

A POWER INDEX AND ITS NORMALIZATION UNDER FUZZY MULTICRITERIA SITUATION

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Abstract. By considering the supreme-utilities among fuzzy level vectors, we propose an index and its normalization in the framework of multicriteria fuzzy transferable-utility (TU) games. We show that there exists a reduced game that could be adopted to analyze these two indexes. Further, alternative formulation for the normalized index is also proposed by applying excess function. Based on reduced game and excess function, we introduce different dynamic processes for the normalized index.

Keywords: multicriteria fuzzy TU games, supreme-utilities, reduced game, excess function, dynamic process.

1. Introduction

In the framework of transferable-utility (TU) games, the power indexes may be defined to measure the political power of each member of a voting system. A member in a voting system is, e.g., a party in a parliament or a country in a confederation. Each member will have a certain number of votes, and so its power will be different. Results of the power indexes may be found in, e.g., Banzhaf [3], Dubey and Shapley [8], Haller [9], Lehrer [11], van den Brink and van der Laan [5] and so on. The theory of *fuzzy TU games* commenced with the investigation of Aubin [1, 2] where the opinions of a fuzzy TU game and the fuzzy core are introduced. Many fuzzy solutions have been applied wildly, e.g.,

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Borkotokey and Mesiar [4], Butnariu and Kroupa [6], Hwang [10], Li and Zhang [12], Meng and Zhang [14], Tijs et al. [19], and so on.

In the axiomatic formulation of cooperative games, *consistency* is a crucial characteristic of viable solutions. Consistency states the independence of a value with respect to fixing some agents with their assigned payoffs. It asserts that the recommendation made for any problem should always agree with the recommendation made in the subproblem that appears when the payoffs of some agents are settled on. It has been introduced in different ways depending upon how the payoffs of the agents that "leave the bargaining" are defined. This property has been investigated in various classes of problems by applying *reduced games* always. In addition to axiomatizations for solutions, *dynamic processes* also can be described that lead the agents to that solution, starting from an arbitrary efficient payoff vector. The foundation of a dynamic theory was laid by Stearns [18].

The above pre-existing results raise one motivation in the framework of fuzzy TU games:

- whether the power indexes could be extended under fuzzy behavior and multicriteria situation simultaneously.

The paper is devoted to answer the question. Different from the framework of fuzzy TU games, we consider the framework of *multicriteria fuzzy TU games* in Section 2. A power index and its normalization on multicriteria fuzzy TU games, the *fuzzy marginal index* and the *fuzzy normalized index*, are further introduced by considering the supreme-utilities among fuzzy level vectors. In order to present the rationality for these two indexes, we propose an extended reduction to analyze these two indexes in Section 3. In order to establish the dynamic processes of the fuzzy normalized index, we present alternative formulation for the fuzzy normalized index in terms of *excess functions*. In Section 4, we adopt reduced game and excess function to show that the fuzzy normalized index can be reached by agents who start from an arbitrary efficient payoff vector.

2. The fuzzy marginal index and its normalization

Let U be the universe of agents. For $i \in U$ and $b_i \in (0, 1]$, $B_i = [0, b_i]$ could be treated as the level (decision) space of agent i and $B_i^+ = (0, b_i]$, where 0 denotes no participation. Let $B^N = \prod_{i \in N} B_i$ be the product set of the level (decision) spaces of all agents of N . For all $T \subseteq N$, we define $\theta^T \in B^N$ is the vector with $\theta_i^T = 1$ if $i \in T$, and $\theta_i^T = 0$ if $i \in N \setminus T$. Denote 0_N the zero vector in $\mathbb{R}^N$. For $m \in \mathbb{N}$, let 0_m be the zero vector in $\mathbb{R}^m$ and $\mathbb{N}_m = \{1, 2, \dots, m\}$.

A **fuzzy TU game**¹ is a triple (N, b, v) , where N is a non-empty and finite set of agents, $b = (b_i)_{i \in N} \in \prod_{i \in N} B_i^+$ is the vector that presents the highest

1. A fuzzy TU game, which is defined by Aubin [1, 2], is a pair (N, v^a) , where N is a coalition and v^a is a mapping such that $v^a : [0, 1]^N \rightarrow \mathbb{R}$ and $v^a(0_N) = 0$. In fact, $(N, v^a) = (N, \theta^N, v)$.

levels for each agent, and $v : B^N \rightarrow \mathbb{R}$ is a characteristic mapping with $v(0_N) = 0$ which assigns to each $\alpha = (\alpha_i)_{i \in N} \in B^N$ the worth that the agents can gain when each agent i participates at level α_i . Given a fuzzy TU game (N, b, v) and $\alpha \in B^N$, we write $A(\alpha) = \{i \in N \mid \alpha_i \neq 0\}$ and α_T to be the restriction of α at T for each $T \subseteq N$. Further, we define $v_*(T) = \sup_{\alpha \in B^N} \{v(\alpha) \mid A(\alpha) = T\}$ is the **supreme-utility**² among all action vector α with $A(\alpha) = T$. A **multicriteria fuzzy TU game** is a triple (N, b, V^m) , where $m \in \mathbb{N}$, $V^m = (v^t)_{t \in \mathbb{N}_m}$ and (N, b, v^t) is a fuzzy TU game for all $t \in \mathbb{N}_m$.

Denote the collection of all multicriteria fuzzy TU games by Γ . Let $(N, b, V^m) \in \Gamma$. A **payoff vector** of (N, b, V^m) is a vector $x = (x^t)_{t \in \mathbb{N}_m}$ and $x^t = (x_i^t)_{i \in N} \in \mathbb{R}^N$, where x_i^t denotes the payoff to agent i in (N, b, v^t) for all $t \in \mathbb{N}_m$ and for all $i \in N$. A payoff vector x of (N, b, V^m) is **multicriteria efficient** if $\sum_{i \in N} x_i^t = v_*^t(N)$ for all $t \in \mathbb{N}_m$. The collection of all multicriteria efficient vector of (N, b, V^m) is denoted by $E(N, b, V^m)$. A **solution** is a map σ assigning to each $(N, b, V^m) \in \Gamma$ an element

$$\sigma(N, b, V^m) = (\sigma^t(N, b, V^m))_{t \in \mathbb{N}_m},$$

where $\sigma^t(N, b, V^m) = (\sigma_i^t(N, b, V^m))_{i \in N} \in \mathbb{R}^N$ and $\sigma_i^t(N, b, V^m)$ is the payoff of the agent i assigned by σ in (N, b, v^t) .

Next, we provide the fuzzy marginal index and the fuzzy normalized index under multicriteria situation.

Definition 1. The **fuzzy marginal index (FMI)**, β , is defined by

$$\beta_i^t(N, b, V^m) = v_*^t(N) - v_*^t(N \setminus \{i\})$$

for all $(N, b, V^m) \in \Gamma$, for all $t \in \mathbb{N}_m$ and for all $i \in N$. Under the solution β , all agents receive their **marginal contributions** related to supreme-utilities in N respectively.

A solution σ satisfies **multicriteria efficiency (MEFF)** if for all $(N, b, V^m) \in \Gamma$ and for all $t \in \mathbb{N}_m$, $\sum_{i \in N} \sigma_i^t(N, b, V^m) = v_*^t(N)$. Property MEFF asserts that all agents allocate all the utility completely. It is easy to check that the FMI violates EFF. Therefore, we consider an efficient normalization as follows.

Definition 2. The **fuzzy normalized index (FNI)**, $\bar{\beta}$, is defined by

$$\bar{\beta}_i^t(N, b, V^m) = \frac{v_*^t(N)}{\sum_{k \in N} \beta_k^t(N, b, V^m)} \cdot \beta_i^t(N, b, V^m)$$

for all $(N, b, V^m) \in \Gamma^*$, for all $t \in \mathbb{N}_m$ and for all $i \in N$, where $\Gamma^* = \{(N, b, V^m) \in \Gamma \mid \sum_{i \in N} \beta_i^t(N, b, V^m) \neq 0 \text{ for all } t \in \mathbb{N}_m\}$.

2. From now on we consider bounded fuzzy TU games, defined as those games (N, b, v) such that, there exists $K_v \in \mathbb{R}$ such that $v(\alpha) \leq K_v$ for all $\alpha \in B^N$. We adopt it to ensure that $v_*(T)$ is well-defined.

Lemma 1. *The FNI satisfies MEFF on Γ^* .*

Proof. For all $(N, b, V^m) \in \Gamma^*$ and for all $t \in \mathbb{N}_m$,

$$\begin{aligned} \sum_{i \in N} \bar{\beta}_i^t(N, b, V^m) &= \sum_{i \in N} \frac{v_*^t(N)}{\sum_{k \in N} \beta_k^t(N, b, V^m)} \cdot \beta_i^t(N, b, V^m) \\ &= \frac{v_*^t(N)}{\sum_{k \in N} \beta_k^t(N, b, V^m)} \sum_{i \in N} \beta_i^t(N, b, V^m) \\ &= v_*^t(N). \end{aligned}$$

Thus, the FNI satisfies MEFF on Γ^* . $\square$

Here we provide a brief application of multicriteria fuzzy TU games in the setting of “management”. This kind of problem can be formulated as follows. Let $N = \{1, 2, \dots, n\}$ be a set of all agents of a grand management system (N, b, V^m) . The function v^t could be treated as an utility function which assigns to each level vector $\alpha = (\alpha_i)_{i \in N} \in B^N$ the worth that the agents can obtain when each agent i participates at operation strategy $\alpha_i \in B_i$ in the sub-management system (N, b, v^t) . Modeled in this way, the grand management system (N, b, V^m) could be considered as a multicriteria fuzzy TU game, with v^t being each characteristic function and B_i being the set of all operation strategies of the agent i . In the following sections, we would like to show that the FMI and the FNI could provide “*optimal allocation mechanisms*” among all agents, in the sense that this organization can get payoff from each combination of operation strategies of all agents under fuzzy behavior and multicriteria situation.

3. Axiomatizations

In this section, we show that there exists a corresponding reduced game that could be adopted to analyze the FMI and the FNI.

First, we present alternative formulation for the FNI in terms of *excess*. Let $(N, b, V^m) \in \Gamma^*$, $S \subseteq N$ and x be a payoff vector in (N, b, V^m) . Define that $x^t(S) = \sum_{i \in S} x_i^t$ for all $t \in \mathbb{N}_m$. The **excess** of a coalition $S \subseteq N$ at x is the real number

$$(1) \quad e(S, V^m, x) = (e(S, v^t, x^t))_{t \in \mathbb{N}_m} \text{ and } e(S, v^t, x^t) = v_*^t(S) - x^t(S).$$

The value $e(S, v^t, x^t)$ can be treated as the **complaint** of coalition S when all agents receive their payoffs from x^t in (N, b, v^t) .

Lemma 2. *Let $(N, b, V^m) \in \Gamma^*$, $t \in \mathbb{N}_m$, $x \in E(N, b, V^m)$ and $a^t = \frac{v_*^t(N)}{\sum_{k \in N} \beta_k^t(N, b, V^m)}$.*

Then

$$e(N \setminus \{i\}, v^t, \frac{x^t}{a^t}) = e(N \setminus \{j\}, v^t, \frac{x^t}{a^t}) \quad \forall i, j \in N \iff x = \bar{\beta}(N, b, V^m).$$

Proof. Let $(N, b, V^m) \in \Gamma^*$ and $x \in E(N, b, V^m)$. For all $t \in \mathbb{N}_m$ and for all $i, j \in N$,

$$\begin{aligned}
 (2) \quad & e(N \setminus \{j\}, v^t, \frac{x^t}{a^t}) = e(N \setminus \{i\}, v^t, \frac{x^t}{a^t}) \\
 & \iff v_*^t(N \setminus \{j\}) - \frac{x^t(N \setminus \{j\})}{a^t} = v_*^t(N \setminus \{i\}) - \frac{x^t(N \setminus \{i\})}{a^t} \\
 & \iff v_*^t(N \setminus \{j\}) - \frac{x_j^t}{a^t} = v_*^t(N \setminus \{i\}) - \frac{x_i^t}{a^t} \\
 & \iff x_i^t - x_j^t = a^t \cdot [v_*^t(N \setminus \{j\}) - v_*^t(N \setminus \{i\})].
 \end{aligned}$$

By definition of $\bar{\beta}$,

$$(3) \quad \bar{\beta}_i^t(N, b, V^m) - \bar{\beta}_j^t(N, b, V^m) = a^t \cdot [v_*^t(N \setminus \{j\}) - v_*^t(N \setminus \{i\})].$$

By equations (2) and (3), for all $i, j \in N$,

$$x_i^t - x_j^t = \bar{\beta}_i^t(N, b, V^m) - \bar{\beta}_j^t(N, b, V^m).$$

Hence,

$$\sum_{j \neq i} [x_i^t - x_j^t] = \sum_{j \neq i} [\bar{\beta}_i^t(N, b, V^m) - \bar{\beta}_j^t(N, b, V^m)].$$

That is, $(|N| - 1) \cdot x_i^t - \sum_{j \neq i} x_j^t = (|N| - 1) \cdot \bar{\beta}_i^t(N, b, V^m) - \sum_{j \neq i} \bar{\beta}_j^t(N, b, V^m)$.

Since $x \in E(N, b, V^m)$ and $\bar{\beta}$ satisfies MEFF, $|N| \cdot x_i^t - v_*^t(N) = |N| \cdot \bar{\beta}_i^t(N, b, V^m) - v_*^t(N)$. Therefore, $x_i^t = \bar{\beta}_i^t(N, b, V^m)$ for all $t \in \mathbb{N}_m$ and for all $i \in N$, i.e., $x = \bar{\beta}(N, b, V^m)$. $\square$

Remark 1. It is easy to check that $e(N \setminus \{i\}, V^m, \beta(N, b, V^m)) = e(N \setminus \{j\}, V^m, \beta(N, b, V^m))$ for all $(N, b, V^m) \in \Gamma$ and for all $i, j \in N$.

Inspired by the complement-reduced game due to Moulin [15], we introduced a fuzzy extension and related consistency as follows. Let ψ be a solution, $(N, b, V^m) \in \Gamma$ and $S \subseteq N$. The **reduced game** $(S, b_S, V_{S,\psi}^m)$ is defined by $V_{S,\psi}^m = (v_{S,\psi}^t)_{t \in \mathbb{N}_m}$ and

$$v_{S,\psi}^t(\alpha) = \begin{cases} 0, & \text{if } \alpha = 0_S, \\ v_*(A(\alpha) \cup (N \setminus S)) - \sum_{i \in N \setminus S} \sigma_i(N, b, v), & \text{otherwise.} \end{cases}$$

ψ satisfies **consistency (CON)** if $\psi_i^t(S, b_S, V_{S,\psi}^m) = \psi_i^t(N, b_S, V^m)$ for all $(N, b, V^m) \in \Gamma$, for all $S \subseteq N$ with $|S| = 2$, for all $t \in \mathbb{N}_m$ and for all $i \in S$. Unfortunately, it is easy to check that $\sum_{k \in S} \beta_k^t(N, b, v) = 0$ for some $(N, b, V^m) \in G$, for some $t \in \mathbb{N}_m$ and for some $S \subseteq N$, i.e., $\bar{\beta}(S, b_S, V_{S,\psi}^m)$ doesn't exist for some $(N, b, V^m) \in \Gamma$ and for some $S \subseteq N$. Therefore, we consider the *resilient consistency* as follows. A solution ψ satisfies **resilient consistency (RCON)** if $(S, b_S, V_{S,\psi}^m)$ and $\psi(S, b_S, V_{S,\psi}^m)$ exist for some $(N, b, V^m) \in \Gamma$ and for some $S \subseteq N$ with $|S| = 2$, it holds that $\psi_i^t(S, b_S, V_{S,\psi}^m) = \psi_i^t(N, b, V^m)$ for all $t \in \mathbb{N}_m$ and for all $i \in S$.

Lemma 3.

1. The FMI satisfies CON on Γ .
2. The FNI satisfies RCON on Γ^* .

Proof. To verify result 1, let $(N, b, V^m) \in \Gamma^*$ and $S \subseteq N$. If $|N| = 1$, then the proof is completed. Assume that $|N| \geq 2$ and $S = \{i, j\}$ for some $i, j \in N$. For all $t \in \mathbb{N}_m$ and for all $i \in S$,

$$\begin{aligned}
 & \beta_i^t(S, b_S, V_{S,\beta}^m) \\
 &= (v_{S,\beta}^t)_*(S) - (v_{S,\beta}^t)_*(S \setminus \{i\}) \\
 (4) \quad &= \sup_{\alpha \in B^S} \{v_{S,\beta}^t(\alpha) | A(\alpha) = S\} - \sup_{\alpha \in B^S} \{v_{S,\beta}^t(\alpha) | A(\alpha) = S \setminus \{i\}\} \\
 &= v_*^t(N) - v_*^t(N \setminus \{i\}) \\
 &= \beta_i^t(N, b, V^m).
 \end{aligned}$$

Thus, the FMI satisfies CON.

To prove result 2, let $(N, b, V^m) \in \Gamma^*$ and $S \subseteq N$. If $|N| = 1$, then the proof is completed. Assume that $|N| \geq 2$. If $S = \{i, j\}$ for some $i, j \in N$ and $(S, b_S, v_{S,\bar{\beta}}^m) \in \Gamma^*$. Similar to the proof of equation (4), for all $t \in \mathbb{N}_m$ and for all $i \in S$,

$$(5) \quad \beta_i^t(S, b_S, V_{S,\bar{\beta}}^m) = \beta_i^t(N, b, V^m).$$

By definition of $\bar{\beta}$ and equation (5),

$$\begin{aligned}
 & \bar{\beta}_i^t(S, b_S, V_{S,\bar{\beta}}^m) \\
 &= \frac{(v_{S,\bar{\beta}}^t)_*(S)}{\sum_{k \in S} \beta_k^t(S, b_S, V_{S,\bar{\beta}}^m)} \cdot \beta_i^t(S, b_S, V_{S,\bar{\beta}}^m) \\
 &= \frac{v_*^t(N) - \sum_{k \in N \setminus S} \bar{\beta}_k^t(N, b, V^m)}{\sum_{k \in S} \beta_k^t(S, b_S, V_{S,\bar{\beta}}^m)} \cdot \beta_i^t(S, b_S, V_{S,\bar{\beta}}^m) \quad (\text{by definition of } V_{S,\bar{\beta}}^m) \\
 &= \frac{v_*^t(N) - \sum_{k \in N \setminus S} \bar{\beta}_k^t(N, b, V^m)}{\sum_{k \in S} \beta_k^t(N, b, V^m)} \cdot \beta_i^t(N, b, V^m) \quad (\text{by equation (5)}) \\
 &= \frac{\sum_{k \in S} \beta_k^t(N, b, V^m)}{\sum_{k \in S} \beta_k^t(N, b, V^m)} \cdot \beta_i^t(N, b, V^m) \quad (\text{by MEFF of } \bar{\beta}) \\
 &= a^t \cdot \beta_i^t(N, b, V^m), \text{ where } a^t = \frac{v_*^t(N)}{\sum_{k \in N} \beta_k^t(N, b, V^m)} \\
 &= \bar{\beta}_i^t(N, b, V^m).
 \end{aligned}$$

Thus, the FNI satisfies RCON on Γ^* . $\square$

Next, we characterize the FMI and the FNI by applying the properties of CON and RCON.

- A solution ψ satisfies **marginal-standard for games (MSG)** if $\psi(N, b, v) = \beta(N, b, v)$ for all $(N, b, v) \in \Gamma$ with $|N| \leq 2$.
- A solution ψ satisfies **normalized-standard for games (NSG)** if $\psi(N, b, v) = \bar{\beta}(N, b, v)$ for all $(N, b, v) \in \Gamma^*$ with $|N| \leq 2$.

Lemma 4. *If a solution ψ satisfies NSG and RCON on Γ^* , then it also satisfies MEFF on Γ^* .*

Proof. Let $(N, b, V^m) \in \Gamma^*$. If $|N| \leq 2$, then ψ satisfies MEFF on Γ^* by NSG. Suppose that $|N| > 2$. Assume, on the contrary, that there exists $(N, b_S, V^m) \in \Gamma^*$ such that $\sum_{i \in N} \psi_i^t(N, b, V^m) \neq v_*^t(N)$ for some $t \in \mathbb{N}_m$. This means that there exist $i \in N$ and $j \in N$ such that $[v_*^t(N) - \sum_{k \in N \setminus \{i, j\}} \psi_k^t(N, b, V^m)] \neq [\psi_i^t(N, b, V^m) + \psi_j^t(N, b, V^m)]$. By RCON and ψ satisfies MEFF for two-person games, this contradicts with

$$\begin{aligned} \psi_i^t(N, b, V^m) + \psi_j^t(N, b, V^m) &= \psi_i^t(\{i, j\}, V_{\{i, j\}, \psi}^m) + \psi_j^t(\{i, j\}, V_{\{i, j\}, \psi}^m) \\ &= v_*^t(N) - \sum_{k \in N \setminus \{i, j\}} \psi_k^t(N, b, V^m). \end{aligned}$$

Hence ψ satisfies MEFF. $\square$

Theorem 1.

1. On Γ , the FMI is the only solution satisfying MSG and CON.
2. On Γ^* , the FNI is the only solution satisfying NSG and RCON.

Proof. By Lemma 3, β and $\bar{\beta}$ satisfy CON and RCON on Γ and Γ^* respectively. Clearly, β and $\bar{\beta}$ satisfy MSG and NSG on Γ and Γ^* respectively.

To prove uniqueness of result 1, suppose ψ satisfies CON and MSG on Γ . Let $(N, b, V^m) \in \Gamma$. If $|N| \leq 2$, then $\psi(N, b, V^m) = \beta(N, b, V^m)$ by MSG. Suppose that $|N| > 2$. Let $t \in \mathbb{N}_m$ and $i \in N$. Assume that $S \subseteq N$ with $|S| = 2$ and $i \in S$. Then,

$$\begin{aligned} \psi_i^t(N, b, V^m) &= \psi_i^t(S, b_S, V_{S, \psi}^m) \quad (\text{by CON of } \psi) \\ &= \beta_i^t(S, b_S, V_{S, \psi}^m) \quad (\text{by MSG of } \psi) \\ &= (v_{S, \psi}^t)_*(S) - (v_{S, \psi}^t)_*(S \setminus \{i\}) \\ &= v_*^t(N) - v_*^t(N \setminus \{i\}) \\ &= \beta_i^t(N, b, V^m). \end{aligned}$$

Hence, $\psi(N, b, V^m) = \beta(N, b, V^m)$ for all $(N, b, V^m) \in \Gamma$.

To prove uniqueness of result 2, suppose ψ satisfies RCON and NSG on Γ^* . By Lemma 4, ψ satisfies MEFF on Γ^* . Let $(N, b, V^m) \in \Gamma^*$. We will complete the proof by induction on $|N|$. If $|N| \leq 2$, it is trivial that $\psi(N, b, V^m) = \bar{\beta}(N, b, V^m)$ by NSG. Assume that it holds if $|N| \leq r - 1$, $r \geq 3$. The case $|N| = r$: Let $t \in \mathbb{N}_m$ and $i, j \in N$ with $i \neq j$. By Definition 2, $\bar{\beta}_k^t(N, b, V^m) =$

$\frac{v_*^t(N)}{\sum_{h \in N} \beta_h^t(N, b, V^m)} \cdot \beta_k^t(N, b, V^m)$ for all $k \in N$. Assume that $\alpha_k^t = \frac{\beta_k^t(N, b, v)}{\sum_{h \in N} \beta_h^t(N, b, v)}$ for all $k \in N$. Therefore,

$$\begin{aligned}
 \psi_i^t(N, b, V^m) &= \psi_i^t(N \setminus \{j\}, V_{N \setminus \{j\}, \psi}^m) && \text{(by RCON of } \psi) \\
 &= \bar{\beta}_i^t(N \setminus \{j\}, V_{N \setminus \{j\}, \psi}^m) && \text{(by NSG of } \psi) \\
 &= \frac{(v_{N \setminus \{j\}, \psi}^t)_*(N \setminus \{j\})}{\sum_{k \in N \setminus \{j\}} \beta_k^t(N \setminus \{j\}, V_{N \setminus \{j\}, \psi}^m)} \cdot \beta_i^t(N \setminus \{j\}, V_{N \setminus \{j\}, \psi}^m) \\
 &= \frac{v_*^t(N) - \psi_i^t(N, b, V^m)}{\sum_{k \in N \setminus \{j\}} \beta_k^t(N, b, V^m)} \cdot \beta_i^t(N, b, V^m) && \text{(by equation (5))} \\
 &= \frac{v_*^t(N) - \psi_i^t(N, b, V^m)}{-\beta_j^t(N, b, V^m) + \sum_{k \in N} \beta_k^t(N, b, V^m)} \cdot \beta_i^t(N, b, V^m).
 \end{aligned}
 \tag{6}$$

By equation (6),

$$\begin{aligned}
 &\psi_i^t(N, b, V^m) \cdot [1 - \alpha_j^t] = [v_*^t(N) - \psi_j^t(N, b, V^m)] \cdot \alpha_j^t \\
 \Rightarrow &\sum_{i \in N} \psi_i^t(N, b, V^m) \cdot [1 - \alpha_j^t] = [v_*^t(N) - \psi_j^t(N, b, V^m)] \cdot \sum_{i \in N} \alpha_j^t \\
 \Rightarrow &v_*^t(N) \cdot [1 - \alpha_j^t] = [v_*^t(N) - \psi_j^t(N, b, V^m)] \cdot 1 \quad \text{(by MEFF of } \psi) \\
 \Rightarrow &v_*^t(N) - v_*^t(N) \cdot \alpha_j^t = v_*^t(N) - \psi_j^t(N, b, V^m) \\
 \Rightarrow &\bar{\beta}_j^t(N, b, V^m) = \psi_j^t(N, b, V^m).
 \end{aligned}$$

The proof is completed. $\square$

The following examples are to show that each of the axioms used in Theorem 1 is logically independent of the remaining axioms.

Example 1. Define a solution ψ by for all $(N, b, V^m) \in \Gamma$, for all $t \in \mathbb{N}_m$ and for all $i \in N$, $\psi_i^t(N, b, V^m) = 0$. Clearly, ψ satisfies CON and RCON on Γ and Γ^* , but it violates MSG and NSG on Γ and Γ^* .

Example 2. Define a solution ψ by for all $(N, b, V^m) \in \Gamma$, for all $t \in \mathbb{N}_m$ and for all $i \in N$,

$$\psi_i^t(N, b, V^m) = \begin{cases} \beta_i^t(N, b, V^m), & \text{if } |N| \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

On Γ , ψ satisfies MSG, but it violates CON.

Example 3. Define a solution ψ by for all $(N, b, V^m) \in \Gamma^*$, for all $t \in \mathbb{N}_m$ and for all $i \in N$,

$$\psi_i^t(N, b, V^m) = \begin{cases} \bar{\beta}_i^t(N, b, V^m), & \text{if } |N| \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

On Γ^* , ψ satisfies NSG, but it violates RCON.

4. Dynamic processes

In this section, we adopt excess function and reduction to propose dynamic processes for the FNI.

In order to establish the dynamic processes of the FNI, we firstly define correction function by means of excess functions. The correction function is based on the idea that, each agent shortens the complaint relating to his own and others' nonparticipation, and adopts these regulations to correct the original payoff.

Definition 3. Let $(N, b, V^m) \in \Gamma^*$ and $i \in N$. The **correction function** is defined to be $f = (f^t)_{t \in \mathbb{N}_m}$, where $f^t = (f_i^t)_{i \in N}$ and $f_i^t : E(N, b, V^m) \rightarrow \mathbb{R}$ is define by

$$f_i^t(x) = x_i^t + w \sum_{j \in N \setminus \{i\}} a^t \cdot \left(e(N \setminus \{j\}, v^t, \frac{x^t}{a^t}) - e(N \setminus \{i\}, v^t, \frac{x^t}{a^t}) \right),$$

where $a^t = \frac{v_*^t(N)}{\sum_{k \in N} \beta_k^t(N, b, V^m)}$ and $w \in \mathbb{R}$ is a fixed postive number, which reflects the assumption that agent i does not ask for full correction (when $w = 1$) but only (usually) a fraction of it. Define $[x]^0 = x$, $[x]^1 = f([x]^0)$, $\dots$, $[x]^q = f([x]^{q-1})$ for all $q \in \mathbb{N}$.

Lemma 5. $f(x) \in E(N, b, V^m)$ for all $(N, b, V^m) \in \Gamma^*$ and for all $x \in E(N, b, V^m)$.

Proof. Let $(N, b, V^m) \in \Gamma^*$, $t \in \mathbb{N}_m$, $i, j \in N$ and $x \in E(N, b, V^m)$.

$$\begin{aligned} (7) \quad & \sum_{j \in N \setminus \{i\}} a^t \cdot \left(e(N \setminus \{j\}, v^t, \frac{x^t}{a^t}) - e(N \setminus \{i\}, v^t, \frac{x^t}{a^t}) \right) \\ &= \sum_{j \in N \setminus \{i\}} a^t \cdot \left(v^t(N \setminus \{j\}) - \frac{x^t(N \setminus \{j\})}{a^t} - v^t(N \setminus \{i\}) + \frac{x^t(N \setminus \{i\})}{a^t} \right) \\ &= \sum_{j \in N \setminus \{i\}} a^t \cdot \left(v^t(N \setminus \{j\}) - v^t(N \setminus \{i\}) - \frac{x_i^t}{a^t} + \frac{x_j^t}{a^t} \right). \end{aligned}$$

By definition of $\bar{\beta}$,

$$(8) \quad \bar{\beta}_i^t(N, b, V^m) - \bar{\beta}_j^t(N, b, V^m) = a^t \cdot \left(v^t(N \setminus \{j\}) - v^t(N \setminus \{i\}) \right).$$

By equations (7) and (8),

$$\begin{aligned} (9) \quad & \sum_{j \in N \setminus \{i\}} a^t \cdot \left(e(N \setminus \{j\}, v^t, \frac{x^t}{a^t}) - e(N \setminus \{i\}, v, \frac{x^t}{a^t}) \right) \\ &= \sum_{j \in N \setminus \{i\}} \left(\bar{\beta}_i^t(N, b, V^m) - \bar{\beta}_j^t(N, b, V^m) - x_i^t + x_j^t \right) \\ &= \left((|N| - 1)(\bar{\beta}_i^t(N, b, V^m) - x_i^t) - \sum_{j \in N \setminus \{i\}} \bar{\beta}_j^t(N, b, V^m) + \sum_{j \in N \setminus \{i\}} x_j^t \right) \\ &= \left(|N|(\bar{\beta}_i^t(N, b, V^m) - x_i^t) - v_*^t(N) + v_*^t(N) \right) \\ & \quad \text{(by MEFF of } \bar{\beta}, x \in E(N, b, V^m)) \\ &= |N| \cdot \left(\bar{\beta}_i^t(N, b, V^m) - x_i^t \right). \end{aligned}$$

Moreover,

$$\begin{aligned}
 & \sum_{i \in N} \sum_{j \in N \setminus \{i\}} a^t \cdot \left(e(N \setminus \{j\}, v^t, \frac{x^t}{a^t}) - e(N \setminus \{i\}, v^t, \frac{x^t}{a^t}) \right) \\
 &= \sum_{i \in N} |N| \cdot \left(\bar{\beta}_i^t(N, b, V^m) - x_i^t \right) \\
 (10) \quad &= |N| \cdot \left(\sum_{i \in N} \bar{\beta}_i^t(N, b, V^m) - \sum_{i \in N} x_i^t \right) \\
 &= |N| \cdot (v_*^t(N) - v_*^t(N)) \quad (\text{by MEFF of } \bar{\beta}, x \in E(N, b, V^m)) \\
 &= 0.
 \end{aligned}$$

So we have that

$$\begin{aligned}
 \sum_{i \in N} f_i^t(x) &= \sum_{i \in N} \left[x_i^t + w \sum_{j \in N \setminus \{i\}} a^t \cdot \left(e(N \setminus \{j\}, v^t, \frac{x^t}{a^t}) - e(N \setminus \{i\}, v^t, \frac{x^t}{a^t}) \right) \right] \\
 &= \sum_{i \in N} x_i^t + w \sum_{i \in N} \sum_{j \in N \setminus \{i\}} a^t \cdot \left(e(N \setminus \{j\}, v^t, \frac{x^t}{a^t}) - e(N \setminus \{i\}, v^t, \frac{x^t}{a^t}) \right) \\
 &= v_*^t(N) + 0 \quad (\text{by equation (10) and } x \in E(N, b, V^m)) \\
 &= v_*^t(N).
 \end{aligned}$$

Hence, $f(x) \in E(N, b, V^m)$ if $x \in E(N, b, V^m)$. $\square$

Theorem 2. Let $(N, b, V^m) \in \Gamma^*$. If $0 < t < \frac{2}{|N|}$, then $\{[x]^q\}_{q=1}^\infty$ converges geometrically to $\bar{\beta}(N, b, V^m)$ for each $x \in E(N, b, V^m)$.

Proof. Let $(N, b, V^m) \in \Gamma^*$, $t \in \mathbb{N}_m$, $i \in N$ and $x \in E(N, b, V^m)$. By equation (9) and definition of f ,

$$\begin{aligned}
 f_i^t(x) - x_i^t &= w \sum_{j \in N \setminus \{i\}} a^t \cdot \left(e(N \setminus \{j\}, v^t, \frac{x^t}{a^t}) - e(N \setminus \{i\}, v^t, \frac{x^t}{a^t}) \right) \\
 &= w \cdot |N| \cdot \left(\bar{\beta}_i^t(N, b, V^m) - x_i^t \right).
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \bar{\beta}_i^t(N, b, V^m) - f_i^t(x) &= \bar{\beta}_i^t(N, b, V^m) - x_i^t + x_i^t - f_i^t(x) \\
 &= \bar{\beta}_i^t(N, b, V^m) - x_i^t - w \cdot |N| \cdot (\bar{\beta}_i^t(N, b, V^m) - x_i^t) \\
 &= (1 - w \cdot |N|) [\bar{\beta}_i^t(N, b, V^m) - x_i^t].
 \end{aligned}$$

So, for all $q \in \mathbb{N}$,

$$\bar{\beta}(N, b, V^m) - [x]^q = (1 - w \cdot |N|)^q [\bar{\beta}(N, b, V^m) - x].$$

If $0 < w < \frac{2}{|N|}$, then $-1 < (1 - w \cdot |N|) < 1$ and $\{[x]^q\}_{q=1}^\infty$ converges geometrically to $\bar{\beta}(N, b, V^m)$. $\square$

Inspired by Maschler and Owen [13], we will find a dynamic process under reductions.

Definition 4. Let ψ be a solution, $(N, b, V^m) \in \Gamma^*$, $S \subseteq N$ and $x \in E(N, b, V^m)$. The (x, ψ) -reduced game $(S, b_S, V_{\psi, S, x}^m)$ is given by $V_{\psi, S, x}^m = (v_{\psi, S, x}^t)_{t \in \mathbb{N}_m}$ and for all $T \subseteq S$,

$$v_{\psi, S, x}^t(\alpha) = \begin{cases} v_*^t(N) - \sum_{i \in N \setminus S} x_i^t, & A(\alpha) = S, \\ v_{S, \psi}^t(\alpha), & \text{otherwise.} \end{cases}$$

Inspired by Maschler and Owen [13], we also define different correction function as follow. The **R-correction function** to be $g = (g^t)_{t \in \mathbb{N}_m}$, where $g^t = (g_i^t)_{i \in N}$ and $g_i^t : E(N, b, V^m) \rightarrow \mathbb{R}$ is define by

$$g_i^t(x) = x_i^t + w \sum_{k \in N \setminus \{i\}} \left(\bar{\beta}_i^t(\{i, k\}, v_{\bar{\beta}, \{i, k\}, x}^t) - x_i^t \right).$$

Define $[\eta]^0 = x$, $[\eta]^1 = g([\eta]^0)$, $\dots$, $[\eta]^q = g([\eta]^{q-1})$ for all $q \in \mathbb{N}$.

Lemma 6. $g(x) \in E(N, b, V^m)$ for all $(N, b, V^m) \in \Gamma^*$ and for all $x \in E(N, b, V^m)$.

Proof. Let $(N, b, V^m) \in \Gamma^*$, $t \in \mathbb{N}_m$, $i, k \in N$ and $x \in X(N, b, v)$. Let $S = \{i, k\}$, by MEFF of $\bar{\beta}$ and Definition 4,

$$\bar{\beta}_i^t(S, b_S, V_{\bar{\beta}, S, x}^m) + \bar{\beta}_k^t(S, b_S, V_{\bar{\beta}, S, x}^m) = x_i^t + x_k^t.$$

By RCON and NSG of $\bar{\beta}$,

$$\begin{aligned} \bar{\beta}_i^t(S, b_S, v_{\bar{\beta}, S, x}^{com}) - \bar{\beta}_k^t(S, b_S, V_{\bar{\beta}, S, x}^m) &= (v_{\bar{\beta}, S, x}^t)_*(\{i\}) - (v_{\bar{\beta}, S, x}^t)_*(\{k\}) \\ &= (v_{S, \bar{\beta}}^t)_*(\{i\}) - (v_{S, \bar{\beta}}^t)_*(\{k\}) \\ &= \bar{\beta}_i^t(S, b_S, V_{S, \bar{\beta}}^m) - \bar{\beta}_k^t(S, b_S, V_{S, \bar{\beta}}^m) \\ &= \bar{\beta}_i^t(N, b, V^m) - \bar{\beta}_k^t(N, b, V^m). \end{aligned}$$

Therefore,

$$(11) \quad 2 \cdot \left[\bar{\beta}_i^t(S, b_S, V_{\bar{\beta}, S, x}^m) - x_i^t \right] = \bar{\beta}_i^t(N, b, V^m) - \bar{\beta}_k^t(N, b, V^m) - x_i^t + x_k^t.$$

By definition of g and equation (11),

$$\begin{aligned} g_i^t(x) &= x_i^t + \frac{w}{2} \cdot \left[\sum_{k \in N \setminus \{i\}} \bar{\beta}_i^t(N, b, V^m) - \sum_{k \in N \setminus \{i\}} x_i^t \right. \\ &\quad \left. - \sum_{k \in N \setminus \{i\}} \bar{\beta}_k^t(N, b, V^m) + \sum_{k \in N \setminus \{i\}} x_k^t \right] \\ &= x_i^t + \frac{w}{2} \cdot \left[\sum_{k \in N \setminus \{i\}} \bar{\beta}_i^t(N, b, V^m) - (|N| - 1)x_i^t \right. \\ (12) \quad &\quad \left. - \sum_{k \in N \setminus \{i\}} \bar{\beta}_k^t(N, b, v) + (v_*^t(N) - x_i^t) \right] \\ &= x_i^t + \frac{w}{2} \cdot \left[(|N| - 1)\bar{\beta}_i^t(N, b, V^m) - (|N| - 1)x_i^t \right. \\ &\quad \left. - (v_*^t(N) - \bar{\beta}_i^t(N, b, V^m)) + (v_*^t(N) - x_i^t) \right] \\ &= x_i^t + \frac{|N| \cdot w}{2} \cdot \left[\bar{\beta}_i^t(N, b, V^m) - x_i^t \right]. \end{aligned}$$

So we have that

$$\begin{aligned}
 \sum_{i \in N} g_i^t(x) &= \sum_{i \in N} \left[x_i^t + \frac{|N| \cdot w}{2} \cdot [\bar{\beta}_i^t(N, b, V^m) - x_i^t] \right] \\
 &= \sum_{i \in N} x_i^t + \frac{|N| \cdot w}{2} \cdot \left[\sum_{i \in N} \bar{\beta}_i^t(N, b, V^m) - \sum_{i \in N} x_i^t \right] \\
 &= v_*^t(N) + \frac{|N| \cdot w}{2} \cdot [v_*^t(N) - v_*^t(N)] \\
 &= v_*^t(N).
 \end{aligned}$$

Thus, $g(x) \in E(N, b, V^m)$ for all $x \in E(N, b, V^m)$. $\square$

Theorem 3. *Let $(N, b, V^m) \in \Gamma^*$. If $0 < \alpha < \frac{4}{|N|}$, then $\{[\eta]^q\}_{q=1}^\infty$ converges to $\bar{\beta}(N, b, V^m)$ for each $x \in E(N, b, V^m)$.*

Proof. Let $(N, b, V^m) \in \Gamma^*$, $t \in \mathbb{N}_m$ and $x \in E(N, b, V^m)$. By equation (12), $g_i^t(x) = x_i^t + \frac{|N| \cdot w}{2} \cdot [\bar{\beta}_i^t(N, b, V^m) - x_i^t]$ for all $i \in N$. Therefore,

$$\left(1 - \frac{|N| \cdot w}{2}\right) \cdot [\bar{\beta}_i^t(N, b, V^m) - x_i^t] = [\bar{\beta}_i^t(N, b, V^m) - g_i^t(x)]$$

So, for all $q \in \mathbb{N}$,

$$\bar{\beta}(N, b, V^m) - [\eta]^q = \left(1 - \frac{|N| \cdot w}{2}\right)^q [\bar{\beta}(N, b, V^m) - x].$$

If $0 < w < \frac{4}{|N|}$, then $-1 < \left(1 - \frac{|N| \cdot w}{2}\right) < 1$ and $\{[\eta]^q\}_{q=1}^\infty$ converges to $\bar{\beta}(N, b, v)$ for all $(N, b, V^m) \in \Gamma^*$, for all $t \in \mathbb{N}_m$ and for all $i \in N$. $\square$

5. Conclusions

In this paper, we investigate the normalizations for the fuzzy marginal index and the fuzzy normalized index. Based on reduced game, two axiomatizations for these two indexes are proposed. By applying reduction and excess function, we also introduce alternative formulations and related dynamic processes for the normalized index. One should compare our results with related pre-existing results:

- The fuzzy marginal index and the fuzzy normalized index are introduced initially in the framework of multicriteria fuzzy TU games.
- The idea of our correction functions in Definitions 3, 4 and related dynamic processes are based on that of Maschler and Owen's [13] dynamic results for the Shapley value [17]. The major difference is that our correction functions in Definition 4 are based on "excess function", and Maschler and Owen's [13] correction function is based on "reduced games".

These mentioned above raise one question:

- Whether there exist other normalizations and related results for some more solutions in the framework of multicriteria fuzzy TU games.

To our knowledge, these issues are still open questions.

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