

SOLITON SOLUTIONS FOR NON-LINEAR DISPERSIVE WAVE EQUATIONS WITH VARIABLE-COEFFICIENTS

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Abstract. In this paper, we study the solitary wave solution for the variable-coefficient non-linear dispersive wave equation. We develop a simplified bilinear method to construct the multi-soliton solutions for such an equation. We prove that the proposed scheme is effective and easy to implement. Finally, effects of the inhomogeneities of media on the soliton behavior are discussed with the aid of the characteristic curve method and graphical analysis.

Keywords: dispersive wave equations, multiple soliton solution, hirota bilinear method.

Introduction

Nowadays, attention has been focused on soliton equations due to its scientific applications. There has been growing interest in studying the variable-coefficient nonlinear evolution equations (NLEEs), which are often considered to be more realistic than their constant-coefficient counterparts in modeling a variety of complex nonlinear phenomena under different physical backgrounds. Since those variable-coefficient NLEEs are of practical importance, it is meaningful to construct various exact analytic solutions, including the soliton solutions and study the integrable properties. However, the inverse scattering transform [1], the Hirota direct method [6, 7], Bäcklund transformation method [12], and the Darboux transformations method [10] are applied for the soliton solutions to NLEEs.

For describing the propagation of solitonic waves in inhomogeneous media, the variable-coefficient non-linear dispersive wave equation

$$(0.1) \quad u_t + \alpha(t)u_x + \beta(t)uu_x + \gamma(t)u_{xxx} = 0$$

where $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are time-dependent functions, has been derived from many physical applications in plasma physics, fluid dynamics and other fields [9, 11, 13].

Obviously, equation (0.1) contains quite a number of variable-coefficient models arising from various branches of physics. A currently important example of which can be seen in [3]. The forced variable-coefficient KdV equation,

which has the relevant physical applications not only in the internal solitary waves dynamics [3], but also in the arterial mechanics [11, 13]. For instance, in [11], the forced variable-coefficient KdV equation has been derived by means of the reductive perturbation as studying the propagation of the weakly nonlinear waves in a composite medium, in which the arteries and blood are treated as the prestressed thin walled elastic tubes with a stenosis and the Newtonian fluid, respectively.

In this paper, by virtue of a newly developed simplified bilinear method, we would like to study equation (0.1) and devote ourselves to its soliton solutions and some important properties under nonlinearly dependent constraints among the variable coefficients.

The structure of the present paper is as follows. In section 1, with certain coefficient conditions, we will construct the analytic one- and two-soliton solutions for equation (0.1) by employing the variable-coefficient simplified bilinear method [2, 8] and Hirota method [6, 7]. Moreover, new singular soliton solutions will be constructed. Section 2 will analyze the effects of the coefficient functions on the stabilities and propagation characteristics of the solitonic waves through figure drawing. Section 3 will offer the conclusion.

1. Solitary wave solutions for equation (0.1)

It is well known that the Hirota method is an important analytic tool for dealing with NLEEs and relevant soliton problems [7]. Through bilinearization for a given NLEE, one can not only construct its multi-soliton solutions, but also derive the bilinear auto-Bäcklund transformation, nonlinear superposition formula, Lax pair and some other remarkable properties.

In this method, a given nonlinear equation is rewritten into the bilinear equation by the dependent variable transformation to enable us deriving the auxiliary function. But it is not easy for us to find such a transformation for many equations. However, Hereman et al. [4, 5], introduced a simplified algorithm to derive the auxiliary functions without using the bilinear forms; instead it assumes that the N -soliton solutions can be expressed as polynomials of exponential functions. The authors in [2] modified Hereman's simplified method to be able to solve variable-coefficient NLEEs. In the rest of this section, we will employ the method in [2, 8] to obtain multiple soliton solutions for equation (0.1).

1.1 Analytic one-soliton solution

Taking as our ansatz the plane wave solution

$$(1.1) \quad u(x, t) = e^{k_i x - \omega_i(t)}.$$

Substituting (1.1) into the linear terms of (0.1), we get the following non-linear dispersion relation

$$(1.2) \quad \omega_i(t) = \phi_i(t) = k_i^2 \int \gamma(t) dt.$$

and as a result we obtain the following phase variables

$$(1.3) \quad \theta_i = k_i x - \int [k_i^3 \gamma(t) + k_i \alpha(t)] dt, \quad i = 1, 2, \dots$$

Our practice with the perturbation method suggests that making the substitution

$$(1.4) \quad u(x, t) = R (\ln f)_{xx}$$

in (0.1) to obtain a bilinear form may be a fruitful undertaking.

To find the one-soliton solution, take

$$(1.5) \quad f(x, t) = 1 + \varepsilon e^{k_1 x - \omega_1(t)},$$

where $\varepsilon = \pm 1$. Substituting (1.4) and (??) into equation (0.1) and solving for R we find

$$R = 12 \frac{\gamma(t)}{\beta(t)}$$

with the constraint on coefficients

$$\gamma(t) = \lambda \beta(t).$$

Here $\lambda \neq 0$ is an arbitrary constant. That is to say, the one-soliton solution of equation (0.1) under condition (1.5) is derived as

$$(1.6) \quad u(x, t) = 12\lambda\varepsilon k_1^2 \frac{e^{\theta_1}}{(1 + \varepsilon e^{\theta_1})^2}.$$

Upon using $\varepsilon = 1$ we obtain the one-soliton solution

$$u(x, t) = 3\lambda k_1^2 \sec^2 h^2 \left(\frac{\theta_1}{2} \right),$$

which can be seen as the bell-shaped profile, while $\varepsilon = -1$ leads to the singular one-soliton solution

$$u(x, t) = 3\lambda k_1^2 \csc^2 h^2 \left(\frac{\theta_1}{2} \right),$$

which gives a profile of an explosive pulse.

1.2 Multiple soliton solutions

To generate the two-soliton solution, we expand the function f as

$$(1.7) \quad f(x, t) = 1 + \varepsilon_1 e^{\theta_1} + \varepsilon_2 e^{\theta_2} + \varepsilon_1 \varepsilon_2 a_{12} e^{\theta_1 + \theta_2},$$

where $\varepsilon = \pm 1$ and θ_i are given by (1.4).

Upon substituting (1.7) and (1.4), into equation (0.1) and solving for the phase shift we obtain

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$$

and hence we can generalize for other phase shifts by

$$(1.8) \quad a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad 1 \leq i < j \leq 3.$$

In this case, the two-soliton solution of equation (0.1) subject to condition (1.5) and using $\varepsilon_1 = \varepsilon_2 = 1$ is found to be

$$(1.9) \quad u(x, t) = 12\gamma \frac{F(\theta_1, \theta_2)}{G(\theta_1, \theta_2)}$$

where

$$\begin{aligned} F(\theta_1, \theta_2) &= k_1^2 e^{\theta_1} \left(1 + a_{12} e^{2\theta_2}\right) + k_2^2 e^{\theta_2} \left(1 + a_{12} e^{2\theta_1}\right) \\ &\quad + \left[(k_1 - k_2)^2 + a_{12} (k_1 + k_2)^2\right] e^{\theta_1 + \theta_2}, \\ G(\theta_1, \theta_2) &= \left(1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2}\right)^2, \end{aligned}$$

where θ_1 and θ_2 are defined in (1.3). It is noted that solution (1.9) represents the interaction of two solitonic waves. On the other hand, using $\varepsilon_1 = \varepsilon_2 = -1$ in (1.7), gives the singular two-soliton solution

$$u(x, t) = -12\gamma \frac{H(\theta_1, \theta_2)}{I(\theta_1, \theta_2)},$$

where

$$\begin{aligned} H(\theta_1, \theta_2) &= k_1^2 e^{\theta_1} \left(1 + a_{12} e^{2\theta_2}\right) + k_2^2 e^{\theta_2} \left(1 + a_{12} e^{2\theta_1}\right) \\ &\quad - \left[(k_1 - k_2)^2 + a_{12} (k_1 + k_2)^2\right] e^{\theta_1 + \theta_2} \end{aligned}$$

and

$$I(\theta_1, \theta_2) = \left(1 - e^{\theta_1} - e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2}\right)^2.$$

For the three kink solutions, we set the auxiliary function

$$(1.10) \quad f(x, t) = 1 + \varepsilon_1 e^{\theta_1} + \varepsilon_2 e^{\theta_2} + \varepsilon_3 e^{\theta_3} + \varepsilon_1 \varepsilon_2 a_{12} e^{\theta_1 + \theta_2} + \varepsilon_1 \varepsilon_3 a_{13} e^{\theta_1 + \theta_3} \\ + \varepsilon_2 \varepsilon_3 a_{23} e^{\theta_2 + \theta_3} + \varepsilon_1 \varepsilon_2 \varepsilon_3 a_{123} e^{\theta_1 + \theta_2 + \theta_3},$$

where $\varepsilon_i = \pm 1$, $i = 1, 2, 3$, and θ_i are given by (1.3). Substituting (1.10) and (1.4) into equation (0.1), we find that

$$a_{123} = a_{12} a_{13} a_{23}.$$

This shows that equation (0.1) has N -soliton solution which can be obtained for finite N , where $N > 1$. Proceeding as before, the three-soliton solutions and the singular three-soliton solutions can be obtained using calculations with Mathematica.

1.3 Propagation characteristics of the soliton solutions

From the one-soliton solution (1.6), it is obvious that the amplitude of the front is $3\lambda k_i^2$ which keeps invariant during the propagation. Following the characteristic-line method [14], the characteristic curve for each solitary solution can be defined by

$$x - \int [k_i^2 \gamma(t) + \alpha(t)] dt = 0, \quad i = 1, 2, \dots,$$

which can be derived from relations (1.2) and (1.3). Correspondingly, the velocity of the front at time t can be expressed as

$$(1.11) \quad \nu_x = k_i^2 \gamma(t) + \alpha(t).$$

From the explicit expression (1.11), we can see that the functions $\alpha(t)$ and $\gamma(t)$ affect the propagation velocity of the solitonic fronts significantly. For example, taking $\gamma(t) = \sin t$ and $\alpha(t) = 0$, solution (1.6) becomes

$$(1.12) \quad u(x, t) = 6 \sec h^2 \left(\frac{x + \cos t}{2} \right),$$

which has the non-zero boundary value 0 as $t, x \rightarrow \infty$. Its velocity along the x direction is given by

$$\nu_x = \sin t$$

and during one periodic time from 0 to 2π , the solitonic wave moves along the positive x direction when $t \in (0, \pi)$ and along the negative x direction when $t \in (\pi, 2\pi)$. Moreover, the wave accelerates in the intervals $(0, \pi/2) \cup (3\pi/2, 2\pi)$ while decelerates in the interval $(\pi/2, 3\pi/2)$.

The characteristic curve in Fig. 1 can be expressed by

$$x + \cos t = 0.$$

The profile figure for solution (1.6) in an inhomogeneous medium in the $x - t$ plane with: $\alpha(t) = 0$, $\gamma(t) = \sin t$, $k_1 = 1$. As a result, the front reveals the snake-like type propagation trajectory with the unalterable amplitude but continuously changeable velocity.

In Fig 2, the characteristic curve is given in the form

$$x + 2 \cos t - \frac{t^3}{3} = 0.$$

When t approaches zero, i.e. $\cos t \gg t^3$, the trajectory is snake-like type with periodic oscillation. Otherwise, when t is far from the origin, the trajectory is t^3 -like type one. From which we conclude that, besides the periodic oscillation of the solitons in the local region, the large scale propagation trajectories for such a structure are the t^3 -typed curves. Likewise, if the variable coefficients are taken as the other forms, the corresponding characteristic curves will present different characters.

The profile figure for solution (1.6) in an inhomogeneous medium in the $x - t$ plane with: $\alpha(t) = t^2$, $\gamma(t) = \sin t$, $\lambda = 2$, $k_1 = 1$.

2. Conclusions

In the present work, we have studied the variable-coefficient non-linear dispersive wave equation (0.1) from the soliton analysis point of view. Of physical interest, we have specially presented the multiple soliton solutions and multiple singular soliton solutions for equation (0.1). Through the figures for some sample solutions, we have discussed the structures of solitonic waves with inhomogeneities of media and nonuniformities of boundaries effects.

Further work for this class of equations is possible to be performed, for example, one can extend the present results to the corresponding three-dimensional variable-coefficient evolution equations.

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