Residual power series technique for solving Fokker-Planck equation

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Abstract. In this paper we used a numerical method to find the solutions of Fokker-Planck equation. The method used the residual power series method to solve the initial value Fokker-Planck problems. Numerous algorithms for linear and nonlinear Fokker-Planck equations are presented. A comparison between residual power series method, Adomian decomposition method (ADM) and the variational iteration method (VIM) is concluded that RPSM is easy to perform and yields very accurate results.
Keywords: RESIDUAL power series method, Fokker-Planck equations, Kolmogorov equation, adomian decomposition method, variational iteration method.

1. Introduction

The Fokker-Planck equation was first introduced by Anthony Fokker [1] and Max Planck [2] to describe the Brownian motion of particles. This equation plays a very important role in a wide range of natural sciences such as statistical physics, solid-state physics, quantum optics, chemical physics, and theoretical biology [3, 4]. The experiment of immersing small particles in a fluid can be characterized by the equation of motion for the distribution function $u(x,t)$ that is given by the linear Fokker-Planck equation

$$\frac{\partial u}{\partial t} = \frac{1}{\gamma}\frac{\partial(x,\, u)}{\partial x} + \frac{\beta}{\gamma}\frac{\partial^2 u}{\partial x^2}.$$

Where γ is the particle relaxation time, x is the velocity of the particle, $\beta = \frac{KT}{m}$ is the thermal velocity, K is the Boltzman's constant, T is the temperature of the fluid, and m is the mass of the particle [3]. The general form of the linear Fokker-Planck equation is given by $\frac{\partial u}{\partial t} = -A(s)\,\frac{\partial u}{\partial s} + B(s)\,\frac{\partial^2 u}{\partial s^2}$ subject to the

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initial condition $u(s,0) = f(s), \quad s \in \mathcal{R}$. Where $A(s)$ and $B(s)$ are positive and represented the drift and diffusion coefficients respectively. In some applications, the drift and diffusion coefficients depend on time and space. Thus, the linear Fokker-Planck equation becomes

$$\frac{\partial u}{\partial t} = -A(s,t)\,\frac{\partial u}{\partial s} + B(s,t)\,\frac{\partial^2 u}{\partial s^2} \tag{1}$$

subject to $u(s,0) = f(s)$.

which is also called the backward Kolmogorov's equation which is a second order partial differential equation of parabolic type (for more details see [3]).

The nonlinear Fokker-Planck equation arises in several applications including plasma physics, surface physics, population dynamics, biophysics, laser physics, pattern formation, psychology and marketing [4]. A one dimensional nonlinear Fokker-Planck equation is given by

$$\frac{\partial u}{\partial t} = -\frac{\partial A(s,t,u)\,u}{\partial s} + \frac{\partial^2 B(s,t,u)\,u}{\partial s^2}.$$

The nth dimensional nonlinear Fokker-Planck equation has the following form:

$$\frac{\partial u}{\partial t} = -\sum_{i=1}^{n} \frac{\partial A_i(s,t,u)\,u}{\partial s_i} + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 B_{ij}(s,t,u)\,u}{\partial s_i \partial s_j},$$

where $S = (s_1, s_2, \cdots, s_n)$.

Recently, vast of studies have been made to solve the Fokker-Planck equation numerically and analytically. Numerous methods have been applied to find approximated solutions of the Fokker-Planck equation [5]-[12]. Tatari et. al. [5] applied the Adomian decomposition method to solve Fokker-Planck equation without discretization. Hesam et. al. [6], applied the differential transform method to obtain an analytical solution of the Fokker-Planck equation. Aminikhah and Jamalian [7], used a combination of Laplace transform and the homotopy perturbation method to solve the nonlinear Fokker-Planck equation. Torvattanabum [8], employed the variational iterational method to obtain approximated analytical solution of the Fokker-Planck equation. Al-Jawary et. al. [12] solved the Fokker-Planck equation by the semi-analytical iterative technique (TAM).

In this work, we applied the residual power series method [13] to obtain analytical solutions for the linear and nonlinear Fokker-Planck equations. The residual power series method (henceforth referred to as RPSM) can be used to obtain a power series solutions for the Fokker-Planck equations without perturbations or discretization. The solutions induced by both the traditional Taylor series method and the RPSM are in polynomials form but the Taylor series method is computationally expensive for higher orders (see Theorem 1 below) [13]. The idea behind the RPSM is to express the solutions of the Fokker-Planck

equation as a power series about the initial point $t = t_0$. The approximated solutions can be written as $u(s,t) = \sum_{n=0}^{\infty} f_n(s)\,(t-t_0)^n$. The solution $u(s,t)$ can be approximated by the kth- truncated solution $u_k(s,t)$ for $k = 1, 2, 3, \cdots$ and hence,

$$u_k(s,t) = f_0(s) + \sum_{n=1}^{k} f_n(s)\,(t-t_0)^k \tag{2}$$

where $f_0(s) = u_0(s,t) = u(s,0)$. Substituting equation 1 into the Fokker-Planck equation yields what is called the kth- residual function which is denoted by $Res_k(s,t)$. The latter is infinitely many times differentiable at $t = 0$ because for each t we have

$$Res_\infty(s,t) = \lim_{k\to\infty} Res_k(s,t) = 0.$$

Moreover, one of a fundamental rules in the construction of the RPS method is

$$\frac{\partial^m}{\partial t^m} Res_\infty(s,0) = \frac{\partial^m}{\partial t^m} Res_k(s,0) = 0,$$

for $m = 1, 2, 3, \cdots, k$. For more details about the method please see [13]-[21]. The following theorem shows the convergence of residual power series method.

Theorem 1 ([13]). *Suppose that $u(s,t)$ is the exact solution of the initial value problem 1, then the approximated solutions obtained by the residual power series method is just the Taylor expansion of $u(s,t)$.*

Corollary 1.1 ([13]). *If $u(s,t)$ is a polynomial, then the exact solution will be obtained by the residual power series method.*

This paper is organized as follows: In section 2, we introduced the algorithms that have been applied on the linear and nonlinear Fokker-Planck equations using the residual power series method. The residual power series method has been illustrated by some examples and we introduced numerical simulations for several examples of Fokker-Planck equations in section 3. In section 4, the conclusion and discussion of the efficiency of the method is presented.

2. Algorithms of the RPSM to solve Fokker-Planck Equation

In this section we will introduce some numerical algorithms to solve several types of the Fokker-Planck equations. These algorithms are based on the RPSM [13]. The backward Kolmogorov equation which is a linear Fokker-Planck equation is introduced by Risken [3]

$$\frac{\partial u}{\partial t} = -A(s,\,t)\frac{\partial u}{\partial s} + B(s,\,t)\frac{\partial^2 u}{\partial s^2} \tag{3}$$

subject to the initial condition $u(s,\,0) = f(s)$.

To apply the RPS method on equation 3, we assume that $u(s,\,t)=\sum_{i=0}^{\infty} f_i(s)t^i$ is a solution of equation 3 and hence the nth-truncated series $u_n(s,\,t)=\sum_{i=0}^{n} f_i(s)t^i$ is the approximated solutions of 3. To determine the coefficient functions $\{f_i\}_{i=0}^{n}$ we define $Res_n = \frac{\partial u_n}{\partial t} + A\frac{\partial u_n}{\partial s} - B\frac{\partial^2 u_n}{\partial s^2}$, and $Res_\infty = \lim_{k\to\infty} Res_k$. Moreover, $Res_\infty|_{t=0} = \frac{\partial^k}{\partial t^k} Res_n|_{t=0} = 0$ for each $k = 0,\, 1\, 2\, 3\, \ldots,\, n$.

Assume $u_0(s,0) = f(s) = f_0(S)$ and hence $u_1(s,t) = f_0(s) + f_1(s) \times t$, to find $f_1(s)$ we use the fact that $Res_1|_{t=0} = 0$ which implies that

$$Res_1 = \frac{\partial u_1}{\partial t} + A\frac{\partial u_1}{\partial s} - B\frac{\partial^2 u_1}{\partial s^2} = f_1(s) + A\left[f'(s) + t\, f_1'(s)\right] - B\left[f''(s) + t\, f_1''(s)\right]$$

Upon substituting $t = 0$ we can solve for $f_1(s)$ and then we can find $u_1(s,t)$. Repeating this procedure to find the nth-truncated solution $u_n(s,t) = f_0(s) + t\, f_1(s) + \ldots + t^n\, f_n(s)$. The above procedure is illustrated in the following algorithm 1

Algorithm 1 RPSM algorithm for solving Kolmogorov equation

procedure RPS($A,\, B,\, t_0,\, f_0,\, n$) ▷ The Kolmogorov equation parameters
  $u_0 = f_0$;
  for $k = 1, \ldots, n$ **do**
    $u_k = u_{k-1} + f_k t^{k-1}$
    $Res_k = \frac{\partial u_k}{\partial t} + A\frac{\partial u_k}{\partial s} - B\frac{\partial^2 u_k}{\partial s^2}$ ▷ Compute the residue
    $\text{equ}_k = \left(\frac{\partial^k}{\partial t^k} Res_k\right)_{t=0}$ ▷ Use the residue condition
    Solve($\text{equ}_k = 0,\, f_k$)
  end for
  end for
  return u_n ▷ The nth approximated solution of Kolmogorov equation
end procedure

The general form of the linear Fokker-Planck Equation is given by

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial s}A(s) + \frac{\partial^2}{\partial s^2}B(s)\right] u \tag{4}$$

subject to the initial condition $u(s,\,0) = f(s)$.

To apply the RPSM on equation 4, we assume that $u(s,\,t) = \sum_{i=0}^{\infty} f_i(s)t^i$ is a solution of equation 4 and hence the nth-truncated series $u_n(s,\,t) = \sum_{i=0}^{n} f_i(s)t^i$ is the approximated solutions of 4. To determine the coefficient functions $\{f_i\}_{i=0}^{n}$ we define $Res_n = \frac{\partial u_n}{\partial t} + \frac{\partial A\, u_n}{\partial s} - \frac{\partial^2 B\, u_n}{\partial s^2}$, and $Res_\infty = \lim_{k\to\infty} Res_k$. Moreover, $Res_\infty|_{t=0} = \frac{\partial^k}{\partial t^k} Res_n|_{t=0} = 0$ for each $k = 0,\, 1\, 2\, 3\, \ldots,\, n$.

Assume $u_0(s,0) = f(s) = f_0(S)$ and hence $u_1(s,t) = f_0(s) + f_1(s) \times t$, to find $f_1(s)$ we use the fact that $Res_1|_{t=0} = 0$ which implies that

$$\begin{aligned} Res_1|_{t=0} &= \frac{\partial u_1}{\partial t} + \frac{\partial}{\partial s} A u_1|_{t=0} - \frac{\partial^2}{\partial s^2} B u_1|_{t=0} \\ &= f_1 + A f'(s) + f(s)\frac{\partial A}{\partial s} - B f''(s) - f'(s)\frac{\partial B}{\partial s} - f(s)\frac{\partial^2 B}{\partial s^2} - f''(s)\frac{\partial B}{\partial s}. \end{aligned}$$

Upon substituting $t = 0$ we can solve for $f_1(s)$ and then we can obtain $u_1(s,t)$ as follows

$$u_1 = f(s) + \left(-A f'(s) - f(s)\frac{\partial A}{\partial s} + B f''(s) + f'(s)\frac{\partial B}{\partial s} + f(s)\frac{\partial^2 B}{\partial s^2} + f''(s)\frac{\partial B}{\partial s} \right) \times t.$$

Repeating this procedure to find the nth-truncated solution $u_n(s,t) = f_0(s) + t\, f_1(s) + \ldots + t^n\, f_n(s)$. The above procedure is illustrated in the following algorithm 2

Algorithm 2 RPSM algorithm for solving Fokker-Planck equation

procedure RPS(A, B, t_0, f_0, n) ▷ The Fokker-Planck Equation parameters
  $u_0 = f_0$;
  for $k = 1, \ldots, n$ **do**
    $u_k = u_{k-1} + f_k t^{k-1}$
    $Res_k = \frac{\partial u_k}{\partial t} + \frac{\partial}{\partial s} A u_k - \frac{\partial^2}{\partial s^2} B u_k$ ▷ Compute the residue
    $\text{equ}_k = \left(\frac{\partial^k}{\partial t^k} Res_k \right)_{t=0}$ ▷ Use the residue condition
    Solve($\text{equ}_k = 0$, f_k)
  end for
  end for
  return u_n ▷ The nth approximate solution of Fokker-Planck Equation
end procedure

The $N-$dimensional linear Fokker-Planck equation is given by

$$\frac{\partial u}{\partial t} = \left[-\sum\nolimits_{i=1}^{N} \frac{\partial}{\partial s_i} A_i(s) + \sum\nolimits_{i,j=1}^{N} \frac{\partial^2}{\partial s_i s_j} B_{i,j}(s) \right] u \tag{5}$$

subject to the the initial condition $u(s, 0) = f(s)$, where $s = (s_1, \ldots, s_N)$. The RPSM procedure to find the approximated solution of equation 5 is described in algorithm 3

Algorithm 3 RPSM algorithm for solving general Fokker-Planck Equation

procedure RPS(A_i, $B_{i,j}$, t_0, f_0, n) ▷ The general Fokker-Planck equation parameters
  $u_0 = f_0$;
  for $k = 1, \ldots, n$ **do**
    $u_k = u_{k-1} + f_k t^{k-1}$
    $Res_k = \frac{\partial u}{\partial t} = -\sum_{i=1}^{N} \frac{\partial}{\partial s_i} A_i(s) u_k + \sum_{i,j=1}^{N} \frac{\partial^2}{\partial s_i s_j} B_{i,j}(s) u_k$ ▷ Compute the residue
    $\text{equ}_k = \left(\frac{\partial^k}{\partial t^k} Res_k \right)_{t=0}$ ▷ Use the residue condition
    Solve($\text{equ}_k = 0$, f_k)
  end for
  end for
  return u_n ▷ The nth approximate solution of general Fokker-Planck Equation
end procedure

The one-dimensional nonlinear Fokker-Planck equation is given by

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial s} A(s, t, u) + \frac{\partial^2}{\partial s^2} B(s, t, u) \right] u. \tag{6}$$

The RPSM procedure to find the approximated solutions of equation 6 is described in a similar way as in algorithm 2.

3. Numerical discussion

In this section, the residue power series method is applied to solve several cases of the Fokker-Planck equations such as in 3, 4, 5 and 6 for different initial conditions.

Example 3.1. Consider the IVP

$$\begin{cases} \dfrac{\partial u}{\partial t} = -A(s, t) \dfrac{\partial u}{\partial s} + B(s, t) \dfrac{\partial^2 u}{\partial s^2}, \\ u(s, 0) = 1 + s \end{cases} \tag{7}$$

where $A(s, t) = -(s + 1)$, $B(s, t) = e^t s^2$.

Assume that the solution $u(s, t)$ has the form $u(s, t) = \sum_{i=0}^{\infty} f_i(s) t^i$ and the approximated solution is $u_n(s, t) = \sum_{i=0}^{n} f_i(s) t^i$, then the coefficient functions $\{f_i(s)\}_{i=0}^{n}$ will be determined By RPSM as follows:

Let $Res_n = \frac{\partial u_n}{\partial t} - (s + 1) \frac{\partial u_n}{\partial s} - e^t s^2 \frac{\partial^2 u_n}{\partial s^2}$, and $Res_\infty = \lim_{k \to \infty} Res_k$.

Since, $Res_\infty|_{t=0} = \frac{\partial^k}{\partial t^k} Res_n|_{t=0} = 0$ for each $k = 0, 1\,2\,3 \ldots, n$, then $f_0(s) = s+1$ and hence, $u_0 = s+1$, $f_1 = s+1$ which implies that $u_1 = (s+1)(1+t)$. For

$n = 2$, $u_2 = f_0 + f_1 \times t + f_2 \times t^2$. Upon applying the condition $\frac{\partial}{\partial t} Res_2|_{t=0} = 0$, we obtained that $f_2 = \frac{s+1}{2}$ and the second term of the RPSM approximated solution is $u_2 = (s+1)(1 + t + \frac{t^2}{2})$.

Similarly the third term of the RPSM approximated solution is given by $u_3 = f_0 + f_1 \times t + f_2 \times t^2 + f_3 \times t^3$ where $f_3 = \frac{s+1}{6}$. Therefore the nth approximated solution can be computed as follows:

$$u_n = (s+1)\left(1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \ldots + \frac{1}{n!}t^n\right).$$

Since $u(s,t) = \lim_{n\to\infty} u_n(s,t)$, then we conclude that $u(s,t) = (s+1)e^t$ which is the exact solution of equation 7. Figure 1 describes the exact solution $u(s,t)$ and the approximated solutions $u_5(s,t)$ and $u_3(s,t)$.

Example 3.2. Consider the IVP

$$\begin{cases} \dfrac{\partial u}{\partial t} = \left[-\dfrac{\partial}{\partial s}A(s) + \dfrac{\partial^2}{\partial s^2}B(s)\right] u, \\ u(s,0) = s \end{cases} \tag{8}$$

where $A(s) = -1$, $B(s) = 1$.

Consider equation 4. Let $A(s) = -1$ and $B(s) = 1$ with $u(s,0) = s$, then by applying algorithm 2 we have:

For $n = 0$, obviously $f_0(s) = f(s) = s$ hence $u_0 = s$. For $n = 1,$, $u_1 = f_0 + f_1 \times t$, then $Res_1|_{t=0} = 0$ which implies that $f_1 = 1$ and $u_1 = s + t$.

Repeating the process $k-$times yields $u_k = s + t$ for all k, which is the exact solution of the problem. Figure 2 describes the approximated solutions $u_0(s,\, t)$ and $u_1(s,\, t)$.

Example 3.3. Consider equation 6. Let $A(s,\, t,\, u) = 4\frac{u}{s} - \frac{s}{3}$ and $B(s,\, t,\, u) = u$ with $u(s,0) = s^2$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial s}\left(\frac{s}{3}u - 4\frac{u^2}{s}\right) + \frac{\partial^2}{\partial s^2}u^2 \tag{9}$$

According to RPSM algorithm 2, we have $u_0 = s^2$, $u_1 = t\, s^2 + s^2$, $u_2 = (t^2\, s^2)/2 + t\, s^2 + s^2$, and $u_3 = s^2(1 + t + \frac{t^2}{2} + \frac{t^3}{6})$.

Repeating this procedure $k-$times we get $u_k = s^2(1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \ldots + \frac{1}{k!}t^k)$, therefore as $k \to \infty$, $u = s^2\, e^t$ which is the exact solution. Figure 3 describes the exact solution $u(s,t)$ and the approximated solutions $u_5(s,t)$ and $u_3(s,t)$.

Example 3.4. Consider equation 5. Let $A_1(s_1,\, s_2) = s_1$, $A_2(s_1,\, s_2) = 5y$, $B_{1,1}(s_1,\, s_2) = s_1^2$, $B_{1,2}(s_1,\, s_2) = 1$, $B_{2,1}(s_1,\, s_2) = 1$, $B_{2,2}(s_1,\, s_2) = s_2^2$ with $u(s_1, s_2, 0) = s_1$, where $s_1,\, s_2 \in \mathbb{R}$.

According to RPSM algorithm 3, we have $u_0(s_1,\, s_2,\, t) = s_1$, $u_1 = t\, s_1 + s_1$, $u_2 = (t^2\, s_1)/2 + t\, s_1 + s_1$ and $u_3 = s_1\,(1 + t + \frac{t^2}{2} + \frac{t^3}{6})$.

Repeating this procedure $k-$times we get $u_k = s_1\,(1+t+\frac{1}{2}t^2+\frac{1}{6}t^3+\ldots+\frac{1}{k!}t^k)$, therefore as $k \to \infty$, $u = s_1\,\mathrm{e}^t$ which is the exact solution.

The level curves of the variable s_i at c will be defined by:

$$l_{c(s_i)}\,[u(s_1,\,\ldots,\,s_N)] = \{(u,\,s_1,\,\ldots,\,s_{i-1},\,s_{i+1},\,\ldots,\,s_N)\,:\,u = u(s_1,\,\ldots,\,s_{i-1},\,c,\,s_{i+1},\,\ldots,\,s_N)\}.$$

These level curves will be used to analyze the high dimensional systems solutions obtained by RPSM. Figures 4, 5, and 6 illustrated the level curves of the solutions of the problem in example (3.4).

Example 3.5. For the sake of comparison, We will study the forward Kolmogorov equation that was considered by Odibat and Momani [10], the authors solved the problem by ADM and VIM methods to obtain numerical solutions. In this example we will solve the same problem by the RPSM method.

Consider the following forward Kolmogorov equation

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial s}\,A(s,t) + \frac{\partial^2}{\partial s^2}\,B(s,t)\right]\,u(s,t),$$

where $A(s,t) = \frac{s}{6}$, $B(s,t) = \frac{s^2}{12}$ and $u(s,0) = s^2$. Upon applying the RPSM method we obtain that $u_1(s,t) = s^2 + \frac{t\,s^2}{2}$, $u_2(s,t) = s^2 + \frac{t\,s^2}{2} + \frac{t^2\,s^2}{8}$, $u_3(s,t) = s^2 + \frac{t\,s^2}{2} + \frac{t^2\,s^2}{8} + \frac{t^3\,s^2}{48}$ and hence, $u(s,t) = s^2\,\mathrm{e}^{t/2}$ which is the exact solution. Figure 7 depicts the plot of the approximated solution $u_3(s,t)$ with several values of time $t = 0.2$(blue), $t = 0.4$(red) and $t = 0.6$(black) and space s (Left) and the plot of the exact solution $u(s,t)$ and the approximated solutions $u_1(s,t), u_2(s,t)$ and $u_3(s,t)$ with time $t = 2$ and space s (Right) .

4. Conclusion

The residual power series method is introduced and applied to find the approximated solutions of Fokker-Planck equation. Numerous algorithms are presented to solve the Fokker-Planck linear and nonlinear equations. Several numerical examples are provided to illustrate the solutions which concluded that the RPSM is efficient in finding numerical solutions for such problems using very simple calculations. A comparison between RPSM and Adomian decomposition method (ADM) and the variational iteration method (VIM) [10] shows the simplicity and the powerful of the RPSM. The following table 1 illustrates the efficiency of the RPSM by introducing a comparison between the RPSM, ADM, VIM and the exact solution of the problem in Example 3.5.

All calculations in the presented examples in this work have been performed using MAPLE.

Table 1: A comparison of the relative error of the approximated solutions in example 3.5 using RPSM with the relative error of ADM and VIM. The values of ADM and VIM are taken from [10]

t	s	E_{relative} RPSM	E_{relative} ADM	E_{relative} VIM
0.2	0.25	0	1.5×10^{-4}	1.5×10^{-4}
	0.5	3.6×10^{-6}	1.2×10^{-4}	1.5×10^{-4}
	0.75	4.8×10^{-6}	1.5×10^{-4}	1.5×10^{-4}
	1	3.6×10^{-6}	1.5×10^{-4}	1.5×10^{-4}
0.4	0.25	6.5×10^{-5}	1.2×10^{-3}	1.2×10^{-3}
	0.5	5.9×10^{-5}	1.1×10^{-3}	1.1×10^{-3}
	0.75	5.7×10^{-5}	1.1×10^{-3}	1.1×10^{-3}
	1	5.7×10^{-5}	1.1×10^{-3}	1.1×10^{-3}
0.6	0.25	3.8×10^{-4}	3.6×10^{-3}	3.6×10^{-3}
	0.5	2.7×10^{-4}	3.6×10^{-3}	3.6×10^{-3}
	0.75	2.7×10^{-4}	3.6×10^{-3}	3.6×10^{-3}
	1	2.7×10^{-4}	3.6×10^{-3}	3.6×10^{-3}

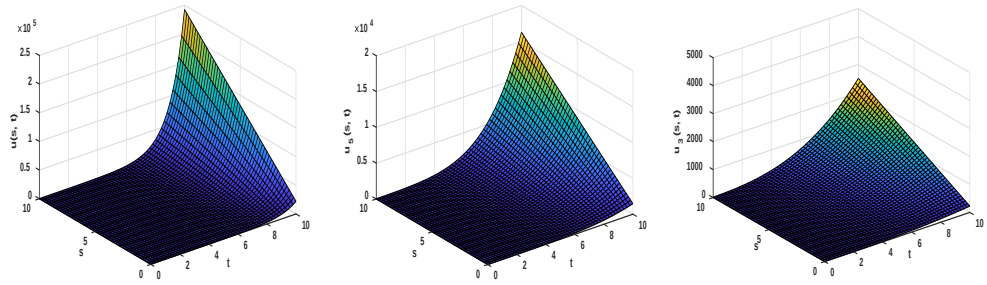

Figure 1: The 3D plot of the exact solution (Left), $u_5(s,t)$ (Middle), and $u_3(s,t)$ (Right) with time t and space s.

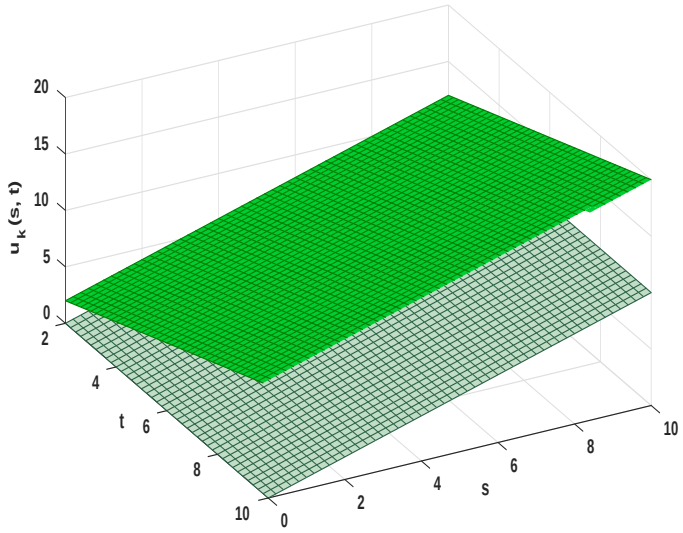

Figure 2: The approximated solutions $u_0(s,\, t)$ and $u_1(s,\, t)$ of problem 8 in Example 3.2

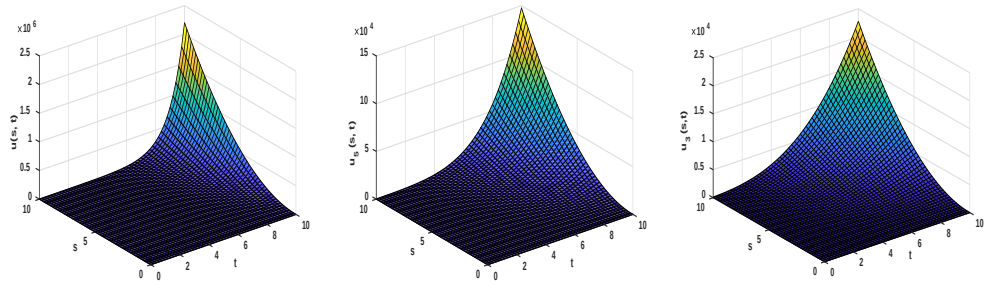

Figure 3: The 3D plot of the exact solution (Left), $u_5(s, t\,)$ (Middle), and $u_3(s, t\,)$ (Right) with time t and space s.

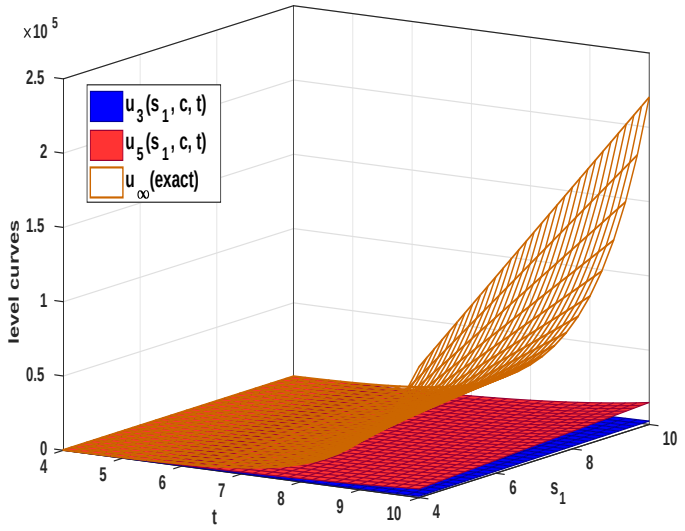

Figure 4: The plot of the level curves $l_{c(s_2)}\,[u(s_1,\, s_2,\, t)]$ for arbitrary value c.

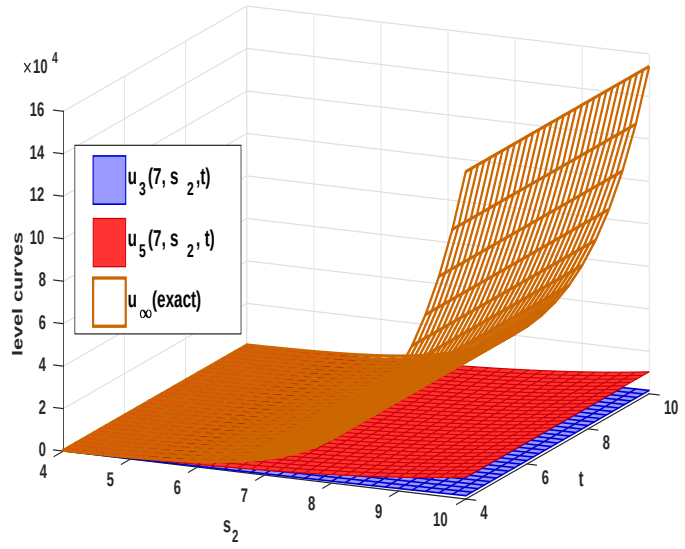

Figure 5: The plot of the level curves $l_{c(s_1)}\,[u(s_1,\, s_2,\, t)]$ for $c = 7$.

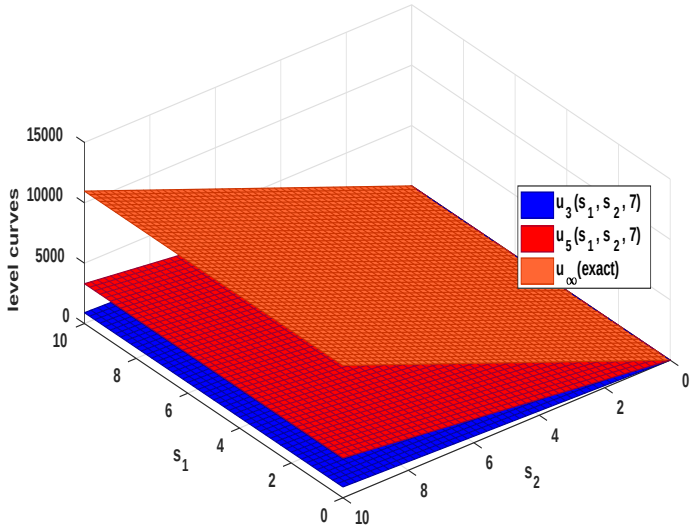

Figure 6: The plot of the level curves $l_{c(t)}\,[u(s_1,\, s_2,\, t)]$ for $c = 7$.

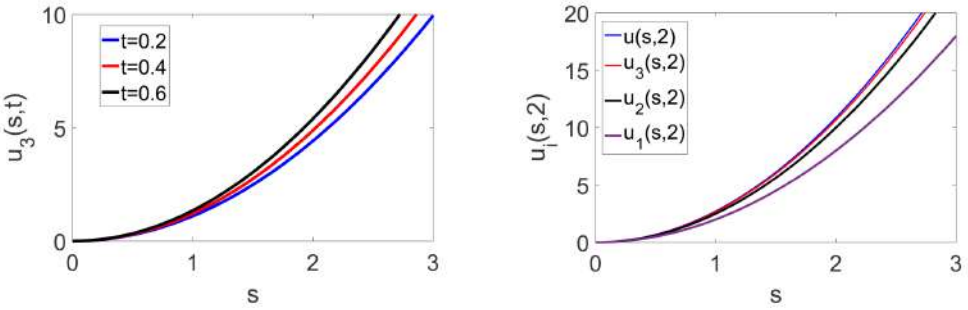

Figure 7: The plot of the approximated solution $u_3(s,t)$ with several values of time t and space s (Left) and the plot of the exact solution $u(s,t)$ and the approximated solutions $u_1(s,t), u_2(s,t)$ and $u_3(s,t)$ with time $t = 2$ and space s (Right).

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