

ON THE ROLE OF THE STOKES PROBLEM IN SECOND GRADE FLUID FLOW IN REGIONS WITH PERMEABLE INTERFACES

R. Maritz

*Department of Mathematical Sciences
University of South Africa
Science Campus, Florida
Unisa 0003
South Africa*

J.M.W. Munganga*

*Department of Mathematical Sciences
University of South Africa
Science Campus, Florida
Unisa 0003
South Africa
mungajmw@unisa.ac.za*

Abstract. The role of the Stokes problem in conducting a theory of existence of solutions, weak as well as classical, is under investigation in this research paper. We model the motion of incompressible flows of non-Newtonian fluids through permeable boundaries in terms of the Stokes equation. The equation of motion in the region $\Omega \in \mathbb{R}^3$ is coupled with the dynamic boundary condition through the permeable boundary Γ , in order to design a transport equation in a product space $\mathbf{Y}_m$. The existence of a unique solution for the Stokes problem is proved by using a special Helmholtz projection and employing the results from a paper of Agmon et al [1, Theorem 10.5 p.78]. The modelling is done for a special case where a "shear flow" is assumed.

Keywords: Stokes problem, Helmholtz projection, weak solution, non-Newtonian fluids, permeable boundary, dynamic boundary conditions, kinematic boundary condition.

1. Introduction

Existence of solutions to the general initial-boundary-value problem for an incompressible second grade fluid in a bounded domain, with no slip, has first been addressed by the pioneer A.P. Oskolkov [13], who proved the global (in time) existence and uniqueness of a *general solution* to a simplified version of the problem by formulating it in terms of

$$\mathbf{u} \equiv \mathbf{v} - \alpha \Delta \mathbf{v},$$

and applying the Faedo-Galerkin method. Cioranescu and Girault [5] followed a similar approach, but used the quantity $\mathbf{curl}(\mathbf{v} - \alpha_1 \Delta \mathbf{v})$ and applied the

*. Corresponding author

Faedo-Galerkin method to the full problem in order to obtain a unique solution, global in time if Ω is in $\mathbb{R}^2$ and local in time if Ω is in $\mathbb{R}^3$, for flows with $\alpha_1 \geq 0$ and $\alpha_1 + \alpha_2 = 0$.

Certain one-dimensional flows of a so-called power-law fluid of grade two, with shear-dependent viscosity, were studied by Man [9], where similar existence results were established.

In the proof of existence the Galdi-Grobbelaar-Sauer method [6] of reducing the equation of conservation of linear momentum to a transport equation and a Stokes-type equation is modified so that the dynamical boundary condition is combined with the transport equation. In [6], Galdi et.al. formulated the problem of existence of classical solutions as a problem of the existence of a fixed point of a certain mapping by considering the change of variable $\mathbf{v} \rightarrow \mathbf{u} = \mathbf{v} - \alpha_1 \Delta \mathbf{v}$. This was the first proof of existence and uniqueness of classical solutions. The restriction that $\alpha_1 > 0$ as well as α_1 sufficiently large was imposed for the global existence result to hold. No restriction on α_2 was imposed. Here, orthogonal projections on solenoidal fields were used to annihilate the pressure term in the equation of motion in the region Ω . They considered a fixed point problem in $\mathbf{w}$, based on the Helmholtz decomposition $\Delta \mathbf{v} = \mathbf{w} + \nabla \pi$.

Stokes flow in region with stretching boundaries was investigated by numerous researchers like Wang [17], Andersson [2] and Ariel et Al [3]. Wang [17] proved an exact solution for a flow due to a stretching boundary, where as Andersson and Ariel et Al [2, 3] gave an analysis of slip flow past stretching surfaces. Makinde et Al [8] investigated MHD steady flow in a channel with slip at permeable boundaries, and Sekhar et Al [4] discussed Stokes flow inside a porous spherical shell stress jump boundary condition.

For traditional boundary conditions, the classical Helmholtz decomposition is often useful. The Helmholtz decomposition results from orthogonal projections from the space of square integrable vector fields onto the subspaces consisting of gradients and solenoidal fields [16, Thm. 1.5, p. 16].

To answer the question: “What will cause the fluid to move through the permeable boundary?”, we consider the boundary as a continuum with its own physical properties. The motion of the boundary material is determined by normal forces exerted on it by the fluid in the container as well as forces due to the contact between the fluid in the container and the boundary surfaces. In section 2 we present the problem where the Frenet-Serret formulae, [11, 19, 18] is used to formulate the permeability through the boundary. In this section two additional assumptions were made concerning the shape of Ω . Section 3 is devoted to definitions of spaces, operators, norms and projections. In section 4 we formulate the auxiliary problems in terms of a Stokes-like expression for the fluid in the region Ω as well as the fluid flow through the boundary. The existence for the Stokes Problem is presented in section 5. The transport problem in terms of the Stokes expression is given in section 6 to be followed by the Conclusion in section 7.

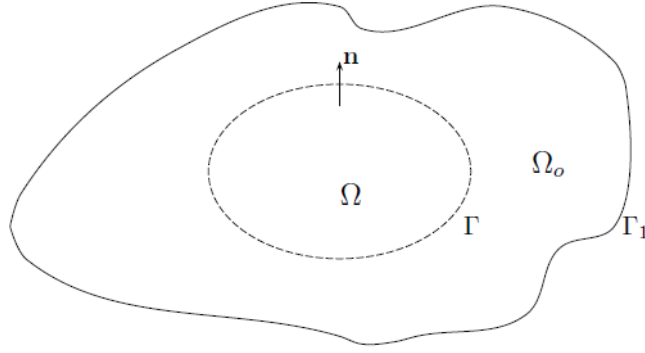


Figure 1: Profile for normal flow through the permeable wall Γ .

2. Presentation

In our case, we consider a canister (Ω), filled with incompressible non-Newtonian fluid, immersed in fluid of the same kind. It is assumed that the wall of the canister admits normal flow through it.

Figure 1 illustrates the situation where the curvature of the boundary Γ of Ω is non negative.

Permeability of the walls of the container is described by assuming that at the boundary Γ , the flow $\mathbf{v}$ has the direction of the normal (see Figure 1):

$$(1) \quad \gamma_o \mathbf{v}(x, t) = -\eta(x, t) \mathbf{n}(x).$$

The velocity component η is treated as an unknown and an evolution equation has to be found for it.

The Frenet-Serret [11, 19, 18], formulae are used to model the dynamic boundary condition, and we ask the reader to familiarize himself with this modelling from [10] or [11].

Modelling of the situation, has led to the equations which are written as follows [10]:

$$(2) \quad \left. \begin{aligned} &\rho^{1/2} \mathbf{v}_t + \rho^{1/2} (\mathbf{v} \cdot \nabla) \mathbf{v} + \rho^{-1/2} \nabla p = \rho^{-1/2} \nabla \cdot \mathbf{T}^* \quad \text{in } \Omega \\ &\nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega \\ &\gamma_o \mathbf{v} = -\eta_v \mathbf{n} \quad \text{on } \Gamma; \quad \mathbf{v} = 0 \quad \text{on } \Gamma_1 \\ &\gamma_o [\mathbf{A}(\mathbf{v})] = -2\eta_v \mathbf{M} \quad \text{on } \Gamma \\ &\rho^{1/2} \partial_t (\rho \eta_v - \alpha \Delta_s \eta_v) + \rho^{-1/2} \frac{\gamma_o p}{\delta} = \rho^{-1/2} \mu \Delta_s \eta_v \quad \text{on } \Gamma. \end{aligned} \right\}$$

where ρ denotes the density of the fluid and δ the "thickness of the boundary" and has a physical dimension of length. $\mathbf{T}^*$ is the part of the stress tensor which depends only on velocity, that is

$$\mathbf{T}^* = \mu \mathbf{A} + \alpha D_t \mathbf{A} + \frac{\alpha}{2} (\mathbf{A} \mathbf{W} - \mathbf{W} \mathbf{A}).$$

In what follows, we assume that the constants μ and α are strictly positive. Please see [11] equation (2.16) for an explanation of the tensor $\mathbf{M}$ which only depends on the geometry of the surface Γ . $\mathbf{M}$ is defined by

$$(3) \quad \mathbf{M} := [K(\mathbf{n} \otimes \mathbf{n}) - \kappa_1(\boldsymbol{\tau}_1 \otimes \boldsymbol{\tau}_1) - \kappa_2(\boldsymbol{\tau}_2 \otimes \boldsymbol{\tau}_2)]$$

We have chosen $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$ and $\mathbf{n}$ so that $\boldsymbol{\tau}_1 \wedge \boldsymbol{\tau}_2 = \mathbf{n}$. Apart from the smoothness of Γ we make two additional assumptions regarding the shape of Ω , namely that the curvatures κ_1 , κ_2 and K are constrained in the following way:

1. There exist constants g and G such that

$$(4) \quad 0 < g \leq K(x) \leq G \text{ for all } x \in \Gamma.$$

2. There exists a constant H such that

$$(5) \quad 0 \leq \kappa_1^2 + \kappa_2^2 \leq H^2 \text{ on } \Gamma.$$

When considering dynamic boundary conditions where interaction between the boundary and the fluid is taken into account, we need a modified projection theorem for the same results. The projection we construct is designed to keep the pair $\langle \rho^{\frac{1}{2}}\mathbf{v}, \rho^{1/2}\eta \rangle$ intact and at the same time eliminate the pressure term $\langle \rho^{-1/2}\nabla p, \rho^{-1/2}\gamma_o p \rangle$ [14]. The following: $\langle \mathbf{v}, \eta \rangle \in \mathbf{L}^2(\Omega) \times L^2(\Gamma)$ has the orthogonal (therefore, unique) decomposition

$$\begin{aligned} \langle \mathbf{v}, \eta \rangle &= \langle \rho^{1/2}\mathbf{w}, -\rho^{1/2}\mathbf{n} \cdot \gamma_o \mathbf{w} \rangle + \langle \rho^{-1/2}\nabla q, \rho^{-1/2}\gamma_o q \rangle \\ \nabla \cdot \mathbf{w} &= 0 \end{aligned}$$

with $\mathbf{w} \in \mathbf{H}^k(\Omega)$, $q \in H^{k+1}(\Omega)$, provided that $\mathbf{v} \in \mathbf{H}^k(\Omega)$ and $\eta \in H^{k-1/2}(\Gamma)$.

The orthogonal projection associated with the term $\langle \rho^{1/2}\mathbf{w}, -\rho^{1/2}\gamma_o \mathbf{w} \cdot \mathbf{n} \rangle$ will be denoted by P .

3. Some spaces, operators and definitions

All spaces of vector fields are denoted by boldface letters. We define the space $\mathbf{V}(\Omega)$ as

$$\mathbf{V}(\Omega) = \{ \mathbf{v} \in \mathbf{C}^\infty(\Omega) : \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega, \gamma_o \mathbf{v} = -\eta_v \mathbf{n} \text{ on } \Gamma,$$

$$\gamma_o[\mathbf{A}(\mathbf{v})] = -2\eta_v \mathbf{M} \text{ on } \Gamma, \gamma_o \mathbf{v} = 0 \text{ on } \Gamma_1 \}.$$

Note that $\mathbf{V}(\Omega)$ is a closed subspace of $\mathbf{H}^2(\Omega)$.

From now on, the notation $\gamma_o \mathbf{v}$ will be used to denote the restriction of $\mathbf{v}$ to Γ and Γ_1 .

We define the following Hilbert spaces:

$$\mathbf{X}_m = Cl_{\mathbf{H}^m(\Omega)}(\mathbf{V}(\Omega)); \quad \mathbf{X}_m^o = \{ \mathbf{v} \in \mathbf{X}_m : \gamma_o \mathbf{v} = \mathbf{0} \}, \quad m = 1, 2, 3, \dots$$

The inner products of the above spaces are the usual inner products defined for the Sobolev space $\mathbf{H}^m(\Omega)$.

The metric of a product space is defined in the usual way: For $\mathbf{X}$ and $\mathbf{Y}$ Hilbert spaces, the scalar product in $\mathbf{X} \times \mathbf{Y}$ is defined as

$$(\langle \mathbf{p}, \mathbf{q} \rangle, \langle \mathbf{r}, \mathbf{s} \rangle)_{X \times Y} = (\mathbf{p}, \mathbf{r})_X + (\mathbf{q}, \mathbf{s})_Y.$$

The corresponding norm is then defined by

$$\|\langle \mathbf{p}, \mathbf{q} \rangle\|_{X \times Y}^2 = \|\mathbf{p}\|_X^2 + \|\mathbf{q}\|_Y^2.$$

We define the canonical operators $\mathbf{C}_o$ and $\mathbf{C}_m$ in the following way:

Let

$$\mathbf{Y}_o = \mathbf{L}^2(\Omega) \times L^2(\Gamma), \quad \text{and} \quad \mathbf{Y}_m = \mathbf{X}_m \times H^{m-\frac{1}{2}}(\Gamma).$$

The canonical operator $\mathbf{C}_o : \mathbf{X}_1 \rightarrow \mathbf{Y}_o := \mathbf{L}^2(\Omega) \times L^2(\Gamma)$ is defined by

$$(6) \quad \mathbf{C}_o \mathbf{v} = \langle \rho^{1/2} \mathbf{v}, -\rho^{1/2} \mathbf{n} \cdot \gamma_o \mathbf{v} \rangle = \langle \rho^{1/2} \mathbf{v}, \rho^{1/2} \eta_v \rangle.$$

According to the definition (6) above, we have

$$(7) \quad \begin{aligned} (\mathbf{C}_o \mathbf{v}, \mathbf{C}_o \mathbf{w})_{Y_o} &:= \rho(\mathbf{v}, \mathbf{w})_{L^2(\Omega)} + (\rho^{1/2} \eta_v, \rho^{1/2} \eta_w)_{L^2(\Gamma)} \\ \|\mathbf{C}_o \mathbf{v}\|_{Y_o}^2 &:= \rho \|\mathbf{v}\|_{L^2(\Omega)}^2 + \rho \|\eta_v\|_{L^2(\Gamma)}^2. \end{aligned}$$

The canonical operator $\mathbf{C}_m : \mathbf{X}_m \rightarrow \mathbf{Y}_m$ ($m \geq 1$) is defined by

$$(8) \quad \mathbf{C}_m \mathbf{v} = \langle \rho^{1/2} \mathbf{v}, -\rho^{1/2} \mathbf{n} \cdot \gamma_o \mathbf{v} \rangle = \langle \rho^{1/2} \mathbf{v}, \rho^{1/2} \eta_v \rangle.$$

With (7) in mind, the norm of $\mathbf{C}_m \mathbf{v}$ in $\mathbf{Y}_m$ is given by

$$\|\mathbf{C}_m \mathbf{v}\|_{Y_m}^2 = \rho \|\mathbf{v}\|_{X_m}^2 + \rho \|\eta_v\|_{H^{m-1/2}(\Gamma)}^2.$$

In terms of the Trace Theorem, $\mathbf{C}_o$ and $\mathbf{C}_m$ are continuous linear mappings.

Let $I = [0, T]$. For Y , a Banach space with norm $\|\cdot\|_Y$, and for $1 \leq p < \infty$ let

$$L^p(I; Y) = \{v : t \rightarrow v(t) \in Y; t \in I, v \text{ measurable, and } \int_0^T \|v(t)\|_Y^p dt < \infty\}.$$

and denote by $W^{m,p}(I; Y)$ the space of functions such that the distributional time derivatives of order up to and including m are in $L^p(I; Y)$. For $p = \infty$, we denote by $L^\infty(I; Y)$ the Banach space of measurable and essentially bounded functions defined on I with values in Y . The norms in $W^{k,\infty}(I; \mathbf{H}^m(\Omega))$ and in $W^{k,\infty}(I; H^{m-1/2}(\Gamma))$, $k \geq 0$, are denoted by $\|\cdot\|_{k,m,T}$ and $\|\cdot\|_{k,m-1/2,T,\Gamma}$, respectively. For $k = 0$, we write $\|\cdot\|_{m,T}$ and $\|\cdot\|_{m-1/2,T,\Gamma}$.

The linear operator $\mathbf{B}_m : \mathbf{X}_{m+2} \rightarrow \mathbf{Y}_m$ is defined by

$$\mathbf{B}_m \mathbf{v} := \langle \rho^{-1/2}(\rho \mathbf{v} - \alpha \Delta \mathbf{v}), \delta \rho^{-1/2}(\rho \eta_v - \alpha \Delta_s \eta_v) \rangle.$$

Now for $\mathbf{v} \in \mathbf{X}_{m+2}$, we define

$$\begin{aligned} \mathbf{u}_1 &= \rho^{-1/2}(\rho\mathbf{v} - \alpha\Delta\mathbf{v}) \\ u_2 &= \delta\rho^{-1/2}(\rho\eta_v - \alpha\Delta_s\eta_v) \\ \mathbf{u} &= \langle \mathbf{u}_1, u_2 \rangle \in \mathbf{Y}_m, \end{aligned}$$

so that

$$\mathbf{B}_m\mathbf{v} = \mathbf{u}.$$

We note that since $\mathbf{X}_m$ is embedded in $\mathbf{X}_1$ and $\mathbf{Y}_m$ is embedded in $\mathbf{Y}_o$, $\mathbf{C}_m\mathbf{v} = \mathbf{C}_o\mathbf{v}$, $\mathbf{v} \in \mathbf{X}_m$ for $m \geq 1$.

In motivation of what is to follow, we obtain a formal energy identity by taking the scalar product in $\mathbf{Y}_o$ of $\mathbf{u} = \mathbf{B}_m\mathbf{v}$ with $\mathbf{C}_m\phi$. This identity holds for $\phi, \mathbf{v} \in \mathbf{X}_m$, $m \geq 1$:

$$\begin{aligned} (\mathbf{u}, \mathbf{C}_m\phi)_{Y_o} &= (\mathbf{B}_m\mathbf{v}, \mathbf{C}_o\phi)_{Y_o} \\ &= (\langle \rho^{-1/2}(\rho\mathbf{v} - \alpha\Delta\mathbf{v}), \delta\rho^{-1/2}(\rho\eta_v - \alpha\Delta_s\eta_v) \rangle, \langle \rho^{1/2}\phi, -\rho^{1/2}\gamma_o\phi \cdot \mathbf{n} \rangle)_{Y_o} \\ &= \rho(\mathbf{v}, \phi)_{L^2(\Omega)} - \alpha(\Delta\mathbf{v}, \phi)_{L^2(\Omega)} + \delta\rho(\eta_v, \eta_\phi)_{L^2(\Gamma)} - \delta\alpha(\Delta_s\eta_v, \eta_\phi)_{L^2(\Gamma)}. \end{aligned}$$

But with $\gamma_o\phi = -\eta_\phi\mathbf{n}$ and $\gamma_o\mathbf{A}(\mathbf{v})\mathbf{n} = -2K\eta_v\mathbf{n}$, we have

$$\begin{aligned} -(\Delta\mathbf{v}, \phi)_{L^2(\Omega)} &= -(\nabla \cdot \mathbf{A}(\mathbf{v}), \phi)_{L^2(\Omega)} \\ &= \frac{1}{2}(\mathbf{A}(\mathbf{v}), \mathbf{A}(\phi))_{L^2(\Omega)} - (\gamma_o\mathbf{A}(\mathbf{v})\mathbf{n}, \gamma_o\phi)_{L^2(\Gamma)} \\ &= \frac{1}{2}(\mathbf{A}(\mathbf{v}), \mathbf{A}(\phi))_{L^2(\Omega)} - (-2K\eta_v\mathbf{n}, -\eta_\phi\mathbf{n})_{L^2(\Gamma)} \\ &= \frac{1}{2}(\mathbf{A}(\mathbf{v}), \mathbf{A}(\phi))_{L^2(\Omega)} - 2(K\eta_v, \eta_\phi)_{L^2(\Gamma)}. \end{aligned}$$

Thus

$$\begin{aligned} (\mathbf{u}, \mathbf{C}_o\phi)_{Y_o} &= \rho(\mathbf{v}, \phi)_{L^2(\Omega)} + \delta\rho(\eta_v, \eta_\phi)_{L^2(\Gamma)} + \frac{\alpha}{2}(\mathbf{A}(\mathbf{v}), \mathbf{A}(\phi))_{L^2(\Omega)} \\ &\quad - 2\alpha(K\eta_v, \eta_\phi)_{L^2(\Gamma)} + \delta\alpha(\nabla_s\eta_v, \nabla_s\eta_\phi)_{L^2(\Gamma)} \end{aligned}$$

for any $\phi, \mathbf{v} \in \mathbf{X}_m$; $m \geq 1$.

For $\phi = \mathbf{v}$,

$$(\mathbf{u}, \mathbf{C}_o\mathbf{v})_{Y_o} = \rho\|\mathbf{v}\|_{L^2(\Omega)}^2 + \|(\delta\rho - 2\alpha K)\eta_v\|_{L^2(\Gamma)}^2 + \frac{\alpha}{2}\|\mathbf{A}(\mathbf{v})\|_{L^2(\Omega)}^2 + \alpha\|\nabla_s\eta_v\|_{L^2(\Gamma)}^2.$$

We define, accordingly, the bilinear form b_1 by

$$\begin{aligned} b_1(\mathbf{v}, \phi) &= \rho(\mathbf{v}, \phi)_{L^2(\Omega)} + ((\delta\rho - 2\alpha K)\gamma_o\mathbf{v}, \gamma_o\phi)_{L^2(\Gamma)} + \frac{\alpha}{2}(\mathbf{A}(\mathbf{v}), \mathbf{A}(\phi))_{L^2(\Omega)} \\ (9) \quad &\quad + \delta\alpha(\nabla_s\eta_v, \nabla_s\eta_\phi)_{L^2(\Gamma)} \quad \text{for } \mathbf{v}, \phi \in \mathbf{X}_1. \end{aligned}$$

We assume that the parameter $p_2 \in (0, 1/2)$, as in [10], and that $p_2 = \frac{\alpha K}{\delta \rho}$.

We define the following two operators: the operator $\mathbf{N}_v$ by

$$(10) \quad \mathbf{N}_v \mathbf{u} = \langle (\mathbf{v} \cdot \nabla) \mathbf{u}_1, 0 \rangle$$

with $\mathbf{u}_1 = \rho^{-1/2}(\rho \mathbf{v} - \alpha \Delta \mathbf{v})$,

and the pressure operator Π is defined by

$$(11) \quad \Pi p = \langle \rho^{-1/2} \nabla p, \rho^{-1/2} \gamma_o p \rangle.$$

According to [15] on Helmholtz projections, there exists a unique $q \in H^{k+1}(\Omega)$ and $\mathbf{w} \in \mathbf{H}^k(\Omega)$ such that

$$\langle \mathbf{v}, \eta_v \rangle = \mathbf{C}_o \mathbf{w} + \Pi q$$

if $\mathbf{v} \in \mathbf{H}^1(\Omega)$, $q \in H^{1/2}(\Gamma)$, i.e. $P \mathbf{C}_o \mathbf{w} = \mathbf{C}_o \mathbf{w}$, $P \Pi q = 0$.

4. The auxiliary problems

Let us write the equations of (2) in the form

$$(12) \quad \left. \begin{aligned} \partial_t [\rho^{1/2} \mathbf{v} - \alpha \rho^{-1/2} \Delta \mathbf{v}] + (\mathbf{v} \cdot \nabla) [\rho^{1/2} \mathbf{v} - \alpha \rho^{-1/2} \Delta \mathbf{v}] \\ + \rho^{-1/2} \nabla p = \mathbf{S}(\mathbf{v}) \quad \text{in } \Omega \times (0, T) \\ \mathbf{v} = 0 \quad \text{on } \Gamma_1 \times (0, T) \\ \gamma_0 \mathbf{v} = -\eta_v \mathbf{n} = -\eta \mathbf{n} \quad \text{on } \Gamma \times (0, T) \\ \gamma_o [\mathbf{A}(\mathbf{v})] = -2\eta \mathbf{M} \quad \text{on } \Gamma \times (0, T) \\ \delta \partial_t (\rho^{1/2} \eta_v - \rho^{-1/2} \alpha \Delta_s \eta_v) + \rho^{-1/2} \gamma_o p = s^*(\eta), \quad \text{on } \Gamma \end{aligned} \right\}$$

with

$$(13) \quad \left. \begin{aligned} \mathbf{S}(\mathbf{v}) &= \rho^{-1/2} \left[\frac{\alpha}{2} \nabla \cdot [\mathbf{A}(\mathbf{v}) \mathbf{W}(\mathbf{v}) - \mathbf{W}(\mathbf{v}) \mathbf{A}(\mathbf{v})] + \right. \\ &\quad \left. \alpha \nabla \cdot (\nabla \mathbf{v} \mathbf{A}(\mathbf{v})) + \mu \Delta \mathbf{v} \right] \\ s^*(\eta) &= \rho^{-1/2} \delta \mu \Delta_s \eta_v \end{aligned} \right\}$$

Note that we refer to $\gamma_o [\mathbf{A}(\mathbf{v}) \mathbf{n}] = -2K \eta_v \mathbf{n}$ as the kinematic boundary condition.

Under the substitution

$$(14) \quad \left. \begin{aligned} \rho^{1/2} \mathbf{v} - \alpha \rho^{-1/2} \Delta \mathbf{v} &= \mathbf{u}_1 \quad \text{in } \Omega \\ \delta (\rho^{1/2} \eta_v - \rho^{-1/2} \alpha \Delta_s \eta_v) &= u_2 \quad \text{on } \Gamma \\ \mathbf{v} &\in \mathbf{X}_{m+2}, \quad m \geq 1 \end{aligned} \right\}$$

the first and last equations of (12) become

$$(15) \quad \left. \begin{aligned} \partial_t \mathbf{u}_1 + (\mathbf{v} \cdot \nabla) \mathbf{u}_1 + \rho^{-1/2} \nabla p &= \mathbf{S}(\mathbf{v}) \quad \text{in } \Omega \\ \partial_t u_2 + \rho^{-1/2} \gamma_o p &= s^*(\eta) \quad \text{on } \Gamma. \end{aligned} \right\}$$

5. The stokes problem ($\mathcal{SP}$)

The problem (14) with $\mathbf{u}_1$ and u_2 given, leads to the Stokes Problem. Since $\mathbf{v}$ has to be divergence-free, we shall write the system (14) as a Stokes-like system. To this end, we use the Helmholtz-decomposition [15] in the following form:

Theorem 1. *Let $\mathbf{u}_1 \in \mathbf{H}^k(\Omega)$ and $u_2 \in H^{k-1/2}(\Gamma)$; $k \geq 1$. Then there exists $q \in H^{k+1}(\Omega)$ and $\mathbf{w} \in \mathbf{H}^k(\Omega)$, such that*

$$\left. \begin{aligned} \mathbf{u}_1 &= \rho^{1/2} \mathbf{w} + \rho^{-1/2} \nabla q \quad \text{in } \Omega \\ u_2 &= -\rho^{1/2} \mathbf{n} \cdot \gamma_o \mathbf{w} + \rho^{-1/2} \gamma_o q \quad \text{on } \Gamma \\ \nabla \cdot \mathbf{w} &= 0 \quad \text{in } \Omega. \end{aligned} \right\}$$

See [15] for proof of Theorem 1.

The operator $P : \langle \mathbf{u}_1, u_2 \rangle \in \mathbf{L}^2(\Omega) \times L^2(\Gamma) \rightarrow \langle \rho^{1/2} \mathbf{w}, -\rho^{1/2} \mathbf{n} \cdot \gamma_o \mathbf{w} \rangle \in \mathbf{L}^2(\Omega) \times L^2(\Gamma)$ is an orthogonal projection.

Remark 2. *Note that for $\mathbf{v} \in \mathbf{X}_m$; $m \geq 1$, $PC_m \mathbf{v} = \mathbf{C}_m \mathbf{v}$, (see (8)).*

Indeed, we have the orthogonal decomposition

$$\begin{aligned} \mathbf{u} = \langle \mathbf{u}_1, u_2 \rangle &= \langle \rho^{1/2} \mathbf{w}, -\rho^{1/2} \gamma_o \mathbf{n} \cdot \mathbf{w} \rangle + \langle \rho^{-1/2} \nabla q, \rho^{-1/2} \gamma_o q \rangle \\ &= \mathbf{C}_o \mathbf{w} + \Pi q. \end{aligned}$$

From this result, we can rewrite (14) in the form

$$(16) \quad \left. \begin{aligned} \rho^{1/2} \mathbf{v} - \alpha \rho^{-1/2} \Delta \mathbf{v} + \rho^{-1/2} \nabla p &= \rho^{1/2} \mathbf{w} \quad \text{in } \Omega \times (0, T) \\ \nabla \cdot \mathbf{v} &= 0 \quad \text{in } \Omega \times (0, T) \\ \gamma_o \mathbf{v} &= -\eta_v \mathbf{n} \quad \text{on } \Gamma \times (0, T); \quad \mathbf{v} = 0 \quad \text{on } \Gamma_1 \times (0, T) \\ \delta(\rho^{1/2} \eta_v - \rho^{-1/2} \alpha \Delta_s \eta_v) + \rho^{-1/2} \gamma_o p &= -\rho^{1/2} \mathbf{n} \cdot \gamma_o \mathbf{w} \quad \text{on } \Gamma \times (0, T) \\ \gamma_o \mathbf{A}(\mathbf{v}) \mathbf{n} &= -2\eta_v K \mathbf{n} \quad \text{on } \Gamma \times (0, T) \end{aligned} \right\}$$

(with $p = -q$), which is Stokes-like. We shall refer to (16) as the Stokes Problem ($\mathcal{SP}$).

Note that the kinematical boundary condition is incorporated in the $\mathcal{SP}$.

According to the definition of the operators $\mathbf{C}_m$, $\mathbf{B}_m$ and Π , (16) may also be written in the form

$$\mathbf{B}_m \mathbf{v} + \Pi p = \mathbf{C}_m \mathbf{w}, \quad \mathbf{v} \in \mathbf{X}_{m+2}, \quad m \geq 1.$$

Led by the discussion above, we show the existence and uniqueness for the $\mathcal{SP}$:

Theorem 3. *Let Γ be of class C^∞ , and suppose that $p_2 < 1/2$. Let $m \geq 0$ and let there be given $\mathbf{w} \in \mathbf{W}^{k,\infty}(I; \mathbf{H}^m(\Omega))$ and $\gamma_o \mathbf{w} \cdot \mathbf{n} \in W^{k,\infty}(I; H^{m+1/2}(\Gamma))$. Then the problem (16) has a solution for which $\mathbf{v}$ is unique and ∇p is unique. $\mathbf{v} \in \mathbf{W}^{k,\infty}(I; \mathbf{X}_{m+2})$, $\eta_v \in W^{k,\infty}(I; H^{m+3/2}(\Gamma))$ and $p \in H^{m+1}(\Omega)$.*

Proof. Consider the system (16) of equations

$$(17) \quad \left. \begin{aligned} \rho^{1/2} \mathbf{v} - \alpha \rho^{-1/2} \Delta \mathbf{v} + \rho^{-1/2} \nabla p &= \rho^{1/2} \mathbf{w} & \text{in } \Omega \times (0, T) \\ \delta \rho^{-1/2} (\rho \eta_v - \alpha \Delta_s \eta_v) + \rho^{-1/2} \gamma_o p &= -\rho^{1/2} \mathbf{n} \cdot \gamma_o \mathbf{w} & \text{on } \Gamma \times (0, T) \\ \gamma_o \mathbf{v} &= -\eta_v \mathbf{n} & \text{on } \Gamma \times (0, T) \\ \gamma_o [\mathbf{A}(\mathbf{v}) \mathbf{n}] &= -2\eta_v K \mathbf{n} & \text{on } \Gamma \times (0, T). \end{aligned} \right\}$$

We prove first that (17) has a unique weak solution: We use the projection P in [15] to eliminate p . If we formally take the $\mathbf{L}^2(\Omega)$ -inner product of the first two equations of (17) with $\mathbf{C}_o \phi$, we obtain the weak formulation of the problem

$$(18) \quad b_1(\mathbf{v}, \phi) = \rho(\mathbf{w}, \phi) + \rho(\mathbf{n} \cdot \gamma_o \mathbf{w}, \eta_\phi)_{L^2(\Gamma)},$$

with b_1 defined in (9).

Since all the spaces $\mathbf{X}_m$ have a common dense subspace $\mathbf{V}(\Omega)$, we define a weak solution of (16) if $\mathbf{v} \in \mathbf{X}_1$ such that (18) holds for all $\phi \in \mathbf{X}_1$. The bilinear form b_1 , being equivalent to the metric in $\mathbf{H}^1(\Omega)$, is clearly positive definite on $\mathbf{X}_1$, and therefore, a unique weak solution exists. $\square$

Regularity results for the $\mathcal{SP}$ need to be investigated. A classical result is that the solution $\mathbf{u} \in \mathbf{H}_o^1(\Omega)$ for the Dirichlet problem $-\Delta \mathbf{u} + \mathbf{u} = \mathbf{f}$ belongs to $\mathbf{H}^{m+2}(\Omega)$ whenever $\mathbf{f} \in \mathbf{H}^m(\Omega)$ (with Ω sufficiently smooth). The question is whether similar results exist for the $\mathcal{SP}$.

A priori estimates for the solution $\mathbf{v}$ of (17) is obtained in a proposition with the use of Theorem 10.5 of [1] in the following way:

Proposition 4. *Let Ω be an open bounded set in $\mathbf{R}^3$, with boundary $\Gamma \cup \Gamma_1$ of class C^{m+3} , m a non-negative integer. Suppose that $\mathbf{v} \in \mathbf{X}_2$ is a weak solution of the Stokes-like problem (16). If $\mathbf{w} \in \mathbf{H}^m(\Omega)$ and $\gamma_o \mathbf{w} \cdot \mathbf{n} \in H^{m-1/2}(\Gamma)$, then $\mathbf{v} \in \mathbf{H}^{m+2}(\Omega)$, and there exists a $p \in H^{m+1}(\Omega)$ and a constant $C_o(\alpha, \rho, m, \Omega)$ such that*

$$(19) \quad \|\mathbf{v}\|_{m+2} + \|p\|_{m+1} \leq C_o \{ \|\mathbf{w}\|_m + \|\gamma_o \mathbf{w} \cdot \mathbf{n}\|_{H^{m-1/2}(\Gamma)} \}.$$

Proof. We write equation (16), after multiplying both sides with $\rho^{1/2}$, in the form

$$\begin{array}{ccccccc} (\Delta - \frac{\rho}{\alpha})v_1 & \cdot & \cdot & -\frac{1}{\alpha}p_{,1} & = & -\frac{\rho^{1/2}}{\alpha}(y_1)_1 \\ \cdot & (\Delta - \frac{\rho}{\alpha})v_2 & \cdot & -\frac{1}{\alpha}p_{,2} & = & -\frac{\rho^{1/2}}{\alpha}(y_1)_2 \\ \cdot & \cdot & (\Delta - \frac{\rho}{\alpha})v_3 & -\frac{1}{\alpha}p_{,3} & = & -\frac{\rho^{1/2}}{\alpha}(y_1)_3 \\ v_{1,1} & +v_{2,2} & +v_{3,3} & & = & 0 \end{array}$$

Let $v_4 = -\frac{\rho}{\alpha}$ and $\mathbf{f} = (-\frac{\rho^{1/2}}{\alpha}(y_1)_1, -\frac{\rho^{1/2}}{\alpha}(y_1)_2, -\frac{\rho^{1/2}}{\alpha}(y_1)_3, 0)$, to become

$$\sum_{j=1}^4 \ell_{ij}(\partial)v_j(\mathbf{x}) = f_i(\mathbf{x}) \quad \text{in } \Omega, \quad i = 1, 2, 3, 4$$

where $\partial = (\partial_1, \partial_2, \partial_3)$ and the matrix $[\ell_{ij}(\xi)]$, $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3$, is given by

$$\begin{aligned}\ell_{ij}(\xi) &= |\xi|^2 \delta_{ij} - \frac{\rho}{\alpha}, \quad |\xi|^2 = \xi_1^2 + \xi_2^2 + \xi_3^2, \quad i, j = 1, 2, 3, \\ \ell_{4j}(\xi) &= -\ell_{j4}(\xi) = \xi_j, \quad j = 1, 2, 3, \\ \ell_{4,4}(\xi) &= 0.\end{aligned}$$

In accordance with the proof of Proposition 2.2 [16, p.34], we define two systems of weights by $s_1 = s_2 = s_3 = 0$, $s_4 = -1$, and $t_1 = t_2 = t_3 = 2$, $t_4 = 1$. Then $s_i \leq 0$ and $\text{degree}(\ell_{ij}(\xi)) \leq s_i + t_j$, as required by [1, p.38]. The matrix $[\ell'_{ij}(\xi)]$, where $\ell'_{ij}(\xi)$ consists of the terms in $\ell_{ij}(\xi)$ that are of order $s_i + t_j$ in ξ , is identical to the corresponding matrix in [16]:

$$[\ell'_{ij}(\xi)] = \begin{bmatrix} |\xi|^2 & 0 & 0 & -\xi_1 \\ 0 & |\xi|^2 & 0 & -\xi_2 \\ 0 & 0 & |\xi|^2 & -\xi_3 \\ \xi_1 & \xi_2 & \xi_3 & 0 \end{bmatrix}$$

It is easily shown that $\mathcal{L}(\xi) \equiv \det[\ell'_{ij}(\xi)] = |\xi|^6$, so that $\mathcal{L}(\xi) \neq 0$ for non-zero real ξ , that is (16) is *elliptic*. Moreover, the *supplementary condition on $\mathcal{L}$* is satisfied: $\mathcal{L}(\xi)$ is of even degree 6, and for every pair of linearly independent real vectors ξ, ξ' , in particular for each point $\mathbf{x}$ on Γ , ξ is a tangent and ξ' is a normal at $\mathbf{x}$; the polynomial $\mathcal{L}(\xi + \tau\xi')$ in τ has exactly 3 roots with positive imaginary part, namely $\tau^+(\xi, \xi') = i|\xi|/|\xi'|$:

$$\begin{aligned}\mathcal{L}(\xi + \tau\xi') &= [(\xi + \tau\xi') \cdot (\xi + \tau\xi')]^3 \\ &= (|\xi|^2 + |\xi'|^2\tau^2)^3 \\ &= |\xi|^6(\tau - i|\xi|/|\xi'|)(\tau + i|\xi|/|\xi'|)(\xi'^6).\end{aligned}$$

Now, concerning the boundary conditions, we set

$$\begin{aligned}\gamma_o v_{1,3} + \gamma_o v_{3,1} &= 0 \\ \gamma_o v_{2,3} + \gamma_o v_{3,2} &= 0 \\ \left(\frac{\rho}{\alpha} - \Delta_s\right)v_3 + \gamma_o p &= y_2.\end{aligned}$$

These boundary conditions can be expressed as

$$[\mathcal{B}_{hj}(\mathbf{x}, \xi)] = \begin{bmatrix} \xi_3 & 0 & \xi_1 & 0 \\ 0 & \xi_3 & \xi_2 & 0 \\ 0 & 0 & \frac{\rho}{\alpha} - \xi_3^2 & 1 \end{bmatrix}$$

in other words

$$\begin{aligned}\mathcal{B}_{hj} &= \xi_3 \delta_{hj} \quad \text{for } h = 1, 2, 3 \quad \text{and } j = 1, 2 \\ \mathcal{B}_{h3} &= \xi_h, \quad \text{for } h = 1, 2, \\ \mathcal{B}_{h4} &= 0, \quad \text{for } h = 1, 2 \\ \mathcal{B}_{34} &= 1 \\ \mathcal{B}_{33} &= \frac{\rho}{\alpha} - \xi_3^2.\end{aligned}$$

Take $r_1 = r_2 = -1, r_3 = 0$ and $t_1 = t_2 = t_3 = 2$, then $\text{degree}(\mathcal{B}_{hj}) \leq r_h + t_j$ and $[\mathcal{B}'_{hj}] = [\mathcal{B}_{hj}]$, where $\mathcal{B}'_{hj}(\mathbf{x}, \xi)$ consists of the terms in $\mathcal{B}_{hj}(\mathbf{x}, \xi)$ that are of order $r_h + t_j$ in ξ .

$$[\mathcal{B}'_{hj}(\mathbf{x}, \xi)] = \begin{bmatrix} \xi_3 & 0 & \xi_1 & 0 \\ 0 & \xi_3 & \xi_2 & 0 \\ 0 & 0 & \xi_3^2 & 0 \end{bmatrix}$$

Now it remains to check the *complementing boundary condition*: For an arbitrary $\mathbf{x} \in \Gamma$, let $\mathbf{n}$ denote the outward unit normal vector at $\mathbf{x}$, let ξ be any non-zero real tangent vector to Γ at $\mathbf{x}$ and define $\mathcal{L}^{jk}(\cdot) \equiv \ell'_{jk}(\cdot)$, $j, k = 1, 2, 3, 4$. Then $[\mathcal{B}'_{hj}(\mathbf{x}, \xi + \tau \mathbf{n}) \mathcal{L}^{jk}(\xi + \tau \mathbf{n})] =$

$$\begin{bmatrix} \tau[|\xi|^2 + \tau^2] & 0 & \xi_1[|\xi|^2 + \tau^2] & -2\xi_1\tau \\ 0 & \tau[|\xi|^2 + \tau^2] & \xi_2[|\xi|^2 + \tau^2] & -2\xi_2\tau \\ 0 & 0 & \tau^2[|\xi|^2 + \tau^2] & -\tau^3 \end{bmatrix}.$$

The rows of the latter matrix are required to be linearly independent modulo $\mathcal{M}^+$. Let $\tau^+ = \tau^+(\xi, \mathbf{n}) = i|\xi|$ and set $\mathcal{M}^+ = (\tau - \tau^+)^3$, and suppose that $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ is a constant vector with the property that, as polynomials in τ ,

$$\sum_{h=1}^3 \mathcal{C}_h \left(\sum_{j=1}^4 \mathcal{B}'_{hj} \mathcal{L}^{jk} \right) \equiv 0 \pmod{\mathcal{M}^+}, \quad k = 1, 2, 3,$$

that is

$$\begin{aligned}\mathcal{C}_1 \tau (|\xi|^2 + \tau^2) &= 0 \\ \mathcal{C}_2 \tau (|\xi|^2 + \tau^2) &= 0 \\ \mathcal{C}_1 \xi_1 (|\xi|^2 + \tau^2) + \mathcal{C}_2 \xi_2 (|\xi|^2 + \tau^2) + \mathcal{C}_3 \tau^2 (|\xi|^2 + \tau^2) &= 0.\end{aligned}$$

Now it is easy to verify that $\mathcal{C} = \mathbf{0}$ and that the *complementing condition* holds.

We then apply Theorem 10.5, on [1, p.78] in order to get the final result. (For a similar application, see [7].) $\square$

Proposition 5. *Let $\Gamma \cup \Gamma_1$ be of class C^∞ . Let $k \geq 0$ and let $\mathbf{w} \in \mathbf{W}^{k,\infty}(I; \mathbf{H}^m(\Omega))$ and $\gamma_o \mathbf{w} \cdot \mathbf{n} \in W^{k,\infty}(I; H^{m-1/2}(\Gamma))$ be given. Then there exists a unique vector field $\mathbf{v} \in \mathbf{W}^{k,\infty}(I; \mathbf{X}_{m+2})$ satisfying (17). Moreover, there is a constant $C_1 = C_{1(\Omega, m, \rho, \alpha)}$ such that*

$$\|\mathbf{v}\|_{k, m+2, T} \leq C_1 \{ \|\mathbf{w}\|_{k, m, T} + \|\gamma_o \mathbf{w} \cdot \mathbf{n}\|_{W^{k,\infty}(I; H^{m-1/2}(\Gamma))} \}.$$

Proof. By Proposition 4, it is only necessary to prove that $\mathbf{v} \in \mathbf{X}_2$. This was proven for the general $\mathbf{L}^p$ case by Miyakawa in [12].

This concludes the proof of Theorem 3. $\square$

Let the mapping $f : \mathbf{Y}_m \rightarrow \mathbf{X}_{m+2}$ be defined as

$$f : \mathbf{u} \rightarrow \mathbf{w} \rightarrow \mathbf{v}^* = f(\mathbf{u}),$$

where $\mathbf{v}^* \in \mathbf{X}_{m+2}$ is the unique solution of (16) for a given $\mathbf{u} \in \mathbf{Y}_m$.

Theorem 6. *f is a bounded linear operator.*

Proof. The result follows from (19) and the fact that $\mathbf{w}$ is obtained from $\mathbf{u}$ by the orthogonal projection P . $\square$

6. Conclusion

We modelled the flow of a second grade fluid in the region Ω which is emerged in fluid of the same kind. The interface of Ω is permeable and fluid flow is modelled through the boundary in the direction of the outer normal. The ultimate goal would be to prove the existence of a weak and classical solution. In this article, the $\mathcal{TP}$ was modelled in terms of the $\mathcal{SP}$, for which the existence of a unique solution was proved. This article lays the foundation for a existence proof.

The existence of a weak and/or classical solution of the $\mathcal{TP}$ can be proven and should roughly be based on the following algorithm:

1. With $\mathbf{v}$ given, at least one solution $\mathbf{u} = \langle \mathbf{u}_1, u_2 \rangle$ of (15) under the initial condition $\mathbf{u}(x, 0) := \mathbf{u}_0(x)$ must be found.
2. With each $\mathbf{u} = \langle \mathbf{u}_1, u_2 \rangle$ in hand, a unique solution $\mathbf{v}^*$ of (14) must be found.
3. It must be proved that the composite mapping

$$\Phi : \mathbf{v} \xrightarrow{g} \mathbf{u} \xrightarrow{f} \mathbf{v}^*$$

has a fixed point, which is a solution of (12).

The rough algorithm described above, should be developed in detail, after some refinement of the equations (14) and (15).

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