

SOME SUFFICIENT CONDITIONS IMPLYING NILPOTENCY OF FINITE GROUPS

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Abstract. Let G be a finite group. For a weak n -Engel condition, we mean that $[x, {}_n y] \in Z(G)$ for two elements x and y of G , where n is a positive integer. In this paper, we mainly study the influence of the weak n -Engel condition on the nilpotence and p -nilpotence of finite groups. Our results generalize some well-known results.

Keywords: weak n -Engel condition, nilpotent groups, p -nilpotent group.

1. Introduction

As we know, the n -Engel condition is that $[x, {}_n y] = 1$ for two elements x and y of G , where n is a positive integer. Obviously, a finite group which satisfies the n -Engel condition must satisfy the weak n -Engel condition, but the inverse is not true. Many scholars investigate the influence of the n -Engel condition on the nilpotence of finite groups. In [1], Huppert. B proved that let x and y be arbitrary element of G , if there is a positive integer n such that $[x, {}_n y] = 1$, then G is nilpotent. In [2], It is proved that let $G = AB$, where A and B are nilpotent subgroups of G . If $[A, B] = 1$, then G is nilpotent. In this paper, we mainly discuss the influence of the weak n -Engel condition on the nilpotence and p -nilpotence of finite groups and generalize some well-known results.

We shall use the following notation for commutators:

$$\begin{aligned} [x, y] &= x^{-1}y^{-1}xy, \\ [x_1, x_2, \dots, x_n] &= [[x_1, \dots, x_{n-1}], x_n] (n \geq 3), \\ [x, {}_0 y] &= x, \\ [x, {}_n y] &= [[x, {}_{n-1} y], y] (n \geq 1) \end{aligned}$$

The rest of the notation is standard (see [2]), all groups mentioned are assumed to be finite groups.

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2. Basic definitions and preliminary results

In this section, we give one definition and some results that are needed in this paper.

Definition 2.1. Let G be a finite group. For a weak n -Engel condition, we mean that $[x, {}_n y] \in Z(G)$ for two elements x and y of G , where n is a positive integer.

Lemma 2.2 ([2]). Let $G = AB$, where A and B are nilpotent subgroups of G . If $[A, B] = 1$, then G is nilpotent.

Lemma 2.3 ([2]). If $G/\Phi(G)$ is nilpotent, then G is nilpotent.

3. Main results

Theorem 3.1. Let $G = AB$, where A is a normal nilpotent subgroup of G and B is a nilpotent subgroup of G with $(|A|, |B|) = 1$. If for each element x in A and each element y in B there is a positive integer n such that $[x, {}_n y] \in Z(G)$, then G is nilpotent.

Proof. Since $[x, y] \in Z(G)$ means $[x, {}_2 y] \in Z(G)$, we may assume $n \geq 2$. Let $a = [x, {}_{n-2} y]$, by the hypothesis we have $[(y^{-1})^a y, y] = [a, y, y] \in Z(G)$. It follows that $(y^{-1})^a y y = y(y^{-1})^a y z$ for some $z \in Z(G)$, that is, $(y^{-1})^a y = y(y^{-1})^a z$, i.e., $[(y^{-1})^a, y] \in Z(G)$. We consider the following cases:

(i) If $|B|_2 = 1$ or $|B|_2 > 1$ and q is an odd prime, then $o(y) | C_{o(y)}^2$ and $((y^{-1})^a)^{o(y)} = ((y^{-1})^a)^{o(y)} y^{o(y)} ([y, (y^{-1})^a]^{o(y)})^{C_{o(y)}^2 / o(y)} = 1$. Hence $(y^{-1})^a y$ is a q -element of G .

(ii) If $|B|_2 > 1$ and q is 2, then $((y^{-1})^a y)^{2o(y)} = 1$ and $(y^{-1})^a y$ is a 2-element of G .

On the other hand, since A is a normal subgroup of G , $(y^{-1})^a y = [a, y] \in A$. And $(y^{-1})^a y$ is a p -element. Therefore, by the above discussion, $[a, y] = (y^{-1})^a y = 1$. That is, $[x, {}_{n-1} y] = 1$. Now, a simple induction on n , we have $[x, y] = 1$. Since $(|A|, |B|) = 1$, it follows that $[A, B] = 1$. By Lemma 2.2 we have G is nilpotent.

If we drop the restriction that A is a normal subgroup of G , we can prove the following result.

Theorem 3.2. Let $G = AB$, where A and B are nilpotent subgroups of G with $(|A|, |B|) = 1$. If for each element x in A and each element y in B there is a positive integer n such that $[x, {}_n y] \in Z(G)$, then G is nilpotent.

Proof. Let M be an arbitrary maximal subgroup of G . Since G is solvable, it follows that $|G : M| = p^n$ for some prime p . Since $(|A|, |B|) = 1$, we can assume that p does not divide, say, $|B|$. Let M_1 be a p' -Hall subgroup of M . By

π -Sylow Theorem [1, Theorem 4.1, p.231], we have $B \leq M_1^x$ for some x in G . Since M^x has the same properties as M , we can replace M by M^x and so we can assume without loss of generality that $B \leq M$. Since $G = AB$, it follows that $M = B(A \cap M)$. Clearly, B and $A \cap M$ are nilpotent subgroups of M , $(|B|, |A \cap M|) = 1$. By the hypothesis, for each element x in $A \cap M$ and each element y in B there is a positive integer n such that $[x, {}_n y] \in Z(G) \cap M \leq Z(M)$. By induction on $|G|$, M is nilpotent. If G is not nilpotent, then G is a minimal non-nilpotent group. By Itô's Theorem [1, Theorem 5.2, p.281], $G = PQ$, where P is normal in G , and P is a Sylow p -subgroup of G , Q is non-normal cyclic Sylow q -subgroup of G , $p \neq q$. We can assume that $P = A$ and $Q = B$, so that $A \triangleleft G$. Due to each element x in A and each element y in B there is a positive integer n such that $[x, {}_n y] \in Z(G)$, Theorem 3.1 implies that G is nilpotent, a contradiction. This is impossible as G is minimal non-nilpotent group. This completes the proof of the Theorem.

Theorem 3.3. *Let $N \trianglelefteq G$, N and G/N are nilpotent groups. If for each element x in N and each element y in G , there is a positive integer n such that $[x, {}_n y] \in Z(G)$, then G is nilpotent.*

Proof. We first prove that there is a nilpotent subgroup A of G such that $G = NA$. Let $\mathcal{F} = \{A | A \subseteq G, G = NA\}$. Clearly, $G \in \mathcal{F}$, then $\mathcal{F}$ is a non-empty set. Now let A be a minimal element in the set $\mathcal{F}$, we show that A is nilpotent. Assume that $N \cap A \not\subseteq \Phi(A)$, there exists a maximal subgroup B of A such that $N \cap A \not\subseteq B$. Since $N \cap A \triangleleft A$, we have $A = (N \cap A)B$. Since $G = NA = N(N \cap A)B = NB$, then $B \in \mathcal{F}$, a contradiction. Hence $N \cap A \subseteq \Phi(A)$. Since $G = NA$, so that $G/N \simeq A/N \cap A$. But $A/N \cap A \simeq (A/N \cap A)/(\Phi(A)/N \cap A)$, then $G/N \simeq (A/N \cap A)/(\Phi(A)/N \cap A) \simeq A/\Phi(A)$. This implies that $G/N \simeq A/\Phi(A)$. G/N is nilpotent implies that $A/\Phi(A)$ is nilpotent. By Lemma 2.3 we have A is nilpotent. By the hypothesis, $G = NA$, N and A are nilpotent subgroups of G , so for each element x in N and each element y in A there is a positive integer n such that $[x, {}_n y] \in Z(G)$. By Theorem 3.1 we know that G is nilpotent.

Theorem 3.4. *Let G be a finite group, x is an arbitrary element of order p or order $4(p=2)$ and y is an arbitrary p' -element of G . If there is positive integer n such that $[x, {}_n y] \in Z(G)$, then G is p -nilpotent.*

Proof. Suppose that the theorem is false and let G be a counter-example of minimal order. Then G is not p -nilpotent. But each of whose proper subgroup of G is p -nilpotent, by Itô's Theorem [1, Theorem 5.2, p.281], $G = PQ$, $P \in \text{Syl}_p(G)$, $Q \in \text{Syl}_q(G)$, $P \trianglelefteq G$, $\exp(P) = p$ or 4 . Let $x \in P$, $y \in Q$. By the hypothesis, there is a positive integer n such that $[x, {}_n y] \in Z(G)$. Theorem 3.1 implies that G is nilpotent.

This Theorem may be considered as a generalization of the following well-known result of Itô: Let G be a finite group. If each element of G of order p or order $4(p=2)$ lies in $Z(G)$, then G is p -nilpotent [1, Theorem 5.5, p.435].

Theorem 3.5. *Let G be a finite group and P be an arbitrary p -subgroup of G . Let x be an arbitrary element of P of order p or order $4(p=2)$ and y be an arbitrary p' -element of $N_G(P)$. If there is a positive integer n such that $[x, {}_n y] \in Z(G)$, then G is p -nilpotent.*

Proof. Let y be an arbitrary p' -element of $N_G(P)$. We can consider the group $P\langle y \rangle$. Let x be an arbitrary element of $P\langle y \rangle$ of order p or order 4 . Clearly, $x \in P$, since $P\langle y \rangle \leq N_G(P)$. Thus an arbitrary p' -element in $P\langle y \rangle$ that it is an arbitrary p' -element in $N_G(P)$. We can assume that y' is an arbitrary p' -element in $N_G(P)$. By the hypothesis, there is a positive integer n such that $[x, {}_n y'] \in Z(G)$. Theorem 3.4 implies that $P\langle y' \rangle$ is p -nilpotent. It follows that $\langle y' \rangle \triangleleft P\langle y' \rangle$. Hence $P\langle y' \rangle = P \times \langle y' \rangle$, $N_G(P)/C_G(P)$ is p -group. By Frobenius' Theorem, G is p -nilpotent. This completes the proof of the Theorem.

This Theorem may be considered as a generalization of Theorem 10.24 [1, Theorem 10.24, p.124].

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