

d -MIXING AND d -UNIVERSAL J -CLASS OPERATORS

A. Tajmouati*

M. El Berrag

Sidi Mohamed Ben Abdellah University

Faculty of Sciences Dhar El Mahraz

Fez

Morocco

abdelaiz.tajmouati@usmba.ac.ma

mohammed.elberrag@usmba.ac.ma

Abstract. In this paper, we characterize the notion of d -universal extended (respectively d -universal extended mixing) limit set and we give a relation between d -universal (or d -hypercyclic sequences) and d -universal extended limit set.

Keywords: tuple of sequence, hypercyclic sequences, d -topologically transitive, d - J^{univer} -class operators, d - J^{u-mix} -class operators.

1. Introduction

For an infinite-dimensional separable complex Banach space X , $\mathcal{B}(X)$ will denote the algebra of all bounded linear operators on X . For $x \in X$, the orbit of x under $(T_n)_n$ is the set $Orb(T_n, x) = \{T_n x : n \in \mathbb{Z}_+\}$. A sequence $(T_n)_n$ of operators is called hypercyclic or universal if there is some x whose orbit under $(T_n)_n$ is dense in X . In such a case, x is called a hypercyclic or universal vector for $(T_n)_n$. A sequence $(T_n)_n$ of operators is called topologically transitive if for every nonempty open subsets U and V of X there is some $n \geq 0$ such that $T_n(U) \cap V \neq \emptyset$. For some sources on these topics; see [1, 6, 7, 8, 10].

The notion of disjoint hypercyclicity, a strengthening of hypercyclicity, concerning a tuple of linear operators, was introduced independently by Bernal [2] and by Bès and Peris [4] in 2007.

For any integer $N \geq 2$, the tuple $(T_1, T_2, \dots, T_N)$ of operators, acting on the same topological vector space X , is said disjoint hypercyclic, or d -hypercyclic for short, provided there exists $(z, \dots, z)$ in X^N , such that $\{(T_1^n z, T_2^n z, \dots, T_N^n z) : n \in \mathbb{Z}_+\}$ is dense in X^N . Such a vector z is called a d -hypercyclic vector for the tuple $(T_1, T_2, \dots, T_N)$.

We say that the operators $T_1, T_2, \dots, T_N$ in $\mathcal{B}(X)$ with $N \geq 2$ are d -topologically transitive if for any non-empty open subsets $V_0, V_1, \dots, V_N$ in X , there exists a positive integer n so that $\emptyset \neq V_0 \cap T_1^{-n}(V_1) \cap T_2^{-n}(V_2) \cap \dots \cap T_N^{-n}(V_N)$.

*. Corresponding author

Definition 1.1 ([4], Definition 2.1). We say that $N \geq 2$ sequences of operators $(T_{1,n})_{n=1}^\infty, (T_{2,n})_{n=1}^\infty, \dots, (T_{N,n})_{n=1}^\infty$ in $\mathcal{B}(X)$ are *d-topologically transitive* (respectively *d-mixing*) provided for every non-empty open subsets $V_0, V_1, \dots, V_N$ of X there exists a positive integer m so that $\emptyset \neq V_0 \cap T_{1,m}^{-1}(V_1) \cap T_{2,m}^{-1}(V_2) \cap \dots \cap T_{N,m}^{-1}(V_N)$ (respectively so that $\emptyset \neq V_0 \cap T_{1,n}^{-1}(V_1) \cap T_{2,n}^{-1}(V_2) \cap \dots \cap T_{N,n}^{-1}(V_N)$ for every $n \geq m$).

Definition 1.2 ([4], Definition 2.2). We say that $N \geq 2$ sequences $(T_{1,n})_{n=1}^\infty, (T_{2,n})_{n=1}^\infty, \dots, (T_{N,n})_{n=1}^\infty$ in $\mathcal{B}(X)$ are *d-universal* if $\{(T_{1,n}z, T_{2,n}z, \dots, T_{N,n}z) : n \in \mathbb{Z}_+\}$ is dense in X^N for some vector $z \in X$. We call vector z a *d-universal* for $(T_{1,n})_{n=1}^\infty, (T_{2,n})_{n=1}^\infty, \dots, (T_{N,n})_{n=1}^\infty$.

Recall that S. Shkarin in [9] gave a short proof of existence of disjoint hypercyclic tuples of operators of any given length on any separable infinite dimensional Fréchet space. Similar argument provides disjoint dual hypercyclic tuples of operators of any length on any infinite dimensional Banach space with separable dual. For recent results on disjoint hypercyclicity (see [3]).

Definition 1.3. Let $T : X \rightarrow X$ be a bounded linear operator on a Banach space X . For every $x \in X$ the sets $J(x) = \{y \in X : \text{there exist a strictly increasing sequence of positive integers } k_n \text{ and a sequence } x_n \subset X \text{ such that } x_n \rightarrow x \text{ and } T^{k_n}x_n \rightarrow y\}$ $J^{mix}(x) = \{y \in X : \text{there exist a sequence } x_n \subset X \text{ such that } x_n \rightarrow x \text{ and } T^n x_n \rightarrow y\}$ will be called the *extended limit set* of x under T and the *extended mixing limit set* of x under T respectively.

The notions of the limit and extended limit sets are well known in the theory of topological dynamics. In [5] G. Costakis and A. Manoussos, defined the J -sets, examined some basic properties of these sets and investigated the relation between hypercyclicity and J -sets. In particular they showed that T is hypercyclic if and only if there exists a cyclic vector $x \in X$ such that $J(x) = X$. Recall that a vector x is cyclic for T if the linear span of the orbit $Orb(T, x)$ is dense in X . In [12] we localized the notion of M -extended semigroup (resp. M -extended semigroup mixing) limit set of x under $\mathcal{T} = (T_t)_{t \geq 0}$ and we gave sufficient conditions of being M -hypercyclic for this semigroup. Then by this result, we proved that $(T_t^{-1})_{t \geq 0}$ is a M -hypercyclic.

On other hand, let $T_1, T_2, \dots, T_N$ in $\mathcal{B}(X)$ with $N \geq 2$. For every $x_0 \in X$ the sets $d\text{-}J_{(T_1, T_2, \dots, T_N)}(x_0) = \{(x_1, \dots, x_N) \in X^N : \text{for every neighborhood } V_0, V_1, \dots, V_N \text{ of } x_0, x_1, \dots, x_N \text{ respectively, there exists a positive integer } n \text{ so that } \emptyset \neq V_0 \cap T_1^{-n}(V_1) \cap T_2^{-n}(V_2) \cap \dots \cap T_N^{-n}(V_N)\}$ and $d\text{-}J_{(T_1, T_2, \dots, T_N)}^{mix}(x_0) = \{(x_1, \dots, x_N) \in X^N : \text{for every neighborhood } V_0, V_1, \dots, V_N \text{ of } x_0, x_1, \dots, x_N \text{ respectively, there exists a positive integer } m \text{ so that } \emptyset \neq V_0 \cap T_1^{-n}(V_1) \cap T_2^{-n}(V_2) \cap \dots \cap T_N^{-n}(V_N) \text{ for every } n \geq m\}$, will be called the *extended limit set* of x_0 under $T_1, T_2, \dots, T_N$ and the *extended mixing limit set* of x_0 under $T_1, T_2, \dots, T_N$ respectively. We have introduced this notion in [11], and we extended some results known for a single operator to a tuple of sequences of operators.

In this paper, we characterize the notion of d -universal extended (respectively d -universal extended mixing) limit set and we give a relation between d -universal (or d -hypercyclic sequences) and d -universal extended limit set.

2. d -universal J-class

Definition 2.1. Let $T_{1,n}, T_{2,n}, \dots, T_{N,n}$ in $\mathcal{B}(X)$ with $N \geq 2$. For every $x_0 \in X$ the sets

$d-L^{univer}(x_0) := \{(x_1, \dots, x_N) \in X^N : \text{there exists a strictly increasing sequence of positive integers } k_n \text{ such that } T_{i,k_n}x_0 \rightarrow x_i \text{ for all } 1 \leq i \leq N\}$

$d-J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{univer}(x_0) = \{(x_1, \dots, x_N) \in X^N : \text{for every neighborhood } V_0, V_1, \dots, V_N \text{ of } x_0, x_1, \dots, x_N \text{ respectively, there exists a positive integer } n \text{ so that } \emptyset \neq V_0 \cap T_{1,n}^{-1}(V_1) \cap T_{2,n}^{-1}(V_2) \cap \dots \cap T_{N,n}^{-1}(V_N)\}$ will be called the d -universal limit set and d -universal extended limit set of x_0 under $T_{1,n}, T_{2,n}, \dots, T_{N,n}$ respectively.

Remark 2.2. An equivalent definition for the set $d-J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{univer}(x_0)$ is the following:

$d-J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{univer}(x_0) = \{(x_1, \dots, x_N) \in X^N : \text{there exist a strictly increasing sequence of positive integers } k_n \text{ and a sequence } (x_n) \subset X \text{ such that } x_n \rightarrow x_0 \text{ and } T_{i,k_n}x_n \rightarrow x_i \text{ for all } 1 \leq i \leq N\}.$

Theorem 2.3. Let $T_{1,n}, T_{2,n}, \dots, T_{N,n}$ in $\mathcal{B}(X)$ with $N \geq 2$. Then the following conditions are equivalent:

1. For every $x_0 \in X$, $d-J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{univer}(x_0) = X^N$.
2. $T_{1,n}, T_{2,n}, \dots, T_{N,n}$ is d -transitive.
3. The set of d -universal vectors for $(T_{1,n})_1^\infty, (T_{2,n})_1^\infty, \dots, (T_{N,n})_1^\infty$ is a dense G_δ .

Proof. We first prove: (1) implies (2). Let $V_0, V_1, \dots, V_N$ the nonempty open. Consider $x_i \in V_i$ ($0 \leq i \leq N$). Since $d-J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{univer}(x_0) = X^N$, so there exists $n \in \mathbb{N}$ such that $\emptyset \neq V_0 \cap \bigcap_{i=1}^N T_{i,n}^{-1}(V_i)$. By definition $T_{1,n}, T_{2,n}, \dots, T_{N,n}$ is d -transitive.

We will show that (2) implies (1). Let $x_i \in V_i$ ($0 \leq i \leq N$) and $V_0, V_1, \dots, V_N$ be relatively open subsets of X . So there exists $n \in \mathbb{N}$ such that $\emptyset \neq V_0 \cap \bigcap_{i=1}^N T_{i,n}^{-1}(V_i)$. So $(x_1, \dots, x_N) \in d-J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{univer}(x_0)$, and consequently $d-J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{univer}(x_0) = X^N$.

For (2) $\Leftrightarrow$ (3) see [4, Proposition 2.3]. □

Proposition 2.4. Let $T_{1,n}, T_{2,n}, \dots, T_{N,n}$ in $\mathcal{B}(X)$ with $N \geq 2$. Then

$$d - J_{(\lambda_1 T_{1,n}, \lambda_2 T_{2,n}, \dots, \lambda_N T_{N,n})}^{univer}(0) = d - J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{univer}(0),$$

for every $|\lambda_i| = 1$ ($1 \leq i \leq N$).

Proof. Let $(x_1, \dots, x_N) \in d\text{-}J_{(\lambda_1 T_{1,n}, \lambda_2 T_{2,n}, \dots, \lambda_N T_{N,n})}^{univer}(0)$. Then there exist a strictly increasing sequence of positive integers k_n and a sequence $(x_n) \subset X$ such that $x_n \rightarrow 0$ and $\lambda_i T_{i,k_n} x_n \rightarrow x_i$ for all $1 \leq i \leq N$. Since $|\lambda_i| = 1$ then $\lambda_i x_n \rightarrow 0$ and since $T_{i,k_n}(\lambda_i x_n) \rightarrow x_i$ it follows that $(x_1, \dots, x_N) \in d\text{-}J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{univer}(0)$.

Let $(x_1, \dots, x_N) \in d\text{-}J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{univer}(0)$. Then there exist a strictly increasing sequence of positive integers k_n and a sequence $(x_n) \subset X$ such that $x_n \rightarrow 0$ and $T_{i,k_n} x_n \rightarrow x_i$ for all $1 \leq i \leq N$. Since $|\lambda_i| = 1$ ($1 \leq i \leq N$), without loss of generality we can assume that $\lambda_i \rightarrow \mu_i$ for some $|\mu_i| = 1$. Hence $\lambda_i T_{i,k_n}(\frac{x_n}{\mu_i}) \rightarrow x_i$ for all $1 \leq i \leq N$ and since $\frac{x_n}{\mu_i} \rightarrow 0$ then $(x_1, \dots, x_N) \in d\text{-}J_{(\lambda_1 T_{1,n}, \lambda_2 T_{2,n}, \dots, \lambda_N T_{N,n})}^{univer}(0)$. $\square$

Lemma 2.5. Let $T_{1,n}, T_{2,n}, \dots, T_{N,n}$ in $\mathcal{B}(X)$ with $N \geq 2$ and $x_{0,n}, x_{1,n}, \dots, x_{N,n}$ be the sequences in X such that $x_{0,n} \rightarrow x_0$ and $x_{i,n} \rightarrow x_i$ for some $x_0, x_i \in X$ ($1 \leq i \leq N$). If $(x_{1,n}, \dots, x_{N,n}) \in d\text{-}J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{univer}(x_{0,n})$ for every $n = 1, 2, \dots$, then $(x_1, \dots, x_N) \in d\text{-}J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{univer}(x_0)$.

Proof. For $n = 1$ there exists a positive integer k_1 such that

$$\|x_{0,k_1} - x_0\| < \frac{1}{2} \quad \text{and} \quad \|x_{i,k_1} - x_i\| < \frac{1}{2} \quad (1 \leq i \leq N).$$

Since $(x_{1,k_1}, \dots, x_{N,k_1}) \in d\text{-}J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}(x_{0,k_1})$ we may find a positive integer l_1 and $z_1 \in X$ such that

$$\|z_1 - x_{0,k_1}\| < \frac{1}{2} \quad \text{and} \quad \|T_{i,l_1} z_1 - x_{i,k_1}\| < \frac{1}{2} \quad (1 \leq i \leq N).$$

Therefore,

$$\|z_1 - x_0\| < 1 \quad \text{and} \quad \|T_{i,l_1} z_1 - x_i\| < 1 \quad (1 \leq i \leq N).$$

Proceeding inductively we find a strictly increasing sequence of positive integers l_n and a sequence z_n in X , such that

$$\|z_n - x_0\| < \frac{1}{n} \quad \text{and} \quad \|T_{i,l_n} z_n - x_i\| < \frac{1}{n} \quad (1 \leq i \leq N).$$

This complete the proof. $\square$

Proposition 2.6. Let $T_{1,n}, T_{2,n}, \dots, T_{N,n}$ in $\mathcal{B}(X)$ with $N \geq 2$. If $\sup_n \|T_{i,n}\| < \infty$ for all $1 \leq i \leq N$, then $d\text{-}J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{univer}(x_0) = d\text{-}L^{univer}(x_0)$ For every $x_0 \in X$.

Proof. Since $\sup_n \|T_{i,n}\| < \infty$ for all $1 \leq i \leq N$ there exists a positive number M such that $\|T_{i,n}\| \leq M$ for every positive integer n and $1 \leq i \leq N$. Let $x_0 \in X$. If $d\text{-}J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{univer}(x_0) = \emptyset$ there is nothing to prove. Therefore assume that $d\text{-}J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{univer}(x_0) \neq \emptyset$.

Since the inclusion $d\text{-}L^{univer}(x_0) \subset d\text{-}J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{univer}(x_0)$ is always true, it suffices to show that $d\text{-}J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{univer}(x_0) \subset d\text{-}L^{univer}(x_0)$. Take $(x_1, \dots, x_N) \in d\text{-}J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{univer}(x_0)$. There exist a strictly increasing sequence of positive

integers k_n and a sequence $(x_n) \subset X$ such that $x_n \rightarrow x_0$ and $T_{i,k_n}x_n \rightarrow x_i$ for all $1 \leq i \leq N$. Then we have

$$\begin{aligned} \|T_{i,k_n}x_0 - x_i\| &\leq \|T_{i,k_n}x_0 - T_{i,k_n}x_n\| + \|T_{i,k_n}x_n - x_i\| \\ &\leq M\|x_0 - x_n\| + \|T_{i,k_n}x_n - x_i\| \end{aligned}$$

and letting n goes to infinity to the above inequality, we get that $(x_1, \dots, x_N) \in d-L^{univer}(x_0)$. $\square$

Proposition 2.7. *Let $T_{1,n}, T_{2,n}, \dots, T_{N,n}$ in $\mathcal{B}(X)$ with $N \geq 2$. Then the set $A = \{x_0 \in X : d-J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{univer}(x_0) = X^N\}$ is closed.*

Proof. By Lemma 2.5 A is closed. $\square$

Theorem 2.8. *Let $T_{1,n}, T_{2,n}, \dots, T_{N,n}$ in $\mathcal{B}(X)$ with $N \geq 2$. Then the following conditions are equivalent:*

1. $(T_{1,n})_1^\infty, (T_{2,n})_1^\infty, \dots, (T_{N,n})_1^\infty$ is d -universal or d -hypercyclic sequences.
2. For every $x_0 \in X$, $d-J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{univer}(x_0) = X^N$.
3. The set $A = \{x_0 \in X : d-J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{univer}(x_0) = X^N\}$ is dense in X^N .

Proof. We first prove that (1) implies (2). Let $x_i \in V_i$ ($0 \leq i \leq N$). Since the set of d -universal vectors for $(T_{1,n})_1^\infty, (T_{2,n})_1^\infty, \dots, (T_{N,n})_1^\infty$ is a dense G_δ in X^N . Hence there exist a d -universal (or sequences of hypercyclic) vectors $(x_n) \subset X$ and a strictly increasing sequence of positive integers k_n such that $x_n \rightarrow x_0$ and $T_{i,k_n}x_n \rightarrow x_i$ for all $1 \leq i \leq N$. Hence $(x_1, \dots, x_N) \in d-J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{univer}(x_0)$.

The implication (2) $\Rightarrow$ (3) is trivial. Conversely (3) $\Rightarrow$ (2) By Lemma 2.5. $\square$

3. d -mixing J-class

Definition 3.1. *Let $T_{1,n}, T_{2,n}, \dots, T_{N,n}$ in $\mathcal{B}(X)$ with $N \geq 2$. For every $x_0 \in X$ the sets $d-J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{u-mix}(x_0) = \{(x_1, \dots, x_N) \in X^N : \text{for every neighborhood } V_0, V_1, \dots, V_N \text{ of } x_0, x_1, \dots, x_N \text{ respectively, there exists a positive integer } m \text{ so that } \emptyset \neq V_0 \cap T_{1,n}^{-1}(V_1) \cap T_{2,n}^{-1}(V_2) \cap \dots \cap T_{N,n}^{-1}(V_N)\} \text{ for every } n \geq m\}$ will be called the d -universal extended mixing limit set of x_0 under $T_{1,n}, T_{2,n}, \dots, T_{N,n}$.*

Proposition 3.2. *An equivalent definition for the set $d-J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{u-mix}(x_0)$ is the following:*

$d-J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{u-mix}(x_0) = \{(x_1, \dots, x_N) \in X^N : \text{there exists a sequence } (x_n) \subset X \text{ such that } x_n \rightarrow x_0 \text{ and } T_{i,n}x_n \rightarrow x_i \text{ for all } 1 \leq i \leq N\}$.

Proof. Let us prove that $d-J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{u-mix}(x_0) \subset \{(x_1, \dots, x_N) \in X^N : \text{there exists a sequence } (x_n) \subset X \text{ such that } x_n \rightarrow x_0 \text{ and } T_{i,n}x_n \rightarrow x_i \text{ for all } 1 \leq i \leq N\}$.

Let $(x_1, \dots, x_N) \in d\text{-}J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{u\text{-}mix}(x_0)$ and consider the open balls $V_{0,n} = B(x_0, \frac{1}{n})$, $V_{i,n} = B(x_i, \frac{1}{n})$ centered at $x_0, x_i \in X$ and with radius $1/n$ for $n = 1, 2, \dots$ and $1 \leq i \leq N$. Then there exists a positive integer m so that $\emptyset \neq V_{0,n} \cap T_{1,n}^{-1}(V_{1,n}) \cap T_{2,n}^{-1}(V_{2,n}) \cap \dots \cap T_{N,n}^{-1}(V_{N,n})$ for every $n \geq m$. Hence there exists $x_n \in V_{0,n} = B(x_0, \frac{1}{n})$ such that $x_n \in \bigcap_{i=1}^N T_{i,n}^{-1}(V_{i,n})$, this implies that $T_{i,n}(x_n) \in V_{i,n}$ for all $i = 1, \dots, N$. Therefore there exists a sequence $(x_n) \subset X$ such that $x_n \rightarrow x_0$ and $T_{i,n}x_n \rightarrow x_i$ for all $1 \leq i \leq N$. The converse is obvious. $\square$

Theorem 3.3. *Let $T_{1,n}, T_{2,n}, \dots, T_{N,n}$ in $\mathcal{B}(X)$ with $N \geq 2$. Then the following conditions are equivalent:*

1. *For every $x_0 \in X$, $d\text{-}J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{u\text{-}mix}(x_0) = X^N$.*
2. *$T_{1,n}, T_{2,n}, \dots, T_{N,n}$ is d-mixing.*

Proof. The proof goes similarly as in Theorem 2.3. $\square$

Proposition 3.4. *Let $T_{1,n}, T_{2,n}, \dots, T_{N,n}$ in $\mathcal{B}(X)$ with $N \geq 2$. Then*

$$d - J_{(\lambda_1 T_{1,n}, \lambda_2 T_{2,n}, \dots, \lambda_N T_{N,n})}^{u\text{-}mix}(0) = d - J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{u\text{-}mix}(0),$$

for every $|\lambda_i| = 1$, $(1 \leq i \leq N)$.

Proof. The proof goes similarly as in Proposition 2.4. $\square$

Lemma 3.5. *Let $T_{1,n}, T_{2,n}, \dots, T_{N,n}$ in $\mathcal{B}(X)$ with $N \geq 2$ and $x_{0,n}, x_{1,n}, \dots, x_{N,n}$ be the sequences in X such that $x_{0,n} \rightarrow x_0$ and $x_{i,n} \rightarrow x_i$ for some $x_0, x_i \in X$ ($1 \leq i \leq N$). If $(x_{1,n}, \dots, x_{N,n}) \in d\text{-}J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{u\text{-}mix}(x_{0,n})$ for every $n = 1, 2, \dots$, then $(x_1, \dots, x_N) \in d\text{-}J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{u\text{-}mix}(x_0)$.*

Proof. For $n = 1$ there exists a positive integer k_1 such that

$$\|x_{0,k_1} - x_0\| < \frac{1}{2} \quad \text{and} \quad \|x_{i,k_1} - x_i\| < \frac{1}{2} \quad (1 \leq i \leq N).$$

Since $(x_{1,k_1}, \dots, x_{N,k_1}) \in d\text{-}J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{u\text{-}mix}(x_{0,k_1})$ we may find a positive integer l_1 and $z_n \in X$ such that

$$\|z_n - x_{0,k_1}\| < \frac{1}{2} \quad \text{and} \quad \|T_{i,n}z_n - x_{i,k_1}\| < \frac{1}{2} \quad (1 \leq i \leq N).$$

For every $n \geq l_1$. Therefore,

$$\|z_n - x_0\| < 1 \quad \text{and} \quad \|T_{i,n}z_n - x_i\| < 1 \quad (1 \leq i \leq N).$$

For every $n \geq l_1$. Proceeding inductively we find a strictly increasing sequence of positive integers $l_2 > l_1$ and a sequence $w_n \subset X$ such that

$$\|w_n - x_0\| < \frac{1}{2} \quad \text{and} \quad \|T_{i,n}w_n - x_i\| < \frac{1}{2} \quad (1 \leq i \leq N).$$

For every $n \geq l_2$. Set $v_n = z_n$ for every $l_1 \leq n \leq l_2$, hence

$$\|v_n - x_0\| < 1 \quad \text{and} \quad \|T_{i,n}v_n - x_i\| < 1 \quad (1 \leq i \leq N).$$

Proceeding inductively we find a strictly increasing sequence of positive integers n_k and a sequence $v_n \subset X$ such that if $n \geq k$ then

$$\|v_n - x_0\| < \frac{1}{k} \quad \text{and} \quad \|T_{i,n}v_n - x_i\| < \frac{1}{k} \quad (1 \leq i \leq N).$$

Take any $\varepsilon > 0$. There exists a positive integer k_0 such that $\frac{1}{k_0} < \varepsilon$. Hence for every $n \geq n_{k_0}$ we get

$$\|v_n - x_0\| < \frac{1}{k_0} < \varepsilon \quad \text{and} \quad \|T_{i,n}v_n - x_i\| < \frac{1}{k_0} < \varepsilon \quad (1 \leq i \leq N).$$

This completes the proof. $\square$

Proposition 3.6. *Let $T_{1,n}, T_{2,n}, \dots, T_{N,n}$ in $\mathcal{B}(X)$ with $N \geq 2$. Then the set $B = \{x_0 \in X : d\text{-}J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{u\text{-}mix}(x_0) = X^N\}$ is closed.*

Proof. By Lemma 3.5 B is closed. $\square$

Theorem 3.7. *Let $T_{1,n}, T_{2,n}, \dots, T_{N,n}$ in $\mathcal{B}(X)$ with $N \geq 2$. Then the following conditions are equivalent:*

1. $(T_{1,n})_1^\infty, (T_{2,n})_1^\infty, \dots, (T_{N,n})_1^\infty$ is d -mixing.
2. For every $x_0 \in X$, $d\text{-}J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{u\text{-}mix}(x_0) = X^N$.
3. The set $B = \{x_0 \in X : d\text{-}J_{(T_{1,n}, T_{2,n}, \dots, T_{N,n})}^{u\text{-}mix}(x_0) = X^N\}$ is dense in X^N

Proof. For (1) implies (2) we use the same argument as in the proof of Theorem 3.3.

The implication (2) $\Rightarrow$ (3) is trivial. Conversely (3) $\Rightarrow$ (2) by Lemma 3.5. $\square$

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