

Approximate approach for solving two points fuzzy boundary value problems

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Abstract. In this paper the Homotopy Perturbation Method (HPM) is employed to solve n 'th order ($n \geq 2$) non linear two point fuzzy boundary value problems (TPFBVP).

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The homotopy perturbation method can be used for solving n 'th order fuzzy differential equations directly without reduction to first order system. The convergence theorem of this method in fuzzy case is presented and proved. We illustrate the method in numerical experiment including second order linear TPFBVP to show the accuracy and capabilities of HPM.

Keywords: fuzzy numbers, fuzzy differential equations, two point fuzzy boundary value problems, homotopy perturbation method.

1. Introduction

Many dynamical real life problems may be formulated as a mathematical model. Many of them can be formulated either as a system of ordinary or partial differential equations. Fuzzy differential equations (FDEs) are a useful tool to model a dynamical system when information about its behavior is inadequate. FDE appears when the modeling of these problems was imperfect and its nature is under uncertainty. FDEs are suitable mathematical models to model dynamical systems in which there exist uncertainties or vagueness. These models are used in various applications including, population models [1, 2, 3], mathematical physics [4], and medicine [5, 6]. In recent year's semi-analytical methods such as the Adomian Decomposition Method (ADM), Homotopy Perturbation Method (HPM), Variational Iteration Method (VIM), Optimal Homotopy asymptotic method (OHAM) and Homotopy Analysis Method (HAM) have been used to solve fuzzy first and n 'th order ordinary differential equations. For n 'th order fuzzy initial value problems, The ADM was employed in [7] to solve second order linear fuzzy initial value problems. Abbasbandy et al. [8] used the VIM to solve linear system of first order fuzzy initial value problems. Moreover, some of these methods have been also used to obtain a semi-analytical solution of TPFBVP. VIM, OHAM and ADM have been used in [9, 10, 11] to solve linear TPFBVP. Other method like undetermined fuzzy coefficients method has been introduced in [12] in order to obtain an approximate solution of second order linear TPF-BVP. The major disadvantage in the traditional perturbation technique is the over dependence on the existence of small parameter. This condition is greatly affects the applications of perturbation techniques because most nonlinear equations do not even contain the small parameter. Moreover, the determination of small parameter is a complicated process and requires special techniques. It is, in fact, a coupling of the traditional perturbation method and homotopy in topology [13]. The HPM, as the previous methods that have been described, does not involve discretization of the variables and linearization, hence is free from rounding of errors and does not require large computer memory or time [14, 15]. The method also provides the solution in a rapidly convergent series with components that are elegantly computed [16].

The Homotopy Perturbation Method has been introduced by He [17] and has been applied to a wide class of deterministic and stochastic problems of mathematical and physical sciences [18, 19, 20]. This method provides the solution

as a rapidly convergent series with components that are elegantly computed. This method can be used to solve all types of linear and nonlinear equations such as differential and integral equations, so it is known as a powerful method. Another important advantage of this method is that it can reduce the size of computations, while increases the accuracy of the approximate solutions so it is known as a powerful method.

In this paper, our aim is to formulate HPM from crisp into fuzzy case in order to solve nonlinear n' th order TPFBVP directly. To the best of our knowledge, this is the first attempt at solving the n' th order TPFBVP using the HPM. The structure of this paper is as follows: In section 2, some basic definitions and notations are given about fuzzy numbers that will be used in other sections we discussed. In section 3, the structure of HPM is formulated for solving high order TPFBVP. In section 4, we present a numerical example and finally, in section 5, we give the conclusion of this study.

2. Preliminaries

Definition 2.1 ([21]). The r -level (or r -cut) set of a fuzzy set $\tilde{A}$, labeled as $\tilde{A}_r$, is the crisp set of all $x \in X$ such that $\mu_{\tilde{A}} \geq r$ i.e.

$$\tilde{A}_r = \{x \in X \mid \mu_{\tilde{A}} > r, \ x, r \in [0, 1]\}.$$

Definition 2.2. Fuzzy numbers are a subset of the real numbers set, and represent uncertain values. Fuzzy numbers are linked to degrees of membership which state how true it is to say if something belongs or not to a determined set. A fuzzy number μ is called a triangular fuzzy number [22] if defined by three numbers $\alpha < \beta < \gamma$ where the graph of $\mu(x)$ is a triangle with the base on the interval $[\alpha, \beta]$ and vertex at $x = \beta$ and its membership function has the following form:

$$\mu(x; \alpha, \beta, \gamma) = \begin{cases} 0, & \text{if } x < \alpha \\ \frac{x - \alpha}{\beta - \alpha}, & \text{if } \alpha \leq x \leq \beta \\ \frac{\gamma - x}{\gamma - \beta}, & \text{if } \beta \leq x \leq \gamma \\ 0, & \text{if } x > \gamma \end{cases}$$

and its r -level is: $[\mu]_r = [\alpha + r(\beta - \alpha), \gamma - r(\gamma - \beta)], \ r \in [0, 1]$.

In this paper the class of all fuzzy subsets of $\mathbb{R}$ will be denoted by $\tilde{E}$ and satisfy the following properties [22, 23]:

1. $\mu(t)$ is normal, i.e $\exists t_0 \in \mathbb{R}$ with $\mu(t_0) = 1$,
2. $\mu(t)$ Is convex fuzzy set, i. e. $\mu(\lambda t + (1 - \lambda)s) \geq \min\{\mu(t), \mu(s)\} \ \forall t, s \in \mathbb{R}, \lambda \in [0, 1]$,
3. μ upper semi-continuous on $\mathbb{R}$,

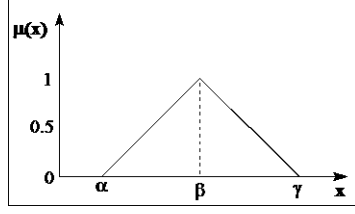


Figure 1: Triangular Fuzzy Number

4. $\overline{\{t \in \mathbb{R}; \mu(t) > 0\}}$ is compact.

$\tilde{E}$ is called the space of fuzzy numbers and $\mathbb{R}$ is a proper subset of $\tilde{E}$. Define the r -level set $x \in \mathbb{R}$, $[\mu]_r = \{x \mid \mu(x) \geq r\}$, $0 \leq r \leq 1$, where $[\mu]_0 = \{x \mid \mu(x) > 0\}$ is compact [28] which is a closed bounded interval and denoted by $[\mu]_r = (\underline{\mu}(t), \bar{\mu}(t))$. In the parametric form [19], a fuzzy number is represented by an ordered pair of functions $(\underline{\mu}(t), \bar{\mu}(t))$, $r \in [0, 1]$ which satisfies:

1. $\underline{\mu}(t)$ is a bounded left continuous non-decreasing function over $[0, 1]$.
2. $\bar{\mu}(t)$ is a bounded left continuous non-increasing function over $[0, 1]$.
3. $\underline{\mu}(t) \leq \bar{\mu}(t)$, $r \in [0, 1]$. A crisp number r is simply represented by $\underline{\mu}(r) = \bar{\mu}(r) = r$, $r \in [0, 1]$.

Definition 2.3 ([24]). If $\tilde{E}$ be the set of all fuzzy numbers, we say that $f(t)$ is a fuzzy function if $f: \mathbb{R} \rightarrow \tilde{E}$

Definition 2.4 ([25]). A mapping $f: T \rightarrow \tilde{E}$ for some interval $T \subseteq \tilde{E}$ is called a fuzzy function process and we denote r -level set by:

$$[\tilde{f}(t)]_r = [\underline{f}(t; r), \bar{f}(t; r)], t \in T, r \in [0, 1].$$

The r -level sets of a fuzzy number are much more effective as representation forms of fuzzy set than the above. Fuzzy sets can be defined by the families of their r -level sets based on the resolution identity theorem.

Definition 2.5 ([26, 27]). Each function $f: X \rightarrow Y$ induces another function $\tilde{f}: F(X) \rightarrow F(Y)$ defined for each fuzzy interval U in X by:

$$\tilde{f}(U)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} U(x), & \text{if } y \in \text{range}(f) \\ 0, & \text{if } y \notin \text{range}(f). \end{cases}$$

This is called the Zadeh extension principle.

Definition 2.6 ([28]). Consider $\tilde{x}, \tilde{y} \in \tilde{E}$. If there exists $\tilde{z} \in \tilde{E}$ such that $\tilde{x} = \tilde{y} + \tilde{z}$, then z is called the H-difference (Hukuhara difference) of x and y and is denoted by $\tilde{z} = \tilde{x} \ominus \tilde{y}$.

Definition 2.7 ([28]). If $\tilde{f} : I \rightarrow \tilde{E}$ and $y_0 \in I$, where $I \in [t_0, T]$. We say that $\tilde{f}$ Hukuhara Differentiable at y_0 , if there exists an element $[\tilde{f}']_r \in \tilde{E}$ such that for all $h > 0$ sufficiently small (near to 0), exists $\tilde{f}(y_0 + h; r) \ominus \tilde{f}(y_0; r)$, $\tilde{f}(y_0; r) \ominus \tilde{f}(y_0 - h; r)$ and the limits are taken in the metric $(\tilde{E}, D)$

$$\lim_{h \rightarrow 0+} \frac{\tilde{f}(y_0 + h; r) \ominus \tilde{f}(y_0; r)}{h} = \lim_{h \rightarrow 0+} \frac{\tilde{f}(y_0; r) \ominus \tilde{f}(y_0 - h; r)}{h}.$$

The fuzzy set $[\tilde{f}'(y_0)]_r$ is called the Hukuhara derivative of $[\tilde{f}']_r$ at y_0 .

These limits are taken in the space $(\tilde{E}, D)$ if t_0 or T , then we consider the corresponding one-side derivation. Recall that $\tilde{x} \ominus \tilde{y} = \tilde{z} \in \tilde{E}$ are defined on r -level set, where $[\tilde{x}]_r \ominus [\tilde{y}]_r = [\tilde{z}]_r$, $\forall r \in [0, 1]$. By consideration of definition of the metric D all the r -level set $[\tilde{f}(0)]_r$ are Hukuhara differentiable at y_0 , with Hukuhara derivatives $[\tilde{f}'(y_0)]_r$, when $\tilde{f} : I \rightarrow \tilde{E}$ is Hukuhara differentiable at y_0 with Hukuhara derivative $[\tilde{f}'(y_0)]_r$, it lead to that $\tilde{f}$ is Hukuhara differentiable for all $r \in [0, 1]$ which satisfies the above limits i.e. if f is differentiable at $t_0 \in [t_0 + \alpha, T]$ then all its r -levels $[\tilde{f}'(t)]_r$ are Hukuhara differentiable at t_0 .

Definition 2.8 ([23]). Define the mapping $\tilde{f}' : I \rightarrow \tilde{E}$ and $y_0 \in I$, where $I \in [t_0, T]$. We say that $\tilde{f}'$ Hukuhara differentiable $t \in \tilde{E}$, if there exists an element $[\tilde{f}^{(n)}]_r \in \tilde{E}$ such that for all $h > 0$ sufficiently small (near to 0), exists $\tilde{f}^{(n-1)}(y_0 + h; r) \ominus \tilde{f}^{(n-1)}(y_0; r)$, $\tilde{f}^{(n-1)}(y_0; r) \ominus \tilde{f}^{(n-1)}(y_0 - h; r)$ and the limits are taken in the metric $(\tilde{E}, D)$

$$\lim_{h \rightarrow 0+} \frac{\tilde{f}^{(n-1)}(y_0 + h; r) \ominus \tilde{f}^{(n-1)}(y_0; r)}{h} = \lim_{h \rightarrow 0+} \frac{\tilde{f}^{(n-1)}(y_0; r) \ominus \tilde{f}^{(n-1)}(y_0 - h; r)}{h}$$

exists and equal to $\tilde{f}^{(n)}$ and for $n = 2$ we have second order Hukuhara derivative.

Theorem 2.9 ([23]). Let $\tilde{f} : [t_0 + \alpha, T] \rightarrow \tilde{E}$ be Hukuhara differentiable and denote by

$$[\tilde{f}'(t)]_r = [\underline{f}'(t), \overline{f}'(t)]_r = [\underline{f}'(t; r), \overline{f}'(t; r)]_r.$$

Then the boundary functions $\underline{f}'(t; r), \overline{f}'(t; r)$ are differentiable

$$[\tilde{f}'(t)]_r = [(\underline{f}'(t; r))', (\overline{f}'(t; r))']_r, \forall r \in [0, 1].$$

Theorem 2.10 ([23]). Let $\tilde{f} : [t_0 + \alpha, T] \rightarrow \tilde{E}$ be Hukuhara differentiable and denote $[\tilde{f}'(t)]_r = [\underline{f}'(t), \overline{f}'(t)]_r = [\underline{f}'(t; r), \overline{f}'(t; r)]_r$.

Then the boundary functions $\underline{f}'(t; r), \overline{f}'(t; r)$ are differentiable we can write for n 'th order fuzzy derivative

$$[\tilde{f}^{(n)}(t)]_r = [(\underline{f}^{(n)}(t; r))', (\overline{f}^{(n)}(t; r))']_r, \forall r \in [0, 1].$$

3. Fuzzification and defuzzification of HPM

The basic idea of this method is to introduce a homotopy parameter, say p , which takes the values from 0 to 1. When $p = 0$, the system of equations usually reduces to a sufficiently simplified form, which normally admits a rather simple solution [29].

The general structure of HPM for solving crisp n' th order two point boundary value problems involving ordinary differential equations are mentioned in [29, 30, 31]. To solve n' th order TPFBVP, we need to fuzzify HPM and then defuzzify it.

Consider the following general n' th order TPFBVP

$$(1) \quad \tilde{y}^{(n)}(t) = f\left(t, \tilde{y}(t), \tilde{y}'(t), \tilde{y}''(t), \dots, \tilde{y}^{(n-1)}(t)\right) + \tilde{w}(t), \quad t \in [t_0, T]$$

subject to two point boundary conditions

$$(2) \quad \begin{cases} \tilde{y}(t_0) = \tilde{\alpha}^{(0)}, \tilde{y}'(t_0) = \tilde{\alpha}^{(1)}, \dots, \tilde{y}^{(k)}(t_0) = \tilde{\alpha}^{(k)}, \\ \tilde{y}(T) = \tilde{\beta}^{(0)}, \tilde{y}'(T) = \tilde{\beta}^{(1)}, \dots, \tilde{y}^{(n-k-2)}(T) = \tilde{\beta}^{(n-k-2)}, \end{cases}$$

where $0 \leq k \leq n-2$ is an integer (For cases when $n = 4$ and $n = 6$, see [32, 33], $\tilde{y}(t)$ is a fuzzy function of the crisp variable t . Also, f is fuzzy function of the crisp variable t and the fuzzy variable $\tilde{y}$. Here $\tilde{y}^{(n)}$ is the n' th order fuzzy H-derivative of $\tilde{y}(t)$ and the H-derivatives $\tilde{y}'(t), \tilde{y}''(t), \dots, \tilde{y}^{(n-1)}(t)$. Moreover the fuzzy boundary conditions $\tilde{y}(t_0), \tilde{y}'(t_0), \dots, \tilde{y}^{(k)}(t_0), \tilde{y}(T), \tilde{y}', \dots, \tilde{y}^{(n-k-s)}(T)$ are convex fuzzy numbers as in section 2. We denote the fuzzy function y by $\tilde{y} = [\underline{y}, \overline{y}]$, for $t \in [t_0, T]$ and $r \in [0, 1]$ it means that the r -level set of $y(t)$ can be defined as:

$$\begin{aligned} [\tilde{y}(t)]_r &= [\underline{y}(t; r), \overline{y}(t; r)], \\ [\tilde{y}'(t)]_r &= [\underline{y}'(t; r), \overline{y}'(t; r)], \dots, [\tilde{y}^{(n-1)}(t)]_r = [\underline{y}^{(n-1)}(t; r), \overline{y}^{(n-1)}(t; r)], \\ [\tilde{y}(t_0)]_r &= [\underline{y}(t_0; r), \overline{y}(t_0; r)], \\ [\tilde{y}'(t_0)]_r &= [\underline{y}'(t_0; r), \overline{y}'(t_0; r)], \dots, [\tilde{y}^{(k)}(t_0)]_r = [\underline{y}^{(k)}(t_0; r), \overline{y}^{(k)}(t_0; r)], \\ [\tilde{y}'(T)]_r &= [\underline{y}'(T; r), \overline{y}'(T; r)], \dots, [\tilde{y}^{(k)}(T)]_r = [\underline{y}^{(k)}(T; r), \overline{y}^{(k)}(T; r)], \end{aligned}$$

where $w(t)$ is crisp or fuzzy inhomogeneous term such that $[\tilde{w}(t)]_r = [\underline{w}(t; r), \overline{w}(t; r)]$. Since

$$y^{(n)}(t) = f\left(t, y(t), y'(t), y''(t), \dots, y^{(n-1)}(t)\right) + w(t).$$

If we let $\mathcal{Y}(t) = y(t), y'(t), y''(t), \dots, y^{(n-1)}(t)$, such that

$$\begin{aligned} \tilde{\mathcal{Y}}(t; r) &= [\underline{\mathcal{Y}}(t; r), \overline{\mathcal{Y}}(t; r)] \\ &= [\underline{y}(t; r), \underline{y}'(t; r), \dots, \underline{y}^{(n-1)}(t; r), \overline{y}(t; r), \overline{y}'(t; r), \dots, \overline{y}^{(n-1)}(t; r)]. \end{aligned}$$

Also, we can write

$$(3) \quad \left[\tilde{f}(t, \tilde{\mathcal{Y}}) \right]_r = \left[\underline{f}(t, \tilde{\mathcal{Y}}; r), \bar{f}(t, \tilde{\mathcal{Y}}; r) \right]$$

by using Zadeh extension principles we have

$$\tilde{f}(t, \tilde{\mathcal{Y}}(t; r)) = \left[\underline{f}(t, \tilde{\mathcal{Y}}(t; r)), \bar{f}(t, \tilde{\mathcal{Y}}(t; r)) \right],$$

such that

$$\begin{cases} \underline{f}(t, \tilde{\mathcal{Y}}(t; r)) = \mathcal{F}(t, \underline{\mathcal{Y}}(t; r), \bar{\mathcal{Y}}(t; r)) = \mathcal{F}(t, \tilde{\mathcal{Y}}(t; r)), \\ \bar{f}(t, \tilde{\mathcal{Y}}(t; r)) = \mathcal{G}(t, \underline{\mathcal{Y}}(t; r), \bar{\mathcal{Y}}(t; r)) = \mathcal{G}(t, \tilde{\mathcal{Y}}(t; r)). \end{cases}$$

Then, we have

$$(4) \quad \begin{cases} \underline{y}^{(n)}(t; r) = \mathcal{F}(t, \tilde{\mathcal{Y}}(t; r)) + \underline{w}(t; r), \\ \bar{y}^{(n)}(t; r) = \mathcal{G}(t, \tilde{\mathcal{Y}}(t; r)) + \bar{w}(t; r), \end{cases}$$

where the membership function of $\mathcal{F}(t, \tilde{\mathcal{Y}}(t; r)) + \underline{w}(t; r)$ and $\mathcal{G}(t, \tilde{\mathcal{Y}}(t; r)) + \bar{w}(t; r)$ can be defined as

$$\begin{aligned} \mathcal{F}(t, \tilde{\mathcal{Y}}(t; r)) + \underline{w}(t; r) &= \min\{\tilde{y}^{(n)}(t, \tilde{\mu}(r)) : \mu | \mu \in [\tilde{\mathcal{Y}}(t; r)]_r\}, \\ \mathcal{G}(t, \tilde{\mathcal{Y}}(t; r)) + \bar{w}(t; r) &= \max\{\tilde{y}^{(n)}(t, \tilde{\mu}(r)) : \mu | \mu \in [\tilde{\mathcal{Y}}(t; r)]_r\}. \end{aligned}$$

Now for all $r \in [0, 1]$ we rewrite Eq. 1 in the following forms

$$(5) \quad \begin{cases} \underline{\mathcal{L}}_n \underline{y}(t; r) = \mathcal{F}(t, \tilde{\mathcal{Y}}(t; r)) + \underline{w}(t; r), & t \in [t_0, T], \\ \underline{y}(t_0; r) = [\underline{\alpha}_0]_r, \underline{y}'(t_0; r) = [\underline{\alpha}_1]_r, \dots, \underline{y}^{(k)}(t_0; r) = [\underline{\alpha}_k]_r, \\ \underline{y}(T; r) = [\underline{\beta}_0]_r, \underline{y}'(T; r) = [\underline{\beta}_1]_r, \dots, \underline{y}^{(n-k-2)}(T; r) = [\underline{\beta}_{n-k-2}]_r, \end{cases}$$

$$(6) \quad \begin{cases} \bar{\mathcal{L}}_n \bar{y}(t; r) = \mathcal{G}(t, \tilde{\mathcal{Y}}(t; r)) + \bar{w}(t; r), & t \in [t_0, T], \\ \bar{y}(t_0; r) = [\bar{\alpha}_0]_r, \bar{y}'(t_0; r) = [\bar{\alpha}_1]_r, \dots, \bar{y}^{(k)}(t_0; r) = [\bar{\alpha}_k]_r, \\ \bar{y}(T; r) = [\bar{\beta}_0]_r, \bar{y}'(T; r) = [\bar{\beta}_1]_r, \dots, \bar{y}^{(n-k-2)}(T; r) = [\bar{\beta}_{n-k-2}]_r, \end{cases}$$

where $\tilde{\mathcal{L}}_n = [\underline{\mathcal{L}}_n, \bar{\mathcal{L}}_n]$ are the linear operators with $\tilde{\mathcal{L}}_n = \frac{d^{(n)}}{dt^{(n)}}$, followed by inverse operator $\tilde{\mathcal{L}}_n^{-1} = \int_0^t \int_0^t \int_0^t \dots [\cdot]_r d\tau d\tau \dots d\tau$ and $\mathcal{F}, \mathcal{G}$ are nonlinear operators. One of the advantages of the HPM is that HPM solution series is only dependent upon two factors: the auxiliary linear operator and initial guess. In fact, the proper choice of the initial condition and the auxiliary linear operator will guarantee the convergence of the HPM solution series [17]. According to the

HPM [29], we construct a homotopy form into Eq. (5) and (6) which satisfies the following relation

$$\begin{aligned}
 \underline{\mathcal{H}}(t, p; r) &= (1 - p)\underline{\mathcal{L}}_n[\underline{y}(t; r) - \underline{y}_0(t; r; \underline{a}_s(r))] \\
 (7) \quad &+ p[\underline{\mathcal{L}}_n \underline{y}_0(t; r; \underline{a}_s(r)) - \mathcal{F}(t, \tilde{\mathcal{Y}}(t; r)) - \underline{w}(t; r)] = 0, \\
 \overline{\mathcal{H}}(t, p; r) &= (1 - p)\overline{\mathcal{L}}_n[\overline{y}(t; r) - \overline{y}_0(t; r; \overline{a}_s(r))] + p[\overline{\mathcal{L}}_n \overline{y}_0(t; r; \overline{a}_s(r)) \\
 &- \mathcal{G}(t, \tilde{\mathcal{Y}}(t; r)) - \overline{w}(t; r)],
 \end{aligned}$$

where $p \in [0, 1]$ is an embedding parameter and $\underline{y}_0(t; r; \underline{a}_s(r))$, and $\overline{y}_0(t; r; \overline{a}_s(r))$ are initials guessing and can be defining as follows:

$$(8) \quad \underline{y}_0(t; r; \underline{a}_s(r)) = \underline{a}_1(r) + \underline{a}_2(r)t + \frac{\underline{a}_3(r)}{2!}t^2 + \dots + \frac{\underline{a}_n(r)}{(n-1)!}t^{n-1},$$

$$(9) \quad \overline{y}_0(t; r; \overline{a}_s(r)) = \overline{a}_1(r) + \overline{a}_2(r)t + \frac{\overline{a}_3(r)}{2!}t^2 + \dots + \frac{\overline{a}_n(r)}{(n-1)!}t^{n-1},$$

where $\underline{a}_s(r)$ and $\overline{a}_s(r)$ are fuzzy constants to be determined. From Eq. (2) we will have

$$(10) \quad \begin{cases} \underline{\mathcal{H}}(t, 0; r) = \underline{\mathcal{L}}_n[\underline{y}(t; r) - \underline{y}_0(t; r; \underline{a}_s(r))] = 0, \\ \underline{\mathcal{H}}(t, 1; r) = [\underline{\mathcal{L}}_n \underline{y}_0(t; r; \underline{a}_s(r)) - \mathcal{F}(t, \tilde{\mathcal{Y}}(t; r)) - \underline{w}(t; r)] = 0. \end{cases}$$

Also for the upper bound we have

$$(11) \quad \begin{cases} \overline{\mathcal{H}}(t, 0; r) = \overline{\mathcal{L}}_n[\overline{y}(t; r) - \overline{y}_0(t; r; \overline{a}_s(r))] = 0, \\ \overline{\mathcal{H}}(t, 1; r) = [\overline{\mathcal{L}}_n \overline{y}_0(t; r; \overline{a}_s(r)) - \mathcal{G}(t, \tilde{\mathcal{Y}}(t; r)) - \overline{w}(t; r)] = 0. \end{cases}$$

As in crisp case we called this deformation, From Eq.(10) and (11) we can represent each y in these equations as follows:

$$(12) \quad \begin{cases} \tilde{y}(t; r; \sum_{i=1}^n \tilde{a}_s(r)) = \sum_{k=0}^{\infty} p^k \tilde{y}_k(t; r; \sum_{s=1}^n \tilde{a}_s(r)), \\ \tilde{y}'(t; r; \sum_{i=1}^n \tilde{a}_s(r)) = \sum_{k=0}^{\infty} p^k \tilde{y}'_k(t; r; \sum_{s=1}^n \tilde{a}_s(r)), \\ \vdots \\ \tilde{y}^{(n)}(t; r; \sum_{i=1}^n \tilde{a}_s(r)) = \sum_{k=0}^{\infty} p^k \tilde{y}_k^{(n)}(t; r; \sum_{s=1}^n \tilde{a}_s(r)). \end{cases}$$

Now substituting equation the above equations into Eq.(10) and (11) and collecting terms of the same powers of p , we have:

$$p^0 : \begin{cases} \underline{\mathcal{L}}_n[\underline{u}_0(t; r) - \underline{y}_0(t; r; \sum_{s=1}^n \underline{a}_s(r))] = 0, \\ \underline{y}(t_0; r) = [\underline{\alpha}_0]_r, \underline{y}'(t_0; r) = [\underline{\alpha}_1]_r, \dots, \underline{y}^{(k)}(t_0; r) = [\underline{\alpha}_k]_r, \\ \underline{y}(T; r) = [\underline{\beta}_0]_r, \underline{y}'(T; r) = [\underline{\beta}_1]_r, \dots, \underline{y}^{(n-k-2)}(T; r) = [\underline{\beta}_{n-k-2}]_r, \end{cases}$$

$$\begin{aligned}
p^1 : & \begin{cases} \mathcal{L}_n [\underline{u}_1(t; r) + \underline{y}_0(t; r; \sum_{s=1}^n \underline{a}_s(r))] \\ -\mathcal{F}\left(t, \tilde{\mathcal{Y}}_0(t; r, \sum_{i=1}^n \underline{a}_s(r))\right) - \underline{u}(t; r) = 0, \\ \underline{y}_1(t_0; r) = 0_r, \underline{y}_1'(t_0; r) = 0, \dots, \underline{y}_1^{(k)}(t_0; r) = 0, \\ \underline{y}_1(T; r) = 0_r, \underline{y}_1'(T; r) = 0, \dots, \underline{y}_1^{(n-k-2)}(T; r) = 0, \end{cases} \\
p^2 : & \begin{cases} \mathcal{L}_n \underline{u}_2(t; r) - \mathcal{F}\left(t, \tilde{\mathcal{Y}}_1(t; r, \sum_{i=1}^n \underline{a}_s(r))\right) = 0, \\ \underline{y}_2(t_0; r) = 0_r, \underline{y}_2'(t_0; r) = 0, \dots, \underline{y}_2^{(k)}(t_0; r) = 0, \\ \underline{y}_2(T; r) = 0_r, \underline{y}_2'(T; r) = 0, \dots, \underline{y}_2^{(n-k-2)}(T; r) = 0, \end{cases} \\
& \vdots \\
p^{n+1} : & \begin{cases} \mathcal{L}_n \underline{u}_n(t; r) - \mathcal{F}\left(t, \tilde{\mathcal{Y}}_n(t; r, \sum_{i=1}^n \underline{a}_s(r))\right) = 0, \\ \underline{y}_n(t_0; r) = 0_r, \underline{y}_n'(t_0; r) = 0, \dots, \underline{y}_n^{(k)}(t_0; r) = 0, \\ \underline{y}_n(T; r) = 0_r, \underline{y}_n'(T; r) = 0, \dots, \underline{y}_n^{(n-k-2)}(T; r) = 0. \end{cases}
\end{aligned}$$

Similarly for the upper bound

$$\begin{aligned}
p^0 : & \begin{cases} \bar{\mathcal{L}}_n [\bar{u}_0(t; r) - \bar{y}_0(t; r; \sum_{s=1}^n \bar{a}_s(r))] = 0, \\ \bar{y}(t_0; r) = [\bar{\alpha}_0]_r, \bar{y}'(t_0; r) = [\bar{\alpha}_1]_r, \dots, \bar{y}^{(k)}(t_0; r) = [\bar{\alpha}_k]_r, \\ \bar{y}(T; r) = [\bar{\beta}_0]_r, \bar{y}'(T; r) = [\bar{\beta}_1]_r, \dots, \bar{y}^{(n-k-2)}(T; r) = [\bar{\beta}_{n-k-2}]_r, \end{cases} \\
p^1 : & \begin{cases} \bar{\mathcal{L}}_n [\bar{y}_1(t; r) + \bar{y}_0(t; r; \sum_{s=1}^n \bar{a}_s(r))] \\ -\mathcal{G}\left(t, \tilde{\mathcal{Y}}_0(t; r, \sum_{i=1}^n \bar{a}_s(r))\right) - \bar{w}(t; r) = 0, \\ \bar{y}_1(t_0; r) = 0, \bar{y}_1'(t_0; r) = 0, \dots, \bar{y}_1^{(k)}(t_0; r) = 0, \\ \bar{y}_1(T; r) = 0, \bar{y}_1'(T; r) = 0, \dots, \bar{y}_1^{(n-k-2)}(T; r) = 0, \end{cases} \\
p^2 : & \begin{cases} \bar{\mathcal{L}}_n \bar{y}_2(t; r) - \mathcal{G}\left(t, \tilde{\mathcal{Y}}_1(t; r, \sum_{i=1}^n \bar{a}_s(r))\right) = 0, \\ \bar{y}_2(t_0; r) = 0, \bar{y}_2'(t_0; r) = 0, \dots, \bar{y}_2^{(k)}(t_0; r) = 0, \\ \bar{y}_2(T; r) = 0, \bar{y}_2'(T; r) = 0, \dots, \bar{y}_2^{(n-k-2)}(T; r) = 0, \end{cases} \\
& \vdots \\
p^{n+1} : & \begin{cases} \bar{\mathcal{L}}_n \bar{y}_n(t; r) - \mathcal{G}\left(t, \tilde{\mathcal{Y}}_n(t; r, \sum_{i=1}^n \bar{a}_s(r))\right) = 0, \\ \bar{y}_n(t_0; r) = 0, \bar{y}_n'(t_0; r) = 0, \dots, \bar{y}_n^{(n-1)}(t_0; r) = 0, \\ \bar{y}_n(T; r) = 0, \bar{y}_n'(T; r) = 0, \dots, \bar{y}_n^{(n-1)}(T; r) = 0, \end{cases}
\end{aligned}$$

for all $r \in [0, 1]$. Then the approximate solution is given by setting $p=1$ as follows:

$$(13) \quad \tilde{y}\left(t; r; \sum_{s=1}^n \tilde{a}_s(r)\right) = \varphi_m\left(t; r; \sum_{s=1}^n \tilde{a}_s(r)\right) = \sum_{i=0}^{m-1} \tilde{\mathcal{Y}}_i\left(t; r; \sum_{s=1}^n \tilde{a}_s(r)\right).$$

Now the exact solution of Eq.(1), therefore, can be obtained by setting $p = 1$ as follows:

$$(14) \quad \begin{aligned} \tilde{Y}(t; r) &= \lim_{p \rightarrow 1} \tilde{y} \left(t; r; \sum_{s=1}^n \tilde{a}_s(r) \right) = \lim_{p \rightarrow 1} \left\{ \sum_{i=0}^{\infty} p^i \tilde{\mathcal{Y}}_i \left(t; r; \sum_{s=1}^n \tilde{a}_s(r) \right) \right\} \\ &= \sum_{i=0}^{\infty} \tilde{\mathcal{Y}}_i \left(t; r; \sum_{s=1}^n \tilde{a}_s(r) \right). \end{aligned}$$

Convergence of HPM in fuzzy environment

According to [34, 35] we can write Eq.(1) in the following forms

$$(15) \quad \mathcal{L}([\tilde{u}(t)]_r) = \mathcal{L}[\tilde{y}_0(t)]_r + p \left[\tilde{f}(s; r) - \mathcal{N}([\tilde{u}(t)]_r) - \mathcal{L}[\tilde{y}_0(t)]_r \right],$$

where $\tilde{f}(s; r)$ is known as analytic fuzzy function. Applying the inverse operator, $\mathcal{L}^{-1}$ as mentioned in section 4 to both sides of Eq.(15), we obtain

$$(16) \quad [\tilde{u}(x; t)]_r = [\tilde{y}_0(t)]_r + p \left[\mathcal{L}^{-1} \tilde{f}(s; r) - \mathcal{L}^{-1} \mathcal{N}([\tilde{u}(t)]_r) - [\tilde{y}_0(t)]_r \right].$$

Suppose that

$$(17) \quad \tilde{u}(t; r) = \sum_{i=0}^{\infty} p^i \tilde{u}_i(t; r).$$

Substituting Eq.(17) into the right-hand side of Eq.(16), we have

$$(18) \quad [\tilde{u}(t)]_r = [\tilde{y}_0(t)]_r + p \left[\mathcal{L}^{-1} \tilde{f}(s; r) - \mathcal{L}^{-1} \mathcal{N} \left(\sum_{i=0}^{\infty} p^i \tilde{u}_i(t; r) \right) - [\tilde{y}_0(t)]_r \right].$$

If $p \rightarrow 1$, the exact solution may be obtained by using the following formula $\tilde{y}(t; r) = \lim_{p \rightarrow 1} \tilde{u}(t; r) = \mathcal{L}^{-1} \tilde{f}(s; r) - \mathcal{L}^{-1} \mathcal{N}(\sum_{i=0}^{\infty} p^i \tilde{u}_i(t; r)) = \mathcal{L}^{-1} \tilde{f}(s; r) - (\sum_{i=0}^{\infty} \mathcal{L}^{-1} \mathcal{N} \tilde{u}_i(t; r))$.

To study the convergence of the method let us state the following Theorem.

Theorem 1. Suppose that $[\tilde{u}]_r$ and $[\tilde{w}]_r$ be Banach spaces and $N : [\tilde{u}]_r \rightarrow [\tilde{w}]_r$ be a contraction mapping such that for all $r \in [0, 1]$

$$(19) \quad [\tilde{u}]_r, [\tilde{u}^*]_r \in [\tilde{U}]_r : \|N([\tilde{u}]_r) - N([\tilde{u}^*]_r)\| \leq \varepsilon \|[\tilde{u}]_r - [\tilde{u}^*]_r\|, \varepsilon \in [0, 1].$$

Then according to Banach's fixed point theorem [36] N has a unique fixed point $[\tilde{y}]_r$ (say) such that $N([\tilde{y}]_r) = [\tilde{y}]_r$. The sequence generated by the homotopy perturbation method will be assumed in the following form as

$$\tilde{U}_n(t; r) = N(\tilde{U}_{n-1}(t; r)), \tilde{U}_{n-1}(t; r) = \sum_{i=0}^{n-1} \tilde{U}_i(t; r), n = 1, 2, 3, \dots$$

and suppose that $\tilde{U}_0(t; r) = \tilde{u}_0(t; r) \in B_s(\tilde{u}_0(t; r))$ where $B_s(\tilde{u}_0(t; r)) = \{[\tilde{u}^*]_r \in [\tilde{U}]_r : \|[\tilde{u}^*]_r - [\tilde{u}]_r\| < s\}$, then we have

1. $\|\tilde{U}_1(t; r) - \tilde{V}(t; r)\| \leq \varepsilon^n \|\tilde{u}_0(t; r) - \tilde{u}(t; r)\|,$
2. $\tilde{U}_n(t; r) \in B_s(\tilde{u}(t; r)),$
3. $\lim_{p \rightarrow 1} \tilde{U}_n(t; r) = \tilde{u}(t; r).$

Proof. As mentioned in section 2 all fuzzy sets are subsets of real numbers $\mathbb{R}$ and the r-level sets is crisp sets for each value of $r \in [0, 1]$, then we can define:

$$(20) \quad \tilde{U}_n(t; r) = [\underline{U}_n(t; r), \bar{U}_n(t; r)] = [\underline{U}_n, \bar{U}_n]_r,$$

$$(21) \quad \tilde{V}(t; r) = [\underline{V}(t; r), \bar{V}(t; r)] = [\underline{V}, \bar{V}]_r.$$

For the lower bound of Eq.(1), according to [36] part (i) of the above theorem is proved by using method of induction.

For $n = 1$, $\|[\underline{U}_1]_r - [\underline{u}]_r\| = \|N([\underline{U}_0]_r) - N([\underline{u}]_r)\| \leq \varepsilon \|[\underline{u}_0]_r - [\underline{u}]_r\|$ (from Eq.(34)). Assume that $\|[\underline{U}_{n-1}]_r - [\underline{u}]_r\| \leq \varepsilon^{n-1} \|[\underline{u}_0]_r - [\underline{u}]_r\|$ as induction hypothesis, then $\|[\underline{U}_n]_r - [\underline{u}]_r\| \leq \|N([\underline{U}_{n-1}]_r) - N([\underline{u}]_r)\| \leq \varepsilon \|[\underline{U}_{n-1}]_r - [\underline{u}]_r\| \leq \varepsilon \varepsilon^{n-1} \|[\underline{u}_0]_r - [\underline{u}]_r\| \leq \varepsilon^n \|[\underline{u}_0]_r - [\underline{u}]_r\|$ (from induction hypothesis).

Hence $\|[\underline{U}_n]_r - [\underline{u}]_r\| \leq \varepsilon^n \|[\underline{u}_0]_r - [\underline{u}]_r\|$.

(ii) Using (i), we have

$$\|[\underline{U}_n]_r - [\underline{u}]_r\| \leq \varepsilon^n \|[\underline{u}_0]_r - [\underline{u}]_r\| \leq \varepsilon^n s \leq s.$$

This implies $[\underline{U}_n]_r \in B_s([\underline{u}]_r)$.

(iii) From (i) we know that $\|[\underline{U}_n]_r - [\underline{u}]_r\| \leq \varepsilon^n \|[\underline{u}_0]_r - [\underline{u}]_r\|$.

Since $0 < \varepsilon < 1$, $\lim_{n \rightarrow \infty} \varepsilon^n = 0$. So $\lim_{n \rightarrow \infty} \|[\underline{U}_n]_r - [\underline{u}]_r\| = 0$. This yield $\lim_{n \rightarrow \infty} [\underline{U}_n]_r = [\underline{u}]_r$.

Similarly in the same way we can prove (i), (ii), (iii) for the upper bound convergence of Eq.(14). Thus the series in Eq. (??) such that $[\underline{y}_i(t; r), \bar{y}_i(t; r)]$ converges for each r-level set.

Numerical example

Let us consider the following fuzzy two-point boundary value problem [9]

$$(22) \quad y''(t) = y'(t) + \gamma_0, \quad y(0) = 0, y(1) = \gamma_1,$$

where γ_0 and γ_1 are triangular fuzzy numbers having r-level sets $[r, 2 - r]$, $[r - 1, 1 - r]$ for all $r \in [0, 1]$. It can be verified the exact solution of Eq.(22) is given in [9]. According to section 3 the linear operator of Eq.(17) is $\tilde{\mathcal{L}}_2$ with the inverse operator $\tilde{\mathcal{L}}_2^{-1}$, and then the initial approximation guesses for all $r \in [0, 1]$ are given by:

$$\underline{y}_0(t; r) = \underline{a}_1(r) + \underline{a}_2(r)t, \quad \bar{y}_0(t; r) = \bar{a}_1(r) + \bar{a}_2(r)t$$

According to the boundary conditions in Eq.(22) the values of $\tilde{a}_1(r) = 0$. According to the HPM in section 3, the n components of $\tilde{y}_k(t; r)$ for $k = 1, 2, \dots, n$ and $r \in [0, 1]$ can be determined by solving the following

$$(23) \quad p^0 : \left\{ \underline{y}_0(t; r) = \underline{a}_2(r) t, \right.$$

$$(24) \quad p^1 : \left\{ \begin{aligned} \underline{y}_1(t; r) &= \underline{\mathcal{L}}_2^{-1} \left[\underline{y}_0'(t; r; \underline{a}_2(r)) + r \right], \\ \underline{y}_1(0; r) &= 0, \underline{y}_1(1; r) = 0, \end{aligned} \right.$$

$$(25) \quad p^2 : \left\{ \begin{aligned} \underline{y}_2(t; r) &= \underline{\mathcal{L}}_2^{-1} \left[\underline{y}_1'(t; r; \underline{a}_2(r)) \right], \\ \underline{y}_2(0; r) &= 0, \underline{y}_2'(1; r) = 0. \end{aligned} \right.$$

And so on till

$$(26) \quad p^{k+1} : \left\{ \begin{aligned} \underline{y}_{k+1}(t; r) &= \underline{\mathcal{L}}_2^{-1} \underline{y}_k'(t; r; \underline{a}_2(r)), \\ \underline{y}_k(0; r) &= 0, \underline{y}_k'(1; r) = 0. \end{aligned} \right.$$

Similarly for the upper bound

$$(27) \quad p^0 : \{ \bar{y}_0(t; r) = \bar{a}_2(r) t, \}$$

$$(28) \quad p^1 : \left\{ \begin{aligned} \bar{y}_1(t; r) &= \bar{\mathcal{L}}_2^{-1} [\bar{y}_0'(t; r; \bar{a}_2(r)) + (2 - r)], \\ \bar{y}_1(0; r) &= 0, \bar{y}_1'(1; r) = 0, \end{aligned} \right.$$

$$(29) \quad p^2 : \left\{ \begin{aligned} \bar{y}_2(t; r) &= \bar{\mathcal{L}}_2^{-1} [\bar{y}_1'(t; r; \bar{a}_2(r))], \\ \bar{y}_2(0; r) &= 0, \bar{y}_2'(1; r) = 0. \end{aligned} \right.$$

And so on till

$$(30) \quad p^{k+1} : \left\{ \begin{aligned} \bar{y}_{k+1}(t; r) &= \bar{\mathcal{L}}_2^{-1} [\bar{y}_k'(t; r; \bar{a}_2(r))], \\ \bar{y}_k(0; r) &= 0, \bar{y}_k'(1; r) = 0. \end{aligned} \right.$$

Evaluating Eqs.(23)-(30) to obtain 10-order HPM series solution in the following form

$$(31) \quad \tilde{\varphi}_{10}(t; r; \tilde{a}_2(r)) = \tilde{y}_0(t; r) + \sum_{i=1}^9 \tilde{y}_i(t; r; \tilde{a}_2(r)).$$

From the boundary conditions in Eq.(21) and Eq.(31), we obtain the values of $\tilde{a}_2(r)$ for all $r \in [0, 1]$ such that:

$$\begin{cases} \underline{a}_2(r) = -0.5819767076349684 + 0.16395341526993693r, \\ \bar{a}_2(r) = -0.2540698770950946 - 0.16395341526993693r. \end{cases}$$

Table 1: Approximate solution and Absolut error of 10-order HPM at $r=0$ and $t \in [0, 1]$

t	$\varphi_{10}(t; 0)$	$[E(t)]_0$	$\bar{\varphi}_{10}(t; 0)$	$[\bar{E}(t)]_0$
0	0	0	0	0
0.2	-0.128851248255356	1.695150×10^{-10}	-0.01344625523392	5.085456×10^{-10}
0.4	-0.286230518266809	3.765406×10^{-10}	0.058691554800427	1.129621×10^{-9}
0.6	-0.478453992733306	6.266763×10^{-10}	0.235361978199918	1.880029×10^{-9}
0.8	-0.713236274547007	8.493847×10^{-10}	0.539708823641023	2.548154×10^{-9}
1	-1	1.110223×10^{-16}	1	4.440892×10^{-16}

Table 2: Approximate solution Absolut error of 10-order HPM at $t=0.5$ and $r \in [0, 1]$

r	$\varphi_{10}(0.5; r)$	$[E(0.5)]_r$	$\bar{\varphi}_{10}(0.5; r)$	$[\bar{E}(0.5)]_r$
0	-0.3775406692945253	4.963798×10^{-10}	0.1326220078835757	1.489139×10^{-9}
0.2	-0.3265244015767151	2.978278×10^{-10}	0.0816057401657656	1.290587×10^{-9}
0.4	-0.2755081338589050	9.927592×10^{-11}	0.0305894724479555	1.092035×10^{-9}
0.6	-0.2244918661410949	9.927592×10^{-11}	-0.0204267952698545	8.934834×10^{-10}
0.8	-0.1734755984232848	2.978279×10^{-10}	-0.0714430629876646	6.949317×10^{-10}
1	-0.1224593307054747	4.963796×10^{-10}	-0.1224593307054747	4.963797×10^{-10}

Substitute these values again in Eq.(31) to get 10- order HPM approximate solution series, and then we can define the absolute error of approximate solution as follows

$$(32) \quad [\tilde{E}(t)]_r = \left| \tilde{Y}(t; r) - \tilde{\varphi}_{10}(t; r) \right|.$$

It can be noted from Figure 2; the exact and 10-order HPM approximate solutions of Eq.(22) when $r = 0$ for all $t \in [0, 1]$ satisfy the fuzzy numbers properties in the form of triangular fuzzy number as all the values of $r \in [0, 1]$. Also Tables 1-2 and Figures 3-4 shows the accuracy of the HPM with 10 order of the approximate solution.

4. Conclusions

In this study, an approximate-analytical method HPM was applied to obtain an approximate solution n' th order nonlinear TPFBVP. A scheme based HPM approximate the solution of n' th order TPFBVP has been formulated to obtain the approximate solution directly with our reduced in to first order system. The convergence theorem of HPM has been presented and proved. From the

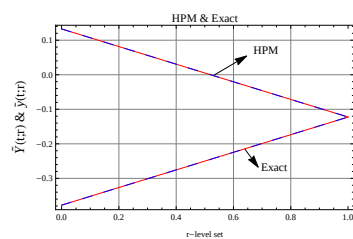


Figure 2: Approximate solution of 10-order of Eq. (25) at $r=0$ and $t \in [0, 1]$

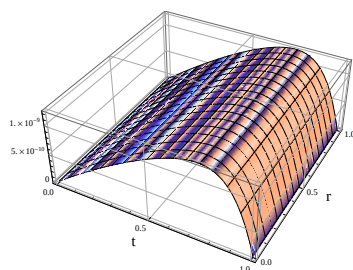


Figure 3: Approximate solution of 10-order of Eq. (25) at $r=0$ and $t \in [0, 1]$

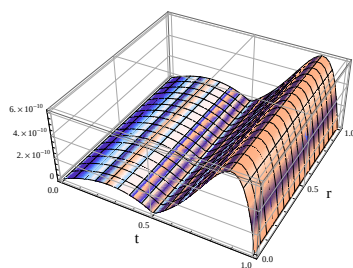


Figure 4: Absolut error of 10-order HPM solution of Eq. (25)
 $\forall r \in [0, 1]$ and $t \in [0, 1]$

nonlinear TPFIVP the accuracy of HPM can be determined even these equations without exact analytical solution. HPM give accurate solution even with low number of terms of the series solution. Numerical example which included linear TPFBVP show the efficiency of the implemented of this approximate- analytical method. The results of the solved problem satisfied the fuzzy numbers properties by taking the triangular fuzzy numbers shape.

References

- [1] A. Omer and O. Omer, *A pray and pretdour model with fuzzy intial values*, Hacettepe Journal of Mathematics and Statistics, 41 (2013), 387-395.
- [2] S. Tapaswini and S. Chakraverty, *Numerical solution of fuzzy arbitrary order predator-prey equations*, Applications and Applied Mathematics, 8 (2013), 647-673.
- [3] M. Zaini, Ahmad Bernard De Baets, *A predator-prey model with fuzzy initial populations*, IFSA-EUSFLAT, 2009.
- [4] M. S. El Naschie, *From experimental quantum optics to quantum gravity via a fuzzy kahler manifold*, Chaos Solution and Fractals, 25 (2005), 969-977.
- [5] M. F. Abbod, D.G. Von Keyserlingk, D.A. Linkens and M. Mahfouf, *Survey of utilization of fuzzy technology in medicine and healthcare*, Fuzzy Sets and System, 120 (2001), 331-3491.
- [6] S. Barro and R. Marin, *Fuzzy logic in medicine*, Heidelberg: Physica-Verlag, 2002.
- [7] L. Wang and S. Guo, *Adomian method for second-order fuzzy differential equation*, World Academy of Science, Engineering and Technology, 5 (2011), 783-786.
- [8] S. Abbasbandy, T. Allahviranloo, P. Darabi and O. Sedaghatfar, *Variational iteration method for solving N-th order fuzzy differential equations*, Mathematical and Computational Applications, 16 (2011), 819-829.
- [9] A. Armand and Z. Gouyandeh, *Solving two-point fuzzy boundary value problem using variational iteration method*, Communications on Advanced Computational Science with Applications, (2013), 1-10.
- [10] A. F. Jameel and A. I. Ismail, *Approximate solution of N'th order two point fuzzy boundary value problems by optimal homotopy asymptotic method*, International Journal of Modern Mathematical Sciences, 6 (2013), 107-120.
- [11] A. F. Jameel, *Semi analytical-solution of nonlinear two points fuzzy boundary value problems by adomian decomposition method*, Applied & Computational Mathematics, 4 (2015), 1-5.

- [12] X. Guo, D. Shang and X. Lu, *Fuzzy approximate solutions of second-order fuzzy linear boundary value problems*, Journal of Boundary Value Problems, (2013), 1-17.
- [13] J.-H. He., *A coupling method of a homotopy technique and a perturbation technique for non-linear problems*, Internat. J. Non-Linear Mech., 35 (2000), 7-43.
- [14] L. Jin, *Analytical approach to the sine-Gordon equation using homotopy perturbation method*, Int. J. Contemp. Math. Sci., 4(5-8) (2009), 225-231.
- [15] C. Chun, H. Jafari and Y.-I. Kim, *Numerical method for the wave and non-linear diffusion equations with the homotopy perturbation method*, Comput. Math. Appl., 57 (2009), 1226-1231.
- [16] Y. Khan and Q. Wu, *Homotopy perturbation transform method for non-linear equations using He's polynomials*, Comput. Math. Appl., 61 (2011), 1963-1967.
- [17] J.-H. He, *Homotopy perturbation technique*, Comput. Methods Appl. Mech. Engrg., 178 (1999), 257-262.
- [18] J.-H. He., *The homotopy perturbation method for nonlinear oscillators with discontinuities*, Applied Mathematics and Computation, 151 (2004), 287-292.
- [19] J.-H. He., *Application of homotopy perturbation method to nonlinear wave equations*, Chaos, Solitons and Fractals, 26 (2005), 695-700.
- [20] J.-H. He, *Limit cycle and bifurcation of nonlinear problems*, Chaos, Solitons and Fractals, 26 (2005), 827-833.
- [21] S. Bodjanova, *Median alpha-levels of a fuzzy number*, Fuzzy. Set. Syst., 157 (2006), 879-891.
- [22] D. Dubois, H. Prade, *Towards fuzzy differential calculus, Part 3: Differentiation*, Fuzzy Sets and Systems, 8 (1982), 225-233.
- [23] S. Mansouri. S , N. Ahmady, *A numerical method for solving n'th-order fuzzy differential equation by using characterization theorem*, Communication in Numerical Analysis, (2012), 1-12.
- [24] O. Kaleva, *Fuzzy differential equations*, Fuzzy Sets Syst, 24 (1987), 301-317.
- [25] O. S. Fard, *An Iterative Scheme for the Solution of Generalized System of Linear Fuzzy Differential Equations*, World. Appl. Sci. J., 7 (2009), 1597-11604.
- [26] L. A. Zadeh, *Fuzzy sets*, Information and Control, 8 (1965), 338-353.

- [27] L. A. Zadeh, *Toward a generalized theory of uncertainty*, Information Sciences, 172 (2005), 1-40.
- [28] S. Salahshour, *N'th-order fuzzy differential equations under generalized differentiability*, Journal of Fuzzy Set Value Analysis, (2011), 1-14.
- [29] A. Saadatmandi, M. Dehghan and A. Eftekhari, *Application of He's homotopy perturbation method for non-linear system of second-order boundary value problems*, Nonlinear Anal. Real World Appl., 10 (2009), 1912-1922.
- [30] Z. Shun-dong, *Homotopy perturbation method for two point boundary value problems*, Journal of the Juliusz Schauder Cente, 31 (2008), 369-372.
- [31] S. S. Shahid and M. Iftikhar, *Use of homotopy perturbation method for solving multi-point boundary value problems*, Research Journal of Applied Sciences, Engineering and Technology, 7 (2014), 778-785.
- [32] S. Liang and D. Jeffrey, *Approximate solutions to a parameterized sixth order boundary value problem*, Comput. Math. Appl, 59 (2010), 247-253.
- [33] S. Liang and D. Jeffrey, *An efficient analytical approach for solving fourth order boundary value problems*, Comput. Phys. Commun, 180 (2009), 2034-2040.
- [34] J. Biazar, and H. Aminikhah, *Study of convergence of homotopy perturbation method for systems of partial differential equations*, Computers & Mathematics with Applications, 58 (2009), 2221-2230.
- [35] J. Biazar, and H. Ghazvini, *Convergence of the homotopy perturbation method for partial differentialequations*, Nonlinear Analysis, 10 (2009), 2633-2640.
- [36] A. K. Gupta, and S. Saha Ray, *Comparison between homotopy perturbation method and optimal homotopy asymptotic method for the solution of Boussinesq-Burgers equations*, Comput. Fluids, 103 (2014), 34-41.

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