

ON WEAK MCCOY MODULES OVER COMMUTATIVE RINGS

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Abstract. Let M be a module over a commutative ring R . In this paper we generalize some annihilator conditions from rings to modules. Denote by $Nil(M)$ the set of all nilpotent elements of M . M is said to be weak McCoy if $f(x)m(x) = 0$, where $f(x) = \sum_{i=0}^k m_i x^i \in R[x] \setminus \{0\}$ and $m(x) = \sum_{j=0}^n a_j x^j \in M[x] \setminus \{0\}$, then $sm_i \in Nil(M)$, for every $1 \leq i \leq k$ and for some $s \in R \setminus \{0\}$. We prove that the class of weak McCoy modules is closed under direct limit, finite direct product and localization. We show that if R is a uniform R -module, then the direct sum of weak McCoy R -modules is again weak McCoy. We prove that over a domain D , the D -module M is weak McCoy provided that if $T(M)$ is weak McCoy, where $T(M)$ is the torsion submodule of M .

Keywords: McCoy ring, McCoy module, weak McCoy ring, weak McCoy module.

1. Introduction

Throughout this paper all rings are considered to be commutative with a nonzero identity and all modules are unitary unless otherwise stated. Let R be a ring, M an R -module and N a submodule of M . We denote by $(N :_R M)$ the set of all r in R such that $rM \subseteq N$. The *annihilator* of M , denoted by $ann_R(M)$, is then $(0 :_R M)$. We denote by $R[x]$ the polynomial ring over the ring R and $M[x]$ the polynomial module over the module M .

An R -module M is called a *multiplication module* if every submodule N of M has the form IM for some ideal I of R (Barnard, 1981). Note that since $I \subseteq (N :_R M)$, then $N = IM \subseteq (N :_R M)M \subseteq N$. So that $N = (N :_R M)M$. A submodule N of M is called *pure* if $IN = N \cap IM$ for every ideal I of R (Ribbenboim, 1972). It is proved in [2, Theorem 1.1] that every pure submodule of a multiplication module is again a multiplication module. A submodule N of

M is *prime* whenever $rm \in N$ for some $r \in R$ and $m \in M$, then either $m \in N$ or $rM \subseteq N$.

We recall that an ideal I of R is *nilpotent* if $I^k = 0$ for some positive integer k and an element r of R is *nilpotent* if $r^k = 0$ for some $k \in \mathbb{N}$. We denote by $Nil(R)$ the set of all nilpotent elements of R . According to [1], a submodule N of M is called *nilpotent* if $(N :_R M)^k N = 0$ for some positive integer k . An element $m \in M$ is said to be *nilpotent* if Rm is a nilpotent submodule of M . Clearly, the zero submodule of M is nilpotent and hence the zero element of M is nilpotent. Denote by $Nil(M)$ the set of all nilpotent elements of M . $Nil(M)$ is not necessarily a submodule of M , but if M is faithful, then $Nil(M)$ is a submodule of M [1, Theorem 6]. Moreover, if M is a faithful multiplication R -module, then $Nil(M) = Nil(R)M = \cap P$, where P runs over all prime submodules of M .

By [1, Proposition 4(2)], if I is a nilpotent ideal of R , then IM is nilpotent in M and the converse is true if M is faithful. Also if $K \subseteq N$ and N is nilpotent in M , then $\frac{N}{K}$ is nilpotent in $\frac{M}{K}$ as an R -module and the converse is true if K is nilpotent in M and M is faithful. Also, a submodule N of M is *idempotent* if $N = (N :_R M)N$ (Ali and Smith, 2004). By [1, Theorem 3], if N is a direct summand of a multiplication module M , then N is multiplication and idempotent. An R module M is called *uniform* if any two nonzero submodules of M have a nonzero intersection (Faith, 1981).

Let R be an associative ring with identity. R is said to be *right McCoy* (respectively *left McCoy*) if for each pair of non-zero polynomials $f(x), g(x) \in R[x]$ with $f(x)g(x) = 0$, then there exists a non-zero element $r \in R$ with $f(x)r = 0$ (respectively $rg(x) = 0$). A ring is *McCoy* if it is both left and right McCoy. Also, we say that a ring R is *right weak McCoy* whenever $f(x) = a_0 + a_1x + \dots + a_mx^m, g(x) = b_0 + b_1x + \dots + b_nx^n \in R[x] \setminus \{0\}$ satisfy $f(x)g(x) = 0$, then $a_is \in Nil(R)$ for each $i = 0, 1, \dots, m$ and for some $s \in R \setminus \{0\}$. *Left weak McCoy* rings are defined similarly. If a ring R is both left and right weak McCoy, then we say that R is a *weak McCoy ring*.

By those definitions, clearly, McCoy rings are weak McCoy. Also, the name "McCoy" was chosen because McCoy in [7] had noted that every commutative ring satisfies this condition.

An R -module M is said to be *McCoy* if $f(x)m(x) = 0$, where $f(x) = \sum_{i=0}^n a_ix^i \in R[x] \setminus \{0\}$ and $m(x) = \sum_{j=0}^k m_jx^j \in M[x]$, implies that there exists a nonzero element $r \in R$ such that $rm(x) = 0$ (Cui and Chen, 2011).

In this paper, we generalize the concept of weak McCoy rings defined on associative rings to modules over commutative rings. Let R be a commutative ring. An R -module M is called a *weak McCoy module* if $f(x)m(x) = 0$, where $f(x) \in R[x] \setminus \{0\}$ and $m(x) = \sum_{i=0}^n m_ix^i \in M[x] \setminus \{0\}$, then $sm_i \in Nil(M)$, for every $0 \leq i \leq n$ and for some $s \in R \setminus \{0\}$.

2. Weak McCoy modules

In this section we study some basic properties of weak McCoy modules and investigate the relations between weak McCoy rings and weak McCoy modules.

Definition 2.1. Let R be a commutative ring. An R -module M is called weak McCoy whenever $f(x) \in R[x] \setminus \{0\}$ and $m(x) = \sum_{i=0}^n m_i x^i \in M[x] \setminus \{0\}$ satisfy $f(x)m(x) = 0$, then $sm_i \in \text{Nil}(M)$, for every $0 \leq i \leq n$ and for some $s \in R \setminus \{0\}$.

Proposition 2.2. Let R be a ring. Then every McCoy R -module M is weak McCoy and the converse is true if M has no nilpotent elements.

Proof. It is trivial. $\square$

Corollary 2.3. Let M be a finitely generated faithful multiplication R -module. Then, the $\frac{R}{\text{Nil}(R)}$ -module $\frac{M}{\text{Nil}(M)}$ is McCoy if and only if it is weak McCoy.

Proof. First note that $\text{Nil}(M)$ is a submodule of M and $\text{Nil}(M) = \text{Nil}(R)M$ by [1, Theorem 6]. So we may form the $\frac{R}{\text{Nil}(R)}$ -module $\frac{M}{\text{Nil}(M)}$. By [1, Corollary 7], $\frac{M}{\text{Nil}(M)}$ has no nilpotent elements. Now the result follows from Proposition 2.2. $\square$

Proposition 2.4. Let M be an R -module. Then the direct limit of every direct system of weak McCoy submodules of M is weak McCoy.

Proof. Let I be a directed set and $\{M_i; i \in I\}$ a direct system of weak McCoy submodules of M . Set $N = \lim_{i \in I} M_i$. We know that $N = \sum_{i \in I} M_i = \bigcup_{i \in I} M_i$, by [8, Example 5.32]. Let $m(x) = \sum_{j=0}^n m_j x^j \in N[x] \setminus \{0\}$ and $f(x) = \sum_{k=0}^m a_k x^k \in R[x] \setminus \{0\}$ with $f(x)m(x) = 0$. There exists $t \in I$ such that $m_j \in M_t$ for each $0 \leq j \leq n$. So $m(x) \in M_t[x]$. Since M_t is weak McCoy, there exists $s \in R \setminus \{0\}$ such that $sm_j \in \text{Nil}(M_t)$. Thus, there exists an $l_j \in \mathbb{N}$ such that $(R(sm_j) :_R M_t)^{l_j} R(sm_j) = 0$, for each $j = 0, 1, \dots, n$. But $(R(sm_j) :_R N) \subseteq (R(sm_j) :_R M_t)$, hence $(R(sm_i) :_R N)^{l_j} R(sm_j) = 0$ and $sm_j \in \text{Nil}(N)$, for each $j = 0, 1, \dots, n$. $\square$

Corollary 2.5. If every finitely generated submodule of an R -module M is weak McCoy, then M is weak McCoy.

Proof. By [8, Example 5.32], every module is the direct limit of its finitely generated submodules. Now the result is trivial by Proposition 2.4. $\square$

Definition 2.6. A ring R is said to be a strongly weak McCoy ring if every R -module is weak McCoy.

It is clear that if an R -module M is not faithful, then it is McCoy, and so is weak McCoy. Hence to study the weak McCoy condition on a module, it is sufficient to consider the faithful case.

Proposition 2.7. *A ring R is strongly weak McCoy if and only if every finitely generated R -module is weak McCoy.*

Proof. $\Rightarrow$) It is trivial.

$\Leftarrow$) Let M be a faithful R -module. We know that M is a direct limit of its finitely generated submodules. Hence by assumption and Proposition 2.4, M is weak McCoy. $\square$

Let M be an R -module and S a multiplicatively closed subset of R . Then $S^{-1}M$ has an $S^{-1}R$ -module structure. M is called *S -torsion free* if $sm \neq 0$, for each $0 \neq m \in M$ and $s \in S$.

Lemma 2.8. *Let R be a ring, S a multiplicatively closed subset of R and M a finitely generated R -module. Then $m \in \text{Nil}_R(M)$ if and only if $\frac{m}{s} \in \text{Nil}_{S^{-1}R}(S^{-1}M)$, for every $s \in S$.*

Proof. Let $m \in M$ and $k \in \mathbb{N}$. Since M is finitely generated, we have

$$\begin{aligned} S^{-1}((Rm :_R M)^k Rm) &= S^{-1}((Rm :_R M)^k) S^{-1}(Rm) \\ &= (S^{-1}(Rm :_R M))^k S^{-1}(Rm) \\ &= (S^{-1}(Rm) :_{S^{-1}R} S^{-1}M)^k S^{-1}(Rm). \end{aligned}$$

On the other hand, $S^{-1}(Rm) = S^{-1}R(\frac{m}{s})$, for each $s \in S$. So the proof is complete. $\square$

Theorem 2.9. *Let R be a ring, S a multiplicatively closed subset of R and M a finitely generated S -torsion free R -module. then M is weak McCoy if and only if $S^{-1}M$ is a weak McCoy $S^{-1}R$ -module.*

Proof. First suppose that M is a weak McCoy R -module.

Let $f(x) = \sum_{i=0}^m \alpha_i x^i \in S^{-1}R[x] \setminus \{0\}$ and $m(x) = \sum_{j=0}^n \beta_j x^j \in S^{-1}M[x] \setminus \{0\}$ with $f(x)m(x) = 0$.

We can assume that $\alpha_i = \frac{r_i}{u}$ and $\beta_j = \frac{m_j}{v}$ for some $r_i \in R$, $m_j \in M$ and $u, v \in S$. Note that if $\alpha_i = \frac{r_i}{u_i}$ and $\beta_j = \frac{m_j}{v_j}$, then we can assume $u = u_1 u_2 \dots u_m$ and $v = v_1 v_2 \dots v_n$.

Therefore $f(x)m(x) = 0$ implies that $0_{S^{-1}R} = \frac{r_0 m_0}{uv} = \frac{r_0 m_1 + r_1 m_0}{uv} = \frac{r_0 m_2 + r_1 m_1 + r_2 m_0}{uv} = \dots$. But M is S -torsion free, so $0 = r_0 m_0 = r_0 m_1 + r_1 m_0 = r_0 m_2 + r_1 m_1 + r_2 m_0 = \dots$.

Now, we set $f_1(x) = \sum_{i=0}^m r_i x^i$ and $m_1(x) = \sum_{j=0}^n m_j x^j$. Clearly $f_1(x) \in R[x] \setminus \{0\}$, $m_1(x) \in M[x] \setminus \{0\}$ and $f_1(x)m_1(x) = 0$. Since M is weak McCoy, there exists $c \in R \setminus \{0\}$ such that $cm_j \in \text{Nil}_R(M)$, for each $0 \leq j \leq n$. It is clear that $c \in S^{-1}R \setminus \{0\}$ and $c(\frac{m_j}{v}) \in \text{Nil}_{S^{-1}R}(S^{-1}M)$, for each $0 \leq j \leq n$, by Lemma 2.8. Hence, $S^{-1}M$ is a weak McCoy $S^{-1}R$ -module.

Now assume that $S^{-1}M$ is a weak McCoy $S^{-1}R$ -module. Let $g(x) = \sum_{i=0}^q a_i x^i \in R[x] \setminus \{0\}$ and $n(x) = \sum_{j=0}^p n_j x^j \in M[x] \setminus \{0\}$ with $g(x)n(x) = 0$. But $n(x) \in S^{-1}M[x] \setminus \{0\}$ and $g(x) \in S^{-1}R[x] \setminus \{0\}$. By assumption, there exists $\frac{r}{s} \in S^{-1}R \setminus \{0\}$ such that $\frac{r}{s}(m_j) \in \text{Nil}_{S^{-1}R}(S^{-1}M)$, for each $0 \leq j \leq p$.

Therefore, $rm_j \in \text{Nil}_R(M)$, for each $0 \leq j \leq p$ and hence M is a weak McCoy R -module. $\square$

We recall that $R[x, x^{-1}]$ denotes the Laurent polynomial ring over R . For an R -module M , let $M[x, x^{-1}] = \{\sum_{i=k}^n m_i x^i; n, k \in \mathbb{Z}, m_i \in M\}$. $M[x, x^{-1}]$ is an $R[x, x^{-1}]$ -module under the addition operation and the following scalar product operation. For $m(x) = \sum_i m_i x^i \in M[x, x^{-1}]$ and $f(x) = \sum_j a_j x^j \in R[x, x^{-1}]$, then $f(x)m(x) = \sum_k (\sum_{i+j=k} a_j m_i) x^k$ (see [6]).

Corollary 2.10. *Let R be a ring and M a finitely generated R -module. Then $M[x]$ is a weak McCoy $R[x]$ -module if and only if $M[x, x^{-1}]$ is a weak McCoy $R[x, x^{-1}]$ -module.*

Proof. Let $S = \{1, x, x^2, \dots\}$. It is clear that S is a multiplicatively closed subset of $R[x]$ and $R[x]$ -module $M[x]$ is S -torsion free. By Theorem 2.9, $M[x]$ is a weak McCoy $R[x]$ -module if and only if $S^{-1}M[x]$ is a weak McCoy $S^{-1}R[x]$ -module. But $S^{-1}R[x] = R[x, x^{-1}]$ and $S^{-1}M[x] = M[x, x^{-1}]$. Now the result follows from Theorem 2.9. $\square$

Corollary 2.11. *Let R be a ring, S a multiplicatively closed subset of R and every R -module is finitely generated and S -torsion free. Then the localization $S^{-1}R$ is strongly weak McCoy if and only if R is strongly weak McCoy.*

Proposition 2.12. *Every pure submodule of a weak McCoy multiplication module is weak McCoy.*

Proof. Let N be a faithful pure submodule of the weak McCoy multiplication R -module M . It is clear that N is multiplication. Let $f(x)m(x) = 0$, where $m(x) = \sum_{i=1}^n m_i x^i \in N[x] \setminus \{0\}$ and $f(x) \in R[x] \setminus \{0\}$. Since $m(x) \in M[x]$ and M is weak McCoy, there exists $s \in R \setminus \{0\}$ such that $sm_i \in \text{Nil}(M)$. By [1, Proposition 4], $(R(sm_i) :_R M)$ is a nilpotent ideal of R and hence $(R(sm_i) :_R M) \subseteq \text{Nil}(R)$, for each $i = 0, 1, \dots, n$. By [1, Theorem 6], $\text{Nil}(R)M = M$ and $\text{Nil}(R)N = N$, since N and M are multiplication. Also, N is a pure submodule of M and $R(sm_i) = (R(sm_i) :_R M)M$. So $sm_i \in N \cap (R(sm_i) :_R M)M = (R(sm_i) :_R M)N \subseteq \text{Nil}(R)N = \text{Nil}(N)$, for each $i = 0, 1, \dots, n$. Therefore, these elements are nilpotent in N and hence N is weak McCoy. $\square$

Proposition 2.13. *Let R be a ring, I a directed set and $\{M_i\}_{i \in I}$ a family of faithful weak McCoy R -modules. If R is uniform as an R -module, then the direct sum $M = \coprod_{i \in I} M_i$ is a weak McCoy R -module.*

Proof. Let $f(x) \in R[x] \setminus \{0\}$ and $m(x) = \sum_{k=0}^n m_k x^k \in M[x] \setminus \{0\}$ satisfy $f(x)m(x) = 0$, where $m_k = (m_{ik})_{i \in I}$, for $k = 0, 1, \dots, n$.

Set $m_i(x) = \sum_{k=0}^n m_{ik} x^k \in M_i[x]$. Clearly, $f(x)m_i(x) = 0$. Since M_i 's are weak McCoy, there exists $s_i \in R \setminus \{0\}$ such that $s_i m_{ik} \in \text{Nil}(M_i)$ for each $i \in I$ and $k = 0, 1, \dots, n$. We know that the set $I' = \{i \in I; m_i(x) \neq 0\}$ is a finite set. Set $U = \bigcap_{i \in I'} R s_i$. Since R is a uniform R -module, $U \neq 0$.

Let $s \in U \setminus \{0\}$. Then $s = r_i s_i$, for each $i \in I'$ and for some $r_i \in R$. Also $R(sm_{ik}) = R(r_i s_i m_{ik}) = (Rr_i)(Rs_i m_{ik})$. But $R(s_i m_{ik})$ is a nilpotent submodule of M_i . Hence $R(sm_{ik})$ is nilpotent in M_i , for each $i \in I'$ and $k = 0, 1, \dots, n$, by [1, Proposition]. Therefore, $(R(sm_{ik}) :_R M_i)$ is a nilpotent ideal of R , for each $i \in I'$, since M_i 's are faithful. But $R(sm_k) \subseteq \prod_{i \in I} R(sm_{ik})$ and also we have

$$\begin{aligned} (R(sm_k) : \prod_{i \in I} M_i) &\subseteq (\prod_{i \in I} R(sm_{ik}) :_R \prod_{i \in I} M_i) \\ &\subseteq (\prod_{i \in I'} R(sm_{ik}) :_R \prod_{i \in I'} M_i) \\ &= \bigcap_{i \in I'} (R(sm_{ik}) :_R M_i). \end{aligned}$$

Thus, $sm_k \in Nil(M)$, for each $k = 0, 1, \dots, n$, since finite intersection of nilpotent ideals is nilpotent. Consequently, $M = \prod_{i \in I} M_i$ is weak McCoy. $\square$

Proposition 2.14. *Every finite direct product of faithful weak McCoy modules is weak McCoy.*

Proof. Let $M = \prod_{k=1}^n M_k$, where M_k 's are faithful weak McCoy R -modules. So M is a faithful R -module. Let $m(x) = \sum_{i=0}^p m_i x^i \in M[x] \setminus \{0\}$ and $f(x) \in R[x] \setminus \{0\}$ with $f(x)m(x) = 0$. Since $m_i \in M$, for each i , $m_i = (m_{i1}, \dots, m_{in})$, for some $m_{ik} \in M_k$.

Set $m_k(x) = \sum_{i=0}^p m_{ik} x^i \in M_k[x]$, for each $1 \leq k \leq n$. Hence $f(x)m_k(x) = 0$, for each $1 \leq k \leq n$. By assumption, there exists $c_k \in R \setminus \{0\}$ such that $c_k m_{ik} \in Nil(M_k)$. Therefore, $R(c_k m_{ik})$ is nilpotent in M_k . So the ideal $(R(c_k m_{ik}) :_R M_k)$ is nilpotent, for each $1 \leq k \leq n$ and $0 \leq i \leq p$, by [1, Proposition 4]. Consider $c = c_1 c_2 \dots c_n$. Clearly, $(R(cm_{ik}) :_R M_k) = (R(c_k m_{ik}) :_R M_k)$. Thus $(R(m_{ik}) :_R M_k)$ is a nilpotent ideal, for each $i = 0, 1, \dots, p$ and $k = 0, 1, \dots, n$.

On the other hand,

$$(R(cm_i) :_R M) = (\prod_{k=1}^n R(cm_{ik}) :_R \prod_{k=1}^n M_k) = \bigcap_{k=1}^n (R(cm_{ik}) :_R M_k).$$

Intersection of finite number of nilpotent ideals is nilpotent. This implies that $(R(cm_i) :_R M)$ is a nilpotent ideal and therefore $cm_i \in Nil(M)$, for each $i = 0, 1, \dots, p$. $\square$

Proposition 2.15. *A ring R is weak McCoy if and only if it is weak McCoy as an R -module.*

Proof. We know R is a multiplication R -module. So by [1, Theorem 6], $Nil(R) = Nil(R)R$. Therefore the proof is obvious. $\square$

Proposition 2.16. *Let R be a ring and N a submodule of an R -module M such that $\frac{M}{N}$ is a faithful weak McCoy R -module. Then M is weak McCoy.*

Proof. Let $m(x) = \sum_{i=0}^n m_i x^i \in M[x] \setminus \{0\}$ and $f(x) = \sum_{j=0}^k a_j x^j \in R[x] \setminus \{0\}$ with $f(x)m(x) = 0$. Set $\bar{m}(x) = \sum_{i=0}^n \bar{m}_i x^i = \sum_{i=0}^n (m_i + N)x^i \in (\frac{M}{N})[x] \setminus \{0\}$. Hence $f(x)\bar{m}(x) = \bar{0} = N$. By hypothesis, there exists $c \in R \setminus \{0\}$ such that

$c\overline{m}_i \in Nil(\frac{M}{N})$, for every $0 \leq i \leq n$. Since R -module $\frac{M}{N}$ is faithful, by [1, Proposition 4], $(R(c\overline{m}_i) :_R \frac{M}{N})$ is a nilpotent ideal of R , for each $0 \leq i \leq n$. But $(R(cm_i) :_R M) \subseteq (R(c\overline{m}_i) :_R \frac{M}{N})$. Thus $(R(cm_i) :_R M)$ is a nilpotent ideal and hence $cm_i \in Nil(M)$ for each i . Therefore, M is weak McCoy. $\square$

For a commutative domain R and an R -module M , the torsion submodule of M is defined by $T(M) = \{x \in M | ann_R(x) \neq 0\}$. In this case, $T(M)$ is a submodule of M , called the torsion part of M . An R -module M is called a *torsion* module if $T(M) = M$.

Proposition 2.17. *Let D be a domain and M a D -module. If D -module $T(M)$ is weak McCoy, then M is also weak McCoy.*

Proof. Suppose that $T(M)$ is weak McCoy. Let $f(x) = \sum_{i=0}^n a_i x^i \in D[x] \setminus \{0\}$ and $m(x) = \sum_{j=0}^n m_j x^j \in M[x]$ satisfy $f(x)m(x) = 0$. We have $a_0 m_0 = a_0 m_1 + a_1 m_0 = a_0 m_2 + a_1 m_1 + a_2 m_0 = \dots = a_k m_n = 0$.

Since $f(x) \neq 0$, we may assume that $a_0 \neq 0$. Multiplying $a_0 m_1 + a_1 m_0 = 0$ by a_0 implies that $a_0^2 m_1 = 0$. Similarly, multiplying $a_0 m_2 + a_1 m_1 + a_2 m_0 = 0$ by a_0^2 implies that $a_0^3 m_2 = 0$. Continuing this process, we have $a_0^{j+1} m_j = 0$ and so $m_j \in T(M)$, for $j = 0, 1, \dots, n$. Hence $m(x) \in T(M)[x]$. Since $T(M)$ is a weak McCoy D -module, there exists $d \in D \setminus \{0\}$ such that $dm_j \in Nil(T(M))$. Therefore, for each $0 \leq j \leq n$, there exists positive integer k_j such that $(D(dm_j) :_D T(M))^{k_j} D(dm_j) = 0$.

But $(D(dm_j) :_D M) \subseteq (D(dm_j) :_D T(M))$. Thus, $(D(dm_j) :_D M)^{k_j} D(dm_j) = 0$, for $j = 0, 1, \dots, n$. Therefore, $dm_j \in Nil(M)$, for $j = 0, 1, \dots, n$ and hence M is a weak McCoy D -module. $\square$

Theorem 2.18. *Let R be an integral domain and Q the quotient field of R . If M is a weak McCoy Q -module, then M is weak McCoy as an R -module and the converse is true if M is finitely generated.*

Proof. Suppose that M is a weak McCoy Q -module. Let $f(x) = \sum_{i=0}^n a_i x^i \in R[x]$ and $m(x) = \sum_{j=0}^n m_j x^j \in M[x]$ satisfy $f(x)m(x) = 0$. First note that every element $q \in Q$ has the form rs^{-1} for some regular element $s \in R$. Since M is weak McCoy as an Q -module, there exists a nonzero element $q \in Q$ such that $qm_j \in Nil_Q(M)$, for every $j = 0, 1, \dots, k$. We have $q = rs^{-1}$. So $rm_j = sqm_j \in Nil_Q(M)$, for every $j = 0, 1, \dots, k$. Moreover, M is a faithful Q -module, since $M \neq 0$. Hence the ideal $(Q(rm_j) :_Q M)$ is a nilpotent ideal of Q , for each $0 \leq j \leq k$, by [1, Theorem 4(1)]. Hence M is a weak McCoy R -module.

For the converse, suppose that M is a finitely generated weak McCoy R -module. Let $f(x) = \sum_{i=0}^n a_i x^i \in Q[x] \setminus \{0\}$ and $m(x) = \sum_{j=0}^n m_j x^j$ belongs to the $Q[x]$ -module $M[x]$. We may assume that $a_i = a'_i s_i^{-1}$, for every $0 \leq i \leq n$ and for some $a'_i \in R$ and regular element $s_i \in R$. Setting $s = s_0 s_1 \dots s_n$, we have $a_i = a'_i s^{-1}$, for every $0 \leq i \leq n$. Then $0 = f(x)m(x) = \sum_{i=0}^n \sum_{j=0}^k s^{-1} a'_i m_j x^{i+j} = s^{-1} f(x)m(x)$. So $f'(x)m(x) = 0$.

Since M is weak McCoy as an R -module, there exists $r \in R \setminus \{0\} (\subseteq Q \setminus \{0\})$ such that $rm_j \in \text{Nil}(M)$. By assumption, M is a faithful R -module. Thus $(R(rm_j) :_R M)$ is a nilpotent ideal of R , for each $j = 0, 1, \dots, k$. But $Q(R(rm_j) :_R M) = (Q(rm_j) :_Q M)$, since M is finitely generated. Hence $(Q(rm_j) :_Q M)$ is a nilpotent ideal of Q and so $rm_j \in \text{Nil}_Q(M)$, for every $0 \leq j \leq k$. Therefore, M is a weak McCoy Q -module. $\square$

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