

F -CONTRACTIVE MAPPINGS OF HARDY-ROGERS-TYPE IN G -METRIC SPACES

Hamed M. Obiedat*

*Department of Mathematics
Hashemite University
P.O.Box150459
Zarqa13115-Jordan
hobiedat@hu.edu.jo*

Ameer A. Jaber

*Department of Mathematics
Hashemite University
P.O.Box150459
Zarqa13115-Jordan
ameerj@hu.edu.jo*

Abstract. In this paper, we prove some fixed point results for F -contractive mappings of Hardy-Rogers-Type in the setting of G -metric spaces.

Keywords: fixed point theorems, G -metric spaces, F -contractive mappings of Hardy-Rogers-type.

1. Introduction

The class of G -metric spaces introduced by Z. Mustafa and B. Sims (See [5]) was to provide a new class of generalized metric spaces and to extend the fixed point theory for a variety of mappings. Moreover, many theorems were proved in this new setting with most of them recognizable as counterparts of a well-known metric space theorems (See [1], [2], [3]).

In [9], a new type of contractive mappings was introduced, called F -contractive mappings, to provide a generalization of Banach's contraction mapping principle. Recently, in ([4]), the authors proved some fixed point results for F -contractive mappings of Hardy-Rogers-Type. Due to the importance of the fixed point results of Hardy-Rogers-Type, it has been improved, generalized, extended and used in many applications such as logic programming semantics, ordinary differential equations of first order, and dislocated topology (see[8], [7], [6]).

In this paper, we prove some fixed point results for F -contractive mappings of Hardy-Rogers-Type in the setting of G -metric spaces. Throughout the article we denote by $\mathbb{R}$ the set of all real numbers, by $\mathbb{R}^+$ the set of all positive real numbers and by $\mathbb{N}$ the set of all positive integers.

*. Corresponding author

Definition 1 ([5]). *G -metric space is a pair (X, G) , where X is a nonempty set, and G is a nonnegative real-valued function defined on $X \times X \times X$ such that for all $x, y, z, a \in X$, we have:*

- (G1) $G(x, y, z) = 0$ if $x = y = z$;
- (G2) $0 < G(x, x, y)$, for all $x, y \in X$, with $x \neq y$;
- (G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$, with $z \neq y$;
- (G4) $G(x, y, z) = G(p\{x, z, y\})$ (symmetry in all three variables);
- (G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, (rectangle inequality).

The function G is called a G -metric on X .

Definition 2 ([5]). *A sequence (x_n) in a G -metric space X is said to converge if there exists $x \in X$ such that $\lim_{n, m \rightarrow \infty} G(x, x_n, x_m) = 0$, and one says that the sequence (x_n) is G -convergent to x .*

Proposition 3. *Let X be G -metric space. Then the following statements are equivalent.*

1. (x_n) is G -convergent to x .
2. $G(x_n, x_n, x) \rightarrow 0$, as $n \rightarrow \infty$.
3. $G(x_n, x, x) \rightarrow 0$, as $n \rightarrow \infty$.
4. $G(x_m, x_n, x) \rightarrow 0$, as $m, n \rightarrow \infty$.

In a G -metric space X , a sequence (x_n) is said to be G -Cauchy if given $\varepsilon > 0$, there is $N_\varepsilon \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq N_\varepsilon$.

Proposition 4 ([5]). *In a G -metric space X , the following statements are equivalent.*

1. The sequence (x_n) is G -Cauchy.
2. For every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq N$.

Proposition 5 ([5]). *Let X be a G -metric space, then the function $G(x, y, z)$ is jointly continuous in all three of its variables. A G -metric space X is said to be complete if every G -Cauchy sequence in X is G -convergent in X .*

2. Preliminaries

In this section, we give definitions and results that we will use later. We start with F -contractive mappings as stated in [9].

Definition 6. Let $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a mapping satisfying:

- (F1) F is strictly increasing;
- (F2) for each sequence $\{\alpha_n\} \subset \mathbb{R}^+$ of positive numbers $\lim \alpha_n = 0$ if and only if $\lim F(\alpha_n) = -\infty$;
- (F3) there exists $k \in (0, 1)$ such that $\lim \alpha_n^k F(\alpha_n) = 0$.

We denote with $\mathcal{F}$ the family of all functions F that satisfy the conditions (F1) – (F3).

Definition 7. Let $(X; G)$ be a G -metric space. A self-mapping T on X is called an F -contraction if there exist $F \in \mathcal{F}$ and $\rho \in \mathbb{R}^+$ such that

$$\rho + F(G(Tx, Ty, Tz)) \leq F(G(x, y, z));$$

for all $x, y \in X$ with $G(Tx, Ty, Tz) > 0$.

Definition 8. Let $(X; G)$ be a G -metric space. A self-mapping T on X is called an F -contraction of Hardy-Rogers-type if there exist $F \in \mathcal{F}$ and $\rho \in \mathbb{R}^+$ such that

$$\begin{aligned} \rho + F(G(Tx, Ty, Tz)) \leq & F(\alpha_1 G(x, y, z) + \alpha_2 G(x, Tx, Tx) \\ & + \alpha_3 G(y, Ty, Ty) + \alpha_4 G(z, Tz, Tz) + \alpha_5 G(x, Ty, Tz) \\ & + \alpha_6 G(y, Tx, Tz) + \alpha_7 G(y, Ty, Tx)), \end{aligned}$$

for all $x, y \in X$ with $G(Tx, Ty, Tz) > 0$, where $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2(\alpha_5 + \alpha_6 + \alpha_7) = 1$ and $\alpha_3 + \alpha_4 \neq 1$.

3. Fixed points for F -contractions of Hardy-Rogers-type

In this section, we give some fixed point results for F -contractions of Hardy-Rogers-type in complete G -metric spaces.

Theorem 9. Let $(X; G)$ be a complete metric space and let T be a self-mapping on X . Assume that there exist $F \in \mathcal{F}$ and $\rho \in \mathbb{R}^+$ such that

$$\begin{aligned} (1) \quad \rho + F(G(Tx, Ty, Tz)) \leq & F(\alpha_1 G(x, y, z) + \alpha_2 G(x, Tx, Tx) \\ & + \alpha_3 G(y, Ty, Ty) + \alpha_4 G(z, Tz, Tz)) \\ & + \alpha_5 G(x, Ty, Tz) + \alpha_6 G(y, Tx, Tz) \\ & + \alpha_7 G(y, Ty, Tx)), \end{aligned}$$

for all $x, y, z \in X$ with $G(Tx, Ty, Tz) \neq 0$, where $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2(\alpha_5 + \alpha_6 + \alpha_7) = 1$ and $\alpha_3 + \alpha_4 \neq 1$. Then T has a fixed point. Moreover, if $\alpha_1 + \alpha_5 + \alpha_6 + \alpha_7 \leq 1$, then the fixed point of T is unique.

Proof. Fix $x_0 \in X$ and let $x_n = T^n x_0 = T(T^{n-1}x_0) = T x_{n-1}$ for $n \in \mathbb{N}$. If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, then it is clear that x_n is a fixed point. Suppose $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$ and set $x = x_{n-1}$ and $y = z = x_n$. Then by applying the inequality (1) with $x = x_{n-1}$ and $y = z = x_n$. We obtain

$$\begin{aligned}
\rho + F(G(Tx_{n-1}, Tx_n, Tx_n)) &= \rho + F(G(x_n, x_{n+1}, x_{n+1})) \\
&\leq F(\alpha_1 G(x_{n-1}, x_n, x_n) + \alpha_2 G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + \alpha_3 G(x_n, Tx_n, Tx_n) \\
&\quad + \alpha_4 G(x_n, Tx_n, Tx_n) + \alpha_5 G(x_{n-1}, Tx_n, Tx_n) + \alpha_6 G(x_n, Tx_{n-1}, Tx_n) \\
&\quad + \alpha_7 G(x_n, Tx_{n-1}, Tx_n)) \\
&= F(\alpha_1 G(x_{n-1}, x_n, x_n) + \alpha_2 G(x_{n-1}, x_n, x_n) + \alpha_3 G(x_n, x_{n+1}, x_{n+1}) \\
&\quad + \alpha_4 G(x_n, x_{n+1}, x_{n+1}) + \alpha_5 G(x_{n-1}, x_{n+1}, x_{n+1}) + \alpha_6 G(x_n, x_n, x_{n+1}) \\
&\quad + \alpha_7 G(x_n, x_n, x_{n+1})) \\
&= F((\alpha_1 + \alpha_2)G(x_{n-1}, x_n, x_n) + (\alpha_3 + \alpha_4)G(x_n, x_{n+1}, x_{n+1}) \\
&\quad + \alpha_5 G(x_{n-1}, x_{n+1}, x_{n+1}) + (\alpha_6 + \alpha_7)G(x_n, x_n, x_{n+1})).
\end{aligned}$$

This implies that

$$\begin{aligned}
0 < \rho &\leq F((\alpha_1 + \alpha_2)G(x_{n-1}, x_n, x_n) + (\alpha_3 + \alpha_4)G(x_n, x_{n+1}, x_{n+1}) \\
&\quad + \alpha_5 G(x_{n-1}, x_{n+1}, x_{n+1}) + (\alpha_6 + \alpha_7)G(x_n, x_n, x_{n+1})) - F(G(x_n, x_{n+1}, x_{n+1})).
\end{aligned}$$

Since F is strictly increasing, which produce

$$\begin{aligned}
G(x_n, x_{n+1}, x_{n+1}) &\leq (\alpha_1 + \alpha_2)G(x_{n-1}, x_n, x_n) + (\alpha_3 + \alpha_4)G(x_n, x_{n+1}, x_{n+1}) \\
&\quad + \alpha_5 G(x_{n-1}, x_{n+1}, x_{n+1}) + (\alpha_6 + \alpha_7)G(x_n, x_n, x_{n+1}).
\end{aligned}$$

This leads to

$$\begin{aligned}
(1 - (\alpha_3 + \alpha_4))G(x_n, x_{n+1}, x_{n+1}) &\leq (\alpha_1 + \alpha_2)G(x_{n-1}, x_n, x_n) \\
&\quad + \alpha_5 G(x_{n-1}, x_{n+1}, x_{n+1}) + (\alpha_6 + \alpha_7)G(x_n, x_n, x_{n+1}) \\
&\leq (\alpha_1 + \alpha_2)G(x_{n-1}, x_n, x_n) + \alpha_5 G(x_{n-1}, x_n, x_n) \\
&\quad + \alpha_5 G(x_n, x_{n+1}, x_{n+1}) + (\alpha_6 + \alpha_7)G(x_n, x_n, x_{n+1}),
\end{aligned}$$

which implies

$$\begin{aligned}
(1 - (\alpha_3 + \alpha_4 + \alpha_5))G(x_n, x_{n+1}, x_{n+1}) &\leq (\alpha_1 + \alpha_2 + \alpha_5)G(x_{n-1}, x_n, x_n) + (\alpha_6 + \alpha_7)G(x_n, x_n, x_{n+1}) \\
&\leq (\alpha_1 + \alpha_2 + \alpha_5)G(x_{n-1}, x_n, x_n) + (\alpha_6 + \alpha_7)G(x_n, x_{n+1}, x_{n+1}) \\
&\quad + (\alpha_6 + \alpha_7)G(x_{n+1}, x_n, x_{n+1}) \\
&= (\alpha_1 + \alpha_2 + \alpha_5)G(x_{n-1}, x_n, x_n) + (\alpha_6 + \alpha_7)G(x_n, x_{n+1}, x_{n+1}) \\
&\quad + (\alpha_6 + \alpha_7)G(x_n, x_{n+1}, x_{n+1}) \\
&= (\alpha_1 + \alpha_2 + \alpha_5)G(x_{n-1}, x_n, x_n) + 2(\alpha_6 + \alpha_7)G(x_n, x_{n+1}, x_{n+1}).
\end{aligned}$$

Therefore,

$$(1 - (\alpha_3 + \alpha_4 + a_5 + 2a_6 + 2a_7))G(x_n, x_{n+1}, x_{n+1}) \leq (\alpha_1 + \alpha_2 + a_5)G(x_{n-1}, x_n, x_n).$$

Hence,

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{(\alpha_1 + \alpha_2 + a_5)}{(1 - (\alpha_3 + \alpha_4 + a_5 + 2a_6 + 2a_7))}G(x_{n-1}, x_n, x_n).$$

Since $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2(a_5 + \alpha_6 + \alpha_7) = 1$ with $\alpha_3 + \alpha_4 \neq 1$, we obtain $1 - (\alpha_3 + \alpha_4 + a_5 + 2a_6 + 2a_7) > 0$. We have

$$\begin{aligned} & G(x_n, x_{n+1}, x_{n+1}) \\ & \leq \frac{(\alpha_1 + \alpha_2 + a_5)}{(1 - (\alpha_3 + \alpha_4 + a_5 + 2a_6 + 2a_7))}G(x_{n-1}, x_n, x_n) = G(x_{n-1}, x_n, x_n). \end{aligned}$$

Therefore,

$$\begin{aligned} F(G(x_n, x_{n+1}, x_{n+1})) & \leq F(G(x_{n-1}, x_n, x_n)) - \rho \\ & \leq F(G(x_{n-2}, x_{n-1}, x_{n-1})) - 2\rho \\ & \leq F(G(x_{n-3}, x_{n-2}, x_{n-2})) - 3\rho. \end{aligned}$$

Continuing in the same argument, we deduce

$$(2) \quad F(G(x_n, x_{n+1}, x_{n+1})) \leq F(G(x_0, x_1, x_1)) - n\rho.$$

This implies that $\lim F(G(x_n, x_{n+1}, x_{n+1})) = -\infty$. By Definition 6 property (F2), we get $\lim G(x_n, x_{n+1}, x_{n+1}) = 0$. Also, if we choose $k \in (0, 1)$ and multiply both sides of (2) by $G^k(x_n, x_{n+1}, x_{n+1})$, we obtain

$$\begin{aligned} & G^k(x_n, x_{n+1}, x_{n+1})F(G(x_n, x_{n+1}, x_{n+1})) - G^k(x_n, x_{n+1}, x_{n+1})F(G(x_0, x_1, x_1)) \\ & \leq -n\rho G^k(x_n, x_{n+1}, x_{n+1}) \leq 0. \end{aligned}$$

Using Definition 6 property (F2), we get

$$\lim G^k(x_n, x_{n+1}, x_{n+1})F(G(x_n, x_{n+1}, x_{n+1})) = 0,$$

which implies that

$$\lim(n\rho - F(G(x_0, x_1, x_1))G^k(x_n, x_{n+1}, x_{n+1})) \leq 0.$$

But $\lim G(x_n, x_{n+1}, x_{n+1}) = 0$. Hence, $\lim(nG^k(x_n, x_{n+1}, x_{n+1})) = 0$, and then we deduce that $\lim(n^{\frac{1}{k}}G(x_n, x_{n+1}, x_{n+1})) = 0$. Now we claim that the sequence $\{x_n\}$ is G -Cauchy. To show this given $\varepsilon > 0$, then there exists a natural number K such that $\left|n^{\frac{1}{k}}G(x_n, x_{n+1}, x_{n+1})\right| < \varepsilon$ for all $n \geq K$. That is, $G(x_n, x_{n+1}, x_{n+1}) < \varepsilon n^{-\frac{1}{k}}$ for all $n \geq K$ which implies that the series

$\sum G(x_n, x_{n+1}, x_{n+1})$ is convergent by comparison test. Hence, the sequence $\{x_n\}$ is G -Cauchy in a complete G -metric space. Then there exists $u \in X$ such that $x_n \rightarrow u$ as $n \rightarrow \infty$. This implies that $u = Tu$ and hence u is a fixed point of T . To show this, assume on the contrary that $u \neq Tu$. Then by inequality(1), we have

$$\begin{aligned} \rho + F(G(x_{n+1}, Tu, Tu)) &\leq F(\alpha_1 G(x_n, u, u) \\ &+ \alpha_2 G(x_n, x_{n+1}, x_{n+1}) + \alpha_3 G(u, Tu, Tu) \\ &+ \alpha_4 G(u, Tu, Tu)) + \alpha_5 G(x_n, x_{n+1}, Tu) \\ &+ \alpha_6 G(u, x_{n+1}, Tu) + \alpha_7 G(u, x_{n+1}, Tu)). \end{aligned}$$

By Definition 6 property (F1), we have

$$\begin{aligned} G(x_{n+1}, Tu, Tu) &\leq \alpha_1 G(x_n, u, u) \\ &+ \alpha_2 G(x_n, x_{n+1}, x_{n+1}) + \alpha_3 G(u, Tu, Tu) + \alpha_4 G(u, Tu, Tu) \\ &+ \alpha_5 G(x_n, x_{n+1}, Tu) + \alpha_6 G(u, x_{n+1}, Tu) + \alpha_7 G(u, x_{n+1}, Tu). \end{aligned}$$

By letting $n \rightarrow \infty$, we have

$$\begin{aligned} G(u, Tu, Tu) &\leq \alpha_1 G(u, u, u) + \alpha_2 G(u, u, u) + \alpha_3 G(u, Tu, Tu) \\ &+ \alpha_4 G(u, Tu, Tu) + \alpha_5 G(u, u, Tu) + \alpha_6 G(u, u, Tu) + \alpha_7 G(u, u, Tu) \\ &\leq \alpha_3 G(u, Tu, Tu) + \alpha_4 G(u, Tu, Tu) + \alpha_5 G(u, u, Tu) \\ &+ \alpha_6 G(u, u, Tu) + \alpha_7 G(u, u, Tu). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} (1 - (\alpha_3 + \alpha_4))G(u, Tu, Tu) &\leq (\alpha_5 + \alpha_6 + \alpha_7)G(u, u, Tu) \\ &= (\alpha_5 + \alpha_6 + \alpha_7)G(Tu, u, u) \\ &\leq (\alpha_5 + \alpha_6 + \alpha_7)G(u, Tu, Tu) + (\alpha_5 + \alpha_6 + \alpha_7)G(Tu, u, Tu) \\ &= 2(\alpha_5 + \alpha_6 + \alpha_7)G(u, Tu, Tu). \end{aligned}$$

Thus

$$(1 - (\alpha_3 + \alpha_4 - 2(\alpha_5 + \alpha_6 + \alpha_7)))G(u, Tu, Tu) \leq 0.$$

But $(1 - (\alpha_3 + \alpha_4 - 2(\alpha_5 + \alpha_6 + \alpha_7))) \geq 0$ which implies that $G(u, Tu, Tu) = 0$, and therefore, $Tu = u$. Now, we prove the uniqueness of the fixed point. Assume that $v \in X$, is another fixed point of T such that $u \neq v$. This means that $G(u, v, v) > 0$. Taking $x = u$ and $y = z = v$ in (1), we have

$$\begin{aligned} \rho + F(G(u, v, v)) &\leq F(\alpha_1 G(u, v, v) + \alpha_2 G(u, u, u) + \alpha_3 G(v, v, v) \\ &+ \alpha_4 G(v, v, v)) + \alpha_5 G(u, v, v) + \alpha_6 G(v, u, v) + \alpha_7 G(v, u, v)) \\ &\leq F(\alpha_1 G(u, v, v) + \alpha_5 G(u, v, v) + \alpha_6 G(v, u, v) + \alpha_7 G(v, u, v)) \\ &= F(\alpha_1 + \alpha_5 + \alpha_6 + \alpha_7)G(u, v, v). \end{aligned}$$

By Definition 6 property (F1), we have

$$G(u, v, v) \leq (\alpha_1 + \alpha_5 + \alpha_6 + \alpha_7)G(u, v, v).$$

The condition $\alpha_1 + \alpha_5 + \alpha_6 + \alpha_7 \leq 1$ implies that $G(u, v, v) = 0$, and hence $u = v$. This completes the proof of Theorem 9. $\square$

If we put $\alpha_5 = \alpha_6 = \alpha_7 = 0$ and $\sum_{i=1}^4 a_i < 1$, we obtain a version of Reich ([2], Theorem 2.1) in G -metric spaces setting.

Corollary 10. *Let $(X; G)$ be a complete metric space and let T be a self-mapping on X . Assume that there exist $F \in \mathcal{F}$ and $\rho \in \mathbb{R}^+$ such that*

$$\begin{aligned} \rho + F(G(Tx, Ty, Tz)) &\leq F(\alpha_1 G(x, y, z) \\ &\quad + \alpha_2 G(x, Tx, Tx) + \alpha_3 G(y, Ty, Ty) + \alpha_4 G(z, Tz, Tz)), \end{aligned}$$

for all $x, y, z \in X$ where $\sum_{i=1}^4 a_i < 1$. Then T has a unique fixed point in X .

Corollary 11. *Let $(X; G)$ be a complete metric space and let T be a self-mapping on X . Assume that there exist $F \in \mathcal{F}$ and $\rho \in \mathbb{R}^+$ such that the mapping T satisfy the following condition for some $m \in \mathbb{N}$*

$$\begin{aligned} \rho + F(G(T^m x, T^m y, T^m z)) &\leq F(\alpha_1 G(x, y, z) + \alpha_2 G(x, T^m x, T^m x) \\ &\quad + \alpha_3 G(y, T^m y, T^m y) + \alpha_4 G(z, T^m z, T^m z)), \end{aligned}$$

for all $x, y, z \in X$ where $\sum_{i=1}^4 a_i < 1$. Then T has a unique fixed point in X .

Proof. From Theorem 9 we deduce that the mapping T^m has a unique fixed point (say u). Then $T^m u = u$ but $Tu = T^{m+1}u = T^m(Tu)$. This implies that Tu is another fixed point of T^m . But T^m has a unique fixed point. Hence, $Tu = u$. $\square$

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Accepted 1.03.2017