

SOME RESULTS ON DISTANCED-BALANCED AND STRONGLY DISTANCE-BALANCED GRAPHS

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Abstract. Distance-balanced graphs are graphs in which for every edge $e = uv$ the number of vertices closer to u than to v is equal to the number of vertices closer to v than to u . The graph is strongly distance-balanced if the relation holds for every distance of i which $0 \leq i \leq \text{diam}(G)$. In this paper, we study some local properties of them and also under some graph operations.

Keywords: distance-balanced graph, strongly distance-balanced graph, graph invariant, graph operation.

1. Introduction and preliminaries

Let G be a simple undirected graph. The distance $d_G(u, v)$ between vertices $u, v \in V(G)$ is the length of a shortest path between u and v in G . For a pair of adjacent vertices $a, b \in V(G)$ let W_{ab} denote the set of all vertices of G closer to a than to b and let ${}_aW_b$ denote the set of all vertices of G that are at the same distance to a and b . For each $i \geq 0$ let W_{ab}^i and ${}_aW_b^i$ be the subsets of W_{ab} and ${}_aW_b$ respectively, of all the vertices at distance i to a . Therefore

$$W_{ab} = \{x \in V(G) | d(a, x) < d(b, x)\}, {}_aW_b = \{x \in V(G) | d(a, x) = d(b, x)\},$$

$$W_{ab}^i = \{x \in W_{ab} | d(a, x) = i\}, {}_aW_b^i = \{x \in {}_aW_b | d(a, x) = i\}.$$

Distance-balanced graphs were introduced as graphs that $|W_{ab}| = |W_{ba}|$ for every pair of adjacent vertices $a, b \in V(G)$. A graph G is strongly distance-balanced if $W_{ab}^i = W_{ba}^i$ holds for every pair of adjacent vertices $a, b \in V(G)$ and every $i \geq 0$. For more information, we recommend the readers to look at [4, 1, 2, 5]. Note that every strongly distance-balanced graph is distance-balanced. Throughout this article, graphs are supposed to be simple and connected. Moreover, we denote the neighbours of v by $N(v)$.

2. Local properties

Let G be a regular graph with v vertices and degree k . G is said to be strongly regular with (v, k, μ, λ) if there are also integers λ and μ such that:

1. Every two adjacent vertices have μ common neighbours.
2. Every two non-adjacent vertices have λ common neighbours.

Theorem 2.1. *Let G is a strongly regular graph with (v, k, μ, λ) , then G is strongly distance-balanced and hence distance-balanced.*

Proof. Let $\lambda = 0$, then every $v \in V(G)$ is connected to the neighbours of $N(v)$, therefore G is a complete graph and strongly distance-balanced. On the other hand, if $\lambda \neq 0$, then for every $uv \in E(G)$, $|W_{uv}^1| = |W_{vu}^1| = k - \mu$ and since $\dim(G) \leq 2$ then $|W_{uv}^2| = |W_{vu}^2| = |V(G)| - (k - \mu) - |{}_uW_v^2|$. Therefore G is strongly distance-balanced and hence distance-balanced.

Let $u \in V(G)$. We define $d(u, G)$ and $d_i(u, G)$ by

$$d(u, G) = \sum_{v \in V(G)} d(u, v)$$

and

$$d_i(u, G) = \sum_{v \in V(G), d(u, v) = i} d(u, v),$$

respectively, which $1 \leq i \leq \text{diam}(G)$. Moreover, $\text{diam}(G)$ is the length of the longest shortest path between any two graph vertices of G .

Theorem 2.2. *G is strongly distance-balanced if and only if $d_i(u, G) = d_i(v, G)$ for each $1 \leq i \leq \text{diam}(G)$ and for each $u, v \in V(G)$.*

Proof. Suppose $d_i(u, G) = d_i(v, G)$ for each $1 \leq i \leq \text{diam}(G)$ and for each $u, v \in V(G)$. It is clear that if $d_1(u, G) = d_1(v, G)$ then $|W_{uv}^1| = |W_{vu}^1|$. Now By induction, suppose the statement holds for $i \leq k$:

$$d_i(u, G) = d_i(v, G) \rightarrow |W_{uv}^i| = |W_{vu}^i|, 1 \leq i \leq k.$$

Let $d_{k+1}(u, G) = d_{k+1}(v, G)$, then

$$d_{k+1}(u, G) = d_{k+1}(u, W_{uv}^{k+1}) + (d_k(v, W_{vu}^k) + |W_{vu}^k|) + d_{k+1}(u, {}_uW_v^{k+1}) =$$

$$d_{k+1}(v, G) = d_{k+1}(v, W_{vu}^{k+1}) + (d_k(u, W_{uv}^k) + |W_{uv}^k|) + d_{k+1}(v, W_u^{k+1}).$$

Therefore

$$d_{k+1}(u, W_{uv}^{k+1}) = d_{k+1}(v, W_{vu}^{k+1}),$$

which implies that

$$|W_{uv}^{k+1}| = |W_{vu}^{k+1}|.$$

For sufficiency, let G be a strongly distance-balanced graph, which means that for each $1 \leq i \leq \text{diam}(G)$ and $uv \in E(G)$, $|W_{uv}^i| = |W_{vu}^i|$. Therefore

$$\begin{aligned} |W_{uv}^i| \times i + |W_{vu}^{i-1}| \times (i-1) + |W_v^i| \times i = \\ |W_{vu}^i| \times i + |W_{uv}^{i-1}| \times (i-1) + |W_u^i| \times i, \end{aligned}$$

which shows that

$$d_i(u, G) = d_i(v, G).$$

Let G be a graph and $e \in E(G)$, then by $G.e$ we mean contraction of e that is the operation which removes the edge from the graph while simultaneously merging the two vertices that it previously joined. By $G.[e_1, \dots, e_n]$, we mean $((G.e_1).e_2) \dots .e_n$.

Theorem 2.3. *If G is regular, then G is distance-balanced if and only if*

$$W_{uv}(G.[up_i]_{uv \neq up_i \in E(G)}) = W_{vu}(G.[vq_j]_{vu \neq vq_j \in E(G)})$$

for each $u, v \in V(G)$, which $[up_i]_{uv \neq up_i \in E(G)}$ and $[vq_j]_{vu \neq vq_j \in E(G)}$ are all edges connected to u and v respectively, except for uv and vu .

Proof. Since

$$|W_{uv}| = |W_{uv}^1| + |W_{p_1u}| + \dots + |W_{p_nu}| - C_{uv},$$

where $p_i (1 \leq i \leq n)$ are vertices connected to u that $p_i \neq v$ and C_{uv} consists of members x such that

1. if $x \in W_{p_iu}$ but $x \in W_{vu}$ or $x \in {}_uW_v$.
2. $x \in W_{p_iu}$ for more than one i .

Since members of C_{uv} and $W_{p_iu} (p_i \in N(u) \setminus v)$ are not related to neighbours of u and v , therefore

$$W_{uv}(G.[up_i]_{uv \neq up_i \in E(G)}) = (|W_{uv}| - |W_{uv}^1|) + |{}_uW_v|$$

Hence

$$\begin{aligned} |W_{uv}| &= |W_{vu}|, \\ |W_{uv}^1| + |W_{up_1}| + \dots + |W_{up_i}| - |C_{uv}| + |{}_uW_v| &= \\ = |W_{vu}^1| + |W_{vq_1}| + \dots + |W_{vq_i}| - |C_{vu}| + |{}_vW_u| &= \\ W_{uv}(G.[up_i]_{uv \neq up_i \in E(G)}) &= W_{vu}(G.[vq_j]_{vu \neq vq_j \in E(G)}). \end{aligned}$$

Graph G is a chordal graph if all its cycles of four or more vertices have a chord, which is an edge that is not part of the cycle but connects two vertices of the cycle.

Theorem 2.4. *Let G be a chordal graph. G is complete if G is distance-balanced.*

Proof. If G be a chordal graph on 3 vertices and distance-balanced, then G is complete. By induction, suppose G is a chordal graph on $n > 3$ vertices and distance-balanced, then G is complete. Now let G be a graph on $n + 1$ vertices, T be the left corner triangle in G , $u \in V(T)$ and x, y be other vertices of T . If there is no other vertex connected to u , therefore G is not distance-balanced ($|W_{ux}| \neq |W_{xu}|$). Since the graph is chordal, suppose p_1 be a vertex such that $up_1, xp_1 \in E(G)$. If there is no vertex connected to p_1 , then G is not distance-balanced ($|W_{p_1u}| \neq |W_{up_1}|$). By chordal property, let p_2 be a vertex on such that $p_1p_2, up_2 \in E(G)$. By continuing this method, u is connected to all vertices of G . By removing u from G , $G - \{u\}$ is again chordal and distance-balanced. Therefore, $G - \{u\}$ is complete which implies that the statement holds for each graph with $n > 3$ vertices.

Let G be a graph, $u \in V(G)$, $N(u) = \{p_1, p_2, \dots, p_n\}$ and $1 \leq i < j \leq n$. We denote the set $\{W_{up_1} \cap \dots \cap W_{up_{i-1}} \cap W_{up_{i+1}} \cap \dots \cap W_{up_{j-1}} \cap W_{up_{j+1}} \cap \dots \cap W_{up_n}\}$ by $S_u(p_i, p_j)$.

Lemma 2.5. *Let G be a distance-balanced graph which $\text{diam}(G) > 1$. Then G is triangle-free if and only if for each $uv, uw \in E(G)$, $|S_u(v, w) \cap W_{wu} \cap_u W_v| = |S_u(v, w) \cap W_{vu} \cap_u W_w|$.*

Proof. Let $uv, uw \in E(G)$, then by the property of triangle-free, we have

$$_u W_v \cap W_{wu} \cap S_u(v, w) = \{x \in S_u(v, w) | d(w, x) = d(v, x) + 1\},$$

$$_u W_w \cap W_{vu} \cap S_u(v, w) = \{x \in S_u(v, w) | d(v, x) = d(w, x) + 1\}.$$

Now, we define the function

$$f : _u W_v \cap W_{wu} \cap S_u(v, w) \rightarrow _u W_w \cap W_{vu} \cap S_u(v, w),$$

$$x \rightarrow y \quad (y \in p(v, \dots, y, x)),$$

where $p(v, \dots, y, x)$ is the shortest path between v and x . Since w is not connected to v (triangle-free), the function is well-defined and bijective. Therefore

$$|W_{wu} \cap_u W_v \cap S_u(v, w)| = |W_{vu} \cap_u W_w \cap S_u(v, w)|.$$

For the sufficiency, suppose $N(u) = \{v, w, p_1, p_2, \dots, p_n\}$ and for each $uv, uw \in E(G)$, $|S_u(v, w) \cap W_{wu} \cap_u W_v| = |S_u(v, w) \cap W_{vu} \cap_u W_w|$. By contrary, if $vw \in E(G)$, then $|S_u(v, p_i) \cap W_{p_iu} \cap_u W_v| = 0$ for each $1 \leq i \leq n$. Therefore $vp_i \in E(G)$ for each $1 \leq i \leq n$, and consequently neighbours of any vertex are connected together and by theorem 2.4, the graph is complete.

Theorem 2.6. *Let G be a distance-balanced graph which $\text{diam}(G) > 1$. Then for each $uv, ab \in E(G)$, $|W_{vu}| = |W_{ab}|$ if and only if G is triangle-free and for each $u \in V(G)$ and $uv, uw \in E(G)$, $|S_u(v, w) \cap W_{vu}| = |S_u(v, w) \cap W_{wu}|$.*

Proof. Let G be a distance-balanced graph which $\text{diam}(G) > 1$, $uv, uw \in E(G)$ and for each $uv, ab \in E(G)$, $|W_{vu}| = |W_{ab}|$. Then

$$|W_{uv}| = \deg(u) + |W_{uv}G.[up_i]_{uv \neq up_i \in E(G)}| - |{}_uW_v|,$$

$$|W_{uw}| = \deg(u) + |W_{uw}G.[up_i]_{uw \neq up_i \in E(G)}| - |{}_uW_w|.$$

Therefore we have

$$|W_{uv}| + |{}_uW_v| = \deg(u) + |W_{uv}G.[up_i]_{uv \neq up_i \in E(G)}|,$$

$$|W_{uw}| + |{}_uW_w| = \deg(u) + |W_{uw}G.[up_i]_{uw \neq up_i \in E(G)}|$$

and

$$|V(G)| - |W_{uv}| = \deg(u) + |W_{uv}G.[up_i]_{uv \neq up_i \in E(G)}|,$$

$$|V(G)| - |W_{uw}| = \deg(u) + |W_{uw}G.[up_i]_{uw \neq up_i \in E(G)}|,$$

which implies that

$$|W_{uv}G.[up_i]_{uv \neq up_i \in E(G)}| = |W_{uw}G.[up_i]_{uw \neq up_i \in E(G)}|.$$

Therefore

$$\begin{aligned} |(W_{p_1u} \cup \dots \cup W_{p_nu} \cup W_{wu}) \cap (W_{uv} \cup {}_uW_v)| = \\ |(W_{p_1u} \cup \dots \cup W_{p_nu} \cup W_{vu}) \cap (W_{uw} \cup {}_uW_w)|, \end{aligned}$$

which shows that

$$\begin{aligned} |(W_{uv} \cup {}_uW_v)| - |(W_{up_1} \cap \dots \cap W_{up_n} \cap W_{uw})| = \\ |(W_{uw} \cup {}_uW_w)| - |(W_{up_1} \cap \dots \cap W_{up_n} \cap W_{uv})|. \end{aligned}$$

Therefore we have

$$\begin{aligned} |(W_{up_1} \cap \dots \cap W_{up_n} \cap W_{uw})| = \\ |(W_{up_1} \cap \dots \cap W_{up_n} \cap W_{uv})|, \end{aligned}$$

which implies that

$$\begin{aligned} |(W_{up_1} \cap \dots \cap W_{up_n} \cap W_{uw} \cap W_{vu})| = \\ |(W_{up_1} \cap \dots \cap W_{up_n} \cap W_{uv} \cap W_{wu})|, \end{aligned}$$

and finally

$$\begin{aligned} |(S_u(v, w) \cap W_{uw} \cap W_{vu})| = \\ |(S_u(v, w) \cap W_{uv} \cap W_{wu})|. \end{aligned}$$

On the other hand,

$$|W_{wu}| = |W_{wu} \cap_u W_v| + |W_{wu} \cap W_{uv}| + |W_{wu} \cap W_{vu}|$$

$$|W_{vu}| = |W_{vu} \cap_u W_w| + |W_{vu} \cap W_{uw}| + |W_{vu} \cap W_{wu}|.$$

Therefore

$$\begin{aligned} |S_u(v, w) \cap W_{wu}| - |S_u(v, w) \cap W_{wu} \cap_u W_v| = \\ |S_u(v, w) \cap W_{vu}| - |S_u(v, w) \cap W_{vu} \cap_u W_w|. \end{aligned}$$

Since $(S_u(v, w) \cap W_{wu} \cap_u W_v) \subseteq (S_u(v, w) \cap W_{wu})$ and $(S_u(v, w) \cap W_{vu} \cap_u W_w) \subseteq (S_u(v, w) \cap W_{vu})$, we have

$$|S_u(v, w) \cap W_{wu} \cap_u W_v| = |S_u(v, w) \cap W_{vu} \cap_u W_w|$$

and

$$|S_u(v, w) \cap W_{wu}| = |S_u(v, w) \cap W_{vu}|.$$

Therefore by lemma 2.5, the graph is triangle-free and $|S_u(v, w) \cap W_{wu}| = |S_u(v, w) \cap W_{vu}|$ for each $u \in V(G)$ and $v, w \in N(u)$. The sufficiency is straightforward with regard to the reverse of the proof and lemma 2.5.

Corollary 2.7. *Let G be a k -regular distance-balanced and triangle-free graph with $\text{diam}(G) = 2$. Then $|W_{ab}| = k$ for each $ab \in E(G)$.*

Lemma 2.8. *Let G be a connected graph with at least one leaf and $n > 2$ vertices. Then G is not distance-balanced.*

Proof. Let $uv \in E(G)$ which v be a leaf of G . Then $|W_{vu}| = 0$ which implies that $|W_{vu}| < |W_{uv}|$.

Lemma 2.9. *If G is a connected distance-balanced graph, then $\kappa(G) \geq 2$.*

Proof. By contrary, suppose that $\kappa(G) = 1$ and $u \in V(G)$ be the cut-vertex, and $C_1, C_2, \dots, C_n$ be components of $G - \{u\}$. Let C_p be the component such that

$$|V(C_p)| \leq |V(C_i)| \quad (1 \leq i \leq n).$$

Then by selecting $e = uv$ such that $v \in V(C_p)$, we have $|W_{uv}| > |W_{vu}|$.

Theorem 2.10. *Let G be a connected distance-balanced graph. Then for each $u, v \in V(G)$, there exists a cycle containing u and v .*

Proof. The proof is clear up to lemma 2.8 and lemma 2.9.

Wiener index W is a graph index defined for a graph on n vertices by

$$W = 1/2 \sum_{i=1}^n \sum_{j=1}^n d_{ij},$$

where d_{ij} is the graph distance matrix. In [7], it has been conjectured that C_n has the most wiener value among distance-balanced graphs.

Theorem 2.11. *Let G be a connected distance-balanced graph with $n > 2$ vertices. Then*

$$W(G) \leq W(C_n).$$

Proof. Since $W(G + e) \leq W(G)$, then suppose G does not contain C_n as its subgraph. Moreover, since $W(G) = \frac{n \times d(u, G)}{2}$, and the fact that between every $u, v \in V(G)$, there exists a cycle in G containing u and v , thus

$$d(u, G) \leq d(x, C_n),$$

where $x \in V(C_n)$ and therefore

$$W(G) \leq W(C_n).$$

3. Strongly distance-balanced graphs under product graphs

All of the graph products constructed from two graphs G and H have vertex set $V(G) \times V(H)$. Let (a, u) and (b, v) be two vertices in $V(G) \times V(H)$. They are adjacent in the Cartesian product $G \square H$ if they are equal in one coordinate and adjacent in the other and are adjacent in the direct product $G \times H$ if they are adjacent in both coordinates. The edge set of the strong product $G \boxtimes H$ is $E(G \times H) \cup E(G \square H)$. These vertices are adjacent in the lexicographic product $G \circ H$ if $ab \in E(G)$ or if $a = b$ and $uv \in E(H)$. The edge set of disjunction $G \vee H$ of graphs G and H is such that (u_1, v_1) is adjacent to (u_2, v_2) whenever $u_1 u_2 \in E(G)$ or $v_1 v_2 \in E(H)$, see [3]. Moreover, the edge corona of two graphs G and H is denoted by $G \diamond H$ is obtained by taking a copy of G and $|E(G)|$ copies of H and joining each end vertices of i -th edge of G to every vertex in the i -th copy of H . We have denoted $ab \in E(G \diamond H)$ if both a and b belong to G and $(ab, x) \in E(G \diamond H)$ if $x \in V(H)$ is connected to $a, b \in V(G)$ ($ab \in E(G)$) in $G \diamond H$. For more information, refer to [3].

By G_i and H_j , we mean i -th copy and j -th copy of G and H , respectively. Moreover, $|W_{ab}|_G$, $|_a W_b|_G$ and $|_a W_b^1|_G$ denotes $|W_{ab}|$ and $|_a W_b|$ in graph G .

Theorem 3.1. *$G \vee H$ is strongly distance-balanced if and only if G and H are strongly distance-balanced.*

Proof. Let G, H be strongly distance-balanced graphs on n and m vertices, respectively. Then

$$\begin{aligned} |W_{(a,x)(b,y)}^1| &= |W_{ab}^1|_{G_1} + \cdots + |W_{ab}^1|_{G_m} + |W_{xy}^1|_{H_1} + \cdots + |W_{xy}^1|_{H_n} \\ &\quad - \deg(a)\deg(x) - |_{(a,x)} W_{(b,y)}^1| \\ |W_{(b,y)(a,x)}^1| &= |W_{ba}^1|_{G_1} + \cdots + |W_{ba}^1|_{G_m} + |W_{yx}^1|_{H_1} + \cdots + |W_{yx}^1|_{H_n} \\ &\quad - \deg(b)\deg(y) - |_{(a,x)} W_{(b,y)}^1|, \end{aligned}$$

and consequently,

$$|W_{(a,x)(b,y)}^2| = |V(G)| \times |V(H)| - |W_{(a,x)(b,y)}^1| - |_{(a,x)} W_{(b,y)}^2|.$$

Therefore,

$$|W_{(a,x)(b,y)}^1| = m(deg(a) - |{}_aW_b^1|_G) + n(deg(x) - |{}_xW_y^1|_H) - deg(a)deg(x) - |{}_{(a,x)}W_{(b,y)}^1|,$$

$$|W_{(b,y)(a,x)}^1| = m(deg(b) - |{}_aW_b^1|_G) + n(deg(y) - |{}_xW_y^1|_H) - deg(b)deg(y) - |{}_{(a,x)}W_{(b,y)}^1|.$$

Hence $G \vee H$ is strongly distance-balanced. Now, suppose $G \vee H$ be a strongly distance-balanced. Since $|W_{(a,x)(b,y)}^1| = |W_{(b,y)(a,x)}^1|$, then by summing up over all $ab \in E(G)$, we have:

$$\begin{aligned} & \sum_{ab \in E(G)} m(deg(a) - deg(b)) + n(deg(x) - deg(y)) \\ &= \sum_{ab \in E(G)} (deg(b)deg(y) - deg(a)deg(x)) \\ & mn \frac{|E(G)|}{2} - mn \frac{|E(G)|}{2} + n^3(deg(x) - deg(y)) \\ &= n \frac{|E(G)|}{2} deg(y) - n \frac{|E(G)|}{2} deg(x) \\ & n^2(deg(x) - deg(y)) = \frac{|E(G)|}{2} (deg(y) - deg(x)). \end{aligned}$$

Since $|E(G)| < 2n^2$, therefore $deg(y) = deg(x)$, $deg(b) = deg(a)$ and G, H are strongly distance-balanced.

Theorem 3.2. $G \diamond H$ is strongly distance-balanced and hence distance-balanced if and only if $G = P_2$ and H is complete.

Proof. Let $G = P_2$ and H be a complete graph, $ab \in E(G)$ and $xy \in E(H)$. Then

$$|W_{(ab,x)(ab,y)}^1| = |W_{xy \in H}^1|,$$

$$|W_{(ab,x)(ab,y)}^2| = |W_{(ab,y)(ab,x)}^2| = 0.$$

Therefore $|W_{(ab,x)(ab,y)}| = |W_{(ab,x)(ab,y)}^1| = deg_H(x) - |{}_xW_y^1|_H = |W_{(ab,y)(ab,x)}| = |W_{(ab,y)(ab,x)}^1| = deg_H(y) - |{}_xW_y^1|_H$. Moreover,

$$|W_{a(ab,x)}^1| = (deg_G(a) - 1) + (deg_G(a) - 1)|V(H)|,$$

$$|W_{a(ab,x)}| = (|V(G)| - 2) + (|E(G)| - 1)|V(H)| + (|V(H)| - deg_H(x) - 1) - |{}_aW_{(ab,x)}|,$$

$$|W_{(ab,x)a}| = 0.$$

Then

$$|W_{a(ab,x)}| = |W_{(ab,x)a}| = 0$$

and finally

$$|W_{ab}|_{G \diamond H} = |W_{ab}|_G + \sum_{c,d \in W_{ab} \text{ in } G, cd \in E(G)} |V(H)|,$$

which implies that

$$|W_{ab}|_{G \diamond H} = |W_{ba}|_{G \diamond H} = 0.$$

On the other hand, suppose that $G \diamond H$ is (strongly) distance-balanced graph. Then

$$|W_{(ab,x)(ab,y)}^1| = \deg_H(x) - |{}_x W_y^1|_H = |W_{(ab,y)(ab,x)}^1| = \deg_H(y) - |{}_y W_x^1|_H,$$

which implies that H is regular and $(\deg_G(a) - 1) + (\deg_G(a) - 1)|V(H)| = 0$ shows that $\deg_G(a) = 1$ for each $a \in V(G)$. Moreover, $|W_{a(ab,x)}| = 0$ implies that $\deg_H(x) = |V(H)| - 1$ for each $x \in V(H)$.

4. Minimum edges for distance-balanced

Let G be a graph and let $b(G)$ be the smallest number of edges that can be added to G such that the obtained graph is distance-balanced.

Lemma 4.1. *Let G, H be graphs. Then*

1. $b(G \square H) = b(G) \times |V(H)| + b(H) \times |V(G)|$.
2. $b(G \circ H) = b(G) \times |V(H)| + r(H) \times |V(G)|$.

where $r(H) = \frac{\deg_{\max}(H)+1 \times (V(H)-1)}{2} + 1$ if $\deg_{\max}(H)$ and $V(H) - 1$ are odd and $r(H) = \frac{\deg_{\max}(H) \times (V(H)-1)}{2}$, otherwise.

Proof.

1. Since $G \square H$ is distance-balanced if and only if G and H are distance-balanced, the proof is clear.
2. Since $G \circ H$ is distance-balanced if and only if G is distance-balanced and H is regular, the proof is clear.

Theorem 4.2. *Let G is a graph on n vertices with $\text{diam}(G) = 2$. then*

$$1/2 \sum_{uv \in E(G)} |\deg(u) - \deg(v)| \leq b(G) \leq |E(K_n)| - E(G).$$

Proof. Let $uv \in E(G)$, then for G to be distanced-balanced, we should have $d(u, G) = d(v, G)$. Therefore,

$$d(u, G) = \deg(u) + 2 \times (|V(G)| - \deg(u) - 1),$$

$$d(v, G) = \deg(v) + 2 \times (|V(G)| - \deg(v) - 1),$$

$$\begin{aligned} b(G) &\geq \sum_{uv \in E(G)} 1/2 |(\deg(u) + 2 \times (|V(G)| - \deg(u) - 1)) \\ &\quad - (\deg(v) + 2 \times (|V(G)| - \deg(v) - 1))| \\ &= 1/2 \sum_{uv \in E(G)} |\deg(u) - \deg(v)|. \end{aligned}$$

On the other hand, it is clear that $b(G) \leq |E(K_n)| - |E(G)|$.

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