

BASIC NOTIONS OF PARTIALLY ORDERED HYPERMODULES

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Abstract. In this paper, we construct the ring-like hyperstructures derived from a (partially) quasi ordered ring R , and we study some basic properties to this class. Then, we introduce the new class of (partially) ordered hypermodules by using of the (partially) ordered modules. Moreover, we study some basic properties of this new class and the essential differences between this class and the earlier one (i.e. ordered modules) are also investigated.

Keywords: (partially) ordered hypermodule, (partially) ordered ring, (partially) ordered module, (good) hyperring, (good) hypermodule

1. Introduction and preliminaries

Hyperstructures theory was born in 1934, when Marty at the 8th congress of Scandinavian mathematicians, gave the definition of a hypergroup and illustrated some applications and showed its utility in the study of groups, algebraic functions and rational functions [16]. He defined the concept of hypergroups, as a natural generalization of groups, based on the notion of hyperoperation. Since then, a number of different hyperstructures have been widely studied by many mathematicians. One of the first books about hypergroups was written

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by P. Corsini in 1993 [6]. Also the book "Hyperstructures and Their Representations" was published in 1994 by T. Vougiouklis [26]. The other book on these topics is "Applications of Hyperstructure Theory", by P. Corsini and V. Leoreanu, published in 2003 [5]. Another book, devoted especially to the study of hyperring theory, is "Hyperring Theory and Applications", written by B. Davvaz and V. Leoreanu-Fotea [9]. A recent book on hyperstructures [8] points out on their applications in fuzzy and rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs. In this book basic definitions and notions concerning hyperstructure theory can be found.

First of all, we present some basic definitions and ideas from the hyperstructures theory. The hyperstructures are algebraic structures equipped with at least one multi-valued operation, called a *hyperoperation*. A nonempty set H , endowed with a hyperoperation, $+: H \times H \longrightarrow \wp^*(H)$ is called a *hypergroupoid*. $\wp^*(H)$ denotes the set of all nonempty subsets of H . A hypergroupoid which verifies the condition $(x + y) + z = x + (y + z)$, for all $x, y, z \in H$, is called a *semihypergroup*. A semihypergroup H which verifies reproduction axioms, $x + H = H = H + x$, for all $x \in H$, is called a *hypergroup* [8].

Since we deal with the theory of ordered structures, we recall that a *quasi ordered (semi)group* is a triple $(G, +, \leq)$, where $(G, +)$ is a (semi)group and $\leq$ is a reflexive and transitive binary relation on G such that for any triple $a, b, c \in G$ with the property $a \leq b$ also $a + c \leq b + c$ and $c + a \leq c + b$ hold. We call the (semi)group *partially ordered* if the relation $\leq$ is moreover antisymmetric [13]. In a partially ordered group, an element x of G is called *positive element* if $0 \leq x$. The set of elements $0 \leq x$ is often denoted with G_+ , and it is called the positive cone of G . So, we have $a \leq b$ if and only if $-a + b \in G_+$.

For a general group G , the existence of a positive cone specifies an order on G [1]. A group G is a *partially ordered group* if and only if there exists a subset H which is G_+ of G such that:

- i) $0 \in G_+$;
- ii) if $a, b \in G_+$ then $a + b \in G_+$;
- iii) if $a \in G_+$ then $-x + a + x \in G_+$ for any x of G ;
- iv) if $a \in G_+$ and $-a \in G_+$ then $a = 0$;
- v) G is totally ordered when it satisfies also $G_+ \cup G_- = G$, in which $G_- = \{-x | x \in G_+\}$.

The (partially) ordered group $(G, \leq)$ is denoted by (G, G_+) . In addition $[a]_{\leq} := \{x \in G | a \leq x\}$ is a principal end generated by $a \in G$. The definition of partially ordered ring and modules will be presented later in sections 2 and 3, respectively.

Recall that a *ring* consists of a set R equipped with two binary operations $+$ and $\cdot$ such that (i) $(R, +)$ is an abelian group and (ii) $(R, \cdot)$ is a semigroup,

and (iii) $a.(b + c) = a.b + a.c$ and $(a + b).c = a.c + b.c$ for all $a, b, c \in R$. A ring R is called *unitary* whenever, there exists $1 \in R$ such that $1.a = a.1 = a$ for all $a \in R$. Let R is a ring and 1 is its multiplicative identity. A *left R -module* M consists of an abelian group $(M, +)$ and an operation $. : R \times M \longrightarrow M$ such that for all $a, b \in R$ and $x, y \in M$, we have (i') $a.(x + y) = a.x + a.y$, (ii') $(a + b).x = a.x + b.x$, (iii') $(ab).x = a.(b.x)$, and (iv') $1.x = x$. The operation of the ring R on M is called *scalar multiplication*.

The relation of ordered sets and algebraic hyperstructures was first studied by Vougiouklis in 1987 [27]. Then the connection between hyperstructures and ordered sets have been analyzed by many researchers, such as Vougiouklis [29, 28], Corsini [7], Hoskova [15], Ghazavi and et al [11, 12, 13], Heidari and Davvaz [14] and Novak [18, 19]. One special aspect of this issue, known as EL-hyperstructures, was touched upon by Chvalina [3]. He investigated quasi ordered sets and hypergroups. Also, Rosenberg in [25], Hoskova in [15], Rackova in [24] and Novak in [17, 20, 21, 22] extended some results on the ordered semigroups and ordered groups connected with EL-hyperstructures. EL-hyperstructures, mainly studied by M. Novak, are hyperstructures constructed from a (partially) quasi-ordered (semi)groups. More exactly, Novak in [21] considered subhyperstructures of EL-hyperstructures and in [18], he discussed some interesting results of important elements in this family of hyperstructures. Then, in [19] Novak studied some basic properties of EL-hyperstructures like invertibility, normality, being closed (ultra closed) and etc.

A number of articles and contributions in the hyperstructures theory discussed about the creation of hyperstructures from a (partially) quasi-ordered (semi)groups. This results known as the "Ends lemma" and first used in [4] as theorems 1.3 and 1.4 (chapter 4; and the remark mentioned in the following), would be presented in the following:

Theorem 1.1 ([4], Theorem 1.3, p. 146). *Let $(G, +, \leq)$ be a partially ordered semigroup. Binary hyperoperation $\oplus : G \times G \longrightarrow \wp^*(G)$ defined by $a \oplus b = [a + b]_{\leq}$ is associative. The semihypergroup $(G, \oplus)$ is commutative if and only if the semigroup $(G, +)$ is commutative.*

Theorem 1.2 ([4], Theorem 1.4, p. 147). *Let $(G, +, \leq)$ be a partially ordered semigroup. The following conditions are equivalent:*

- i) *For any pair $a, b \in G$, there exists a pair $c, c' \in G$ such that $b + c \leq a$ and $c' + b \leq a$.*
- ii) *The associated semihypergroup $(G, \oplus)$ is a hypergroup.*

Remark 1 ([4]). If $(G, +, \leq)$ is a partially ordered group, then if we take $c = -b + a$ and $c' = a - b$, then condition (i) is valid. Therefore, if $(G, +, \leq)$ is a partially ordered group, then its associated hyperstructure is a hypergroup.

If the condition of quasi ordered is replaced by the condition of partially ordered, then the proofs are valid.

In this paper, we intend to build the ring-like hyperstructures using of a ring H . Also, the ring H with order relation, on which two operations $+$ and $.$ and the quasi ordered semigroups (groups) $(H, +, \leq)$ and $(H, ., \leq)$ has studied. In addition, we introduce a new class of hypermodules from a given (partially) quasi-ordered modules as a generalization of Ends lemma based on hyperstructures. Then, the essential differences between this class and the ordered modules are also investigated.

2. (Partially) ordered (semi)hyperrings

Since a ring R is a algebraic structure that endow two operations $+$ and $.$, such that $(R, +)$ is an abelian group and $(R, .)$ is a semigroup, therefore an ordered ring is defined as:

Definition 2.1 ([10]). Let R be a ring with unit element $1 \neq 0$. We say that R is *partially ordered* when there exists a partial order $\leq$ on the underlying set R such that for any a, b, c in R :

- i) $a \leq b$ implies $a + c \leq b + c$
- ii) $0 \leq a$ and $0 \leq b$ imply that $0 \leq a.b$.

If any two elements $a, b \in R$ are comparable, then R is *ordered*.

The additive group of a partially ordered ring is a partially ordered group. Furthermore, The set of elements x for which $0 \leq x$ (the set of non-negative elements) of a partially ordered ring, is closed under addition and multiplication, i.e. if P is the set of non-negative elements of a partially ordered ring, then $P + P \subseteq P$, and $P.P \subseteq P$. Furthermore, $P \cap (-P) = \{0\}$.

If there exists a subset R_+ of R such that:

- i) $0 \in R_+$;
- ii) $R_+ \cap (R_-) = \{0\}$, in which $R_- = \{-x | x \in R_+\}$;
- iii) $R_+ + R_+ \subseteq R_+$;
- iv) $R_+.R_+ \subseteq R_+$;
- v) R is ordered when it satisfies also $R_+ \cup R_- = R$,

then the relation $\leq$ where $a \leq b$ if and only if $b - a \in R_+$ defines a compatible partial order on R (i.e. $(R, \leq)$ is a partially ordered ring). Also, R is ordered when it satisfies also $R_+ \cup R_- = R$.

Example 1. i) The ring $(\mathbb{Z}, \leq)$ with the usually order relation is an ordered ring.

- ii) The ring $\mathbb{Z}^I$ of integral-valued functions on a set, with pointwise order, is partially ordered (when I has at least two elements).

In any ring R , the *absolute value* $|x|$ of an element x can be defined as following:

$$|x| = \begin{cases} x, & 0_R \leq x \\ -x, & x \leq 0_R. \end{cases}$$

The order relation of partially ordered ring R is compatible with the addition of the abelian group $(R, +)$. Thus for the construction of hyperstructures based Ends Lemma, the hyperoperations are defined:

Definition 2.2. Let R be an ordered ring. For all $a, b \in R$, we define:

$$\begin{aligned} (1) \quad a \oplus b &= [|a| + |b|]_{\leq} \\ (2) \quad a \odot b &= [|a| |b|]_{\leq}. \end{aligned}$$

Lemma 2.3. Let R be an ordered ring. By definitions are presented in (1) and (2), $(R, \oplus)$ is an commutative semihypergroup, and $(R, \odot)$ is a semihypergroup.

Proof. Let $a_1, a_2, a_3 \in R$ such that $a_1, a_2 \in R_-$ and $a_3 \in R_+$. Now, we show that $a_1 \oplus (a_2 \oplus a_3) = (a_1 \oplus a_2) \oplus a_3$. Therefore, by Definition 2.2. we have:

$$\begin{aligned} a_1 \oplus (a_2 \oplus a_3) &= \cup \{a_1 \oplus a' | a' \geq |a_2| + |a_3|\} \\ &= \cup \{a_1 \oplus a' | a' \geq -a_2 + a_3\} \\ &= \{a | a \geq |a_1| + |a'|, a' \geq -a_2 + a_3\} \\ &= \{a | a \geq -a_1 + a', a' \geq -a_2 + a_3\} \\ &= \{a | a \geq -a_1 + a' \geq -a_1 + (-a_2 + a_3)\} \\ &= \{a | a \geq -a_1 - a_2 + a_3\}. \end{aligned}$$

On the other hand

$$\begin{aligned} (a_1 \oplus a_2) \oplus a_3 &= \cup \{a' \oplus a_3 | a' \geq |a_1| + |a_2|\} \\ &= \cup \{a' \oplus a_3 | a' \geq -a_1 - a_2\} \\ &= \{a | a \geq |a'| + |a_3|, a' \geq -a_1 - a_2\} \\ &= \{a | a \geq a' + a_3, a' \geq -a_1 - a_2\} \\ &= \{a | a \geq a' + a_3 \geq (-a_1 - a_2) + a_3\} \\ &= \{a | a \geq -a_1 - a_2 + a_3\}. \end{aligned}$$

For $(R, \odot)$, we have:

$$\begin{aligned} a_1 \odot (a_2 \odot a_3) &= \cup \{a_1 \odot a' | a' \geq |a_2| |a_3|\} \\ &= \cup \{a_1 \odot a' | a' \geq (-a_2)a_3\} \\ &= \{a | a \geq |a_1| |a'|, a' \geq -a_2a_3\} \\ &= \{a | a \geq (-a_1)a', a' \geq -a_2a_3\} \\ &= \{a | a \geq -a_1a' \geq -a_1(-a_2a_3)\} \\ &= \{a | a \geq a_1a_2a_3\}. \end{aligned}$$

On the other hand

$$\begin{aligned}
(a_1 \odot a_2) \odot a_3 &= \cup \{a' \odot a_3 | a' \geq |a_1| |a_2|\} \\
&= \cup \{a' \odot a_3 | a' \geq (-a_1)(-a_2)\} \\
&= \{a | a \geq |a'| |a_3|, a' \geq a_1 a_2\} \\
&= \{a | a \geq a' a_3, a' \geq a_1 a_2\} \\
&= \{a | a \geq a' a_3 \geq (a_1 a_2) a_3\} \\
&= \{a | a \geq a_1 a_2 a_3\}.
\end{aligned}$$

Therefore, in this case $(R, \oplus)$ and $(R, \odot)$ are semihypergroup. The other cases are proved, similarly. The commutativity of hyperoperation $\oplus$ is the direct consequence of the commutativity of operation $+$, according to Theorem 1.1. $\square$

The hyperstructures endowed with two internal hyperoperations, is called a *hyperringoid*. Recall that $(R, +, \cdot)$ is a *hyperring* (*semihyperring*) in the general sense if; (1) $(R, +)$ is a commutative hypergroup (semihypergroup); (2) $\cdot$ is associative hyperoperation and; (3) the distributive law $a \cdot (b + c) \subseteq (a \cdot b) + (a \cdot c)$ and $(a + b) \cdot c \subseteq (a \cdot c) + (b \cdot c)$ is satisfied for every a, b, c of R . If the equality in the distributive law is valid, then the hyperring (semihyperring) is called *good*.

Theorem 2.4. *Let R be an ordered ring. Then $(R, \oplus, \odot)$ is a semihyperring.*

Proof. It is sufficient the condition (3) of definition of semihyperring is checked. So, let $a, b \in R_-$ and $c \in R_+$. Then

$$\begin{aligned}
a_1 \odot (a_2 \oplus a_3) &= \cup \{a_1 \odot a' | a' \geq |a_2| + |a_3|\} \\
&= \cup \{a_1 \odot a' | a' \geq -a_2 + a_3\} \\
&= \{a | a \geq |a_1| |a'|, a' \geq -a_2 + a_3\} \\
&= \{a | a \geq -a_1 a', a' \geq -a_2 + a_3\} \\
&= \{a | a \geq -a_1(-a_2 + a_3)\} \\
&= \{a | a \geq a_1 a_2 - a_1 a_3\}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
(a_1 \odot a_2) \oplus (a_1 \odot a_3) &= \cup \{a \oplus b | a \geq |a_1| |a_2|, b \geq |a_1| |a_3|\} \\
&= \cup \{a \oplus b | a \geq a_1 a_2, b \geq -a_1 a_3\} \\
&= \{a' | a' \geq |a| + |b|, a \geq a_1 a_2, b \geq -a_1 a_3\} \\
&= \{a' | a' \geq a + b, a \geq a_1 a_2, b \geq -a_1 a_3\} \\
&= \{a' | a' \geq a_1 a_2 - a_1 a_3\}.
\end{aligned}$$

The proof of distributive property on the right is done, similarly. The proof of the other cases is done easily, too. $\square$

Notice that if R is not ordered, then there exists an element $a_0 \in R$ such that $a_0 \notin R_+ \cup R_-$, so $|a_0|$ is meaningless. Therefore, the definitions of hyperoperations $\oplus$ and $\odot$ in (1) and (2) respectively, is not efficient for $a \oplus b$ or $a \odot b$ when, at least one of a or b is not belonging to $R_+ \cup R_-$. So, we modify the definitions of $\oplus$ and $\odot$ in (1) and (2) respectively, in the following way:

Definition 2.5. Let $(R, +, \cdot, \leq)$ be a partially ordered ring. For $a, b \in R$, we define:

$$\begin{aligned} (3) \quad a \oplus_1 b &= [|a| + |b|]_{\leq} \cup \{a, b\}, \\ (4) \quad a \odot_1 b &= [|a| |b|]_{\leq} \cup \{a, b\}. \end{aligned}$$

With the hyperoperations $\oplus_1$ and $\odot_1$ presented in (3) and (4), for any a, b of the partially ordered ring R , we have $a \oplus_1 b \subseteq \wp^*(R)$ and $a \odot_1 b \subseteq \wp^*(R)$.

Recall that $(R, +, \cdot)$ is a hyperring in the general sense if $(R, +)$ is a commutative hypergroup, $\cdot$ is associative hyperoperation and the distributive law $a \cdot (b + c) \subseteq (a \cdot b) + (a \cdot c)$, $(a + b) \cdot c \subseteq (a \cdot c) + (b \cdot c)$ is satisfied for any a, b, c of R . If the equality in the distributive law is valid, then the hyperring is called *good*.

Example 2. The hyperstructure $R = (\{a, b\}, \oplus, \odot)$ defined as follows:

$\oplus$	a	b	$\odot$	a	b
a	a	$\{a, b\}$	a	a	$\{a, b\}$
b	$\{a, b\}$	$\{a, b\}$	b	a	$\{a, b\}$

is a good hyperring.

Theorem 2.6. Let $(R, +, \cdot, \leq)$ be an ordered ring. Then $(R, \oplus_1, \odot_1)$ is a good hyperring.

Proof. Let $a, b, c \in R$. First of all, we show that $(a \oplus_1 b) \oplus_1 c = a \oplus_1 (b \oplus_1 c)$. To prove the associative property of $\oplus$, there are eight different cases. All cases, particularly the cases in which $a, b, c \in R_+$ or $a, b, c \in R_-$, can easily be proved. Here we prove the case in which $a \in R_+$ and $b, c \in R_-$.

$$\begin{aligned} (a \oplus_1 b) \oplus_1 c &= ([|a| + |b|]_{\leq} \cup \{a, b\}) \oplus_1 c \\ &= (\{r' | r' \geq |a| + |b|\} \cup \{a, b\}) \oplus_1 c \\ &= (\{r' | r' \geq a - b\} \cup \{a, b\}) \oplus_1 c \\ &= \cup \{r' \oplus_1 c | r' \geq a - b\} \cup a \oplus_1 c \cup b \oplus_1 c \\ &= \{r | r \geq |r'| + |c|, r' \geq a - b\} \cup \{r' | r' \geq a - b\} \cup \{c\} \\ &\quad \cup \{a, c\} \cup \{r | r \geq |a| + |c|\} \cup \{b, c\} \cup \{r | r \geq |b| + |c|\} \\ &= \{r | r \geq r' - c, r' \geq a - b\} \cup \{r' | r' \geq a - b\} \\ &\quad \cup \{r | r \geq a - c\} \cup \{r | r \geq -b - c\} \cup \{a, b, c\} \\ &= \{r | r \geq (a - b) - c\} \cup \{r' | r' \geq a - b\} \cup \{r | r \geq a - c\} \\ &\quad \cup \{r | r \geq -b - c\} \cup \{a, b, c\}. \end{aligned}$$

On the other hand

$$\begin{aligned}
a \oplus_1 (b \oplus_1 c) &= a \oplus_1 ([|b| + |c|]_{\leq} \cup \{b, c\}) \\
&= a \oplus_1 (\{r'|r' \geq |b| + |c|\} \cup \{b, c\}) \\
&= a \oplus_1 (\{r'|r' \geq -b - c\} \cup \{b, c\}) \\
&= \cup\{a \oplus_1 r'|r' \geq -b - c\} \cup a \oplus_1 b \cup a \oplus_1 c \\
&= \{r|r \geq |a| + |r'|, r' \geq -b - c\} \cup \{r'|r' \geq -b - c\} \\
&\quad \cup \{a\} \cup \{a, b\} \cup \{r|r \geq |a| + |b|\} \cup \{a, c\} \cup \{r|r \geq |a| + |c|\} \\
&= \{r|r \geq a + r', r' \geq -b - c\} \cup \{r'|r' \geq -b - c\} \\
&\quad \cup \{r|r \geq a - b\} \cup \{r|r \geq a - c\} \cup \{a, b, c\} \\
&= \{r|r \geq a + (-b - c)\} \cup \{r'|r' \geq -b - c\} \cup \{r|r \geq a - b\} \\
&\quad \cup \{r|r \geq a - c\} \cup \{a, b, c\}.
\end{aligned}$$

In the following, we show that the reproduction principles. Let $a \in R$, then we have:

$$R \supseteq a \oplus_1 R = \cup\{a \oplus_1 b | b \in R\} \supseteq \cup\{a, b | b \in R\} = R,$$

as well as

$$R \supseteq R \oplus_1 a = \cup\{b \oplus_1 a | b \in R\} \supseteq \cup\{b, a | b \in R\} = R.$$

So $R \oplus_1 a = a \oplus_1 R = R$, for all $a \in R$. Therefore, $(R, \oplus_1)$ is a hypergroup. Now, we show that $(R, \odot)$ is a semihypergroup. Again, consider the case in which $a \in R_+$ and $b, c \in R_-$, then

$$\begin{aligned}
(a \odot b) \odot c &= ([|a| |b|]_{\leq}) \odot c \\
&= (\{r'|r' \geq -ab\}) \odot c \\
&= \cup\{r' \odot c | r' \geq -ab\} \\
&= \{r|r \geq |r'| |c|, r' \geq -ab\} \\
&= \{r|r \geq -r'c, r' \geq -ab\} \\
&= \{r|r \geq r'(-c) \geq (-ab)(-c)\} \\
&= \{r|r \geq (ab)c\}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
a \odot (b \odot c) &= a \odot ([|b| |c|]_{\leq}) \\
&= a \odot \{r'|r' \geq (-b)(-c)\} \\
&= \cup\{a \odot r'|r' \geq bc\} \\
&= \{r|r \geq |a| |r'|, r' \geq bc\} \\
&= \{r|r \geq ar', r' \geq bc\} \\
&= \{r|r \geq ar' \geq a(bc)\} \\
&= \{r|r \geq a(bc)\}.
\end{aligned}$$

To investigate the distribution $\odot$ by the ratio of $\oplus_1$ of the left, the case $a \in R_+$ and $b, c \in R_-$ are considered. So we have:

$$\begin{aligned}
a \odot (b \oplus_1 c) &= a \odot (\{r'|r' \geq |b| + |c|\} \cup \{b, c\}) \\
&= \cup\{a \odot r'|r' \geq -b - c\} \cup a \odot b \cup a \odot c \\
&= \{r|r \geq |a| \parallel r', r' \geq (-b - c)\} \\
&\quad \cup\{r|r \geq |a| \parallel b\} \cup \{r|r \geq |a| \parallel c\} \\
&= \{r|r \geq ar', r' \geq -(b + c)\} \cup \{r|r \geq -ab\} \cup \{r|r \geq -ac\} \\
&= \{r|r \geq -ab - ac\} \cup \{r|r \geq -ab\} \cup \{r|r \geq -ac\}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
(a \odot b) \oplus_1 (a \odot c) &= \cup\{r \oplus_1 r'|r \geq |a| \parallel b, r' \geq |a| \parallel c\} \\
&= \{r|r \geq -ab\} \cup \{r|r' \geq -ac\} \\
&\quad \cup\{r''|r'' \geq |r| + |r'|, r \geq -ab, r' \geq -ac\} \\
&= \{r|r \geq -ab\} \cup \{r|r' \geq -ac\} \\
&\quad \cup\{r''|r'' \geq r + r', r \geq -ab, r' \geq -ac\} \\
&= \{r|r \geq -ab\} \cup \{r|r' \geq -ac\} \cup \{r''|r'' \geq -ab - ac\}.
\end{aligned}$$

The distribution $\odot$ by the ratio of $\oplus_1$ of the right is proved similarly. Therefore, $(R, \oplus_1, \odot)$ is a good hyperring. $\square$

Theorem 2.7. *Let $(R, +, \leq)$ be a partially ordered group. Then $(R, \oplus_1)$ is a hypergroup.*

Proof. According to the previous theorems, it is sufficient to show the associative property of hyperoperation $\oplus_1$ and the reproduction principles for the case in which at least one element is not belonging to $R_+ \cup R_-$. Let $a \notin R_+ \cup R_-$, and $b, c \in R_+$. Then we have:

$$\begin{aligned}
(a \oplus_1 b) \oplus_1 c &= ([|a| + |b|]_{\leq} \cup \{a, b\}) \oplus_1 c \\
&= (\emptyset \cup \{a, b\}) \oplus_1 c \\
&= a \oplus_1 c \cup b \oplus_1 c \\
&= \{a, c\} \cup \{r|r \geq |a| + |c|\} \cup \{b, c\} \cup \{r|r \geq |b| + |c|\} \\
&= \emptyset \cup \{r|r \geq b + c\} \cup \{a, b, c\} \\
&= \{r|r \geq b + c\} \cup \{a, b, c\}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
a \oplus_1 (b \oplus_1 c) &= a \oplus_1 ([|b| + |c|]_{\leq} \cup \{b, c\}) \\
&= a \oplus_1 (\{r'|r' \geq |b| + |c|\} \cup \{b, c\}) \\
&= a \oplus_1 (\{r'|r' \geq b + c\} \cup \{b, c\}) \\
&= \cup\{a \oplus_1 r'|r' \geq b + c\} \cup a \oplus_1 b \cup a \oplus_1 c
\end{aligned}$$

$$\begin{aligned}
&= \{r|r \geq |a| + |r'|, r' \geq b + c\} \cup \{r'|r' \geq b + c\} \\
&\quad \cup \{a\} \cup \{a, b\} \cup \{r|r \geq |a| + |b|\} \cup \{a, c\} \cup \{r|r \geq |a| + |c|\} \\
&= \emptyset \cup \{r'|r' \geq b + c\} \cup \emptyset \cup \{a, b\} \cup \emptyset \cup \{a, c\} \\
&= \{r|r \geq b + c\} \cup \{a, b, c\}.
\end{aligned}$$

But, for $a \in R$ such that $a \notin R_+ \cup R_-$, we have:

$$a \oplus_1 R = \cup \{a \oplus_1 b | b \in R\} = \cup \{a, b | b \in R\} = R,$$

as well as

$$R \oplus_1 a = \cup \{b \oplus_1 a | b \in R\} = \cup \{b, a | b \in R\} = R.$$

So $R \oplus_1 a = a \oplus_1 R = R$, for all $a \in R$. Therefore, $(R, \oplus_1)$ is a hypergroup. $\square$

Example 3. Let $G = (\mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\}, +)$ with ordinary addition of complex numbers. Put $\mathbb{Z}[i]_+ = \{a | a \in \mathbb{Z}, a \geq 0\} \cup \{bi | b \in \mathbb{Z}, b \leq 0\}$. Then $(\mathbb{Z}[i], \mathbb{Z}[i]_+)$ is a partially ordered group.

Also, we can see that:

Theorem 2.8. *Let $(R, \cdot, \leq)$ be a partially ordered group. Then $(R, \odot_1)$ is a hyperring.*

According to the previous lemmas, The following theorems satisfying:

Theorem 2.9. *Let $(R, +, \cdot, \leq)$ be a partially ordered ring. Then $(R, \oplus_1, \odot_1)$ is a hyperring.*

Proof. First, we show the distributivity $\odot_1$ by the ratio of $\oplus_1$ of the left, for the case in which $a \in R_+$ and $b, c \in R_-$.

$$\begin{aligned}
a \odot_1 (b \oplus_1 c) &= a \odot_1 (\{r'|r' \geq |b| + |c|\} \cup \{b, c\}) \\
&= \cup \{a \odot_1 r' | r' \geq -b - c\} \cup a \odot_1 b \cup a \odot_1 c \\
&= \{a\} \cup \{r'|r' \geq -b - c\} \cup \{r|r \geq |a| \mid r', r' \geq (-b - c)\} \\
&\quad \cup \{a, b\} \cup \{r|r \geq |a| \mid b\} \cup \{a, c\} \cup \{r|r \geq |a| \mid c\} \\
&= \{r|r \geq ar', r' \geq -(b + c)\} \cup \{r|r \geq -ab\} \cup \{r|r \geq -ac\} \\
&\quad \cup \{a, b, c\} \cup \{r'|r' \geq -b - c\} \\
&= \{r|r \geq -ab - ac\} \cup \{r|r \geq -ab\} \cup \{r|r \geq -ac\} \\
&\quad \cup \{a, b, c\} \cup \{r'|r' \geq -b - c\}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
(a \odot_1 b) \oplus_1 (a \odot_1 c) &= \cup \{r \oplus_1 r' | r \in a \odot_1 b, r' \in a \odot_1 c\} \\
&= \cup \{r \oplus_1 r' | r \geq |a||b|, r = a, b, r' \geq |a||c|, r' = a, c\} \\
&\supseteq \cup \{r \oplus_1 r' | r \geq -ab, r' \geq -ac\} \cup b \oplus_1 c \cup \{a\} \\
&= \{r|r \geq -ab\} \cup \{r'|r' \geq -ac\} \\
&\quad \cup \{r'' | r'' \geq |r| + |r'|, r \geq -ab, r' \geq -ac\}
\end{aligned}$$

$$\begin{aligned}
& \cup \{r | r \geq |b| + |c|\} \cup \{b, c\} \cup \{a\} \\
= & \{r | r \geq -ab\} \cup \{r' | r' \geq -ac\} \cup \{r'' | r'' \geq -ab - ac\} \\
& \cup \{r | r \geq -b - c\} \cup \{a, b, c\}.
\end{aligned}$$

Then, consider the case in which $a, c \in R_+$ and $b \notin R_+ \cup R_-$.

$$\begin{aligned}
a \odot_1 (b \oplus_1 c) &= a \odot_1 (\{r' | r' \geq |b| + |c|\} \cup \{b, c\}) \\
&= a \odot_1 (\emptyset \cup \{b, c\}) \\
&= a \odot_1 b \cup a \odot_1 c \\
&= \{a, b\} \cup \{r | r \geq |a||c|\} \cup \{a, c\} \\
&= \{a, b, c\} \cup \{r | r \geq ac\}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
(a \odot_1 b) \oplus_1 (a \odot_1 c) &= \cup \{r \oplus_1 r' | r \in a \odot_1 b, r' \in a \odot_1 c\} \\
&= \cup \{r \oplus_1 r' | r \geq |a||b|, r = a, b, r' \geq |a||c|, r' = a, c\} \\
&= \cup \{r \oplus_1 r' | r = a, b, r' \geq ac, r' = a, c\} \\
&\supseteq \{a, b, c\} \cup \{r' | r' \geq ac\}.
\end{aligned}$$

The other cases are proved, similarly. therefore $(R, \oplus_1, \odot_1)$ is a hyperring. $\square$

Example 4. Consider the partially ordered ring $\mathbb{Z}^I$ of integral-valued functions on a set $I = \{a, b\}$, with pointwise order. Then $(\mathbb{Z}^I, \oplus_1, \odot)$ is a good hyperring.

3. (Partially) ordered hypermodules

In this section, we intend to build module-like hyperstructures using the definitions presented in previous section and the definitions would be presented in following.

Definition 3.1 ([23]). Let R be a partially ordered ring. A *partially ordered* (left) R -module is a (left) R -module $(M, \cdot)$ (by the function $\cdot : R \times M \rightarrow M$), together with a compatible partial order, i.e. a partial order $\leq_M$ on the underlying set M that is compatible with the operation of the abelian group M and the operation $\cdot$, in the sense that it satisfies:

- i) $x \leq y$ implies $x + z \leq y + z$
- ii) $0_R \leq a$ and $0_M \leq x$ imply that $0_M \leq ax$,

for any $x, y, z \in M$ and $a \in R$. If any two elements $a, b \in R$ are comparable and any two elements $x, y \in M$ too, then M is *ordered*.

If there exists a subset N (which is M_+) of M such that:

- i) $0_M \in M_+$;

- ii) $M_+ \cap M_- = \{0\}$ in which $M_- = \{-m | m \in M_+\}$;
- iii) $M_+ + M_+ \subseteq M_+$;
- iv) $R_+.M_+ \subseteq M_+$;
- v) M is ordered when it satisfies also $M_+ \cup M_- = M$.

Then the relation $\leq_M$ where $x \leq y$ if and only if $y - x \in M$ defines a compatible partial order on M (i.e. $(M, \leq_M)$ is a partially ordered R -module).

Example 5. Consider $\mathbb{Z}$ -module $\mathbb{Z}$ with the usually order relation. Then M_+ equals to all non-negative integer numbers, and M_- equals to all non-positive integer numbers. Therefore $M = M_+ \cup M_-$.

In the following, we present an example of an ordered module in which the order relation is partially only.

Example 6. Let I be a non-empty set and $M := \mathbb{Z}^I$ be the $\mathbb{Z}$ - module of all functions from I to $\mathbb{Z}$, with pointwise order, where the order relation is the usually order relation on integer numbers. Then M_+ consists of all functions from I to $\mathbb{Z}^+ \cup \{0\}$. Now, Let me put $I := \{a, b\}$. Then, assuming $f(a) = 1$ and $f(b) = -1$, we have $f \in M$, while $f \notin M_+ \cup M_-$.

In any module, the *absolute value* $|x|$ of an element x can be defined as following:

$$|x| = \begin{cases} x, & 0_M \leq x \\ -x, & 0_M \geq x. \end{cases}$$

Now we are trying to present an appropriate hyperoperations associated to the operations and order relation of the R -module M . The order relation of partially ordered R -module M is compatible with the addition of the abelian group M and abelian group R . Thus for the construction of hyperstructures based Ends Lemma, the hyperoperations are defined:

Definition 3.2. Let R be a partially ordered ring and M be a partially ordered (left) R -module. For all $a \in R$ and $x, y \in M$, we define the hyperoperations as follows:

- (5) $x \star y = [|x| + |y|]_{\leq}$;
- (6) $x \star_1 y = [|x| + |y|]_{\leq} \cup \{x, y\}$;
- (7) $a \circ x = [|a| \cdot |x|]_{\leq}$;
- (8) $a \circ_1 x = [|a| \cdot |x|]_{\leq} \cup \{x\}$.

Let $(R, +, \cdot)$ be a hyperring and $\circ : R \times M \longrightarrow \wp^*(M)$ be the scalar hyperoperation. Then M is a *left R -hypermodule* whenever, $(M, +)$ is a commutative hypergroup and for all $a, b \in R$ and $x, y \in M$, i) $a \circ (x + y) \subseteq (a \circ x) + (a \circ y)$; ii) $(a + b) \circ x \subseteq (a \circ x) + (b \circ x)$, and iii) $(a \cdot b) \circ x = a \circ (b \circ x)$. If R is a good hyperring and the equalities in the (i) and (ii) are valid, then the hypermodule is called *good* [2].

Example 7 ([2]). Let A be a ring and M be an R -module. If M' is a submodule of M and one defines the following scalar hyperoperation

$$\forall(a, x) \in A \times M, a \circ x = ax + M'.$$

Then M is an R -hypermodule.

Remark 2. With an argument similar to that presented in the previous section, we can see that $(M, \star)$ is a commutative semihypergroup and $(M, \star_1)$ is a commutative hypergroup.

Theorem 3.3. Let R be an ordered ring, and M be an ordered module over R . We apply hyperoperations $\oplus_1, \odot, \star_1$ and $\circ$ on the ordered R -module M . Then M is an good R -hypermodule.

Proof. It is sufficient that we show the validity of equality in conditions (i), (ii) and (iii) of definition good R -hypermodule. To show equality in conditions (i), (ii) and (iii); we consider cases in which $a \in R_-, x \in M_+, y \in M_-, a, b \in R_-, x \in M_-$ and $a \in R_+, b \in R_-, x \in M_-$, respectively. To prove (i), we have:

$$\begin{aligned} a \circ (x \oplus_1 y) &= a \circ (\{z | z \geq |x| + |y|\} \cup \{x, y\}) \\ &= a \circ (\{z | z \geq x - y\} \cup \{x, y\}) \\ &= \cup \{a \circ z | z \geq x - y\} \cup a \circ x \cup a \circ y \\ &= \{z' | z' \geq |a| \cdot |z|, z \geq x - y\} \cup \{z' | z' \geq |a| \cdot |x|\} \cup \{z' | z' \geq |a| \cdot |y|\} \\ &= \{z' | z' \geq -a \cdot z, z \geq x - y\} \cup \{z' | z' \geq -a \cdot x\} \cup \{z' | z' \geq a \cdot y\} \\ &= \{z' | z' \geq a \cdot (y - x)\} \cup \{z' | z' \geq -a \cdot x\} \cup \{z' | z' \geq a \cdot y\}. \end{aligned}$$

On the other hand

$$\begin{aligned} a \circ x \star_1 a \circ y &= \cup \{z_1 \star_1 z_2 | z_1 \geq |a| \cdot |x|, z_2 \geq |a| \cdot |y|\} \\ &= \{z' | z' \geq |z_1| + |z_2|, z_1 \geq -a \cdot x, z_2 \geq a \cdot y\} \cup \{z_1 | z_1 \geq -a \cdot x\} \\ &\quad \cup \{z_2 | z_2 \geq a \cdot y\} \\ &= \{z' | z' \geq z_1 + z_2, z_1 \geq -a \cdot x, z_2 \geq a \cdot y\} \cup \{z_1 | z_1 \geq -a \cdot x\} \\ &\quad \cup \{z_2 | z_2 \geq a \cdot y\} \\ &= \{z' | z' \geq z_1 + z_2 \geq -a \cdot x + a \cdot y\} \cup \{z_1 | z_1 \geq -a \cdot x\} \cup \{z_2 | z_2 \geq a \cdot y\} \\ &= \{z' | z' \geq a \cdot (y - x)\} \cup \{z_1 | z_1 \geq -a \cdot x\} \cup \{z_2 | z_2 \geq a \cdot y\}. \end{aligned}$$

To prove (ii),

$$\begin{aligned} (a \oplus_1 b) \circ x &= (\{c | c \geq |a| + |b|\} \cup \{a, b\}) \circ x \\ &= \cup \{c \circ x | c \geq -a - b\} \cup a \circ x \cup b \circ x \\ &= \{y | y \geq |c| \cdot |x|, c \geq -a - b\} \cup \{y | y \geq a \cdot x\} \cup \{y | y \geq b \cdot x\} \\ &= \{y | y \geq -c \cdot x, c \geq -a - b\} \cup \{y | y \geq a \cdot x\} \cup \{y | y \geq b \cdot x\} \\ &= \{y | y \geq (a + b) \cdot x\} \cup \{y | y \geq a \cdot x\} \cup \{y | y \geq b \cdot x\}, \end{aligned}$$

and

$$\begin{aligned}
a \circ x \star_1 b \circ x &= \cup \{y_1 \star_1 y_2 | y_1 \geq |a| \cdot |x|, y_2 \geq |b| \cdot |x|\} \\
&= \{y' | y' \geq |y_1| + |y_2|, y_1 \geq a \cdot x, y_2 \geq b \cdot x\} \cup \{y_1 | y_1 \geq a \cdot x\} \\
&\quad \cup \{y_2 | y_2 \geq b \cdot x\} \\
&= \{y' | y' \geq y_1 + y_2, y_1 \geq a \cdot x, y_2 \geq b \cdot x\} \cup \{y_1 | y_1 \geq a \cdot x\} \\
&\quad \cup \{y_2 | y_2 \geq b \cdot x\} \\
&= \{y' | y' \geq y_1 + y_2 \geq a \cdot x + b \cdot x\} \cup \{y_1 | y_1 \geq a \cdot x\} \cup \{y_2 | y_2 \geq b \cdot x\} \\
&= \{y' | y' \geq a \cdot x + b \cdot x\} \cup \{y_1 | y_1 \geq a \cdot x\} \cup \{y_2 | y_2 \geq b \cdot x\}.
\end{aligned}$$

In the following for proving (iii), we have

$$\begin{aligned}
(a \odot b) \circ x &= \{c | c \geq |a||b|\} \circ x \\
&= \cup \{c \circ x | c \geq -ab\} \\
&= \{y | y \geq |c| \cdot |x|, c \geq -ab\} \\
&= \{y | y \geq c \cdot (-x), c \geq -ab\} \\
&= \{y | y \geq c \cdot (-x) \geq (-ab) \cdot (-x)\} \\
&= \{y | y \geq (ab) \cdot x\},
\end{aligned}$$

and

$$\begin{aligned}
a \circ (b \circ x) &= a \circ \{y | y \geq |b| \cdot |x|\} \\
&= \cup \{a \circ y | y \geq b \cdot x\} \\
&= \{y' | y' \geq |a| \cdot |y|, y \geq b \cdot x\} \\
&= \{y' | y' \geq a \cdot y, y \geq b \cdot x\} \\
&= \{y' | y' \geq a \cdot (b \cdot x)\}.
\end{aligned}$$

The other cases in (i), (ii) and (iii) are proved, similarly. $\square$

Theorem 3.4. *Let R be an ordered ring, and M be an ordered module over R . We apply hyperoperations $\oplus_1$, $\odot_1$, $\star_1$ and $\circ_1$ on the ordered R -module M . Then M is an R -hypermodule.*

Proof. We showed that $(R, \oplus_1, \odot_1)$ is a hyperring, and we know that $(M, \star_1)$ is a commutative hypergroup. Now, we prove that for all $a, b \in R$ and $x, y \in M$, (i) $a \circ_1 (x \star_1 y) \subseteq (a \circ_1 x) + (a \circ_1 y)$; (ii) $(a \oplus_1 b) \circ_1 x \subseteq (a \circ_1 x) + (b \circ_1 x)$, and (iii) $(a \odot_1 b) \circ_1 x = a \circ_1 (b \circ_1 x)$. For the proving of the case (i), suppose that $x \in M_+$, $y \in M_-$ and $a \in R_-$. So, we have:

$$\begin{aligned}
a \circ_1 (x \star_1 y) &= a \circ_1 (\{z : z \geq |x| + |y|\} \cup \{x, y\}) \\
&= \cup \{a \circ_1 z : z \geq x - y\} \cup a \circ_1 x \cup a \circ_1 y \\
&= \{z' : z' \geq |a| \cdot |z|, z : z \geq x - y\} \cup \{z : z \geq x - y\}
\end{aligned}$$

$$\begin{aligned}
& \cup \{z' : z' \geq |a|.|x|\} \cup \{z' : z' \geq |a|.|y|\} \cup \{x, y\} \\
= & \{z' : z' \geq (-a).z, (-a).z \geq (-a).(x - y)\} \cup \{z : z \geq x - y\} \\
& \cup \{z' : z' \geq (-a).x\} \cup \{z' : z' \geq (-a).(-y)\} \cup \{x, y\} \\
= & \{z' : z' \geq (-a).z \geq (-a).(x - y)\} \cup \{z : z \geq x - y\} \\
& \cup \{z' : z' \geq (-a).x\} \cup \{z' : z' \geq (-a).(-y)\} \cup \{x, y\} \\
= & \{z' : z' \geq a.(y - x)\} \cup \{z : z \geq x - y\} \cup \{z' : z' \geq -a.x\} \\
(9) \quad & \cup \{z' : z' \geq a.y\} \cup \{x, y\},
\end{aligned}$$

and

$$\begin{aligned}
a \circ_1 x \star_1 a \circ_1 y &= \cup \{z_1 \star_1 z_2 | z_1 \geq |a|.|x|, z_1 = x, z_2 \geq |a|.|y|, z_2 = y\} \\
&\supseteq \{z' | z' \geq |z_1| + |z_2|, z_1 \geq -a.x, z_2 \geq a.y\} \cup \{z_1 | z_1 \geq -a.x\} \\
&\quad \cup \{z_2 | z_2 \geq a.y\} \cup x \star_1 y \\
&= \{z' | z' \geq z_1 + z_2, z_1 \geq -a.x, z_2 \geq a.y\} \cup \{z_1 | z_1 \geq -a.x\} \\
&\quad \cup \{z_2 | z_2 \geq a.y\} \cup \{z | z \geq |x| + |y|\} \cup \{x, y\} \\
&= \{z' | z' \geq z_1 + z_2, z_1 + z_2 \geq (-a.x + a.y)\} \cup \{z_1 | z_1 \geq -a.x\} \\
&\quad \cup \{z_2 | z_2 \geq a.y\} \cup \{z | z \geq x - y\} \cup \{x, y\} \\
&= \{z' | z' \geq a.(y - x)\} \cup \{z_1 | z_1 \geq -a.x\} \cup \{z_2 | z_2 \geq a.y\} \\
(10) \quad &\cup \{z | z \geq x - y\} \cup \{x, y\}.
\end{aligned}$$

The proof of (ii), for $a \in R_+$, $b \in R_-$ and $x \in M_-$,

$$\begin{aligned}
(a \oplus_1 b) \circ_1 x &= (\{c | c \geq |a| + |b|\} \cup \{a, b\}) \circ_1 x \\
&= \cup \{c \circ_1 x | c \geq a - b\} \cup a \circ_1 x \cup b \circ_1 x \\
&= \{y | y \geq |c|.|x|, c \geq a - b\} \cup \{y | y \geq -a.x\} \cup \{y | y \geq b.x\} \cup \{x\} \\
&= \{y | y \geq c.(-x), c.(-x) \geq (a - b).(-x)\} \cup \{y | y \geq -a.x\} \\
&\quad \cup \{y | y \geq b.x\} \cup \{x\} \\
&= \{y | y \geq (b - a).x\} \cup \{y | y \geq -a.x\} \cup \{y | y \geq b.x\} \cup \{x\},
\end{aligned}$$

and

$$\begin{aligned}
a \circ_1 x \star_1 b \circ_1 x &= \cup \{y_1 \star_1 y_2 | y_1 \geq |a|.|x|, y_1 = x, y_2 \geq |b|.|x|, y_2 = x\} \\
&\supseteq \{y' | y' \geq |y_1| + |y_2|, y_1 \geq a.(-x), y_2 \geq (-b).(-x)\} \\
&\quad \cup x \star_1 x \cup \{y_1 | y_1 \geq a.(-x)\} \cup \{y_2 | y_2 \geq (-b).(-x)\} \\
&\supseteq \{y' | y' \geq y_1 + y_2, y_1 + y_2 \geq -a.x + b.x\} \cup \{x\} \\
&\quad \cup \{y_1 | y_1 \geq -a.x\} \cup \{y_2 | y_2 \geq b.x\} \\
&= \{y' | y' \geq (-a + b).x\} \cup \{x\} \cup \{y_1 | y_1 \geq -a.x\} \cup \{y_2 | y_2 \geq b.x\} \\
&= \{y' | y' \geq (b - a).x\} \cup \{y_1 | y_1 \geq -a.x\} \cup \{y_2 | y_2 \geq b.x\} \cup \{x\}.
\end{aligned}$$

Finally, for $a, b \in R_+$ and $x \in M_-$, we have

$$\begin{aligned}
(a \odot_1 b) \circ_1 x &= (\{c | c \geq |a||b|\} \cup \{a, b\}) \circ_1 x \\
&= \cup \{c \circ_1 x | c \geq ab\} \cup a \circ_1 x \cup b \circ_1 x \\
&= \{y | y \geq |c|.|x|, c \geq ab\} \cup \{y | y \geq |a|.|x|\} \cup \{y | y \geq |b|.|x|\} \cup \{x\} \\
&= \{y | y \geq c.(-x), c.(-x) \geq (ab).(-x)\} \cup \{y | y \geq a.(-x)\} \\
&\quad \cup \{y | y \geq b.(-x)\} \cup \{x\} \\
&= \{y | y \geq (ab).(-x)\} \cup \{y | y \geq -a.x\} \cup \{y | y \geq -b.x\} \cup \{x\},
\end{aligned}$$

and

$$\begin{aligned}
a \circ_1 (b \circ_1 x) &= a \circ_1 (\{y | y \geq |b|.|x|\} \cup \{x\}) \\
&= \cup \{a \circ_1 y | y \geq b.(-x)\} \cup a \circ_1 x \\
&= \{y' | y' \geq |a|.|y|, y \geq b.(-x)\} \cup \{y | y \geq b.(-x)\} \\
&\quad \cup \{y | y \geq a.(-x)\} \cup \{x\} \\
&= \{y' | y' \geq a.y, a.y \geq a.(-b.x)\} \cup \{y | y \geq -b.x\} \\
&\quad \cup \{y | y \geq -a.x\} \cup \{x\} \\
&= \{y' | y' \geq -a.(b.x)\} \cup \{y | y \geq -b.x\} \cup \{y | y \geq -a.x\} \cup \{x\}.
\end{aligned}$$

The other cases in (i), (ii) and (iii) are proved, similarly. $\square$

Example 8. Consider the abelian group $(\mathbb{Z}[x], +)$ with the ordinary addition of polynomials. Let $p(x) = a_m x^m + a_{m+1} x^{m+1} + \dots + a_n x^n$ be a polynomial with $a_m, a_n \neq 0$ and $m \leq n$. Define $p(x) \in \mathbb{Z}[x]_+$ if and only if $a_m \in \mathbb{Z}_+$. Then $(\mathbb{Z}[x], \mathbb{Z}[x]_+)$ is an ordered abelian group, and $\mathbb{Z}$ -module $\mathbb{Z}[x]$ is an ordered module. Applying Theorem 3.3, the resulting hyperstructure $(\mathbb{Z}[x], \oplus, \star_1)$ will be a good hypermodule.

Theorem 3.5. Let R be a partially ordered ring, and M be a partially ordered module over R . We apply hyperoperations $\oplus_1, \odot_1, \star_1$ and $\circ_1$ on the partially ordered R -module M . Then M is an R -hypermodule.

Proof. It is sufficient that we show inclusion in scalar conditions for the case in which there is at least an element $a \in R$ or $x \in M$ such that $a \notin R_+ \cup R_-$ or $x \notin M_+ \cup M_-$, respectively. So for condition (i), in the case $a \in R_+, x \in M_-$ and $y \notin M_+ \cup M_-$, we have:

$$\begin{aligned}
a \circ_1 (x \star_1 y) &= a \circ_1 (\{z | z \geq |x| + |y|\} \cup \{x, y\}) \\
&= a \circ_1 (\emptyset \cup \{x, y\}) \\
&= a \circ_1 x \cup a \circ_1 y \\
&= \{z | z \geq |a|.|x|\} \cup \{z | z \geq |a|.|y|\} \cup \{x, y\} \\
&= \{z | z \geq a.(-x)\} \cup \emptyset \cup \{x, y\} \\
&= \{z | z \geq -a.x\} \cup \{x, y\}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
a \circ_1 x \star_1 a \circ_1 y &= (\{z|z \geq |a|.|x|\} \cup \{x\}) \star_1 (\{z|z \geq |a|.|y|\} \cup \{y\}) \\
&= (\{z|z \geq a.(-x)\} \cup \{x\}) \star_1 (\emptyset \cup \{y\}) \\
&= \cup\{z \star_1 y|z \geq a.(-x)\} \cup x \star_1 y \\
&= \{z'|z' \geq |z| + |y|, z \geq -a.x\} \cup \{z|z \geq -a.x\} \\
&\quad \cup \{z'|z' \geq |x| + |y|\} \cup \{x, y\} \\
&= \emptyset \cup \{z|z \geq -a.x\} \cup \emptyset \cup \{x, y\} \\
&= \{z|z \geq -a.x\} \cup \{x, y\}.
\end{aligned}$$

In condition (ii), for the case in which $a \notin R_+ \cup R_-$, $b \in R_-$ and $x \in M_+$, we have

$$\begin{aligned}
(a \oplus_1 b) \circ_1 x &= (\{c|c \geq |a| + |b|\} \cup \{a, b\}) \circ_1 x \\
&= (\emptyset \cup \{a, b\}) \circ_1 x \\
&= a \circ_1 x \cup b \circ_1 x \\
&= \{y|y \geq |a|.|x|\} \cup \{y|y \geq |b|.|x|\} \cup \{x\} \\
&= \emptyset \cup \{y|y \geq -b.x\} \cup \{x\} \\
&= \{y|y \geq -b.x\} \cup \{x\}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
a \circ_1 x \star_1 b \circ_1 x &= (\{y_1|y_1 \geq |a|.|x|\} \cup \{x\}) \star_1 (\{y_2|y_2 \geq |b|.|x|\} \cup \{x\}) \\
&= (\emptyset \cup \{x\}) \star_1 (\{y_2|y_2 \geq |b|.|x|\} \cup \{x\}) \\
&= \cup\{x \star_1 y_2|y_2 \geq (-b).x\} \cup x \star_1 x \\
&\supseteq \{y|y \geq |x| + |y_2|, y_2 \geq -b.x\} \cup \{y_2|y_2 \geq -b.x\} \cup \{x\} \\
&\supseteq \{y_2|y_2 \geq -b.x\} \cup \{x\}.
\end{aligned}$$

Finally, for $a \in R_+$, $b \notin R_- \cup R_+$ and $x \in M_-$, we have

$$\begin{aligned}
(a \odot_1 b) \circ_1 x &= (\{c|c \geq |a||b|\} \cup \{a, b\}) \circ_1 x \\
&= (\emptyset \cup \{a, b\}) \circ_1 x \\
&= a \circ_1 x \cup b \circ_1 x \\
&= \{y|y \geq |a|.|x|\} \cup \{y|y \geq |b|.|x|\} \cup \{x\} \\
&= \{y|y \geq a.(-x)\} \cup \emptyset \cup \{x\} \\
&= \{y|y \geq -a.x\} \cup \{x\},
\end{aligned}$$

and

$$\begin{aligned}
a \circ_1 (b \circ_1 x) &= a \circ_1 (\{y|y \geq |b|.|x|\} \cup \{x\}) \\
&= a \circ_1 (\emptyset \cup \{x\}) \\
&= a \circ_1 x
\end{aligned}$$

$$\begin{aligned}
&= \{y|y \geq |a|.|x|\} \cup \{x\} \\
&= \{y|y \geq a.(-x)\} \cup \{x\} \\
&= \{y|y \geq -a.x\} \cup \{x\}.
\end{aligned}$$

The other cases in (i), (ii) and (iii) are proved, similarly. $\square$

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