

EXISTENCE OF MANY NON-RADIAL SOLUTIONS OF AN ELLIPTIC SYSTEM

Zhenluo Lou

*School of Mathematics and Statistics
Henan University of Science and Technology
Luoyang, Henan, 471023
P.R. China
louzhenluo@amss.ac.cn*

Abstract. In this paper, we consider the following elliptic system

$$\begin{cases} -\Delta u = \mu_1 u^{p-1} + \beta u^{\frac{p}{2}-1} v^{\frac{p}{2}}, & x \in \Omega \\ -\Delta v = \mu_2 v^{p-1} + \beta u^{\frac{p}{2}} v^{\frac{p}{2}-1}, & x \in \Omega \\ u, v > 0, \quad x \in \Omega, \quad u = v = 0, \quad x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ ($N > 4$) is an annulus, $\mu_1, \mu_2 > 0$, $\beta > 0$ and $2 < p < \frac{2N-2}{N-3}$. By variational and rescaling method, we prove that the system has many non-radial solutions.

Keywords: elliptic system, variational method, non-radial solutions.

1. Introduction

In this paper, we study the following elliptic system

$$(1.1) \quad \begin{cases} -\Delta u = \mu_1 u^{p-1} + \beta u^{\frac{p}{2}-1} v^{\frac{p}{2}}, & x \in \Omega \\ -\Delta v = \mu_2 v^{p-1} + \beta u^{\frac{p}{2}} v^{\frac{p}{2}-1}, & x \in \Omega \\ u, v > 0, \quad x \in \Omega, \quad u = v = 0, \quad x \in \partial\Omega, \end{cases}$$

where $\Omega = \{x \in \mathbb{R}^N : r^2 < |x|^2 < (r+d)^2\}$ is an annulus, $N > 4$, $\mu_1, \mu_2 > 0$, $\beta > 0$, $2 < p < \frac{2N-2}{N-3}$, and d is a fixed positive number.

In [6, 7], Gidas-Ni-Nirenberg studied the scalar equation

$$(1.2) \quad -\Delta u + u = u^p, \quad x \in \Omega, \quad u > 0, \quad x \in \Omega, \quad u = 0, \quad x \in \partial\Omega,$$

where $1 < p < 2^* = \frac{2N}{N-2}$. By moving plane method, they proved that the positive solutions of equation (1.2) are radially symmetric. After that there are many results about the symmetric properties of solutions. In fact, if moving plane method can not be used, then the solutions can be symmetry-breaking. We point out that if the domain Ω is an annulus, then it is not convex, thus the moving plane method can not be used. In [9], Y. Li considered the non-radial solutions of equation (1.2), the author also studied the supercritical case, see also [11, 15] and the reference therein.

In recently decades, several researchers have studied the symmetric results of elliptic systems. In [3], by maximal principle and Morse index, the authors studied the symmetric and symmetry-breaking results of elliptic system, see [14, 17, 18] for symmetry-breaking results of system and the references therein. In particular, for $p = 3$ of system, the author [12] studied the non-radial solution of system (1.1), where they proved that the system has at least a non-radial solution.

Noticing that $\frac{2N-2}{N-3} > 2^*$, the exponent p may be super critical exponent. In this paper, by a compact embedding in [9], we study the multiple solutions of system (1.1), we have that

Theorem 1.1. *Let d be a fixed positive number, $2 < p < \frac{2N-2}{N-3}(N > 4)$. If $r > d$ is large enough, then system (1.1) has at least $[\frac{N}{2}] - 1$ non-radially symmetric solutions.*

This paper is organized as follows. In section 2, we give several preliminaries and lemmas; in section 3, by variational and rescaling method, we give the proof of Theorem 1.1.

2. Some preliminaries and lemmas

Firstly, we define the working space $H = H_0^1(\Omega) \times H_0^1(\Omega)$ with the norm as

$$\|(u, v)\| = \left(\int_{\Omega} |\nabla u|^2 + |\nabla v|^2 dx \right)^{1/2},$$

where $H_0^1(\Omega)$ is the standard Sobolev space.

We define the functional of system (1.1) as

$$I(u, v) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + |\nabla v|^2 dx - \frac{1}{p} \int_{\Omega} \mu_1 u^p + \mu_2 v^p dx - \frac{2}{p} \beta \int_{\Omega} u^{\frac{p}{2}} v^{\frac{p}{2}} dx,$$

and the following quotation

$$J(u, v) = \frac{(\int_{\Omega} |\nabla u|^2 + |\nabla v|^2 dx)^{p/2}}{\int_{\Omega} \mu_1 u^p + \mu_2 v^p dx + \frac{p}{2} \int_{\Omega} u^{\frac{p}{2}} v^{\frac{p}{2}} dx}.$$

Now we define

$$\Lambda_0 = \{(u, v) \in H \setminus \{(0, 0)\} : u, v \text{ are radial functions}\},$$

and

$$\Lambda_k = \{(u, v) \in H \setminus \{(0, 0)\} : u(x) = u(|y_1|, |y_2|), v(x) = v(|y_1|, |y_2|)\},$$

where $y_1 = (x_1, \dots, x_k)$, $y_2 = (x_{k+1}, \dots, x_N)$, $k = 0, 1, \dots, [\frac{N}{2}]$.

We define the minimum value as

$$\lambda_k = \inf_{(u, v) \in \Lambda_k} J(u, v), \quad k = 0, 1, \dots, [\frac{N}{2}].$$

Now we list a inequality, see [9].

Lemma 2.1. *Let $\Omega = \{x \in \mathbb{R}^N : r^2 < |x|^2 < (r+d)^2\}$ be an annulus, $N > 4$, $2 < p < \frac{2N-2}{N-3}$ and d is a fixed positive number, let*

$$\bar{\Lambda}_k = \{u \in H_0^1(\Omega) : u(x) = u(|y_1|, |y_2|)\},$$

where $y_1 = (x_1, \dots, x_k)$, $y_2 = (x_{k+1}, \dots, x_N)$. Then for $r > 10d$ large enough, we have

$$\frac{(\int_{\Omega} |\nabla u|^2 dx)^{p/2}}{\int_{\Omega} u^p dx} \geq Cr^{\frac{p-2}{2}},$$

where C is a positive constant independent of r .

Remark 2.2. *If $1 < p < 2^*$, by Sobolev embedding theorem, we have that $\bar{\Lambda}_k \hookrightarrow L^p$ is compact, see [8, 20, 22], by previous lemma, we have that $\bar{\Lambda}_k \hookrightarrow L^{\frac{2N-2}{N-3}}$ is continuous, thus for $2 < p < \frac{2N-2}{N-3}$, we have that $\bar{\Lambda}_k \hookrightarrow L^p$ is compact.*

Lemma 2.3. *Suppose $2 \leq k < \bar{k} \leq [\frac{N}{2}]$, then*

$$\bar{\Lambda}_k \cap \bar{\Lambda}_{\bar{k}} \subset \bar{\Lambda}_0.$$

Next section we give the proof of the main result.

3. The proof of Theorem 1.1

We divide the proof into two parts. Firstly we have that

Lemma 3.1. *There exists a positive constant C depending on μ_1, μ_2, β, d , such that*

$$(3.1) \quad \lambda_0 \geq Cr^{(N-1)(\frac{p}{2}-1)}.$$

Proof. Let $(u, v) \in \Lambda_0$, i.e. $u(x) = u(|x|)$, $v(x) = v(|x|)$, then

$$(3.2) \quad \begin{aligned} \int_{\Omega} |\nabla u|^2 dx &= |\partial\Omega| \int_r^{r+d} |u'_{\rho}|^2 \rho^{N-1} d\rho \\ &\geq C(\Omega) r^{N-1} \int_r^{r+d} |u'_{\rho}|^2 d\rho. \end{aligned}$$

Similarly we have that

$$\int_{\Omega} |\nabla v|^2 dx \geq C(\Omega) r^{N-1} \int_r^{r+d} |v'_{\rho}|^2 d\rho.$$

By Sobolev embedding results, we have that

$$(3.3) \quad \begin{aligned} \int_{\Omega} u^p dx &= |\partial\Omega| \int_r^{r+d} u^p(\rho) \rho^{N-1} d\rho \\ &\leq C(\Omega, d) r^{N-1} \int_r^{r+d} u^p d\rho \\ &\leq C(\Omega, d) r^{N-1} \left(\int_r^{r+d} |u'_{\rho}|^2 d\rho \right)^{p/2}, \end{aligned}$$

and

$$\int_{\Omega} v^p dx \leq C(\Omega, d) r^{N-1} \left(\int_r^{r+d} |v'_\rho|^2 d\rho \right)^{p/2}.$$

By Hölder and Sobolev inequalities, we have

$$\begin{aligned} \int_{\Omega} u^{\frac{p}{2}} v^{\frac{p}{2}} dx &\leq \left(\int_{\Omega} u^p dx \right)^{1/2} \left(\int_{\Omega} v^p dx \right)^{1/2} \\ (3.4) \quad &\leq C(\Omega, d) r^{N-1} \left(\int_r^{r+d} |u'_\rho|^2 d\rho \right)^{\frac{p}{4}} \left(\int_r^{r+d} |v'_\rho|^2 d\rho \right)^{\frac{p}{4}}. \end{aligned}$$

Then we obtain that

$$\begin{aligned} (3.5) \quad \int_{\Omega} \mu_1 u^p + \mu_2 v^p + \frac{p\beta}{2} u^{\frac{p}{2}} v^{\frac{p}{2}} dx &\leq C r^{N-1} \left\{ \left(\int_r^{r+d} |u'_\rho|^2 d\rho \right)^{\frac{p}{2}} + \left(\int_r^{r+d} |v'_\rho|^2 d\rho \right)^{\frac{p}{2}} \right. \\ &\quad \left. + 2 \left(\int_r^{r+d} |u'_\rho|^2 d\rho \right)^{\frac{p}{4}} \left(\int_r^{r+d} |v'_\rho|^2 d\rho \right)^{\frac{p}{4}} \right\} \\ &\leq C r^{N-1} \left\{ \left(\int_r^{r+d} |u'_\rho|^2 d\rho \right)^{\frac{p}{4}} + \left(\int_r^{r+d} |v'_\rho|^2 d\rho \right)^{\frac{p}{4}} \right\}^2 \\ &\leq 4C r^{N-1} \left\{ \left(\int_r^{r+d} |u'_\rho|^2 d\rho \right)^{\frac{p}{2}} + \left(\int_r^{r+d} |v'_\rho|^2 d\rho \right)^{\frac{p}{2}} \right\}, \end{aligned}$$

where C is a positive constant depending on μ_1, μ_2, β, d .

Noticing that $2 < p < \frac{2N-2}{N-3}$, then we obtain that

$$\left(\int_r^{r+d} |u'_\rho|^2 d\rho \right)^{\frac{p}{2}} + \left(\int_r^{r+d} |v'_\rho|^2 d\rho \right)^{\frac{p}{2}} \leq \left(\int_r^{r+d} |u'_\rho|^2 d\rho + \int_r^{r+d} |v'_\rho|^2 d\rho \right)^{\frac{p}{2}}.$$

Then we have that

$$\begin{aligned} (3.6) \quad \lambda_0 &\geq \frac{C r^{\frac{p}{2}(N-1)} \left(\int_r^{r+d} |u'_\rho|^2 d\rho + \int_r^{r+d} |v'_\rho|^2 d\rho \right)^{\frac{p}{2}}}{4C r^{N-1} \left(\int_r^{r+d} |u'_\rho|^2 d\rho + \int_r^{r+d} |v'_\rho|^2 d\rho \right)^{\frac{p}{2}}} \\ &\sim C(\mu_1, \mu_2, \beta, d) r^{(N-1)(\frac{p}{2}-1)}. \end{aligned}$$

In conclusion, we complete the proof. $\square$

Following the idea of Li [9], we have that

Lemma 3.2. *There exists a constant $C > 0$ depending on μ_1, μ_2, β, d , such that*

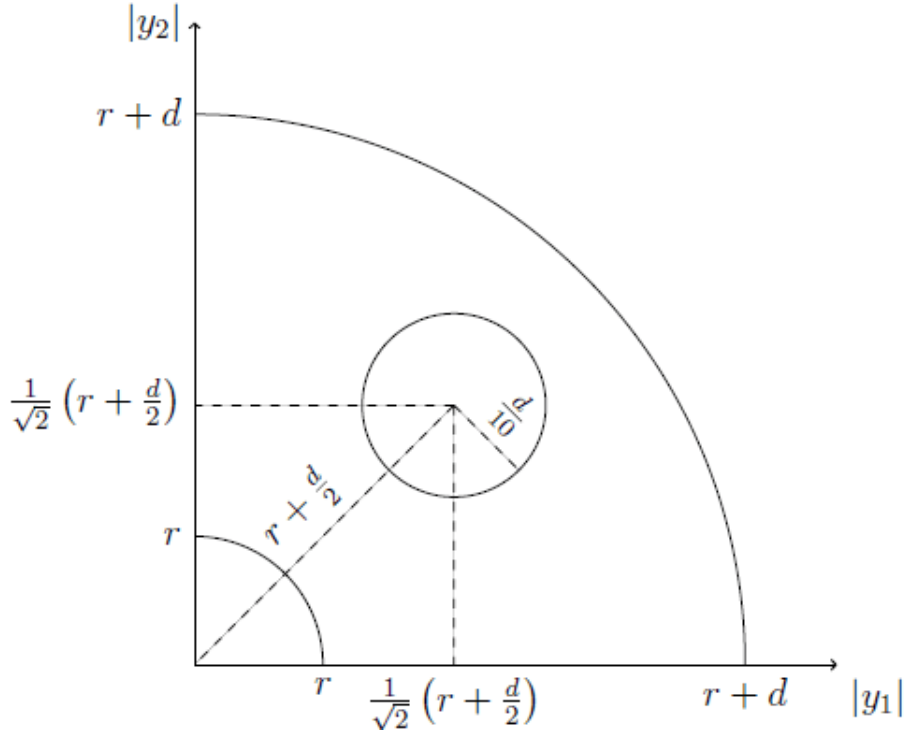
$$(3.7) \quad \lambda_k \leq C r^{(N-2)(\frac{p}{2}-1)}, \text{ for } r > 10d \text{ large enough,}$$

where $k = 2, \dots, [\frac{N}{2}]$.

Proof. Choose a nonnegative function $\omega(x) \in C_0^\infty(B_{d/10}) \setminus \{0\}$, where $B_{d/10} \subset \mathbb{R}^2$, and let

$$u_k(x) = v_k(x) = \omega(|y_1| - \frac{1}{\sqrt{2}}(r + \frac{d}{2}), |y_2| - \frac{1}{\sqrt{2}}(r + \frac{d}{2})),$$

where $y_1 = (x_1, \dots, x_k)$, $y_2 = (x_{k+1}, \dots, x_N)$. One can prove that $(u_k(x), v_k(x)) \in H$, see the following figure



Then we have that

$$\begin{aligned}
 \int_{\Omega} |\nabla u_k|^2 dx &\leq C \int_{r^2 \leq t^2 + s^2 \leq (r+d)^2} |\nabla \omega|^2 t^{k-1} s^{N-k-1} dt ds \\
 (3.8) \quad &\leq C(r+d)^{N-2} \int_{r^2 \leq t^2 + s^2 \leq (r+d)^2} |\nabla \omega|^2 dt ds \\
 &\leq C(d, \Omega) r^{N-2} \int_{r^2 \leq t^2 + s^2 \leq (r+d)^2} |\nabla \omega|^2 dt ds,
 \end{aligned}$$

and

$$\int_{\Omega} |\nabla v_k|^2 dx \leq C(d, \Omega) r^{N-2} \int_{r^2 \leq t^2 + s^2 \leq (r+d)^2} |\nabla \omega|^2 dt ds.$$

Next we have that

$$(3.9) \quad \begin{aligned} \int_{\Omega} u_k^p dx &= |\partial\Omega| \int_{r^2 \leq t^2 + s^2 \leq (r+d)^2} u_k^p t^{k-1} s^{N-k-1} dt ds \\ &\geq Cr^{N-2} \int_{r^2 \leq t^2 + s^2 \leq (r+d)^2} \omega^p dt ds, \end{aligned}$$

and

$$(3.10) \quad \int_{\Omega} v_k^p dx \geq Cr^{N-2} \int_{r^2 \leq t^2 + s^2 \leq (r+d)^2} \omega^p dt ds.$$

Then

$$(3.11) \quad \int_{\Omega} u_k^{\frac{p}{2}} v_k^{\frac{p}{2}} dx = \int_{\Omega} \omega^p dx \geq Cr^{N-2} \int_{r^2 \leq t^2 + s^2 \leq (r+d)^2} \omega^p dt ds.$$

By the previous inequalities and noticing $\text{supp } \omega(t, s) \subset \mathbb{R}^2$, we have that

$$\begin{aligned} \lambda_k &\leq \frac{Cr^{\frac{p}{2}(N-2)} \left(\int_{r^2 \leq t^2 + s^2 \leq (r+d)^2} |\nabla \omega|^2 dt ds \right)^{\frac{p}{2}}}{C(\beta, \mu_1, \mu_2) \int_{r^2 \leq t^2 + s^2 \leq (r+d)^2} \omega^p dt ds} \\ &\leq Cr^{(\frac{p}{2}-1)(N-2)}, \end{aligned}$$

where $C > 0$ is constant depending on β, μ_1, μ_2 . In conclusion, we complete the proof. $\square$

Next we give the proof of main result.

The Proof of Theorem 1.1. By Lemma 2.1, we have that $\lambda_k, k = 0, 1, \dots, [\frac{N}{2}] - 1$ is achieved, then by standard rescaling method and critical theory, we can find $(u_0, v_0) \in \Lambda_0$ and $(u_k, v_k) \in \Lambda_k$ which are solutions of system (1.1). If $r > 0$ is large enough, by (3.1) and (3.7), it is a contraction, thus (u_k, v_k) is non-radial, $k = 2, \dots, [\frac{N}{2}]$. Notice that Lemma 2.3, we prove that system (1.1) has at least $[\frac{N}{2}] - 1$ non-radial solutions. In conclusion, we complete the proof. $\square$

Remark 3.3. If $y_1 = (x_1, \dots, x_k)$ $y_2 = (x_{k+1}, \dots, x_N)$ and $\bar{y}_1 = (x_{i_1}, \dots, x_{i_k})$ $\bar{y}_2 = (x_{i_{k+1}}, \dots, x_{i_N})$, $i_1, \dots, i_N$ is a permutation of $1, \dots, N$, we call that $u(|y_1|, |y_2|) \sim u(|\bar{y}_1|, |\bar{y}_2|)$.

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