States on hyper equality algebras

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Abstract. In this paper, states on hyper equality algebras are introduced and studied. Two types of states, namely, Bosbach states and Riečan states are considered and some related properties of them are obtained. Also, the existence of two types of states are given by some examples and the relationships between them are discussed. One of important results is that Bosbach states and Riečan states coincide on bounded involutive hyper equality algebras. Moreover, Bosbach states on quotient hyper equality algebras are investigated. By use of θ-invariant Bosbach states, it follows that the induced function $\overline{s}$ by a a Bosbach state s on H is also a Bosbach state on H/θ and the corresponding example is constructed.

Keywords: hyper equality algebra, quotient hyper equality algebra, Bosbach state, Riečan state.

1. Introduction

The hyper structure theory was introduced by Marty [13], at the 8th Congress of Scandinavian Mathematicians. In an algebraic hyper structure, the composition of two elements is not an element but a set. Since then hyper structure theory has been intensively researched such as hyper BCK-algebras [10, 11], hyper MV-algebras [9], hyper BE-algebras [3, 15] and hyper EQ-algebras [1, 4], etc.

Now, hyper structure theory has been applied to many disciplines such as geometry, graphs, automata, cryptography, artificial intelligence, probability

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theory, dismutation reactions and inheritance, etc (see [7, 8]). Recently, Cheng, Xin and Jun [2] have applied the hyper theory to equality algebras in order to introduce hyper equality algebras which are a generalization of equality algebras. They defined some types of hyper filters and hyper deductive systems, moreover discussed the relations between hyper EQ-algebras, hyper BCK-algebras and hyper equality algebras. In addition, they also constructed quotient hyper equality algebras via regular congruence relations.

The notion of states on MV-algebras was introduced by Mundici [14] in 1995 with the intent of capturing the notion of average degree of truth of a proposition in Łukasiewicz logic, and so the states have been used as a semantical interpretation of the probability of fuzzy events a. That is, if s is a state and a is a fuzzy event, then $s(a)$ is presented as averaging of appearing the event a. Hence it is meaningful to extend the notion of states to other algebraic structures. Different approaches to the generalization mainly gave rise to two different notions, namely, Bosbach states and Riečan states. Consequently, Liu studied the existence of Bosbach states and Riečan states on MTL-algebra in [12]. Some examples showed that there exist MTL-algebras having no Bosbach states and Riečan states. Ciungu [6] extended the notions of states to the case of pseudo-BCK algebras, and proved that, in any good pseudo-BCK algebra with pseudo-double negation, Riečan states coincide with Bosbach states. Cheng, Xin and He [5] introduced states on pseudo equality algebras and proved that each Bosbach state is a Riečan state in any good bounded pseudo equality algebra, but the converse may be not true. Recently, Xin and Wang [16] have applied the state theory to hyper structures introducing and studying inf-Bosbach states and hyper measures on hyper BCK-algebras. Furthermore, they considered states on quotient hyper BCK-algebras and got some important results. Then Xin, He and Fu [17] investigated states on hyper MV-algebras and gave some equivalent conditions that a quotient hyper MV-algebra becomes a Boolean algebra. The above are our motivation of studying states on hyper equality algebras.

This paper is organized as follows: in Section 2, we review some basic concepts and results on hyper equality algebras. In Section 3, we introduce Bosbach states and Riečan states on hyper equality algebras and investigate some related properties of them. Also, we give a characterization of Bosbach states and discuss the relationships between Bosbach states and Riečan states. In Section 4, we study states on quotient hyper equality algebras and induce Bosbach states on quotient hyper equality algebras by use of θ-invariant Bosbach states. Moreover we construct a corresponding example.

2. Preliminaries

In [2] hyper equality algebras have been introduced and studied. We recollect some definitions and results from [2] which will be used in the following.

Definition 2.1 ([2]). *A hyper equality algebra $(H;\sim,\wedge,1)$ is a nonempty set H endowed with a binary operation $\wedge$, a binary hyper operation $\sim$ and a top element 1 satisfying the following axioms, for all $x,y,z\in H$:*

$(HE1)$ $(E,\wedge,1)$ *is a meet-semilattice with top element* 1;

$(HE2)$ $x\sim y\ll y\sim x$;

$(HE3)$ $1\in x\sim x$;

$(HE4)$ $x\in 1\sim x$;

$(HE5)$ $x\ll y\ll z$ *implies* $x\sim z\ll y\sim z$ *and* $x\sim z\ll x\sim y$;

$(HE6)$ $x\sim y\ll (x\wedge z)\sim (y\wedge z)$;

$(HE7)$ $x\sim y\ll (x\sim z)\sim (y\sim z)$,

where $x\ll y$ iff $x\wedge y=x$ and $A\ll B$ means that for any $x\in A$, there exists $y\in B$ such that $x\ll y$.

Clearly, an equality algebra can be seen as a hyper equality algebra. In any hyper equality algebra $(H,\sim,\wedge,1)$, define the operation $\to$ by $x\to y:=x\sim (x\wedge y)$ for any $x,y\in H$. Moreover for any nonempty subsets $A,B\subseteq H$, we write $A\wedge B=\{a\wedge b:a\in A,b\in B\}$, $A\circ B=\bigcup_{a\in A,b\in B}a\circ b$, where $\circ\in\{\sim,\to\}$.

Example 2.2 ([2]). Let $H=\{0,a,1\}$ with $0\ll a\ll 1$. Define the operations $\wedge$ and $\sim_1,\sim_2$ on H as follows: $x\wedge y=\min\{x,y\}$ and

$\sim_1$	0	a	1
0	$\{1\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{1\}$	$\{a,1\}$
1	$\{0\}$	$\{a,1\}$	$\{1\}$

$\sim_2$	0	a	1
0	$\{1\}$	$\{0,a\}$	$\{0,a\}$
a	$\{0,a\}$	$\{1\}$	$\{a\}$
1	$\{0,a\}$	$\{0,a\}$	$\{1\}$

Then $(H,\sim_1,\wedge,1)$ and $(H,\sim_2,\wedge,1)$ are hyper equality algebras.

Proposition 2.1 ([2]). *Let $(H,\sim,\wedge,1)$ be a hyper equality algebra. Then for all $x,y,z\in H,A,B,C\subseteq H$, we have:*

$(P1)$ $A\sim B\ll B\sim A$;

$(P2)$ $A\ll B$ *and* $B\ll C$ *imply* $A\ll C$;

$(P3)$ $A\subseteq B$ *implies* $A\ll B$;

$(P4)$ $x\ll y$ *implies* $y\to x=y\sim x$;

$(P5)$ $1\in x\to x$, $1\in x\to 1$, $x\ll x\sim 1$, $x\in 1\to x$;

$(P6)$ $x\sim y\ll x\to y$, $x\sim y\ll y\to x$;

$(P7)$ $x \ll y$ *implies* $1 \in x \to y$;

$(P8)$ $x \ll y \to x$, $A \ll B \to A$;

$(P9)$ $x \ll y$ *implies* $z \to x \ll z \to y$ *and* $y \to z \ll x \to z$;

$(P10)$ $x \to y = x \to x \wedge y$;

$(P11)$ $x \to y \ll (y \to z) \to (x \to z)$.

Definition 2.3 ([2]). *A hyper equality algebra* $(H, \sim, \wedge, 1)$ *is called:*

- *good if* $x = 1 \sim x$ *for any* $x \in H$;
- *bounded if there exists an element* 0 *such that* $0 \ll x$ *for any* $x \in H$.

Proposition 2.2 ([2]). *Let* $(H, \sim, \wedge, 1)$ *be a good hyper equality algebra. Then for any* $x, y \in H$:

(1) $x \ll (x \sim y) \sim y$;

(2) $x \ll (x \to y) \to y$;

(3) $x \ll y \to z$ *iff* $y \ll x \to z$;

(4) $x \to (y \to z) \ll y \to (x \to z)$.

In any bounded hyper equality algebra $(H, \sim, \wedge, 1)$, we denote $x^* := x \sim 0$ for any $x \in H$. Then $x^* = x \to 0$.

Proposition 2.3 ([2]). *Let* $(H, \sim, \wedge, 0, 1)$ *be a bounded hyper equality algebra. Then for all* $x, y, z \in H$:

(1) $0 \in 1^*$ *and* $1 \in 1^{**}$;

(2) $1 \in 0^*$ *and* $0 \in 0^{**}$;

(3) $x \sim y \ll x^* \sim y^*$;

(4) $x \to y \ll y^* \to x^*$;

(5) $x \ll y$ *implies* $y^* \ll x^*$.

Definition 2.4 ([2]). *Let* $(H, \sim, \wedge, 1)$ *be a hyper equality algebra. A subset* D *containing* 1 *of* H *is called a strong hyper deductive system, if* $x \in D$ *and* $x \to y \cap D \neq \emptyset$ *imply* $y \in D$ *for all* $x, y \in H$.

Notice that in any hyper equality algebra, each strong hyper deductive system is an upset.

Lemma 2.1 ([2]). *Let* D *be an upset of a hyper equality algebra* $(H, \sim, \wedge, 1)$. *Then for any nonempty subset* A, B *of* H, $A \cap D \neq \emptyset$ *and* $A \ll B$ *imply* $B \cap D \neq \emptyset$.

Definition 2.5 ([2]). *Let $(H, \sim, \wedge, 1)$ be a hyper equality algebra and θ be an equivalence relation on H.*

(1) *for any $A, B \subseteq H$, $A\overline{\theta}B$ means for any $a \in A$ there exists $b \in B$ such that $a\theta b$ and for any $b \in B$ there exists $a \in A$ such that $a\theta b$;*

(2) *θ is called a congruence relation if for any $x, y, u, v \in H$, $x\theta y$ and $u\theta v$ imply $x \sim u\overline{\theta}y \sim v$ and $x \wedge u\theta y \wedge v$.*

Let θ be a congruence relation on a hyper equality algebra $(H, \sim, \wedge, 1)$. Denote $H/\theta = \{[x]_\theta : x \in H\}$, where $[x]_\theta = \{y \in H : y\theta x\}$ and denote $[A]_\theta = \{[a]_\theta : a \in A\}$. $\overline{\sim}$, $\overline{\Rightarrow}$ and $\overline{\wedge}$ on H/θ are defined by $[x]_\theta\overline{\sim}[y]_\theta = [x \sim y] = \{[a]_\theta : a \in x \sim y\}$, $[x]_\theta\overline{\Rightarrow}[y]_\theta = [x \to y]_\theta = \{[a]_\theta : a \in x \to y\}$ and $[x]_\theta\overline{\wedge}[y]_\theta = [x \wedge y]_\theta$, respectively. $[x]_\theta \ll_\theta [y]_\theta$ iff $[x]_\theta = [x]_\theta\overline{\wedge}[y]_\theta$ iff $x \wedge y\theta x$ for any $[x]_\theta, [y]_\theta \in H/\theta$. For all $A, B \subseteq H/\theta$, $A \ll_\theta B$ means for any $[a]_\theta \in A$, there exists $[b]_\theta \in B$ such that $[a]_\theta \ll_\theta [b]_\theta$. Clearly, $x \ll y$ implies that $[x]_\theta \ll_\theta [y]_\theta$.

Definition 2.6 ([2]). *A congruence relation θ on a hyper equality algebra $(H, \sim, \wedge, 1)$ is called regular if $[x]_\theta \ll_\theta [y]_\theta$ implies $[x]_\theta \ll [y]_\theta$ for all $x, y \in H$.*

Theorem 2.1 ([2]). *Let θ be a regular congruence relation on a hyper equality algebra $(H, \sim, \wedge, 1)$. Then:*

(1) *$[x]_\theta \ll_\theta [y]_\theta$ implies that $[1]_\theta \in [x]_\theta\overline{\Rightarrow}[y]_\theta$ for any $x, y \in H$;*

(2) *$(H/\theta, \overline{\sim}, \overline{\wedge}, [1]_\theta)$ is a hyper equality algebra, which is called as the quotient hyper equality algebra with respect to θ.*

3. States on hyper equality algebras

In this section, we introduce Bosbach states and Riečan states on hyper equality algebras and investigate some related properties of them. Also, we give a characterization of Bosbach states, and discuss the relationships of Bosbach states and Riečan states. By H denote a bounded hyper equality algebra $(H, \sim, \wedge, 0, 1)$, unless otherwise specified.

Definition 3.1. *A function $s : H \to [0, 1]$ is said to be a Bosbach state on H, if it satisfies the following conditions:*

$(BS1)$ *$s(0) = 0$ and $s(1) = 1$;*

$(BS2)$ *$s(x) + s(x \to y) = s(y) + s(y \to x)$ for all $x, y \in H$,*

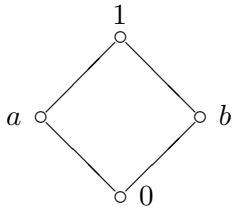
where $s(A) = \sup\{s(a) : a \in A\}$ for all $A \subseteq H$.

Example 3.2. (1) Let $(H, \sim_1, \wedge, 0, 1)$ be a bounded hyper equality algebra in Example 2.2. Define a function $s : H \to [0, 1]$ by $s(0) = 0, s(a) = s(1) = 1$. Then one can calculate that s is a unique Bosbach state on H.

(2) Let $(H; \sim_2, \wedge, 1)$ be a bounded hyper equality algebra in Example 2.2. Define a function $s : H \to [0, 1]$ by $s(0) = 0, s(a) = 0.5, s(1) = 1$. Then one can check that s is a unique Bosbach state on H.

The following example indicates that there exist hyper equality algebras not possessing Bosbach states.

Example 3.3. Let $H = \{0, a, b, 1\}$ in which the Hasse diagram and the operations $\wedge, \sim$ are as follows:



$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\sim$	0	a	b	1
0	$\{1\}$	$\{1\}$	$\{b, 1\}$	$\{0, a\}$
a	$\{1\}$	$\{1\}$	$\{a, 1\}$	$\{a\}$
b	$\{b, 1\}$	$\{a, 1\}$	$\{1\}$	$\{b, 1\}$
1	$\{0, a\}$	$\{a\}$	$\{b, 1\}$	$\{1\}$

Then $(H, \sim, \wedge, 0, 1)$ is a bounded hyper equality algebra [2]. Assume that the function $s : H \to [0, 1]$ defined by $s(0) = 0, s(a) = \alpha, s(b) = \beta, s(1) = 1$ is a Bosbach state on H. In (BS2), taking $x = 1, y = b$ and $x = 0, y = b$, we obtain $\beta = 1$ and $\beta = 0$, respectively, which is a contraction. This shows that H admits no Bosbach state.

Proposition 3.1. *Let s be a Bosbach state on H. Then for any $x, y \in H$, the following hold:*

(1) *If $x \ll y$, then $s(x) \leq s(y)$;*

(2) *If $x \ll y$, then $s(y \to x) = 1 + s(x) - s(y) = s(y \sim x) = s(x \sim y)$;*

(3) *$s(x^*) = 1 - s(x)$;*

(4) *$s(A^*) = 1 - s_*(A)$, where $s_*(A) = \inf\{s(a) : a \in A\}$;*

(5) *$s(x \to y) = 1 + s(x \wedge y) - s(x)$;*

(6) *$Ker(s) = \{x \in H : s(x) = 1\}$ is a strong hyper deductive system of H.*

Proof. (1) Let $x \ll y$. Then $1 \in x \to y$ and so $s(x \to y) = s(1) = 1$. Hence by the definition of the Bosbach state, $s(x) + 1 = s(y) + s(y \to x)$. This implies that $s(x) - s(y) = s(y \to x) - 1 \leq 0$ and thus $s(x) \leq s(y)$.

(2) Let $x \ll y$. Then $y \to x = y \sim x$. Using the proof of (1), we have $s(y \sim x) = s(y \to x) = 1 + s(x) - s(y)$.

(3) Since $0 \ll x$, we have $1 \in 0 \to x$ and thus $s(0 \to x) = 1$. Considering $x^* = x \to 0$ and $s(x) + s(x \to 0) = s(0) + s(0 \to x)$, we can get that $s(x^*) = s(0 \to x) - s(x) = s(1) - s(x) = 1 - s(x)$.

(4) Using (3), we get that $s(A^*) = \sup\{s(a^*) : a \in A\} = \sup\{1 - s(a) : a \in A\} = 1 - \inf\{s(a) : a \in A\} = 1 - s_*(A)$.

(5) Since $x \wedge y \ll x$, then from (2) we obtain that $s(x \to y) = s(x \to x \wedge y) = 1 + s(x \wedge y) - s(x)$.

(6) Let $x \to y \cap Ker(s) \neq \emptyset$ and $x \in Ker(s)$ for any $x, y \in H$. Then $s(x) = 1$ and there exists $a \in x \to y$ such that $a \in Ker(s)$. Hence $s(a) = 1$, which implies $s(x \to y) = \sup\{s(t) : t \in x \to y\} = 1$. Since s is a Bosbach state on H, we have $s(y) = s(x) + s(x \to y) - s(y \to x) = 2 - s(y \to x) \geq 1$. Therefore $s(y) = 1$ and thus $y \in Ker(s)$. □

In what follows, we give a characterization of Bosbach states on hyper equality algebras.

Theorem 3.1. *Let $s : H \to [0, 1]$ be a function with $s(0) = 0$. Then the following are equivalent:*

(1) *s is a Bosbach state;*

(2) *If $x \ll y$, then $s(y \sim x) = 1 + s(x) - s(y) = s(y \to x)$;*

(3) *$s(y \to x) = 1 - s(y) + s(x \wedge y)$.*

Proof. (1) ⇒ (2) By Proposition 3.1.

(2) ⇒ (3) Assume (2) holds. Since $x \wedge y \ll y$, then $s(y \to x) = s(y \to x \wedge y) = 1 + s(x \wedge y) - s(y)$.

(3) ⇒ (1) Assume (3) holds. Since $1 \in x \to 1$, then $s(1) = s(x \to 1) = 1 - s(x) + s(x \wedge 1) = 1 - s(x) + s(x) = 1$. Also, $s(x) + s(x \to y) = s(x) + 1 - s(x) + s(x \wedge y) = 1 + s(x \wedge y) = s(y) + 1 - s(y) + s(x \wedge y) = s(y) + s(y \to x)$. Thus s is a Bosbach state on H. □

Next we introduce the notion of Riečan states on hyper equality algebras.

Definition 3.4. *Two elements $x, y \in H$ is said to be orthogonal, if $1 \in x^{**} \to y^*$, we write by $x \perp y$. If $x, y \in H$ is orthogonal, we define a hyper operation $\oplus$ on H by $x \oplus y := y^* \to x^{**}$.*

Proposition 3.2. *In any bounded hyper equality algebra H, the following hold for all $x, y \in H$:*

(1) *$0 \perp x$ and $x^{**} \subseteq 0 \oplus x$;*

(2) *$x \perp 0$ and $x^{**} \subseteq x \oplus 0$.*

Proof. (1) Since $1 \in 0 \to x^* \subseteq 0^{**} \to x^*$, then $0 \perp x$. From Proposition 2.3 $0 \in 0^{**}$, it follows that $x^{**} = x^* \to 0 \subseteq x^* \to 0^{**} = 0 \oplus x$.

(2) Since $1 \in 0^*$ by Proposition 2.3, we have $1 \in x^{**} \to 1 \subseteq x^{**} \to 0^*$. Hence $x \perp 0$. $x^{**} \subseteq x \oplus 0$ can be obtained by $x^{**} \subseteq 1 \to x^{**} \subseteq 0^* \to x^{**}$. □

Definition 3.5. *A Riečan state on a bounded hyper equality algebra H is a function $s : H \to [0,1]$ such that:*

$(RS1)$ $s(1) = 1$;

$(RS2)$ $s(x \oplus y) = s(x) + s(y)$ *whenever* $x \perp y$ *for all* $x, y \in H$, *where* $s(A) = \sup\{s(a) : a \in A\}$ *for any* $A \subseteq H$. *In particular, if* $s(x^*) = 1 - s(x)$ *for all* $x \in H$, *the Riečan state* s *is said to be normal.*

Example 3.6. Let $H = \{0, a, 1\}$. Define the operations $\wedge$ and $\sim$ on H as follows: $x \wedge y = \min\{x, y\}$ and

$\sim$	0	a	1
0	$\{1\}$	$\{0, a\}$	$\{0\}$
a	$\{0, a\}$	$\{1\}$	$\{a\}$
1	$\{0\}$	$\{a\}$	$\{1\}$

Then $(H, \sim, \wedge, 0, 1)$ is a bounded hyper equality algebra [2]. Now we define a function $s : H \to [0,1]$ by $s(0) = 0, s(a) = s(1) = 1$. One can check that s is a Riečan state, but it is not normal since $s(a^*) = s(\{0, a\}) = s(a) = 1 \neq 0 = 1 - s(a)$.

Example 3.7. Let $(H, \sim_1, \wedge, 0, 1)$ be a bounded hyper equality algebra given by Example 2.2, and the function $s : H \to [0,1]$ be defined by $s(0) = 0, s(a) = s(1) = 1$. Then one can calculate that s is a unique normal Riečan state on H.

The following example shows that there exist hyper equality algebras having no (normal) Riečan states.

Example 3.8. Let $(H, \sim, \wedge, 0, 1)$ be a bounded hyper equality algebra given by Example 3.3, and the function $s : H \to [0,1]$ defined by $s(0) = \alpha, s(a) = \beta, s(b) = \gamma, s(1) = 1$ be a Riečan state on H. In (RS2), taking $x = 0, y = a$ and $x = b, y = 0$, we can obtain that $s(a) = s(0 \oplus a) = s(0) + s(a)$ and $1 = s(b \oplus 0) = s(b) + s(0)$, respectively. This implies $\beta = 0$ and $\beta = 1$. On the other hand, taking $x = 1, y = b$ in (RS2), we get that $1 = s(1 \oplus b) = s(1) + s(b)$ and so $\beta = 0$, which is a contraction. Therefore H admits no (normal) Riečan state.

Proposition 3.3. *Let s be a normal Riečan state on H. Then for all $x, y \in H$ and $A \subseteq H$, we have:*

(1) $s(0) = 0$;

(2) $s(x^{**}) \leq s(x)$;

(3) $s(A^*) = 1 - s_*(A)$, *where* $s_*(A) = \inf\{s(a) : a \in A\}$;

(4) $x \ll y$ *implies* $s(x) \leq s(y)$ *and* $s(y^{**} \to x^{**}) = 1 + s(x) - s(y)$.

Proof. (1) Since $0 \in 1^*$, we have $s(\{0\}) \leq s(1^*) = 1 - s(1) = 1 - 1 = 0$.

(2) Since $x^{**} \subseteq x \oplus 0$ from Proposition 3.2, then $s(x^{**}) \leq s(x \oplus 0) = s(x) + s(0) = 0 + s(x) = s(x)$.

(3) Similar to the proof of Proposition 3.1 (4).

(4) Let $x \ll y$ for any $x, y \in H$. Then it follows from Proposition 2.3 that $x^{**} \ll y^{**}$. Hence $1 \in x^{**} \to y^{**}$ and so there is $t \in y^*$ such that $1 \in x^{**} \to t^*$. This implies $x \perp t$ and thus $s(t^* \to x^{**}) = s(x \oplus t) = s(x) + s(t)$. Then we can get that $\sup\{s(t^* \to x^{**}) : t \in y^*\} = s(x) + \sup\{s(t) : t \in y^*\}$. That is $s(y^{**} \to x^{**}) = s(x) + s(y^*) = 1 + s(x) - s(y)$. Furthermore $s(x) - s(y) = s(y^{**} \to x^{**}) - 1 \leq 0$, which implies $s(x) \leq s(y)$. □

Remark 3.9. There exist hyper equality algebras admitting Bosbach states but not possessing Riečan states. Conversely, it is also true that there exist hyper equality algebras admitting Riečan states but not possessing Bosbach states. Indeed, let $(H; \sim_2, \wedge, 1)$ be the bounded hyper equality algebras given by Example 2.2. From (2) of Example 3.2 we know that the function $s_1 : H \to [0, 1]$ defined by $s_1(0) = 0, s_1(a) = 0.5, s_1(1) = 1$ is a unique Bosbach state on H. Also one can calculate that the function $s_2 : H \to [0, 1]$ defined by $s_2(0) = 0, s_2(a) = s_2(1) = 1$ is a unique Riečan state on H.

Definition 3.10. *A hyper equality algebra $(H, \sim, \wedge, 1)$ is said to be involutive if $x^{**} = x$ for all $x \in H$.*

Example 3.11. Let $H = \{0, a, 1\}$ with $0 \ll a \ll 1$. Define the operations $\wedge$ and $\sim$ on H as follows: $x \wedge y = \min\{x, y\}$ and

$\sim$	0	a	1
0	$\{1\}$	$\{0, a\}$	$\{0\}$
a	$\{a\}$	$\{1\}$	$\{a\}$
1	$\{0\}$	$\{a\}$	$\{1\}$

One can easily check that $(H; \sim, \wedge, 1)$ is an involutive hyper equality algebra.

Lemma 3.1. *Let s be a Riečan state (Bosbach state) on H. Then $A \ll B$ implies $s(A) \leq s(B)$ for any $A, B \subseteq H$.*

Proof. Since $A \ll B$, then for any $a \in A$, there exists $b \in B$ such that $a \ll b$. Hence $\sup\{s(a) : a \in A\} \leq \sup\{s(b) : b \in B\}$, namely, $s(A) \leq s(B)$. □

Theorem 3.2. *Let s be a Riečan state on H. Then the following are equivalent:*

(1) *s is a Bosbach state on H;*

(2) *$x \ll y$ implies $s(y \to x) \geq 1 + s(x) - s(y)$ for any $x, y \in H$.*

Proof. (1) $\Rightarrow$ (2) By Theorem 3.1.

(2) $\Rightarrow$ (1) Assume (2) holds and s is a Riečan state on H. Then $s(1) = 1$ and $s(0) = 0$. Now let $x \ll y$ for any $x, y \in H$. Then by Proposition 3.3 we

have $s(y^{**} \to x^{**}) = 1 + s(x) - s(y)$. On the other hand, it follows from $x \ll y$ and Proposition 2.3 that $y \to x \ll x^* \to y^* \ll y^{**} \to x^{**}$. Hence from (P2) $y \to x \ll y^{**} \to x^{**}$. Again using Lemma 3.1, we obtain $s(y \to x) \leq s(y^{**} \to x^{**})$. Therefore $s(y \to x) \leq 1 + s(x) - s(y)$. Considering (2) and Theorem 3.1, we show that s is a Bosbach state on H. □

Theorem 3.3. *Riečan states and Bosbach states coincide on bounded involutive hyper equality algebras.*

Proof. Assume that $(H; \sim, \wedge, 1)$ is a bounded involutive hyper equality algebra. If s is a Riečan state on H and $x \ll y$ for any $x, y \in H$, then $s(y \to x) = s(y^{**} \to x^{**}) = 1 + s(x) - s(y)$ from the proof of Theorem 3.2 and the involutivity of H. It follows from Theorem 3.5 that s is a Bosbach state. Conversely, if s is a Bosbach state on H and $x \perp y$, namely, $1 \in x^{**} \to y^*$ for any $x, y \in H$. Since $s(t) + s(t \to x) = s(x) + s(x \to t)$ for any $t \in y^*$, we get that $sup\{s(t) : t \in y^*\} + sup\{s(t \to x : t \in y^*\} = s(x) + sup\{s(x \to t : t \in y^*\}$. According to the involutivity of H, $s(y^*) + s(y^* \to x^{**}) = s(x) + s(x^{**} \to y^*)$. It follows from $1 \in x^{**} \to y^*$ and $s(y^*) = 1 - s(y)$ that $s(x \oplus y) = s(y^* \to x^{**}) = s(x) + s(y)$. Therefore s is a Riečan state on H. □

Example 3.12. In Example 3.11, define a function $s : H \to [0, 1]$ by $s(0) = 0, s(a) = 0.5, s(1) = 1$. Then one can calculate that s is both a Riečan state and a Bosbach state on H.

4. Bosbach states on quotient hyper equality algebras

In this section, we investigate Bosbach states on quotient hyper equality algebras. By use of θ-invariant Bosbach states, we get the result that the induced function $\overline{s}$ by a a Bosbach state s on H is also a Bosbach state on H/θ and furthermore we construct a corresponding example of the above result.

Definition 4.1. *Let θ be a congruence relation and s be a Bosbach state on H. Then s is called θ-invariant provided that $x\theta y$ implies $s(x) = s(y)$ for all $x, y \in H$.*

Lemma 4.1. *Let θ be a regular congruence relation and s be a θ-invariant Bosbach state on H. Then in the bounded quotient hyper equality algebra $(H/\theta; \overline{\wedge}, \overline{\sim}, [0]_\theta, [1]_\theta)$, the following hold for any $x, y \in H$:*

(1) $[x]_\theta = [y]_\theta$ *implies* $s(x) = s(y)$;

(2) $[x]_\theta \leq_\theta [y]_\theta$ *implies* $s(x \to y) = 1, s(y \to x) = 1$ *and* $s(x) = s(x \wedge y)$.

Proof. (1) It is evident by Definition 4.1.

(2) Let $[x]_\theta \ll_\theta [y]_\theta$ for any $x, y \in H$. Then $[1]_\theta \in [x]_\theta \overline{\Rightarrow} [y]_\theta$ from Proposition 2.1. Hence there exists $a \in x \to y$ such that $a\theta 1$. Since s is a θ-invariant Bosbach state, then $s(a) = s(1) = 1$ and so $s(x \to y) = sup\{s(t) : t \in x \to y\} = 1$. $s(x) = s(x \wedge y)$ follows from $x\theta x \wedge y$. □

Theorem 4.1. *Let θ be a regular congruence relation and s be a θ-invariant Bosbach state on H. Define $\overline{s} : H/\theta \to [0,1]$ by $\overline{s}([x]_\theta) = s(x)$, where $\overline{s}(\{[x]_\theta : x \in A\}) = \sup\{\overline{s}([x]_\theta) : x \in A\}$, for any $x \in H, A \subseteq H$. Then $\overline{s}$ is a Bosbach state on the bounded quotient hyper equality algebra $(H/\theta; \overline{\wedge}, \overline{\sim}, [0]_\theta, [1]_\theta)$.*

Proof. By Lemma 4.1 $\overline{s}$ is well-defined. $\overline{s}([0]_\theta) = s(0) = 0$ and $\overline{s}([1]_\theta) = s(1) = 0$. Since $\overline{s}([x]_\theta \overline{\Rightarrow} [y]_\theta) = \sup\{\overline{s}([t]_\theta) : t \in x \to y\} = \sup\{s(t) : t \in x \to y\} = s(x \to y)$. Hence by (BS2) we have $\overline{s}([x]_\theta) + \overline{s}([x]_\theta \overline{\Rightarrow} [y]_\theta) = \overline{s}([y]_\theta) + \overline{s}([y]_\theta \overline{\Rightarrow} [x]_\theta)$. That is, $\overline{s}$ is a Bosbach state on H/θ. □

In what follows, we construct an example of Theorem 4.1.

Definition 4.2. *A nonempty subset S of H is called $\sim$-reflexive provided that $x \sim y \cap S \neq \emptyset$ implies $x \sim y \subseteq S$.*

Let s be a Bosbach state on H and $Ker(s)$ be $\sim$-reflexive, we define the relation θ_K as follows: $x\theta_K y$ iff $x \sim y \cap Ker(\mu) \neq \emptyset$ for any $x, y \in H$.

Proposition 4.1. *Let s be a Bosbach state on H and $Ker(s)$ be $\sim$-reflexive. Then:*

(1) *θ_K is an equivalence relation on H;*

(2) *θ_K preserves the operation $\wedge$.*

Proof. (1) Let $x, y, z \in H$. Since $1 \in x \sim x$ we have $x \sim x \cap Ker(s) \neq \emptyset$ and thus θ_K is reflexive. The symmetry of θ_K follows from (HE2) and Lemma 2.1. Now we prove the transitivity of θ_K. If $x\theta_K y$ and $y\theta_K z$, then $x \sim y \cap Ker(\mu) \neq \emptyset$ and $y \sim x \cap Ker(s) \neq \emptyset$. From (HE7),(HE2) and (P6) $x \sim y \ll (x \sim z) \sim (y \sim z) \ll (y \sim z) \sim (x \sim z) \ll (y \sim z) \to (x \sim z)$. It follow from (P2) that $x \sim y \ll (y \sim z) \to (x \sim z)$. Using Lemma 2.1 and $x \sim y \cap Ker(\mu) \neq \emptyset$ we have $(y \sim z) \sim (x \sim z) \cap Ker(s) \neq \emptyset$. Hence there exist $a \in x \sim y$ and $b \in y \sim z \subseteq Ker(s)$ such that $a \to b \cap Ker(s) \neq \emptyset$, which implies $b \in Ker(s)$. This shows that $x \sim z \cap Ker(s) \neq \emptyset$, that is, $x\theta_K z$.

(2) Suppose $x\theta_K y$ and $u\theta_K v$ for any $x, y, u, v \in H$. Then $x \sim y \cap Ker(s) \neq \emptyset$ and $u \sim v \cap Ker(s) \neq \emptyset$. By (HE6) we have $x \sim y \ll (x \wedge v) \sim (y \wedge v)$ and $u \sim v \ll (x \wedge u) \sim (x \wedge v)$. Hence $(x \wedge v) \sim (y \wedge v) \cap Ker(s) \neq \emptyset$ and $(x \wedge u) \sim (x \wedge v) \cap Ker(s) \neq \emptyset$. That is, $x \wedge v \theta_K y \wedge v$ and $x \wedge u \theta_K x \wedge v$, which implies $x \wedge u \theta_K y \wedge v$. □

We shall denote the equivalence class of $x \in H$ w.r.t. θ_K by $[x]_K$ and the quotient set associated with θ_K by H/K.

Example 4.3. Assume that $(H, \sim_1, \wedge, 0, 1)$ is a bounded hyper equality algebra given in Example 2.2. One can check that $\theta_K = \{(0,0), (a,a), (1,1), (1,a), (a,1)\}$ is a regular congruence relation and the function s defined by Example 3.2 is a θ_K-invariant Bosbach state on H. The quotient hyper equality algebra $H/K = \{[0]_K, [1]_K\}$, where $[0]_K = \{0\}, [a]_K = [1]_K = \{a, 1\}$ and $\overline{\sim}$ is as follows:

$\overline{\sim_1}$	$[0]_K$	$[1]_K$
$[0]_K$	$\{[1]_K\}$	$\{[0]_K\}$
$[1]_K$	$\{[0]_K\}$	$\{[1]_K\}$

One can easily calculate that $\overline{s} : H/\theta \to [0,1]$ defined by $\overline{s}([0]_K) = s(0) = 0, \overline{s}([1]_K) = s(1) = 1$ is a Bosbach state on H/K.

By Theorem 4.1, we can obtain the following theorem.

Theorem 4.2. *Assume that θ is a regular congruence relation, and s is a θ-invariant Bosbach state on H. Then there is a unique Bosbach state $\overline{s} : H/\theta \to [0,1]$ such that $s = \overline{s} \circ \pi_\theta$, where $\pi_\theta : H \to H/\theta$ defined by $\pi_\theta(x) = [x]_\theta$ is the canonical map.*

Proof. Let $[t]_K \in H/\theta$ for any $t \in H$. Then $\pi_\theta(t) = [t]_\theta$. Setting $s(t) = \overline{s}([t]_\theta)$, where $\overline{s}(\{[t]_\theta : t \in A\}) = \sup\{\overline{s}([t]_\theta) : t \in A\}$ for any $x \in H, A \subseteq H$, the function $\overline{s}$ is a Bosbach state from Theorem 4.1 and further $s(t) = \overline{s}([t]_\theta) = \overline{s}(\pi_\theta(t)) = \overline{s} \circ \pi_\theta(t)$. That is $s = \overline{s} \circ \pi_\theta$. Now we show that $\overline{s} \circ \pi_\theta(t)$ is also a Bosbach state on H. $\overline{s} \circ \pi_\theta(0) = \overline{s}(\pi_\theta(0)) = \overline{s}([0]_\theta) = s(0) = 0$ and similarly $\overline{s} \circ \pi_\theta(1) = s(1) = 1$. $\overline{s} \circ \pi_\theta(x) + \overline{s} \circ \pi_\theta(x \to y) = \overline{s}([x]_\theta) + \overline{s}([x \to y]_\theta) = \overline{s}([x]_\theta) + \overline{s}([x]_\theta \overrightarrow{\to} [y]_\theta) = \overline{s}([y]_\theta) + \overline{s}([y]_\theta \overrightarrow{\to} [x]_\theta) = \overline{s}([y]_\theta) + \overline{s}([y \to x]_\theta) = \overline{s} \circ \pi_\theta(y) + \overline{s} \circ \pi_K(y \to x)$. Finally, we prove that $\overline{s}$ is unique. Suppose there is another Bosbach state ϑ on H/θ such that $s = \vartheta \circ \pi_\theta$. Then for any $[a]_\theta \in H/\theta$, $\vartheta([a]_\theta) = \vartheta(\pi_\theta(a)) = \vartheta \circ \pi_\theta(a) = s(a) = \overline{s}([a]_\theta)$, which implies $\vartheta = \overline{s}$. Combining the above arguments, this proof is finished. □

5. Conclusions

States play an important role in studying fuzzy logics and the related algebraic structures. Recently, states on hyper BCK-algebras and hyper MV-algebras have been studied. In this paper, we introduce states including Bosbach states and Riečan states on hyper equality algebras and obtain some important results. In future work, we shall further study state theory, especially on quotient hyper equality algebras.

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