Granular reduction based matrix in crisp-fuzzy variable threshold concept lattices

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Abstract. Attribute reduction is a crucial issue in fuzzy formal concept analysis (FFCA for short). Essentially, the volume of the large database can be decreased by the representation of knowledge as far as possible. With respect to reduction, the discernibility matrix and the discernibility function have played a meaningful affect. Nevertheless, the common approach needs to consider the construction of fuzzy concept lattice before searching reducts, and this invariably leads to the large amount of computation during constructing lattice. Considering adequately this problem, granular concepts are taken into account in this paper. First, the intensions of object set, the extensions of attribute set, and object granular matrix based on matrix operations are investigated in detail. Subsequently, the properties of fuzzy sub-contexts and join-irreducible elements derived from a given crisp-fuzzy variable threshold concept lattice are systematically determined, respectively. Furthermore, the notations of granular consistent set, granular reduct, and discernibility attribute matrix, with a given threshold level, are proposed. In the process of reduction, the construction of a variable threshold concept lattice is averted. Hence, the computational complexity is largely reduced, and our discussion is employed by an algorithm of granular reduction on the basis of matrix theory.

Keywords: discernibility attribute matrix, formal fuzzy context, granular reduction, variable threshold.

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1. Introduction

Wille [1, 2] pioneered an order-theoretic mathematical model to show the formal concept analysis (FCA) induced by a context $F = (U, A, I)$. This context F contains an object set U, an attribute set A, and a binary relation $I \subseteq U \times A$ derived from objects and attributes which should be in principle necessary. Focusing on FCA, the essential notion of its is a concept hierarchy denoted as a concept lattice, where a formal concept called a pair (X, B) is composed of extent X (object set) and intent B (attribute set) holden Galois connection. In recent years, it already has a widespread and successful application in the domains of knowledge acquisition, data mining and other orientations [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

Nevertheless, in real-word applications, the fuzzy binary relation is more comomn than its crisp counterpart. Therefore, a great many of scholars have applied the theory of classical formal contexts to the formal fuzzy contexts based on fuzzy sets [16]. For instance, R. Bêlohlávek [17, 18, 19] formulated some theories and properties without changing the structure of formal fuzzy contexts from the perspectives of fuzzy logic. After that, Burusco et al. [20, 21] firstly illustrated L-fuzzy concept lattice, and made a comparison of the fuzziness in a given L-fuzzy formal context by utilizing the fuzzy implications of Łukasiewicz, Kleene [22], and Gödel [23]. And concluding that the fuzzy implication of Kleene is fuzzier than the Łukasiewicz and the Gödel applications. Formica [24] proposed an approach to evaluate concept similarity in fuzzy formal contexts. And further, it should be noted that defining the fuzzy operations to obtain fuzzy concepts has been highly successful from different angles. Krajci [25], Yahia and Jaoua [26] all-sidedly investigated an "one-sided fuzzy concept", made identical to a variable threshold concept lattice with a determined threshold $\delta = 1$. Correspondingly, from the granular computing, with a given formal fuzzy context based on the notion of "crisp-fuzzy fuzzy concept", Shao et al. [27] and Shi et al. [28], respectively, demonstrated attribute and object granular reduct approaches in the sense of removing the non-essential attributes and objects while ensuring the lattice structure unchanged. And the proposed method improved greatly the computational efficiency and saved available storage space. Afterwards, Li et al. [30] further studied the judgment theorems of attribute characteristics, i.e. indispensable attributes, relative necessary attributes and dispensable attributes in fuzzy formal contexts for "fuzzy-crisp fuzzy concept". Then the algorithms which are used to compute three types of attributes were proposed. Subsequently, based on formal fuzzy contexts proposed by Shao et al. [27], Mao et al. [31] discussed a new framework to determine all irreducible elements and classify the attribute characteristics applying with the directed graph.

As is well known, Zhang et al. [32] developed three kinds of "variable threshold concept lattices", that is, "crisp-crisp", "crisp-fuzzy", and "fuzzy-crisp" variable threshold concept lattice. If the threshold is $\delta = 1$, the above lat-

tices become "one-sided fuzzy concept lattices". In Zhang et al. [32] notations, the properties are similar to that of a classical concept lattice, and can also be derived from a formal fuzzy context. J Konecny [33] studied a new method to adjust the clarification and reduction approach as an extension of variable threshold concept based on a simple scaling. In addition, Shao et al. [34] generalized the approach of attribute reduction in classical formal contexts proposed by Zhang et al. [35] to variable threshold concept lattices based on the literature [32]. And he proposed knowledge reduction frameworks to construct discernibility matrices which are removed the non-essential attributes and objects without loss of the granular structures. Furthermore, the essential approach of enumerating all the attribute reducts based on discernibility matrix introduced in [35] was applied to namely the three-way concept lattices [37] and the generalized one-sided formal contexts [38].

Compared to "one-sided fuzzy concept" in formal fuzzy contexts, knowledge reduction has always been a meaningful issue. However, little work about computing attribute reduction from the perspectives of granular computing within variable threshold concept lattices parallel to that developed by Shao et al. [27]. In a formal fuzzy context, an object concept is a join of some other object concepts. More importantly, Shao et al. [27] presented a method of granular reduction in the sense of maintaining the extensions of all object concepts. Afterwards, Lin et al. [40] introduced two based-matrix heuristic algorithms of granular reduction by the similar degree about attributes. It should be noticed that constructing the corresponding lattice in variable threshold concept lattices is more complicated. Therefore, the reductions based on the granular computing are more interesting. In this article, based on the matrix theory, a novel framework to calculate reducts is necessarily formulated based on the object granular matrix. Specially, employing the proposed method can be weaken to "one-sided fuzzy concept" when the threshold $\delta = 1$.

The rest of this paper is arranged as follows. Section 2 reviews the notations of variable threshold concept lattices and matrix operations. In the next section, based on the matrix theory, the intensions of object set, extensions of attribute set, and the object granular matrix are developed, respectively. Section 4 and 5 discussed the properties of join-irreducible elements, the discernibility attribute matrix, and the corresponding algorithm for reducing the size of a formal fuzzy context. In the final section, we draw a conclusion with a summary.

2. Preliminaries

In this section, we first briefly review some notions and terminologies of the variable threshold concept lattices and fuzzy logic, and then give some results of matrix operations. Specific information can be found in [32, 34].

2.1 Variable threshold concept lattices and crisp-fuzzy variable threshold concept

Let U be a nonempty and finite set called the universe of discourse and L be a complete lattice. A mapping $\tilde{X} : U \to L$ is described as a L-set of U, and $\tilde{X}(x)$ is denoted as the truth degree of the fact "x belongs to $\tilde{X}$" for all $x \in U$. The set of all L-sets in U is denoted as L^U. For any $\tilde{X}_1$, $\tilde{X}_2 \in L^U$, $\tilde{X}_1 \subseteq \tilde{X}_2$ is given by the expression:

$$\tilde{X}_1 \subseteq \tilde{X}_2 \Leftrightarrow \tilde{X}_1(x) \leq \tilde{X}_2(x), \forall x \in U;$$

and operators $\vee$ and $\wedge$ on L^U are denoted by:

$$(\tilde{X}_1 \vee \tilde{X}_2)(x) = \tilde{X}_1(x) \vee \tilde{X}_2(x);\ (\tilde{X}_1 \wedge \tilde{X}_2)(x) = \tilde{X}_1(x) \wedge \tilde{X}_2(x).$$

Definition 2.1 ([17, 18]). *A structure $\boldsymbol{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$ is deemed as a complete residuated lattice if*

1. *$(L, \vee, \wedge, 0, 1)$ is a complete lattice with the least element 0 and the greatest element 1;*
2. *$(L, \otimes, 1)$ is a commutative monoid, i.e. $\otimes$ is commutative, associative, and $x \otimes 1 = 1 \otimes x = x$ for any $x \in L$;*
3. *$(\otimes, \to)$ forms a residuated pair in L, i.e. $x \leq y \to c$ if and only if $x \otimes y \leq z$ for all $x, y, z \in L$.*

There are some identities holden in any complete residuated lattice:

$$(a \otimes b) \to c = a \to (b \to c) = b \to (a \to c); \tag{1}$$

$$b \to \wedge_{i \in I} a_i = \wedge_{i \in I}(b \to a_i), \vee_{i \in I} b_i \to a = \wedge_{i \in I}(b_i \to a). \tag{2}$$

According to the above definition, $\vee$ and $\wedge$ are supremum and infimum, $\otimes$ and $\to$ are "fuzzy conjunction" and "fuzzy implication", respectively. A common choice of $\mathbf{L}$ which usually used in fuzzy logic applications is complete residuated lattice with the unit interval $L = [0, 1]$, $\wedge = \min$, $\vee = \max$. And here are three significant pairs of adjoint operators: Łukasiewicz one ($a \otimes b = max(a + b - 1, 0)$, $a \to b = min(1 - a + b, 1)$), Gödel one ($a \otimes b = min(a, b)$, $a \to b = 1$ if $a \leq b$ and $= b$ else), Goguen (product) one ($a \otimes b = a \cdot b$, $a \to b = 1$ if $a \leq b$ *and* $= \frac{b}{a}$ else).

Hereinafter, for convenience of description, let $\otimes$ and $\to$ be the Gödel adjoint operator. A structure $\mathbf{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$ is a complete residuated lattice, and a formal fuzzy context is a triple $(U, A, \tilde{I})$, where U and A are, respectively, non-empty finite set of objects and attributes, and $\tilde{I} : U \times A \to L$ is a L-fuzzy relation between U and A in [41].

Definition 2.2 ([32]). *Let $(U, A, \tilde{I})$ be a formal fuzzy context with $X \in \mathcal{P}(U)$, $\tilde{B} \in L^A$ and $0 < \delta \leq 1$. Two operators $\triangledown_\delta : \mathcal{P}(U) \to L^A$ and $\triangle_\delta : L^A \to \mathcal{P}(U)$ are denoted as:*

$$X^{\triangledown_\delta}(a) = \delta \to \bigwedge_{x \in X} \tilde{I}(x, a), a \in A; \tag{3}$$

$$\tilde{B}^{\triangle_\delta} = \{x \in U : \bigwedge_{a \in A} (\tilde{B}(a) \to \tilde{I}(x, a)) \geq \delta\}. \tag{4}$$

To simplify notation, if $X = \{x\}$ for any $x \in U$, we will use $x^{\triangledown_\delta}$ instead of $\{x\}^{\triangledown_\delta}$.

By Definition 2.2, its properties are analogous to that of the classical formal context, and it is straightforward to obtain the following properties induced from the formal fuzzy context.

Property 2.1 ([32]). *Let $(U, A, \tilde{I})$ be a formal fuzzy context, $\tilde{X}, \tilde{X}_1, \tilde{X}_2 \in \mathcal{P}(U)$ and $\tilde{B}, \tilde{B}_1, \tilde{B}_2 \in L^A$. Then the following properties hold:*

1. $X_1 \subseteq X_2 \Rightarrow X_2^{\triangledown_\delta} \subseteq X_1^{\triangledown_\delta}$; $\tilde{B}_1 \subseteq \tilde{B}_2 \Rightarrow \tilde{B}_2^{\triangle_\delta} \subseteq \tilde{B}_1^{\triangle_\delta}$.
2. $X \subseteq X^{\triangledown_\delta \triangle_\delta}$; $\tilde{B} \subseteq \tilde{B}^{\triangle_\delta \triangledown_\delta}$.
3. $X^{\triangledown_\delta} = X^{\triangledown_\delta \triangle_\delta \triangledown_\delta}$; $\tilde{B} = \tilde{B}^{\triangle_\delta \triangledown_\delta \triangle_\delta}$.
4. $(X_1 \cup X_2)^{\triangledown_\delta} = X_1^{\triangledown_\delta} \cap X_2^{\triangledown_\delta}$; $(\tilde{B}_1 \cup \tilde{B}_2)^{\triangle_\delta} = \tilde{B}_1^{\triangle_\delta} \cap \tilde{B}_2^{\triangle_\delta}$.

In a formal fuzzy context $(U, A, \tilde{I})$, a pair $(X, \tilde{B}) \in \mathcal{P}(U) \times L^A$ is called a crisp-fuzzy variable threshold concept of $(U, A, \tilde{I})$ such that $X = \tilde{B}^{\triangle_\delta}$ and $\tilde{B} = X^{\triangledown_\delta}$ (see [32]), where X and $\tilde{B}$ are referred to as the extent and intent of $(X, \tilde{B})$, respectively. We denote the set of all crisp-fuzzy variable threshold concepts in $(U, A, \tilde{I})$ as $L_\delta(U, \tilde{A}, \tilde{I})$. From Property 2.1.3, we can see that both $(X^{\triangledown_\delta \triangle_\delta}, X^{\triangledown_\delta})$ and $(\tilde{B}^{\triangle_\delta}, \tilde{B}^{\triangle_\delta \triangledown_\delta})$ are crisp-fuzzy variable threshold concepts. Especially, $(x^{\triangledown_\delta \triangle_\delta}, x^{\triangledown_\delta})$ is interpreted as an object concept for any $x \in U$. For two crisp-fuzzy variable threshold concepts $(X_1, \tilde{B}_1)$ and $(X_2, \tilde{B}_2)$, $(X_1, \tilde{B}_1) \leq (X_2, \tilde{B}_2)$ iff $X_1 \subseteq X_2$ (equivalently $\tilde{B}_2 \subseteq \tilde{B}_1$). Obviously, $L_\delta(U, \tilde{A}, \tilde{I})$ forms a complete lattice, in which the join $\vee$ and the meet $\wedge$ are respectively given by:

$$(X_1, \tilde{B}_1) \vee (X_2, \tilde{B}_2) = ((X_1 \cup X_2)^{\triangledown_\delta \triangle_\delta}, \tilde{B}_1 \cap \tilde{B}_2); \tag{5}$$

$$(X_1, \tilde{B}_1) \wedge (X_2, \tilde{B}_2) = (X_1 \cap X_2, (\tilde{B}_1 \cup \tilde{B}_2)^{\triangle_\delta \triangledown_\delta}). \tag{6}$$

More specifically, a formal fuzzy context can be considered as a data table which is combined of rows and columns corresponding to objects and attributes. In particular, for any $x \in U$ and $a \in A$, a degree $\tilde{I}(x, a) \in L$ is represented as a level to which object x owns attribute a. That is to say, such data table filled with all truth degrees $\tilde{I}(x, a)$ can be seen as a matrix $M_{\tilde{I}} = (c_{ij})_{n \times m}$ called a fuzzy relation matrix, where $|U| = n$ and $|A| = m$. In addition, some matrix operations will be introduced in the following.

Lemma 2.1. *Let $A = (a_{ij})_{n\times m}$, $B = (b_{ij})_{n\times m}$ and $C = (c_{ij})_{m\times p}$ be fuzzy relation matrices, then the following properties hold:*

1. *$A \le B$ iff $a_{ij} \le b_{ij}$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$;*
2. *$A \cup B = (a_{ij} \vee b_{ij})_{n\times m}$;*
3. *$A \cap B = (a_{ij} \wedge b_{ij})_{n\times m}$;*
4. *$A \cdot C = (e_{ij})_{n\times p}$, where $e_{ij} = \bigvee_{1\le k\le m}(a_{ik} \wedge c_{kj})$;*
5. *$A - B = (a_{ij} \wedge (1 - b_{ij}))_{n\times m}$;*
6. *$\sim A = (1 - a_{ij})_{n\times m}$.*

in which $\vee$ and $\wedge$ are the supremum and the infimum, respectively. Meanwhile, we denote the transposition of a matrix A as A^T.

3. Matrix representation of crisp-fuzzy variable threshold concept lattices

Based on the matrix operations, we will discuss the matrix representation of crisp-fuzzy variable threshold concept lattices in this section. Furthermore, the relationship between derivation operators and the fuzzy sub-contexts is investigated.

Let $U = \{x_1, x_2, \cdots, x_n\}$ and $X \subseteq U$. Denote the characteristic function of X as $\lambda(X)$, where

$$\lambda_X(x_i) \begin{cases} 1, & x_i \in X, \\ 0, & x_i \notin X. \end{cases}$$

Then $\lambda(X) = (\lambda_X(x_1), \lambda_X(x_2), \cdots, \lambda_X(x_n))$.

To facilitate our discussion, let $\otimes$ and $\to$ be the Gödel adjoint operator, i.e. $a \otimes b = min(a, b)$, and $a \to b = 1$ if $a \le b$ and $= b$ else.

Definition 3.1. *Let $A = (a_{ij})_{n\times m}$, $B = (b_{ij})_{m\times p}$ and $C = (c_{ij})_{n\times m}$ be three matrices. Two matrix operators $\odot$ and $\bullet$ are defined as follows:*

1. *$A \odot B = (d_{ij})_{n\times p}$, where $d_{ij} = \bigwedge_{1\le k\le m}(a_{ik} \odot b_{kj})$, and*

$$a_{ik} \odot b_{kj} = \begin{cases} b_{kj}, & a_{ik} > b_{kj}, \\ 1, & a_{ik} \le b_{kj}. \end{cases}$$

2. *$A \bullet C = (e_{ij})_{n\times m}$, where $e_{ij} = 1$ if $a_{ij} \ge c_{ij}$, and $e_{ij} = 0$ else.*

Property 3.1. *Let $A = (a_{ij})_{n\times m}$, $B = (b_{ij})_{n\times m}$, $C = (c_{ij})_{p\times n}$, $D = (d_{ij})_{n\times m}$, $E = (e_{ij})_{m\times q}$, and $F = (f_{ij})_{m\times q}$ are fuzzy matrices. Then the following properties hold:*

1. *If* $A \leq B$, *then* $C \odot A \leq C \odot B$.
2. *If* $A \leq B$, *then* $B \odot E \leq A \odot E$.
3. *If* $A \leq B$,*then* $A \bullet D \leq B \bullet D$.
4. *If* $A \leq B$ *and* $E \leq F$, *then* $C \odot B \odot E \leq C \odot A \odot F$.

Proof. The proof of (1)-(3) is obvious by applying Definition 3.1. Hence, the following item (4) will be illustrated.

(4) Since $A \leq B$, we can have that $C \odot B \odot E \leq C \odot A \odot E$ by item (1) and (2). Then, as $E \leq F$, we get $C \odot A \odot E \leq C \odot A \odot F$. Thus, $C \odot B \odot E \leq C \odot A \odot F$. □

The following theorem shows that the intensions of object set can be represented by matrix operations.

Theorem 3.1. *Let* $K = (U, A, \tilde{I})$ *be a formal fuzzy context and* $0 < \delta \leq 1$. $M_{\tilde{I}}$ *is a fuzzy relation matrix and* $X \in \mathcal{P}(U)$. *Then*

$$X^{\nabla_\delta} = \delta \odot (\sim (\lambda_X \cdot (\sim M_{\tilde{I}}))). \tag{7}$$

Proof. Assume that $\lambda(X) = (\lambda_X(x_1), \lambda_X(x_2), \cdots, \lambda_X(x_n))$. Then $(\alpha_i)_{1\times m} = \sim (\lambda_X \cdot (\sim M_{\tilde{I}}))$, we have that $\alpha_i = 1 - \bigvee_{k=1}^{n} (\lambda_X(x_k) \wedge (1 - c_{ki})) = 1 - \bigvee_{\lambda_X(x_k)=1} (1 - c_{ki}) = 1 - \bigvee_{x\in X} (1 - \tilde{I}(x, a_i)) = \bigwedge_{x\in X} \tilde{I}(x, a_i)$.

Hence, $X^{\nabla_\delta}(a_i) = \delta \to \bigwedge_{x\in X} \tilde{I}(x, a_i)$
$= \delta \odot (\sim (\lambda_X \cdot (\sim M_{\tilde{I}})))$. □

By the following theorem, one can also obtain the extensions of attribute set in a formal fuzzy context. For simplicity, we denote a threshold vector with the size $1 \times n$ as λ_δ. For example, $\delta = 0.6$, then $\lambda_\delta = (0.6, 0.6, \cdots, 0.6)_{1\times n}$.

Theorem 3.2. *Let* $K = (U, A, \tilde{I})$ *be a formal fuzzy context and* $0 < \delta \leq 1$. $M_{\tilde{I}}$ *is a fuzzy relation matrix and* $\tilde{B} \in L^A$. *Then*

$$\lambda(\tilde{B}^{\triangle_\delta}) = (\tilde{B} \odot M_{\tilde{I}}^T) \bullet \lambda_\delta. \tag{8}$$

Proof. It follows from Definition 2.2 that the proof is immediate. □

We denote the following matrix to show the granular matrix of objects.

Definition 3.2. *Let* $K = (U, A, \tilde{I})$ *be a formal fuzzy context and* $0 < \delta \leq 1$. *Each row of* M *corresponds to* $x^{\nabla_\delta \triangle_\delta}$ *of* $x \in U$, *that is to say,* $M(i, :)$ *presents that* $\lambda(x_i^{\nabla_\delta \triangle_\delta})$ *for* $i \leq n$. *Then* $M = (m_{ij})_{n\times n}$ *is called an object granular matrix.*

For convenience of the following description, we write M instead of M_A, and define E_δ and M_δ the diagonal threshold matrix with the size $n \times n$, and the all threshold matrix with the size $n \times n$, respectively. For example, $\delta = 0.6$, $n = 3$, then E_δ and M_δ are obtained:

$$E_\delta = \begin{bmatrix} 0.6 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.6 \end{bmatrix}, M_\delta = \begin{bmatrix} 0.6 & 0.6 & 0.6 \\ 0.6 & 0.6 & 0.6 \\ 0.6 & 0.6 & 0.6 \end{bmatrix}.$$

Theorem 3.3. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$ and $M_{\tilde{I}}$ is a fuzzy relation matrix of K. Then $M = E_\delta \odot M_{\tilde{I}} \odot M_{\tilde{I}}^T \bullet M_\delta$.*

Proof. It follows easily from Eq. (7) and (8). □

To make a clear comprehension on the structure of formal fuzzy contexts. An example in literatures [27, 34] is used to illustrate our discussion.

Example 3.1. Table 1 narrates a formal fuzzy context $K = (U, A, \tilde{I})$ with $U = \{x_1, x_2, x_3, x_4\}$, $A = \{a_1, a_2, a_3, a_4, a_5\}$, and a fuzzy relation is denoted in Table 1. Suppose that $X = \{x_1, x_3, x_4\}$ and $\delta = 0.8$, according to Eq.(7) and (8), we can have:

Table 1: Formal fuzzy context $(U, A, \tilde{I})$

$\tilde{I}$	a_1	a_2	a_3	a_4	a_5
x_1	0.5	0.7	0.7	0.6	0.5
x_2	0.6	0.7	0.9	0.7	1.0
x_3	1.0	0.9	1.0	0.1	0.9
x_4	1.0	0.9	0.9	0.1	0.9

$$X^{\nabla_\delta} = (0.8) \odot (\sim ((1,0,1,1) \cdot \begin{bmatrix} 0.5 & 0.3 & 0.3 & 0.4 & 0.5 \\ 0.4 & 0.3 & 0.1 & 0.3 & 0 \\ 0 & 0.1 & 0 & 0.9 & 0.1 \\ 0 & 0.1 & 0.1 & 0.9 & 0.1 \end{bmatrix}))$$
$$= (0.5, 0.7, 0.7, 0.1, 0.5).$$

$$\lambda(X^{\nabla_\delta \triangle_\delta}) = (0.5, 0.7, 0.7, 0.1, 0.5) \odot \begin{bmatrix} 0.5 & 0.6 & 1.0 & 1.0 \\ 0.7 & 0.7 & 0.9 & 0.9 \\ 0.7 & 0.9 & 1.0 & 0.9 \\ 0.6 & 0.7 & 0.1 & 0.1 \\ 0.5 & 1.0 & 0.9 & 0.9 \end{bmatrix} \bullet (0.8, 0.8, 0.8, 0.8)$$
$$= (1, 1, 1, 1).$$

Then by Theorem 3.3, we can calculate that:

$$M = E_\delta \odot M_{\tilde{I}} \odot M_{\tilde{I}}^T \bullet M_\delta = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

4. Fuzzy sub-contexts and join-irreducible elements

To refer to literatures [32, 34], Zhang et al. and Shao et al. proposed the concept of variable threshold concept lattices and fuzzy sub-contexts based on attribute subsets, respectively. In this section, the relationship between the derivation operators of a formal fuzzy context and that of its fuzzy sub-contexts is researched based on matrix theory.

Definition 4.1 ([34]). *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$ and $C \subseteq A$. Here is a formal fuzzy context $K_C = (U, C, \tilde{I_C})$, called a fuzzy sub-context of K, where $\tilde{I}_C = \tilde{I} \cap (U \times C)$. For $X \in \mathcal{P}(U)$ and $\tilde{B} \in L^C$, the operators $\bigtriangledown_\delta^C : \mathcal{P}(U) \to L^C$ and $\bigtriangleup_\delta^C : L^C \to \mathcal{P}(U)$ are given by:*

$$X^{\bigtriangledown_\delta^C}(a) = \delta \to \bigwedge_{x \in X} \tilde{I}(x, a), a \in C; \tag{9}$$

$$\tilde{B}^{\bigtriangleup_\delta^C} = \{x \in U : \bigwedge_{a \in C} (\tilde{B}(a) \to \tilde{I}(x, a)) \geq \delta\}. \tag{10}$$

In Eq.(9) and (10), $X^{\bigtriangledown_\delta^C}$ is the restriction of $X^{\bigtriangledown\delta}$ on the set C, and $\tilde{B}^{\bigtriangleup_\delta^C}$ is the restriction of $\tilde{B}^{\bigtriangleup_\delta}$ on the set C, i.e. $X^{\bigtriangledown_\delta^C}(a) = (X^{\bigtriangledown\delta}|_C)(a)$ and $\tilde{B}^{\bigtriangleup_\delta^C}(x) = (\tilde{B}^{\bigtriangleup_\delta}|_C)(x)$. For brevity, we write $X^{\bigtriangledown\delta}$ the intension of X and $\tilde{B}^{\bigtriangleup_\delta}$ the extension of $\tilde{B}$ instead of $X^{\bigtriangledown_\delta^A}$ and $\tilde{B}^{\bigtriangleup_\delta^A}$, respectively.

Let $K_C = (U, C, \tilde{I}_C)$ be a fuzzy sub-context of the formal fuzzy context K. For $C \subseteq A$, its characteristic function is deemed as $\lambda(C)$. Obviously, $M_{\lambda(C)}$ is called a copy matrix with the size $n \times m$ where each row of $M_{\lambda(C)}$ is characteristic function $\lambda(C)$. According to Example 3.1, suppose $C = \{a_1, a_3, a_5\}$, then we have:

$$M_{\lambda(C)} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Then by Theorem 3.3, we can also have an object granular matrix $M_C = E_\delta \odot (M_{\tilde{I}} \wedge M_{\lambda(C)}) \odot (M_{\tilde{I}} \wedge M_{\lambda(C)})^T \bullet M_\delta$ in K_C. Based on the above definition, the following property is discussed.

Property 4.1. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$, $C \subseteq A$, and $D = A - C$. For any $X \subseteq U$, then*

1. $X^{\bigtriangledown_\delta^C} = X^{\bigtriangledown\delta} \cap \lambda(C)$;
2. $M \leq M_C$;
3. $X^{\bigtriangledown\delta} = X^{\bigtriangledown_\delta^C} \cup X^{\bigtriangledown_\delta^D}$;

4. $M = M_C \cap M_D$.

Proof. 1. It follows immediately from Eq. (3) and (9).

2. Note that $M_{\tilde{I}} \wedge M_{\lambda(C)} \leq M_{\tilde{I}}$, then by Property 3.1.(1)-(2), we have $E_\delta \odot (M_{\tilde{I}} \wedge M_{\lambda(C)}) \leq E_\delta \odot M_{\tilde{I}}$, and then $E_\delta \odot M_{\tilde{I}} \odot M_{\tilde{I}}^T \leq E_\delta \odot (M_{\tilde{I}} \wedge M_{\lambda(C)}) \odot M_{\tilde{I}}^T$. Since, $E_\delta \odot (M_{\tilde{I}} \wedge M_{\lambda(C)}) \odot (M_{\tilde{I}} \wedge M_{\lambda(C)})^T \leq E_\delta \odot (M_{\tilde{I}} \wedge M_{\lambda(C)}) \odot M_{\tilde{I}}^T$. Assume that $E_\delta \odot (M_{\tilde{I}} \wedge M_{\lambda(C)}) \odot (M_{\tilde{I}} \wedge M_{\lambda(C)})^T = (a_{ij})_{n\times n} < E_\delta \odot (M_{\tilde{I}} \wedge M_{\lambda(C)}) \odot M_{\tilde{I}}^T = (b_{ij})_{n\times n}$. Further, suppose that $E_\delta \odot (M_{\tilde{I}} \wedge M_{\lambda(C)}) = (e_{ij})_{n\times m}$, $(M_{\tilde{I}} \wedge M_{\lambda(C)})^T = (c_{ij})_{m\times n}$, $M_{\tilde{I}}^T = (m_{ij})_{m\times n}$. From Definition 3.1, if $a_{ij} = \wedge_{1\leq k\leq m}(e_{ik} \odot c_{kj}) = 1$, then we can have $b_{ij} = \wedge_{1\leq k\leq m}(e_{ik} \odot m_{kj}) = 1$ which is a contradiction to the assumption and then $E_\delta \odot (M_{\tilde{I}} \wedge M_{\lambda(C)}) \odot (M_{\tilde{I}} \wedge M_{\lambda(C)})^T < E_\delta \odot (M_{\tilde{I}} \wedge M_{\lambda(C)}) \odot M_{\tilde{I}}^T$ does not hold. Therefore, we can obtain that $E_\delta \odot (M_{\tilde{I}} \wedge M_{\lambda(C)}) \odot (M_{\tilde{I}} \wedge M_{\lambda(C)})^T = E_\delta \odot (M_{\tilde{I}} \wedge M_{\lambda(C)}) \odot M_{\tilde{I}}^T$. Thus, an induction complete the proof that $E_\delta \odot M_{\tilde{I}} \odot M_{\tilde{I}}^T \leq E_\delta \odot (M_{\tilde{I}} \wedge M_{\lambda(C)}) \odot (M_{\tilde{I}} \wedge M_{\lambda(C)})^T$, i.e. $M \leq M_C$.

3. From item (1), we obtain $X^{\nabla_\delta^C} \cup X^{\nabla_\delta^D} = (X^{\nabla_\delta} \cap \lambda(C)) \cup (X^{\nabla_\delta} \cap \lambda(D)) = X^{\nabla_\delta} \cap (\lambda(C) \cup \lambda(D)) = X^{\nabla_\delta}$.

4. Follows readily by applying Property 2.1(4) and item (3).

□

On this basis, the following is an example to confirm the fuzzy sub-context and Property 4.1.

Example 4.1. Continued from Example 3.1, let $K_C = (U, C, \tilde{I}_C)$ be a fuzzy sub-context of the formal fuzzy context K, where $C = \{a_1, a_3, a_5\}$, and a fuzzy relation $\tilde{I}_C$ is depicted in Table 2. Then according to Definition 4.1, we can

Table 2: Formal fuzzy context $K = (U, C, \tilde{I}_C)$

$\tilde{I}$	a_1	a_3	a_5
x_1	0.5	0.7	0.5
x_2	0.6	0.9	1.0
x_3	1.0	1.0	0.9
x_4	1.0	0.9	0.9

compute that

$$M_C = E_\delta \odot (M_{\tilde{I}} \wedge M_{\lambda(C)}) \odot (M_{\tilde{I}} \wedge M_{\lambda(C)})^T \bullet M_\delta = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Consequently, we have $M \leq M_C$, which is in accordance with Property 4.1.(2).

Lemma 4.1. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$, $C \subseteq A$, $X \subseteq U$, and $X_i \subseteq U$, $i = 1, 2, \cdots, k$. If $X^{\nabla_\delta} = \bigcap_{i=1}^k X_i^{\nabla_\delta}$, then $X^{\nabla_\delta^C} = \bigcap_{i=1}^k X_i^{\nabla_\delta^C}$.*

Proof.

$$X^{\nabla_\delta^C} = X^{\nabla_\delta} \cap \lambda(C) = (\bigcap_{i=1}^{k} X_i^{\nabla_\delta}) \cap \lambda(C)$$
$$= \bigcap_{i=1}^{k} (X_i^{\nabla_\delta} \cap \lambda(C)) = \bigcap_{i=1}^{k} X_i^{\nabla_\delta^C}.$$

□

Corollary 4.1. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$, $C \subseteq A$, $x \in U$, and $x_i \in U$, $i = 1, 2, \cdots, k$. If $x^{\nabla_\delta} = \bigcap_{i=1}^{k} x_i^{\nabla_\delta}$, then $x^{\nabla_\delta^C} = \bigcap_{i=1}^{k} x_i^{\nabla_\delta^C}$.*

Lemma 4.2. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$, $C \subseteq A$, $X \subseteq U$, and $X_i \subseteq U$, $i = 1, 2, \cdots, k$. If*

$$(X^{\nabla_\delta \triangle_\delta}, X^{\nabla_\delta}) = \bigvee_{i=1}^{k} (X_i^{\nabla_\delta \triangle_\delta}, X_i^{\nabla_\delta}), \tag{11}$$

then

$$(X^{\nabla_\delta^C \triangle_\delta^C}, X^{\nabla_\delta^C}) = \bigvee_{i=1}^{k} (X_i^{\nabla_\delta^C \triangle_\delta^C}, X_i^{\nabla_\delta^C}). \tag{12}$$

Proof. From Eq.(11), we know $X^{\nabla_\delta} = \bigcap_{i=1}^{k} X_i^{\nabla_\delta}$. Then by Lemma 4.1, $X^{\nabla_\delta^C} = \bigcap_{i=1}^{k} X_i^{\nabla_\delta^C}$. Additionally, we have $(\bigcup_{i=1}^{k} X_i^{\nabla_\delta^C \triangle_\delta^C})^{\nabla_\delta^C \triangle_\delta^C} = (\bigcap_{i=1}^{k} X_i^{\nabla_\delta^C \triangle_\delta^C \nabla_\delta^C})^{\triangle_\delta^C}$ $= (\bigcap_{i=1}^{k} X_i^{\nabla_\delta^C})^{\triangle_\delta^C} = X^{\nabla_\delta^C \triangle_\delta^C}$.

□

According to Lemma 4.2, we can know that if a crisp-fuzzy variable threshold concept stemmed from an object set is a join of some crisp-fuzzy variable threshold concepts originated from some finite object sets on U. Therefore, then in any fuzzy sub-context, the corresponding crisp-fuzzy variable threshold concept stemmed from the identical object set can also be expressed as a join of crisp-fuzzy variable threshold concepts originated from these object sets on U, too.

Corollary 4.2. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$, $C \subseteq A$, $x \in U$, and $x_i \in U$, $i = 1, 2, \cdots, k$. If*

$$(x^{\nabla_\delta \triangle_\delta}, x^{\nabla_\delta}) = \bigvee_{i=1}^{k} (x_i^{\nabla_\delta \triangle_\delta}, x_i^{\nabla_\delta}), \tag{13}$$

then

$$(x^{\nabla_\delta^C \triangle_\delta^C}, x^{\nabla_\delta^C}) = \bigvee_{i=1}^{k} (x_i^{\nabla_\delta^C \triangle_\delta^C}, x_i^{\nabla_\delta^C}). \tag{14}$$

Note that any crisp-fuzzy variable threshold concept $(X, \tilde{B})$ can be shown as a join of object concepts of its extension, i.e. $(X, \tilde{B}) = \bigvee_{x \in X}(x^{\nabla_\delta \triangle_\delta}, x^{\nabla_\delta})$.

Shao et al. [27] searched the join-irreducible elements in a formal fuzzy context, i.e. crisp-fuzzy variable threshold concept lattice with threshold $\delta = 1$. In what follows, we will investigate granules and order relations based on objects in crisp-fuzzy variable threshold concept lattices with threshold varied from 0 to 1, and discuss the properties of join-irreducible elements.

Definition 4.2 ([12]). *Let L be a finite lattice and $v \in L$. An element v_* is called a join-irreducible in the case of $v \neq v_*$ if $v_* = \bigvee\{x \in L : x < v\}$.*

Lemma 4.3 ([12]). *Let L be a finite lattice. Then every element in L is composed of a join of some join-irreducible elements.*

Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$, for any $C \subseteq A$, a binary relation R_C is denoted as follows:

$$R_C = \{(x, y) \in U \times U : x^{\nabla_\delta}(a) \leq y^{\nabla_\delta}(a), \forall a \in C\}. \tag{15}$$

R_C is interpreted as an order relation based on the object set, in which $(x, y) \in R_C$ represents that y is not less than x about attributes C. That is to say, Eq.(15) is evident that $R_C = \bigcap_{a \in C} R_a$. For brevity, if $C = \{a\}$, we will write R_a instead of $R_{\{a\}}$.

For any $x \in U$, we denote the object set whose attribute value includes the intension of object x about all attributes in C as $[x]_{R_C}$: that is, $[x]_{R_C} = \{y \in U : (x, y) \in R_C\}$. Then by Eq.(15), $[x]_{R_C}$ can also be represented as

$$[x]_{R_C} = \{y \in U : x^{\nabla_\delta}(a) \leq y^{\nabla_\delta}(a), \forall a \in C\}. \tag{16}$$

Lemma 4.4. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$, $C \subseteq A$. For any $x \in U$, we can obtain: $[x]^{\nabla_\delta}_{R_C}(a) = x^{\nabla^C_\delta}(a)$.*

Proof. By Eq.(2) and (9), we have

$$[x]^{\nabla_\delta}_{R_C}(a) = \delta \to \bigwedge_{y \in [x]_{R_C}} \tilde{I}(y, a) = \bigwedge_{y \in [x]_{R_C}} (\delta \to \tilde{I}(y, a)) = \bigwedge_{y \in [x]_{R_C}} y^{\nabla_\delta}(a) = x^{\nabla^C_\delta}(a).$$

□

Theorem 4.1. *Let $K_C = (U, C, \tilde{I}_C)$ be a fuzzy sub-context of the formal fuzzy context K, $0 < \delta \leq 1$, and $x \in U$. Afterwards, $([x]_{R_C}, x^{\nabla^C_\delta})$ is a crisp-fuzzy variable threshold concept of K_C and $[x]_{R_C} = x^{\nabla^C_\delta \triangle^C_\delta}$.*

Proof. By Lemma 4.4, we have $[x]^{\nabla_\delta \triangle_\delta}_{R_C} = x^{\nabla^C_\delta \triangle^C_\delta}$. On the one hand, according to Property 2.1(2), we obtain $[x]_{R_C} \subseteq [x]^{\nabla_\delta \triangle_\delta}_{R_C} = x^{\nabla^C_\delta \triangle^C_\delta}$.

On the other hand, for any $y \in x^{\nabla^C_\delta \triangle^C_\delta}$, we can easily obtain $\{y\} \subseteq x^{\nabla^C_\delta \triangle^C_\delta}$. By Property 2.1(1) and (3), we can draw a conclusion that $x^{\nabla^C_\delta} \subseteq y^{\nabla^C_\delta}$. Thus,

$y \in [x]_{R_C}$, and from which $x^{\nabla_\delta^C \triangle_\delta^C} = [x]_{R_C}^{\nabla_\delta \triangle_\delta} \subseteq [x]_{R_C}$. Obviously, $x^{\nabla_\delta^C \triangle_\delta^C} = [x]_{R_C}^{\nabla_\delta \triangle_\delta} = [x]_{R_C}$. Then $([x]_{R_C}, x^{\nabla_\delta^C})$ is a crisp-fuzzy variable threshold concept. □

Notice that $[x]_{R_C} = x^{\nabla_\delta^C \triangle_\delta^C}$ in fuzzy sub-context K_C of K. To avoid confusion, we use m_{ij}^C instead of m_{ij} in the object granular matrix M_C for $x_i, x_j \in U$. By Definition 4.1, we can see that $m_{ij}^C = 1 \Leftrightarrow x_i^{\nabla_\delta} \subseteq x_j^{\nabla_\delta}, \forall a \in C$. That is to say, Eq.(15) is equivalent that

$$R_C = \{(x_i, x_j) \in U \times U : m_{ij}^C = 1\}. \tag{17}$$

Specifically, we write m_{ij} instead of $m_{ij}^A = 1$ in M. Moreover, it is easily obtained that

$$[x_i]_{R_C} = \{x_j \in U : m_{ij}^C = 1\}. \tag{18}$$

Theorem 4.2. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$, $\forall C \subseteq A$, and $x \in U$. If $(x^{\nabla_\delta^C \triangle_\delta^C}, x^{\nabla_\delta^C})$ is a join-irreducible element in $L_\delta(U, \tilde{C}, \tilde{I}_C)$, then $(x^{\nabla_\delta \triangle_\delta}, x^{\nabla_\delta})$ is also a join-irreducible element in $L_\delta(U, \tilde{A}, \tilde{I})$.*

Proof. Suppose that $(x^{\nabla_\delta \triangle_\delta}, x^{\nabla_\delta})$ is not a join-irreducible element in $L_\delta(U, \tilde{A}, \tilde{I})$. By Lemma 4.3, $(x^{\nabla_\delta \triangle_\delta}, x^{\nabla_\delta})$ is a join of some join-irreducible elements in $L_\delta(U, \tilde{A}, \tilde{I})$. That is, there exists $x_i \in U$, $i = 1, 2, \cdots, k$ $(k \geq 2)$ such that $(x^{\nabla_\delta \triangle_\delta}, x^{\nabla_\delta}) = \bigvee_{i=1}^k (x_i^{\nabla_\delta \triangle_\delta}, x_i^{\nabla_\delta})$. Then according to Corollary 4.2, we know that $(x^{\nabla_\delta^C \triangle_\delta^C}, x^{\nabla_\delta^C}) = \bigvee_{i=1}^k (x_i^{\nabla_\delta^C \triangle_\delta^C}, x_i^{\nabla_\delta^C})$. Hence, $(x^{\nabla_\delta^C \triangle_\delta^C}, x^{\nabla_\delta^C})$ is not a join-irreducible element in $L_\delta(U, \tilde{C}, \tilde{I}_C)$, which comes a contradiction. Therefore, we assert that $(x^{\nabla_\delta \triangle_\delta}, x^{\nabla_\delta})$ is a join-irreducible element in $L_\delta(U, \tilde{A}, \tilde{I})$. □

5. Granular reduction based on matrix operations in formal fuzzy contexts

Based on the literature [27], Shao et al. investigated granular reducts based on granular computing in formal fuzzy contexts. In this section, we generalize Shao et al. and Lin et al.'s work [27, 39] to the crisp-fuzzy variable threshold concept lattices and formulate the granular reduction via matrix operations. Thus intuitively speaking, we construct the discernibility matrix and the discernibility function which are employed to calculate attribute reduction.

Definition 5.1. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$, and $\forall C \subseteq A$. The attribute set C is called a granular consistent set of K if $M = M_C$. And further, C is referred to as a granular reduct of K if $M \neq M_B$ for any $B \subset C$.*

Note that M_C is a matrix of extensions of object concepts in the fuzzy sub-context $L_\delta(U, C, \tilde{I}_C)$. According to Definition 5.1, a granular reduct is a minimal attribute subset that preserves the object granules of variable threshold concept lattice unchanged.

Example 5.1. Continued from Example 3.1. Suppose that $C = \{a_1, a_3, a_4\}$, it can easily be concluded that

$$M_C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

i.e. $M = M_C$, hence, C is a granular consistent set of K.

Property 5.1. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$, and $\forall C \subseteq A$. C is a granular consistent set iff $M_C \leq M$.*

Proof. It follows immediately from Property 4.1(2) and Definition 5.1. □

Property 5.2. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$, $\forall C \subseteq A$, and $D = A - C$. C is a granular consistent set iff $M_C \leq M_D$.*

Proof. It follows immediately from Property 4.1(4) and Definition 5.1. □

All granular reducts of K are denoted as $Red(K)$, in which the attribute set A is classified into three categories based on the significance of the attributes:

- Indispensable (core) attribute set C_K: $C_K = \bigcap Red(K)$;
- Relative necessary attribute set K_K: $K_K = \bigcup Red(K) - \bigcap Red(K)$;
- Dispensable (unnecessary) attribute set I_K: $I_K = A - \bigcup Red(K)$.

In what follows, judgment of indispensable attribute set is discussed based on matrix operations with the purpose of deleting the redundant attributes in a crisp-fuzzy variable threshold concept lattice.

Corollary 5.1. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$. An attribute $a \in A$ is an indispensable attribute if and only if $M \neq M_{A-\{a\}}$.*

Proof. It is easy to prove from Definition 5.1 and the concept of indispensable attribute. □

Example 5.2. Continued from Example 3.1, we can compute that

$$M_{A-\{a_4\}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Consequently, a_4 is an indispensable attribute because of $M \neq M_{A-\{a_4\}}$.

Definition 5.2. *Let $A = (a_{ij})_{n\times m}$ and $B = (b_{ij})_{n\times m}$ be matrices. The operation is denoted as follows: $A \ominus B = (c_{ij})_{n\times m}$, where*

$$c_{ij} = \begin{cases} 1, & a_{ij} > b_{ij}, \\ 0, & a_{ij} \leq b_{ij}. \end{cases}$$

Hereinafter, for a formal fuzzy context $K = (U, A, \tilde{I})$ with threshold δ, we know that a granular reduct is in fact a minimal granular consistent set of K. In order to build a new framework of attribute reduction, we denote a discernibility attribute matrix similar to [14], which can be given by:

Definition 5.3. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$, and $M = (m_{ij})_{n\times n}$ is an object granular matrix. For any $(x_i, x_j) \in U \times U$, if $m_{ij} = 0$, denote*

$$\lambda(\mathcal{D}(x_i, x_j)) = \lambda(x_i^{\nabla\delta}) \ominus \lambda(x_j^{\nabla\delta}).$$

$\lambda(\mathcal{D}(x_i, x_j))$ is interpreted as the discernibility attribute vector of x_i and x_j with threshold δ in K, and $\boldsymbol{D^*} = (\lambda(\mathcal{D}(x_i, x_j)) : (x_i, x_j) \in U \times U)$ is defined as the discernibility attribute matrix of K with threshold δ.

In the above definition, number 1 of discernibility attribute vector $\lambda(\mathcal{D}(x_i, x_j))$ represents that the ordered pair (x_i, x_j) can be distinguished when $m_{ij} = 0$. More precisely, the more the total numbers of 1 in $\boldsymbol{D^*}$ are, the stronger discernibility ability is.

Theorem 5.1. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$, and $D \subseteq A$. D is a granular consistent set iff $\lambda(D) \cap \boldsymbol{D^*}(k, :) \neq (0, 0, \cdots, 0)_{1\times m}$ for any $\lambda(\mathcal{D}(x_i, x_j)) \neq (0, 0, \cdots, 0)_{1\times m}$, that is, $k \leq q$ (q means the number of rows of $\boldsymbol{D^*}$).*

Proof. ($\Rightarrow$) If D is a granular consistent set, then by Property 5.1, we know that $M_D \leq M$. From Theorem 4.1, we deduce that $[x_i]_{R_D} \subseteq [x_i]_{R_A}, \forall x_i \in U$. For any $\lambda(\mathcal{D}(x_i, x_j)) \in \boldsymbol{D^*}$ with $\lambda(\mathcal{D}(x_i, x_j)) \neq (0, 0, \cdots, 0)_{1\times m}$, then there exists $a \in A$ such that $m_{ij} = 0$. Thus, $x_j \notin [x_i]_{R_A}$, that is to say, $x_j \notin [x_i]_{R_D}$, which represents that $m_{ij}^D = 0$. By Eq.(17), there exists $a \in D$ such that $(x_i, x_j) \notin R_a$, i.e. $x_i^{\nabla\delta}(a) \not\leq x_j^{\nabla\delta}(a)$. Then, that is: $x_i^{\nabla\delta}(a) > x_j^{\nabla\delta}(a)$. From Definition 5.3, we have $\lambda(x_i^{\nabla\delta}(a)) \ominus \lambda(x_j^{\nabla\delta}(a)) = 1$. Therefore, $\lambda(D) \cap \boldsymbol{D^*}(k, :) \neq (0, 0, \cdots, 0)_{1\times m}$.

($\Leftarrow$) Suppose that $\lambda(D) \cap \boldsymbol{D^*}(k, :) \neq (0, 0, \cdots, 0)_{1\times m}$ for any $\lambda(\mathcal{D}(x_i, x_j)) \in \boldsymbol{D^*}$ with $\lambda(\mathcal{D}(x_i, x_j)) \neq (0, 0, \cdots, 0)_{1\times m}$. For any $(x_i, x_j) \in U \times U$, if $x_j \notin [x_i]_{R_A}$, that implies $m_{ij} = 0$. Hence, $\lambda(\mathcal{D}(x_i, x_j)) \neq (0, 0, \cdots, 0)_{1\times m}$. Since $\lambda(D) \cap \boldsymbol{D^*}(k, :) \neq (0, 0, \cdots, 0)_{1\times m}$, there exists $a \in D$ such that $\lambda(x_i^{\nabla\delta}(a)) \ominus \lambda(x_j^{\nabla\delta}(a)) = 1$, i.e. $m_{ij}^D = 0$, and this means that $x_j \notin [x_i]_{R_D}$. Thus, we can obtain that $[x_i]_{R_D} \subseteq [x_i]_{R_A}$. From Property 5.1, it follows that $M_D \leq M$. Consequently, D is a granular consistent set. □

Theorem 5.1 shows a method to judge whether an attribute set is consistent or not.

Theorem 5.2. *Let $K = (U, A, \tilde{I})$ be a formal fuzzy context, $0 < \delta \leq 1$, and an attribute $a_k \in A$ is an indispensable attribute iff there exists $(x, y) \in U \times U$, that is, $i \leq q$ (q means the number of rows of $\boldsymbol{D}^*$) such that $|\boldsymbol{D}^*(i,:)| = 1$ and $\boldsymbol{D}^*(i,k) = 1$.*

Proof. ($\Rightarrow$) Since $a_k \in A$ is an indispensable attribute of K with threshold δ, then $A - \{a_k\}$ is not a granular consistent set. From Theorem 5.1, we have $\lambda(A - \{a_k\}) \cap \boldsymbol{D}^*(i,:) = (0, 0, \cdots, 0)_{1\times m}$. As a consequence, $|\boldsymbol{D}^*(i,:)| = 1$ and $\boldsymbol{D}^*(i,k) = 1$ with 1 representing the position of $a_k \in A$.

($\Leftarrow$) If there exists $i \leq q$ such that $|\boldsymbol{D}^*(i,:)| = 1$ and $\boldsymbol{D}^*(i,k) = 1$ with 1 indicating the position of $a_k \in A$. By Definition 5.2, we can know that $x^{\nabla\delta}(a_k) \ominus y^{\nabla\delta}(a_k) = 1$, and $x^{\nabla\delta}(b) \ominus y^{\nabla\delta}(b) = 0$ for all $b \in A - \{a_k\}$. That is, $(x, y) \notin R_{a_k}$, and $(x, y) \in R_{A-\{a_k\}}$. And from which we have $m_{ij} = 0$ and $m_{ij}^{A-\{a\}} = 1$. We deduce that $m_{ij} \neq m_{ij}^{A-\{a\}}$. That is to say, $M \neq M_{A-\{a_k\}}$, i.e. $a_k \in A$ is an indispensable attribute of K with threshold δ. □

According to Theorem 5.2, by $C_K = \{a_i :| \boldsymbol{D}^*(k,:) |= 1 \wedge \boldsymbol{D}^*(k,i) = 1\}$ we denote the set of indispensable attributes which find the row k-th, column i-th of the discernibility attribute matrix $\boldsymbol{D}^*$. Based on the above discussion, we denote $R(a_i) = \{\boldsymbol{D}^*(j.:) : \boldsymbol{D}^*(:,i) = 1\}$.

Algorithm 1 provides a method to calculate a granular reduct in K. In Step 1, initialization is conducted: *Red* is used to store a reduct, and $\boldsymbol{D}^*$ is stored a discernibility attribute matrix. In Step 2, the diagonal threshold matrix E_δ and the all threshold matrix M_δ are given based on δ. Steps 3-17 are to calculate the discernibility attribute matrix and its time complexity is $O(|U||A| + |U|^2)$. Steps 18-21 are to find all indispensable attributes. And further, if present attribute a is a core attribute, the whole row and column where attribute a is located are deleted, respectively. In Steps 22-25, an attribute with the maximal discernibility ability is added to the above *Red* in each repeat until the repeat fulfills the end condition. Then notice that running time is not than $O(|U|^2|A|)$. Therefore, it is easily seen that the time complexity of Algorithm 1 (ARTM) is $max\{O(|U||A| + |U|^2), O(|A||U|^2)\}$.

In what follows, an example is to confirm our discussion with respect to computing a granular reduct as shown in Algorithm 1.

Example 5.3. Continued from Example 3.1. According to Definition 5.3, we can have the discernibility attribute matrix is

$$\boldsymbol{D}^* = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Algorithm 1: A matrix-based algorithm for finding a granular reduct of K with threshold δ based on matrix theory (ARTM)

Require: A formal fuzzy context $K = (U, A, \tilde{I})$ with threshold δ, where $U = \{x_1, x_2, \cdots, x_n\}$ and $A = \{a_1, \cdots, a_m\}$;
Ensure: A granular reduct Red.
1: Let $M_3 \leftarrow \phi$, $\boldsymbol{D^*} \leftarrow \phi$ and $Red \leftarrow \phi$;
2: Generate the diagonal threshold matrix E_δ and the all threshold matrix M_δ according to δ;
3: **for** $i = 1 : |U|$ **do**
4: Generate copy matrix B inducing from an $|A|$-by-1 tilling of copies of $E_\delta(i, :)$;
5: Let $A = B - M_{\tilde{I}}^T$;
6: **if** $A \leq= 0$ **then**
7: $D = min(C, F) + A$;
8: $E = min(D, [], 2)$ which means that E is used to stored the minimum value of each row in D;
9: $M3 = [M3; E^T]$;
10: **end if**
11: **end for**
12: **for** $i = 1 : |U|$ **do**
13: Generate copy matrix C inducing from an $|U|$-by-1 tilling of copies of $M_3(i, :)$;
14: **if** $H = C - M_3 > 0$ **then**
15: $\boldsymbol{D^*} = [\boldsymbol{D^*}; H]$;
16: **end if**
17: **end for**
18: **if** $\mid \boldsymbol{D^*}(i, :) \mid = 1$ and $\boldsymbol{D^*}(i, j) = 1$ **then**
19: $Red \longleftarrow Red \cup \{a_j\}$ and compute $R(a_j)$. Then all rows of $\boldsymbol{D^*}$ that the sum is 0 and 1 are removed;
20: $\boldsymbol{D^*} = \boldsymbol{D^*} - \cup_{a_j \in Red} R(a_j)$ and updated $\boldsymbol{D^*}$ by deleting columns $\boldsymbol{D^*}(:, j)$, $a_j \in Red$;
21: **end if**
22: **while** $\boldsymbol{D^*} \neq \phi$ **do**
23: $Red = Red \cup \{a_t\}$, where a_i satisfies $|\boldsymbol{D^*}(:, t)| = max\{|\boldsymbol{D^*}(:, j)| : a_j \in A - Red\}$;
24: $\boldsymbol{D^*} = \boldsymbol{D^*} - R(t)$ and then delete corresponding columns in $\boldsymbol{D^*}$;
25: **end while**
26: Output Red.

It should be noted that $Red = \{a_4\}$ in Step 18 according to $\boldsymbol{D^*}(1, 4) = 1$ and $|\boldsymbol{D^*}(1, :)| = 1$ for attribute a_4. and $R(a_4)$ is depicted automatically as shown:

$$R(a_4) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

In Step 20, the discernibility attribute matrix is renewed as follows:

$$\boldsymbol{D}^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}. \tag{19}$$

It is easy to see that $\boldsymbol{D}^*(:,1) = \boldsymbol{D}^*(:,2) = 2$ is maximum value in Steps 22-25. A corresponding attribute a_1 is selected automatically. In addition, $\boldsymbol{D}^*$ is updated to empty matrix. Therefore, a granular reduct of K with threshold δ is $Red = \{a_1, a_4\}$.

6. Numerical experiments

In this section, to further explain the effectiveness of the proposed algorithm, we compare it with some existing algorithm with respect to attribute reduction in FFCA. The experiments are performed on a personal computer with Windows 10 and Intel (R) Core (TM) i5-6200U CPU @2.3GHz with 8.0 GB of memory. And these algorithms are performed using Matlab 9.3.

6.1 Real-world datasets

The first set of experiments focuses on the real-word data. All data sets are downloaded from UCI machine learning repository [42] depicted in Table 3, in which $|\cdot|$ means the cardinality of sets.

Table 3: Description of data sets

No.	Data sets	$\lvert U\rvert$	$\lvert A\rvert$
1	Wine	178	14
2	Wpbc	198	34
3	Yeast	1484	8
4	Australian	690	15
5	Crx	690	16
6	Derm	366	35
7	ICU	200	21
8	Sonar	208	61
9	Yale	165	1025
10	Iono	351	35
11	Glass	214	10
12	Wdbc	569	31

6.2 Comparison on finding a granular reduct in crisp-fuzzy variable concept lattices

In the following, we compare Algorithm 1 (called ARTM) with other algorithms from two aspects: the number of reducts and the whole running time. For the first aspect, we compare ARTM with the algorithms COAM [6] and HGRM [39]. For another aspect, we compare ARTM with the algorithm [27] (called TSGR). The results are shown in Table 4 and Table 5. For the ease of discussing, the algorithm ARTM is tested based on the threshold value $\delta = 0.8$.

Table 4: The cardinality of reduct

No.	Data sets	ARTM	COAM	HGRM
1	Wine	13	12	14
2	Wpbc	20	22	21
3	Yeast	7	8	8
4	Australian	9	9	14
5	Crx	6	6	13
6	Derm	33	35	32
7	ICU	5	5	19
8	Glass	7	9	8

Table 4 shows the cardinality of reducts under different algorithms. Intuitively speaking, the cardinality of reducts of ARTM is generally less than COAM and HGRM, respectively.

Table 5: Comparision of the running time

	The time of DAM		The time of reduct	
	ARTM	TSGR	ARTM	TSGR
Glass	0.122	41.871	0.127	41.885
Wine	0.197	27.948	0.201	27.956
Wpbc	0.890	109.484	0.896	109.510
Sonar	1.486	245.081	1.595	245.151
Yale	12.811	1714.114	13.154	1714.588
Iono	3.140	1095.685	3.170	1095.743
Wdbc	11.497	7402.319	11.563	7402.483

Table 5 shows that the running time to compute a discernibility attribute matrix (DAM for short) and find a granular reduct in different algorithms. More precisely, we see that the running time of ARTM is less than that of TSGR. Hence, the proposed algorithm ARTM is more efficient based on Table 4 and 5.

7. Conclusions

As we know, attribute reduction from the viewpoint of granular computing is an important research in FFCA. In this paper, we have introduced systematically a matrix-based approach of granular reduction in crisp-fuzzy variable threshold concept lattices which does not change the extensions of all object concepts. At first, some new matrix representations, such as, intensions of object set, extensions of attribute set, and the object granular matrix have been proposed based on the matrix operations. Subsequently, several notions and properties of fuzzy sub-contexts and join-irreducible elements have been illustrated. Then the attribute reduction method in the sense of constructing the discernibility matrix has been presented by adapting to the extensions of object concepts. With respect to the proposed approach, we have demonstrated how to delete the reductant attributes and search for an optimal attribute subset in a crisp-fuzzy variable threshold concept lattice. In further work, the heuristic approach will be discussed to obtain a granular reduct such that the process and time computation can be speeded up. On the other hand, we will apply this theory to fuzzy-crisp variable threshold concept lattice, three-way concept lattices or generalized one-sided concept lattices. Besides, we should also develop new attribute reduct methods under different requirements in variable threshold concept lattices.

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