{aligned}|\Phi u(t_2)-\Phi u(t_1)| &\leq \frac{1}{\Gamma(\alpha)}\Big(|\int_0^{t_2}(t_2-s)^{\alpha-1}\Big[u(s-\tau))+f(s)\Big]ds\\ &- \int_0^{t_1}(t_1-s)^{\alpha-1}\Big[u(s-\tau))+f(s)\Big]ds|\Big)\\ &+ \frac{1}{\Gamma(\alpha)}\Big(|\int_0^{t_2}(t_2-s)^{\alpha-1}\Big[\int_0^s K(s,x)F(u(x))dx\Big]ds\\ &- \int_0^{t_1}(t_1-s)^{\alpha-1}\Big[\int_0^s K(s,x)F(u(x))dx\Big]ds|\Big)\\ &+ \frac{1}{\Gamma(\alpha)}\Big(|\int_0^{t_2}(t_2-s)^{\alpha-1}\Big[\int_0^1 M(s,x)G(u(x))dx\Big]ds\\ &- \int_0^{t_1}(t_1-s)^{\alpha-1}\Big[\int_0^1 M(s,x)G(u(x))dx\Big]ds|\Big).\end{aligned}$$

By some simple calculations,

$$\begin{aligned}|\Phi u(t_2)-\Phi u(t_1)| &\leq \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\Big[|u(s-\tau))|+|f(s)|\Big]ds\\ &+ \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\Big[|u(s-\tau))|+|f(s)|\Big]ds\\ &- \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_2-s)^{\alpha-1}\Big[|u(s-\tau))|+|f(s)|\Big]ds\\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\Big[\int_0^s |K(s,x)||F(u(x))|dx\Big]ds\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \left[\int_0^1 |M(s,x)||G(u(x))|dx \right] ds \\
&+ \quad \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} \left[\int_0^1 |M(s,x)||G(u(x))|dx \right] ds \\
&+ \quad \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} \left[\int_0^s |K(s,x)||F(u(x))|dx \right] ds \\
&- \quad \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_2 - s)^{\alpha-1} \left[\int_0^1 |M(s,x)||G(u(x))|dx \right] ds \\
&- \quad \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_2 - s)^{\alpha-1} \left[\int_0^s |K(s,x)||F(u(x))|dx \right] ds \\
&\to \quad 0,
\end{aligned}
$$

as $t_2 \to t_1$, that is, u is the solution of (2.1). □

On the other hand, we prove the uniqueness of the existed solution Eq. (2.1). Our result is based on the Banach contraction theorem [21].

For $u, v \in S$, we have

$$
\begin{aligned}
|\Phi u(t) - \Phi v(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[|u(s-\tau) - v(s-\tau)| \Big] ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[\int_0^s |K(s,x)||F(u(x)) - F(v(x))|dx \\
&+ \int_0^1 |M(s,x)||G(u(x)) - G(v(x))|dx \Big] ds \\
&\leq \frac{\Upsilon}{\Gamma(\alpha)} \|u - v\| \int_0^t s^{\alpha-1} ds \\
&\leq \frac{\Upsilon}{\Gamma(\alpha+1)} \|u - v\| t^{\alpha}.
\end{aligned}
$$

Thus for $\frac{\Upsilon}{\Gamma(\alpha+1)} t^{\alpha} < 1$, we deduce that

$$\|\Phi u(t) - \Phi v(t)\| < \|u(t) - v(t)\|.$$

By the contraction mapping principle, we conclude that Φ has a unique fixed point in S. Consequently, the integro-differential equation (2.1) has a unique solution $u(t)$. In [13], give a definition to check the asymptote stable system.

3. Stability of Caputo fractional integro-differential equations

Definition 3.1. The autonomous system (2.1) with $u(0) = u_0$, is said to be asymptotically stable if and only if

$$\lim_{t \to \infty} \|u(t)\| = 0. \tag{3.1}$$

Theorem 1.1 permits to check stability. First, we investigate the stability as follows:

For $m, n \in \mathbb{N}$

$$\begin{aligned}\|\Phi u_m(t) - \Phi u_n(t)\| = \; & \|\lambda_1 \int_0^t K(t,x)G(u_m(x))\; dx + \lambda_2 \int_0^t M(t,x)H(u_m(x))\; dx \\ & - \lambda_1 \int_0^t K(t,x)G(u_n(x))\; dx - \lambda_2 \int_0^t M(t,x)H(u_n(x))\; dx \\ & + u_m(t-\tau) - u_n(t-\tau)\| \\ & \leq |\lambda_1| \left[\int_0^t \int_0^t K^2(t,s)dsdt\right]^{\frac{1}{2}} \|u_m(t) - u_n(t)\| \\ & + |\lambda_2| \left[\int_0^t \int_0^1 L^2(t,s)dsdt\right]^{\frac{1}{2}} \|u_m(t) - u_n(t)\| \\ & + \|u_m(t-\tau) - u_n(t-\tau)\|,\end{aligned}$$

where

$$\|u_m(t-\tau) - u_n(t-\tau)\| \;\; = \;\; c\|u_m(t) - u_n(t)\|,$$

this leads to

$$\begin{aligned}\|\Phi u_m(t) - \Phi u_n(t)\| \;\; \leq \;\; & \left[|\lambda_1| \left[\int_0^t \int_0^t K^2(t,s)dsdt\right]^{\frac{1}{2}} + c \right. \\ + \;\; & \left. |\lambda_2| \left[\int_0^1 \int_0^1 M^2(t,s)dsdt\right]^{\frac{1}{2}}\right] \|u_m(t) - u_n(t)\|.\end{aligned}$$

Therefore, if $|\lambda_1| < \frac{1}{\zeta_1}$, $|\lambda_2| < \frac{1}{\zeta_2}$, where

$$\zeta_1 = \left[\int_0^t \int_0^t K^2(t,s)dsdt\right]^{\frac{1}{2}}, \; \zeta_2 = \left[\int_0^1 \int_0^1 M^2(t,s)dsdt\right]^{\frac{1}{2}},$$

then, the nonlinear mapping Φ has a fixed point. In particular, for $L = 0$ and

$$\chi = |\lambda_1| \left[\int_0^t \int_0^t K^2(t,s)dsdt\right]^{\frac{1}{2}} + |\lambda_2| \left[\int_0^1 \int_0^1 M^2(t,s)dsdt\right]^{\frac{1}{2}} + c.$$

Theorem 1.1 hold for the nonlinear mapping Φ and hence, it is $\Phi-$ stable. Similarly as in the previous steps, it is easy to indicate that the system is asymptote stable by (3.1).

4. Numerical method for Eq. (2.1)

In this section, we discuss the numerical method with examples.

4.1 Modified Laplace decomposition method [3]

Operating with the Laplace operator L to both sides of Eq. (2.1), yields

$$L[D^{\alpha}u(t)] = L\Bigg[u(t-\tau) + \lambda_1 \int_0^t K(t,x)G(u(x))\,dx + f(t) + \lambda_2 \int_0^1 M(t,x)H(u(x))\,dx\Bigg]. \tag{4.1}$$

By virtue of the linearity property of Laplace transform, we have

$$L[D^{\alpha}u(t)] = L[u(t-\tau)] + \lambda_1 L\Bigg[\int_0^t K(t,x)G(u(x))\,dx\Bigg] + L[f(t)] + \lambda_2 L\Bigg[\int_0^1 M(t,x)H(u(x))\,dx\Bigg], \tag{4.2}$$

hence, from the differentiation rule and formulas of Laplace transform

$$s^{\alpha}U(s) - c = L[u(t-\tau)] + \lambda_1 L\Bigg[\int_0^t K(t,x)G(u(x))\,dx\Bigg] + L[f(t)] + \lambda_2 L\Bigg[\int_0^1 M(t,x)H(u(x))\,dx\Bigg], \tag{4.3}$$

where $c = \sum_{k=1}^{n-1} s^{\alpha-k-1}u^{(k)}(0)$,

on simplifying the previous equation, we note that

$$U(s) = \frac{L[f(t)]+c}{s^{\alpha}} + \frac{\lambda_1}{s^{\alpha}} L\Bigg[\int_0^t K(t,x)G(u(x))\,dx\Bigg] + \frac{\lambda_2}{s^{\alpha}} L\Bigg[\int_0^1 M(t,x)H(u(x))\,dx\Bigg] + \frac{L[u(t-\tau)]}{s^{\alpha}}, \tag{4.4}$$

moreover, the solution $u(t)$ can be rewritten by the infinite series

$$u(t) = \sum_{n=1}^{\infty} u_n. \tag{4.5}$$

Also, the nonlinear functions are decomposed as

$$G = \sum_{n=1}^{\infty} B_n,\ H = \sum_{n=1}^{\infty} D_n, \tag{4.6}$$

where B_n, D_n are the Adomain polynomials of u_0, u_1, ..., u_n. For the nonlinear functions, the Adomain polynomials are given by

$$B_n = \frac{1}{\Gamma(n+1)} \frac{d^n}{d\ell^n}[G(\sum_{i=1}^{\infty} \ell^i u_i)]. \tag{4.7}$$

$$D_n = \frac{1}{\Gamma(n+1)} \frac{d^n}{d\ell^n}[H(\sum_{i=1}^{\infty} \ell^i u_i)], \ n = 0, 1, 2, ... \tag{4.8}$$

The new recursive scheme becomes

$$\begin{aligned} B_0 &= G(u_0), \\ B_1 &= u_1 G'(u_0), \\ B_1 &= u_2 G'(u_0) + \frac{1}{2} u_1^2 G''(u_0), \\ B_2 &= u_3 G'(u_0) + u_1 u_2 G''(u_0) + \frac{1}{3} u_1^3 G'''(u_0), \end{aligned}$$

and so on

$$\begin{aligned} D_0 &= H(u_0), \\ D_1 &= u_1 H'(u_0), \\ D_1 &= u_2 H'(u_0) + \frac{1}{2} u_1^2 H''(u_0), \\ D_2 &= u_3 H'(u_0) + u_1 u_2 H''(u_0) + \frac{1}{3} u_1^3 H'''(u_0), \end{aligned}$$

substituting (4.6), (4.7) into (4.5), we will obtain

$$\begin{aligned} L(\sum_{n=1}^{\infty} u_n) = \ & \frac{L[f(t)] + c}{s^\alpha} + \frac{\lambda_1}{s^\alpha} L\left[\int_0^t K(t,x) \sum_{n=1}^{\infty} B_n \ dx\right] \\ & + \frac{\lambda_2}{s^\alpha} L\left[\int_0^1 M(t,x) \sum_{n=1}^{\infty} D_n \ dx\right] + \frac{L[\sum_{n=1}^{\infty} u_{n-1}(t-\tau)]}{s^\alpha}. \end{aligned} \tag{4.9}$$

Comparing two both sides of (4.9). Then, we have the following iterative algorithm:

$$\begin{aligned} L(u_0) &= \frac{L[f(t)] + c}{s^\alpha}, \ u_0 = T(t), \\ L(u_1) &= \frac{\lambda_1}{s^\alpha} L[\int_0^t K(t,x) B_0 \ dx] + \frac{\lambda_2}{s^\alpha} L[\int_0^1 M(t,x) D_0 \ dx] + \frac{L[u_0(t-\tau)]}{s^\alpha}, \end{aligned}$$

therefore, the general term is given by

$$L(u_n) = \frac{\lambda_1}{s^\alpha} L[\int_0^t K(t,x) B_{n-1} \ dx] + \frac{\lambda_2}{s^\alpha} L[\int_0^1 M(t,x) D_{n-1} \ dx] + \frac{L[u_{n-1}(t-\tau)]}{s^\alpha}.$$

Applying the inverse Laplace transform, the modified Laplace decomposition method suggests that the function $T(t)$ is decomposed into two parts $T(t) = T_1(t) + T_2(t)$, $u_0(t) = T_1(t)$, $n \geq 1$,

$$\begin{aligned} u_1(t) = \ & T_2(t) + L^{-1}[\frac{\lambda_1}{s^\alpha} L(\int_0^t K(t,x) B_0 \ dx)] + L^{-1}[\frac{\lambda_2}{s^\alpha} L(\int_0^1 M(t,x) D_0 \ dx)] \\ & + L^{-1}[\frac{L(u_1(t-\tau))}{s^\alpha}]. \end{aligned}$$

In general,

$$u_n(t) = L^{-1}[\frac{\lambda_1}{s^\alpha} L(\int_0^t K(t,x) B_{n-1}\ dx)] + L^{-1}[\frac{\lambda_2}{s^\alpha} L(\int_0^1 M(t,x) D_{n-1}\ dx)] + L^{-1}[\frac{L[u_{n-1}(t-\tau)]}{s^\alpha}].$$

4.2 Convergence analysis for MLADM

Due to [11], we suppose the following two hypotheses for convergence of MLADM:

i. $(T(u(t)) - T(v(t)), u(t) - v(t)) \geq \beta \|u(t) - v(t)\|^2$; $\beta > 0$, $\forall u,\ v \in C[0,1]$,

ii. for $\varphi > 0$, $u,\ v \in C[0,1]$, there exists a constant $C(\varphi) > 0$ such that $\|u\| \leq \varphi$, $\|v\| \leq \varphi$ we get:

$$(T(u) - T(v), w) \leq C(\varphi) \|u - v\| \|w\| \tag{4.10}$$

for every $w \in C[0,1]$.

Theorem 4.1. *The MLADM applied to the fractional Volterra-Fredholm integro-differential equation without initial condition converges to a particular solution.*

Proof. In the first hand, for the fractional differential operator $T(u)$, $\exists k > 0$, $\forall u, v \in C[0,1]$, we have

$$\begin{aligned} T(u) - T(v) = \ & u(t-\tau) - v(t-\tau) + \lambda_1 \int_0^t K(t,x)[G(u(x)) - G(v(x))]\ dx \\ & + \lambda_2 \int_0^1 M(t,x)[H(u(x)) - H(v(x))]\ dx. \end{aligned}$$

Then, we get

$$\begin{aligned} (T(u) - T(v), u - v) = \ & \Big(u(t-\tau) - v(t-\tau) \\ & + \lambda_1 \int_0^t K(t,x)[G(u(x)) - G(v(x))]\ dx \\ & + \lambda_2 \int_0^1 M(t,x)[H(u(x)) - H(v(x))]\ dx, u - v \Big). \end{aligned}$$

Hence, by using the Schwartz inequality and Eq.1.7 we obtain

$$\begin{aligned} & \Big(u(t-\tau) - v(t-\tau) + \lambda_1 \int_0^t K(t,x)[G(u(x)) - G(v(x))]\ dx \\ & \qquad + \lambda_2 \int_0^1 M(t,x)[H(u(x)) - H(v(x))]\ dx, u - v \Big) \\ & \qquad \leq \Big(\wp \|u - v\| + |\lambda_1| \sup_{0 \leq t \leq 1} K(t,x) \xi \|u - v\| \Big) \Big(\|u - v\| \Big) \\ & \qquad + |\lambda_2| \sup_{0 \leq t \leq 1} M(t,x) \mu \|u - v\| \Big(\|u - v\| \Big). \end{aligned} \tag{4.11}$$

It follows that,

$$(T(u) - T(v), u - v) \geq k\|u - v\|^2.$$

On the other hand,

$$\begin{aligned}(T(u) - T(v), w) = \Big(& u(t-\tau) - v(t-\tau) \\ & + \lambda_1 \int_0^t K(t,x)[G(u(x)) - G(v(x))]\, dx \\ & + \lambda_2 \int_0^1 M(t,x)[H(u(x)) - H(v(x))]\, dx, w\Big).\end{aligned}$$

This implies that

$$(T(u) - T(v), w) \leq C(\varphi)\|u - v\|\|w\|.$$

This completes the proof. □

Now, we will study the convergence theorem of solutions based on the MLADM.

Theorem 4.2. *If the series solution is convergent, then it converges to the exact solution of the fractional Volterra-Fredholm integro-differential equation with constant time delay.*

Proof. For $n \geq m$, assume that S_n, S_m are arbitrary partial sum. We want to prove that S_n is a cauchy sequence, i.e, S_n becomes a convergent series:

$$\|S_n - S_m\|_{L^\infty[0,1]} = \max_{t\in[0,1]} |S_n - S_m| = \max_{t\in[0,1]} |\sum_{k=m+1}^{n} u_k(t)|,$$

where $S_j = \sum_{k=0}^{j} u_k(t)$, $j = 1, 2$.

$$\begin{aligned}\|S_n - S_m\|_{L^\infty[0,1]} = \max_{t\in[0,1]} | & \sum_{k=m+1}^{j} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[u_k(s-\tau)\Big] ds \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[\int_0^s K(s,x) B_k dx + \int_0^1 M(s,x) D_k dx\Big] ds|.\end{aligned}$$

From (4.7) and (4.8), we have

$$\begin{aligned}\sum_{k=m}^{j-1} B_k &= G(S_{n-1}) - G(S_{m-1}), \\ \sum_{k=m}^{j-1} D_k &= H(S_{n-1}) - H(S_{m-1}), \\ \sum_{k=m}^{j-1} u_k &= u(S_{n-1}) - u(S_{m-1}).\end{aligned}$$

Then,

$$\|S_n - S_m\|_{L^\infty[0,1]} = \max_{t\in[0,1]} |\sum_{k=m+1}^{j} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\Big[u_k(s-\tau)\Big]ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\Big[\int_0^s K(s,x)(G(S_{n-1}) - G(S_{m-1}))dx\Big]ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\Big[\int_0^1 M(s,x)(H(S_{n-1}) - H(S_{m-1}))dx\Big]ds|.$$

Since

$$\begin{aligned}\|S_n - S_m\|_{L^\infty[0,1]} &\le \|S_{m+1} - S_m\|_{L^\infty[0,1]}\| + \|\|S_{m+2} - S_{m+1}\|_{L^\infty[0,1]} + ... \\ &\le [\gamma^m + \gamma^{m+1} + ... + \gamma^{n-1}]\|S_1 - S_0\|_{L^\infty[0,1]} \\ &\le \gamma^m[1 + \gamma + \gamma^2 + ... + \gamma^{n-m-1}]\|S_1 - S_0\|_{L^\infty[0,1]} \\ &\le \gamma^m(\frac{1-\gamma^{n-m}}{1-\gamma})\|u_1\|_{L^\infty[0,1]}.\end{aligned}$$

Since $0 < \gamma < 1$, $(1 - \gamma^{n-m}) < 1$, but $|u_1(t)| < \infty$, so $\|S_n - S_m\| \to 0$, as m, $n \to \infty$. This implies that S_n is a cauchy sequence in $C[0,1]$, i.e., the series is also convergent. □

Now, we give examples to illustrate the obtained results.

4.3 Test examples

Consider the following fractional integro-differential equations with time delay.

Example 1.

$$D^\alpha u(t) = u(t-\tau) + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + 3\int_0^1 u^2(t)dt - 4\int_0^t \frac{u^3(s)}{t^3}ds, \tag{4.12}$$

subject to the initial conditions $u(0) = 0$, with the exact solution $u(t) = t$ at $\tau = 1$ and $\alpha = 0.75$.

Solution. First, we apply Laplace transform on both sides of (4.12)

$$\begin{aligned}L[D^\alpha u(t)] = L[u(t-\tau)] + L[\frac{t^{0.25}}{\Gamma(1.25)}] + 3L[\int_0^1 u^2(t)dt] \\ - 4L[\int_0^t \frac{u^3(s)}{t^3}ds],\end{aligned} \tag{4.13}$$

using the properties of Laplace transform,

$$\begin{aligned}s^\alpha L[u(t)] - c = L[u(t-\tau)] + \frac{1}{\Gamma(1.25)}L[t^{0.25}] + 3L[\int_0^1 u^2(t)dt] \\ - 4L[\int_0^t \frac{u^3(s)}{t^3}ds].\end{aligned} \tag{4.14}$$

Thus, (4.14) is equivalent to

$$L[u(t)] = \frac{1}{s^{0.75}}L[u(t-\tau)] + \frac{1}{\Gamma(1.25)s^{0.75}}\frac{\Gamma(1.25)}{s^{1.25}} + \frac{3}{s^{0.75}}L[\int_0^1 u^2(t)dt]$$
$$- \frac{4}{s^{0.75}}L[\int_0^t \frac{u^3(s)}{t^3}ds], \tag{4.15}$$

by means of the Laplace decomposition method

$$L[\sum_{n=0}^{\infty} u_n(t)] = \frac{e^{-s}}{s^{0.75}}L[\sum_{n=0}^{\infty} u_n(t)] + \frac{\Gamma(2)}{s^2} + \frac{3}{s^{0.75}}L[\int_0^1 \sum_{n=0}^{\infty} A_n(t)dt]$$
$$- \frac{4}{s^{0.75}}L[\int_0^t \frac{1}{t^3}(\sum_{n=0}^{\infty} B_n(s))ds]. \tag{4.16}$$

This leads to $L(u_0) = \frac{\Gamma(2)}{s^2}$

$$\begin{aligned} L[u_1(t)] &= \frac{e^{-s}}{s^{0.75}}L[t] + \frac{3}{s^{0.75}}L[\int_0^1 t^2dt] - \frac{4}{s^{0.75}}L[\int_0^t \frac{1}{t^3}s^3ds] \\ &= \frac{e^{-s}}{s^{0.75}}L[t] + \frac{1}{s^{0.75}}L[1] - \frac{1}{s^{0.75}}L[t] \\ &= \frac{1}{s^{0.75}}L[t-1] + \frac{1}{s^{0.75}}L(1-t) \\ &= 0 \end{aligned} \tag{4.17}$$

$$L[u_2(t)] = \frac{e^{-s}}{s^{0.75}}L[u_1(t)] + \frac{3}{s^{0.75}}L[\int_0^1 A_1(t)dt] - \frac{4}{s^{0.75}}L[\int_0^t \frac{1}{t^3}(B_1(s))ds]$$
$$\cdots$$
$$\cdots$$
$$\cdots$$

Taking the inverse Laplace transform in the preceding iterative algorithm,

$$\begin{aligned} u_0 &= L^{-1}[\frac{\Gamma(2)}{s^2}] = t, \\ u_1(t) &= L^{-1}[\frac{e^{-s}}{s^{0.75}}L[u_0(t)]] + L^{-1}[\frac{3}{s^{0.75}}L[\int_0^1 A_0(t)dt]] \\ &\quad - L^{-1}[\frac{4}{s^{0.75}}L[\int_0^t (B_0(s))ds]] = 0, \\ &\vdots \end{aligned}$$

Therefore, the solution $u(t) = \sum_{n=0}^{\infty} u_n = t$.

Remark 4.3 For arbitrary constants a, b, c, $d \in \mathbb{R}$, if the non-linear functions $F(u(t))$ and $G(u(t))$ are replaced by $au(t)+b$, $cu(t)+d$ respectively, then Eq. (2.1) becomes a linear Caputo fractional Volterra-Fredholm integro-differential equation.

Example 2.

$$D^{\alpha}u(t) = u(t-\tau) + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + 2\int_0^1 u(t)dt - 2\int_0^t \frac{u(s)-1}{t}ds \tag{4.18}$$

with the initial condition $u(0)=0$, with the exact solution $u(t)=t$ at $\tau=3$ and $\alpha=0.5$.

Solution. Utilizing transform of Laplace to (4.18)

$$\begin{aligned} L[D^{\alpha}u(t)] &= L[u(t-\tau)] + L[\frac{t^{0.5}}{\Gamma(1.5)}] + 2L[\int_0^1 u(t)dt] \\ &\quad - 2L[\int_0^t \frac{u(s)-1}{t}ds], \end{aligned} \tag{4.19}$$

hence, by (1.8), (1.10) and the initial conditions

$$\begin{aligned} s^{0.5}L[u(t)] &= e^{-3s}L[u(t)] + \frac{1}{\Gamma(1.5)}L[t^{0.5}] + 2L[\int_0^1 u(t)dt] \\ &\quad - 2L[\int_0^t \frac{u(s)-1}{t}ds]. \end{aligned} \tag{4.20}$$

In order to simplify the above equation, we have

$$\begin{aligned} L[u(t)] &= \frac{e^{-3s}}{s^{0.5}}L[u(t)] + \frac{1!}{s^2} + \frac{2}{s^{0.5}}L[\int_0^1 u(t)dt] \\ &\quad - \frac{2}{s^{0.5}}L[\int_0^t \frac{u(s)-1}{t}ds]. \end{aligned} \tag{4.21}$$

According to MADM

$$\begin{aligned} L[\sum_{i=0}^{\infty} u_i(t)] &= \frac{e^{-3s}}{s^{0.5}}L[\sum_{n=0}^{\infty} u_i(t)] + \frac{1!}{s^2} + \frac{2}{s^{0.5}}L[\int_0^1 \sum_{i=0}^{\infty} u(t)dt] \\ &\quad - \frac{2}{s^{0.5}}L[\int_0^t (\sum_{i=0}^{\infty} \frac{u(s)-1}{t}ds]. \end{aligned} \tag{4.22}$$

That is, $L(u_0) = \frac{1!}{s^2} \Rightarrow u_0 = t$. Consequently,

$$\begin{aligned} u_1(t) &= L^{-1}\left[\frac{e^{-3s}}{s^{0.5}}L[t] + \frac{1}{s^{1.5}} + \frac{2}{s^{1.5}} + \frac{1!}{s^{1.5}}\right] \\ &= L^{-1}\left[\frac{1}{s^{0.5}}L[t-3] + \frac{1}{s^{0.5}}L(3-t)\right] \\ &= 0 \end{aligned} \tag{4.23}$$

Similarly, for $i > 1$, $u_i = 0$. So, the approximate solution is given by

$$u(t) = \sum_{i=0}^{\infty} u_i = t,$$

which is the exact solution.

5. Conclusion

It is obvious that this work represents essential development in modeling the linear and non-linear fractional integro-differential equations. It provides us with an accurate analysis of the solution to the mentioned problem. Throughout it, we can get different results easily. Such previous results will arise, if any terms of Eq. (2.1) are vanished as in [11, 23, 24] and etc.

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