Synchronization of two three-species food chain system with Beddington-DeAngelis functional response using active controllers based on the Lyapunov function

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Abstract. In this paper master-slave scheme is used to investigate the problem of synchronization of two three species models with Beddington-DeAngelis type of functional response. Using Lyapunov stability theory the analytical expression for nonlinear active controllers have been deduced for the synchronization of (I) two identical three-species web chain but with different growth rate and (II) two identical chaotic three-species food chain with different initial conditions. The bifurcation diagram of the food chain system has been used to find the intrinsic growth rate parameter for which the dynamics of three-species food chain exhibit either regular or chaotic behavior in phase space. Based on the chosen intrinsic growth parameters that characterizes the complex dynamics of the master and slave system, the active controller functions, have been obtained, which causes global asymptotic stability of synchronization errors, for the case I and II respectively. The efficacy of the controller functions, in the synchronization process, is demonstrated using the time series plot of species population and phase plots before and after the activation.

Keywords: Bennington-DeAngelis response, active control, Lyapunov function, Chaos control.

1. Introduction

Chaotic systems and attractors have been investigated in a variety of fields including physics [18], biology [14], chemistry [8], marketing and finance [28,2]. Chaotic systems are sensitive to initial conditions and despite being identical

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can lead to different trajectories as a result of slightly different initial conditions. As a result the problem of synchronization of chaotic system becomes very challenging. The possibility of synchronization of two identical chaotic system with different initial conditions was first demonstrated by Pecora and Carroll [30].

Soon after several methods like adaptive control method[43], sliding mode control method [17,40], nonlinear feedback method [24], active control method [1,4,12,39,41], back-stepping design [45], impulsive control [44], passive control [42] and anti-synchronization method [38] have been proposed for synchronization of chaotic system in different areas. Further, in recent years the problems of chaos control dynamics and synchronization has been investigated in specific systems i.e., Rossler and Chen system [22], Hasting Powell food chain model [23], coupled electrical oscillators [27], tritrophic food chain meta-communities [6], supply chain systems [16,47], delay coupled heterogeneous networks [32], Chua and Lure system [34]. The interdisciplinary nature of the problem therefore led to significant developments in the area of synchronization of chaotic dynamical systems in network of oscillators [31].

Several investigations in ecology and animal sciences [7,10,19,21,26,33,5] have revealed plant and animal species showing synchronous behavior over large areas as a consequence of population dispersal and meteorological driving forces. Therefore in ecological systems, study of chaotic behavior and its control through synchronization is a subject of great interest. Modeling of ecological food chains is significant to promote mutual coexistence and sustainable growth of species. In modeling of ecological systems one of the central goals is to understand the predator-prey interactions and its dynamics. The functional response of the predator is one of the most significant aspect of prey-predator relationship [5,13,15,20,46]. In recent years the complex dynamics of predator prey system has been extensively studied using various forms of function response [3,11,48]. Further, chaotic dynamics is observed in a system involving three-species over a wide range of parameter space [9,11,25,29,36,48].

The three species food chain considered involves at the top level super predators preying on the middle level predators which subsequently preys on the prey at the lowest level. The model further incorporates a growth rate term that includes the resource utilization efficiency [46]. However, the population interaction between the species is described by Beddington-DeAngelis functional response [43]. In section 2, we briefly outline the details of various features of the model including the linear stability analysis of the model [48]. The bifurcation diagram of this model exhibit complex chaotic dynamics i.e., a limit cycle , period-doubling and chaos depending on the value of the chosen control parameter, say, the intrinsic growth rate parameter of the species at the bottom level. In section 3, the problem of synchronization of two such identical three-species food chains is investigated using the Lyapunov stability theory for stabilizing the synchronizations errors. The controllers have been designed using the method of active control in the master-slave framework for synchronization involving (I)

two identical three-species food web chain but with different intrinsic growth rate parameter.

Numerical results are presented for the settings when the master three-species food chain behavior is described by a (i) limit cycle and (ii) chaotic attractor while the slave system is chaotic in the absence of any control. In both the cases, the application of the designed controller function is shown to result in synchronization of the slave system with the master. The mathematical analysis is also extended to study the synchronization of (II) two identical but chaotic three-species food chains. The efficacy of the derived nonlinear controller is shown using the time series and the phase plots before and after the onset of synchronization. At the end, in section 4, we summarizes the main results of our work.

2. Mathematical model

The basic equation governing the dynamics of three species food chain model comprising of prey X, predator Y and a super predator Z with Beddington-DeAngelis type functional response can be described by the following equations[48]:

$$\dot{X} = \frac{R_1(J_0 - X)X}{J_1 - X} - K_1(X,Y)Y, \tag{1}$$

$$\dot{Y} = L_1K_1(X,Y)Y - K_2(Y,Z)Z - M_1Y, \tag{2}$$

$$\dot{Z} = L_2K_2(Y,Z)Z - M_2Z, \tag{3}$$

where $K_1(X,Y) = \frac{E_1X}{F_1Y+G_1X+H_1}$ and $K_2(Y,Z) = \frac{E_2Y}{F_2Z+G_2Y+H_2}$ are the Beddington-DeAngelis type functional response.

For the two responses E_i, F_i and G_i $(i = 1,2)$ are the saturating parameters. The maximum harvest rate of predator from prey is given by $\frac{E_i}{H_i}$ while constants $\frac{H_i}{F_i}$ and $\frac{H_i}{G_i}$ $(i = 1,2)$ represent half saturation. Intrinsic growth rate of prey species X is denoted by R_1 and R_1J_0 is the carrying capacity. For the available resources J_1 is the limiting value. It becomes equivalent to carrying capacity if no resources are wasted in the system under ideal circumstances which is not possible in real systems where resources get wasted. Efficiency of nutrient utilization in species X is given by the ration $\frac{J_0}{J_1}$ and is either zero or one. For all times $X < J_0 < J_1$ and when the ratio $\frac{J_0}{J_1}$ is close to one it implies higher efficiency of nutrient utilization while if the ratio is closer to zero it implies lower efficiency indicating restricting of increment in population by limiting of resources. L_i and M_i represent the prey to predator conversion rate and species death rate of Y and Z respectively. The system parameters R_0, J_0, J_1, E_i, F_i, G_i, H_i and M_i, $(i = 1,2)$ have positive values and state space of the system is given as ${R_t}^3 = \{(X,Y,Z) \in R^3 : X \geq 0, Y \geq 0, Z \geq 0\}$.

In terms of new variables x, y and z defined as,

$$x = \frac{X}{H_1},\ y = \frac{Y}{H_1},\ z = \frac{Z}{H_1} \text{ and } t = M_1 T,$$

we may rewrite the system equations (1)-(3) in the following form,

$$\dot{x} = \frac{r_1 x(k_0 - x)}{(k_1 - x)} - \frac{\alpha_1 xy}{\alpha_2 y + \alpha_3 x + 1}, \tag{4}$$

$$\dot{y} = \frac{\alpha_4 xy}{\alpha_2 y + \alpha_3 x + 1} - \frac{\alpha_5 yz}{\alpha_6 z + \alpha_7 y + \alpha_8} - y, \tag{5}$$

$$\dot{z} = \frac{\alpha_9 yz}{\alpha_6 z + \alpha_7 y + \alpha_8} - \alpha_{10} z, \tag{6}$$

where $r_1 = \frac{R_1}{M_1}$, $k_0 = \frac{J_0}{H_1}$, $k_1 = \frac{J_1}{H_1}$, $\alpha_1 = \frac{E_1}{M_1}$, $\alpha_2 = F_2$, $\alpha_3 = G_1$, $\alpha_4 = \frac{E_1 L_1}{M_1}$, $\alpha_5 = \frac{E_2}{M_1}$, $\alpha_6 = F_2$, $\alpha_7 = G_2$, $\alpha_8 = \frac{H_2}{H_1}$, $\alpha_9 = \frac{E_2 L_2}{M_1}$ and $\alpha_{10} = \frac{M_2}{M_1}$.

The Jacobian for system eqs.(4)-(6) may be written as,

$$J = \begin{bmatrix} x\frac{\partial \Gamma}{\partial x} + \Gamma & x\frac{\partial \Gamma}{\partial y} & x\frac{\partial \Gamma}{\partial z} \\ y\frac{\partial \Omega}{\partial x} & y\frac{\partial \Omega}{\partial y} + g & y\frac{\partial \Omega}{\partial z} \\ z\frac{\partial \Theta}{\partial x} & z\frac{\partial \Theta}{\partial y} & z\frac{\partial \Theta}{\partial z} + \Theta \end{bmatrix}, \tag{7}$$

where $\Gamma(x,y,z) = \frac{r_1(k_0 - x)}{(k_1 - x)} - \frac{\alpha_1 y}{\alpha_2 y + \alpha_3 x + 1}$, $\Omega(x,y,z) = \frac{\alpha_4 x}{\alpha_2 y + \alpha_3 x + 1} - \frac{\alpha_5 z}{\alpha_6 z + \alpha_7 y + \alpha_8} - 1$ and $\Theta(x,y,z) = \frac{\alpha_9 y}{\alpha_6 z + \alpha_7 x + \alpha_8} - \alpha_{10}$.

The system eqs.(4)-(6) has four positive equilibrium points (i) $E_1(0,0,0)$, (ii) $E_2(k_0,0,0)$, (iii) $E_3(x^*,y^*,0)$ and (iv) $E_4(\bar{x},\bar{y},\bar{z})$.

Following [48], we briefly consider the problem of linear stability of the equilibrium points. For linear stability analysis at any of these equilibrium points, the Jacobian matrix of the system needs to be evaluated using the following values of the various biological relevant parameters:

$$k_0 = 3,\ k_1 = 4,\ \alpha_1 = 4.0,\ \alpha_2 = 0.20,\ \alpha_3 = 0.8,\ \alpha_4 = 3.0,\ \alpha_5 = 2.0,\ \alpha_6 = 0.15,$$

$$\alpha_7 = 0.80, \alpha_8 = 0.50,\ \alpha_9 = 1.5,\ \alpha_{10} = 0.20.$$

(i) The equilibrium point $E_1 = (0,0,0)$ is found to be a saddle point as the eigenvalues of the Jacobian matrix are $\lambda_1 = \frac{r_0 k_0}{k_1} > 0$, $\lambda_2 = -1 < 0$ and $\lambda_3 = -\alpha_{10} < 0$.

(ii) For the equilibrium point $E_2 = (k_0,0,0)$, the eigenvalues of the corresponding Jacobian matrix are $\lambda_1 = -\frac{r_0 k_0}{k_1 - k_0} < 0$, $\lambda_2 = \frac{k_0(\alpha_4 - \alpha_3) - 1}{k_0 \alpha_3 + 1} > 0$ and $\lambda_3 = -\alpha_{10} < 0$. Therefore the equilibrium point is a saddle point.

(iii) The equilibrium point $E_3(x^*,y^*,0)$ suggest the possibility of survival of the predator y in the absence of the super-predator z. In the absence of the super-predator, the subsystem comprising of eqs.(4)-(5) only results in the

following positive solutions,

$$y^* = \frac{(\alpha_4 - \alpha_3)x^* - 1}{\alpha_2}, \quad x^*(\alpha_4 - \alpha_3) > 1, \tag{8}$$

$$x^* = \frac{\chi_2 \pm \sqrt{\chi_2^2 - 4\chi_1\chi_3}}{2\chi_1}, \quad \chi_1 > 0 \quad \text{and} \quad (\chi_2^2 - 4\chi_1\chi_3) > 0, \tag{9}$$

where $\chi_1 = [\alpha_1(\alpha_4 - \alpha_3) - r_0\alpha_2\alpha_4]$, $\chi_2 = [\alpha_1(\alpha_4 - \alpha_3)k_1 - r_0k_0\alpha_2\alpha_4 + \alpha_1]$ and $\chi_3 = \alpha_1k_1$. The stability of the equilibrium point (x^*, y^*) corresponding to the subsystem may be investigated using the following Jacobian matrix,

$$J(x^*, y^*) = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}, \tag{10}$$

where

$$\begin{aligned} J_{11} &= x^* \left[\frac{r_1(k_0 - k_1)}{(k_1 - x^*)^2} + \frac{\alpha_3\alpha_1 y^*}{(\alpha_2 y^* + \alpha_3 x^* + 1)^2} \right], \\ J_{12} &= \left[\frac{-\alpha_1 x^*(\alpha_3 x^* + 1)}{(\alpha_2 y^* + \alpha_3 x^* + 1)^2} \right], \\ J_{21} &= \left[\frac{\alpha_4 y^*(\alpha_2 y^* + 1)}{(\alpha_2 y^* + \alpha_3 x^* + 1)^2} \right], \\ J_{22} &= \left[\frac{-\alpha_2\alpha_4 x^* y^*}{(\alpha_2 * y^* + \alpha_3 * x^* + 1)^2} \right]. \end{aligned} \tag{11}$$

The condition for stability of the equilibrium point for the subsystem is that $\Delta_1 = -(J_{11} + J_{22}) > 0$ and $\Delta_2 = (J_{11}J_{22} - J_{12}J_{21}) > 0$.

The stability of the third equilibrium point $E_3(x^*, y^*, 0)$ can be ensured only if the equilibrium solution of the subsystem $E(x^*, y^*)$ is stable. The three eigenvalues of the Jacobian matrix for the equilibrium solution $E_3(x^*, y^*, 0)$ satisfies the following constraints:

$$\begin{aligned} \lambda_1 + \lambda_2 &= x^* \left[\frac{r_1(k_0 - k_1)}{(k_1 - x^*)^2} + \frac{(\alpha_3\alpha_1 - \alpha_2\alpha_4)x^* y^*}{(\alpha_2 y^* + \alpha_3 x^* + 1)^2} \right], \\ \lambda_1\lambda_2 &= -\frac{r_1(k_0 - k_1)\alpha_2 x^* y^*}{(k_1 - x^*)^2(\alpha_2 y^* + \alpha_3 x^* + 1)} + \frac{\alpha_1 y^*}{(\alpha_2 y^* + \alpha_3 x^* + 1)^2}, \\ \lambda_3 &= \frac{\alpha_9 y^*}{(\alpha_7 y + \alpha_8)} - \alpha_{10}. \end{aligned} \tag{12}$$

The stability of the equilibrium solution $E_3(x^*, y^*, 0)$ in the direction of increasing z requires $\lambda_3 < 0$ or $\alpha_{10} > \frac{\alpha_9 y^*}{\alpha_7 y^* + \alpha_8}$.

(*iv*) The equilibrium point $E_4(\bar{x}, \bar{y}, \bar{z})$ corresponds to the coexistence of the species. From the system eqs.(4)-(6), it is readily observed that

$$\begin{aligned} \bar{y} &= \frac{\alpha_{10}(\alpha_8 + \alpha_6\bar{z})}{\alpha_9 - \alpha_{10}\alpha_7}, \\ \bar{z} &= \left[\frac{u_4(k_0 - \bar{x})\bar{x}}{u_1(k_1 - x)} - \frac{u_2}{u_1}\right], \\ \bar{x} &= \frac{u_3 \pm (u_3^2 - 4u_2k_1u_4)^{1/2}}{2u_4}, \end{aligned} \tag{13}$$

where $u_1 = \alpha_1\alpha_6\alpha_9\alpha_{10} + \alpha_1\alpha_5\alpha_{10}(\alpha_9 - \alpha_7\alpha_{10})$, $u_2 = \alpha_1\alpha_8\alpha_9\alpha_{10}$, $u_3 = u_2 + r_1\alpha_4\alpha_9(\alpha_9 - \alpha_{10}\alpha_7)k_0$, and $u_4 = r_1\alpha_4\alpha_9(\alpha_9 - \alpha_{10}\alpha_7)$.

The linear stability analysis of the equilibrium point $E_4(\bar{x}, \bar{y}, \bar{z})$ of the system results in the following characteristic equation for the Jacobian $J(\bar{x}, \bar{y}, \bar{z})$ matrix,

$$\lambda^3 + \omega_1\lambda^2 + \omega_2\lambda + \omega_3 = 0. \tag{14}$$

where the coefficients $\omega_1 = -(P_{11} + P_{22} + P_{33})$, $\omega_2 = P_{11}P_{22} - P_{12}P_{21} + P_{22}P_{33} - P_{23}P_{32} + P_{11}P_{33}$, $\omega_3 = P_{11}P_{23}P_{32} - P_{11}P_{22}P_{33} + P_{12}P_{21}P_{33}$ and further the various elements $P_{ij}, i, j = 1, 2, 3$ of the Jacobian matrix are,

$$\begin{aligned} P_{11} &= \bar{x}\left[\frac{r_1(k_0 - k_1)}{(k_1 - \bar{x})^2} + \frac{\alpha_3\alpha_1\bar{y}}{(\alpha_2\bar{y} + \alpha_3\bar{x} + 1)^2}\right], \\ P_{12} &= -\frac{\alpha_1\bar{x}(\alpha_3\bar{x} + 1)}{(\alpha_2\bar{y} + \alpha_3\bar{x} + 1)^2} < 0, \\ P_{21} &= \frac{\alpha_4\bar{y}(\alpha_2\bar{y} + 1)}{(\alpha_2\bar{y} + \alpha_3\bar{x} + 1)^2} > 0, \\ P_{22} &= \bar{y}\left[-\frac{\alpha_2\alpha_4\bar{x}}{(\alpha_2\bar{y} + \alpha_3\bar{x} + 1)^2} + \frac{\alpha_5\alpha_7\bar{z}}{(\alpha_6\bar{z} + \alpha_7\bar{y} + \alpha_8)^2}\right], \\ P_{23} &= -\frac{\alpha_5\bar{y}(\alpha_7\bar{y} + \alpha_8)}{(\alpha_6\bar{z} + \alpha_7\bar{y} + \alpha_8)^2} < 0, \quad P_{13} = 0, \quad P_{31} = 0 \\ P_{32} &= \frac{\alpha_9\bar{z}(\alpha_6\bar{z} + \alpha_8)}{((\alpha_6\bar{z} + \alpha_7\bar{y} + \alpha_8)^2)} > 0, \quad P_{33} = -\frac{\alpha_6\alpha_9\bar{z}\bar{y}}{(\alpha_6\bar{z} + \alpha_7\bar{y} + \alpha_8)^2} < 0. \end{aligned} \tag{15}$$

Following Routh-Hurwitz criterion, the equilibrium point $E_4(\bar{x}, \bar{y}, \bar{z})$ is asymptotically stable if $\omega_1 > 0$, $\omega_2 > 0$, $\omega_1\omega_2 > \omega_3$. Violation of these conditions results in limit cycle, period doubling bifurcation leading to chaotic behavior of the system. In order to find the parameter range over which dynamical transition occur, we have computed the bifurcation diagram (Fig.1) of the system viz. eqs.(4)-(6). The bifurcation diagram reveals the dynamical complexity of the system based on simulation results on varying the intrinsic growth rate parameter in the range $1.8 \leq r_0 \leq 8.0$ while retaining the values of other parameters of the system. The transition from limit cycle behavior for $r_0 < 2.5$ to period doubling cascade is initiated in the region $2.5 \leq r_0 \leq 4.2$. The system dynamics is observed to be chaotic for $4.2 \leq r_0 \leq 6.8$. The phenomena of reverse period doubling is observed beyond $r_0 = 6.8$.

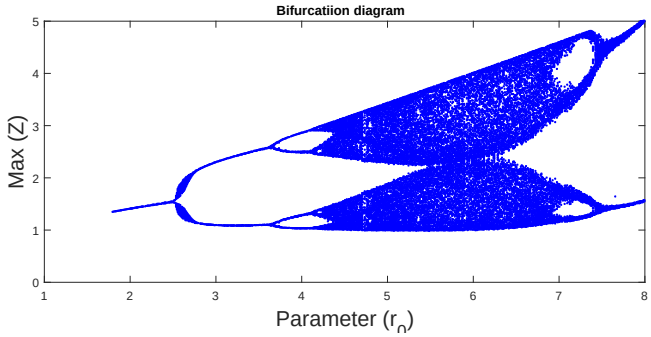


Figure 1: Bifurcation Diagram for Three species food chain model with Beddington-DeAngelis type functional response.

3. Synchronization of two identical type of three species food chains but with different growth rate

In this section, we consider the problem of synchronization of (a) two different three-species food web chains and (b) two identical three-species food web chain with different initial conditions.

3.1 Synchronization of two different three-species food chains

The three-species food web models considered here are similar to the system, eqs.(4)-(6). The master and slave system are therefore described by the following sets of coupled nonlinear differential equations,

Master system:

$$
\begin{aligned}
\dot{x_1} &= \frac{r_1 x_1(k_0 - x_1)}{(k_1 - x_1)} - \frac{\alpha_1 x_1 y_1}{\alpha_2 y_1 + \alpha_3 x_1 + 1}, \\
\dot{y_1} &= \frac{\alpha_4 x_1 y_1}{\alpha_2 y_1 + \alpha_3 x_1 + 1} - \frac{\alpha_5 y_1 z_1}{\alpha_6 z_1 + \alpha_7 y_1 + \alpha_8} - y_1, \\
\dot{z_1} &= \frac{\alpha_9 y_1 z_1}{\alpha_6 z_1 + \alpha_7 y_1 + \alpha_8} - \alpha_{10} z_1.
\end{aligned} \tag{16}
$$

Slave system:

$$
\begin{aligned}
\dot{x_2} &= \frac{r_2 x_2(k_0 - x_2)}{(k_1 - x_2)} - \frac{\alpha_1 x_2 y_2}{\alpha_2 y_2 + \alpha_3 x_2 + 1} + \nu_1, \\
\dot{y_2} &= \frac{\alpha_4 x_2 y_2}{\alpha_2 y_2 + \alpha_3 x_2 + 1} - \frac{\alpha_5 y_2 z_2}{\alpha_6 z_2 + \alpha_7 y_2 + \alpha_8} - y_2 + \nu_2, \\
\dot{z_2} &= \frac{\alpha_9 y_2 z_2}{\alpha_6 z_2 + \alpha_7 y_2 + \alpha_8} - \alpha_{10} z_2 + \nu_3..
\end{aligned} \tag{17}
$$

where r_1 and r_2 corresponds to intrinsic growth rate parameter of the master and slave system respectively. However, all other biologically relevant parameters are having the same meaning as in section 2 and their values are kept same

in the two food web models. The variables $\nu_i,\ i = 1, 2, 3$ refers to the control functions to be determined for effective synchronization of the two three-species food web with different intrinsic growth parameter. Here the master and slave system for the three species food web chains differs only with respect to different intrinsic growth parameters in the absence of controller functions. However, mathematical analysis remain valid if we change the master or slave system by any other model of three species food chain [21].

The master system, eq.(16), when deducted from slave system, eq.(17), allow us to determine the dynamical system involving the state errors $e_1,\ e_2$ and e_3 for designing the controllers. The state errors $e_1,\ e_2$ and e_3 are thus defined as,

$$\begin{aligned} e_1 &= x_2 - x_1, \quad \dot{e_1} = \dot{x_2} - \dot{x_1} \\ e_2 &= y_2 - y_1, \quad \dot{e_2} = \dot{y_2} - \dot{y_1} \\ e_3 &= z_2 - z_1, \quad \dot{e_3} = \dot{z_2} - \dot{z_1} \end{aligned} \tag{18}$$

Therefore from eqs.(16)-(18), the differentrial equations for the state errors $e_i,\ i = 1, 2, 3$ could be written as,

$$\begin{aligned} \dot{e_1} &= \left[\frac{(r_1k_0k_1 - r_1k_1(x_1 + x_2) + r_1x_1x_2+)}{(k_1 - x_2)(k_1 - x_1)} - \frac{\alpha_1\alpha_2y_1y_2}{\beta_1}\right] e_1 - \left[\frac{\alpha_1\alpha_3x_1x_2}{\beta_1}\right] e_2 \\ &\quad + \epsilon\frac{\left[k_0k_1x_2 - k_1x_2^2 - k_0x_1x_2 + x_1x_2^2\right]}{(k_1 - x_2)(k_1 - x_1)} - \left[\frac{\alpha_1(x_2y_2 - x_1y_1}{\beta_1}\right] + \nu_1, \\ \dot{e_2} &= \left[\frac{\alpha_4\alpha_2y_1y_2}{\beta_1}\right] e_1 + \left[\frac{\alpha_4\alpha_3x_1x_2}{\beta_1} - \frac{\alpha_5\alpha_6z_1z_2}{\beta_2} - 1\right] e_2 - \left[\frac{\alpha_5a_7y_1y_2}{\beta_2}\right] e_3 \\ &\quad + \left[\frac{\alpha_4(x_2y_2 - y_2y_1)}{\beta_2} - \frac{\alpha_5\alpha_8(y_2z_2 - y_1z_1)}{\beta_2}\right] + \nu_2, \\ \dot{e_3} &= \left[\frac{\alpha_9\alpha_6z_1z_2}{\beta_2}\right] e_2 + \left[\frac{\alpha_9\alpha_7y_1y_2}{\beta_2} - \alpha_{10}\right] e_3 + \left[\frac{\alpha_9\alpha_8(y_2z_2 - y_1z_1)}{\beta_2}\right] + \nu_3 \end{aligned} \tag{19}$$

where we have defined the parameters $\beta_1,\ \beta_2$ and the difference between the growth rate of the two food chains, ϵ, as:

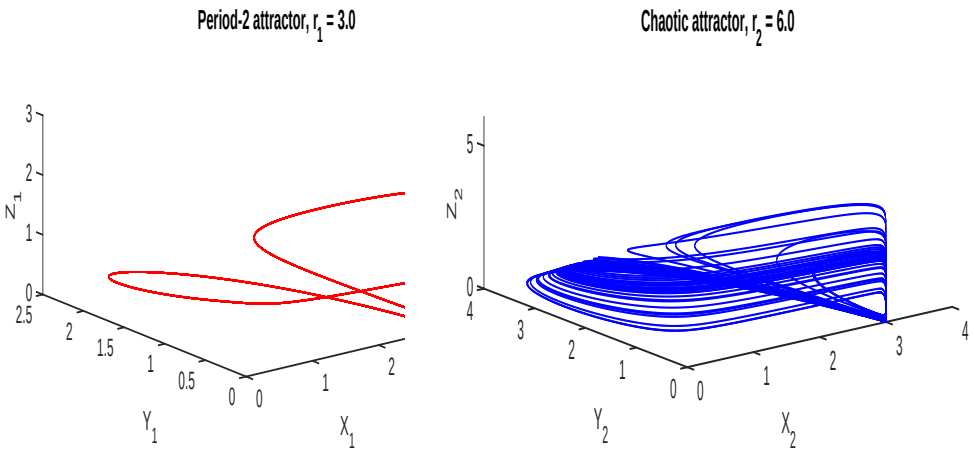
$$\begin{aligned} \beta_1 &= (\alpha_2y_2 + \alpha_3x_2 + 1)(\alpha_2y_1 + \alpha_3x_1 + 1) \\ \beta_2 &= (\alpha_6z_2 + \alpha_7y_2 + a_8)(\alpha_6z_1 + \alpha_7y_1 + \alpha_8) \\ \epsilon &= r_2 - r_1 \end{aligned} \tag{20}$$

The system defined by eq.(19) is called error system and to ensure the asymptotic stability of error system about origin, we define the Lyapunov function ϕ as,

$$\phi(e_1, e_2, e_3) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2). \tag{21}$$

From eq.(19) and eq.(21), it is observed that

$$
\begin{aligned}
\dot{\phi} &= e_1^2\left[\frac{(r_1k_0k_1-r_1k_1(x_1+x_2)+r_1x_1x_2)}{(k_1-x_2)(k_1-x_1)}-\frac{\alpha_1\alpha_2y_1y_2}{\beta_1}\right]-\left[\frac{\alpha_1\alpha_3x_1x_2}{\beta_1}\right]e_1e_2 \\
&-\left[\frac{\alpha_1(x_2y_2-x_1y_1}{\beta_1}\right]e_1+\epsilon\frac{\left[k_0k_1x_2-k_1x_2^2-k_0x_1x_2+x_1x_2^2\right]}{(k_1-x_2)(k_1-x_1)} \\
&+\nu_1e_1+\left[\frac{\alpha_4\alpha_2y_1y_2}{\beta_1}\right]e_1e_2-\left[\frac{\alpha_5\alpha_7y_1y_2}{\beta_2}\right]e_3e_2 \\
&+\left[\frac{\alpha_4\alpha_3x_1x_2}{\beta_1}-\frac{\alpha_5\alpha_6z_1z_2}{\beta_2}-1\right]e_2^2 \\
&+\left[\frac{\alpha_4(x_2y_2-x_1y_1)}{\beta_1}-\frac{\alpha_5\alpha_8(y_2z_2-y_1z_1)}{\beta_2}\right]e_2 \\
&+\nu_2e_2+\left[\frac{\alpha_9\alpha_6z_1z_2}{\beta_2}\right]e_2e_3+\left[\frac{\alpha_9\alpha_7y_1y_2}{\beta_2}-\alpha_{10}\right]e_3^2 \\
&+\left[\frac{\alpha_9\alpha_8(y_2z_2-y_1z_1)}{\beta_2}\right]e_3+\nu_3e_3
\end{aligned}
\tag{22}
$$



(a) (b)

Figure 2: Phase portrait of three-species food chain (a) regular master system with $r_1 = 3.0$, (b) chaotic slave system with $r_2 = 6.0$, with $k_0 = 3$, $k_1 = 4.0$, $\alpha_1 = 4.0$, $\alpha_2 = 0.20$, $\alpha_3 = 0.8$, $\alpha_4 = 3.0, \alpha_5 = 2.0$, $\alpha_6 = 0.15$, $\alpha_7 = 0.80$, $\alpha_8 = 0.50$, $\alpha_9 = 1.5$ and $\alpha_{10} = 0.20$

3.1.1 Active controller functions for two three-species food chain with different intrinsic growth rate

Under active control synchronization scheme, we find the functional form of controller ν_1, ν_2 and ν_3 such that Lyapunov function $\phi(e_1, e_2, e_3)$ satisfy the

condition $\dot{\phi} < 0$ [13,21]. There could be several control functions which possibly yield different synchronization times and satisfy the requirement that $\dot{\phi} < 0$. However, in the following, we chose the the following functional form of the active controller ν_1, ν_2 and ν_3 such that $\dot{\phi} < 0$.

Controller function:

$$\begin{aligned}
\nu_1 &= -\left[\frac{(r_1k_0k_1 - r_1k_1(x_1+x_2) + r_1x_1x_2)}{(k_1-x_2)(k_1-x_1)} - \frac{\alpha_1\alpha_2y_1y_2}{\beta_1} + 1\right]e_1 \\
&+ \left[\frac{\alpha_1\alpha_3x_1x_2}{\beta_1}\right]e_2 + \left[\frac{\alpha_1(x_2y_2 - x_1y_1)}{\beta_1}\right] \\
&- \epsilon\frac{[k_0k_1x_2 - k_1x_2^2 - k_0x_1x_2 + x_1x_2^2]}{(k_1-x_2)(k_1-x_1)} \\
\nu_2 &= -\left[\frac{\alpha_4\alpha_2y_1y_2}{\beta_1}\right]e_1 + \left[\frac{\alpha_5\alpha_7y_1y_2}{\beta_2}\right]e_3 - \left[\frac{\alpha_4\alpha_3x_1x_2}{\beta_1} - \frac{\alpha_5\alpha_6z_1z_2}{\beta_2}\right]e_2 \\
&+ \left[-\frac{\alpha_4(x_2y_2 - x_1y_1)}{\beta_1} + \frac{\alpha_5\alpha_8(y_2z_2 - y_1z_1)}{\beta_2}\right] \\
\nu_3 &= -\left[\frac{\alpha_9\alpha_6z_1z_2}{\beta_2}\right]e_2 - \left[\frac{\alpha_9\alpha_7y_1y_2}{\beta_2} - \alpha_{10} + 1\right]e_3 - \left[\frac{\alpha_9\alpha_8(y_2z_2 - y_1z_1)}{\beta_2}\right]
\end{aligned} \tag{23}$$

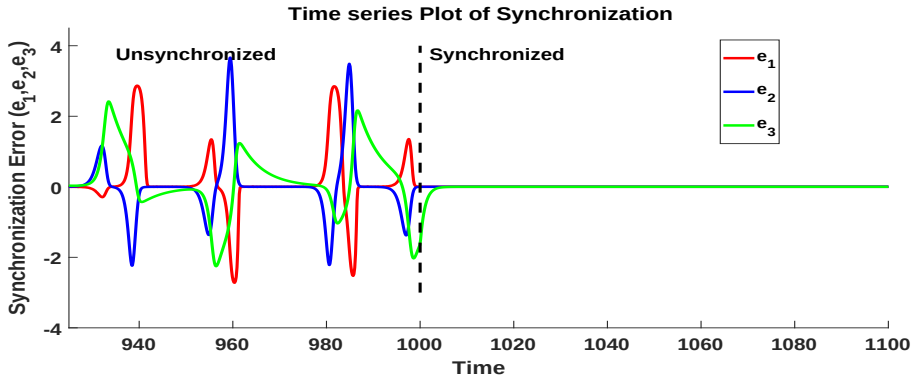


Figure 3: The evolution of state errors before and after the application of controller, eq.20, when the regular master system with $r_1 = 3.0$ drives the chaotic slave system with $r_2 = 6.0$, with $k_0 = 3$, $k_1 = 4.0$, $\alpha_1 = 4.0$, $\alpha_2 = 0.20$, $\alpha_3 = 0.8$, $\alpha_4 = 3.0, \alpha_5 = 2.0$, $\alpha_6 = 0.15$, $\alpha_7 = 0.80$, $\alpha_8 = 0.50$, $\alpha_9 = 1.5$ and $\alpha_{10} = 0.20$

For the problem of synchronization of two different three-species food web chain, the simulation of the master system is effected with initial condition $(0.3, 0.3, 0.22)$ and $(0.15, 0.15, 0.11)$ for the slave system in the absence of controller functions. We further chose the intrinsic growth rate $r_1 = 3.0$ for the master or drive system so that its response is that of a period-2 attractor in phase space (Fig.2a). We further take $\epsilon = 3.0$ such that the intrinsic growth

rate parameter of the driven system is $r_2 = 6.0$ which results in a chaotic response in the absence of the controller function (Fig.2b).

The effect of the application of controller functions, $\nu_i,\ i = 1, 2, 3$ given in eq.(23), on the state errors $e_i,\ i = 1, 2, 3$, is illustrated in Fig.3. It is observed that the state errors for the unsynchronized three-species food chains are significant whereas the application of controller function at time $t = 1000$ and thereafter results in the synchronization of the two food chains. In Fig.4, we further display the simulation results regarding the species population before and after the synchronization of the two system take place. Fig.5a shows the phase plot of the master food chain system with $r_1 = 3.0$ while Fig.5b illustrates the behavior of chaotic slave system with $r_2 = 6.0$ before and after switching on the controller function, eq.(23). The transition of the chaotic slave system with $r_2 = 6.0$ to regular behavior as a result of the application of active controller function, eq.(23), is shown in red color.

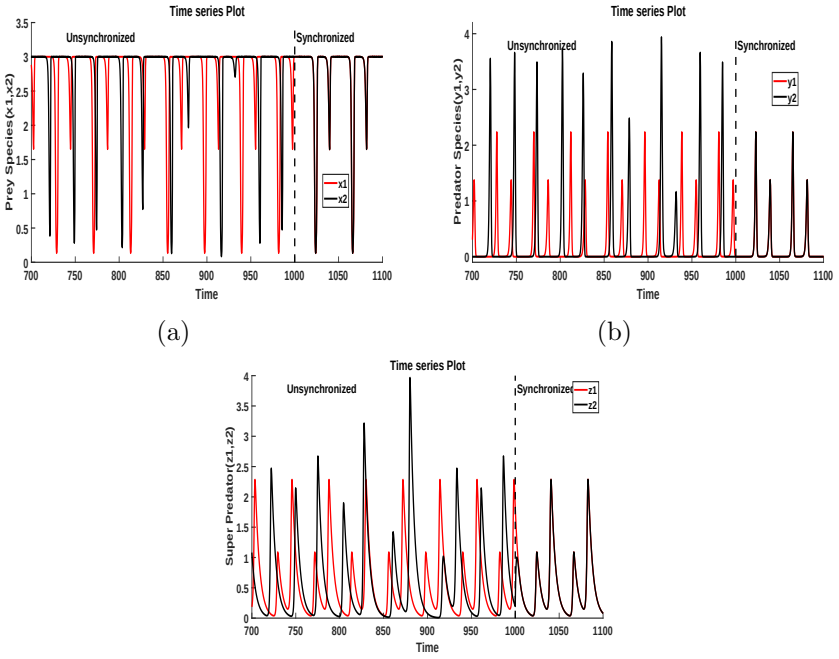
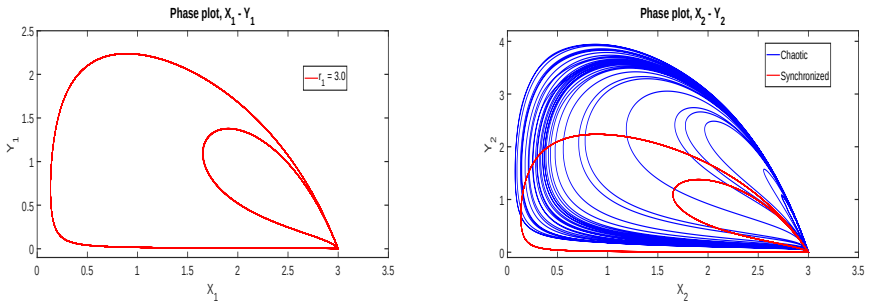


Figure 4: The evolution of species population before and after the application of active controller, eq.(20), when the regular master system with $r_1 = 3.0$ drives the chaotic slave system with $r_2 = 6.0$, with $r_1 = 3.0$, (b) chaotic slave system with $r_2 = 6.0$, with $k_0 = 3,\ k_1 = 4.0,\ \alpha_1 = 4.0,\ \alpha_2 = 0.20,\ \alpha_3 = 0.8,\ \alpha_4 = 3.0, \alpha_5 = 2.0,\ \alpha_6 = 0.15,\ \alpha_7 = 0.80,\ \alpha_8 = 0.50,\ \alpha_9 = 1.5$ and $\alpha_{10} = 0.20$

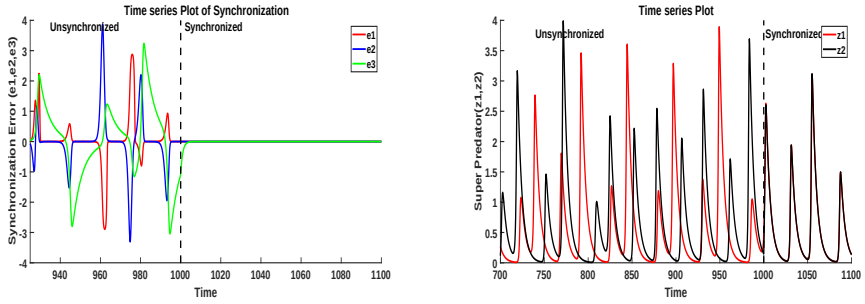
Simulation results are also obtained for the situation when the master three-species food chain is chaotic say with $r_1 = 5.0$ and the slave system is also chaotic with $r_2 = 6.0$. Fig.6a shows time series of the state errors before and

after switching the active controller. Fig.6b shows the time series of evolution of top-predator before and after synchronization. Fig.7a,b illustrates the phase plot of the chaotic master food chain with $r_1 = 5.0$ and the slave chaotic system system with $r_2 = 6.0$ before and after synchronization.



(a) (b)

Figure 5: The phase plot of (a) regular master system with $r_1 = 3.0$, (b) chaotic slave system with $r_2 = 6.0$ (blue) and the synchronized state (red), with $k_0 = 3$, $k_1 = 4.0$, $\alpha_1 = 4.0$, $\alpha_2 = 0.20$, $\alpha_3 = 0.8$, $\alpha_4 = 3.0, \alpha_5 = 2.0$, $\alpha_6 = 0.15$, $\alpha_7 = 0.80$, $\alpha_8 = 0.50$, $\alpha_9 = 1.5$ and $\alpha_{10} = 0.20$

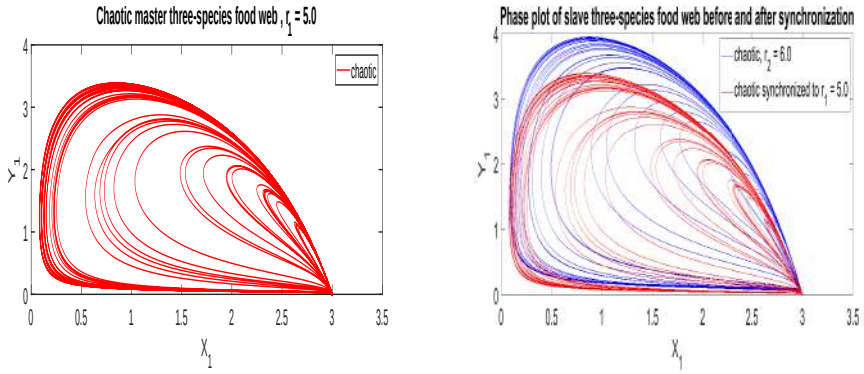


(a) (b)

Figure 6: The evolution of (a) state errors and (b) top-predator population before and after the application of controller, eq.(20), when the chaotic master system with $r_1 = 5.0$ drives the chaotic slave system with $r_2 = 6.0$ with $k_0 = 3$, $k_1 = 4.0$, $\alpha_1 = 4.0$, $\alpha_2 = 0.20$, $\alpha_3 = 0.8$, $\alpha_4 = 3.0$, $\alpha_5 = 2.0$, $\alpha_6 = 0.15$, $\alpha_7 = 0.80$, $\alpha_8 = 0.50$, $\alpha_9 = 1.5$ and $\alpha_{10} = 0.20$

3.2 Synchronization of two identical chaotic three-species food chains with same intrinsic growth rate ($\epsilon = 0$) but different initial conditions

The time evolution of a trajectory, in the phase space of a system exhibiting chaotic dynamics, is very sensitively dependent on initial conditions. Two



(a) (b)

Figure 7: The phase plot of (a) chaotic master system with $r_1 = 5.0$, (b) chaotic slave system with $r_2 = 6.0$ (blue) and the synchronized state (red), with $k_0 = 3$, $k_1 = 4.0$, $\alpha_1 = 4.0$, $\alpha_2 = 0.20$, $\alpha_3 = 0.8$, $\alpha_4 = 3.0$, $\alpha_5 = 2.0$, $\alpha_6 = 0.15$, $\alpha_7 = 0.80$, $\alpha_8 = 0.50$, $\alpha_9 = 1.5$ and $\alpha_{10} = 0.20$

very close trajectories asymptotically diverges from each other depending on the magnitude of the positive Lyapunov characteristic exponent of the system. Therefore, two identical chaotic three-species food web chain evolves differently in phase space for different initial conditions that characterizes evolution of each of the systems. It is within this scenario, we investigate the problem of synchronization of the following two identical three species food chain models. We consider the following master chaotic system with initial conditions $(0.30, 0.30, 0.22)$

Master system:

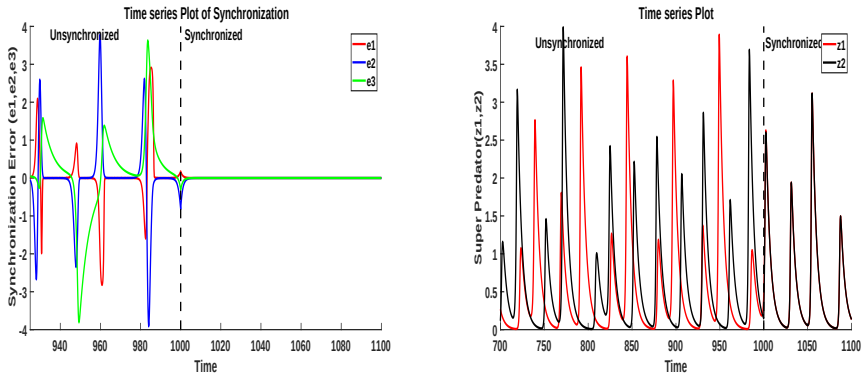
$$
\begin{aligned}
\dot{x_1} &= \frac{r_1 x_1(k_0 - x_1)}{(k_1 - x_1)} - \frac{\alpha_1 x_1 y_1}{\alpha_2 y_1 + \alpha_3 x_1 + 1}, \\
\dot{y_1} &= \frac{\alpha_4 x_1 y_1}{\alpha_2 y_1 + \alpha_3 x_1 + 1} - \frac{\alpha_5 y_1 z_1}{\alpha_6 z_1 + \alpha_7 y_1 + \alpha_8} - y_1, \\
\dot{z_1} &= \frac{\alpha_9 y_1 z_1}{\alpha_6 z_1 + \alpha_7 y_1 + \alpha_8} - \alpha_{10} z_1.
\end{aligned} \tag{24}
$$

Slave system:

$$
\begin{aligned}
\dot{x_2} &= \frac{r_1 x_2(k_0 - x_2)}{(k_1 - x_2)} - \frac{\alpha_1 x_2 y_2}{\alpha_2 y_2 + \alpha_3 x_2 + 1} + \nu_1, \\
\dot{y_2} &= \frac{\alpha_4 x_2 y_2}{\alpha_2 y_2 + \alpha_3 x_2 + 1} - \frac{\alpha_5 y_2 z_2}{\alpha_6 z_2 + \alpha_7 y_2 + \alpha_8} - y_2 + \nu_2, \\
\dot{z_2} &= \frac{\alpha_9 y_2 z_2}{\alpha_6 z_2 + \alpha_7 y_2 + \alpha_8} - \alpha_{10} z_2 + \nu_3..
\end{aligned} \tag{25}
$$

Master drives the slave system with initial condition $(0.15, 0.15, 0.11)$ while retaining the values of all other parameters of the two systems. As earlier,

ν_1, ν_2 and ν_3 in the slave system corresponds to unknown control functions. In view of the different initial conditions, the chaotic dynamics of the master system differs from that of the slave system in phase space in the absence of the controlling functions, for values of the intrinsic growth variable r_1 in the range $4.2 \leq r_0 \leq 6.8$.



(a) (b)

Figure 8: The evolution of (a) state errors and (b) top-predator population before and after the application of controller, eq.(20), in case of identical three-species food web with different initial conditions. Initial condition for the master system is $(0.30, 0.30, 0.22)$ whereas for the slave system initial condition is $(0.15, 0.15, 0.11)$. Other parameter values are $k_0 = 3,\ k_1 = 4.0,\ \alpha_1 = 4.0,\ \alpha_2 = 0.20,\ \alpha_3 = 0.8,\ \alpha_4 = 3.0,\ \alpha_5 = 2.0,\ \alpha_6 = 0.15,\ \alpha_7 = 0.80,\ \alpha_8 = 0.50,\ \alpha_9 = 1.5$ and $\alpha_{10} = 0.20$

Defining the state errors e_1, e_2 and e_3 as in eq.(18), we obtain the following nonlinear differential equations for the evolution of state errors,

$$
\begin{aligned}
\dot{e_1} &= \left[\frac{(r_1k_0k_1 - r_1k_1(x_1+x_2) + r_1x_1x_2)}{(k_1-x_2)(k_1-x_1)} - \frac{\alpha_1\alpha_2y_1y_2}{\beta_1}\right]e_1 \\
&- \left[\frac{\alpha_1\alpha_3x_1x_2}{\beta_1}\right]e_2 - \left[\frac{\alpha_1(x_2y_2 - x_1y_1}{\beta_1}\right] + \nu_1 \qquad (26)\\
\dot{e_2} &= \left[\frac{\alpha_4\alpha_2y_1y_2}{\beta_1}\right]e_1 + \left[\frac{\alpha_4\alpha_3x_1x_2}{\beta_1} - \frac{\alpha_5\alpha_6z_1z_2}{\beta_2} - 1\right]e_2 - \left[\frac{\alpha_5a_7y_1y_2}{\beta_2}\right]e_3 \\
&+ \left[\frac{\alpha_4(x_2y_2 - y_2y_1)}{\beta_2} - \frac{\alpha_5\alpha_8(y_2z_2 - y_1z_1)}{\beta_2}\right] + \nu_2 \\
\dot{e_3} &= \left[\frac{\alpha_9\alpha_6z_1z_2}{\beta_2}\right]e_2 + \left[\frac{\alpha_9\alpha_7y_1y_2}{\beta_2} - \alpha_{10}\right]e_3 + \left[\frac{\alpha_9\alpha_8(y_2z_2 - y_1z_1)}{\beta_2}\right] + \nu_3
\end{aligned}
$$

where β_1 and β_2 are defined in eq.(20).

As earlier, we define the Lyapunov function ϕ as,

$$\phi = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2). \qquad (27)$$

Following the method described in section 3.1, the asymptotic stability of error system about origin is ensured if

$$\begin{aligned}\dot{\phi} = e_1^2 &\left[\frac{(r_1k_0k_1 - r_1k_1(x_1+x_2) + r_1x_1x_2)}{(k_1-x_2)(k_1-x_1)} - \frac{\alpha_1\alpha_2y_1y_2}{\beta_1}\right] \\ &- \left[\frac{\alpha_1\alpha_3x_1x_2}{\beta_1}\right]e_1e_2 + \nu_1e_1 + \left[\frac{\alpha_4\alpha_2y_1y_2}{\beta_1}\right]e_1e_2 \\ &- \left[\frac{\alpha_5\alpha_7y_1y_2}{\beta_2}\right]e_3e_2 + \left[\frac{\alpha_4\alpha_3x_1x_2}{\beta_1} - \frac{\alpha_5\alpha_6z_1z_2}{\beta_2} - 1\right]e_2^2 \\ &+ \left[\frac{\alpha_4(x_2y_2 - x_1y_1)}{\beta_1} - \frac{\alpha_5\alpha_8(y_2z_2 - y_1z_1)}{\beta_2}\right]e_2 \\ &- \left[\frac{\alpha_1(x_2y_2 - x_1y_1}{\beta_1}\right]e_1 + \nu_2e_2 + \left[\frac{\alpha_9\alpha_6z_1z_2}{\beta_2}\right]e_2e_3 \\ &+ \left[\frac{\alpha_9\alpha_7y_1y_2}{\beta_2} - \alpha_{10}\right]e_3^2 + \left[\frac{\alpha_9\alpha_8(y_2z_2 - y_1z_1)}{\beta_2}\right]e_3 + \nu_3e_3.\end{aligned}$$

3.2.1 Active controller functions for two identical system with different initial conditions

In the following, we deduce the following functional form of the active controller $\nu_1,\ \nu_2$ and ν_3 such that $\dot{\phi} < 0$.

Controller function:

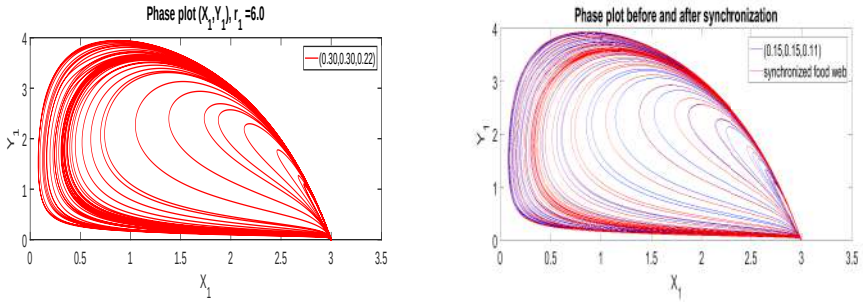
$$\begin{aligned}\nu_1 = &-\left[\frac{(r_1k_0k_1 - r_1k_1(x_1+x_2) + r_1x_1x_2)}{(k_1-x_2)(k_1-x_1)} - \frac{\alpha_1\alpha_2y_1y_2}{\beta_1} + 1\right]e_1 \\ &+ \left[\frac{\alpha_1\alpha_3x_1x_2}{\beta_1}\right]e_2 + \left[\frac{\alpha_1(x_2y_2 - x_1y_1)}{\beta_1}\right] \\ \nu_2 = &-\left[\frac{\alpha_4\alpha_2y_1y_2}{\beta_1}\right]e_1 + \left[\frac{\alpha_5\alpha_7y_1y_2}{\beta_2}\right]e_3 \\ &- \left[\frac{\alpha_4\alpha_3x_1x_2}{\beta_1} - \frac{\alpha_5\alpha_6z_1z_2}{\beta_2}\right]e_2 \\ &+ \left[-\frac{\alpha_4(x_2y_2 - x_1y_1)}{\beta_1} + \frac{\alpha_5\alpha_8(y_2z_2 - y_1z_1)}{\beta_2}\right] \\ \nu_3 = &-\left[\frac{\alpha_9\alpha_6z_1z_2}{\beta_2}\right]e_2 - \left[\frac{\alpha_9\alpha_7y_1y_2}{\beta_2} - \alpha_{10} + 1\right]e_3 \\ &- \left[\frac{\alpha_9\alpha_8(y_2z_2 - y_1z_1)}{\beta_2}\right].\end{aligned} \tag{28}$$

As a result, we find that the state errors $e_i,\ i = 1, 2, 3$ satisfy the following set of equations

$$\dot{e}_i \ = \ -e_i,\ i = 1, \cdots, 3 \tag{29}$$

which suggest the synchronization of the two identical systems with different initial conditions.

For numerical computation regarding synchronization of two identical system, we consider both the master and slave chaotic system with $r_1 = 6.0$ but with different initial conditions i.e, $(0.30, 0.30, 0.22)$ for the master and $(0.15, 0.15, 0.11)$ for the slave system respectively. Using the fourth order Runge-Kutta scheme with the foregoing initial conditions, eq.(24) and (25) are integrated to provide the state error functions before and after the activation of controller, eq.(29). Fig.8a illustrates the behavior of error functions whereas Fig.8b describes the oscillations of top-predator population before and after switching on the controller at time $t = 1000$. Fig.9a,b describes the phase plot of the master three-species food web chain with $r_1 = 6.0$ and its synchronization with the identical chaotic slave system, however with different initial condition as above before and after the controller is activated at time $t = 1000$, respectively.



(a) (b)

Figure 9: The phase plot of (a) chaotic master system with $r_1 = 6.0$ with initial conditions $(0.30, 0.30, 0.22)$, (b) chaotic slave system with $r_1 = 6.0$ (blue) with initial conditions $(0.15, 0.15, 0.11)$ and the synchronized state (red), with $k_0 = 3,\ k_1 = 4.0,\ \alpha_1 = 4.0,\ \alpha_2 = 0.20,\ \alpha_3 = 0.8,\ \alpha_4 = 3.0, \alpha_5 = 2.0,\ \alpha_6 = 0.15,\ \alpha_7 = 0.80,\ \alpha_8 = 0.50,\ \alpha_9 = 1.5$ and $\alpha_{10} = 0.20$

4. Conclusion

In this paper two three-species food chain models with Beddington-DeAngelis type functional response of prey-predator interaction has been considered for the study of synchronization under master-slave scheme. For our investigation, we consider the model of three species food web chain that incorporates the effects of resource utilization [43] and Beddington-DeAngelis type functional response. The details of analytical results concerning the linear stability of various possible equilibrium points of such a system has been provided along with appropriate conditions for their stability. The bifurcation diagram of the model has been constructed by noting the maximum of the top-predator population as the intrinsic growth parameter r_1 is varied (Fig.1). It reveals the dynamical

transition of from limit cycle behavior to period doubling bifurcation and chaos with interspersed thin regular windows in the region $2 \leq r_1 \leq 6.8$. However, for the range $6.8 \leq r_1 \leq 8.0$ the model behavior switches over to limit cycle behavior following reverse period-doubling scenario. The problem of synchronization of two such three-species food chain is investigated by establishing active controllers using the Lyapunov stability theory. The effectiveness of the analytically derived active controller function has been tested numerically for the case of (I) two three-species food web chains with different intrinsic growth rate and (II) two identical three-species food web chain with same intrinsic growth but with different initial conditions. We, however, take the same values of other biologically relevant parameters in the two models.

For case I, we first consider the master three-species food chain with intrinsic growth parameter $r_1 = 3.0$ that drives the slave three-species model with intrinsic growth $r_2 = 6.0$. Accordingly, the population dynamics of the master and slave system corresponds to that of period-2 and chaotic attractor in phase space (Fig. 2). Fig. 3 describes the behavior of state error functions before and after the controller function is switched on at time $t = 1000$. The efficacy of the non-linear active controller function, eq.(23), with regard to synchronization of master-slave system is further shown in terms of the time series of species population (Fig.4) and the phase plot (Fig.5). The same controller function is observed to be effective in synchronizing the master chaotic three-species model with $r_1 = 5.0$ and the slave system with $r_2 = 6.0$ (Fig. 6,7).

For case II, we consider investigate the synchronization of two identical three-species food chain with same value of the intrinsic growth parameter i.e. $r_1 = r_2$ or $\epsilon = 0$ but with different initial conditions. In this case the active controller function is different from the case I and as earlier is observed to be effective in the synchronization of the two systems (Fig.8,9). The process of control of response system using the nonlinear active controller function for synchronization of the two three-species food chain involves basically the state error functions representing the difference in the growth of the species population in the two model. As a result, the diffusion of the species tends to balance the growth rate of master with that of the slave system causing synchronization. It is noted that the method is suitable for synchronization of any two models of the three-species food chains treated as master-slave system. We intend to extend the technique of finding the active controller function in this paper to the setting where several parameters of the master three-species food differ from that of the slave system subsequently.

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