Vague soft separation axioms and vague soft continuous functions in vague soft topological spaces

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Abstract. In this present study, we introduce the notions of vague soft points, vague soft neighborhoods and vague soft functions which pave a way to define vague soft separation axioms in vague soft topological spaces. This research work brings out the following interesting results in vague soft set theory, which are contrary to their respective basic results in set theory and the interrelation between separation axioms in topological spaces.

1. The intersection of a vague soft set and its complement need not be a null vague soft set.
2. The union of a vague soft set and its complement need not be an absolute (universal) vague soft set.
3. Every vague soft normal space need not be a vague soft regular space.
4. Every vague soft regular space need not be a vague soft Hausdorff space.

Counter examples are given to illustrate the above results. Also, we provided an application in the field of medicine using the concept of vague soft functions.

Keywords: vague soft set, vague soft topology, vague soft point, vague soft neighborhood, vague soft $T_i (i = 0, 1, 2, 3, 4)$ spaces, vague soft continuous functions.

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1. Introduction

The most appropriate theory, for dealing with **uncertainties** is the Theory of Fuzzy sets introduced by Zadeh [22]. A fuzzy set F is described by its membership function μ_F. This membership function μ_F associates a real number $\mu_F(x)$ in the interval $[0, 1]$ for every point $x \in X$. The number $\mu_F(x)$ is interpreted for the point as a degree of belonging x to the fuzzy set F. As a generalization of fuzzy sets, Intuitionistic fuzzy (IF) sets [7] and Vague sets [10] are developed. Instead of using point-base membership as in Fuzzy sets, interval-based membership is used in Vague sets which is more expressive in capturing vagueness of data. Vague sets are regarded as a special case of context-dependent fuzzy sets.

A vague set V is defined by a truth-membership function t_V and a false-membership function f_V, where $t_V(x)$ is a lower bound on the grade of membership of x derived from the evidence for x, and $f_V(x)$ is a lower bound on the negation of x derived from the evidence against x. These true membership function value and false membership function value noted as $t_V(x)$ and $f_V(x)$ are associated as a real number in $[0, 1]$ with each point in a basic set X, which satisfies the condition $0 \leq t_V(x) + f_V(x) \leq 1$. The most interesting feature about Vague set V is the approach of bounds for each grade of membership of an object x to a subinterval $[t_V(x), 1 - f_V(x)]$ of [0,1]. The value of the interval characterizes favor, against and abstentions of an object x which belong to the vague set V, whereas in Intuitionistic fuzzy set A, each object has a representation $< t_A(x), f_A(x) >$ which interprets only favor and against of an object x belong to the IF set A. The way of handling the notion of inclusion, union and complement of vague sets in [10] is realistic in nature. When measuring vagueness in practice, it is proved with example that using a Vague set is more nature than using an IF set.

Recently, several authors Bhargavi and Eswarla (2016)[8], Borumand Saeid and Zarandi (2011)[9], Jothi (2017)[13], Mariapresenti and Arockiarani (2016)[14], Mary Margaret and Arockiarani (2016)[15] extended the study of vague sets and they have given many applications in the field of vague set theory. Molodtsov [16] introduced soft set theory as an effective mathematical tool to deal with uncertainties especially in the respective fields like economics, engineering and environment. By combining the notions of vague sets and soft sets, Xu et al. [21] introduced the notion of vague soft sets and illustrated its potential applications. This combination of Vague soft set theory makes descriptions of the object world more realistic, practical and accurate. Vague soft set was applied by Alhazaymeh and Hassan [2, 3, 6] in decision making and further explored interval-valued vague soft set [4], generalized vague soft expert set [5] and Nautrosophic vague soft set [1]. In due course C. Wang and Li [19] introduced the topological structure of vague soft sets to carry further study. Further, C. Wang and Qu [20] examined several properties of vague soft sets and its applications. Adding to it, C. Wang [18] contributed vague soft sets applications in

decision making problems. Recently C. Wang [17] developed the decomposition theorems and representation theorems of vague soft sets.

In this paper, we the authors define new concepts such as vague soft points, vague soft neighborhoods, vague soft T_i $(i = 0, 1, 2, 3, 4)$ spaces in vague soft topological spaces. This research work brings out the characterization of vague soft continuous functions. Finally, we present an application of vague soft functions in the field of medicine to show it's potential use in general situation. It is important to note that most of the results given in this research paper are very basic and introductory to the theory of vague soft topological spaces. This will open new vistas in this area of research.

2. Preliminaries

Definition 2.1 ([16]). *Let X be an initial universe set, $P(X)$ the set of all subsets of X, E a set of parameters, and $A \subseteq E$. A pair (F, A) is called a soft set over X, where F is a mapping given by F: $A \to P(X)$.*

Definition 2.2 ([10]). *A vague set $A = \{(x_i, [t_A(x_i), 1 - f_A(x_i)]) \mid x_i \in X\}$ in the universe $X = \{x_1, x_2, ..., x_n\}$ is characterized by a truth-membership function $t_A : X \to [0, 1]$, and a false-membership function $f_A : X \to [0, 1]$, where $t_A(x_i)$ is a lower bound on the grade of membership of x_i derived from the evidence of x_i, $f_A(x_i)$ is the lower bound on the negation of x_i derived from the evidence against x_i such that $0 \leq t_A(x_i) + f_A(x_i) \leq 1$ for any $x_i \in X$. The grade of membership of x_i in the vague set is bounded to a subinterval $[\, t_A(x_i), 1-f_A(x_i)\,]$ of $[0, 1]$. The vague value $[t_A(x_i), 1 - f_A(x_i)]$ (simply, $V_A(x_i)$) indicates that the exact grade of membership $\mu_A(x_i)$ of x_i may be unknown, but it is bounded by $t_A(x_i) \leq \mu_A(x_i) \leq 1 - f_A(x_i)$, where $0 \leq t_A(x_i) + f_A(x_i) \leq 1$.*

Definition 2.3 ([10]). *A vague set $A = \{(x, [t_A(x), 1 - f_A(x)]) \mid x \in X\}$ with $t_A(x) = 0$, $1 - f_A(x) = 0$ for all $x \in X$ is called the zero vague set.*

Definition 2.4 ([10]). *A vague set $A = \{(x, [t_A(x), 1 - f_A(x)]) \mid x \in X\}$ with $t_A(x) = 1$, $1 - f_A(x) = 1$ for all $x \in X$ is called the unit vague set.*

Remark 2.1. Throughout this paper, we use the notation $\tilde{\emptyset}$ and $\tilde{X}$ for zero vague set and unit vague set respectively.

Definition 2.5 ([21]). *Let X be an initial universe set, $V(X)$ the set of all vague sets on X, E be a set of parameters, and $A \subseteq E$. A pair (F, A) is called a vague soft set over X, where F is a mapping given by $F : A \to V(X)$. The set of vague soft sets on X is denoted by $V\tilde{S}(X, E)$, called vague soft classes.*

Definition 2.6 ([21]). *Let (F, A) and (G, B) be two vague soft sets over X. If $A \subseteq B$ and for all $e \in A$, $F(e)$ is a vague subset of $G(e)$, then (F, A) is called a vague soft subset of (G, B). This relation is denoted by $(F, A) \subseteq (G, B)$.*

Definition 2.7 ([21]). *Two vague soft sets (F, A) and (G, B) over X are said to be vague soft equal if (F, A) is a vague soft subset of (G, B) and (G, B) is a vague soft subset of (F, A). This relation is denoted by $(F, A) = (G, B)$.*

Definition 2.8 ([21]). *The complement of a vague soft set (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, A)$ and is given by $t_{F^c(e)}(x) = f_{F(e)}(x)$, $1 - f_{F^c(e)}(x) = 1 - t_{F(e)}(x)$, for all $e \in A,\ x \in X$.*

Definition 2.9 ([21]). *A vague soft set (F, A) over X is said to be a null vague soft set denoted by $\hat{\emptyset}$, if $\forall\ e \in A$, $t_{F(e)}(x) = 0$, $1 - f_{F(e)}(x) = 0$, $x \in X$.*

Definition 2.10 ([21]). *A vague soft set (F, A) over X is said to be an absolute (Universal) vague soft set denoted by $\hat{X}$, if $\forall\ e \in A$, $t_{F(e)}(x) = 1$, $1 - f_{F(e)}(x) = 1$, $x \in X$.*

Definition 2.11 ([21]). *The union of two vague soft sets (F, A) and (G, B) over a universe X is a vague soft set (H, C), where $C = A \cup B$ and $\forall\ e \in C$,*

$$t_{H(e)}(x) = \begin{cases} t_{F(e)}(x), & e \in A - B,\ x \in X, \\ t_{G(e)}(x), & e \in B - A,\ x \in X, \\ t_{F(e)}(x) \vee t_{G(e)}(x), & e \in A \cap B,\ x \in X, \end{cases}$$

$$1 - f_{H(e)}(x) = \begin{cases} 1 - f_{F(e)}(x), & e \in A - B,\ x \in X, \\ 1 - f_{G(e)}(x), & e \in B - A,\ x \in X, \\ (1 - f_{F(e)}(x) \vee 1 - f_{G(e)}(x)), & e \in A \cap B,\ x \in X. \end{cases}$$

We denote it by $(F, A) \cup (G, B) = (H, C)$.

Definition 2.12 ([21]). *The intersection of two vague soft sets (F, A) and (G, B) over a universe X is a vague soft set (H, C), where $C = A \cup B$ and $\forall\ e \in C$,*

$$t_{H(e)}(x) = \begin{cases} t_{F(e)}(x), & e \in A - B,\ x \in X, \\ t_{G(e)}(x), & e \in B - A,\ x \in X, \\ t_{F(e)}(x) \wedge t_{G(e)}(x), & e \in A \cap B,\ x \in X, \end{cases}$$

$$1 - f_{H(e)}(x) = \begin{cases} 1 - f_{F(e)}(x), & e \in A - B,\ x \in X, \\ 1 - f_{G(e)}(x), & e \in B - A,\ x \in X, \\ (1 - f_{F(e)}(x) \wedge 1 - f_{G(e)}(x)), & e \in A \cap B,\ x \in X. \end{cases}$$

We denote it by $(F, A) \cap (G, B) = (H, C)$.

Definition 2.13 ([19]). *Let X be an initial universe set, E be the nonempty fixed set of parameters and τ be the collection of vague soft sets over X, then τ is said to be a vague soft topology on X if:*

1. *$\hat{\emptyset}_E,\ \hat{X}_E$ belongs to τ.*
2. *the union of any number of vague soft sets in τ belongs to τ.*
3. *the intersection of any two vague soft sets in τ belongs to τ.*

The triplet (X, τ, E) is called a vague soft topological space over X.

Definition 2.14 ([19]). *Let (X, τ, E) be a vague soft topological space over X, then the members of τ are said to be vague soft open set in X. A vague soft set (F, E) over X is said to be a vague soft closed set in X, if its relative complement $(F, E)^c$ belongs to τ.*

Definition 2.15 ([12]). *Let (X, τ, E) be a vague soft topological space and (F, E) be a vague soft set over X. Then vague soft interior of (F, E) is defined by $v\tilde{s}int(F, E) = \cup\{(G, E)/(G, E) \in \tau$ and $(G, E) \subseteq (F, E)\}$.*

Definition 2.16 ([12]). *Let (X, τ, E) be a vague soft topological space and (F, E) be a vague soft set over X. Then vague soft closure of (F, E) is defined by $v\tilde{s}cl(F, E) = \cap\{(H, E)/(H, E) \in \tau^c$ and $(F, E) \subseteq (H, E)\}$.*

3. Vague soft separation axioms

Definition 3.1. *A vague soft set $(F, E) \in V\tilde{S}(X, E)$ is called a **vague soft point**, denoted by e_F if for the element $e \in E$, $F(e) \neq \tilde{\emptyset}$ and $F(e') = \tilde{\emptyset}$ for all $e' \in E \setminus \{e\}$.*

Definition 3.2. *A vague soft point e_F is said to be in a vague soft set (G, E) (or e_F is a vague soft point of (G, E)) if $F(e) \subseteq G(e)$ (i.e, if $e_F \subseteq (G, E)$) and it is denoted by $e_F \in (G, E)$.*

Definition 3.3. *A vague soft point e_F is said to be in an another vague soft point e_G (or e_F is a vague soft point of e_G) if $F(e) \subseteq G(e)$ and it is denoted by $e_F \in e_G$.*

Theorem 3.1. *Let (F, E), (G, E), (H, E), and (S, E) be vague soft sets over X. Then, the following properties hold:*

1. If $(F, E) \cap (G, E) = \hat{\emptyset}_E$ then $(F, E) \subseteq (G, E)^c$.

2. If $(F, E) \subseteq (G, E)$ and $(G, E) \subseteq (H, E)$ then $(F, E) \subseteq (H, E)$.

3. If $(F, E) \subseteq (G, E)$ and $(H, E) \subseteq (S, E)$ then $(F, E) \cap (H, E) \subseteq (G, E) \cap (S, E)$.

4. $(F, E) \subseteq (G, E)$ iff $(G, E)^c \subseteq (F, E)^c$.

Proof. It is obvious from the above Definitions 2.6, 2.8. □

Result 3.1. It is observed that,

1. The union of a vague soft set and its complement need not be the absolute (Universal) vague soft set.

2. The intersection of a vague soft set and its complement need not be the null vague soft set.

The above two results are illustrate in the following Example.

Example 3.2. Let $X=\{x_1, x_2\}$, $E=\{e_1, e_2\}$ and (F, E) be vague soft set defined as follows: $(F,E) = \left\{\left\langle e_1, \frac{[0.2,0.9]}{x_1}, \frac{[0.1,0.5]}{x_2}\right\rangle, \left\langle e_2, \frac{[0.4,0.9]}{x_1}, \frac{[0.3,0.6]}{x_2}\right\rangle\right\}$. Its complement $(F,E)^c$ is given by $(F,E)^c = \left\{\left\langle e_1, \frac{[0.1,0.8]}{x_1}, \frac{[0.5,0.9]}{x_2}\right\rangle, \left\langle e_2, \frac{[0.1,0.6]}{x_1}, \frac{[0.4,0.7]}{x_2}\right\rangle\right\}$. Then $(F,E)\cup(F,E)^c = \left\{\left\langle e_1, \frac{[0.2,0.9]}{x_1}, \frac{[0.5,0.9]}{x_2}\right\rangle, \left\langle e_2, \frac{[0.4,0.9]}{x_1}, \frac{[0.4,0.7]}{x_2}\right\rangle\right\} \neq \hat{X}_E$. and $(F,E) \cap (F,E)^c = \left\{\left\langle e_1, \frac{[0.1,0.8]}{x_1}, \frac{[0.1,0.5]}{x_2}\right\rangle, \left\langle e_2, \frac{[0.1,0.6]}{x_1}, \frac{[0.3,0.6]}{x_2}\right\rangle\right\} \neq \hat{\emptyset}_E$

Proposition 3.1 ([23]). *Let e_F be a soft point and (G, A) be a soft set. If $e_F \in (G, A)$, then $e_F \notin (G, A)^c$.*

Remark 3.3. The following Example 3.4 illustrates that the above Proposition 3.1 fails to satisfy in vague soft set theory.

Example 3.4. Let $X=\{x_1, x_2\}$, $E=\{e_1, e_2\}$, $e_{_1F}=\{< e_1, \frac{[0.2,0.5]}{x_1}, \frac{[0.1,0.3]}{x_2} >\}$ be a vague soft point and (G, E) be vague soft set defined as follows:

$$(G,E) = \left\{\left\langle e_1, \frac{[0.3,0.6]}{x_1}, \frac{[0.4,0.5]}{x_2}\right\rangle, \left\langle e_2, \frac{[0.8,0.9]}{x_1}, \frac{[0.1,0.8]}{x_2}\right\rangle\right\}.$$

And its complement $(G, E)^c$ is given by

$$(G,E)^c = \left\{\left\langle e_1, \frac{[0.4,0.7]}{x_1}, \frac{[0.5,0.6]}{x_2}\right\rangle, \left\langle e_2, \frac{[0.1,0.2]}{x_1}, \frac{[0.2,0.9]}{x_2}\right\rangle\right\}.$$

Then, we note that $e_{_1F} \in (G, E)$ and $e_{_1F} \in (G, E)^c$.

Definition 3.4. *A vague soft point e_F is said to satisfy the condition*

1. *($\mathcal{P}1$), if $t_{F(e)}(x) \leq f_{F(e)}(x)$ for all $x \in X$.*

Proposition 3.2. *If e_F is a vague soft point satisfies the condition ($\mathcal{P}1$), then there exist at-least one vague soft set (G, E) such that $e_F \in (G, E)$ and $e_F \in (G, E)^c$.*

Proof. Since e_F is a vague soft point satisfies the condition ($\mathcal{P}1$), then $t_{F(e)}(x) \leq f_{F(e)}(x)$ for all $x \in X$. Now we construct a vague soft set (G, E) with $G(e)$ as follows: Let us take $t_{G(e)}(x)$ and $f_{G(e)}(x)$ be such that $t_{F(e)}(x) \leq t_{G(e)}(x) \leq f_{F(e)}(x)$ and $t_{F(e)}(x) \leq f_{G(e)}(x) \leq f_{F(e)}(x)$ for all $x \in X$. Since $t_{F(e)}(x) \leq t_{G(e)}(x)$ and $1 - f_{F(e)}(x) \leq 1 - f_{G(e)}(x)$ for all $x \in X$, we have $e_F \in (G, E)$. Also we note that $t_{F(e)}(x) \leq f_{G(e)}(x)$ and $1 - f_{F(e)}(x) \leq 1 - t_{G(e)}(x)$ for all $x \in X$, then $e_F \in (G, E)^c$. □

Corollary 3.1. *If $e_F = (F, E)$ is a vague soft point satisfies the condition ($\mathcal{P}1$), then $e_F \in (F, E)^c$.*

Proof. Since e_F is a vague soft point satisfies the condition ($\mathcal{P}1$), then we have $t_{F(e)}(x) \leq f_{F(e)}(x)$ for all $x \in X$. That is, $1 - f_{F(e)}(x) \leq 1 - t_{F(e)}(x)$ for all $x \in X$. Hence, $e_F \in (F, E)^c$. □

Remark 3.5. In vague soft set theory, let e_F be a vague soft point and (G, E) be a vague soft set. If $e_F \in (G, E)$, then $e_F \cap (G, E)^c = \hat{\emptyset}_E$ is not hold always. It is possible only if $G(e) = \tilde{X}$.

Theorem 3.6. *Every non-null vague soft set can be written as the union of all its vague soft points.*

Proof. Let X be an initial universal set, E be the set of parameters, and (F, E) be a non-null vague soft set. Assume that for each $e \in E$, $\{e_{F_k}\}_{k\in\Lambda}$ is a family of all vague soft points of (F, E). For a fixed $e_i \in E$, $F(e_i) = \bigcup_{k\in\Lambda} F_k(e_i)$ we have $(F, E) = \{(e_i, F(e_i)) : e_i \in E\} = \bigcup_{k\in\Lambda,\ e\in E} e_{F_k}$. □

Theorem 3.7. *For vague soft sets (F, E) and (G, E), $(F, E) \subseteq (G, E)$ if and only if $e_H \in (F, E)$ implies $e_H \in (G, E)$.*

Proof. Let $(F, E) \subseteq (G, E)$. Then, for all e, $F(e) \subseteq G(e)$. If $e_H \in (F, E)$ then $H(e) \subseteq F(e)$. Thus $H(e) \subseteq G(e)$ and so $e_H \in (G, E)$.

Converse is obvious by above Theorem 3.6. □

Definition 3.5. *Let (X, τ, E) be a vague soft topological space. A vague soft set $(G, E) \in V\tilde{S}(X, E)$ is called* ***vague soft neighborhood of the vague soft point*** *$e_F \in (G, E)$ if there exists a vague soft open set (H, E) such that $e_F \in (H, E) \subseteq (G, E)$. The set of all vague soft neighborhood of e_F is called its neighborhood system and is denoted by $\mathcal{N}_\tau(e_F)$.*

Example 3.8. Let $X=\{x_1, x_2, x_3\}$, $E=\{e_1, e_2\}$ and $\tau = \{\hat{\emptyset}_E, (H, E), \hat{X}_E\}$ where $(H, E)=\{< e_1, \frac{[0.1.0.6]}{x_1}, \frac{[0.4,0.4]}{x_2}, \frac{[0,0.5]}{x_3} >, < e_2, \frac{[0.8,0.9]}{x_1}, \frac{[0.2,0.3]}{x_2}, \frac{[0.6,0.7]}{x_3} >\}$ and (X, τ, E) be a vague soft topological space over X. If we assume that the vague soft set $(G, E)=\{< e_1, \frac{[0.3.0.6]}{x_1}, \frac{[0.4,0.6]}{x_2}, \frac{[0.4,0.9]}{x_3} >, < e_2, \frac{[0.8,1]}{x_1}, \frac{[0.4,0.7]}{x_2}, \frac{[0.8,0.9]}{x_3} >\}$ and a vague soft point $e_{_2F}=\{< e_2, \frac{[0,0.1]}{x_1}, \frac{[0.2,0.2]}{x_2}, \frac{[0.5,0.6]}{x_3} >\}$ of (G, E). Then (G, E) is also a vague soft neighborhood of vague soft point $e_{_2F}$, since there exists a vague soft open set (H, E) such that $e_{_2F} \in (H, E) \subseteq (G, E)$.

Definition 3.6. *Let (X, τ, E) be a vague soft topological space. A vague soft set $(F, E) \in V\tilde{S}(X, E)$ is called vague soft neighborhood of a vague soft set (G, E) if there exists a vague soft open set (H, E) such that $(G, E) \subseteq (H, E) \subseteq (F, E)$. The set of all vague soft neighborhood of (G, E) is called its neighborhood system and is denoted by $\mathcal{N}_\tau(G, E)$.*

Theorem 3.9. *Let (X, τ, E) be a vague soft topological space and e_F be a vague soft point. Then $\mathcal{N}_\tau(e_F)$ has the following properties:*

1. *If $(G, E) \in \mathcal{N}_\tau(e_F)$, then $e_F \in (G, E)$.*
2. *If $(G, E) \in \mathcal{N}_\tau(e_F)$ and $(G, E) \subseteq (H, E)$, then $(H, E) \in \mathcal{N}_\tau(e_F)$.*
3. *If $(G, E), (H, E) \in \mathcal{N}_\tau(e_F)$, then $(G, E) \cap (H, E) \in \mathcal{N}_\tau(e_F)$.*
4. *If $(G, E), (H, E) \in \mathcal{N}_\tau(e_F)$, then $(G, E) \cup (H, E) \in \mathcal{N}_\tau(e_F)$.*

Proof. It is obvious from the Definitions 3.5 and 2.13. □

Theorem 3.10. *Let (X, τ, E) be a vague soft topological space. A vague soft set $(F, E) \in V\tilde{S}(X, E)$ is vague soft open if and only if (F, E) is vague soft neighborhood of all its vague soft points.*

Proof. Necessary part is obvious.

Conversely, let (F, E) be a vague soft neighborhood of all its vague soft points in the family $\{e_{F_k}\}_{k\in\Lambda, e\in E}$. Then, for each $e \in E$ and $k \in \Lambda$, there exists a vague soft open set (G_{e_k}, E) such that $e_{F_k} \in (G_{e_k}, E) \subseteq (F, E)$ implying $e_{F_k} \subseteq (G_{e_k}, E) \subseteq (F, E)$. Using Theorem 3.6 we obtain $(F, E) = \bigcup_{k\in\Lambda, e\in E} e_{F_i} \subseteq \bigcup_{k\in\Lambda, e\in E}(G_{e_k}, E) \subseteq (F, E)$.

So, $\bigcup_{k\in\Lambda, e\in E}(G_{e_k}, E) = (F, E)$ is a arbitrary union of vague soft open sets. Hence, (F, E) is vague soft open set. □

Definition 3.7. *Let X be an initial universe set and E be the set of parameters. If $\tau = \{\hat{\emptyset}_E, \hat{X}_E\}$, then τ is called vague soft indiscrete topology on X and (X, τ, E) is **vague soft indiscrete topological space** over X.*

Definition 3.8. *Let X be an initial universe set and E be the set of parameters. If τ be the collection of all vague soft sets which can be defined over X, then τ is called the vague soft discrete topology on X and (X, τ, E) is said to be a **vague soft discrete topological space** over X.*

Definition 3.9. *Two vague soft sets (F, E) and (G, E) in $V\tilde{S}(X, E)$ are said to be vague soft disjoint (written $(F, E) \neq (G, E)$), if $(F, E) \cap (G, E) = \hat{\emptyset}_E$.*

Definition 3.10. *Two vague soft points e_F and e_G are said to be vague soft distinct (written $e_F \neq e_G$), if its corresponding vague soft sets (F, E) and (G, E) are disjoint.*

Remark 3.11. If $e_F \neq e_G$, then neither $F(e) \neq \tilde{X}$ nor $G(e) \neq \tilde{X}$.

Definition 3.11. *A vague soft topological space (X, τ, E) is said to be a **vague soft T_0-space** iff for any two vague soft points $e_M \neq e_N$ there exist vague soft open sets (F, E) and (G, E) such that $e_M \in (F, E)$ and $e_N \notin (F, E)$ or $e_N \in (G, E)$ and $e_M \notin (G, E)$.*

Example 3.12. A vague soft indiscrete topological space over X is not vague soft T_0-space.

Example 3.13. Let X=$\{x_1, x_2\}$, E=$\{e_1, e_2\}$. If we consider the vague soft topologies $\tau_1 = \{\hat{\emptyset}_E, (F_1, E), \hat{X}_E\}$ and $\tau_2 = \{\hat{\emptyset}_E, (F_2, E), \hat{X}_E\}$, where

$$(F_1, E) = \{< e_1, \frac{[1,1]}{x_1}, \frac{[0,0]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >$$

and

$$(F_2, E) = \{< e_1, \frac{[0.5,0.6]}{x_1}, \frac{[0,0]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >\}.$$

Then, the triplet (X, τ_1, E) is a vague soft T_0-space. But in τ_2, there is no such a vague soft open set (at-least one vague soft open set containing one vague soft point not containing another vague soft point) for the distinct vague soft points $e_{_1M}$=$\{< e_1, \frac{[0.7,1]}{x_1}, \frac{[0,0]}{x_2} >\}$, $e_{_1N}$=$\{< e_1, \frac{[0,0]}{x_1}, \frac{[0.2,0.4]}{x_2} >\}$. Hence (X, τ_2, E) is not a vague soft T_0-space.

Remark 3.14. If (X, τ, E) is vague soft T_0-space and $\tau \subset \tau'$, then (X, τ', E) is also a vague soft T_0-space.

Example 3.15. Let X=$\{x_1, x_2\}$, E=$\{e_1, e_2\}$ and $\tau = \{\hat{\emptyset}_E, (F_1, E), (F_2, E), \hat{X}_E\}$ where

$$(F_1, E) = \{< e_1, \frac{[1,1]}{x_1}, \frac{[0,0]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >\},$$
$$(F_2, E) = \{< e_1, \frac{[0.5,0.6]}{x_1}, \frac{[0,0]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >\}.$$

Since (X, τ_1, E) is a vague soft T_0-space as from the above Example 3.13 and $\tau_1 \subset \tau$, we have (X, τ, E) is also a vague soft T_0-space.

Definition 3.12. *A vague soft topological space (X, τ, E) is said to be a* ***vague soft T_1-space*** *iff for any two vague soft points $e_M \neq e_N$ there exist vague soft open sets (F, E) and (G, E) such that $e_M \in (F, E)$, $e_N \notin (F, E)$ and $e_N \in (G, E)$, $e_M \notin (G, E)$.*

Example 3.16. Let $X = \{x_1, x_2\}, E = \{e_1, e_2\}$ and $\tau = \{\hat{\emptyset}_E, (F_1, E), \ldots, (F_9, E), \hat{X}_E\}$ where

$$(F_1, E) = \{< e_1, \frac{[1,1]}{x_1}, \frac{[0,0]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[0.2,0.8]}{x_2} >\},$$
$$(F_2, E) = \{< e_1, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[0,0]}{x_2} >\},$$
$$(F_3, E) = \{< e_1, \frac{[1,1]}{x_1}, \frac{[1,1]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >\},$$
$$(F_4, E) = \{< e_1, \frac{[1,1]}{x_1}, \frac{[1,1]}{x_2} >, < e_2, \frac{[1,1]}{x_1}, \frac{[0,0]}{x_2} >\},$$
$$(F_5, E) = \{< e_1, \frac{[1,1]}{x_1}, \frac{[0,0]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[0.2,0.8]}{x_2} >\},$$
$$(F_6, E) = \{< e_1, \frac{[1,1]}{x_1}, \frac{[0,0]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[0,0]}{x_2} >\},$$
$$(F_7, E) = \{< e_1, \frac{[1,1]}{x_1}, \frac{[1,1]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[0,0]}{x_2} >\},$$
$$(F_8, E) = \{< e_1, \frac{[1,1]}{x_1}, \frac{[1,1]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[0.2,0.8]}{x_2} >\},$$
$$(F_9, E) = \{< e_1, \frac{[1,1]}{x_1}, \frac{[1,1]}{x_2} >, < e_2, \frac{[1,1]}{x_1}, \frac{[0.2,0.8]}{x_2} >\}.$$

Then the vague soft topological space (X, τ, E) is a vague soft T_1-space.

Proposition 3.3. *Every vague soft T_1-space is vague soft T_0-space.*

Proof. If (X, τ, E) is a vague soft T_2-space, then for the distinct vague soft points e_M and e_N, there exist vague soft open sets (F, E) and (G, E) such that $e_M \in (F, E), e_N \notin (F, E)$ and $e_N \in (G, E), e_M \notin (G, E)$. Hence (X, τ, E) is a vague soft T_1-space. □

Remark 3.17. The converse of the above Proposition 3.3 is need not be true as shown in the following example.

Example 3.18. Let us consider the vague soft T_0-space (X, τ, E) as in Example 3.13. Then the vague soft topological space (X, τ, E) is not vague soft T_1-space, since for the pair of distinct vague soft points e_{1M}=$\{< e_1, \frac{[0.4,0.6]}{x_1}, \frac{[0,0]}{x_2} >\}$, e_{1N}=$\{< e_1, \frac{[0,0]}{x_1}, \frac{[0.6,0.7]}{x_2} >\}$ there is no vague soft open set containing e_{1N} and not containing e_{1M}.

Remark 3.19. Based on Proposition 3.2 and Corollary 3.1, the following Theorem 3.20 due to S. Hussain and B. Ahmad need not be true in vague soft topological spaces.

Theorem 3.20 ([11])**.** *A soft topological space (X, τ, E) is a soft T_1-space iff every soft point of X is soft closed.*

Definition 3.13. *Let (X, τ, E) be a vague soft topological space and e_M and e_N be any two vague soft points such that $e_M \neq e_N$. If there exist vague soft open sets (F, E) and (G, E) such that $e_M \in (F, E)$, $e_N \in (G, E)$ and $(F, E) \cap (G, E) = \hat{\emptyset}_E$, then (X, τ, E) is called a* ***vague soft Hausdorff (or vague soft T_2) space.***

Example 3.21. Let X=$\{x_1, x_2\}$, E=$\{e_1, e_2\}$ and $\tau = \{\hat{\emptyset}_E, (F_1, E), \ldots, (F_9, E), \hat{X}_E\}$ where

$$(F_1, E) = \{< e_1, \frac{[1,1]}{x_1}, \frac{[0,0]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >\},$$
$$(F_2, E) = \{< e_1, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >\},$$
$$(F_3, E) = \{< e_1, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >, < e_2, \frac{[1,1]}{x_1}, \frac{[0,0]}{x_2} >\},$$
$$(F_4, E) = \{< e_1, \frac{[0,0]}{x_1}, \frac{[0,0]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >\},$$
$$(F_5, E) = \{< e_1, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[0,0]}{x_2} >\},$$
$$(F_6, E) = \{< e_1, \frac{[1,1]}{x_1}, \frac{[1,1]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >\},$$
$$(F_7, E) = \{< e_1, \frac{[1,1]}{x_1}, \frac{[1,1]}{x_2} >, < e_2, \frac{[1,1]}{x_1}, \frac{[0,0]}{x_2} >\},$$

$$(F_8, E) = \{< e_1, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >, < e_2, \frac{[1,1]}{x_1}, \frac{[1,1]}{x_2} >\},$$
$$(F_9, E) = \{< e_1, \frac{[1,1]}{x_1}, \frac{[1,1]}{x_2} >, < e_2, \frac{[0,0]}{x_1}, \frac{[0,0]}{x_2} >\}.$$

Then, the vague soft topological space (X, τ, E) is a vague soft T_2-space.

Example 3.22. A vague soft discrete topological space form a vague soft T_2-space, since every vague soft set is vague soft open.

Proposition 3.4. *Every vague soft T_2-space is vague soft T_1-space.*

Proof. If (X, τ, E) is a vague soft T_2-space, then for the distinct vague soft points e_M and e_N, there exist vague soft open sets (F, E) and (G, E) such that $e_M \in (F, E)$, $e_N \in (G, E)$ and $(F, E) \cap (G, E) = \hat{\emptyset}_E$. Since $(F, E) \cap (G, E) = \hat{\emptyset}_E$, we have $e_M \notin (G, E)$, $e_N \notin (F, E)$. Hence (X, τ, E) is a vague soft T_1-space. □

Remark 3.23. The converse of the above Proposition 3.4 is need not be true as shown in the following example.

Example 3.24. Let us consider the vague soft T_1-space (X, τ, E) as in Example 3.16. Then the vague soft topological space (X, τ, E) is not vague soft T_2-space, since for the pair of vague soft points $e_{2M} = \{< e_2, \frac{[0,0]}{x_1}, \frac{[0.8,0.9]}{x_2} >\}$, $e_{2N} = \{< e_2, \frac{[0.2,0.6]}{x_1}, \frac{[0,0]}{x_2} >\}$ there exist no disjoint vague soft open sets (F, E) and (G, E) containing e_{2M}, e_{2N} respectively.

Remark 3.25. If (X, τ, E) is a vague soft $T_i (i = 1, 2)$-space and $\tau \subset \tau'$, then (X, τ', E) is also a vague soft $T_i (i = 1, 2)$-space.

Definition 3.14. *Let (X, τ, E) be a vague soft topological space, (G, E) be a vague soft closed set in X and $e_K \in V\tilde{S}(X, E)$ such that $e_K \notin (G, E)$. If there exist vague soft open sets (F_1, E) and (F_2, E) such that $e_K \in (F_1, E)$, $(G, E) \subseteq (F_2, E)$ and $(F_1, E) \cap (F_2, E) = \hat{\emptyset}_E$, then (X, τ, E) is called a* ***vague soft regular-space****.*

Definition 3.15. *Let (X, τ, E) be a vague soft topological space. Then (X, τ, E) is said to be a vague soft T_3-space if it is vague soft regular and vague soft T_1-space.*

Remark 3.26. Every vague soft Hausdorff (or vague soft T_2) space need not be vague soft regular-space and vague soft T_3 space.

Example 3.27. Let us consider the vague soft Hausdorff(or vague soft T_2)-space (X, τ, E) as in Example 3.21 and consider a vague soft point $e_K = \{< e_1, \frac{[0,0]}{x_1}, \frac{[0.7,0.8]}{x_2} >\}$ disjoint with vague soft closed set $(F_5, E)^c = \{< e_1, \frac{[1,1]}{x_1}, \frac{[0,0]}{x_2} >, < e_2, \frac{[1,1]}{x_1}, \frac{[1,1]}{x_2} >\}$. Since $\hat{X}_E$ is the only vague soft open set containing $(F_5, E)^c$, which shows that (X, τ, E) is not vague soft regular-space. Hence, (X, τ, E) is not vague soft T_3-space.

Definition 3.16. *Let (X, τ, E) be a vague soft topological space, (F, E) and (G, E) be vague soft closed sets in X such that $(F, E) \cap (G, E) = \hat{\emptyset}_E$. If there exist vague soft open sets (F_1, E) and (F_2, E) such that $(F, E) \subseteq (F_1, E)$, $(G, E) \subseteq (F_2, E)$ and $(F_1, E) \cap (F_2, E) = \hat{\emptyset}_E$, then (X, τ, E) is called a* ***vague soft normal-space****.*

Definition 3.17. *Let (X, τ, E) be a vague soft topological space. Then (X, τ, E) is said to be a vague soft T_4-space if it is vague soft normal and vague soft T_1-space.*

Remark 3.28. Due to the Corollary 3.1, we obtain the following results

1. Every vague soft normal space need not be a vague soft regular space.
2. Every vague soft regular space need not be a vague soft Hausdorff space.

which are illustrated by the following examples.

Example 3.29. Let X=$\{x_1, x_2\}$, E=$\{e_1, e_2\}$ and $\tau = \{\hat{\emptyset}_E, (F_1, E), \ldots, (F_5, E), \hat{X}_E\}$ where

$$(F_1, E) = \{< e_1, \frac{[0.2, 0.8]}{x_1}, \frac{[1, 1]}{x_2} >, < e_2, \frac{[1, 1]}{x_1}, \frac{[0.4, 0.6]}{x_2} >\},$$
$$(F_2, E) = \{< e_1, \frac{[1, 1]}{x_1}, \frac{[0.4, 0.6]}{x_2} >, < e_2, \frac{[0.2, 0.8]}{x_1}, \frac{[1, 1]}{x_2} >\},$$
$$(F_3, E) = \{< e_1, \frac{[0, 0]}{x_1}, \frac{[0.4, 0.6]}{x_2} >, < e_2, \frac{[0.2, 0.8]}{x_1}, \frac{[0, 0]}{x_2} >\},$$
$$(F_4, E) = \{< e_1, \frac{[0.2, 0.8]}{x_1}, \frac{[0, 0]}{x_2} >, < e_2, \frac{[0, 0]}{x_1}, \frac{[0.4, 0.6]}{x_2} >\},$$
$$(F_5, E) = \{< e_1, \frac{[0.2, 0.8]}{x_1}, \frac{[0.4, 0.6]}{x_2} >, < e_2, \frac{[0.2, 0.8]}{x_1}, \frac{[0.4, 0.6]}{x_2} >\}.$$

Here, we note that each vague soft set in τ is both vague soft open as well as vague soft closed. Now for the disjoint vague soft closed sets $(F_3, E) \neq (F_4, E)$, there exist a vague soft open sets (F_3, E) and (F_4, E) such that $(F_3, E) \subseteq (F_3, E)$, $(F_4, E) \subseteq (F_4, E)$ and $(F_3, E) \cap (F_4, E) = \hat{\emptyset}_E$. Therefore the vague soft topological space (X, τ, E) is a vague soft normal-space.

Next we consider a vague soft point e_K=$\{< e_1, \frac{[0.5,0.6]}{x_1}, \frac{[0,0]}{x_2} >\}$ and vague soft closed set (F_3, E) with $e_K \notin (F_3, E)$. Since for this vague soft point e_K and vague soft closed set (F_3, E), there exist no vague soft open sets (G, E) and (H, E) such that $e_K \in (G, E)$, $(F_3, E) \subseteq (H, E)$ and $(G, E) \cap (H, E) = \hat{\emptyset}_E$. Hence (X, τ, E) is not a vague soft regular-space.

Example 3.30. Let X=$\{x_1, x_2\}$, E=$\{e_1, e_2\}$ and $\tau = \{\hat{\emptyset}_E, (F_1, E), (F_2, E), \hat{X}_E\}$ where (F_1, E)=$\{< e_1, \frac{[0,0]}{x_1}, \frac{[1,1]}{x_2} >, < e_2, \frac{[1,1]}{x_1}, \frac{[1,1]}{x_2} >\}$, (F_2, E)=$\{< e_1, \frac{[1,1]}{x_1}, \frac{[0,0]}{x_2} >$, $< e_2, \frac{[0,0]}{x_1}, \frac{[0,0]}{x_2} >\}$ which are both vague soft open and vague soft closed. Then (X, τ, E) is a vague soft regular-space but not a vague soft Hausdorff space.

4. Vague soft continuous functions via vague soft neighborhoods

Definition 4.1. *Let $V\tilde{S}(X,E)$ and $V\tilde{S}(Y,K)$ be two vague soft classes, and let $u : X \to Y$ and $p : E \to K$ be mappings. Then a vague soft function $g_{pu} = (u,p) : V\tilde{S}(X,E) \to V\tilde{S}(Y,K)$ is defined as: for $(F,A) \in V\tilde{S}(X,E)$, the image of (F,A) under g_{pu} denoted by $g_{pu}(F,A) = (u(F),p(A))$, is a vague soft set in $V\tilde{S}(Y,K)$ given by*

$$t_{u(F)(\beta)}(y) = \begin{cases} \sup_{\alpha \in p^{-1}(\beta)\cap A,\ x \in u^{-1}(y)} t_{F(\alpha)}(x), & \text{if } u^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$1 - f_{u(F)(\beta)}(y) = \begin{cases} \sup_{\alpha \in p^{-1}(\beta)\cap A,\ x \in u^{-1}(y)} 1 - f_{F(\alpha)}(x), & \text{if } u^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

for all $\beta \in p(A)$ and $y \in Y$.

Definition 4.2. *Let $V\tilde{S}(X,E)$ and $V\tilde{S}(Y,K)$ be two vague soft classes, and let $g_{pu} = (u,p) : V\tilde{S}(X,E) \to V\tilde{S}(Y,K)$ be a vague soft function and (G,B) be a vague soft set in $V\tilde{S}(Y,K)$. Then the inverse image of (G,B) under g_{pu}, denoted by $g_{pu}^{-1}(G,B) = (u^{-1}(G),\ p^{-1}(B))$ is a vague soft set in $V\tilde{S}(X,E)$ given by $t_{u^{-1}(G)(\alpha)}(x) = t_{G(p(\alpha))}(u(x))$ and $1 - f_{u^{-1}(G)(\alpha)}(x) = 1 - f_{G(p(\alpha))}(u(x))$, for all $\alpha \in p^{-1}(B)$ and $x \in X$.*

Example 4.1. For vague soft classes $V\tilde{S}$(X,E) , $V\tilde{S}(Y,K)$, Let X=$\{x_1,x_2,x_3\}$, E=$\{e_1,e_2,e_3,e_4\}$, Y=$\{y_1,y_2,y_3\}$, K=$\{k_1,k_2,k_3\}$. Define $u : X \to Y$ and $p : E \to K$ as follows:

$u(x_1) = y_1$, $u(x_2) = y_2$, $u(x_3) = y_1$, $p(e_1) = k_1$, $p(e_2) = k_3$, $p(e_3) = k_3$, $p(e_4) = k_1$.

Consider the two vague soft sets (F,A) and (G,K) over X and Y respectively as follows: $(F,A) = \left\{ \left\langle e_1, \frac{[0.3,0.6]}{x_1}, \frac{[0.2,0.5]}{x_2}, \frac{[0.1,0.7]}{x_3} \right\rangle, \left\langle e_2, \frac{[0.4,0.9]}{x_1}, \frac{[0.6,0.8]}{x_2}, \frac{[0.2,0.7]}{x_3} \right\rangle, \left\langle e_3, \frac{[0.8,0.9]}{x_1}, \frac{[0.1,0.8]}{x_2}, \frac{[0.9,1]}{x_3} \right\rangle \right\}$ and $(G,K) = \left\{ \left\langle k_1, \frac{[0.6,0.7]}{y_1}, \frac{[0.9,1]}{y_2}, \frac{[0.5,0.8]}{y_3} \right\rangle, \left\langle k_2, \frac{[0.4,0.8]}{y_1}, \frac{[0.2,0.6]}{y_2}, \frac{[0.7,0.8]}{y_3} \right\rangle, \left\langle k_3, \frac{[0.1,0.8]}{y_1}, \frac{[0.2,0.6]}{y_2}, \frac{[0.2,0.5]}{y_3} \right\rangle \right\}$. Then, the vague soft function $g_{pu} : V\tilde{S}(X,E) \to V\tilde{S}(Y,K)$ is given as: for a vague soft set $(F,E) \in V\tilde{S}(X,E)$, $g_{pu}(F,A) = (u(F),p(A))$ is a vague soft set in $V\tilde{S}(Y,K)$ and is given by $g_{pu}(F,A) = \left\{ \left\langle k_1, \frac{[0.3,0.7]}{y_1}, \frac{[0.2,0.5]}{y_2}, \frac{[0,0]}{y_3} \right\rangle, \left\langle k_3, \frac{[0.9,1]}{y_1}, \frac{[0.6,0.8]}{y_2}, \frac{[0,0]}{y_3} \right\rangle \right\}$.

Next, for the vague soft set (G,K) in $V\tilde{S}(Y,K)$, the vague soft inverse image is given by

$g_{pu}^{-1}(G,K) = (u^{-1}(G),p^{-1}(B))$

$$= \left\{ \begin{array}{l} \left\langle e_1, \frac{[0.6,0.7]}{x_1}, \frac{[0.9,1]}{x_2}, \frac{[0.6,0.7]}{x_3} \right\rangle, \\ \left\langle e_2, \frac{[0.1,0.8]}{x_1}, \frac{[0.2,0.6]}{x_2}, \frac{[0.1,0.8]}{x_3} \right\rangle, \left\langle e_3, \frac{[0.1,0.8]}{x_1}, \frac{[0.2,0.6]}{x_2}, \frac{[0.1,0.8]}{x_3} \right\rangle, \\ \left\langle e_4, \frac{[0.6,0.7]}{x_1}, \frac{[0.9,1]}{x_2}, \frac{[0.6,0.7]}{x_3} \right\rangle \end{array} \right\}.$$

Remark 4.2. The vague soft function g_{pu} is called surjective (resp. injective and constant) if p and u are surjective (resp. injective and constant).

Theorem 4.3. *Let $V\tilde{S}(X,E)$ and $V\tilde{S}(Y,K)$ be two vague soft classes. For the vague soft function $g_{pu}: V\tilde{S}(X,E) \to V\tilde{S}(Y,K)$, the following statements hold:*

1. *$g_{pu}(\hat{\emptyset}_E) \subseteq \hat{\emptyset}_K$ and if p is surjective, the equality holds.*
2. *$g_{pu}(\hat{X}_E) \subset \hat{Y}_K$ and if g_{pu} is surjective the equality holds.*
3. *$(g_{pu}(F,A))^c \subseteq g_{pu}((F,A)^c)$ if g_{pu} is surjective. And if g_{pu} is bijective then its equality holds.*
4. *If $(F_1,A) \subseteq (F_2,A)$, then $g_{pu}(F_1,A) \subseteq g_{pu}(F_2,A)$, $\forall$ $(F_1,A),(F_2,A) \in V\tilde{S}(X,E)$.*

Proof. 1, 2 and 3 are obvious.

4. Since $(F_1,A) \subseteq (F_2,A)$, we have for each $\alpha \in A$ and for all $x \in X$ $t_{F_1(\alpha)}(x) \leq t_{F_2(\alpha)}(x)$, and $1 - f_{F_1(\alpha)}(x) \leq 1 - f_{F_2(\alpha)}(x)$.

Now, for all $\beta \in p(A)$ and $y \in Y$,

$$t_{u(F_1)(\beta)}(y) = \sup_{\alpha \in p^{-1}(\beta)\cap A,\ x \in u^{-1}(y)} t_{F_1(\alpha)}(x) \leq \sup_{\alpha \in p^{-1}(\beta)\cap A,\ x \in u^{-1}(y)} t_{F_2(\alpha)}(x),$$

and

$$\begin{aligned} 1 - f_{u(F_1)(\beta)}(y) &= \sup_{\alpha \in p^{-1}(\beta)\cap A,\ x \in u^{-1}(y)} 1 - f_{F_1(\alpha)}(x) \\ &\leq \sup_{\alpha \in p^{-1}(\beta)\cap A,\ x \in u^{-1}(y)} 1 - f_{F_2(\alpha)}(x) \end{aligned}$$

and hence, $t_{u(F_1)(\beta)}(y) \leq t_{u(F_2)(\beta)}(y)$ and $1 - f_{u(F_1)(\beta)}(y) \leq 1 - f_{u(F_2)(\beta)}(y)$.

This implies that, $g_{pu}(F_1,A) \subseteq g_{pu}(F_2,A)$. □

Theorem 4.4. *Let $V\tilde{S}(X,E)$ and $V\tilde{S}(Y,K)$ be two vague soft classes. For the vague soft function $g_{pu}: V\tilde{S}(X,E) \to V\tilde{S}(Y,K)$, the following statements hold,*

1. *$g_{pu}^{-1}(\hat{\emptyset}_K) = \hat{\emptyset}_E$.*
2. *$g_{pu}^{-1}(\hat{Y}_K) = \hat{X}_E$*
3. *$(g_{pu}^{-1}(G,B))^c = g_{pu}^{-1}((G,B)^c)$*
4. *If $(G_1,B) \subseteq (G_2,B)$, then $g_{pu}^{-1}(G_1,B) \subseteq g_{pu}^{-1}(G_2,B)$, for all $(G_1,B),(G_2,B) \in V\tilde{S}(Y,K)$.*

Proof. 1, 2 and 3 are obvious.

4. Since $(G_1,B) \subseteq (G_2,B)$, we have for each $\beta \in B$ and for all $y \in Y$ $t_{G_1(\beta)}(y) \leq t_{G_2(\beta)}(y)$, and $1 - f_{G_1(\beta)}(y) \leq 1 - f_{G_2(\beta)}(y)$.

Now, for all $\alpha \in p^{-1}(B)$ and $x \in X$, we have $t_{u^{-1}(G_1)(\alpha)}(x) = t_{G_1(p(\alpha))}(u(x)) \leq t_{G_2(p(\alpha))}(u(x)) = t_{u^{-1}(G_2)(\alpha)}(x)$.

Similarly, $1 - f_{u^{-1}(G_1)(\alpha)}(x) = 1 - f_{G_1(p(\alpha))}(u(x)) \leq 1 - f_{G_2(p(\alpha))}(u(x)) = 1 - f_{u^{-1}(G_2)(\alpha)}(x)$. Hence, $g_{pu}^{-1}(G_1, B) \subseteq g_{pu}^{-1}(G_2, B)$. □

Theorem 4.5. *Let (F, A), (G, B) be two vague soft sets in $V\tilde{S}(X, E)$ and $V\tilde{S}(Y, K)$ respectively, and $g_{pu} : V\tilde{S}(X, E) \to V\tilde{S}(Y, K)$ be a vague soft function. Then*

1. *$(F, A) \subseteq g_{pu}^{-1}(g_{pu}(F, A))$ and if g_{pu} is injective, the equality holds.*
2. *$g_{pu}(g_{pu}^{-1}(G, B)) \subseteq (G, B)$ and if g_{pu} is surjective, the equality holds.*

Proof. The proof is obvious from the Definitions 4.1 and 4.2. □

Theorem 4.6. *Let $\{(F_i, E)\}_{i\in I}$ be a family of vague soft sets over X and $g_{pu} : V\tilde{S}(X, E) \to V\tilde{S}(Y, K)$ be a vague soft function. Then 1. $g_{pu}((F_1, E) \cup (F_2, E) = g_{pu}(F_1, E) \cup g_{pu}(F_2, E)$. In general, $g_{pu}(\bigcup_{i\in I}(F_i, E)) = \bigcup_{i\in I} g_{pu}(F_i, E)$.*

2. $g_{pu}((F_1, E) \cap (F_2, E) \subseteq g_{pu}(F_1, E) \cap g_{pu}(F_2, E)$. In general, $g_{pu}(\bigcap_{i\in I}(F_i, E)) \subseteq \bigcap_{i\in I} g_{pu}(F_i, E)$ and if g_{pu} is injective, the equalities holds.

Proof. Obvious. □

Theorem 4.7. *Let $\{(G_j, K)\}_{j\in J}$ be a family of vague soft sets over Y and $g_{pu} : V\tilde{S}(X, E) \to V\tilde{S}(Y, K)$ be a vague soft function. Then:*

1. $g_{pu}^{-1}((G_1, K) \cup (G_2, K)) = g_{pu}^{-1}(G_1, K) \cup g_{pu}^{-1}(G_2, K)$.
In general, $g_{pu}^{-1}(\bigcup_{j\in J}(G_j, K)) = \bigcup_{j\in J} g_{pu}^{-1}(G_j, K)$.
2. $g_{pu}^{-1}((G_1, K) \cap (G_2, K)) = g_{pu}^{-1}(G_1, K) \cap g_{pu}^{-1}(G_2, K)$.
In general, $g_{pu}^{-1}(\bigcap_{j\in J}(G_j, K)) = \bigcap_{j\in J} g_{pu}^{-1}(G_j, K)$.

Proof. 1. Let $(G_1, K) \cup (G_2, K) = (S, K)$. For $\alpha \in p^{-1}(K)$ and $x \in X$, $t_{u^{-1}(S)(\alpha)}(x) = t_{S(p(\alpha))}(u(x))$ and $1 - f_{u^{-1}(G)(\alpha)}(x) = 1 - f_{G(p(\alpha))}(u(x))$.

Since $(S, K) = (G_1, K) \cup (G_2, K)$ we have $t_{S(p(\alpha))}(u(x)) = t_{G_1(p(\alpha))}(u(x)) \vee t_{G_2(p(\alpha))}(u(x)) = t_{u^{-1}(G_1)(\alpha)}(x) \vee t_{u^{-1}(G_2)(\alpha)}(x)$.

Thus, $t_{u^{-1}(S)(\alpha)}(x) = t_{u^{-1}(G_1)(\alpha)}(x) \vee t_{u^{-1}(G_2)(\alpha)}(x)$.

Similarly, $1 - f_{u^{-1}(S)(\alpha)}(x) = 1 - f_{u^{-1}(G_1)(\alpha)}(x) \vee 1 - f_{u^{-1}(G_2)(\alpha)}(x)$ for all $\alpha \in p^{-1}(K)$ and $x \in X$. Hence $g_{pu}^{-1}(S, K) = g_{pu}^{-1}((G_1, K) \cup (G_2, K)) = g_{pu}^{-1}(G_1, K) \cup g_{pu}^{-1}(G_2, K)$.

2. Proof is as similar to the proof of 1. □

Definition 4.3. *Let (X, τ, E) and (Y, σ, K) be two vague soft topological spaces. The vague soft function $g_{pu} : V\tilde{S}(X, E) \to V\tilde{S}(Y, K)$ is called **vague soft continuous at vague soft point** e_F of $V\tilde{S}(X, E)$ if for each $(G, K) \in N_\sigma(g_{pu}(e_F))$, there exist a $(H, E) \in N_\tau(e_F)$ such that $g_{pu}(H, E) \subseteq (G, K)$.*

*If g_{pu} is vague soft continuous at each vague soft point of $V\tilde{S}(X, E)$, then g_{pu} is called **vague soft continuous function**.*

Theorem 4.8. *Let $g_{pu} : (X, \tau, E) \to (Y, \sigma, K)$ be a vague soft function and e_F be a vague soft point. Then g_{pu} is vague soft continuous at e_F if and only if for each $(G, K) \in N_\sigma(g_{pu}(e_F))$, there exist a $(H, E) \in N_\tau(e_F)$ such that $(H, E) \subseteq g_{pu}^{-1}(G, K)$.*

Proof. If g_{pu} is vague soft continuous at e_F, then for each $(G, K) \in N_\sigma(g_{pu}(e_F))$, there exist a $(H, E) \in N_\tau(e_F)$ such that $g_{pu}(H, E) \subseteq (G, K)$. Thus, from the Theorem 4.5 we have $(H, E) \subseteq g_{pu}^{-1}(g_{pu}(H, E)) \subseteq g_{pu}^{-1}(G, K)$.

Conversely, assume that for each $(G, K) \in N_\sigma(g_{pu}(e_F))$, there exist a $(H, E) \in N_\tau(e_F)$ such that $(H, E) \subseteq g_{pu}^{-1}(G, K)$. Now by Theorem 4.5, we get $g_{pu}(H, E) \subseteq g_{pu}(g_{pu}^{-1}(G, K)) \subseteq (G, K)$. Hence from the Definition 4.8, g_{pu} is vague soft continuous at e_F. □

Theorem 4.9. *Let $g_{pu} : (X, \tau, E) \to (Y, \sigma, K)$ be a vague soft function and e_F be a vague soft point. Then g_{pu} is called vague soft continuous at e_F if and only if for each $(G, K) \in N_\sigma(g_{pu}(e_F))$, $g_{pu}^{-1}(G, K) \in \tau$.*

Proof. It is clear from Theorem 4.8. □

Theorem 4.10. *The vague soft function $g_{pu} : (X, \tau, E) \to (Y, \sigma, K)$ is called vague soft continuous if and only if for all $(G, K) \in \sigma$, $g_{pu}^{-1}(G, K) \in \tau$.*

Proof. It is follows from Theorem 4.9. □

Example 4.11. Let $g_{pu} : (X, \tau, E) \to (Y, \sigma, K)$ be a vague soft function.

1. If σ is the vague soft indiscrete topology on Y, then g_{pu} is a vague soft continuous function.

2. If τ is the vague soft discrete topology on X, then g_{pu} is a vague soft continuous function.

Theorem 4.12. *Let $g_{pu} : (X, \tau, E) \to (Y, \sigma, K)$ be a vague soft function. Then the following conditions are equivalent:*

1. *g_{pu} is vague soft continuous function.*

2. *For each vague soft closed set (G, K) over Y, $g_{pu}^{-1}(G, K)$ is a vague soft closed set over X.*

3. *For each vague soft set (F, E) over X, $g_{pu}(v\tilde{s}cl(F, E)) \subseteq v\tilde{s}cl(g_{pu}(F, E))$.*

4. *For each vague soft set (G, K) over Y, $v\tilde{s}cl(g_{pu}^{-1}(G, K)) \subseteq g_{pu}^{-1}(v\tilde{s}cl(G, K))$.*

5. *For each vague soft set (G, K) over Y, $g_{pu}^{-1}(v\tilde{s}int(G, K)) \subseteq v\tilde{s}int(g_{pu}^{-1}(G, K))$.*

Proof. 1 ⇒ 2: Let (G,K) be a vague soft closed set over Y. Then $(G,K)^c \in \sigma$. By 1, we have $g_{pu}^{-1}((G,K)^c) \in \tau$. Now from the Theorem 4.4 (3) we have $g_{pu}^{-1}((G,K)^c) = (g_{pu}^{-1}(G,K))^c \in \tau$. And hence $g_{pu}^{-1}(G,K)$ is a vague soft closed set over X.

2 ⇒ 3: Let (F,E) be a vague soft set over X. Then $(F,E) \subseteq g_{pu}^{-1}(g_{pu}(F,E)) \subseteq g_{pu}^{-1}(v\tilde{s}cl(g_{pu}(F,E)))$.

Since $v\tilde{s}cl(g_{pu}(F,E))$ is vague soft closed set over Y, by 2 $g_{pu}^{-1}(v\tilde{s}cl(g_{pu}(F,E)))$ is vague soft closed set over X. Then, $v\tilde{s}cl(F,E) \subseteq g_{pu}^{-1}(v\tilde{s}cl(g_{pu}(F,E))) \Rightarrow g_{pu}(v\tilde{s}cl(F,E)) \subseteq g_{pu}(g_{pu}^{-1}(v\tilde{s}cl(g_{pu}(F,E)))) \Rightarrow\subseteq v\tilde{s}cl(g_{pu}(F,E))$.

Hence, $g_{pu}(v\tilde{s}cl(F,E)) \subseteq v\tilde{s}cl(g_{pu}(F,E))$.

3 ⇒ 4: Let (G,K) be a vague soft set over Y and let $g_{pu}^{-1}(G,K) = (F,E)$. By 3, we have $g_{pu}(v\tilde{s}cl(F,E)) = g_{pu}(v\tilde{s}cl(g_{pu}^{-1}(G,K))) \subseteq v\tilde{s}cl(g_{pu}(g_{pu}^{-1}(G,K))) \subseteq v\tilde{s}cl((G,K))$ (i.e) $g_{pu}(v\tilde{s}cl(F,E)) \subseteq v\tilde{s}cl(g_{pu}((F,E))) \subseteq v\tilde{s}cl((G,K))$.

$\Rightarrow v\tilde{s}cl(F,E) \subseteq g_{pu}^{-1}(g_{pu}(v\tilde{s}cl(F,E))) \subseteq g_{pu}^{-1}(v\tilde{s}cl((G,K)))$.

Then, $v\tilde{s}cl(g_{pu}^{-1}(G,K)) = v\tilde{s}cl(F,E) \subseteq g_{pu}^{-1}(v\tilde{s}cl((G,K)))$.

Hence, $v\tilde{s}cl(g_{pu}^{-1}(G,K)) \subseteq g_{pu}^{-1}(v\tilde{s}cl(G,K))$.

4 ⇒ 5: Let (G,K) be a vague soft set over Y. Substituting $(G,K)=(G,K)^c$ in 4, $v\tilde{s}cl(g_{pu}^{-1}((G,K)^c)) \subseteq g_{pu}^{-1}(v\tilde{s}cl((G,K)^c))$.

Since $v\tilde{s}int(G,K) = (v\tilde{s}cl((G,K)^c))^c$, we have

$$\begin{aligned} g_{pu}^{-1}(v\tilde{s}int(G,K)) &= g_{pu}^{-1}(v\tilde{s}cl((G,K)^c))^c \\ &= (g_{pu}^{-1}(v\tilde{s}cl((G,K)^c)))^c \qquad \text{by 3 of Theorem 4.4} \\ &\subseteq (v\tilde{s}cl(g_{pu}^{-1}((G,K)^c)))^c \qquad \text{by 4} \\ &= (v\tilde{s}cl((g_{pu}^{-1}(G,K))^c))^c \qquad \text{by 3 of Theorem 4.4} \\ &= v\tilde{s}int(g_{pu}^{-1}(G,K)). \end{aligned}$$

Hence, $g_{pu}^{-1}(v\tilde{s}int(G,K)) \subseteq v\tilde{s}int(g_{pu}^{-1}(G,K))$.

5 ⇒ 1: Let (G,K) be a vague soft open set over Y. Then

$$\begin{aligned} v\tilde{s}int(g_{pu}^{-1}(G,K)) &\subseteq g_{pu}^{-1}(G,K) \\ &= g_{pu}^{-1}(v\tilde{s}int(G,K)) \\ &\subseteq v\tilde{s}int(g_{pu}^{-1}(G,K)) \\ \Rightarrow v\tilde{s}int(g_{pu}^{-1}(G,K)) &= g_{pu}^{-1}(G,K). \end{aligned}$$

That is, $g_{pu}^{-1}(G,K)$ is vague soft open set over X.

Hence, g_{pu} is a vague soft continuous function. □

Theorem 4.13. *Let $g_{pu} : (X,\tau,E) \to (Y,\sigma,K)$ be a bijective vague soft function. Then the following conditions are equivalent:*
1. $g_{pu}^{-1}(v\tilde{s}int(G,K)) \subseteq v\tilde{s}int(g_{pu}^{-1}(G,K))$, for each vague soft set (G,K) over Y.
2. $v\tilde{s}int(g_{pu}(F,E)) \subseteq g_{pu}(v\tilde{s}int(F,E))$, for each vague soft set (F,E) over X.

Proof. Necessary part: Let (F,E) be any vague soft set over X. Then $g_{pu}(F,E)$ is a vague soft set of Y. So by 1, $g_{pu}^{-1}(v\tilde{s}int(g_{pu}(F,E))) \subseteq v\tilde{s}int(g_{pu}^{-1}(g_{pu}(F,E)))$.

Since g_{pu} is one-to-one, $g_{pu}^{-1}(v\tilde{s}int(g_{pu}(F,E))) \subseteq v\tilde{s}int(g_{pu}^{-1}(g_{pu}(F,E))) = v\tilde{s}int(F,E)$. Thus, $g_{pu}(g_{pu}^{-1}(v\tilde{s}int(g_{pu}(F,E)))) \subseteq g_{pu}(v\tilde{s}int(F,E))$. Since g_{pu} is onto, $v\tilde{s}int(g_{pu}(F,E)) = g_{pu}(g_{pu}^{-1}(v\tilde{s}int(g_{pu}(F,E)))) \subseteq g_{pu}(v\tilde{s}int(F,E))$. Hence, $v\tilde{s}int(g_{pu}(F,E)) \subseteq g_{pu}(v\tilde{s}int(F,E))$.

Conversely, let (G,K) be any vague soft set over Y. Then $g_{pu}^{-1}(G,K)$ is a vague soft set of X. Since g_{pu} is onto, $v\tilde{s}int(G,K) = v\tilde{s}int(g_{pu}(g_{pu}^{-1}(G,K))) \subseteq g_{pu}(v\tilde{s}int(g_{pu}^{-1}(G,K)))$.

Since g_{pu} is one-to-one $g_{pu}^{-1}(v\tilde{s}int(G,K)) \subseteq g_{pu}^{-1}(g_{pu}(v\tilde{s}int(g_{pu}^{-1}(G,K)))) = v\tilde{s}int(g_{pu}^{-1}(G,K))$.

Hence, $g_{pu}^{-1}(v\tilde{s}int(G,K)) \subseteq v\tilde{s}int(g_{pu}^{-1}(G,K))$. □

5. Application

In this section, we illustrate an application of vague soft function in the field of medicine. Let us consider a patient with his/her complaints. It can be easily transformed into vague soft sets of possible symptoms and their respective importance. There are six complaints according to the patient. It can be represented by the universal set X={ a (burning in stomach), b (sleeplessness), c (indigestion), d (joint pain), e (depression)}. And it's importance taken to be the set of parameters E={e_1(high importance), e_2(low importance) }.

We know it from the medical experts,

u(burning in stomach) = acidity; u(sleeplessness) = stress;

u(indigestion) = acidity; u(joint pain) = rheumatoid; u(depression) = stress; and

p(high importance) = infrequent high potency;

p(low importance) = frequent low potency.

For the purpose of achieving mathematical manipulation, we denote causes and medical preference for treatment by symbols as follows:

α= acidity; β= stress; γ=rheumatoid;

e_1' = infrequent high potency; e_2' = frequent low potency.

Let Y={α, β, γ} and E'={e_1', e_2'}. Now, there are two vague soft classes (X,E) and (Y,E') denotes the symptoms according to their importance for the patient, the causes with its medical preference for treatments respectively. And we have the mappings $u : X \to Y$ and $p : E \to E'$ as follows,

$u(a) = \alpha$; $u(b) = \beta$; $u(c) = \alpha$; $u(d) = \gamma$; $u(e) = \beta$; and $p(e_1) = e_1'$; $p(e_2) = e_2'$.

Consider Patient's narration into vague soft set (F,E) as follows:

$$(F,E) = \left\{ \begin{array}{l} \left\langle e_1, \frac{[0.6,0.9]}{a}, \frac{[0.5,0.6]}{b}, \frac{[0.8,0.8]}{c}, \frac{[0.3,0.4]}{d}, \frac{[0.2,0.3]}{e} \right\rangle, \\ \left\langle e_2, \frac{[0.3,0.5]}{a}, \frac{[0.4,0.9]}{b}, \frac{[0.4,0.6]}{c}, \frac{[0.1,0.2]}{d}, \frac{[0.7,0.8]}{e} \right\rangle \end{array} \right\}.$$

By calculating the vague soft image of (F,E) under g_{pu} using the Definition 4.1,

we observe

$$g_{pu}(F,E) = \left\{ \left\langle e_1', \frac{[0.8,0.9]}{\alpha}, \frac{[0.5,0.6]}{\beta}, \frac{[0.3,0.4]}{\gamma} \right\rangle, \left\langle e_2', \frac{[0.4,0.6]}{\alpha}, \frac{[0.7,0.9]}{\beta}, \frac{[0.1,0.2]}{\gamma} \right\rangle \right\}.$$

According to the results we have to give medical treatments for the patients having **acidity** and **stress** with their medical preferences infrequent high potency and frequent low potency respectively.

6. Conclusion

In this paper, we have introduced the notions of vague soft points, vague soft neighborhood and vague soft separation axioms. We have investigated the algebraic properties of vague soft sets and obtained an interesting result which is contrary to the fundamental complement laws of set algebra. We have also presented the interconnections between vague soft separation axioms which are also contrary to the properties of the separation axioms in topological spaces. Counter examples are given to illustrate the above results. Also, we provided an application in the field of medicine using the concept of vague soft functions. This research work could be extended by developing computational algorithms to give more applications in the fields of medicine, Engineering and social work, the mysterious results could even be resolved.

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References

[1] A. Al-Quran, N. Hassan, *Nautrosophic vague soft sets and its applications*, Malaysian Journal of Mathematical Sciences, 11 (2017), 141-163.

[2] K. Alhazaymeh, N. Hassan, *Generalized vague soft set and its application*, International Journal of Pure and Applied Mathematics, 77 (2012), 391-401.

[3] K. Alhazaymeh, N. Hassan, *Possibility vague soft sets and its application in decision making*, International Journal of Pure and Applied Mathematics, 77 (2012), 549-563.

[4] K. Alhazaymeh, N. Hassan, *Generalized interval-valued vague soft set*, Applied Mathematical Sciences, 7 (2013), 6983-6988.

[5] K. Alhazaymeh, N. Hassan, *Generalized vague soft expert set*, International Journal of Pure and Applied Mathematics, 93 (2014), 351-360.

[6] K. Alhazaymeh, N. Hassan, *Vague soft expert sets and its application in decision making*, Malaysian Journal of Mathematical Sciences, 11 (2017), 23-39.

[7] K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 20 (1986), 87-96.

[8] Y. Bhargavi, T. Eswarla, *Application of vague set in medical diagnosis*, International Journal of Chemical Sciences, 14 (2016), 842-846.

[9] A. Borumand Saeid, A. Zarandi, *Vague set theory applied to bm-algebras*, International Journal of Algebra, 5 (2011), 207-222.

[10] W. Gau, D. Buehrer, *Vague sets*, IEEE Transactions on Systems Man and Cybernetics, 23 (1993), 610-614.

[11] S. Hussain, B. Ahmad, *Soft separation axioms in soft topological spaces*, Hacettepe Journal of Mathematics and Statistics, 44 (2015), 559-568.

[12] V. Inthumathi, M. Pavithra, *Decomposition of vague α-soft open sets in vague soft topological spaces*, Global Journal of Pure and Applied Mathematics, 14 (2018), 501-515.

[13] P. Jothi, *Vague generalized semi-pre-closed sets in topological spaces*, International Journal of Multidisciplinary Research and Development, 4 (2017), 182-187.

[14] L. Mariapresenti, I. Arockiarani, *Vague generalized alpha closed sets in topological spaces*, International Journal of Mathematical Archive, 7 (2016), 23-39.

[15] A. Mary Margaret, I. Arockiarani, *Generalized pre-closed sets in vague topological spaces,* International Journal of Applied Research, 2 (2016), 893-900.

[16] D. Molodtsov, *Soft set theory-first results*, Computers and Mathematics with Applications, 37 (1999), 19-31.

[17] C. Wang, *Decomposition theorems and representation theorems of vague soft sets*, Journal of Intelligent and Fuzzy Systems, 32 (2017), 85-95.

[18] C. Wang, *Vague parameterized vague soft set theory and its decision making*, Journal of Intelligent and Fuzzy Systems, 33 (2017), 2341-2350.

[19] C. Wang, Y. Li, *Topological structure of vague soft sets*, Abstract and Applied Analysis, Volume 2014, Article ID 504021, (2014), 8 pages.

[20] C. Wang, A. Qu, *The applications of vague soft sets and generalized vague soft sets*, Acta Mathematicae Applicatae Sinica, 31 (2015), 977-990.

[21] W. Xu, J. Ma, S. Wang, G. Hao, *Vague soft set and their properties*, Computers and Mathematics with Application, 59 (2010), 787-794.

[22] L. Zadeh, *Fuzzy sets*, Information and Control, 8 (1965), 338-353.

[23] I. Zorlutuna, M. Akdag, W. Min, S. Atmaca, *Remarks on soft topological spaces*, Annals of Fuzzy Mathematics and Informatics, 3 (2012), 171-185.

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