Conformal η-Ricci solitons in Lorentzian para-Sasakian manifold admitting semi-symmetric metric connection

G. Somashekhara
Department of Mathematics and Statistics
M. S. Ramaiah University of Applied Sciences
Bangalore-560 054, Karnataka
India
somashekhara.mt.mp@msruas.ac.in

S. Girish Babu
Department of Mathematics
Acharya institute of Technology
Bangalore-560 107, Karnataka
India
sgirishbabu84@gmail.com

P. Siva Kota Reddy*
Department of Mathematics
Sri Jayachamarajendra College of Engineering
JSS Science and Technology University
Mysuru-570 006, Karnataka
India
pskreddy@jssstuniv.in

Abstract. The object of the present research is to study conformal η-Ricci solitons on a semi-symmetric metric connection in a LP-Sasakian manifold of some curvature tensor like conharmonic curvature, W_2 curvature and projective curvature.

Keywords: conformal η-Ricci solitons, LP-Sasakian manifold, semi-symmetry, curvature tensors.

1. Introduction

A Ricci soliton is a natural generalization of an Einstein metric introduced by Hamilton [5] in 1982. A pseudo-Riemannian manifold (M, g) is called a Ricci soliton if it admits a smooth vector field V on (M, g) such that

$$\frac{1}{2}(L_V g(X_1, Y_1)) + S(X_1, Y_1) + \lambda g(X_1, Y_1) = 0, \tag{1}$$

where L_V denotes the lie derivative in the direction of V, S is the Ricci tensor field, λ is a constant and X_1, Y_1 are arbitrary vector fields on M.

The study of η-Ricci soliton was initiated by Cho and Kimura [3]. For a given 1-form η a η-Ricci soliton is a data (g, V, λ, μ) on pseudo-Riemannian manifold

*. Corresponding author

(M, g) satisfying

$$\frac{1}{2}(L_V g(X_1, Y_1)) + S(X_1, Y_1) + \lambda g(X_1, Y_1) + \mu\eta \otimes \eta = 0, \tag{2}$$

where L_V denotes the lie derivative in the direction of V, S is the Ricci tensor field, λ is a constant and X_1, Y_1 are arbitrary vector fields on M.

In [2] the authors studied the η-Ricci solitons on Hopf hypersurfaces in complex space forms. Later in the year 2015, Basu and Bhattacharya [1] introduced the notion of conformal Ricci soliton as a soliton satisfying

$$L_V g(X_1, Y_1) + 2S(X_1, Y_1) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X_1, Y_1) = 0. \tag{3}$$

In [4], the authors studied the properties of conformal Ricci soliton in Lorentzian α-Sasakian manifolds. From (2) and (3), the conformal η-Ricci soliton given by

$$L_V g(X_1, Y_1) + 2S(X_1, Y_1) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X_1, Y_1) + 2\mu\eta \otimes \eta = 0. \tag{4}$$

Particularly if $\mu = 0$ the the data (g, V, λ) is a conformal Ricci soliton. Clearly, the conformal η-Ricci soliton is a generalization of conformal Ricci soliton.

Somashekhara et al. [6], studied some results on invariant sub-manifolds of LP-Sasakian manifolds endowed with semi-symmetric metric conection. Also, they have obtained a condition for totally geodesic by using certain geometrical conditions. In [7], we studied the C-Bochner curvature tensor under D-homothetic deformation in LP-Sasakian manifolds.

In this paper, we study the conformal η-Ricci solitons in LP-Sasakian manifold with semi-symmetric metric connection.

2. Preliminaries

A smooth manifold (M^n, g) is called Lorentzian para-Sasakian manifold, if it admits $(1, 1)$ tensor field ϕ, contravariant vector field ξ, 1-form η and Lorentzian g such that

$$\eta(\xi) = -1, \tag{5}$$

$$\phi^2(X_1) = X_1 + \eta(X_1)\xi, \tag{6}$$

$$g(\phi X_1, \phi Y_1) = g(X_1, Y_1) + \eta(X_1)\eta(Y_1), \tag{7}$$

$$g(X_1, \phi Y_1) = g(\phi X_1, Y_1), \tag{8}$$

$$g(X_1, \xi) = \eta(X_1), \tag{9}$$

$$\nabla_{X_1}\xi = \phi X_1, \tag{10}$$

$$(\nabla_{X_1}\phi)(Y_1) = g(X_1, Y_1)\xi + \eta(Y_1)X_1 + 2\eta(X_1)\eta(Y_1)\xi, \tag{11}$$

where ∇ denotes the covariant differentiation with respect to the Lorentzian metric g.

In an LP-Sasakian manifold the following relations holds:

$$\phi\xi = 0,\ \eta(\phi X_1) = 0,\ \text{Rank}(\phi) = n-1. \tag{12}$$

If we put

$$\Phi(X_1, Y_1) = g(X_1, \phi Y_1), \tag{13}$$

for any vector fields X_1 and Y_1, then the tensor field $\Phi(X_1, Y_1)$ is a symmetric $(0,2)$ tensor field. Also since the 1-form η is closed in a LP-Sasakian manifold, then we have:

$$(\nabla_{X_1}\eta)(Y_1) = \Phi(X_1, Y_1), \Phi(X_1, \xi) = 0, \tag{14}$$

$\forall\ X_1, Y_1\ \in \text{TM}$.

In a LP-Sasakian manifold, the following relations holds:

$$g(R(X_1, Y_1)Z_1, \xi) = \eta(R(X_1, Y_1)Z_1) = g(Y_1, Z_1)\eta(X_1) - g(X_1, Z_1)\eta(Y_1), \tag{15}$$

$$R(\xi, X_1)Y_1 = g(X_1, Y_1)Z_1 - \eta(Y_1)X_1, \tag{16}$$

$$R(X_1, Y_1)\xi = \eta(Y_1)X_1 - \eta(X_1)Y_1, \tag{17}$$

$$R(\xi, X_1)\xi = X_1 + \eta(X_1)\xi, \tag{18}$$

$$S(X_1, \xi) = (n-1)\eta(X_1), \tag{19}$$

$$S(\phi X_1, \phi Y_1) = S(X_1, Y_1) + (n-1)\eta(X_1)\eta(Y_1). \tag{20}$$

for any vector fields X_1, Y, Z_1 where R and S are the Riemannian curvature and Ricci tensors of M, respectively.

A semi-symmetric metric connection $\tilde{\nabla}$ in a LP-Sasakian manifold can be defined by

$$\tilde{\nabla_{X_1}} Y_1 = \nabla_{X_1} Y_1 + \eta(Y_1)X_1 - g(X_1, Y_1)\xi, \tag{21}$$

where ∇ is the Levi-Civita connection on M.

Let M^n, g be a LP-Sasakian manifold. The curvature tensor $\tilde{R}$ of M with respect t0 the semi-symmetric metric connection, the following relations holds:

$$\begin{aligned}\tilde{R}(X_1, Y_1)Z_1 = R(X_1, Y_1)Z_1 &- \left[g(\phi Y_1, Z_1) - \eta(Y_1)\eta(Z_1) - \frac{1}{2}g(Y_1, Z_1)\right] \\ &X_1 + \left[g(\phi X_1, Z_1) - \eta(X_1)\eta(Z_1) - \frac{1}{2}g(X_1, Z_1)\right] Y_1 \\ &- g(Y_1, Z_1)\left[\phi X_1 - \eta(X_1)\xi - \frac{1}{2}X_1\right] \\ &+ g(X_1, Z_1)\left[\phi Y_1 - \eta, (Y_1)\xi - \frac{1}{2}Y_1\right],\end{aligned} \tag{22}$$

$$\tilde{S}(X_1,Y_1)=S(X_1,Y_1)-(n-2)\left[g(\phi X_1,Y_1)-\eta(X_1)\eta(Y_1)-\frac{1}{2}g(X_1,Y_1)\right] - \text{trace}(\alpha)g(X_1,Y_1), \tag{23}$$

$$\tilde{r} = r - 2(n-1)\text{trace}(\alpha), \tag{24}$$

$$\tilde{S}(\phi X_1,\phi Y_1) = S(X_1,Y_1) + \left[\frac{1}{2}\text{trace}(\alpha)\right]\eta(X_1)\eta(Y_1) - (n-2) \left[g(\phi X_1,Y_1) - \eta(X_1)\eta(Y_1) - \frac{1}{2}g(X_1,Y_1)\right] - \text{trace}(\alpha)g(X_1,Y_1). \tag{25}$$

We can use the notion of conformal η-Ricci soliton on LP-Sasakian manifold and we get the following:

$$L_\xi g(X_1,Y_1) + 2S(X_1,Y_1) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X_1,Y_1) + 2\mu\eta\otimes\eta = 0, \tag{26}$$

using (10) in (26), we get

$$S(X_1,Y_1) = \left(\frac{p}{2} + \frac{1}{n} - \lambda\right) g(X_1,Y_1) - \mu\eta(X_1)\eta(Y_1) - g(\phi X_1,Y_1), \tag{27}$$

$$QX_1 = \left(\frac{p}{2} + \frac{1}{n} - \lambda\right) X_1 - \mu\eta(X_1)\xi - \phi X_1, \tag{28}$$

$$S(X_1,\xi) = \left(\frac{p}{2} + \frac{1}{n} - \lambda + \mu\right)\eta(X_1), \tag{29}$$

$$Q\xi = \left(\frac{p}{2} + \frac{1}{n} - \lambda + \mu\right)\xi. \tag{30}$$

Also the idea of conformal η-Ricci soliton on LP-Sasakian manifold with semi-symmetric metric connection, we have:

$$\tilde{S}(X_1,Y_1) = A_1 g(X_1,Y_1) + A_2\eta(X_1)\eta(Y_1) - (n-1)g(\phi X_1,Y_1), \tag{31}$$

$$\tilde{Q}X_1 = A_1 X_1 + A_2\eta(X_1)\xi - (n-1)\phi X_1, \tag{32}$$

$$\tilde{S}(X_1,\xi) = (A_1 - A_2)\eta(X_1), \tag{33}$$

$$\tilde{Q}\xi = (A_1 - A_2)\xi, \tag{34}$$

where

$$A_1 = \left(\frac{p}{2} + \frac{1}{n} - \lambda + \frac{n-2}{2} - \text{trace}(\alpha)\right), A_2 = (n-2-\mu).$$

3. Conformal η-Ricci soliton in LP-Sasakian manifolds satisfying $\tilde{H}(\xi, Y_1).\tilde{S} = 0$ with respect to semi-symmetric metric connection

Let M be an n-dimensional conformal η-Ricci in LP-Sasakian manifold admitting semi-symmetric metric connection satisfying $\tilde{H}(\xi, Y_1).\tilde{S} = 0$, where $\tilde{H}$ and $\tilde{S}$ are conharmonic curvature tensor and Ricci tensor respectively with semi-symmetric metric connection. The conharmonic curvature tensor $\tilde{H}$ is defined by

$$\tilde{H}(X_1, Y_1)Z_1 = \tilde{R}(X_1, Y_1)Z_1 - \frac{1}{n-2}\left(\tilde{S}(Y_1, Z_1)X_1 - \tilde{S}(X_1, Z_1)Y_1 + g(Y_1, Z_1)\tilde{Q}X_1 - g(X_1, Z_1)\tilde{Q}Y_1\right), \tag{35}$$

we assume that the condition $\tilde{H}(\xi, Y_1).\tilde{S} = 0$. It implies that

$$\tilde{S}(\tilde{H}(\xi, X_1)Y_1, Z_1) + \tilde{S}(Y_1, \tilde{H}(\xi, X_1)Z_1) = 0, \tag{36}$$

substituting $X_1 = \xi$ in (35), we get

$$\tilde{H}(\xi, Y_1)Z_1 = \tilde{R}(\xi, Y_1)Z_1 - \frac{1}{n-2}\left(\tilde{S}(Y_1, Z_1)\xi - \tilde{S}(\xi, Z_1)Y_1 + g(Y_1, Z_1)\tilde{Q}\xi - g(\xi, Z_1)\tilde{Q}Y_1\right), \tag{37}$$

using equation (33) and (37), equation (36) reduces to

$$\begin{aligned}
&(A_1 - A_2)g(X_1, Y_1)\eta(Z_1) - \eta(Y_1)\tilde{S}(X_1, Z_1) \\
&- (A_1 - A_2)g(\phi X_1, Y_1)\eta(Z_1) + \eta(Y_1) \\
&\tilde{S}(\phi X_1, Z_1) - \frac{1}{n-2}(A_1 - A_2)\eta(Z_1)\tilde{S}(X_1, Y_1) \\
&+ \frac{1}{n-2}(A_1 - A_2)\eta(Y_1)\tilde{S}(X_1, Z_1) - \\
&\frac{1}{n-2}(A_1 - A_2)^2\eta(Z_1)g(X_1, Y_1) + \frac{1}{n-2}A_1\eta(Y_1)\tilde{S}(X_1, Z_1) \\
&+ \frac{1}{n-2}A_2(A_1 - A_2) \\
&\eta(X_1)\eta(Y_1)\eta(Z_1) - \frac{n-1}{n-2}\tilde{S}(\phi X_1, Z_1)\eta(Y_1) \\
&+ (A_1 - A_2)g(X_1, Z_1)\eta(Y_1) - \eta(Z_1)\tilde{S}(X_1, Y_1) \\
&- (A_1 - A_2)g(\phi X_1, Z_1)\eta(Y_1) + \eta(Z_1)\tilde{S}(\phi X_1, Y_1) \\
&- \frac{1}{n-2}(A_1 - A_2)\eta(Y_1)\tilde{S}(X_1, Z_1) + \\
&\frac{1}{n-2}(A_1 - A_2)\eta(Z_1)\tilde{S}(X_1, Y_1) - \frac{1}{n-2}(A_1 - A_2)^2\eta(Y_1)g(X_1, Z_1)
\end{aligned} \tag{38}$$

$$+\frac{1}{n-2}A_1\eta(Z_1)$$
$$\tilde{S}(X_1,Y_1)+\frac{1}{n-2}A_2(A_1-A_2)\eta(X_1)\eta(Y_1)\eta(Z_1)$$
$$-\frac{n-1}{n-2}\tilde{S}(\phi X_1,Y_1)\eta(Z_1)=0.$$

Substituting $Z_1=\xi$ and using (5) and (33), we get

$$A_3g(X_1,Y_1)+A_4g(\phi X_1,Y_1)+A_5\eta(X_1)\eta(Y_1)+A_6\tilde{S}(X_1,Y_1)$$
$$+A_7\tilde{S}(\phi X_1,Y_1)=0, \tag{39}$$

where

$$A_3=\left[\frac{(A_1-A_2)^2}{n-2}-(A_1-A_2)\right],$$
$$A_4=(A_1-A_2),$$
$$A_5=\left[\frac{A_1(A_1-A_2)}{n-2}-\frac{(A_1-A_2)^2}{n-2}\right],$$
$$A_6=\left[\frac{A_1}{n-2}+1\right],$$
$$A_7=\left[\frac{1}{n-2}\right].$$

In view of (31), we get

$$A_8g(X_1,Y_1)+A_9\eta(X_1)\eta(Y_1)+A_{10}g(\phi X_1,Y_1)$$
$$+A_6S(X_1,Y_1)+A_7S(\phi X_1,Y_1)=0, \tag{40}$$

where

$$A_8=\left(A_3+\frac{A_6(n-2)}{2}-A_6\text{trace}(\alpha)-A_7(n-2)\right),$$
$$A_9=(A_5+A_6(n-2)-A_7(n-2)),$$
$$A_{10}=\left(A_4-A_6(n-2)+\frac{A_7(n-2)}{2}-A_7\right).$$

Using (27) in (40), we get

$$g(\phi X_1,Y_1)=A_{11}g(X_1,Y_1)+A_{12}\eta(X_1)\eta(Y_1). \tag{41}$$

where

$$A_{11}=-\left[\frac{A_8-A_7+A_6\left(\frac{p}{2}+\frac{1}{n}-\lambda\right)}{A_{10}-A_6+A_6\left(\frac{p}{2}+\frac{1}{n}-\lambda\right)}\right],$$
$$A_{12}=-\left[\frac{A_9-A_7-\mu A_6}{A_{10}-A_6+A_6\left(\frac{p}{2}+\frac{1}{n}-\lambda\right)}\right].$$

Now, again using (27) and (41) in equation (40), we get

$$S(X_1, Y_1) = G_1 g(X_1, Y_1) + G_2 \eta(X_1)\eta(Y_1), \tag{42}$$

where

$$G_1 = -\left[\frac{A_8 - A_7 + A_{11}\left(A_{10} + A_7\left(\frac{p}{2} + \frac{1}{n} - \lambda\right)\right)}{A_6}\right],$$

$$G_2 = -\left[\frac{A_9 - A_7 + A_{12}\left(A_{10} + A_7\left(\frac{p}{2} + \frac{1}{n} - \lambda\right)\right)}{A_6}\right].$$

Hence, we can state the following theorem

Theorem 3.1. *An n-dimensional conformal η-Ricci soliton in LP-Sasakian manifold with semi-symmetric metric connection satisfying $\tilde{H}(\xi, Y_1).\tilde{S} = 0$ is an η-Einstein manifold.*

4. Conformal η-Ricci soliton in LP-Sasakian manifolds satisfying $\tilde{W}_2(\xi, Y_1).\tilde{S} = 0$ with respect to semi-symmetric metric connection

Let M be an n-dimensional conformal η-Ricci soliton in LP-Sasakian manifold admitting semi-symmetric metric connection satisfying $\tilde{W}_2(\xi, Y_1).\tilde{S} = 0$, where $\tilde{W}_2$ and $\tilde{S}$ are W_2 curvature tensor and Ricci tensor respectively with semi-symmetric metric connection. Then the W_2-curvature tensor is defined by

$$\tilde{W}_2(X_1, Y_1)Z_1 = \tilde{R}(X_1, Y_1)Z_1 - \frac{1}{n-1}\left(g(X_1, Z_1)\tilde{Q}Y_1 - g(Y_1, Z_1)\tilde{Q}X_1\right). \tag{43}$$

We assume that the condition $\tilde{W}_2(\xi, Y_1).\tilde{S} = 0$. It implies that

$$\tilde{S}(\tilde{W}_2(\xi, X_1)Y_1, Z_1) + \tilde{S}(Y_1, \tilde{W}_2(\xi, X_1)Z_1) = 0. \tag{44}$$

Substituting $X_1 = \xi$ in (43), we get

$$\tilde{W}_2(\xi, Y_1)Z_1 = \tilde{R}(\xi, Y_1)Z_1 - \frac{1}{n-1}\left(g(\xi, Z_1)\tilde{Q}Y_1 - g(Y_1, Z_1)\tilde{Q}\xi\right). \tag{45}$$

Using (45) in (44), we get

$$\begin{aligned}
&- \eta(Y_1)\tilde{S}(X_1, Z_1) - g(\phi X_1, Y_1)\tilde{S}(\xi, Z_1) \\
&+ \frac{A_1}{n-1}\eta(Y_1)\tilde{S}(X_1, Z_1) + \frac{A_2}{n-1}\eta(X_1)\eta(Y_1) \\
&\tilde{S}(\xi, Z_1) - \frac{(A_1 - A_2)}{n-1}g(X_1, Y_1)\tilde{S}(\xi, Z_1) \\
&- \eta(Z_1)\tilde{S}(X_1, Y_1) - g(\phi X_1, Z_1)\tilde{S}(Y_1, \xi) + \frac{A_1}{n-1} \\
&\eta(Z_1)\tilde{S}(X_1, Y_1) + \frac{A_2}{n-1}\eta(X_1)\eta(Z_1)\tilde{S}(Y_1, \xi) \\
&- \frac{(A_1 - A_2)}{n-1}g(X_1, Z_1)\tilde{S}(\xi, Y_1) = 0.
\end{aligned} \tag{46}$$

Substituting $Z_1 = \xi$ and using (33), we get

$$A_{13}\eta(X_1)\eta(Y_1) + A_{14}g(X_1,Y_1) + A_{15}g(\phi X_1,Y_1) + A_{16}\tilde{S}(X_1,Y_1) = 0, \tag{47}$$

where

$$\begin{aligned}
A_{13} &= \left[(A_2 - A_1) + \frac{A_1(A_1-A_2)}{n-1} - 2\frac{A_2(A_1-A_2)}{n-1} - \frac{(A_1-A_2)^2}{n-1}\right],\\
A_{14} &= \frac{(A_1-A_2)^2}{n-1},\\
A_{15} &= (A_1 - A_2),\\
A_{16} &= \left(1 - \frac{A_1}{n-1}\right).
\end{aligned}$$

Using (31) , we get

$$\begin{aligned}
&(A_{13} - A_{16}(n-2))\,\eta(X_1)\eta(Y_1)\\
&+ g(X_1,Y_1)\left(A_{14} + A_{16}(\frac{n-2}{2} - \text{trace}(\alpha))\right) g(\phi X_1,Y_1)\\
&(A_{15} - A_{16}(n-2)) + A_{16}S(X_1,Y_1) = 0.
\end{aligned} \tag{48}$$

On rearranging and simplification, we have

$$S(X_1,Y_1) = G_3 g(X_1,Y_1) + G_4\eta(X_1)\eta(Y_1), \tag{49}$$

where

$$\begin{aligned}
G_3 &= -\frac{1}{A_{16}}\left[A_{14} - A_{16}\left(\frac{n-2}{2} + \text{trace}(\alpha)\right)\right] + \frac{1}{A_{16}}\left[\frac{A_{15} - A_{16}(n-2)}{A_{15} - A_{16}(n-1)}\right]\\
&\quad \left[A_{14} + A_{16}\left(\frac{n-2}{2} - \text{trace}(\alpha) + \left(\frac{p}{2} + \frac{1}{n} - \lambda\right)\right)\right],\\
G_4 &= -\frac{1}{A_{16}}(A_{13} - (n-2)A_{16})\\
&\quad + \frac{1}{A_{16}}\left[\frac{(A_{15} - A_{16}(n-2))}{A_{15} - A_{16}(n-1)}\right][A_{13} - A_{16}(n-2+\mu)].
\end{aligned}$$

Hence, we can state the following theorem

Theorem 4.1. *An n-dimensional conformal η-Ricci soliton in LP-Sasakian manifold with semi-symmetric metric connection satisfying $\tilde{W}_2(\xi,Y_1).\tilde{S} = 0$ is an η-Einstein manifold.*

5. Conformal η-Ricci soliton in LP-Sasakian manifolds satisfying $\tilde{P}(\xi,Y_1).\tilde{S} = 0$ with respect to semi-symmetric metric connection

Let M be an n-dimensional conformal η-Ricci soliton in LP-Sasakian manifold admitting semi-symmetric metric connection satisfying $\tilde{P}(\xi,Y_1).\tilde{S} = 0$, where $\tilde{P}$

and $\tilde{S}$ are projective curvature tensor and Ricci tensor respectively with semi-symmetric metric connection. Then the Projective curvature tensor is defined by

$$\tilde{P}(X_1,Y_1)Z_1 = \tilde{R}(X_1,Y_1)Z_1 - \frac{1}{n-1}[\tilde{S}(Y_1,Z_1)X_1 - \tilde{S}(X_1,Z_1)Y_1]. \tag{50}$$

We assume that the condition $\tilde{P}(\xi,Y_1).\tilde{S}(Z_1,U) = 0$. It implies that

$$\tilde{S}(\tilde{P}(\xi,Y_1)Z_1,U) + \tilde{S}(Z_1,\tilde{P}(\xi,Y_1)U) = 0. \tag{51}$$

Substituting $X_1 = \xi$ in (50), we get

$$\tilde{P}(\xi,Y_1)Z_1 = \tilde{R}(\xi,Y_1)Z_1 - \frac{1}{n-1}[\tilde{S}(Y_1,Z_1)\xi - \tilde{S}(\xi,Z_1)Y_1]. \tag{52}$$

Using (52) in (51), we get

$$\begin{aligned} &g(Y_1,Z_1)\tilde{S}(\xi,U) - g(\phi Y_1,Z_1)\tilde{S}(\xi,U) \\ &- \eta(Z_1)\tilde{S}(U,Y_1) + \eta(Z_1)\tilde{S}(U,\phi Y_1) - \frac{1}{n-1}\tilde{S}(Y_1,Z_1) \\ &\tilde{S}(\xi,U) + \frac{1}{n-1}\tilde{S}(U,Y_1)\tilde{S}(\xi,Z_1) + g(Y_1,U)\tilde{S}(\xi,Z_1) \\ &- g(\phi Y_1,U)\tilde{S}(\xi,Z_1) - \eta(U)\tilde{S}(Z_1,Y_1) \\ &+ \eta(U)\tilde{S}(Z_1,\phi Y_1) - \frac{1}{n-1}\tilde{S}(Y_1,U)\tilde{S}(\xi,Z_1) + \frac{1}{n-1}\tilde{S}(Z_1,Y_1)\tilde{S}(\xi,U). \end{aligned} \tag{53}$$

Substituting $U = \xi$ and using (33) in (53), we have

$$(A_1 - A_2)g(\phi Y_1,Z_1) - (A_1 - A_2)g(Y_1,Z_1) + \tilde{S}(Y_1,Z_1) - \tilde{S}(\phi Y_1,Z_1) = 0. \tag{54}$$

Using (23) and (27), we get

$$\begin{aligned} &[A_{17}+1]g(Y_1,Z_1) + [A_{19}+1]\eta(Y_1)\eta(Z_1) + \\ &\left[A_{18} - \left(\frac{p}{2} + \frac{1}{n} - \lambda\right)\right] g(\phi Y_1,Z_1) + S(Y_1,Z_1) = 0, \end{aligned} \tag{55}$$

where

$$\begin{aligned} A_{17} &= \left[(A_2 - A_1) + \frac{3}{2}(n-2) - \text{trace}(\alpha)\right], \\ A_{18} &= \left[(A_1 - A_2) - \frac{3}{2}(n-2) + \text{trace}(\alpha)\right], \\ A_{19} &= 2(n-2). \end{aligned}$$

Using (27) in equation (55), we get:

$$g(\phi X_1, Z_1) = -\left(\frac{A_{17}+1+\left(\frac{p}{2}+\frac{1}{n}-\lambda\right)}{A_{18}-1-\left(\frac{p}{2}+\frac{1}{n}-\lambda\right)}\right) g(Y_1, Z_1) - \left(\frac{A_{19}+1-\mu}{A_{18}-1-\left(\frac{p}{2}+\frac{1}{n}-\lambda\right)}\right) \eta(Y_1)\eta(Z_1). \tag{56}$$

In view of (56), equation (55) reduce to

$$S(Y_1, Z_1) = G_5 g(Y_1, Z_1) + G_6 \eta(Y_1)\eta(Z_1), \tag{57}$$

where

$$G_5 = \left(\frac{A_{18}-\left(\frac{p}{2}+\frac{1}{n}-\lambda\right)}{A_{18}-1-\left(\frac{p}{2}+\frac{1}{n}-\lambda\right)}\right)\left(A_{17}+1+\left(\frac{p}{2}+\frac{1}{n}-\lambda\right)\right)-(A_{17}+1),$$

$$G_6 = \left(\frac{A_{18}-\left(\frac{p}{2}+\frac{1}{n}-\lambda\right)}{A_{18}-1-\left(\frac{p}{2}+\frac{1}{n}-\lambda\right)}\right)(A_{19}+1-\mu)-(A_{19}+1).$$

Hence, we can state the following theorem

Theorem 5.1. *An n-dimensional conformal η-Ricci soliton in LP-Sasakian manifold with semi-symmetric metric connection satisfying $\tilde{P}(\xi, Y_1).\tilde{S} = 0$ is an η-Einstein manifold.*

6. Conformal η-Ricci soliton in LP-Sasakian manifolds satisfying $\tilde{R}.\tilde{S} = 0$ with respect to semi-symmetric metric connection

Let M be a n-dimensional conformal η-Ricci soliton in LP-Sasakian manifold admitting semi-symmetric metric connection satisfying $\tilde{R}.\tilde{S} = 0$, where $\tilde{R}$ and $\tilde{S}$ are Riemannian curvature and Ricci tensor respectively with semi-symmetric metric connection. We assume that the condition $\tilde{R}(\xi, Y_1).\tilde{S}(Z_1, U) = 0$. Then

$$\tilde{S}(\tilde{R}(\xi, Y_1)Z_1, U) + \tilde{S}(Z_1, \tilde{R}(\xi, Y_1)U) = 0. \tag{58}$$

Using (22) in equation (58), we get

$$\begin{aligned} &g(Y_1, Z_1)\tilde{S}(U, \xi) - g(\phi Y_1, Z_1)\tilde{S}(U, \xi) - \eta(Z_1)\tilde{S}(U, Y_1) + \eta(Z_1)\tilde{S}(U, \phi Y_1) + \\ &g(Y_1, U)\tilde{S}(Z_1, \xi) - g(\phi Y_1, U)\tilde{S}(Z_1, \xi) - \eta(U)\tilde{S}(Z_1, Y_1) + \eta(U)\tilde{S}(Z_1, \phi Y_1). \end{aligned} \tag{59}$$

Substituting $U = \xi$ and using (33), we get

$$(A_1 - A_2)g(\phi Y_1, Z_1) - (A_1 - A_2)g(Y_1, Z_1) + \tilde{S}(Y_1, Z_1) - \tilde{S}(\phi Y_1, Z_1) = 0. \tag{60}$$

Using (23) in (60), we have

$$A_{20}g(Y_1, Z_1) + A_{21}g(\phi Y_1, Z_1) + A_{22}\eta(Y_1)\eta(Z_1) + S(Y_1, Z_1) - S(\phi Y_1, Z_1) = 0, \tag{61}$$

where

$$A_{20} = \left[(A_2 - A_1) + \frac{3}{2}(n-2) - \text{trace}(\alpha)\right],$$
$$A_{21} = \left[(A_1 - A_2) - \frac{3}{2}(n-2) + \text{trace}(\alpha)\right],$$
$$A_{22} = 2(n-2).$$

Using (27) in equation (61), we get

$$g(\phi Y_1, Z_1) = -g(Y_1, Z_1)\left(\frac{A_{20} + 1 + \left(\frac{p}{2} + \frac{1}{n} - \lambda\right)}{A_{21} - 1 - \left(\frac{p}{2} + \frac{1}{n} - \lambda\right)}\right) - \eta(Y_1)\eta(Z_1)\left(\frac{A_{22} + 1 - \mu}{A_{21} - 1 - \left(\frac{p}{2} + \frac{1}{n} - \lambda\right)}\right). \tag{62}$$

In view of (62) and (27), the equation (61) reduces to

$$S(Y_1, Z_1) = G_7 g(Y_1, Z_1) + G_8 \eta(Y)\eta(Z_1), \tag{63}$$

where

$$G_7 = \left(\frac{A_{21} - \left(\frac{p}{2} + \frac{1}{n} - \lambda\right)}{A_{21} - 1 - \left(\frac{p}{2} + \frac{1}{n} - \lambda\right)}\right)\left(A_{20} + 1 + \left(\frac{p}{2} + \frac{1}{n} - \lambda\right)\right) - (A_{20} + 1),$$
$$G_8 = \left(\frac{A_{21} - \left(\frac{p}{2} + \frac{1}{n} - \lambda\right)}{A_{21} - 1 - \left(\frac{p}{2} + \frac{1}{n} - \lambda\right)}\right)(A_{22} + 1 - \mu) - (A_{22} + 1).$$

Hence, we can state the following theorem

Theorem 6.1. *An n-dimensional conformal η-Ricci soliton in LP-Sasakian manifold with semi-symmetric metric connection satisfying* $\tilde{R}.\tilde{S} = 0$ *is an η-Einstein manifold.*

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