Hesitant fuzzy sets approach to ideal theory in ordered Γ-semigroups

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Abstract. The aim of this paper is to apply hesitant fuzzy sets theory in ordered Γ- semigroups. The hesitant fuzzy ideals in ordered Γ-semigroups are introduced and some related properties are explored. Using this concept, some characterizations on hesitant fuzzy left (right and bi-) ideals are given. The hesitant fuzzy interior ideal in oredered Γ-semigroup are defined and their related properties are studied. Finally, the characterization of a simple ordered Γ-semigroup in terms of a hesitant fuzzy simple ordered Γ-semigroup is presented.

Keywords: ordered Γ-semigroup, hesitant fuzzy ideal, hesitant fuzzy interior ideal.

1. Introduction

In 1981, the concept and notion of the Γ-semigroups was introduced by Sen [32]. Later on, this notion was futher studied by Sen and Saha in ([31], [30], [33]). In 1963, Fuchs had completed the first monograph on ordered algebras, in which ordered group, ordered semigroup, ordered ring and ordered field were involved

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[7]. Since ordered semigroup has a close relation with theoretical computer science, especially with formal language and theory of automata, it has been extensively investigated by many researcher (see e.g. [3], [8]). For the general development of gamma ordered semigroups, the ideal theory plays an important role.

In the year 1997, Kwon and Lee [19] introduced the concepts of the weakly prime ideals and the weakly semiprime ideals in ordered Γ-semigroups and gave some characterizations of the weakly prime ideals and the weakly semiprime ideals in ordered Γ-semigroups.

Since the concept of fuzzy sets was introduced by Zadeh in 1965[42], the theories of fuzzy sets and fuzzy systems developed rapidly. The study of the fuzzy algebraic structures has started in the pioneering paper of Rosenfeld [29] in 1971. Rosenfeld introduced the notion of fuzzy groups and showed that many results in groups can be extended in an elementary manner to develop the theory of fuzzy group. Since then the literature of various fuzzy algebraic concepts has been growing very rapidly.

Kuroki ([16], [17]) applied the fuzzy sets theory to semigroups and Fuzzy sets in ordered semigroups were first considered by Kehayopulu and Tsingelis in ([13], [14], [15]).

Recently, to deal with hesitant and incongruous problems, Torra and Narukawa [38] and Torra [37] proposed the concept of hesitant fuzzy set(s) [briefly, HFS(s)], a new generalization of fuzzy sets, which allows the membership of an element of a set to be represented by several possible values. They also discussed relationships among hesitant fuzzy sets and other generalizations of fuzzy sets such as intuitionistic fuzzy sets, type-2 fuzzy sets, and fuzzy multisets. Some set theoretic operations such as union, intersection and complement on hesitant fuzzy sets have also been proposed by Torra [37]. Hesitant fuzzy sets can be used as an efficient mathematical tool for modeling peoples hesitancy in daily life than the other classical extensions of fuzzy sets. The motivation for introducing these sets is that it is sometimes difficult to determine the membership of an element into a set and in some circumstances this difficulty is caused by a doubt between a few different values. For example, two experts discuss the membership of x into A, and one wants to assign 0.3 and the other 0.4. So, the uncertainty on the possible values is somehow limited.

After the pioneering work of Torra, the HFS has received much attention from many authors and has been used in clustering analysis and decision-making (see [4], [28], [39], [40], [41]). For example, Chen et al. [4] systematically investigated the correlation coefficients of HFSs and applied them to clustering analysis, Xia and Xu [40] studied the aggregation operators of hesitant fuzzy sets and applied them to decision making.

Recently, hesitant fuzzy sets theory have been applied to different algebraic structures on various aspects viz., Jun et al. have applied the hesitant fuzzy sets theory to the theory of semigroups (see [9, 10, 11, 12]). Muhiuddin et al. have applied the hesitant fuzzy sets theory to different algebraic (see [20, 21, 22, 23,

24, 25, 26]). As follow-up, we have studied the ideals of ordered Γ-semigroups in terms of HFSs. In the present analysis, we first give the equivalent condition on hesitant fuzzy bi-ideal in a ordered Γ-semigroups. Moreover, we prove that a non-empty hesitant fuzzy subset H of an ordered Γ-semigroup S is hesitant fuzzy ideal of S if and only if the T-cut H_T of H is an ideal of S for every $T \subseteq \mathcal{P}([0,1])$, provided H_T is non-empty. Then we characterize simple ordered Γ-semigroup by means of a hesitant fuzzy simple ordered Γ-semigroup. We prove that in a simple ordered Γ-semigroup, every hesitant fuzzy interior ideal of S is a constant function and in a left zero (right zero) ordered Γ-semigroup, every hesitant fuzzy left (resp., hesitant fuzzy right) ideal is a constant function.

2. Preliminaries

Throughout the present paper, ordered Γ-semigroup will be denoted by po-Γ-semigroup. To develop our main results, we need the following notions.

Definition 2.1 ([5])**.** Let S and Γ be two non-empty sets. Then S is called a Γ-semigroup if there exist mappings $S \times \Gamma \times S$ to S, written as $(a, \alpha, b) \to a\alpha b$ and $\Gamma \times S \times \Gamma$ to Γ, written as $(\alpha, a, \beta) \to \alpha a \beta$ satisfying the following associative propeerties $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$ and $\alpha(a\beta b)\gamma = (\alpha a\beta)b\gamma = \alpha a(\beta b\gamma)$, for all $a, b, c \in S$ and for all $\alpha, \beta, \gamma \in \Gamma$.

Definition 2.2 ([6])**.** A Γ-semigroup S is a po-Γ-semigroup if S and Γ are posets such that:

(1) $a \leq b$ in S $\Rightarrow$ aαc $\leq$ bαc, cαa $\leq$ cαb in S, for all $c \in S$ and $\alpha \in \Gamma$.

(2) $\alpha \leq \beta$ in $\Gamma \Rightarrow$ aαb $\leq$ aβb, for all $a, b \in S$.

In our further discussion, unless otherwise mentioned S will stand for a ordered Γ-semigroup. Torra [37] defined hesitant fuzzy sets in terms of a function that returns a set of membership values for each element in the domain which can be defined as follows:

Definition 2.3 ([37])**.** Let S be a reference set, a hesitant fuzzy set on S is a function H that returns a subset of values in [0,1]:

$$H : S \to \mathcal{P}([0,1]),$$

where $\mathcal{P}([0,1])$ denotes the set of all subsets of [0,1].

Mathematically, we can represent HFS H on S as:

$H = \{(a, h(a)) \ \forall a \in S : h(a)$ is the set of some different values in $[0,1]\}$.

Let H and G be any two HFSs on S, we define $H \sqsubseteq G$ if $H(a) \subseteq G(a)$, for all $a \in S$. For any element a of S, we define

$$A_a = \{(b, c) \in S \times S \text{ and } \gamma \in \Gamma : \text{a} \leq \text{b}\gamma\text{c}\}.$$

For any HFSs H and G in S. The hesitant fuzzy product of H and G is defined to be the HFS $H\tilde{o}G$ on S as follows:

$$(H\tilde{o}G)_a = \begin{cases} \bigcup\limits_{a\leq b\gamma c} (H_b \cap G_c) & \text{if } \ \mathrm{A_a} \neq \emptyset \\ \emptyset & \text{otherwise.} \end{cases}$$

For any two HFSs H and G on S, let $\mathcal{P}([0,1])$ denotes the set of all subsets of [0,1], the hesitant union $H \sqcup G$ of H and G is defined to be HFS on S as follows:

$$H \sqcup G : S \to \mathcal{P}([0,1]), a \longmapsto H(a) \cup G(a)$$

and the hesitant intersection $H \sqcap G$ of H and G is defined to be HFS on S as follows:

$$H \sqcap G : S \to \mathcal{P}([0,1]), a \longmapsto H(a) \cap G(a).$$

Let A be any non empty subset of S. Recall that, we denote by H_A the characteristic HFS on S as follows:

$$H_A : S \ \to \ (\mathcal{P}[0,1]) \ , \ a \ \mapsto \ \begin{cases} [0,1], & \text{if } \ a \ \in \ A \\ \emptyset, & \text{if } \ a \ \notin \ A. \end{cases}$$

Let S be a po-Γ-semigroup, we define a hesitant fuzzy subset I of S as follows:

$$I : S \ \to \ (\mathcal{P}[0,1]) \mid x \to \ I_x := [0,1].$$

3. Main results

Definition 3.1. A hesitant fuzzy set H on po-Γ-semigroup S is called a hesitant fuzzy semigroup on S if it satisfies :

$$(\forall \ x, y \ \in S, \gamma \in \Gamma)(H(x\gamma y) \supseteq H(x) \cap H(y)).$$

Definition 3.2. A hesitant fuzzy set H on po-Γ-semigroup S is called a hesitant fuzzy left (resp., right) ideal on S if it satisfies:

(1) $(\forall \ x, y \in S) \ \ x \leq y \Rightarrow H(x) \supseteq H(y)$

(2) $(\forall \ x, y \ \in S, \gamma \in \Gamma)(H(x\gamma y) \supseteq H(y)(resp., H(x\gamma y) \supseteq H(x)).$

If a hesitant fuzzy set H on S is both a hesitant fuzzy left ideal and a hesitant fuzzy right ideal on S, we say that H is a hesitant fuzzy (two-sided) ideal on S.

Example 3.3. Let $S = M_{1\times 2}(\mathbb{Z}_2)$ be a set of all 1×2 matrices over the field $\mathbb{Z}_2$ and $\Gamma = M_{2\times 1}(\mathbb{Z}_2)$ be a set of all 2×1 matrices over the field $\mathbb{Z}_2$. Then S is a Γ-semigroup with respect to usual matrix multiplication. Also S and Γ are posts with respect to $\leq$ defined by $(a_{ij}) \leq (b_{ij})$ if and only if $a_{ij} \leq b_{ij} \ \forall i, j$. Then, S is a po-Γ-semigroup.

Define a hesitant fuzzy subset H of S such that

$$H(x) = \begin{cases} [0,1], & \text{if} \quad x = (0,0) \\ \{0.1\}, & \text{otherwise.} \end{cases}$$

Clearly, H is a hesitant fuzzy ideal of S.

Definition 3.4. A hesitant fuzzy subsemigroup H on po-Γ-semigroup S is called a hesitant fuzzy bi-ideal on S if it satisfies:

(1) $(\forall\ x, y \in S)\ \ x \leq y \Rightarrow H(x) \supseteq H(y)$;

(2) $(\forall\ x, y, z \in S, \alpha, \beta \in \Gamma)(H(x\alpha y\beta z) \supseteq H(x) \cap H(z))$.

Example 3.5. Let $S = \{a, b, c, d\}$ and $\Gamma = \{\alpha, \beta, \gamma\}$ be two non empty sets. Then S and Γ are posets with respect to the ordered relation $\leq$ given by $a \leq b \leq c \leq d$ and $\gamma \leq \alpha \leq \beta$ respectively. Clearly S is a po-Γ-semigroup with respect to the operations defined below:

α	a	b	c	d
a	a	a	a	a
b	a	c	a	c
c	a	b	a	c
d	c	c	c	d

β	a	b	c	d
a	a	a	a	a
b	a	a	a	c
c	a	a	a	c
d	c	c	c	c

γ	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	a	a	a

Let H be a hesitant fuzzy subset of S such that

$$H_a = [0,1];\ H_b = [0.1, 0.9];\ H_c = [0.2, 0.8];\ H_d = [0.3, 0.7]$$

Clearly H is a hesitant fuzzy bi-ideal of S. But H is neither a hesitant fuzzy right ideal nor a hesitant fuzzy left ideal of S, since $H(b\alpha d) = H(c) \not\supseteq H(b)$ and $H(d\alpha a) = H(c) \not\supseteq H(a)$ respectively.

Definition 3.6. A hesitant fuzzy subsemigroup H on po-Γ-semigroup S is called a hesitant fuzzy interior ideal on S if it satisfies:

(1) $(\forall\ x, y \in S)\ \ x \leq y \Rightarrow H(x) \supseteq H(y)$;

(2) $(\forall\ x, y, z \in S, \alpha, \beta \in \Gamma)(H(x\alpha y\beta z) \supseteq H(y))$.

Theorem 3.7. *Let H be a non empty hesitant fuzzy subset of a po-Γ-semigroup S. Then the following conditions are equivalent:*

(1) H is a hesitant fuzzy bi-ideal of S;

(2) $H \tilde{o} H \sqsubseteq H$, $H \tilde{o} I \tilde{o} H \sqsubseteq H$ and $a \leq b \Rightarrow H(a) \supseteq H(b)\ \forall\ a, b \in S$.

Proof. (1) ⇒ (2) Let H be a hesitant fuzzy bi-ideal of S. Then H is a hesitant fuzzy subsemigroup of S. Again, let $a \in S$. If for some $b, c \in S$ and $\gamma \in \Gamma$such that $a \leq b\gamma c$. Then, we have $(H \tilde{o} H)(a) = \bigcup_{a \leq b\gamma c} \{H(b) \cap H(c)\} \subseteq \bigcup_{a \leq b\gamma c} H(b\gamma c) \subseteq H(a)\ \forall\ a \in S$ implies $H \tilde{o} H \sqsubseteq H$. Other wise, $(H \tilde{o} H)(a) = \emptyset \subseteq H(a)$. Consequenly, $H \tilde{o} H \sqsubseteq H$.

Further to prove $H\tilde{o}I\tilde{o}H \sqsubseteq H$, let $a \in S$. Suppose $\exists\ u, v, p, q \in S$ and $\alpha, \beta \in \Gamma$ such that $a \leq u\alpha v$ and $u \leq p\beta q$. Again, since H is a hesitant fuzzy bi-ideal of S, we have

$$\begin{aligned}
(H\tilde{o}I\tilde{o}H)(a) &= \bigcup_{a\leq u\alpha v} \{(H\tilde{o}I)(u) \cap H(v)\} \\
&= \bigcup_{a\leq u\alpha v} \{ \bigcup_{u\leq p\beta q} \{H(p) \cap I(q)\} \cap H(v)\} \\
&= \bigcup_{a\leq u\alpha v} \{ \bigcup_{u\leq p\beta q} \{H(p) \cap [0,1]\} \cap H(v)\} \\
&= \bigcup_{a\leq p\beta q\alpha v} \{H(p) \cap H(v)\} \\
&\subseteq \bigcup_{a\leq p\beta q\alpha v} \{H(p\beta q\alpha v)\} (cf. Definition\ 3.4) \\
&\subseteq H(a)\ \forall\ a \in S.
\end{aligned}$$

implies $H\tilde{o}I\tilde{o}H \sqsubseteq H$. Other wise, $H\tilde{o}I\tilde{o}H(a) = \emptyset \subseteq H(a)$. Consequently, $H\tilde{o}I\tilde{o}H \sqsubseteq H$.

Moreover, for all $a, b \in S$ such that $a \leq b \Rightarrow H(a) \supseteq H(b)$.

(2) $\Rightarrow$ (1). Assume that $H\tilde{o}H \sqsubseteq H$. Let $a, b \in S$ and $\gamma \in \Gamma$ we have $H(a\gamma b) \supseteq (H\tilde{o}H)(a\gamma b) \supseteq \{H(a) \cap H(b)\}$. Thus, H is a hesitant fuzzy subsemigroup of S.

Again, let $a, b, c \in S$ and $\alpha, \beta \in \Gamma$ such that $x \leq a(\alpha b\beta)c := a\gamma c$, where $\gamma =: \alpha b\beta \in \Gamma$. Since $H\tilde{o}I\tilde{o}H \sqsubseteq H$, it follows that

$$\begin{aligned}
H(a(\alpha b\beta)c) &:= H(a\gamma c) \supseteq H(x) \\
&\supseteq \big((H\tilde{o}I)\tilde{o}H\big)(x) \\
&= \bigcup_{x\leq m\delta n} \{(H\tilde{o}I)(m) \cap H(n)\} \\
&= \bigcup_{x\leq m\delta n} \{ \bigcup_{m\leq p\eta q} \{H(p) \cap I(q)\} \cap H(n)\} \\
&= \bigcup_{x\leq m\delta n} \{ \bigcup_{m\leq p\eta q} \{H(p) \cap [0,1]\} \cap H(n)\} \\
&= \bigcup_{x\leq p(\eta q\delta)n} \{H(p) \cap H(n)\} \supseteq \{H(a) \cap H(c)\}.
\end{aligned}$$

Hence, H is a hesitant fuzzy bi-ideal of S. □

Theorem 3.8. *Let H be a non empty hesitant fuzzy subsemigroup of a po-Γ-semigroup S. Then the following conditions are equivalent:*

(1) *H is a hesitant fuzzy left (respecively right) ideal of S;*

(2) *$I\tilde{o}H \sqsubseteq H$. (resp; $H\tilde{o}I \sqsubseteq H$) and $a \leq b \Rightarrow H(a) \supseteq H(b)\ \forall\ a, b \in S$.*

Proof. (1) $\Rightarrow$ (2) Let H be a hesitant fuzzy left ideal of S. Again, let $a \in S$ and suppose there exist $b, c \in S$ and $\gamma \in \Gamma$ such that $a \leq b\gamma c$, then

$$\begin{aligned}(I\tilde{o}H)(a) &= \bigcup_{a\leq b\gamma c}\{I(b)\cap H(c)\}\\ &\subseteq \bigcup_{a\leq b\gamma c}\{[0,1]\cap H(c)\}\\ &\subseteq \bigcup_{a\leq b\gamma c}\{H(c)\}\\ &\subseteq \bigcup_{a\leq b\gamma c}\{H(b\gamma c)\}\ \ (cf. Definition\ 3.2)\\ &\subseteq H(a)\ \forall\ a\in S\ \ (\text{since } a\leq b\gamma c).\end{aligned}$$

implies $I\tilde{o}H \sqsubseteq H$. Similar arguments for the right ideal. If there do not exist $b, c \in S$ and $\gamma \in \Gamma$ such that $a \leq b\gamma c$, then $(I\tilde{o}H)(a) = \emptyset \subseteq H(a)$. Hence $I\tilde{o}H \sqsubseteq H$. Moreover, for all $a, b \in S$ such that $a \leq b \Rightarrow H(a) \supseteq H(b)$, for all $a \in S$ implying that $I\tilde{o}H \sqsubseteq H$.

Furthermore, by definition of HF left ideal, we have for all $a, b \in S$ such that $a \leq b \Rightarrow H(a) \supseteq H(b)$.

(2) $\Rightarrow$ (1). Assume that $I\tilde{o}H \subseteq H$ and $a \leq b \Rightarrow H(a) \supseteq H(b)$. Again, let $a, b \in S$ and $\gamma \in \Gamma$, then we have

$$\begin{aligned}H(a\gamma b) &= (I\tilde{o}H)(a\gamma b)\\ &\supseteq I(a)\cap H(b)\\ &= [0,1]\cap H(b)\\ &= H(b).\end{aligned}$$

Consequently, H is a hesitant fuzzy left ideal of S. In the similar way, we can prove the another case. □

Definition 3.9. Let H be a non-empty hesitant fuzzy subset of a S and $T \subseteq \mathcal{P}([0,1])$. Then the set $H_T := \{x \in S : H(x) \supseteq T\}$ is called the T-cut of H.

Theorem 3.10. *Let H be a non-empty hesitant fuzzy subset of a po-Γ-semigroup S. Then the T-cut H_T of H is a left (respectively right) ideal of S for every $T \subseteq \mathcal{P}([0,1])$, provided it is non-empty if and only if H is a hesitant fuzzy left (resp., right) ideal of S.*

Proof. For every $T \subseteq \mathcal{P}([0,1])$, let H_T be a left ideal of S. We first show, H is a hesitant fuzzy subsemigroup of S. If possible $\exists\ x_0, y_0 \in S, \gamma_0 \in \Gamma$ such that $H(x_0\gamma_0 y_0) \subset \{H(x_0) \cap H(y_0)\}$. Let $\{H(x_0) \cap H(y_0)\} = T_0$. Then $H(x_0) \supseteq T_0, H(y_0) \supseteq T_0$. Thus $x_0, y_0 \in H_{T_0}$ but $x_0\gamma_0 y_0 \notin H_{T_0}$, a contradiction. Thus, $H(x_0\gamma_0 y_0) \supseteq \{H(x_0) \cap H(y_0)\}$. Hence H is an hesitant fuzzy subsemigroup of

S. Again suppose that $\exists\ x_0, y_0 \in S, \gamma_0 \in \Gamma$ such that $H(x_0\gamma_0 y_0) \subset H(y_0)$. Since $H(y_0) \subseteq \mathcal{P}([0,1])$. Let $H(y_0) := T_0 \subseteq \mathcal{P}([0,1])$. Thus, $y_0 \in H_{T_0}$ but $x_0\gamma_0 y_0 \notin H_{T_0}$, a contradiction. Thus, $H(x_0\gamma_0 y_0) \supseteq H(y_0)$. Moreover let $x_0, y_0 \in S$, be such that $x_0 \leq y_0$, then $H(x_0) \supseteq H(y_0)$. Indeed: since $H_T \neq \emptyset$, let $U = H(y_0)$ for any subset U of [0, 1]. Then $y_0 \in H(U)$. Since $H(U)$ is a left ideal of S, we have $x_0 \in H_U$. Therefore $H(x_0) \supseteq U = H(y_0)$. Hence H is a hesitant fuzzy left ideal of S.

Conversely, suppose that H is a hesitant fuzzy left ideal of S and $T \subseteq \mathcal{P}([0,1])$ such that H_T is non empty. Let $a, b \in H_T$ and $\gamma \in \Gamma$. Then $H(a) \supseteq T$ and $H(b) \supseteq T$ which implies $H(a) \cap H(b) \supseteq T$. Since H is a hesitant fuzzy left ideal, it is a hesitant fuzzy subsemigroup, and hence $H(a\gamma b) \supseteq H(a) \cap H(b) \supseteq T$. Consequently, $a\gamma b \in H_T$. Hence H_T is a subsemigroup of S. Now, let $a \in S; \gamma \in \Gamma$ and $b \in H_T$. Then $H(a\gamma b) \supseteq H(b) \supseteq T$ and so $a\gamma b \in H_T$. Furthermore, let $a, b \in S, a \in H_T$ such that $b \leq a$. Then $b \in H_T$. Indeed: since $a \in H_T, H(a) \supseteq T$ and H is a hesitant fuzzy left ideal of S, we have $H(b) \supseteq H(a) \supseteq T$, so $b \in H_T$. Hence H_T is a left ideal of S. Similarly we can prove for another case. □

Theorem 3.11. *Let H be a non-empty hesitant fuzzy subset of a po-Γ-semigroup S. Then the T-cut H_T of H is an interior ideal of S for every $T \subseteq \mathcal{P}([0,1])$, provided it is non-empty if and only if H is a hesitant fuzzy interior ideal of S.*

Proof. For every $T \subseteq \mathcal{P}([0,1])$, let H_T be an interior ideal of S. Assume that $\exists\ a_0, x_0, y_0 \in S, \beta_0, \gamma_0 \in \Gamma$ such that $H(x_0\beta_0 a_0\gamma_0 y_0) \subset H(a_0)$. Since $H(a_0) \subseteq \mathcal{P}([0,1])$. Let $H(a_0) := T_0 \subseteq \mathcal{P}([0,1])$. Thus, $a_0 \in H_{T_0}$ but $x_0\beta_0 a_0\gamma_0 y_0 \notin H_{T_0}$, a contradiction.

Hence, $H(x_0\beta_0 a_0\gamma_0 y_0) \supseteq H(a_0)$.

Conversely, Assume that H is a hesitant fuzzy interior ideal of S and $T \subseteq \mathcal{P}([0,1])$ such that H_T is non empty. Let $a, b \in S; \beta, \gamma \in \Gamma$ and $x \in H_T$. Then $H(a\beta x\gamma b) \supseteq H(x) \supseteq T$ and so $a\beta x\gamma b \in H_T$.

The rest of the proof is a consequence of Theorem 3.10 □

Remark 3.12. Let H be a hesitant fuzzy ideal of a po-Γ-semigroup S and $T_1, T_2 \subseteq \mathcal{P}([0,1])$ such that $T_1 \subset T_2$, then $H_{T_1} \supseteq H_{T_2}$.

Definition 3.13. [27] A po-Γ-semigroup S is said to be regular if for each element $a \in S$, there exist $s \in S$ and $\alpha, \beta \in \Gamma$ such that $a \leq a\alpha s\beta a$.

Theorem 3.14. *Let H be a hesitant fuzzy set in a regular po-Γ-semigroup S then H is a hesitant fuzzy ideal of S if and only if H is a hesitant fuzzy interior ideal of S.*

Proof. Let H be a hesitant fuzzy ideal of S. For any x, y, $z \in S$ and $\alpha, \beta \in \Gamma$, we have $H(x\alpha y\beta z) = H\big((x\alpha(y\beta z)\big) \supseteq H(y\beta z) \supseteq Hy$. Hence H is a hesitant fuzzy interior ideal of S.

Conversely, let $a, b \in S$ and $\gamma \in \Gamma$. Since S is regular, there exist elements $x \in S, \alpha, \beta \in \Gamma$ such that $a \leq a\alpha x\beta a$. Since H is a hesitant fuzzy interior ideal

of S, we have $H(a\gamma b) \supseteq H(a\alpha x\beta a\gamma b) \supseteq H_a$. So H is a hesitant fuzzy right ideal of S. Similarly, We can prove that H is a hesitant fuzzy left ideal of S. Hence H is a hesitant fuzzy ideal of S. □

Theorem 3.15. *Let S be a regular po-Γ-semigroup and H be a hesitant fuzzy right ideal and G be a hesitant fuzzy left ideal S. Then $H\tilde{o}G = H \cap G$.*

Proof. Let H be a hesitant fuzzy right ideal and G be a hesitant fuzzy left ideal of S. Let $a \in S$. Suppose there exist $b, c \in S$ and $\gamma \in \Gamma$ such that $a \leq b\gamma c$.

Then we have,

$$(H\tilde{o}G)(a) = \bigcup_{a \leq b\gamma c} \big\{H(b) \cap G(c)\big\} \subseteq \bigcup_{a \leq b\gamma c} \big\{H(b\gamma c) \cap G(b\gamma c)\big\}$$
$$\subseteq H(a) \cap G(a) = \big(H \cap G\big)(a) \ \forall \ a \in S.$$

implies $H\tilde{o}G \subseteq H \cap G$. If there do not exist $b, c \in S$ such that $a \leq b\gamma c$. Then $(H\tilde{o}G)(a) = \emptyset \subseteq \big(H \cap G\big)(a)$. Thus $H\tilde{o}G \sqsubseteq H \cap G$.

Again, Let $a \in S$. Since S is regular, then there exists an element $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a \leq a\alpha x\beta a := a\gamma a$ where $\gamma := \alpha x\beta \in \Gamma$. Therefore,

$$(H\tilde{o}G)(a) = \bigcup_{a \leq p\delta q} \big\{H(p) \cap G(q)\big\} \supseteq H(a) \cap G(a) = \big(H \cap G\big)(a)$$

Thus $H\tilde{o}G \sqsupseteq H \cap G$. Hence $H\tilde{o}G = H \cap G$. □

In order to conclude the paper we obtain the following characterization of a simple po-Γ-semigroup by means of a hesitant fuzzy simple po-Γ-semigroup.

Definition 3.16 ([27])**.** A po-Γ-semigroup S is said to be simple if it does not contain any proper ideal. A po-Γ-semigroup S is said to be left (respectively right) simple if S has no proper left (respectively right) ideals.

Definition 3.17. A po-Γ-semigroup S is said to be hesitant fuzzy simple if every hesitant fuzzy ideal of S is a constant function. A po-Γ-semigroup S is called hesitant fuzzy left (respectively right) simple if every hesitant fuzzy left (respectively right) ideal of S is a constant function.

Theorem 3.18. *A po-Γ-semigroup S is simple if and only if it is hesitant fuzzy simple.*

Proof. Suppose that the po-Γ-semigroup S is simple. Let H be a hesitant fuzzy ideal of S and $x, y \in S$. Then, by Theorem 3.9, $H_{H(x)}$ and $H_{H(y)}$ are ideals of S. Since S is simple, $H_{H(x)} = S = H_{H(y)}$. Therefore, $x, y \in H_{H(x)}$ and $x, y \in H_{H(y)}$. In particular, $y \in H_{H(x)}$ and $x \in H_{H(y)}$, hence $H(y) \supseteq H(x)$ and $H(x) \supseteq H(y)$ implying $H(x) = H(y)$, for all $x, y \in S$. Hence H is a constant function. Consequently, S is a hesitant fuzzy simple.

Conversely, assume that S is a hesitant fuzzy simple. Let A be any ideal of S. Then its characteristic function H_A is a hesitant fuzzy ideal of S. Then H_A is a constant function. Let $a \in S$. Since A is non empty, therefore $H_A(a) = [0, 1]$ and so $a \in A$. Thus we obtain $S = A$. Hence S is simple. □

Theorem 3.19. *For a left (resp., right) simple po-Γ-semigroup S, every hesitant fuzzy bi-ideal of S is a hesitant fuzzy right ideal (resp., hesitant fuzzy left ideal) of S.*

Proof. Let H be any hesitant fuzzy bi-ideal of S and $a, b \in S$. Then, $(S\Gamma a]$ is a left ideal of S and S is left simple po-Γ-semigroup, so $(S\Gamma a] = S$. So there exist $a_o \in S, \gamma \in \Gamma$ such that $b \leq a_o\gamma a$. Since H is hesitant fuzzy bi-ideal of S, we have for any $\alpha \in \Gamma$

$$H(a\alpha b) \supseteq H(a\alpha a_o\gamma a) \supseteq H(a) \cap H(a) = H(a).$$

Hence, H is a hesitant fuzzy right ideal of S. Similarly, we can prove the other case also. □

Theorem 3.20. *Let S be a left simple po-Γ-semigroup. Then every hesitant fuzzy left ideal of S is hesitant fuzzy left simple.*

Proof. Let H be any hesitant fuzzy left ideal of S and $a, b \in S$. Then, $(S\Gamma a]$, $(S\Gamma b]$ are left ideals of S and S is left simple po-Γ-semigroup, so $(S\Gamma a] = S = (S\Gamma b]$. So there exist $a_o, b_o \in S, \alpha, \beta \in \Gamma$ such that $b \leq a_o\alpha a$, $a \leq b_o\beta b$. Since H is hesitant fuzzy left ideal of S, we have

$$H(b) \supseteq H(a_o\alpha a) \supseteq H(a) \text{ and } \mathrm{H(a)} \supseteq \mathrm{H(b_o\beta b)} \supseteq \mathrm{H(b)}.$$

Thus, $H(a) = H(b)$. Since a and b are any elements of S, which implies that H is a constant function. Hence H is a hesitant fuzzy left simple. □

Theorem 3.21. *Let S be a simple po-Γ-semigroup. Then every hesitant fuzzy interior ideal of S is a constant function.*

Proof. Let H be any hesitant fuzzy interior ideal of S and a and b any elements of S. Then, $(S\Gamma b\Gamma S]$ is an ideal of S and S is simple po-Γ-semigroup, so $(S\Gamma b\Gamma S] = S$. So, there exist elements a_o, b_o in S, $\alpha, \beta \in \Gamma$ such that $a \leq a_o\alpha b\beta b_o$. Since H is hesitant fuzzy interior ideal of S, we have

$$H(a) \supseteq H(a_o\alpha b\beta b_o) \supseteq H(b).$$

Similarly, we can see that $H(b) \supseteq H(a)$. Thus $H(a) = H(b)$. Since a and b are arbitrary elements of S, hence H is a constant function. □

Definition 3.22. A po-Γ-semigroup S is called left zero (resp., right zero), if $a \leq a\gamma b$ (resp., $\mathrm{b} \leq \mathrm{a\gamma b}$), for all $a, b \in S, \gamma \in \Gamma$.

Theorem 3.23. *Let S be a left zero (resp., right zero) po-Γ-semigroup. Then every hesitant fuzzy left (resp., hesitant fuzzy right) ideal is a constant function.*

Proof. Let S be a left zero po-Γ-semigroup and H be a hesitant fuzzy left ideal of S. Let $a, b \in S$, then $a \leq a\gamma b$ and $b \leq b\gamma a$, for all $\gamma \in \Gamma$. Since H is a hesitant fuzzy left ideal of S, we have $H(a) \supseteq H(a\gamma b) \supseteq H(b)$ and $H(b) \supseteq H(b\gamma a) \supseteq H(a)$. Therefore $H(a) = H(b)$. Hence H is a constant function. □

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