The small intersection graph of ideals of a lattice

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Abstract. In this paper, we introduce the small intersection graph of ideals of a lattice $\mathfrak{L}$, denoted by $SG(\mathfrak{L})$, in which the vertex set is being the set of all non-trivial ideals of $\mathfrak{L}$, and two distinct vertices I and J are adjacent if $I \wedge J \ll \mathfrak{L}$. We study basic properties of $SG(\mathfrak{L})$. Also, we investigate the properties of $SG(\mathfrak{L})$ as connectivity, diameter, grith and orthogonal vertices.

Keywords: lattice, small element, small intersection graph.

1. Introduction

The study of algebraic structures, using the properties of graph theory, tends to an exciting research topic in the last decade. There are many papers on assigning a graph to algebraic structures, see for example [3], [4], [5], [9]. The intersection graph of algebraic structures has been investigated by several authors [1], [3], [10], [12]. In 1973, Zelinka [13] introduced the concept of intersection graph of a lattice. In 2014, the intersection graph of ideals of a lattice was considered by Afkhami and Khashyarmanesh [2]. In this paper, we introduce and study the small intersection graph of a lattice, a new kind of intersection graph of lattices.

Now, we recall some definition and notation of graphs and lattices (see [6], [11], [8]). Let G be a graph with vertex-set $V(G)$ and edge-set $E(G)$. The *degree* of a vertex x in the graph G is the number of edges of G incident with x and denoted by $deg(x)$. Also for two distinct vertices x and y in G, the notation $x - y$ means that x and y are adjacent. A *path* from x to y is a series of adjacent vertices $x - x_1 - x_2 - ... - x_n - y$. A graph G is called *connected*, if for any

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vertices x and y of G there is a path between x and y. Otherwise, G is called disconnected. The distance between two vertices x and y, denoted by $d(x, y)$, is the length of shortest path connecting them (if such a path does not exist, then $d(x, y) := \infty$). The *diameter* of a graph G, denoted by $diam(G)$, is equal to $\sup\{d(x, y) : x, y \in V(G)\}$. A graph is *complete* if it is connected with diameter less than or equal to one. We use K_n to denote the *complete* graph with n vertices. The *girth* of a graph G, denoted by $gr(G)$, is the length of a shortest cycle in G, provided G contains a cycle; otherwise, $gr(G) := \infty$. A *clique* of a graph is its maximal complete subgraph and the number of vertices in the largest clique of G, denoted by $\omega(G)$, is called the *clique number* of G. Two vertices x and y of the graph G are *orthogonal*, denote by $x \perp y$, if x and y are adjacent in G and there is no vertex $z \in G$ which is adjacent to both x and y. A graph G is called *complemented* if for each vertex v of G, there is a vertex w of G (called a *complement* of v) such that $v \perp w$.

For r a nonnegative integer, an *r-partite* graph is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A *complete r-partite* graph is one in which each vertex is joined to every vertex that is not in the same subset. The *complete bipartite* graph with part sizes m and n is denoted by $K_{m,n}$. We will sometimes call $K_{1,n}$ a *star* graph. Note that a graph whose vertices-set is empty is a *null* graph and a graph whose edge-set is empty is an *empty* graph.

Recall that a *lattice* $\mathfrak{L}$ is a poset $(\mathfrak{L}, \leq)$ in which every pair of elements $a, b \in \mathfrak{L}$ has a *g.l.b.* (called the meet of a and b, and written $a \wedge b$) and a *l.u.b.* (called the join of a and b, and written $a \vee b$). In this paper, we assume that $\mathfrak{L}$ is a a finite lattice. A lattice $\mathfrak{L}$ is called *bounded* if there are elements 0 and 1 such that $0 \wedge a = 0$ and $a \vee 1 = 1$, for all $a \in \mathfrak{L}$. Thus every finite lattice is bounded.

A non-empty subset I of a lattice $\mathfrak{L}$ is called a *semi-ideal* if for $a \in I$ and $b \in L, b \leq a$ implies $b \in I$. A semi-ideal I of a lattice $\mathfrak{L}$ is called an *ideal* of $\mathfrak{L}$ if $a, b \in I$ implies $a \vee b \in I$. An ideal of $\mathfrak{L}$ which consists of all $x \leq a$, for some $a \in \mathfrak{L}$ is called a *principle ideal* generated by a and denoted by $[a]^l$. Clearly in a finite lattice every ideal is principal (see [8]).

An element a of a lattice $\mathfrak{L}$ is said to be *small* (written $a \ll \mathfrak{L}$) if $a \vee b \neq 1$ for every $b \neq 1$. A lattice $\mathfrak{L}$ is called *hollow* if every non-zero element is small in $\mathfrak{L}$. An element $a \in \mathfrak{L}$ is called *compact* if for every subset X of $\mathfrak{L}$ and $a \leq \bigvee X$, there is a finite subset $F \subseteq X$ such that $a \leq \bigvee F$ and $\mathfrak{L}$ is said to be compact if 1 is compact.

Let a and b be two distinct elements of $\mathfrak{L}$. Whenever $a \leq b$ and there is no element l in $\mathfrak{L}$ such that $a \leq l \leq b$, we say b covers a, in notation $a \prec b$. An element $a \in \mathfrak{L}$ is an *atom* of lattice $\mathfrak{L}$ if $0 \prec a$. We denote the set of all atoms in a lattice $\mathfrak{L}$ by $A(\mathfrak{L})$. A lattice $\mathfrak{L}$ is called *semiatomic* if 1 is a join of atoms in $\mathfrak{L}$.

A subset D of $\mathfrak{L}$ is called *upper directed* if each finite subset of D has an upper bound in D. The lattice $\mathfrak{L}$ is called *upper continuous* if $a \wedge (\bigvee D) = \bigvee_{d \in D}(a \wedge d))$

holds for every $a \in \mathfrak{L}$ and upper directed subset $D \subseteq \mathfrak{L}$. A lattice $\mathfrak{L}$ is called *local* if it has a largest element different from 1.

2. Some basic properties of $SG(\mathfrak{L})$

We define the small intersection graph of ideals of a lattice $\mathfrak{L}$, denoted by $SG(\mathfrak{L})$, as an undirected graph with all elements of $\mathfrak{L} \setminus \{0, 1\}$ being the vertices, and two distinct vertices a and b are adjacent if and only if $a \wedge b \ll \mathfrak{L}$. Now, we begin with the following Proposition.

Proposition 2.1. *The small intersection graph $SG(\mathfrak{L})$ is a null graph if and only if 1 is an atom.*

Since all definitions of graph theory are for non-null graph, in this paper all graphs are considered non-null.

Theorem 2.1. *Let a and b be two vertices of $SG(\mathfrak{L})$ such that $a \leq b$, then $deg(b) \leq deg(a)$.*

Proof. Let a and b be two vertices of $SG(\mathfrak{L})$ such that $a \leq b$. Let d be vertex adjacent to b. So $b \wedge d \ll \mathfrak{L}$, which implies $a \wedge d \ll \mathfrak{L}$. Thus d is adjacent to a. Hence $deg(b) \leq deg(a)$. □

Corollary 2.1. *If $SG(\mathfrak{L})$ is a small intersection graph, then $\Delta(SG(\mathfrak{L})) = max\{deg(a) : a \in A(\mathfrak{L})\}$, where $\Delta(SG(\mathfrak{L}))$ is the maximum degree of vertices in $SG(\mathfrak{L})$.*

Proof. Suppose that $b \in \mathfrak{L}$ and $a \in A(\mathfrak{L})$ such that $a \leq b$. Then, by Theorem 2.2, $deg(b) \leq deg(a)$. Thus the result holds. □

It is well-known that, if Q is an ordered set with power set $\mathcal{P}(Q)$ and $\varphi : Q \to \mathcal{P}(Q)$ is given by $\varphi(p) = \{q \in Q : q \leq p\}$, then one can identify Q by the image of φ ordered by inclusion. So every lattice $(\mathfrak{L}, \leq)$ can be embedded to the lattice $(\mathcal{P}(\mathfrak{L}), \subseteq)$. Thus we can consider the graph $SG(\mathfrak{L})$ as a subgraph of $SG(\mathcal{P}(\mathfrak{L}))$. Clearly, in $SG(\mathcal{P}(\mathfrak{L}))$, two distinct elements A and B in $\mathcal{P}(\mathfrak{L})$ are adjacent if and only if $(A \cap B) \cup C \neq \mathfrak{L}$ for every element C in $\mathcal{P}(\mathfrak{L}) \setminus \{\emptyset, \mathfrak{L}\}$.

Proposition 2.2. *Let X be a finite set that contains exactly n elements and S be a subset of X with m elements. Then the neighborhood of S in the small intersection graph of the lattice $(\mathcal{P}(X), \subseteq)$, where $\mathcal{P}(X)$ is the power set of X, is $\mathcal{P}(S^c) \setminus \{\emptyset\}$ and so, the degree of the vertex S in the $SG(\mathcal{P}(X))$ is $2^{n-m} - 1$.*

Proof. For every element A in $\mathcal{P}(X) \setminus \{\emptyset, X\}$, A is adjacent to S if and only if $A \cap S \ll \mathcal{P}(X)$. Clearly, $A \cap S \ll \mathcal{P}(X)$ if and only if $A \cap S = \emptyset$, which is equivalent to $A \subseteq S^c$. Hence the result follows. □

In the next Theorem, we determine the degree of a small element in $SG(\mathfrak{L})$.

Theorem 2.2. *If a is a non-zero small element of $\mathfrak{L}$ and $|SG(\mathfrak{L})| = n$, then $deg(a) = n - 1$, for all $n \in \mathbb{N}$.*

Proof. Suppose that a is a non zero small element of $\mathfrak{L}$. Since for every element $x \neq a$, $a \wedge x \leq a$ and since $a \ll \mathfrak{L}$, $a \wedge x \ll \mathfrak{L}$. Then there is an edge between a and x, for every non-zero element $x \neq a$. Therefore, if $|SG(\mathfrak{L})| = n$, then $deg(a) = n - 1$, for all $n \in \mathbb{N}$. □

The following Proposition is an immediate consequence of Theorem 2.5.

Proposition 2.3. *If $\mathfrak{L}$ is hollow, then $SG(\mathfrak{L})$ is complete graph.*

Corollary 2.2. *Let $\mathfrak{L}$ be a lattice. Then the following hold:*

(i) If $\mathfrak{L}$ is local, then the graph $SG(\mathfrak{L})$ is complete.

(ii) If $\mathfrak{L}$ has at least one non-zero small element, then the induced subgraph on the sets of small elements are clique in $SG(\mathfrak{L})$.

Proof. (i) Since local lattices are hollow, the result follows from Proposition 2.6.

(ii) It follows easily. □

Recall that the one-to-one correspondence $f : \mathfrak{L} \to \acute{\mathfrak{L}}$, for two lattices $\mathfrak{L}$ and $\acute{\mathfrak{L}}$, is a lattice isomorphism if, $f(a \vee b) = f(a) \vee f(b)$ and $f(a \wedge b) = f(a) \wedge f(b)$, for any $a, b \in \mathfrak{L}$.

Proposition 2.4. *Let for two lattices $\mathfrak{L}$ and $\acute{\mathfrak{L}}$, $f : \mathfrak{L} \to \acute{\mathfrak{L}}$ is a lattice isomorphism. Then there exists a graph isomorphism between the small intersection graphs $SG(\mathfrak{L})$ and $SG(\acute{\mathfrak{L}})$.*

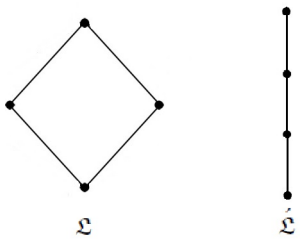
Proof. First Suppose that $\bar{f} : V(SG(\mathfrak{L})) \to V(SG(\acute{\mathfrak{L}}))$ such that for each $x \in V(SG(\mathfrak{L}))$, $\bar{f}(x) = f(x)$. Now, assume that x and y are adjacent in $SG(\mathfrak{L})$, then for each $z \in \mathfrak{L}$, $x \wedge y \ll \mathfrak{L}$ and $(x \wedge y) \vee z \neq 1$. Thus by the lattice isomorphism definition, $(x \wedge y) \vee z \neq 1$ if and only if $f(x \wedge y) \vee f(z) \neq 1$ if and only if $(f(x) \wedge f(y)) \vee f(z) \neq 1$, for every element $z \in \mathfrak{L}$. So $\bar{f}(x)$ is adjacent to $\bar{f}(y)$ in $SG(\acute{\mathfrak{L}})$ and the result holds. □

In the following example shows that, in general, the converse of Proposition 2.8 is not true.

Example 2.1. Assume that $\mathfrak{L}$ and $\acute{\mathfrak{L}}$ are the lattices in Figure 1. Then $\mathfrak{L}$ and $\acute{\mathfrak{L}}$ are not isomorphic, but $SG(\mathfrak{L})$ and $SG(\acute{\mathfrak{L}})$ are isomorphic.

3. Connectivity, diameter, girth and orthogonal vertices of $SG(\mathfrak{L})$

In this section, we study some properties of the graph $SG(\mathfrak{L})$ such as connectivity, diameter, girth and orthogonal vertices. Now, we start with the following Theorem.



Theorem 3.1. *Let $\mathfrak{L}$ be a lattice with at least a non-zero small element. Then $SG(\mathfrak{L})$ is a connected graph with $diam(SG(\mathfrak{L})) \leq 2$.*

Proof. Suppose that a_1 and a_2 are two distinct vertices of the graph $SG(\mathfrak{L})$. Assume that $\mathfrak{L}$ has at least a non-zero small element as b. Now we have the following cases:

Case 1. If $b = a_i$, for $i = 1$ or $i = 2$. Since $a_1 \wedge a_2 \leq a_i$ for $i = 1, 2$, $a_1 \wedge a_2 \ll \mathfrak{L}$. Then there is an edge between a_1 and a_2, thus $d(a_1, a_2) = 1$.

Case 2. If $b \neq a_i$, for $i = 1, 2$. Since $a_i \wedge b \leq b$, $a_i \wedge b \ll \mathfrak{L}$, for $i = 1, 2$, then there exists an a_1, a_2-path to form $a_1 - b - a_2$ and $d(a_1, a_2) = 2$. As the diameter of graph is the maximum distance between any two vertices, consequently $diam(SG(\mathfrak{L})) \leq 2$ and so $SG(\mathfrak{L})$ is a connected graph. □

In the following example, we show that, the above Theorem may not be hold for any lattice without small element.

Example 3.1. Let $X = \{1, 2, 3\}$. Set $\mathfrak{L} := (\mathcal{P}(X), \subseteq)$, where $\mathcal{P}(X)$is the power set of X. Then $\mathfrak{L}$ has no small element and $d(\{1, 3\}, \{2, 3\}) = 3$.

In a lattice $\mathfrak{L}$, $R(\mathfrak{L})$ will indicate the radical of $\mathfrak{L}$, where $R(\mathfrak{L})$ is the intersection of all the maximal elements $\neq 1$ in $\mathfrak{L}$.

Corollary 3.1. *The graph $SG(\mathfrak{L})$ is a connected graph, if one of the following holds:*

(i) The lattice $\mathfrak{L}$ is compact.

(ii) There exists a compact element of $\mathfrak{L}$ contained in $R(\mathfrak{L})$.

Proof. (i) Since $\mathfrak{L}$ is a compact lattice, $R(\mathfrak{L}) \ll \mathfrak{L}$ and results follows from Theorem 3.1.

(ii) If a is a compact element in $\mathfrak{L}$ and $a \leq R(\mathfrak{L})$, then $a \ll \mathfrak{L}$. Hence we can easily check that $SG(\mathfrak{L})$ is connected. □

We say that two vertices x and y of the graph G are orthogonal, written $x \perp y$, if x and y are adjacent in G and there is no vertex $z \in SG(\mathfrak{L})$ which is adjacent to both x and y. In the other word $x \perp y$ if $x \wedge y \ll \mathfrak{L}$ and for every vertex $z \in SG(\mathfrak{L})$, either $x \wedge z$ is not small in $\mathfrak{L}$ or $z \wedge y$ is not small in $\mathfrak{L}$. Also a graph G is called complemented if for each vertex v of G, there is a vertex w of G (called a complement of v) such that $v \perp w$.

Recall that, in a lattice $\mathfrak{L}$, $S(\mathfrak{L})$ denotes the set of all the non-zero small elements of $\mathfrak{L}$.

Proposition 3.1. *Let $\mathfrak{L}$ be a lattice with $|S(\mathfrak{L})| \geq 2$ and $|SG(\mathfrak{L})| \geq 3$, then the following conditions holds:*

(i) Every two small elements of $\mathfrak{L}$ cannot be orthogonal vertices to each other.

(ii) The graph $SG(\mathfrak{L})$ has at least a triangle.

(iii) For $|SG(\mathfrak{L})| = n$ and $|S(\mathfrak{L})| = 2$, if we take m equal to sum of the number of the triangles of this graph, then $(n-2) \leq m \leq \frac{1}{6}n(n-1)(n-2)$.

Proof. (i) Suppose that x and y are two small elements of $\mathfrak{L}$, then $x \wedge y \ll \mathfrak{L}$ and for any vertex $v \in SG(\mathfrak{L})$, since $x \wedge v \leq x$ and $y \wedge v \leq y$, $x \wedge v \ll \mathfrak{L}$ and $y \wedge v \ll \mathfrak{L}$. Thus v is adjacent to both x and y. Therefore, they cannot be orthogonal vertices each other.

(ii) It is clear.

(iii) Let $|SG(\mathfrak{L})| = n$, where $n \geq 3$ and $|S(\mathfrak{L})| = 2$. Then the number of the elements which are not small is equal to $n-2$. If we choose two vertices of the triangles are small elements, then the sum of the triangles is at least equal to $C_2^2 \times C_1^{n-2} = n-2$. Also the sum of the number of all triangles in this graph is at most equal to $C_2^2 \times C_1^{n-2} + C_1^2 \times C_1^{n-2} + C_3^{n-2} = \frac{1}{6}n(n-1)(n-2)$. Therefore, $(n-2) \leq m \leq \frac{1}{6}n(n-1)(n-2)$. □

Theorem 3.2. *Let $\mathfrak{L}$ be a lattice. Then the following statements are equivalent:*

(i) The graph $SG(\mathfrak{L})$ has no triangle.

(ii) Every two adjacent vertices of the graph $SG(\mathfrak{L})$ are orthogonal vertices.

(iii) The lattice $\mathfrak{L}$ has at most a non-zero small element such that the meet of each pair of the non-small element of $\mathfrak{L}$ is not small in $\mathfrak{L}$.

Proof. (i)⇒ (ii) Let a and b be two adjacent vertices of the graph $SG(\mathfrak{L})$ which are not orthogonal vertices. Then by the definition of orthogonality, there exists another vertex c of the graph $SG(\mathfrak{L})$ which is adjacent to both a and b, then there exists a cycle to form $a - b - c$. This means that there is a triangle in the graph $SG(\mathfrak{L})$, a contradiction.

(ii)⇒ (iii) Assume that every two adjacent vertices of the graph $SG(\mathfrak{L})$ are orthogonal vertices. Let there exist at least two non-zero small elements a_1 and a_2 of $\mathfrak{L}$. Since $a_1 \wedge a_2 \ll \mathfrak{L}$, they are adjacent vertices of the graph $SG(\mathfrak{L})$ and also $a_1 \perp a_2$. However, by Proposition 3.4, they cannot be orthogonal to each other, a contradiction.

(iii)⇒ (i) Suppose that $\mathfrak{L}$ has no non-zero small element. Since the meet of each pair of the non-small non-zero element of $\mathfrak{L}$ is not small in $\mathfrak{L}$, $SG(\mathfrak{L})$ has no triangle. Moreover, let z be only non-zero small element of $\mathfrak{L}$. Then for every three arbitrary vertices x_1, x_2 and x_3 of the graph $SG(\mathfrak{L})$, at least two of them are not small in $\mathfrak{L}$. Let $z = x_1$. Since $x_2 \wedge x_3$ is not small in $\mathfrak{L}$, $x_2 - z - x_3$ is a path. Also if $z \neq x_i$ for $i = 1, 2, 3$. Since $x_i \wedge x_j$ is not small in $\mathfrak{L}$, for $i, j = 1, 2, 3$ and $i \neq j$, then x_1, x_2 and x_3 are not adjacent vertices in graph $SG(\mathfrak{L})$. Therefore, there is no triangle in the graph $SG(\mathfrak{L})$. □

For any lattice $\mathfrak{L}$, two elements a and b are comaximal if $a \vee b = 1$.

Proposition 3.2. *Let $\mathfrak{L}$ be a lattice and a, b be two comaximal elements of $\mathfrak{L}$. If $R(\mathfrak{L}) = 0$ and $a \perp b$ in $SG(\mathfrak{L})$, then $1 = a \oplus b$.*

Proof. Since a, b be two comaximal elements of $\mathfrak{L}$ and $a \perp b$ in the graph $SG(\mathfrak{L})$, then $1 = a \vee b$, $a \wedge b \ll \mathfrak{L}$. Hence $a \wedge b \leq R(\mathfrak{L}) = 0$. Consequently, the result follows.. □

We say that a lattice $\mathfrak{L}$ satisfies the covering condition if for all $a, b \in \mathfrak{L}$, $a \wedge b \prec a$ if and only if $b \prec a \vee b$. For the rest of paper, we assume that $\mathfrak{L}$ satisfies the covering condition.

Proposition 3.3. *If $1 = a_1 \vee a_2$, where a_1 and a_2 are two atoms in $\mathfrak{L}$ and it satisfies the covering condition, then $SG(\mathfrak{L})$ is a complete graph.*

Proof. Suppose that $1 = a_1 \vee a_2$ such that a_1 and a_2 are two atoms in $\mathfrak{L}$. Then by covering condition, any non-trivial element of $\mathfrak{L}$ is an atom. Let x and y be two distinct vertices of graph $SG(\mathfrak{L})$. Then they are the non-trivial elements of $\mathfrak{L}$ which are atoms. Moreover, $x \wedge y \leq x, y$ and if $x \wedge y \neq 0$ then $x \wedge y = x = y$, which is a contradiction. Therefore, $x \wedge y = 0 \ll \mathfrak{L}$, thus $SG(\mathfrak{L})$ is a complete graph. □

In the following example, we state that the converse of Proposition 3.7 is not true, in general.

Example 3.2. Assume that $L = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}\}$ with two operation $\cup$ and $\cap$. Then, it is easy to see that the graph $SG(\mathfrak{L})$ is complete, but $\{1, 2, 3\}$ can not be written as the disjoint union.

Proposition 3.4. *Let $\mathfrak{L}$ be a lattice with $R(\mathfrak{L}) \neq 0$. Then the following conditions hold:*

(i) If a is a non-zero element of lattice $\mathfrak{L}$ which is a direct summand of 1 with $0 \neq R([a]^l) \ll \mathfrak{L}$, then the graph $SG(\mathfrak{L})$ contains at least one cycle of length 3.

(ii) If x is compact element of $\mathfrak{L}$ contained in $R(\mathfrak{L})$,then $d(x, R(\mathfrak{L})) = 1$ and $d(x, z) = 1$ for every non-zero element z of $\mathfrak{L}$.

Proof. (i) Since a is a direct summand of $\mathfrak{L}$, there exists an element b of $\mathfrak{L}$ such that $a \oplus b = 1$. Then by, $R(\mathfrak{L}) = R([a]^l) \oplus R([b]^l)$. Since $R([a]^l) \leq a$ and $a \wedge R([b]^l) \leq a \wedge b = 0$, by covering condition, $a \wedge R(\mathfrak{L}) = R([a]^l)$. Then $a \wedge R(\mathfrak{L}) \ll \mathfrak{L}$. Also, $R([a]^l) = a \wedge R([a]^l) \ll \mathfrak{L}$ and $R([a]^l) = R([a]^l) \wedge R(\mathfrak{L}) \ll \mathfrak{L}$ and there exist edges $a - R(\mathfrak{L})$, $a - R([a]^l)$ and $R([a]^l) - R(\mathfrak{L})$ in graph $SG(\mathfrak{L})$. Hence $(a, R([a]^l), R(\mathfrak{L}))$ is a cycle of length 3. Consequently, the graph $SG(\mathfrak{L})$ contains at least one cycle of length 3.

(ii) Let x be a compact element of $\mathfrak{L}$, then $x \ll \mathfrak{L}$. Since $x \leq R(\mathfrak{L})$, $x = x \wedge R(\mathfrak{L}) \ll \mathfrak{L}$ and since $x \wedge z \leq x, x \wedge z \ll \mathfrak{L}$ for every non-zero element z of $\mathfrak{L}$. Hence $d(x, R(\mathfrak{L})) = 1$ and $d(x, z) = 1$. □

Theorem 3.3. *If $|S(\mathfrak{L})| \geq 2$, then $SG(\mathfrak{L})$ contains at least a cycle and $gr(SG(\mathfrak{L})) = 3$.*

Proof. Suppose that $|S(\mathfrak{L})| \geq 2$. Then $\mathfrak{L}$ has at least two different small elements, say a_1 and a_2. Since $a_1 \wedge a_2 \leq a_i$ for $i = 1, 2$, $a_1 \wedge a_2 \ll \mathfrak{L}$. Moreover $a_1 \wedge (a_1 \wedge a_2) \ll \mathfrak{L}$ and $a_2 \wedge (a_1 \wedge a_2) \ll \mathfrak{L}$. We consider the following cases:

Case 1. If $a_1 \wedge a_2 \neq 0$, then there exists three edges $a_1 - a_2$, $a_1 - (a_1 \wedge a_2)$ and $a_2 - (a_1 \wedge a_2)$ of graph $SG(\mathfrak{L})$. Hence $a_1 - (a_1 \wedge a_2) - a_2$ is a cycle of length 3. Also $a_1 \vee a_2 \ll \mathfrak{L}$ and since $a_1 \wedge (a_1 \vee a_2) \ll \mathfrak{L}$ and $a_2 \wedge (a_1 \vee a_2) \ll \mathfrak{L}$, $a_1 - (a_1 \vee a_2) - a_2$ is a cycle of length 3. Similarly $a_1 \wedge a_2, a_1, a_1 \vee a_2$ and $a_1 \wedge a_2, a_2, a_1 \vee a_2$ are cycles of length 3 and is a cycle of length 4.

Case 2. If $a_1 \wedge a_2 = 0$, then there exists at least a cycle $a_1 - (a_1 \vee a_2) - a_2$ of length 3 in graph $SG(\mathfrak{L})$. Therefore, the graph $SG(\mathfrak{L})$ contains at least a cycle and since the girth of a graph with cycles is the length of the shortest cycle, $gr(SG(\mathfrak{L})) = 3$. □

Theorem 3.4. *Let $\mathfrak{L}$ be a lattice. Then the following statements hold:*

(i) If $\mathfrak{L}$ is a upper continuous semiatomic lattice which is not simple, then the graph $SG(\mathfrak{L})$ is complemented.

(ii) If $\mathfrak{L}$ is a compact lattice with $R(\mathfrak{L}) \neq 0$, then the graph $SG(\mathfrak{L})$ is not complemented.

Proof. (i) Suppose that $\mathfrak{L}$ is upper continuous semiatomic lattice. Let x be a vertex of the graph $SG(\mathfrak{L})$. Then by [7, Theorem 6.6], every element of $\mathfrak{L}$ is a direct summand of 1. Thus there exists a element y of $\mathfrak{L}$ such that $1 = x \oplus y$. Hence $x \wedge y = 0 \ll \mathfrak{L}$. Since $\mathfrak{L}$ has no non-zero small element, there is no vertex z of $SG(\mathfrak{L})$ such that $z \wedge x \ll \mathfrak{L}$ and $z \wedge y \ll \mathfrak{L}$. Therefore, $SG(\mathfrak{L})$ is complemented.

(ii) Since $\mathfrak{L}$ is a compact lattice, $0 \neq R(L) \ll \mathfrak{L}$. So we have the following cases:

Case 1. If $R(L)$ is an atom, since $R(\mathfrak{L}) = \bigwedge_{i \in I} m_i$ where m_i is the maximal element of $\mathfrak{L}$, for all $i \in I$. We choose $s = \bigwedge_{i \in I - \{1\}} m_i$. Then $m_1 - s - R(\mathfrak{L})$ is a triangle in the graph $SG(\mathfrak{L})$. Then the graph $SG(\mathfrak{L})$ is not complemented

Case 2. If $R(\mathfrak{L})$ is not an atom of $\mathfrak{L}$, then there exists an element x of $\mathfrak{L}$ which $x \leq R(\mathfrak{L})$, then $x \ll \mathfrak{L}$. Thus for each vertex y of $SG(\mathfrak{L})$, $x - y - R(\mathfrak{L})$ is a triangle of this graph. Consequently, the graph $SG(\mathfrak{L})$ is not complemented. □

Proposition 3.5. *Let $1 = a \oplus b$ be a compact lattice such that $R([a]^l) \neq 0$ and $R([b]^l) \neq 0$. Then the following statements hold:*

(i) The vertices a and b are not orthogonal in the graph $SG(\mathfrak{L})$.

(ii) The complete graph K_5 is an induced subgraph of $SG(\mathfrak{L})$.

(iii) The graph $SG(\mathfrak{L})$ is not complemented.

Proof. (i) Suppose that $1 = a \oplus b$. Then $a \wedge b = 0 \ll \mathfrak{L}$ and since $\mathfrak{L}$ is a compact lattice, $R(\mathfrak{L}) \ll \mathfrak{L}$. Hence $a \wedge R(\mathfrak{L}) \ll \mathfrak{L}$ and $b \wedge R(\mathfrak{L}) \ll L$. Thus $SG(\mathfrak{L})$ has a triangle to form $(a, R(\mathfrak{L}), b)$. The result now follows from Theorem 3.5.

(ii) Let $1 = a \oplus b$. Then $R(\mathfrak{L}) = R([a]^l) \oplus R([b]^l)$ and $R([a]^l) \oplus R([b]^l) = 0 \ll \mathfrak{L}$. Also $R([a]^l) \leq a$ and $a \wedge R([b]^l) \leq a \wedge b = 0$, then the covering condition, implies that $a \wedge R(\mathfrak{L}) = R([a]^l)$ and similarly, $b \wedge R(\mathfrak{L}) = R([b]^l)$. Moreover, $a \wedge R([a]^l) = R([a]^l) \ll \mathfrak{L}$, $b \wedge R([b]^l) = R([b]^l) \ll \mathfrak{L}$, $a \wedge R(\mathfrak{L}) = R([a]^l) \ll \mathfrak{L}$, $b \wedge R(\mathfrak{L}) = R([b]^l) \ll \mathfrak{L}$, $R([a]^l) \wedge R(\mathfrak{L}) = R([a]^l) \ll \mathfrak{L}$ and $R([b]^l) \wedge R(\mathfrak{L}) = R([b]^l) \ll \mathfrak{L}$. Hence $a, b, R([a]^l), R([b]^l)$ and $R(\mathfrak{L})$ are adjacent vertices in the graph $SG(\mathfrak{L})$. Therefore the subgraph induced on the vertex set of $\{a, b, R([a]^l), R([b]^l), R(\mathfrak{L})\}$ is the complete graph K_5.

(iii) Let $1 = a \oplus b$. Then, by Theorem 3.11, we are done. □

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