A study on weak pseudoorders in hyper S-posets

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Abstract. In this paper, the concept of weak pseudoorders in a hyper S-poset is introduced, and several related properties are given. Especially, we characterize the structure of factor hyper S-posets by weak pseudoorders. Furthermore, we discuss the relationship between order-congruences and weak pseudoorders on a hyper S-poset, and give some characterizations of order-congruences by the properties of weak pseudoorders. In the sequel, we give some homomorphism theorems of hyper S-posets by weak pseudoorders, which are generalizations of similar results in S-posets and ordered semigroups.

Keywords: ordered semihypergroup, hyper S-poset, order-congruence, weak pseudoorder.

1. Introduction

As we know, an ordered semigroup $(S, \cdot, \leq)$ is a semigroup $(S, \cdot)$ together with an order relation " $\leq$ " which is compatible with the binary operation. Recall that if $(S, \cdot, \leq)$ is an ordered semigroup then a nonempty poset $(A, \leq_A)$ together with a mapping $A \times S \to A, (a, s) \mapsto as$, where as denotes the result of action of s on a, is called a *right S-poset* if (1) $(as)t = a(st)$, (2) $a \leq_A b$ implies $as \leq_A bs$, and (3) $s \leq t$ implies $as \leq_A at$, for all $a, b \in A, s, t \in S$. Left S-posets are defined analogously. S-posets play an important role not only in studying properties of ordered semigroups but also in other mathematical areas, such as graph theory and algebraic automata theory (for example, see [1]). Nowadays many researchers studied different aspects of S-posets, for instance, Bulman-Fleming et al. [2], Laan [11], Liang and Khosravi [12], Liang and Luo [14], Shi [17] and Shi et al. [18], also see [13].

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Algebraic hyperstructures (or hypersystems) represent a natural extension of classical algebraic structures and they were originally proposed in 1934 by a French mathematician Marty [16], at the 8th Congress of Scandinavian Mathematicians. Since then, hyperstructures are widely investigated from the theoretical point of view and for their applications to many branches of pure and applied mathematics, for example, see [4, 5, 7, 22]. One of the main reason which attracts researches towards hyperstructures is its unique property that in hyperstructures composition of two elements is a set, while in classical algebraic structures the composition of two elements is an element. Thus algebraic hyperstructures are natural extension of classical algebraic structures. It is worth pointing out that Heidari and Davvaz [9] applied the theory of hyperstructures to ordered semigroups and introduced the concept of ordered semihypergroups, which is a generalization of the concept of ordered semigroups. The work on ordered semihypergroup theory can be found in [3, 6, 19, 20, 21, 25, 26]. In particular, Tang et al. [20] defined and studied the hyper S-posets over an ordered semihypergroup, and extended some results on S-posets to hyper S-posets. They introduced the concepts of pseudoorders and (strong) order-congruences on a hyper S-poset A, and obtained the relationship between strongly order-congruences and pseudoorders on A. Especially, they proved that there must exist a strong congruence ρ on a hyper S-poset A such that A/ρ is a hyper S-poset (see Theorem 3.7 in [20]). A natural question about hyper S-posets is raised: Is there a congruence ρ on a hyper S-poset A for which A/ρ is a hyper S-poset? As a further study, in the present paper we study the weak pseudoorders in a hyper S-poset in detail and answer the above question.

The rest of this paper is organized as follows. After an introduction, in Section 2 we recall some basic notions and results from the hyperstructure theory which will be used throughout this paper. In Section 3, we introduce the concept of weak pseudoorders on a hyper S-poset A, and illustrate this notion is a generalization of the concept of pseudoorders on A by some examples. Furthermore, the properties of weak pseudoorders on a hyper S-poset are investigated. In particular, we construct a hyper S-poset $A/\underline{\rho}$ by a weak pseudoorder ρ. In Section 4, the relationships between order-congruences and weak pseudoorders on a hyper S-poset is established, and some homomorphism theorems of hyper S-posets by weak pseudoorders are given.

2. Preliminaries and some notations

In this section, we present some definitions and results which will be used throughout this paper.

Let S be a nonempty set. A mapping $\circ : S \times S \to P^\star(S)$, where $P^\star(S)$ denotes the family of all nonempty subsets of S, is called a *hyperoperation* or *hypercomposition* on S. The image of the pair (x, y) is denoted by $x \circ y$. The couple $(S, \circ)$ is called a *hyperstructure*. In the above definition, if $x \in S$ and

A, B are nonempty subsets of S, then $A \circ B$ is defined by

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b.$$

Also $A \circ x$ is used for $A \circ \{x\}$ and $x \circ A$ for $\{x\} \circ A$. A hyperstructure $(S, \circ)$ is a *semihypergroup* [4] if for all $x, y, z \in S$, $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

Definition 2.1 ([9])**.** An *ordered semihypergroup* (also called *po-semihypergroup* in [9]) $(S, \circ, \leq)$ is a semihypergroup $(S, \circ)$ with an order relation $\leq$ which is compatible with the hyperoperation $\circ$, meaning that for any $x, y, a \in S$, $x \leq y$ implies $a \circ x \leq a \circ y$ and $x \circ a \leq y \circ a$. Here, if $A, B \in P^\star(S)$, then we say that $A \leq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$. In particular, if $A = \{a\}$, then we write $a \leq B$ instead of $\{a\} \leq B$.

Clearly, every ordered semigroup can be regarded as an ordered semihypergroup. Also see [19]. We now recall the notion of hyper S-acts over semihypergroups from [8].

Definition 2.2 ([8])**.** Let $(S, \circ)$ be a semihypergroup and A a nonempty set. If we have a mapping $\mu : A \times S \to P^\star(A) \mid (a, s) \mapsto \mu(a, s) := a * s \in P^\star(A)$, called the *hyper action of* S (or the S*-hyperaction*) on A, such that $a*(s \circ t) = (a*s)*t$, for all $a \in A, s, t \in S$, where

$$(\forall T \subseteq S)\ a * T = \bigcup_{t \in T} a * t;\ (\forall B \subseteq A)\ B * t = \bigcup_{b \in B} b * t,$$

then we call A a *right hyper* S*-act* (also called *right* S*-hypersystem* in [8]).

Left hyper S-acts can be defined analogously, and in this paper we will often use the term hyper S-act to mean right hyper S-act. Furthermore, Tang et al. introduced the concept of hyper S-posets over an ordered semihypergroup.

Definition 2.3 ([20])**.** Let $(S, \circ, \leq)$ be an ordered semihypergroup. A *right hyper* S*-poset* $(A, \leq_A)$, often denoted $A_{\mathcal{H}}$ (or briefly A), is a right hyper S-act A equipped with a partial order $\leq_A$ and, in addition, for all $s, t \in S$ and $a, b \in A$, if $s \leq t$ then $a * s \leq_A a * t$, and if $a \leq_A b$ then $a * s \leq_A b * s$. Here, $a * s$ stands for the result of the hyper action of s on a, and if $A_1, A_2 \in P^\star(A)$, then we say that $A_1 \leq_A A_2$ if for every $a_1 \in A_1$ there exists $a_2 \in A_2$ such that $a_1 \leq_A a_2$.

Analogously, we can define a left hyper S-poset ${}_{\mathcal{H}}A$. Throughout this paper we shall use the term hyper S-poset to mean right hyper S-poset. Clearly, every S-poset over an ordered semigroup can be regarded as a hyper S-poset over an ordered semihypergroup.

Let A be a hyper S-act over a semihypergroup $(S, \circ)$ and ρ a relation on A. If B and C are both nonempty subsets of A, then we set

$$\begin{aligned} B\overrightarrow{\rho}C &\Leftrightarrow (\forall b \in B)(\exists c \in C)\ b\rho c, \\ B\overleftarrow{\rho}C &\Leftrightarrow (\forall c' \in C)(\exists b' \in B)\ b'\rho c', \\ B\overline{\rho}C &\Leftrightarrow B\overrightarrow{\rho}C \text{ and } B\overleftarrow{\rho}C, \\ B\overline{\overline{\rho}}C &\Leftrightarrow (\forall b \in B, \forall c \in C)\ b\rho c. \end{aligned}$$

An equivalence relation ρ on A is called *congruence* [20] if $(\forall x, y \in A)\ x\rho y \Rightarrow x * s\ \overline{\rho}\ y * s$ for all $s \in S$; ρ is called *strong congruence* [20] if

$$(\forall x, y \in A)\ x\rho y \Rightarrow x * s\ \overline{\overline{\rho}}\ y * s \text{ for all } s \in S.$$

We denote by $C(A)$ (resp. $SC(A)$) the set of all congruences (resp. strong congruences) on a hyper S-act A.

Lemma 2.4 ([20]). *Let A be a hyper S-act over a semihypergroup $(S, \circ)$ and ρ an equivalence relation on A. Then*

(1) *If ρ is a congruence, then A/ρ is a hyper S-act with respect to the following hyper S-action: $(a)_\rho \otimes s = \{(x)_\rho \mid x \in a * s\}$, and it is called a factor hyper S-act.*

(2) *If ρ is a strong congruence, then A/ρ is a hyper S-act with respect to the following (hyper) S-action: $(a)_\rho \otimes s = (x)_\rho$ for all $x \in a * s$, and it is called a factor hyper S-act. In particular, if S is a semigroup and the operation on S is defined by $s \circ t := \{st\}$ for all $s, t \in S$, then A/ρ is an S-act.*

Definition 2.5 ([20]). Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$. A relation ρ on A is called *pseudoorder* if it satisfies the following conditions:

(1) $\leq_A \subseteq \rho$.

(2) $a\rho b$ and $b\rho c$ imply $a\rho c$, i.e., $\rho \circ \rho \subseteq \rho$.

(3) $a\rho b$ implies $a * s\ \overline{\overline{\rho}}\ b * s$, for all $s \in S$.

Definition 2.6 ([20]). Let $(A, \leq_A)$ and $(B, \leq_B)$ be two hyper S-posets over an ordered semihypergroup $(S, \circ, \leq)$, the "$*$" and "$\diamond$" are hyper S-actions on A and B, respectively, $f : A \to B$ a mapping from A to B. f is called *isotone* if $x \leq_A y$ implies $f(x) \leq_B f(y)$, for all $x, y \in A$. f is called *reverse isotone* if $x, y \in A$, $f(x) \leq_B f(y)$ implies $x \leq_A y$. f is called *homomorphism* if it is isotone and satisfies $f(a) \diamond s = \{f(x) \mid x \in a * s\}$ for all $a \in A, s \in S$. f is called *isomorphism* if it is homomorphism, onto and reverse isotone. The hyper S-posets A and B are called *isomorphic*, in symbol $A \cong B$, if there exists an isomorphism between them.

Remark 2.7. Let $(A, \leq_A)$ and $(B, \leq_B)$ be two hyper S-posets over an ordered semihypergroup $(S, \circ, \leq)$. If f is a homomorphism and reverse isotone mapping from A to B, then $A \cong Im(f)$.

In [20], Tang et al. have illustrated that, for a (strong) congruence ρ on a hyper S-poset A, the factor hyper S-act A/ρ is not necessarily a hyper S-poset, in general. To characterize the structure of factor hyper S-posets, they introduced the concept of (strongly) order-congruences on a hyper S-poset.

Definition 2.8 ([20])**.** Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$. A congruence (resp. strong congruence) ρ is called an *order-congruence* (resp. a *strong order-congruence*) if there exists an order relation " $\preceq$ " on A/ρ such that:

(1) $(A/\rho, \preceq)$ is a hyper S-poset, where the S-hyperaction " $\otimes$ " on A/ρ is defined as one in Lemma 2.4.

(2) The canonical epimorphism $\varphi : A \to A/\rho, x \mapsto (x)_\rho$ is isotone, that is, φ is a homomorphism (resp. strong homomorphism) from A onto A/ρ.

The reader is referred to [15, 20, 23] for notation and terminology not defined in this paper.

3. Weak pseudoorders in hyper S-posets

In this section, we shall introduce the concept of weak pseudoorders in a hyper S-poset, and investigate their related properties. In particular, we characterize the structure of factor hyper S-posets by weak pseudoorders.

Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$ and ρ a relation on A. If $B, C \in P^\star(A)$, then we set

$$B \ \widetilde{\rho}\ C \ \Leftrightarrow\ (\forall b \in B)(\exists c \in C)\ b\rho c \text{ and } c\rho b \ \&\ (\forall c' \in C)(\exists b' \in B)\ b'\rho c' \text{ and } c'\rho b'.$$

Definition 3.1. Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$. A relation ρ on A is called *weak pseudoorder* if it satisfies the following conditions:

(1) $\leq_A \subseteq \rho$.

(2) $a\rho b$ and $b\rho c$ imply $a\rho c$, i.e., $\rho \circ \rho \subseteq \rho$.

(3) $a\rho b$ implies $a * s\ \overrightarrow{\rho}\ b * s$, for all $s \in S$.

(4) $a\rho b$ and $b\rho a$ imply $a * s\ \widetilde{\rho}\ b * s$, for all $s \in S$.

Obviously, the order relation " $\leq_A$ " on a hyper S-poset A is a weak pseudoorder relation on A. Furthermore, one can easily observe that every pseudoorder on a hyper S-poset A is a weak pseudoorder on A. However, the converse is not true, in general, as shown in the following example.

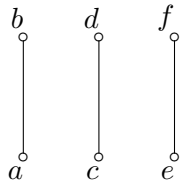
Example 3.2. We consider a set $S := \{a, b, c, d, e, f\}$ with the following hyperoperation " $\circ$ " and the order " $\leq$ ":

$\circ$	a	b	c	d	e	f
a	$\{a\}$	$\{a, b\}$	$\{c\}$	$\{c, d\}$	$\{e\}$	$\{e, f\}$
b	$\{b\}$	$\{b\}$	$\{d\}$	$\{d\}$	$\{f\}$	$\{f\}$
c	$\{c\}$	$\{c, d\}$	$\{c\}$	$\{c, d\}$	$\{c\}$	$\{c, d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$
e	$\{e\}$	$\{e, f\}$	$\{c\}$	$\{c, d\}$	$\{e\}$	$\{e, f\}$
f	$\{f\}$	$\{f\}$	$\{d\}$	$\{d\}$	$\{f\}$	$\{f\}$

$$\leq := \{(a, a), (a, b), (b, b), (c, c), (c, d), (d, d), (e, e), (e, f), (f, f)\}.$$

We give the covering relation "$\prec$" and the figure of S as follows:

$$\prec = \{(a, b), (c, d), (e, f)\}.$$

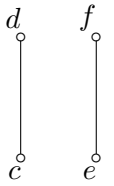


Then $(S, \circ, \leq)$ is an ordered semihypergroup. We now consider the partially ordered set $A = \{c, d, e, f\}$ defined by the order below:

$$\leq_A := \{(c, c), (d, d), (e, e), (f, f), (c, d), (e, f)\}.$$

We give the covering relation "$\prec_A$" and the figure of A

$$\prec_A = \{(c, d), (e, f)\}.$$



Then $(A, \leq_A)$ is a hyper S-poset over S with respect to S-hyperaction on A as above hyperoperation table (see [20]). Let ρ be a relation on S defined as follows:

$$\rho := \{(c, c), (d, d), (e, e), (f, f), (c, d), (c, f), (d, f), (e, f)\}.$$

It is a routine matter to verify that ρ is a weak pseudoorder on A, but it is not a pseudoorder on A. In fact, since $c\rho c$, while $c * b \,\overline{\overline{\rho}}\, c * b$ doesn't hold.

In the following, we shall construct a congruence on a hyper S-poset by a weak pseudoorder, from which we can characterize the structure of factor hyper S-posets.

Theorem 3.3. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$ and ρ a weak pseudoorder on A. Then, there exists a congruence $\underline{\rho}$ on A such that $A/\underline{\rho}$ is a hyper S-poset over S.*

Proof. We denote by $\underline{\rho}$ the relation on A defined by

$$\underline{\rho} := \{(a, b) \in A \times A \mid a\rho b \text{ and } b\rho a\} \ (= \rho \cap \rho^{-1}).$$

First, we claim that $\underline{\rho}$ is a congruence on A. In fact, for any $a \in A$, clearly, $(a, a) \in \leq_A \subseteq \rho$, so $a\underline{\rho}a$. If $(a, b) \in \underline{\rho}$, then $a\rho b$ and $b\rho a$. Thus $(b, a) \in \underline{\rho}$. Let $(a, b) \in \underline{\rho}$ and $(b, c) \in \underline{\rho}$. Then $a\rho b, b\rho a, b\rho c$ and $c\rho b$. Hence $a\rho c$ and $c\rho a$, which imply that $(a, c) \in \underline{\rho}$. Thus $\underline{\rho}$ is an equivalence relation on A. Now, let $a\underline{\rho}b$ and $s \in S$. Then $a\rho b$ and $b\rho a$. Since ρ is a weak pseudoorder on A, by condition (4) of Definition 3.1, we have $a * s \ \widetilde{\rho} \ b * s$. Thus, for every $x \in a * s$, there exists $y \in b * s$ such that $x\rho y$ and $y\rho x$ which imply that $x\underline{\rho}y$, and for every $y' \in b * s$, there exists $x' \in a * s$ such that $x'\rho y'$ and $y'\rho x'$ which mean that $x'\underline{\rho}y'$. It thus follows that $a * s \ \overrightarrow{\rho} \ b * s$ and $a * s \ \overleftarrow{\rho} \ b * s$. Hence $a * s \ \overline{\rho} \ b * s$. Therefore, $\underline{\rho}$ is indeed a congruence on A. By Lemma 2.4, $A/\underline{\rho}$ is a hyper S-act over S with respect to the following hyper S-action:

$$(\forall (a)_{\underline{\rho}} \in A/\underline{\rho}, \forall s \in S) \ (a)_{\underline{\rho}} \otimes s = \{(x)_{\underline{\rho}} \mid x \in a * s\}.$$

Now, we define a relation $\preceq_\rho$ on $A/\underline{\rho}$ as follows:

$$\preceq_\rho := \{((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in A/\underline{\rho} \times A/\underline{\rho} \mid (x, y) \in \rho\}.$$

Then $(A/\underline{\rho}, \preceq_\rho)$ is a poset. Indeed, suppose that $(x)_{\underline{\rho}} \in A/\underline{\rho}$, where $x \in A$. Then $(x, x) \in \leq_A \subseteq \rho$. Hence, $(x)_{\underline{\rho}} \preceq_\rho (x)_{\underline{\rho}}$. Let $(x)_{\underline{\rho}} \preceq_\rho (y)_{\underline{\rho}}$ and $(y)_{\underline{\rho}} \preceq_\rho (x)_{\underline{\rho}}$. Then $x\rho y$ and $y\rho x$. Thus $x\underline{\rho}y$, and we have $(x)_{\underline{\rho}} = (y)_{\underline{\rho}}$. Now, if $(x)_{\underline{\rho}} \preceq_\rho (y)_{\underline{\rho}}$ and $(y)_{\underline{\rho}} \preceq_\rho (z)_{\underline{\rho}}$, then $x\rho y$ and $y\rho z$. Hence, by hypothesis, $x\rho z$, and we conclude that $(x)_{\underline{\rho}} \preceq_\rho (z)_{\underline{\rho}}$.

Furthermore, let $(x)_{\underline{\rho}}, (y)_{\underline{\rho}} \in A/\underline{\rho}$, $(x)_{\underline{\rho}} \preceq_\rho (y)_{\underline{\rho}}$ and $s \in S$. Then $x\rho y$. Since ρ is a weak pseudoorder on A, by condition (3) of Definition 3.1, we have $x * s \ \overrightarrow{\rho} \ y * s$. Thus, for any $a \in x * s$ there exists $b \in y * s$ such that $a\rho b$. This implies that $(a)_{\underline{\rho}} \preceq_\rho (b)_{\underline{\rho}}$. Hence we have

$$(x)_{\underline{\rho}} \otimes s = \{(a)_{\underline{\rho}} \mid a \in x * s\} \preceq_\rho \{(b)_{\underline{\rho}} \mid b \in y * s\} = (y)_{\underline{\rho}} \otimes s.$$

Moreover, let $s, t \in S, s \leq t$ and $(a)_{\underline{\rho}} \in A/\underline{\rho}$. Then, similar to discuss as above, we have $(a)_{\underline{\rho}} \otimes s \preceq_\rho (a)_{\underline{\rho}} \otimes t$. Therefore, $A/\underline{\rho}$ is a hyper S-poset over S. □

Furthermore, we have the following proposition:

Proposition 3.4. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$ and ρ a weak pseudoorder on A. Then $\underline{\rho}$ is an order-congruence on A, where $\underline{\rho} = \rho \cap \rho^{-1}$.*

Proof. By Theorem 3.3, $(A/\underline{\rho}, \preceq_\rho)$ is a hyper S-poset over S, where the order relation $\preceq_\rho$ is defined as follows:

$$\preceq_\rho := \{((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in A/\underline{\rho} \times A/\underline{\rho} \mid (x, y) \in \rho\}.$$

Also, let $x, y \in A$ and $x \leq_A y$. Then, since ρ is a weak pseudoorder on A, $(x, y) \in \leq_A \subseteq \rho$, and thus $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \preceq_\rho$, i.e., $(x)_{\underline{\rho}} \preceq_\rho (y)_{\underline{\rho}}$. Therefore, $\underline{\rho}$ is an order-congruence on A. □

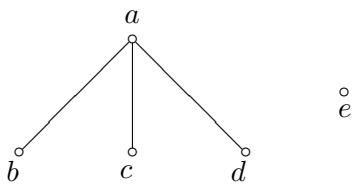
Example 3.5. We consider a set $S := \{a, b, c, d, e\}$ with the following hyperoperation " $\circ$ " and the order " $\leq$ ":

$\circ$	a	b	c	d	e
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$
c	$\{a, c\}$	$\{a, c\}$	$\{a, c\}$	$\{a, c\}$	$\{a, c\}$
d	$\{a, d\}$	$\{a, d\}$	$\{a, d\}$	$\{a, d\}$	$\{a, d\}$
e	$\{e\}$	$\{e\}$	$\{e\}$	$\{e\}$	$\{e\}$

$$\leq := \{(a, a), (b, b), (b, a), (c, c), (c, a), (d, a), (d, d), (e, e)\}.$$

The covering relation "$\prec$" and the figure of S are given by:

$$\prec = \{(b, a), (c, a), (d, a)\}.$$

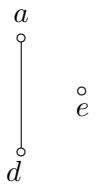


Then $(S, \circ, \leq)$ is an ordered semihypergroup. We now consider the partially ordered set $A = \{a, d, e\}$ defined by the order below:

$$\leq_A := \{(a, a), (d, d), (e, e), (d, a)\}.$$

We give the covering relation "$\prec_A$" and the figure of A.

$$\prec_A = \{(d, a)\}.$$



Then $(A, \leq_A)$ is a hyper S-poset over S with respect to S-hyperaction on A as above hyperoperation table (see [20]).

Let ρ be a weak pseudoorder on A defined as follows:

$$\rho := \{(a,a),(d,d),(e,e),(a,e),(d,a),(d,e),(e,a)\}.$$

Applying Theorem 3.3, we get

$$\underline{\rho} := \{(a,a),(d,d),(e,e),(a,e),(e,a)\}.$$

Then, by Proposition 3.4, $\underline{\rho}$ is an order-congruence on A. Furthermore, $A/\underline{\rho} = \{w_1, w_2\}$, where $w_1 = \{a, e\}, w_2 = \{d\}$. Moreover, $(A/\underline{\rho}, \preceq_\rho)$ is a hyper S-poset over S with the hyper S-action " $\otimes$ " and the order " $\preceq_\rho$ " are given below:

$\otimes$	a	b	c	d	e
w_1	$\{w_1\}$	$\{w_1\}$	$\{w_1\}$	$\{w_1\}$	$\{w_1\}$
w_2	$\{w_1, w_2\}$	$\{w_1, w_2\}$	$\{w_1, w_2\}$	$\{w_1, w_2\}$	$\{w_1, w_2\}$

$$\preceq_\rho := \{(w_1,w_1),(w_2,w_1),(w_2,w_2)\}.$$

Theorem 3.6. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$, ρ, θ be weak pseudoorders on A such that $\rho \subseteq \theta$. We define a relation θ/ρ on $A/\underline{\rho}$ as follows:*

$$\theta/\rho := \{((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in A/\underline{\rho} \times A/\underline{\rho} \mid (x,y) \in \theta\}.$$

Then θ/ρ is a weak pseudoorder on $A/\underline{\rho}$.

Proof. For the simplicity's sake, let

$$\theta' := \{((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in A/\underline{\rho} \times A/\underline{\rho} \mid (x,y) \in \theta\}.$$

We claim that θ' is a weak pseudoorder on $A/\underline{\rho}$. To prove our claim, let $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \preceq_\rho$. Then, by Theorem 3.3, $(x,y) \in \rho \subseteq \theta$, which implies that $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \theta'$. Thus, $\preceq_\rho \subseteq \theta'$. Now, assume that $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \theta'$ and $((y)_{\underline{\rho}}, (z)_{\underline{\rho}}) \in \theta'$. Then, $(x,y) \in \theta$ and $(y,z) \in \theta$. It implies that $(x,z) \in \theta$. Hence, $((x)_{\underline{\rho}}, (z)_{\underline{\rho}}) \in \theta'$. Also, let $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \theta'$ and $s \in S$. Then, $(x,y) \in \theta$ and $s \in S$. Since θ is a weak pseudoorder on A, we have $x * s \overrightarrow{\theta} y * s$. Thus, for every $a \in x * s$, there exists $b \in y * s$ such that $(a,b) \in \theta$. This implies that $((a)_{\underline{\rho}}, (b)_{\underline{\rho}}) \in \theta'$, and thus we have

$$(x)_{\underline{\rho}} \otimes s = \{(a)_{\underline{\rho}} \mid a \in x * s\} \overrightarrow{\theta'} \{(b)_{\underline{\rho}} \mid b \in y * s\} (y)_{\underline{\rho}} \otimes s.$$

Furthermore, let $s \in S$ and $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in A/\underline{\rho}$ be such that $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \theta'$ and $((y)_{\underline{\rho}}, (x)_{\underline{\rho}}) \in \theta'$. Then $(x,y) \in \theta, (y,x) \in \theta$ and $s \in S$. By hypothesis, we have $x * s \widetilde{\theta} y * s$. Then, similar to the above proof, it can be obtained that $(x)_{\underline{\rho}} \otimes s \widetilde{\theta'} (y)_{\underline{\rho}} \otimes s$. Therefore, $\theta'(= \theta/\rho)$ is indeed a weak pseudoorder on $A/\underline{\rho}$. □

4. Homomorphism theorems of hyper S-posets

In the current section, we establish the relationships between order-congruences and weak pseudoorders on a hyper S-poset, and discuss homomorphism theorems of hyper S-posets by weak pseudoorders.

In order to establish the relationships between order-congruences and weak pseudoorders on a hyper S-poset, the following lemma is essential.

Lemma 4.1. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$ and σ a relation on A. Then the following statements are equivalent:*

(1) *σ is a weak pseudoorder on A.*

(2) *There exist a hyper S-poset $(B, \leq_B)$ over S and a homomorphism $\varphi : A \to B$ such that*

$$\overrightarrow{ker}\varphi := \{(a, b) \in A \times A \mid \varphi(a) \leq_B \varphi(b)\} = \sigma.$$

Proof. (1) $\Rightarrow$ (2). Let σ be a weak pseudoorder on A. We denote by $\underline{\sigma}$ the congruence on A defined by

$$\underline{\sigma} := \{(a, b) \in A \times A \mid (a, b) \in \sigma, (b, a) \in \sigma\} (= \sigma \cap \sigma^{-1}).$$

Then, by Theorem 3.3, the set $A/\underline{\sigma} := \{(a)_{\underline{\sigma}} \mid a \in A\}$ with the S-hyperaction $(a)_{\underline{\sigma}} \otimes s = \{(x)_{\underline{\sigma}} \mid x \in a * s\}$ for all $a \in A, s \in S$ and the order

$$\preceq_\sigma := \{((x)_{\underline{\sigma}}, (y)_{\underline{\sigma}}) \in A/\underline{\sigma} \times A/\underline{\sigma} \mid (x, y) \in \sigma\}$$

is a hyper S-poset. Let $B = (A/\underline{\sigma}, \preceq_\sigma)$ and φ be the mapping of A onto $A/\underline{\sigma}$ defined by $\varphi : A \to A/\underline{\sigma} \mid a \mapsto (a)_{\underline{\sigma}}$. Then, by Proposition 3.4, φ is a homomorphism from A onto $A/\underline{\sigma}$ and clearly, $\overrightarrow{ker}\varphi = \sigma$.

(2) $\Rightarrow$ (1). Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$. If there exist a hyper S-poset $(B, \leq_B)$ over S and a homomorphism $\varphi : A \to B$ such that $\overrightarrow{ker}\varphi = \sigma$, then σ is a weak pseudoorder on A. Indeed, let $(a, b) \in \leq_A$. Then, by hypothesis, $\varphi(a) \leq_B \varphi(b)$. Thus $(a, b) \in \overrightarrow{ker}\varphi = \sigma$, and we have $\leq_A \subseteq \sigma$. Moreover, let $(a, b) \in \sigma$ and $(b, c) \in \sigma$. Then $\varphi(a) \leq_B \varphi(b) \leq_B \varphi(c)$. Hence $\varphi(a) \leq_B \varphi(c)$, i.e., $(a, c) \in \overrightarrow{ker}\varphi = \sigma$. Also, if $(a, b) \in \sigma$, then $\varphi(a) \leq_B \varphi(b)$. Since $(B, \leq_B)$ is a hyper S-poset over S, for any $s \in S$ we have $\varphi(a) \diamond s \leq_B \varphi(b) \diamond s$, where " $\diamond$ " is the S-hyperaction on B. Since φ is a homomorphism from A to B, we have

$$\{\varphi(x) \mid x \in a * s\} = \varphi(a) \diamond s \leq_B \varphi(b) \diamond s = \{\varphi(y) \mid y \in b * s\}.$$

Thus, for every $x \in a*s$ there exists $y \in b*s$ such that $\varphi(x) \leq_B \varphi(y)$, and we have $(x, y) \in \overrightarrow{ker}\varphi = \sigma$, and thus $a * s \ \overrightarrow{\sigma}\ b * s$. Furthermore, let $(a, b) \in \sigma, (b, a) \in \sigma$ and $s \in S$. Then $\varphi(a) \leq_B \varphi(b)$ and $\varphi(b) \leq_B \varphi(a)$. Thus $\varphi(a) = \varphi(b)$, which implies that $\varphi(a) \diamond s = \varphi(b) \diamond s$, i.e., $\{\varphi(x) \mid x \in a * s\} = \{\varphi(y) \mid y \in b * s\}$. Hence, for any $x \in a * s$, there exists $y \in b * s$ such that $\varphi(x) = \varphi(y)$, and

we have $\varphi(x) \leq_B \varphi(y)$ and $\varphi(y) \leq_B \varphi(x)$. It thus follows that $(x, y) \in \sigma$ and $(y, x) \in \sigma$. On the other hand, for any $y' \in b * s$, there exists $x' \in a * s$ such that $\varphi(x') = \varphi(y')$. Hence we have $\varphi(x') \leq_B \varphi(y')$ and $\varphi(y') \leq_B \varphi(x')$, which imply that $(x', y') \in \sigma$ and $(y', x') \in \sigma$. Thus, $a * s \ \widetilde{\sigma} \ b * s$, exactly as required. Therefore, σ is a weak pseudoorder on A. □

In the following, we give a characterization of order-congruences in terms of weak pseudoorders.

Theorem 4.2. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$ and $\rho \in C(A)$. Then the following statements are equivalent:*

(1) *ρ is an order-congruence on A.*

(2) *There exists a weak pseudoorder σ on A such that $\rho = \sigma \cap \sigma^{-1}$.*

(3) *There exist a hyper S-poset B over S and a homomorphism $\varphi : A \to B$ such that $\rho = ker(\varphi)$, where $ker\varphi = \{(a, b) \in A \times A \mid \varphi(a) = \varphi(b)\}$ is the kernel of φ.*

Proof. (1) ⇒ (2). Let ρ be an order-congruence on A. Then there exist an order relation "$\preceq$" on the factor hyper S-act A/ρ such that $(A/\rho, \preceq)$ is a hyper S-poset over S, and $\varphi : A \to A/\rho$ is a homomorphism. Let $\sigma = \overrightarrow{ker\varphi}$. By Lemma 4.1, σ is a weak pseudoorder on A and it is easy to check that $\rho = \sigma \cap \sigma^{-1}$.

(2) ⇒ (3). For a weak pseudoorder σ on A, by Lemma 4.1, there exist a hyper S-poset B over S and a homomorphism $\varphi : A \to B$ such that $\sigma = \overrightarrow{ker\varphi}$. Then, we have

$$ker\varphi = \overrightarrow{ker\varphi} \cap (\overrightarrow{ker\varphi})^{-1} = \sigma \cap \sigma^{-1} = \rho.$$

(3) ⇒ (1). By hypothesis and Lemma 4.1, $\overrightarrow{ker\varphi}$ is a weak pseudoorder on A. Then, by Theorem 3.3, $\rho = \overrightarrow{ker\varphi} \cap (\overrightarrow{ker\varphi})^{-1}$ is a congruence on A. Thus, by the proof of Lemma 4.1, ρ is an order-congruence on A. □

Remark 4.3. (1) For an order-congruence ρ on A, since the order "$\preceq$" such that $(A/\rho, \preceq)$ is a hyper S-poset is not unique in general, we have the weak pseudoorder σ containing ρ such that $\rho = \sigma \cap \sigma^{-1}$ is not unique.

(2) If σ is a weak pseudoorder on a hyper S-poset A, then $\rho = \sigma \cap \sigma^{-1}$ is the greatest order-congruence on A contained in σ. In fact, if ρ_1 is an order-congruence on A contained in σ, then $\rho_1 = \rho_1 \cap \rho_1^{-1} \subseteq \sigma \cap \sigma^{-1} = \rho$.

Let σ be a weak pseudoorder on a hyper S-poset $(A, \leq_A)$. Then, by Theorem 4.2, $\rho = \sigma \cap \sigma^{-1}$ is an order-congruence on A. We denote by $\rho^\sharp$ the canonical epimorphism from A onto A/ρ, i.e., $\rho^\sharp : A \to A/\rho \mid x \mapsto (x)_\rho$, which is a homomorphism. In the following, we give out a homomorphism theorem of hyper S-posets by weak pseudoorders, which is a generalization of Theorem 12 in [24]. For a similar result about ordered semigroups we refer the readers to Theorem 1 in [10].

Theorem 4.4. *Let $(A, \leq_A)$ and $(B, \leq_B)$ be two hyper S-posets over an ordered semihypergroup $(S, \circ, \leq)$, $\varphi : A \to B$ a homomorphism. Then, if σ is a weak pseudoorder on A such that $\sigma \subseteq \overrightarrow{ker}\varphi$, then there exists the unique homomorphism $f : A/\rho \to B \mid (a)_\rho \mapsto \varphi(a)$ such that the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle \rho^\sharp}\downarrow & \nearrow_{f} & \\ A/\rho & & \end{array}$$

commutes, where $\rho = \sigma \cap \sigma^{-1}$. Moreover, $Im(\varphi) = Im(f)$. Conversely, if σ is a weak pseudoorder on A for which there exists a homomorphism $f : (A/\rho, \preceq_\sigma) \to (B, \leq_B)$ $(\rho = \sigma \cap \sigma^{-1})$ such that the above diagram commutes, then $\sigma \subseteq \overrightarrow{ker}\varphi$.

Proof. Let σ be a weak pseudoorder on A such that $\sigma \subseteq \overrightarrow{ker}\varphi$, $f : A/\rho \to B \mid (a)_\rho \mapsto \varphi(a)$. Then

(1) f is well defined. Indeed, if $(a)_\rho = (b)_\rho$, then $(a, b) \in \rho \subseteq \sigma$. Since $\sigma \subseteq \overrightarrow{ker}\varphi$, we have $(\varphi(a), \varphi(b)) \in \leq_B$. Furthermore, since $(b, a) \in \sigma \subseteq \overrightarrow{ker}\varphi$, we have $(\varphi(b), \varphi(a)) \in \leq_B$. Therefore, $\varphi(a) = \varphi(b)$.

(2) f is a homomorphism and $\varphi = f \circ \rho^\sharp$. In fact, by Lemma 4.1, there exist an order relation "$\preceq_\sigma$" on the factor hyper S-act A/ρ such that $(A/\rho, \preceq_\sigma)$ is a hyper S-poset and the canonical epimorphism $\rho^\sharp$ is a homomorphism. Moreover, we have

$$\begin{array}{rcl} (a)_\rho \preceq_\sigma (b)_\rho & \Rightarrow & (a, b) \in \sigma \subseteq \overrightarrow{ker}\varphi \\ & \Rightarrow & \varphi(a) \leq_B \varphi(b) \\ & \Rightarrow & f((a)_\rho) \leq_B f((b)_\rho). \end{array}$$

Also, let $(a)_\rho \in A/\rho$ and $s \in S$. Since φ is a homomorphism from A to B, we have

$$f((a)_\rho) \diamond s = \varphi(a) \diamond s = \{\varphi(x) \mid x \in a * s\} = \{f((x)_\rho) \mid (x)_\rho \in (a)_\rho \otimes s\},$$

where "$\diamond$" is the S-hyperaction on B. Furthermore, for any $a \in A$, $(f \circ \rho^\sharp)(a) = f((a)_\rho) = \varphi(a)$, and thus $\varphi = f \circ \rho^\sharp$.

We claim that f is the unique homomorphism from A/ρ to B. To prove our claim, let g is a homomorphism from A/ρ to B such that $\varphi = g \circ \rho^\sharp$. Then, for any $(a)_\rho \in A/\rho$, we have

$$f((a)_\rho) = \varphi(a) = (g \circ \rho^\sharp)(a) = g((a)_\rho).$$

Moreover, $Im(f) = \{f((a)_\rho) \mid a \in A\} = \{\varphi(a) \mid a \in A\} = Im(\varphi)$.

Conversely, let σ be a weak pseudoorder on A, $f : A/\rho \to B$ is a homomorphism and $\varphi = f \circ \rho^\sharp$. Then $\sigma \subseteq \overrightarrow{ker}\varphi$. Indeed, by hypothesis, we have

$$\begin{array}{rcl} (a, b) \in \sigma & \Leftrightarrow & (a)_\rho \preceq_\sigma (b)_\rho \Rightarrow f((a)_\rho) \leq_B f((b)_\rho) \\ & \Rightarrow & (f \circ \rho^\sharp)(a) \leq_B (f \circ \rho^\sharp)(b) \\ & \Rightarrow & \varphi(a) \leq_B \varphi(b) \Rightarrow (a, b) \in \overrightarrow{ker}\varphi, \end{array}$$

where the order $\preceq_\sigma$ on A/ρ is defined as in the proof of Lemma 4.1, that is $\preceq_\sigma := \{((x)_\rho, (y)_\rho) \in A/\rho \times A/\rho \mid (x,y) \in \sigma\}$. □

Corollary 4.5. *Let $(A, \leq_A)$ and $(B, \leq_B)$ be two hyper S-posets over an ordered semihypergroup $(S, \circ, \leq)$ and $\varphi : A \to B$ a homomorphism. Then $A/ker\varphi \cong Im(\varphi)$, where $ker\varphi$ is the kernel of φ.*

Proof. Let $\sigma = \overrightarrow{ker}\varphi$ and $\rho = \overrightarrow{ker}\varphi \cap (\overrightarrow{ker}\varphi)^{-1}$. Then, by Theorems 4.2 and 4.4, ρ is an order-congruence on A and $f : A/\rho \to B \mid (a)_\rho \mapsto \varphi(a)$ is a homomorphism. Moreover, f is inverse isotone. In fact, let $(a)_\rho, (b)_\rho$ be two elements of A/ρ such that $f((a)_\rho) \leq_B f((b)_\rho)$. Then $\varphi(a) \leq_B \varphi(b)$, and we have $(a,b) \in \overrightarrow{ker}\varphi$. Thus, by Lemma 4.1, $((a)_\rho, (b)_\rho) \in \preceq_\sigma$, i.e., $(a)_\rho \preceq_\sigma (b)_\rho$. Clearly, $\rho = ker\varphi$. By Remark 2.7, $A/ker\varphi \cong Im(f)$. Also, by Theorem 4.4, $Im(f) = Im(\varphi)$. Therefore, $A/ker\varphi \cong Im(\varphi)$. □

Remark 4.6. Note that if $(A, \leq_A)$ and $(B, \leq_B)$ are both S-posets, then Corollary 4.5 coincides with Corollary 13 in [24].

Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$, ρ and θ weak pseudoorders on A and $\rho \subseteq \theta$. We define a relation on the hyper S-poset $(A/\underline{\rho}, \preceq_\rho)$ denoted by θ/ρ as follows:

$$\theta/\rho := \{((a)_{\underline{\rho}}, (b)_{\underline{\rho}}) \in A/\underline{\rho} \times A/\underline{\rho} \mid (a,b) \in \theta\},$$

where $\preceq_\rho := \{((a)_{\underline{\rho}}, (b)_{\underline{\rho}}) \mid (a,b) \in \rho\}, \underline{\rho} = \rho \cap \rho^{-1}$. By Theorem 3.6, θ/ρ is a weak pseudoorder on $(A/\underline{\rho}, \preceq_\rho)$.

Theorem 4.7. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$, ρ and θ weak pseudoorders on A and $\rho \subseteq \theta$. Then $(A/\underline{\rho})/\underline{\theta/\rho} \cong A/\underline{\theta}$.*

Proof. Since θ/ρ is a weak pseudoorder on $A/\underline{\rho}$, it can be obtained that the mapping $\varphi : A/\underline{\rho} \to A/\underline{\theta} \mid (a)_{\underline{\rho}} \mapsto (a)_{\underline{\theta}}$ is a homomorphism. In fact:

(1) φ is well-defined. Indeed, let $(a)_{\underline{\rho}} = (b)_{\underline{\rho}}$. Then $(a,b) \in \underline{\rho}$. Thus, by the definition of $\underline{\rho}$, $(a,b) \in \rho \subseteq \theta$ and $(b,a) \in \rho \subseteq \theta$. This implies that $(a,b) \in \underline{\theta}$, and thus $(a)_{\underline{\theta}} = (b)_{\underline{\theta}}$.

(2) φ is a homomorphism. In fact, let $(a)_{\underline{\rho}} \in A/\underline{\rho}$ and $s \in S$. Then, since $\underline{\rho}, \underline{\theta} \in C(A)$, we have

$$(a)_{\underline{\rho}} \otimes_\rho s = \{(x)_{\underline{\rho}} \mid x \in a * s\}, (a)_{\underline{\theta}} \otimes_\theta s = \{(x)_{\underline{\theta}} \mid x \in a * s\},$$

where " $\otimes_\rho$ " and " $\otimes_\theta$ " are the S-hyperaction on $A/\underline{\rho}$ and $A/\underline{\theta}$, respectively. Thus,

$$\varphi((a)_{\underline{\rho}}) \otimes_\theta s = (a)_{\underline{\theta}} \otimes_\theta s = \{(x)_{\underline{\theta}} \mid x \in a * s\} = \{\varphi((x)_{\underline{\rho}}) \mid (x)_{\underline{\rho}} \in (a)_{\underline{\rho}} \otimes_\rho s\}.$$

Also, if $(a)_{\underline{\rho}} \preceq_\rho (b)_{\underline{\rho}}$, then $(a,b) \in \rho \subseteq \theta$. It implies that $(a)_{\underline{\theta}} \preceq_\theta (b)_{\underline{\theta}}$, and thus φ is isotone.

On the other hand, it is easy to see that φ is onto, since

$$Im(\varphi) = \{\varphi((a)_{\underline{\rho}}) \mid a \in A\} = \{(a)_{\underline{\theta}} \mid a \in A\} = A/\underline{\theta}.$$

It thus follows from Corollary 4.5 that $A/\underline{\rho}/Ker\varphi \cong Im(\varphi) = A/\underline{\theta}$.

Furthermore, let $\overrightarrow{ker}\varphi := \{((a)_{\underline{\rho}}, (b)_{\underline{\rho}}) \mid \varphi((a)_{\underline{\rho}}) \preceq_\theta \varphi((b)_{\underline{\rho}})\}$. Then

$$\begin{aligned} ((a)_{\underline{\rho}}, (b)_{\underline{\rho}}) \in \overrightarrow{ker}\varphi &\iff (a)_{\underline{\theta}} \preceq_\theta (b)_{\underline{\theta}} \iff (a,b) \in \theta \\ &\iff ((a)_{\underline{\rho}}, (b)_{\underline{\rho}}) \in \theta/\rho. \end{aligned}$$

Therefore, $Ker\varphi = \overrightarrow{ker}\varphi \cap (\overrightarrow{ker}\varphi)^{-1} = (\theta/\rho) \cap (\theta/\rho)^{-1} = \underline{\theta/\rho}$. We have thus shown that $(A/\underline{\rho})/\underline{\theta/\rho} \cong A/\underline{\theta}$. □

Definition 4.8. Let $(A, \leq_A)$ and $(B, \leq_B)$ be two hyper S-posets over an ordered semihypergroup $(S, \circ, \leq)$, ρ, θ be two weak pseudoorders on A and B, respectively, and the mapping $f : A \to B$ a homomorphism. Then, f is called a (ρ, θ)-*homomorphism* if $(x, y) \in \rho$ implies $(f(x), f(y)) \in \theta$, for all $x, y \in A$.

Lemma 4.9. *Let $(A, \leq_A)$ and $(B, \leq_B)$ be two hyper S-posets over an ordered semihypergroup $(S, \circ, \leq)$, ρ, θ be two weak pseudoorders on A and B, respectively, and the mapping $f : A \to B$ a (ρ, θ)-homomorphism. Then, the mapping $\overline{f} : (A/\underline{\rho}, \preceq_\rho) \to (B/\underline{\theta}, \preceq_\theta)$ defined by*

$$(\forall x \in A)\ \overline{f}((x)_{\underline{\rho}}) := (f(x))_{\underline{\theta}}$$

is a homomorphism of hyper S-posets, where the orders $\preceq_\rho$, $\preceq_\theta$ on $A/\underline{\rho}$ and $B/\underline{\theta}$, respectively, are both defined as in the proof of Lemma 4.1.

Proof. Let $f : A \to B$ be a (ρ, θ)-homomorphism and $\overline{f} : A/\underline{\rho} \to B/\underline{\theta} \mid (x)_{\underline{\rho}} \mapsto (f(x))_{\underline{\theta}}$. Then

(1) $\overline{f}$ is well defined. In fact, let $(x)_{\underline{\rho}}, (y)_{\underline{\rho}} \in A/\underline{\rho}$ be such that $(x)_{\underline{\rho}} = (y)_{\underline{\rho}}$. Then $(x, y) \in \underline{\rho} \subseteq \rho$. Since f is a (ρ, θ)-homomorphism, we have $(f(x), f(y)) \in \theta$. It implies that $((f(x))_{\underline{\theta}}, (f(y))_{\underline{\theta}}) \in \preceq_\theta$. Similarly, since $(y, x) \in \rho$, we have $((f(y))_{\underline{\theta}}, (f(x))_{\underline{\theta}}) \in \preceq_\theta$. Therefore, $((f(x))_{\underline{\theta}} = (f(y))_{\underline{\theta}}$, i.e., $\overline{f}((x)_{\underline{\rho}}) = \overline{f}((y)_{\underline{\rho}})$.

(2) $\overline{f}$ is a homomorphism. Indeed, let $(x)_{\underline{\rho}} \in A/\underline{\rho}$ and $s \in S$. By Theorem 4.2, $\underline{\rho}, \underline{\theta}$ are order-congruences on A and B, respectively. Since f is a homomorphism, by Lemma 2.4 we have

$$\overline{f}((x)_{\underline{\rho}}) \otimes_\theta s = (f(x))_{\underline{\theta}} \otimes_\theta s = \{(f(a))_{\underline{\theta}} \mid a \in x * s\} = \{\overline{f}((a)_{\underline{\rho}}) \mid (a)_{\underline{\rho}} \in (x)_{\underline{\rho}} \otimes_\rho s\}.$$

Also, since f is a (ρ, θ)-homomorphism, we have

$$\begin{aligned} (x)_{\underline{\rho}} \preceq_\rho (y)_{\underline{\rho}} &\Rightarrow (x, y) \in \rho \Rightarrow (f(x), f(y)) \in \theta \\ &\Rightarrow (f(x))_{\underline{\theta}} \preceq_\theta (f(y))_{\underline{\theta}} \Rightarrow \overline{f}((x)_{\underline{\rho}}) \preceq_\theta \overline{f}((y)_{\underline{\rho}}). \end{aligned}$$

Hence $\overline{f}$ is isotone. We have thus shown that $\overline{f}$ is a homomorphism, as required. □

Lemma 4.10. *Let $(A, \leq_A)$ and $(B, \leq_B)$ be two hyper S-posets over an ordered semihypergroup $(S, \circ, \leq)$, ρ, θ be two weak pseudoorders on A and B, respectively, and the mapping $f : A \to B$ a (ρ, θ)-homomorphism. We define a relation on the hyper S-poset $(A/\underline{\rho}, \preceq_\rho)$ denoted by ρ_f as follows:*

$$\rho_f := \{((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in A/\underline{\rho} \times A/\underline{\rho} \mid (f(x))_{\underline{\theta}} \preceq_\theta (f(y))_{\underline{\theta}}\}.$$

Then ρ_f is a weak pseudoorder on $A/\underline{\rho}$.

Proof. Assume that $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \preceq_\rho$. By Lemma 4.9, $\overline{f}$ is a homomorphism. Then $\overline{f}((x)_{\underline{\rho}}) \preceq_\theta \overline{f}((y)_{\underline{\rho}})$, i.e., $(f(x))_{\underline{\theta}} \preceq_\theta (f(y))_{\underline{\theta}}$. It implies that $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \rho_f$, and thus $\preceq_\rho \subseteq \rho_f$. Now, let $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \rho_f$ and $((y)_{\underline{\rho}}, (z)_{\underline{\rho}}) \in \rho_f$. Then $(f(x))_{\underline{\theta}} \preceq_\theta (f(y))_{\underline{\theta}}$ and $(f(y))_{\underline{\theta}} \preceq_\theta (f(z))_{\underline{\theta}}$. Thus, by the transitivity of $\preceq_\theta$, $(f(x))_{\underline{\theta}} \preceq_\theta (f(z))_{\underline{\theta}}$. This implies that $((x)_{\underline{\rho}}, (z)_{\underline{\rho}}) \in \rho_f$. Moreover, let $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \rho_f$ and $s \in S$. Then $(f(x))_{\underline{\theta}} \preceq_\theta (f(y))_{\underline{\theta}}$. Since $(B/\underline{\theta}, \preceq_\theta)$ is a hyper S-poset over S, it can be obtained that $(f(x))_{\underline{\theta}} \otimes_\theta s \preceq_\theta (f(y))_{\underline{\theta}} \otimes_\theta s$, that is, $\overline{f}((x)_{\underline{\rho}}) \otimes_\theta s \preceq_\theta \overline{f}((y)_{\underline{\rho}}) \otimes_\theta s$. Then, since $\overline{f}$ is a homomorphism, we have

$$\{\overline{f}((a)_{\underline{\rho}}) \mid (a)_{\underline{\rho}} \in (x)_{\underline{\rho}} \otimes_\rho s\} \preceq_\theta \{\overline{f}((b)_{\underline{\rho}}) \mid (b)_{\underline{\rho}} \in (y)_{\underline{\rho}} \otimes_\rho s\},$$

which means that $\{(f(a))_{\underline{\theta}} \mid a \in x * s\} \preceq_\theta \{(f(b))_{\underline{\theta}} \mid b \in y * s\}$. Thus, for any $a \in x * s$, there exists $b \in y * s$ such that $(f(x))_{\underline{\theta}} \preceq_\theta (f(y))_{\underline{\theta}}$. Hence $((a)_{\underline{\rho}}, (b)_{\underline{\rho}}) \in \rho_f$. It thus implies that $(x)_{\underline{\rho}} \otimes_\rho s \ \overrightarrow{\rho}_f \ (y)_{\underline{\rho}} \otimes_\rho s$. Furthermore, let $(x)_{\underline{\rho}}, (y)_{\underline{\rho}} \in S/\underline{\rho}$ and $s \in S$ be such that $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \rho_f$ and $((y)_{\underline{\rho}}, (x)_{\underline{\rho}}) \in \rho_f$. Then, similar to the above proof, it can be verified that $(x)_{\underline{\rho}} \otimes_\rho s \ \widetilde{\rho}_f \ (y)_{\underline{\rho}} \otimes_\rho s$. Therefore, ρ_f is a weak pseudoorder on $A/\underline{\rho}$. □

By Lemmas 4.9 and 4.10, we immediately obtain the following corollary:

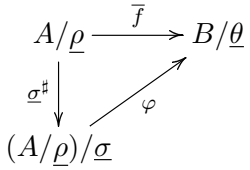
Corollary 4.11. *Let $(A, \leq_A)$, $(B, \leq_B)$ be two hyper S-posets over an ordered semihypergroup $(S, \circ, \leq)$, ρ, θ be two weak pseudoorders on A and B, respectively, and the mapping $f : A \to B$ a (ρ, θ)-homomorphism. Then, the following diagram*

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ {\scriptstyle \underline{\rho}^\sharp}\downarrow & & \downarrow{\scriptstyle \underline{\theta}^\sharp} \\ A/\underline{\rho} & \xrightarrow[\overline{f}]{} & B/\underline{\theta} \end{array}$$

commutates.

Theorem 4.12. *Let $(A, \leq_A)$, $(B, \leq_B)$ be two hyper S-posets over an ordered semihypergroup $(S, \circ, \leq)$, ρ, θ be two weak pseudoorders on A and B, respectively, and the mapping $f : A \to B$ a (ρ, θ)-homomorphism. If σ is a weak pseudoorder on $A/\underline{\rho}$ such that $\sigma \subseteq \rho_f$, then there exists the unique homomorphism*

$\varphi : (A/\underline{\rho})/\underline{\sigma} \to B/\underline{\theta} \mid ((a)_{\underline{\rho}})_{\underline{\sigma}} \mapsto \underline{f}((a)_{\underline{\rho}})$ *such that the diagram*



commutes.

Conversely, if σ is a weak pseudoorder on $A/\underline{\rho}$ for which there exists a homomorphism $\varphi : (A/\underline{\rho})/\underline{\sigma} \to B/\underline{\theta}$ such that the above diagram commutes, then $\sigma \subseteq \rho_f$.

Proof. The proof is straightforward by Lemmas 4.9, 4.10 and Theorem 4.4, and we omit the details. □

5. Conclusions

In [20], Tang et al. defined and studied the hyper S-posets over an ordered semihypergroup, and introduced the concepts of pseudoorders and (strong) order-congruences on a hyper S-poset A. In particular, they proved that there must exist a strong congruence ρ on a hyper S-poset A such that A/ρ is a hyper S-poset (see Theorem 3.7 in [20]). A natural question about hyper S-posets is raised: Is there a congruence ρ on a hyper S-poset A for which A/ρ is a hyper S-poset? In this paper we studied the weak pseudoorders in a hyper S-poset in detail and answered the above question. To begin with, the concept of weak pseudoorders in a hyper S-poset was introduced, and several related properties was given. Especially, we constructed a hyper S-poset $A/\underline{\rho}$ by a weak pseudoorder ρ.. Furthermore, we provided some homomorphism theorems of hyper S-posets by weak pseudoorders, which are generalizations of similar results in S-posets and ordered semigroups. The foundations that we made through this paper can offer foundation for further study of the hyper S-poset theory.

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