

A DISSIPATIVE HYPERBOLIC SYSTEMS APPROACH TO IMAGE RESTORATION

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Abstract. We present here a new dissipative hyperbolic systems to image restoration. The existence of global dissipative solutions of this system under the Dirichlet boundary conditions and initial condition is shown. To this end, an experimental results are provided to show the efficiency of this kind of model.

Keywords: image restoration, wave equation, dissipative solution.

1. Introduction

In this paper, we consider the following dissipative hyperbolic systems

$$(1.1) \quad \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \operatorname{div}(g(|\nabla G_\rho * u|)\nabla u) = 0,$$

$$(1.2) \quad \frac{\partial^2 v}{\partial t^2} + \frac{\partial v}{\partial t} - \lambda \operatorname{div}(\nabla v) - (1 - \lambda)(|\nabla u| - v) = 0,$$

subject to the initial condition and Dirichlet boundary conditions

$$(1.3) \quad u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = 0, \quad \frac{\partial v}{\partial t}(x, 0) = 0 \quad x \in \Omega$$

$$(1.4) \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \quad \frac{\partial v}{\partial n}|_{\partial\Omega} = 0 \quad 0 < t < T,$$

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where Ω is a bounded domain of $\mathbf{R}^n$ with appropriately smooth boundary, n is the unit outer normal to Ω , $T > 0$ and $\lambda > 0$. The nonlinear term $g(s)$,

$$(1.5) \quad g(s) = \frac{1}{1 + (\frac{s}{K})^2}, \quad \text{or} \quad g(s) = |s|^{-1}$$

with $K > 0$.

Here $u = u(t, x)$ denotes the original image describing a real scene, $u_0(x)$ denotes the observed image, $G_\rho(x)$ is the Gaussian kernel

$$G_\rho(x) = \frac{1}{(4\pi\rho)^{\frac{M}{2}}} e^{-\frac{|x|^2}{4\rho}},$$

and

$$|\nabla G_\rho * u| = \left[\sum_{i=1}^M \left(\frac{\partial G_\rho}{\partial x_i} * u \right)^2 \right]^{\frac{1}{2}}.$$

Partial differential equations based image restoration is a powerful method to deal with the trade-off between noise removal and edge preservation. This method is now a well researched area within the image processing community. One of powerful models is the following nonlinear parabolic model with variable coefficient

$$\frac{\partial u}{\partial t} - \operatorname{div}(c(x, y) \nabla u) = 0,$$

where the degree of denoising and preservation of singularities can be determined by changing $c(x, y)$. There exist other type parabolic equation, such as anisotropic diffusion models [1, 18], complex diffusion models [13], fourth order equation models [7, 20], total variation models [9, 14, 24]. In Perona-Malik [18] the denoising capabilities of the linear diffusion can be better, let $c(x, y) = g(|\nabla u|)$ and initial data $u(0) = u_0$. Here the diffusion smooth function $g : [0, \infty) \rightarrow [0, \infty)$ is important in controlling the smoothing and even enhancement of edges. They mainly considered the following two diffusion functions

$$g(s) = \frac{1}{1 + (\frac{s}{K})^2}, \quad \text{or} \quad g(s) = e^{-(\frac{s}{K})^2} \quad \text{with } K > 0.$$

Catte et al. [3] first introduced a new modification and proved its wellposedness to make the gradient computation robust outliers and provide a smooth edge map for the diffusion operator. This makes the Perona-Malik type PDE better.

$$\frac{\partial u}{\partial t} - \operatorname{div}(g(|G_\sigma * \nabla u|)) \nabla u = 0,$$

where

$$G_\sigma(x) = (2\pi\sigma)^{-1} e^{-\frac{|x|^2}{2\sigma}}$$

is the Gaussian kernel and the operation means convolution.

A coupled parabolic equations was introduced to create better edge maps (see [5, 6]), which has the following form:

$$\begin{aligned}\frac{\partial u}{\partial t} - \operatorname{div}(g(v)\nabla u) &= 0, \\ \frac{\partial v}{\partial t} - \Delta v &= 0.\end{aligned}$$

In order to localizing denoising effects in the diffusion process based scheme, Nitzberg and Shiotani [17] introduced the following relaxation model:

$$\begin{aligned}\frac{\partial u}{\partial t} - \operatorname{div}(g(v)\nabla u) &= 0, \\ \frac{\partial v}{\partial t} - \lambda G_\sigma \star |\nabla u|^2 - \lambda v &= 0,\end{aligned}$$

where $\lambda > 0$ is the relaxation parameter.

Surya Prasath and Vorotnikov [19] improved the above model and provided some new modifications. One of them has caught our attention, as follows:

$$\begin{aligned}\frac{\partial u}{\partial t} - \operatorname{div}(g(v)\nabla u) &= 0, \\ \frac{\partial v}{\partial t} - \lambda \operatorname{div}(\nabla v) - (1 - \lambda)(|\nabla u| - v) &= 0,\end{aligned}$$

where $g(s) = \frac{1}{1+(\frac{s}{K})^2}$ (Perona-Malik type diffusion function) or $g(s) = |s|^{-1}$ (total variation diffusion function). $0 \leq \lambda \leq 1$ is a balancing parameter. The first equation is usually used in the Perona-Malik type PDEs. In their discussion, above model is in favor of preservation of edges. However, when the noise is very large, the preservation of edges will be unstable, which is similar to that of the Perona-Malik model.

Another of powerful models is the following nonlinear hyperbolic model with variable coefficient

$$\frac{\partial^2 u}{\partial t^2} - \operatorname{div}(c(x, y)\nabla u) = 0,$$

where the degree of denoising and preservation of singularities can be determined by changing $c(x, y)$. Ratner and Zeevi [21] introduced a new telegraph-diffusion model

$$\frac{\partial^2 u}{\partial t^2} + \lambda \frac{\partial u}{\partial t} - \operatorname{div}(c(x, y)\nabla u) = 0$$

to describe the contraction and fluctuation of the image create denoising and edge preserving effect. This model is based on viewing the image as an elastic sheet.

Inspired by the work of [19, 21], we consider a coupled hyperbolic systems (1.1)-(1.2) as a method based on viewing the image as an elastic sheet to improve the quality of the detected edges. As we known, most of these schemes use the absolute value of the gradient image as a guiding road map in the diffusion process to restore noisy images. One can see [2, 3, 4, 8, 12, 16, 23] for more details.

This paper is organized as follows. In section 2 we study the existence and uniqueness of solutions of the problem (1.1)-(1.5). In section 3 we give some numerical experiments.

2. Existence of dissipative solutions and weak solutions

In this section, we establish the existence, uniqueness, regularity of dissipative solutions to the problem (1.1)-(1.5) following the arguments in [19]. Recall that the standard notations used throughout the paper. Let $L^p(\Omega)$, $W_p^m(\Omega)$ and $H^m(\Omega)$ for the Lebesgue and Sobolev spaces. We will keep the function space symbol and omit Ω . The Euclidean norm in finite-dimensional spaces is denoted by $|\cdot|$. The symbol $\|\cdot\|$ will stand for the Euclidean norm in $L^2(\Omega)$. The corresponding scalar products is denoted by a $\cdot$ and parentheses $(\cdot, \cdot)$. Let $H_0^1(\Omega)$ be the closure of the smooth set, and it is compactly supported in Ω . By virtue of Friedrichs inequality, the Euclidean norm $\|\cdot\|_1$ corresponding to the scalar product $(u, v)_1 = (\nabla u, \nabla v)$ is a norm in H_0^1 . Then there are standard Sobolev inequality

The usual Sobolev inequality

$$\|u\|_{L_\infty} \leq C(\Omega)\|u\|_2, \quad \forall u \in V_2,$$

and the Ladyzhenskaya inequality

$$\|u^2\| \leq \sqrt{2}\|u\|\|\nabla u\|, \quad \forall u \in H_0^1.$$

Let V_r be the closure of V_2 in W_r^1 with $1 < r < 2$, where $V_2 = H_0^1(\Omega) \cap H^2(\Omega)$ is a Hilbert space with the scalar product

$$(u, v)_2 = (u, v)_1 + \sum_{|\alpha|=2} (D^\alpha u, D^\alpha v).$$

We will consider our problem in the following space

$$W_1 = W_1(\Omega, T) = \{u \in L_2(0, T; V_2), u' \in L_2(0, T; V_2^*)\}$$

with the norm

$$\|u\|_{W_1} = \|u\|_{L_2(0, T; V_2)} + \|u'\|_{L_2(0, T; V_2^*)},$$

and

$$W_2 = W_2(\Omega, T) = \{u \in L_2(0, T; H_0^1), u' \in L_2(0, T; H^{-1})\}$$

with the norm

$$\|u\|_{W_2} = \|u\|_{L_2(0,T;H_0^1)} + \|u'\|_{L_2(0,T;H^{-1})}.$$

We also need the following class of pairs of functions

$$\begin{aligned} \mathbb{R} &= L_{4,loc}(0, \infty; V_2) \cap L_\infty(0, \infty; W_\infty^1) \cap W_{4,loc}^1(0, \infty; L_2) \times L_{2,loc}(0, \infty; V_2) \\ &\cap L_\infty(0, \infty; L_\infty) \cap W_{2,loc}^1(0, \infty; L_2). \end{aligned}$$

Define

$$\begin{aligned} E_1(u, v, \nu) &= -\frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial t} - \nu \operatorname{div}(g(|\nabla G_\rho * u|)\nabla u), \\ E_2(u, v, \nu) &= -\frac{\partial^2 v}{\partial t^2} - \frac{\partial v}{\partial t} + \lambda \operatorname{div}(\nabla v) + \nu(1 - \lambda)(|\nabla u| - v) + (1 - \delta)(\nabla v, \nabla \lambda), \\ E_1(u, v) &= E_1(u, v, 1), \\ E_2(u, v) &= E_2(u, v, 1), \end{aligned}$$

where ν is a positive constant and $\forall (u, v) \in \mathbb{R}$.

Definition 2.1. A pair of functions $(u, v) \in C_w([0, \infty); L_2)$ is called a dissipative solutions of the problem (1.1)-(1.2), if $\forall$ test functions $(\psi, \phi) \in \mathbb{R}$, one has

$$\begin{aligned} &\gamma^{\|u(t)\|^2} [\|u(t) - \psi(t)\|^2 + \|u'(t) - \psi'(t)\|^2 + \|v(t) - \phi(t)\|^2 + \|v'(t) - \phi'(t)\|^2] \\ &\leq \gamma^{2t+\|u_0\|^2} \{\|u(0) - \psi(0)\|^2 + \|u'(0) - \psi'(0)\|^2 + \|v(0) - \phi(0)\|^2 \\ &+ \|v'(0) - \phi'(0)\|^2 + \int_0^t 2\gamma^{-s} |(E_1(\psi, \phi)(s), u(s) - \psi(s)) \\ &+ (E_2(\psi, \phi)(s), v(s) - \phi(s))|\}, \end{aligned}$$

where $u'(t) = \frac{du}{dt}$, $v'(t) = \frac{dv}{dt}$, $u_0, v_0, v'_0 \in L_2(\Omega)$ and $\gamma > 1$ is a certain function of Ω , g , λ , ψ and ϕ .

Definition 2.2. A pair of functions $(u, v) \in W_1 \times W_2$ is called a weak solutions of the problem (1.1)-(1.2), if $\forall$ test functions $(\psi, \phi) \in V_2 \times H_0^1$,

$$(2.1) \quad \frac{d}{dt}(u', \psi') + \frac{d}{dt}(u, \psi) + \epsilon(u, \psi)_2 + \nu(g(|\nabla G_\rho * u|)\nabla u, \nabla \psi) = 0,$$

$$\frac{d}{dt}(v', \phi') + \frac{d}{dt}(v, \phi) + \lambda(\nabla v, \nabla \phi) + \nu(\nabla v, \phi \nabla \nu)$$

$$(2.2) \quad -\nu(1 - \lambda)(|\nabla u| - v, \phi) = 0$$

holds almost everywhere in $(0, T)$.

Now we state our main result.

Theorem 2.3. *The problem (1.1)-(1.2) with conditions (1.3)-(1.4) admits a dissipative solution $(u, v) \in H^1(0, \infty; H_0^1) \times H^1(0, \infty; H_0^1)$ with the initial data $u_0, u'_0, v_0, v'_0 \in L_2$ and $0 < \epsilon < \frac{1}{3}$. Moreover, if there exists a strong solution (u_T, v_T) of problem (1.1)-(1.2), then the restriction of any dissipative solution to $(0, T)$ coincides with (u_T, v_T) ($T > 0$). Every strong solution $(u, v) \in \mathbb{R}$ is a unique dissipative solution.*

To prove our main result, we introduce the following auxiliary problem

$$(2.3) \quad \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \nu \operatorname{div}(g(|\nabla G_\rho * u|) \nabla u),$$

$$(2.4) \quad \frac{\partial^2 v}{\partial t^2} + \frac{\partial v}{\partial t} - \lambda \operatorname{div}(\nabla v) = \nu(1 - \lambda)(|\nabla u| - v) + (1 - \delta)(\nabla v, \nabla \lambda),$$

with the condition

$$(2.5) \quad \begin{aligned} u(x, 0) &= \nu u_0(x), \quad u'(x, 0) = \nu u'_0(x) \quad v(x, 0) = \nu v_0(x), \\ v'(x, 0) &= \nu v'_0(x), \quad x \in \Omega, \end{aligned}$$

$$(2.6) \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \quad \frac{\partial v}{\partial n}|_{\partial\Omega} = 0, \quad 0 < t < T.$$

Lemma 2.4. *Let $(u_0, v_0, u'_0, v'_0) \in L_2 \times L_2$ and T be positive constant. Then the problem (2.3)-(2.4) admits a weak solution with $\nu = 1$.*

Proof. We define the operators $\mathcal{A}$ and $\mathcal{B}$ from $W_1 \times W_2$ to $L_2(0, T; V_2^*) \times H^1(0, T; H^{-1}) \times L_2 \times L_2 \times L_2$ by

$$\begin{aligned} (\mathcal{A}(u, v), (\psi, \phi)) &= \left(\frac{d}{dt}(u', \psi') + \frac{d}{dt}(u, \psi) + \epsilon(u, \psi)_2, \frac{d}{dt}(v', \phi') + \frac{d}{dt}(v, \phi) \right. \\ &\quad \left. + (\lambda \nabla v, \nabla \phi), u|_{t=0}, v|_{t=0}, v'|_{t=0} \right), \\ (\mathcal{B}(u, v), (\psi, \phi)) &= \left(-(g(|\nabla G_\rho * u|) \nabla u, \nabla \psi), -(c(x) \nabla v, \phi \nabla \lambda) \right. \\ &\quad \left. + (1 - \lambda)(|\nabla u| - v, \phi), u_0, v_0 \right), \end{aligned}$$

where $(\psi, \phi) \in H_0^1 \times H_0^1$ is a pair of test function.

Then we can rewrite the problem (2.3)-(2.4) as weak statement

$$(2.7) \quad \mathcal{A}(u, v) = \nu \mathcal{B}(u, v).$$

Note that $\mathcal{B}$ is continuous and compact, $W_1 \subset L_p(0, T; W_p^1)$ is compact for some $p > 2$ and $W_2 \subset L_p(0, T; L_2^1)$ (see [22]). If

$$(u_n, v_n) \longrightarrow (u_0, v_0), \quad \text{weakly in } W_1 \times W_2,$$

then we have

$$(u_n, v_n) \longrightarrow (u_0, v_0), \quad \text{strongly in } H^1(0, T; L_2) \times H^1(0, T; L_2).$$

By Krasnoselskii's theorem ([15]), we have

$$g(v_n) \longrightarrow g(v_0), \quad \text{in } L_p(0, T; L_q) \quad \forall q < +\infty.$$

So there holds

$$g(|\nabla G_\rho * u_n|) \nabla u_n \longrightarrow g(|\nabla G_\rho * u_0|) \nabla u_0, \quad \text{in } L_p(0, T; L_2).$$

Since the linear operator $\mathcal{A}$ is continuous and invertible (see [19]), we can rewrite (2.7) as

$$(2.8) \quad (u, v) = \nu \mathcal{A}^{-1} \mathcal{B}(u, v) \quad \text{in } W_1 \times W_2.$$

Now we derive the following estimate.

$$(2.9) \quad \begin{aligned} & \gamma^{\|u(t)\|^2} [\|u(t) - \psi(t)\|^2 + \|u'(t) - \psi'(t)\|^2 \\ & + \|v(t) - \phi(t)\|^2 + \|v'(t) - \phi'(t)\|^2 \\ & + 2\epsilon \int_0^t \|u(s) - \psi(s)\|_2^2 ds + \lambda_0 \int_0^t \|v(s) - \phi(s)\|_1^2 ds] \\ & \leq \gamma^{2t+\nu\|u_0\|^2} \{ \|\nu u(0) - \psi(0)\|^2 + \|\nu u'(0) - \psi'(0)\|^2 \\ & + \|\nu v(0) - \phi(0)\|^2 + \|\nu v'(0) - \phi'(0)\|^2 \\ & + \int_0^t 2\gamma^{-s} |(E_1(\psi, \phi, \nu)(s), u(s) - \psi(s)) \\ & + (E_2(\psi, \phi, \nu)(s), v(s) - \phi(s)) - \epsilon(\psi(s), u(s) - \psi(s))_2| ds \}, \end{aligned}$$

where $\gamma > 1$ is a certain function of $\Omega, g, \lambda, \mu, \psi$ and ϕ .

To prove above estimate, we need to carry out energy estimate. Let $\psi(t) = u(t)$ and $\phi(t) = v(t)$ in (2.1)-(2.2), respectively, and we have

$$(2.10) \quad \frac{d}{dt}(u', u') + \frac{1}{2} \frac{d}{dt}(u, u) + \epsilon(u, u)_2 + \nu(g(|\nabla G_\rho * u|) \nabla u, \nabla u) = 0,$$

$$(2.11) \quad \begin{aligned} & \frac{d}{dt}(v', v') + \frac{1}{2} \frac{d}{dt}(v, v) + \lambda(\nabla v, \nabla v) + \nu(\nabla v, v \nabla \nu) \\ & - \nu(1 - \lambda)(|\nabla u| - v, v) = 0. \end{aligned}$$

Summing up (2.12)-(2.13) and integrating over $(0, t)$, we get

$$(2.12) \quad \begin{aligned} & \frac{1}{2}(\|u\|^2 + \|u'\|^2 + \|v\|^2 + \|v'\|^2) + \int_0^t \nu(g(|\nabla G_\rho * u|) \nabla u, \nabla u) ds \\ & + \int_0^t \lambda(\nabla v, \nabla v) + \nu(\nabla v, v \nabla \nu) - \nu(1 - \lambda)(|\nabla u| - v, v) ds \\ & \leq \frac{\nu}{2}(\|u_0\|^2 + \|u'_0\|^2 + \|v_0\|^2 + \|v'_0\|^2). \end{aligned}$$

On the other hand, $\forall(\eta, \theta)$ test function in $H_0^1 \times H_0^1$, we have

$$(2.13) \quad \begin{aligned} & \frac{d}{dt}(\psi', \eta') + \frac{d}{dt}(\psi, \eta) + \nu(g(|\nabla G_\rho * u|)\nabla u, \nabla \eta) \\ & + (E_1(\psi, \phi, \nu), \eta) + \epsilon(\psi, \eta)_2 = \epsilon(\psi, \eta), \end{aligned}$$

$$(2.14) \quad \begin{aligned} & \frac{d}{dt}(\phi', \theta') + \frac{d}{dt}(\phi, \theta) + (\lambda \nabla \phi, \theta) + \nu(\nabla \phi, \theta \nabla \lambda) \\ & - \nu(1 - \lambda)(|\nabla \psi| - \phi, \theta) + (E_2(\psi, \phi, \lambda), \theta) = 0. \end{aligned}$$

Let $\eta = u - \psi$ and $\theta = v - \phi$. Summing up (2.13)-(2.14) and noticing (2.12)-(2.13), we get

$$(2.15) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt}((\eta, \eta) + (\theta, \theta) + (\eta', \eta') + (\theta', \theta')) + \nu(g(|\nabla G_\rho * v|)\nabla \eta, \nabla \eta) \\ & + \epsilon(\eta, \eta)_2 + (\lambda \nabla \theta, \nabla \theta) + \nu((1 - \lambda)\theta, \theta) \\ & = -\nu([g(|\nabla G_\rho * v|) - g(|\nabla G_\rho * \phi|)]\nabla \theta, \nabla \eta) \\ & + \nu(1 - \lambda)(|\nabla u| - |\nabla \theta|, \theta) - \nu(\nabla \theta, \theta \nabla \lambda) \\ & + (E_1(\psi, \phi, \nu), \eta) + (E_2(\psi, \phi, \lambda), \theta) - \epsilon(\eta, \theta)_2. \end{aligned}$$

It is easy to derive that

$$(2.16) \quad \begin{aligned} & -\nu([g(|\nabla G_\rho * v|) - g(|\nabla G_\rho * \phi|)]\nabla \theta, \nabla \eta) + \nu(1 - \lambda)(|\nabla u| - |\nabla \theta|, \theta) \\ & \leq C(\psi, g)\nu(|v - \phi|, |\nabla \eta|) \\ & \leq \|\sqrt{\nu g(|\nabla G_\rho * v|)}\nabla \eta\|^2 + C(\psi, \phi, g)\|\theta\|^2 \\ & + C(\psi, g)(\theta^2, \sqrt{\nu g(|\nabla G_\rho * v|)}|\nabla u|), \end{aligned}$$

and

$$(2.17) \quad -\nu(\nabla \theta, \theta \nabla \lambda) \leq \frac{\lambda_0}{4}\|\theta\|_1^2 + C(\lambda)\|\theta\|^2.$$

Thus, applying (2.16)-(2.17) to (2.15), we derive

$$(2.18) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt}((\eta, \eta) + (\eta', \eta') + (\theta, \theta) + (\theta', \theta')) + \epsilon(\eta, \eta)_2 + \frac{3\lambda_0}{4}\mu^{-1}\|\theta\|_1^2 \\ & \leq C(\theta, \phi, \lambda, g)(\theta^2, 1 + \sqrt{\nu g(|\nabla G_\rho * v|)}|\nabla u|) \\ & + (E_1(\psi, \phi, \nu), \eta) + (E_2(\psi, \phi, \lambda), \theta) - \epsilon(\eta, \theta)_2. \end{aligned}$$

Denote $\Phi(t) = \|1 + \sqrt{\nu g(|\nabla G_\rho * v|)}|\nabla u|\|$. Then it follows from (2.18) that

$$(2.19) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt}(\|\eta\|^2 + \|\theta\|^2 + \|\theta'\|^2) + 2\epsilon\|\eta\|_2^2 + \lambda_0\mu^{-1}\|\nabla \theta\|^2 \\ & \leq C(\theta, \phi, \lambda, g)\Phi^2\|\theta\|^2 + 2(E_1(\psi, \phi, \nu), \eta) + 2(E_2(\psi, \phi, \lambda), \theta) \\ & - 2\epsilon(\eta, \theta)_2. \end{aligned}$$

By (2.13), we have

$$(2.20) \quad t|\Omega| \leq \int_0^t \Phi^2(s)ds \leq 2t|\Omega| + \nu\|u_0\|^2 - \|u(t)\|^2.$$

Hence, using (2.19)-(2.20) and a Growall-type inequality, we obtain (2.9).

Let $\eta = \theta = 0$. It is easy to see that

$$(2.21) \quad \|u\|_{L_\infty(0,T;L_2)} + \|u'\|_{L_2(0,T;H_0^1)} + \|v\|_{L_\infty(0,T;L_2)} + \|v'\|_{L_2(0,T;H_0^1)} \leq C,$$

$$(2.22) \quad \|u\|_{L_\infty(0,T;V_2)} + \|u'\|_{L_2(0,T;H_0^1)} \leq C\epsilon^{-\frac{1}{2}},$$

where C is a constant independent of ϵ and ν .

It follows from (1.5) that

$$(2.23) \quad \frac{1}{\sqrt{g(s)}} \leq \left| \frac{1}{\sqrt{g(s)}} - \frac{1}{\sqrt{g(0)}} \right| + \frac{1}{\sqrt{g(0)}} \leq C(g)(1 + |s|).$$

So by (2.23) and (2.15), we have

$$\|\nabla u\|_{L_2(0,T;L_1)} \leq \left\| \sqrt{\nu g(|\nabla G_\rho * v|)} \nabla u \right\|_{L_2(0,T;L_2)} \left\| \sqrt{g(|\nabla G_\rho * v|)}^{-1} \right\|_{L_\infty(0,T;L_2)} \leq C.$$

By Sobolev embedding $H_0^1 \subset L_p$ for any $p < \infty$ and Holder inequality, we derive

$$\|\nabla u\|_{L_2(0,T;L_1)} + \|\nabla u\|_{L_1(0,T;L_r)} + \|\nabla u\|_{L_{\frac{4}{3}}(0,T;L_{-\epsilon+\frac{4}{3}})} \leq C.$$

Furthermore, by (2.13)-(2.14) and (2.15), for $1 < r < 2$ and $0 < \epsilon < \frac{1}{3}$, we have the following estimates

$$\begin{aligned} \|\nabla u\|_{L_2(0,T;V_2^*)} + \|v\|_{L_2(0,T;H^{-2})} &\leq C(1 + \sqrt{\epsilon}), \\ \|u\|_{H^1(0,T;H^{-1})} &\leq C(1 + \frac{1}{\sqrt{\epsilon}}), \\ \|v\|_{H^1(0,T;H^{-1})} &\leq C(1 + \frac{1}{\sqrt{\epsilon}}), \end{aligned}$$

where C is a constant independent of ϵ and ν .

Hence the above estimates imply that

$$\|u\|_{W_1} + \|u'\|_{W_1} + \|v\|_{W_2} + \|v'\|_{W_2} \leq C,$$

where C depends on ϵ but not ν .

Applying Schaeffer's theorem (see [11], p. 539), we know that there exists a fixed point of (2.8), which is a solution of (2.7). This completes the proof. $\square$

The following convergence Proposition is taken from [19].

Proposition 2.5. *Let G be a measurable set in a finite dimensional space, $y_n : G \rightarrow R$ be a sequence of functions and $\mathcal{X} : R \rightarrow R$ be a continuous function. Assume that $\{y_n\}$ is uniformly bounded in $L_\infty(G)$ and $y_m \rightarrow y_0$ in $L_q(G)$ with $q \geq 1$. Then*

$$\mathcal{X}(y_n) \rightarrow \mathcal{X}(y_0)$$

in $L_p(G)$ for any $p < \infty$.

Now we are ready to prove Theorem 2.1. The proof is similar to that of theorem 1 in [19]. For completeness of our paper, we sketch the proof. Based on Lemma 2.1, we can proceed with the sketch of the proof of Theorem 2.1. We refer to [19] for the details of the technique, and mainly focus on the new issues. The existence of dissipative solutions, one passes the limit in (2.9) with $\nu = 1$ as $\epsilon = \epsilon_m \rightarrow 0$ on every interval $(0, T)$ with $T > 0$. Let (u_n, v_n) be the weak solution to problem (2.5)-(2.6) with ϵ_n in Lemma 2.1. Using Sobolev embedding $W_1 \subset L_2$, we derive

$$\begin{aligned} u_m &\rightarrow u \text{ in } H^1(0, T; L_2), \\ v_m &\rightarrow v \text{ in } H^1(0, T; L_2). \end{aligned}$$

Then (2.21) and Proposition 2.1,

$$\begin{aligned} \gamma^{\|u_n(t)\|^2} &\rightarrow \gamma^{\|u(t)\|^2} \text{ in } L_2(0, T), \\ \|u_n(t) - \psi(t)\|^2 &\rightarrow \|u(t) - \psi(t)\|^2 \text{ in } L_2(0, T), \\ \|u'_n(t) - \psi'(t)\|^2 &\rightarrow \|u'(t) - \psi'(t)\|^2 \text{ in } L_2(0, T), \\ \|v_n(t) - \phi(t)\|^2 &\rightarrow \|v(t) - \phi(t)\|^2 \text{ in } L_2(0, T), \\ \|v'_n(t) - \phi'(t)\|^2 &\rightarrow \|v'(t) - \phi'(t)\|^2 \text{ in } L_2(0, T). \end{aligned}$$

So we have

$$\begin{aligned} &\gamma^{\|u_n(t)\|^2} (\|u_n(t) - \psi(t)\|^2 + \|u'_n(t) - \psi'(t)\|^2 + \|v_n(t) - \phi(t)\|^2 + \|v'_n(t) - \phi'(t)\|^2) \\ &\rightarrow \gamma^{\|u(t)\|^2} (\|u(t) - \psi(t)\|^2 + \|u'(t) - \psi'(t)\|^2 + \|v(t) - \phi(t)\|^2 + \|v'(t) - \phi'(t)\|^2) \end{aligned}$$

in $L_1(0, T)$. Thus, we can pass to the limit in the right-hand side of (2.9) as well and the last summand (the one with ϵ) goes to zero. Therefore, we conclude that Theorem 2.1 holds.

3. Numerical experiments

In this section we present some experimental results on pictures in the two dimensions case by using an explicit numerical scheme based on Rothe's method in time discretization and finite difference method in spatial discretization. To proceed with the discrete numerical algorithm, we subdivide the time interval $(0, T)$ by points $t_n = n\tau$ with $\tau = \frac{T}{L}$, $n = 0, 1, 2, \dots, N$, where N is

a positive integer. Assume that the image $(u(t), v(t))$ is defined in the lattice $\{1h, 2h, \dots, Mh\} \times \{1h, 2h, \dots, Lh\}$, where h is the space stepsize. Denote $(u_{i,j}^n, v_{i,j}^n)$ an approximation of $(u(n\tau, ih, jh), v(n\tau, ih, jh))$. We define the discrete approximation

$$\begin{aligned} \nabla_x^+ u_{i,j}^n &= \frac{u_{i+1,j}^n - u_{i,j}^n}{h}, \quad \nabla_x^+ v_{i,j}^n = \frac{v_{i+1,j}^n - v_{i,j}^n}{h}, \\ \nabla_x^- u_{i,j}^n &= \frac{u_{i-1,j}^n - u_{i,j}^n}{h}, \quad \nabla_x^- v_{i,j}^n = \frac{v_{i-1,j}^n - v_{i,j}^n}{h}, \\ \nabla_y^+ u_{i,j}^n &= \frac{u_{i,j+1}^n - u_{i,j}^n}{h}, \quad \nabla_y^+ v_{i,j}^n = \frac{v_{i,j+1}^n - v_{i,j}^n}{h}, \\ \nabla_y^- u_{i,j}^n &= \frac{u_{i,j-1}^n - u_{i,j}^n}{h}, \quad \nabla_y^- v_{i,j}^n = \frac{v_{i,j-1}^n - v_{i,j}^n}{h}, \\ \delta u_{i,j}^n &= \frac{u_{i,j}^n - u_{i,j}^{n-1}}{\tau}, \quad \delta v_{i,j}^n = \frac{v_{i,j}^n - v_{i,j}^{n-1}}{\tau}, \\ \delta^2 u_{i,j}^n &= \frac{\delta u_{i,j}^n - \delta u_{i,j}^{n-1}}{\tau}, \quad \delta^2 v_{i,j}^n = \frac{\delta v_{i,j}^n - \delta v_{i,j}^{n-1}}{\tau}, \\ g_{i,j}^n &= \frac{1}{1 + (\frac{v_{i,j}^n}{K})^2}, \quad |\nabla G_\rho * u_{i,j}^n| = [\sum_{i=1}^M (\delta G_\rho * u_{i,j}^n)^2]^{\frac{1}{2}}. \end{aligned}$$

Then the discrete explicit scheme of the problem (1.1)-(1.2) can be obtained

$$\begin{aligned} \delta^2 u_{i,j}^{n+1} + \delta u_{i,j}^{n+1} &- \frac{1}{2} [(g_{i+1,j}^n + g_{i,j}^n) \nabla_x^+ u_{i,j}^n + (g_{i-1,j}^n + g_{i,j}^n) \nabla_x^- u_{i,j}^n \\ &+ (g_{i,j+1}^n + g_{i,j}^n) \nabla_y^+ u_{i,j}^n + (g_{i,j-1}^n + g_{i,j}^n) \nabla_y^- u_{i,j}^n] = 0, \\ \delta^2 v_{i,j}^{n+1} + \delta v_{i,j}^{n+1} &- \frac{1}{2} [(c_{i+1,j}^n + c_{i,j}^n) \nabla_x^+ v_{i,j}^n + (c_{i-1,j}^n + c_{i,j}^n) \nabla_x^- v_{i,j}^n \\ &+ (c_{i,j+1}^n + c_{i,j}^n) \nabla_y^+ v_{i,j}^n + (c_{i,j-1}^n + c_{i,j}^n) \nabla_y^- v_{i,j}^n] \\ &- (1 - \lambda)(|\nabla_x^+ u_{i,j}^n| + |\nabla_x^- u_{i,j}^n| - v_{i,j}^n) = 0, \end{aligned}$$

with the condition

$$\begin{aligned} u_{i,j}^0 &= u_0(i, j), \quad \delta u_{i,j}^0 = \delta u_0(i, j), \quad v_{i,j}^0 = v_0(i, j), \\ \delta v_{i,j}^0 &= \delta v_0(i, j) \quad 1 \leq i \leq M, \quad 1 \leq j \leq L, \\ u_{0,j}^n &= u_{1,j}^n, \quad u_{M,j}^n = u_{M+1,j}^n, \quad u_{i,0}^n = u_{i,1}^n, \quad u_{i,L}^n = u_{i,L+1}^n, \\ v_{0,j}^n &= v_{1,j}^n, \quad v_{M,j}^n = v_{M+1,j}^n, \quad v_{i,0}^n = v_{i,1}^n, \quad v_{i,L}^n = v_{i,L+1}^n. \end{aligned}$$

We show numerical results which are obtained by applying the above scheme to two artificial heavily noised images. In order to stabilize the numerical scheme, we rescale the images so that the theoretical results obtained in the previous section can be applied. Our experiments depend on three parameters: the “scale” of diffusion λ , the diffusive coefficient $c(x)$ and the threshold K . In

general, we measure the amount of noise by its standard deviation. The better quality image will have a higher SNR. Define the signal noise ratio (SNR) as

$$SNR = 10 \log_{10} \left(\frac{\sum_{\Omega} (u_{i,j} - \bar{u})^2}{\sum_{\Omega} (n_{i,j})^2} \right),$$

where $\bar{u}$ denotes the mean of the signal $u_{i,j}$ and $n_{i,j}$ is the noise.

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