A characterization of simple group $S_4(7)$

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Abstract. In this paper, we will use nse to give a new characterization of simple group $S_4(7)$.
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1. Introduction

In this paper, all groups are finite. Let G be a group. We denote by $\pi(G)$ the set of prime divisors of $|G|$, $\pi_e(G)$ the set of element orders of G and $n_p(G)$ the number of Sylow p-subgroups of G, for $p \in \pi(G)$. Recall that G is a simple K_n-group if G is simple with $|\pi(G)| = n$. Write $M_t(G) := \{g \in G \mid g^t = 1\}$ with integer t. We call that groups G_1 and G_2 are of the same order type if and only if $|M_t(G_1)| = |M_t(G_2)|$ for all t. All further unexplained notation is standard, readers may refer to [1].

This paper is related to Thompson's Problem.

Thompson's problem. Suppose that G_1 and G_2 are two groups of the same order type. If G_1 is solvable, is it true that G_2 is also necessarily solvable?

Unfortunately, up to now, no one can prove the Thompson's Problem, even give a counterexample. Let $k \in \pi_e(G)$ and $m_k(G)$ be the number of elements

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of order k in G. Set $nse(G) := \{m_k(G)|k \in \pi_e(G)\}$. We call $nse(G)$ the set of numbers of elements with the same order. It is clear that if the assumption on Thompson's Problem holds, we always have $nse(G_1) = nse(G_2)$. So it is natural to investigate the Thompson's Problem by $nse(G)$, for instance, Asboei A.K, and Amiri S.S.S characterized PSL$(2, p)$ by the set nse in [6]. Recently, Liu characterized A_{26} by the set nse in [7].

In this present paper we provide a new method in characterizing the simple group $S_4(7)$. Our result is:

Main theorem. Let G be a group. Then $G \cong S_4(7)$ if and only if:
(1) $|G| = |S_4(7)|$;
(2) $nse(G) = nse(S_4(7))$.

2. Preliminaries

In this section, we give some lemmas which will be used in the sequel.

Lemma 2.1 ([2, Theorem 9.1.2]). *Let G be a group and t be a positive integer dividing $|G|$. Then $t||M_t(G)|$.*

Lemma 2.2 ([2, Theorem 9.3.1]). *Let G be a solvable group of order mn, where $(m, n) = 1$. Then the number h_m of subgroups of order m may be expressed as a product of factors, each of which (a) is congruent to 1 modulo some prime factor of m, and (b) is a power of a prime and divides one of the chief factors of G.*

Lemma 2.3 ([3, Theorem 2]). *If G is a simple K_3-group, then G is isomorphic to one of the following groups: $A_5(2^2\cdot3\cdot5), A_6(2^2\cdot3^2\cdot5), L_2(7)(2^3\cdot3\cdot7), L_2(8)(2^3\cdot3^2\cdot7), L_2(17)(2^3\cdot3\cdot7\cdot17), L_3(3)(2^4\cdot3^3\cdot13), U_3(3)(2^5\cdot3^3\cdot7)$ or $U_4(2)(2^6\cdot3^4\cdot5)$.*

Lemma 2.4 ([5, Lemma 2.3]). *Let G be a simple group of order $2^a\cdot3^b\cdot5^c\cdot7^d$, then $G \cong A_7(2^3\cdot3^2\cdot5\cdot7), A_8(2^6\cdot3^2\cdot5\cdot7), A_9(2^6\cdot3^4\cdot5\cdot7), A_{10}(2^7\cdot3^4\cdot5^2\cdot7), L_2(49)(2^4\cdot3\cdot5^2\cdot7^2), L_3(4)(2^6\cdot3^2\cdot5\cdot7), S_4(7)(2^8\cdot3^2\cdot5^2\cdot7^4), S_6(2)(2^9\cdot3^4\cdot5\cdot7), U_3(5)(2^4\cdot3^2\cdot5^3\cdot7), U_4(3)(2^7\cdot3^6\cdot5\cdot7), J_2(2^7\cdot3^3\cdot5^2\cdot7)$ or $O_8^+(2)(2^{12}\cdot3^5\cdot5^2\cdot7)$.*

Lemma 2.5 ([5, Lemma 2.5]). *Let G be a group, $P \in Syl_p(G)$, where $p \in \pi(G)$. If G has a normal series: $K \trianglelefteq L \trianglelefteq G$ such that $P \leq L$ and $p \nmid |K|$, then the following hold:*
(1) $\boldsymbol{N}_{G/K}(PK/K) = \boldsymbol{N}_G(P)K/K$;
(2) $|G : \boldsymbol{N}_G(P)| = |L : \boldsymbol{N}_L(P)|$, *that is,* $n_p(G) = n_p(L)$;
(3) $|L/K : \boldsymbol{N}_{L/K}(PK/K)|t = |G : \boldsymbol{N}_G(P)| = |L : \boldsymbol{N}_L(P)|$, *that is,* $n_p(L/K)t = n_p(G) = n_p(L)$ *for some positive integer t. And $|\boldsymbol{N}_K(P)|t = |K|$.*

3. Proof of the main theorem

Proof. The necessity of the proof is trivial, we only prove the other direction. By [1], we see that $|S_4(7)| = 2^8\cdot3^2\cdot5^2\cdot7^4$ and $nse(S_4(7)) = \{1, 52675, 274400,$

$2366700, 5531904, 6860000, 5764800, 10290000, 11524800, 5409600, 13171200, 23049600, 27659520, 9878400, 6585600\}$.

Let $t \in \pi_e(G)$ and k be the number of cyclic subgroups of G with order t. Then $m_t(G) = k\varphi(t)$. If $t > 2$, then $\varphi(t)$ is even, so is $m_t(G)$. This indicates that $m_2(G) = 52675$. Moreover, $m_3(G) = 274400$ or 6860000, $m_5(G) = 5531904$ and $m_7(G) = 5764800$ by Lemma 2.1.

We claim that G is non-solvable. Suppose false, then G is solvable. Let $\pi = \{3, 5, 7\}$ and H be a Hall π-subgroup of G. By Lemma 2.2, we see that $n_7(H) = 3^{a_0} \cdot 5^{b_0}$, where $3^{a_0} \equiv 1(\text{mod } 7)$, $5^{b_0} \equiv 1(\text{mod } 7)$ with $a_0 \leq 2$ and $b_0 \leq 2$, implying $n_7(H) = 1$. As a result, $m_7(G) \leq (7^4 - 1) \cdot 2^8 = 614400$ contradicting with $m_7(G) = 5764800$. Hence G is non-solvable, as claimed.

Let N be a maximal solvable normal subgroup of G and M/N be a minimal normal subgroup of G/N. Then M/N is non-solvable.

Assume that M/N is non-simple. Then $M/N = S_1 \times \cdots \times S_l$ is a direct product of isomorphic simple groups S_i, where $l \geq 2$. Moreover, by comparing the group orders, it follows by Lemma 2.3 and Lemma 2.4 that $l = 2$ and that $S_1 \cong A_5$ or $L_2(7)$. Assume that $S_1 \cong A_5$. Then $M/N = A_5 \times A_5$. Let $A/N := \mathbf{C}_{G/N}(M/N)$. Then $G/A \cong (G/N)/(A/N)$ is a subgroup of $\text{Aut}(M/N)$, the automorphic group of M/N. Notice that $\text{Aut}(A_5 \times A_5) = \text{Aut}(A_5) \wr S_2$. Then $|A| \big| 2^4 \cdot 7^4$, which implies that A is solvable. Thus $A = N$ as N is the maximal solvable normal subgroup of G. Consequently, $G/N \leq \text{Aut}(A_5) \wr S_2$, implying and $7^4 \big| |N|$ and $|N| \big| 2^4 \cdot 7^4$, this forces $m_7(G) = m_7(N) < |N|$, contrary to $m_7 = 5764800$. Hence $S_1 \ncong A_5$. Similarly, $S_1 \ncong L_2(7)$.

Consequently, M/N is a non-abelian simple group. By Lemma 2.3 and 2.4, it follows that $M/N \cong A_5, A_6, L_2(7), L_2(8), A_7, A_8, L_2(49), L_3(4)$ or $S_4(7)$. Next we prove that $\mathbf{C}_{G/N}(M/N) = 1$.

Suppose that $A/N := \mathbf{C}_{G/N}(M/N) > 1$. Since $A/N \cap M/N = Z(M/N) = 1$, we see that $A/N \times M/N \leq G/N$. This indicates that $|M/N| \big| |G/A|$. On the other hand, $G/A \cong (G/N)/(A/N) \leq \text{Aut}(M/N)$ implies that $|G/A| \big| |\text{Aut}(M/N)|$. That is, $|G|/|\text{Aut}(M/N)|$ divides $|A|$ and $|A|$ divides $|G|/|M/N|$.

If $M/N \cong A_5$, then the argument above forces $2^5 \cdot 3 \cdot 5 \cdot 7^4 \big| |A|$ and $|A| \big| 2^6 \cdot 3 \cdot 5 \cdot 7^4$. Moreover, $A \trianglelefteq G$ implies that $m_7 = m_7(A) \leq |A| \leq 2^6 \cdot 3 \cdot 5 \cdot 7^4 = 2304960$, contrary to $m_7 = 5764800$. Similarly, $M/N \ncong A_6$.

Now we consider the case $M/N \cong L_2(7)$. The the same argument above gives $2^4 \cdot 3 \cdot 5^2 \cdot 7^3 \big| |A|$ and $|A| \big| 2^5 \cdot 3 \cdot 5^2 \cdot 7^3$. Notice that $A \trianglelefteq G$. It follows that $m_5(G) = m_5(A) < |A| \leq 2^5 \cdot 3 \cdot 5^2 \cdot (7^3 - 1) = 823200$, contradicting with $m_5 = 5531904$. Analogously, $M/N \ncong L_2(8)$.

As a result, M/N must be a simple K_4-group. Since A/N is non-solvable, we see that A/N has a section which is a simple group. Note that $A/N \times M/N \leq G/N$, in comparison the order of $A/N \times M/N$ with $|G|$, then only possibility is A/N has a section isomorphic to $L_2(7)$ and $M/N \cong L_2(49)$. In this case, $A/N \times M/N \leq G/N$ implies that $|N| \big| 14$. Note that $|M/N| = 2^4 \cdot 3 \cdot 5^2 \cdot 7^2$ and $M \trianglelefteq G$, we obtain $m_5(G) = m_5(M)$. Recall that $m_5(M/N) = 4704$. Then

$m_5(M) \leq m_5(M/N) \cdot 14 = 65856 < m_5(G)$. This contradiction shows that $\mathbf{C}_{G/N}(M/N) = 1$, as required.

Furthermore, $G/N \leq \mathrm{Aut}(M/N)$. Recall that $M/N \cong A_5, A_6, L_2(7), L_2(8), A_7, A_8, L_2(49)$, $L_3(4)$ or $S_4(7)$, we will discuss it case by case.

Assume first that $M/N \cong A_5$. Then $G/N \leq S_5$, which implies that $G/N = A_5$ or S_5. Moreover, $|N| = 2^5 \cdot 3 \cdot 5 \cdot 7^4$ or $2^4 \cdot 3 \cdot 5 \cdot 7^4$. Since $N \trianglelefteq G$, we see that $m_7(G) = m_7(N)$. Moreover, by Lemma 2.2 we obtain that $n_7(N) = 1$ or 8. Therefore, $m_7 \leq 19200$, contrary to $m_7 = 5764800$. Similarly, $M/N \ncong A_6$.

If $M/N \cong L_2(7)$, then $G/N \leq \mathrm{Aut}(L_2(7))$ and thus $|N| = 2^5 \cdot 3 \cdot 5^2 \cdot 7^3$ or $2^4 \cdot 3 \cdot 5^2 \cdot 7^3$. As $N \trianglelefteq G$, then $m_5(G) = m_5(N)$. By Lemma 2.2, we have $n_5(N) = 1$ or 2^4, which implies $m_5 \leq 400$, again contradiction. Similarly, $M/N \ncong L_2(8)$.

Now we consider the case that $M/N \cong A_7$. Then $G/N \leq S_7$, which implies that $G/N \cong A_7$ or S_7. Hence $|N| = 2^5 \cdot 5 \cdot 7^3$ or $|N| = 2^4 \cdot 5 \cdot 7^3$. By Lemma 2.5, we see that $n_3(G) = n_3(G/N)t$ for some integer t. As a result, $n_3(G) = 70t$ since $n_3(G/N) = 70$. Let P_3 be a Sylow 3-subgroup of G. As there is no element of order 9 in G/N, we see that P_3 is elementary abelian. This implies that $140t \leq m_3 \leq 560t$. If $m_3 = 274400$, then $490 \leq t \leq 1960$. On the other hand, $|\mathbf{N}_N(P_3)|t = |N|$ by Lemma 2.5, which indicates that $t \big| |N|$. Along with the fact that $3 \big| (70t - 1)$ by Sylow's Theorem, we obtain that $t = 490$ or 1960, yielding $|\mathbf{N}_N(P_3)| = 2^4 \cdot 7$. Notice that the action of P_3 on N by conjugation is coprime, there is a P_3-invariant group $N_5 \in \mathrm{Syl}_5(N)$. In N_5P_3, by applying Sylow's Theorem, we have $P_3 \trianglelefteq G$. Hence $N_5P_3 = N_5 \times P_3 \leq G$. This implies that $N_5 \leq N_N(P_3)$, a contradiction. Similarly, if $m_3 = 6860000$, we also get a contradiction.

Assume then that $M/N \cong L_2(49)$. In this case, $G/N \leq \mathrm{Aut}(L_2(49))$. Let P_5 be a Sylow 5-subgroup of G. Then P_5 is a cyclic subgroup of order 25 because M/N has an element of order 25. Moreover, $5 \nmid |G/M|$ and $M \trianglelefteq G$ imply $m_{25}(G) = m_{25}(M)$. By Lemma 2.5, we have $n_5(G) = n_5(M/N)t$ and $|\mathbf{N}_M(P_5)|t = |M|$ for some integer t. Furthermore, by [1], we see that $n_5(M/N) = 2^3 \cdot 3 \cdot 7^2$. As a result, $m_{25} = \varphi(25) \cdot n_5(G) = 20tn_5(M/N) = 2^5 \cdot 3 \cdot 5 \cdot 7^2 t = 27659520$, implying $t = 1176 = 2^3 \cdot 3 \cdot 7^2$. Thus $n_5(M) = 2^6 \cdot 3^2 \cdot 7^4$, yielding $|\mathbf{N}_M(P_5)| \big| 2^2 \cdot 5^2$. Since $|\mathbf{N}_N(P_5)|t = |N| \big| 2^4 \cdot 3 \cdot 7^2$, we have that $|\mathbf{N}_N(P_5)| \big| 2$. By a similar argument above, there is also contradiction.

Now we show that $M/N \ncong L_3(4)$. If not, then $M/N \cong L_3(4)$, implying $G/N \leq \mathrm{Aut}(L_3(4))$. Moreover, $|N| \big| 2^2 \cdot 5 \cdot 7^3$. By [1], we obtain $n_3(L_3(4)) = 2^3 \cdot 5 \cdot 7$. Moreover, $n_3(M) = n_3(M/N)t$ according to Lemma 2.5. Assume that $m_3 = 274400$. Then $2n_3(M) \leq m_3 \leq 8n_3(M)$, which implies that $122 < t \leq 490$. If $5 \big| t$, then $t = 140$, 245 or 490. Note that $3 \big| (n_3(G) - 1)$, we obtain $t = 490$ and thus $n_3(G) = 137200$. Hence $|\mathbf{N}_G(P_3)| = 2^4 \cdot 3^2 \cdot 7$. The similar argument in the case $G \cong A_7$ also gives a contradiction. Hence $5 \nmid t$. Moreover, $t = 196$ or 343. If the former holds, then $5 \big| |\mathbf{N}_N(P_3)|$. We assert that $N_5 \leq \mathbf{C}_G(P_3)$, where $N_5 \in \mathrm{Syl}_5(N)$. It follows that $m_{15} = 4 \cdot m_3 = 1646400$ since $N_5 \trianglelefteq G$, a contradiction. If $t = 343$, then $|\mathbf{N}_N(P_3)| \big| 2^4 \cdot 5$. Similarly, we get a contradiction. Therefore, $m_3 = 6860000$ and $3062 < t \leq 12250$, which yields that $t = 3430$ or 6860. As

$|G : \mathbf{N}_G(P_3)| = 2^3 \cdot 5 \cdot 7t$, $t = 3430$ and $|\mathbf{N}_N(P_3)| = 2^4$. Similar to the statement in the case $G \cong A_7$, we get a contradiction again.

Now we deal with the case $M/N \cong A_8$. In this situation, we have $G/N \leq S_8$. By [1], we have that $n_3(A_8) = 280$. Assume $m_3(G) = 274400$, then $490 > t \geq 122$. On the other hand, $|\mathbf{N}_N(P_3)|t = |N|\big|2^2 \cdot 5 \cdot 7^3$ gives $t = 196$ or 343. If $t = 196$, then $|\mathbf{N}_G(P_3)|\big|2^3 \cdot 3^2 \cdot 5 \cdot 7$. Since $\mathbf{N}_G(P_3)/C_G(P_3)$ is isomorphic to a subgroup of $\mathrm{Aut}(P_3)$, we have $35\big||\mathbf{C}_G(P_3)|$, which forces $C_{35} \times P_3 \leq G$ and thus $m_{15}(G) = 4 \cdot m_3(G) = 1646400$, a contradiction. If $t = 343$, then $|\mathbf{N}_G(P_3)|\big|2^5 \cdot 3^2 \cdot 5$. Similarly, we also get a contradiction. Assume $m_3(G) = 6860000$, then $12250 \geq t > 3062$. Note that $|\mathbf{N}_N(P_3)|t = |N|\big|2^2 \cdot 5 \cdot 7^3$, we have that $t = 3430$ or 6860 and $|\mathbf{N}_N(P_3)|\big|2$. This final contradiction shows that $M/N \cong S_4(7)$. Consequently, $G = M \cong S_4(7)$, the proof is completed. □

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