

# AN EQUIVALENT DEFINITION OF A $\mathcal{C}$ -GROUP

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**Abstract.** In 1974, Fattahi [2] classified finite non-nilpotent groups in which every subgroup is either normal or abnormal. Surely, it is an interesting topic to define and classify a bigger class of groups containing those classified by Fattahi. In 2012, the first author etc defined a  $\mathcal{C}$ -group. A finite group  $G$  is said to be a  $\mathcal{C}$ -group if for each divisor  $d$  of the order of  $G$ ,  $G$  always contains a subgroup  $H$  of order  $d$  such that  $H$  is either normal or abnormal in  $G$ . The class of  $\mathcal{C}$ -groups is really a class of groups bigger than those classified by Fattahi. During the first author etc investigate the structure of  $\mathcal{C}$ -groups, they found that some property cannot hold if 'normal' is changed into 'subnormal'. So the authors of this paper is motivated to find an equivalent definition of a  $\mathcal{C}$ -group, which can be described by 'subnormal' and 'abnormal'. A  $\mathcal{C}_1$ -group is defined, some good properties of a  $\mathcal{C}_1$ -group are given, then it is proved that a  $\mathcal{C}_1$ -group is equivalent to a  $\mathcal{C}$ -group. At last, as an example of effectiveness of a  $\mathcal{C}_1$ -group, the necessary and sufficient conditions of a semi-product of a  $p$ -group and a  $p'$ -group to be a  $\mathcal{C}$ -group is given.

**Keywords:** normal, abnormal, subnormal, a  $\mathcal{C}$ -group, a  $\mathcal{C}_1$ -group.

## 1. Introduction

All groups considered in this paper are finite. Our notation and terminology are standard, see, for example, Robinson [6].

In 1974, Fattahi [2] classified finite non-nilpotent groups in which every subgroup is either normal or abnormal. Surely, it is an interesting topic to define and classify a bigger class of groups containing those classified by Fattahi. In 2012, Liu, Li and He [5] was motivated by this and introduced the following definition: A group  $G$  is called a  $\mathcal{C}$ -group if for each divisor  $d$  of the order of  $G$ ,  $G$  contains a subgroup  $H$  of order  $d$  such that  $H$  is either normal or abnormal in  $G$ . Of course, a finite group with every subgroup is either normal or abnormal is a  $\mathcal{C}$ -group. So the class of  $\mathcal{C}$ -groups is really a class of groups bigger than those classified by Fattahi. In [5], it is investigated the structure of finite groups

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based on the assumption that some subgroups are  $\mathcal{C}$ -groups and the following equivalence is proved:

**Theorem 1.1** ([5, Corollary 3.2]). *The following statements for a group  $G$  are equivalent:*

- (1)  $G$  is a  $\mathcal{C}$ -group.
- (2) For each subgroup  $B$  of  $G$ , there exists a subgroup  $H$  of  $G$  of order  $|B|$  in  $G$  such that  $H$  is either normal or abnormal in  $G$ .

Notice that the above theorem doesn't hold if "normal" is replaced by "subnormal" in Statement (2). For example, the alternating group of degree 4 is a counterexample. This motivates us to investigate if there exists a kind of equivalent definition of  $\mathcal{C}$ -group, which is described just by using "subnormal" and "abnormal". To reach this target, we give the following definition:

**Definition 1.2.** *A group  $G$  is called a  $\mathcal{C}_1$ -group if for each subgroup  $B$  of a group  $G$ , there exists a subgroup  $H$  of order  $|G : B|$  in  $G$  such that  $H$  is either subnormal or abnormal in  $G$ .*

By using the definition of a  $\mathcal{C}_1$ -group, some good properties of a  $\mathcal{C}_1$ -group are given as follows:

**Theorem 1.3.** *The following statements for a group  $G$  are equivalent:*

- (a)  $G$  is a  $\mathcal{C}_1$ -group.
- (b) Either  $G$  is nilpotent or  $G$  satisfies the following three conditions:
  - (b-1)  $G$  is supersolvable.
  - (b-2)  $G/F(G)$  is cyclic of order  $p$ , where  $p$  is the smallest prime dividing the order of  $G$ .
  - (b-3)  $G/O_p(G)$  is a Frobenius group with Frobenius complement  $P/O_p(G)$ , where  $P$  is a Sylow  $p$ -subgroup of  $G$  and  $P/O_p(G)$  is cyclic of order  $p$ .

Then, we get the following equivalence by using above theorem:

**Theorem 1.4.** *A group  $G$  is a  $\mathcal{C}$ -group if and only if  $G$  is a  $\mathcal{C}_1$ -group.*

Although a  $\mathcal{C}_1$ -group and a  $\mathcal{C}$ -group are equivalent, we find that it is easier to get structure of a  $\mathcal{C}$ -group by using definition of  $\mathcal{C}_1$ -group. At last, as an example of effectiveness of a  $\mathcal{C}_1$ -group, the necessary and sufficient conditions of a semi-product of a  $p$ -group and a  $p'$ -group to be a  $\mathcal{C}_1$ -group is given as follows:

**Theorem 1.5.** *Let  $P$  be a  $p$ -group that acts non-trivially on a  $p'$ -group  $M$ . Then the semidirect product  $P \ltimes M$  is a  $\mathcal{C}$ -group if the following conditions hold:*

- (1) *The automorphism group induced by  $P$  on  $M$  is a cyclic group  $\langle \alpha \rangle$  of order  $p$ .*
- (2)  *$\alpha$  is a fixed-point-free automorphism.*
- (3)  *$\alpha$  is a semi-power-automorphism.*

*The converse is true for a non-nilpotent  $\mathcal{C}_1$ -group.*

## 2. Preliminaries

In this section we present some lemmas which are required in Section 3. A subgroup  $H$  of a group  $G$  is said to be abnormal in  $G$  if  $g \in \langle H, H^g \rangle$  for all  $g$  in  $G$ . A subgroup  $H$  is said to be pronormal in a group  $G$  if for all  $g$  in  $G$  the subgroups  $H$  and  $H^g$  are conjugate in  $\langle H, H^g \rangle$ .

**Lemma 2.1.** *Let  $H$  be a subgroup of a group  $G$ . Then the following statements are true:*

- (a) *If  $H$  is abnormal in  $G$  and  $H \leq K \leq G$ , then  $K$  is also abnormal in  $G$ .*
- (b)  *$H$  is abnormal in  $G$  if and only if  $H$  is self-normalizing and pronormal in  $G$ .*
- (c) *Let  $N \trianglelefteq G$  and  $N \leq H$ . Then  $H$  is abnormal in  $G$  if and only if  $H/N$  is abnormal in  $G/N$ .*
- (d) *If  $A \leq B < G$  and  $B$  is subnormal in  $G$ , then  $A$  is not abnormal in  $G$ .*
- (e) *A subnormal Sylow  $p$ -subgroup is normal and in a solvable group a subnormal Hall subgroup is normal.*

**Proof.** Statements (a) and (b) hold by [1, Chapter 1, 6.20 and 6.21]. Statement (c) follows by directly verifying the definition. Statement (d) also follows from Statements (a) and (b).  $\square$

**Lemma 2.2.** *Let  $G = H \ltimes M$  be the semidirect product of the non-trivial subgroup  $H$  with the normal subgroup  $M$ . Then the following statements are equivalent:*

- (a)  *$G$  is a Frobenius group with Frobenius kernel  $M$ .*
- (b)  *$H$  is a group of fixed-point-free automorphisms of  $M$ .*

**Proof.** See [4, V, 8.5 Satz].  $\square$

**Lemma 2.3** ([5, Theorem 3.1]). *The following statements for a group  $G$  are equivalent:*

- (a)  *$G$  is a  $\mathcal{C}$ -group.*

- (b) *Either  $G$  is nilpotent or  $G$  satisfies the following three conditions:*
- (b-1)  *$G$  is supersolvable,*
  - (b-2)  *$G/F(G)$  is cyclic of order  $p$ , where  $p$  is the smallest prime divisor of the order of  $G$ , and*
  - (b-3)  *$G/O_p(G)$  is a Frobenius group whose Frobenius complement  $P/O_p(G)$  is cyclic of order  $p$ , where  $P$  is a Sylow  $p$ -subgroup of  $G$ .*

**Lemma 2.4** ([3, Theorem 1]). *A group  $G$  is nilpotent if and only if for each divisor  $d$  of the order of  $G$  there exists a normal subgroup of order  $d$ .*

Note that the above lemma doesn't hold if the word "nilpotent" is replaced by "supersolvable". But we have:

**Lemma 2.5** ([5, Lemma 2.5]). *Let  $M$  be a normal nilpotent subgroup of a group  $G$ . If  $G$  is supersolvable, then for every divisor  $d$  of the order of  $M$  there exists a subgroup  $H$  of  $M$  of order  $d$  such that  $H$  is normal in  $G$ .*

Let  $\alpha$  be an automorphism of a group  $G$ . Then we say that  $\alpha$  is a semi-power-automorphism of  $G$  if there exist elements  $a_1, a_2, \dots, a_n$  which generate  $G$  such that  $\alpha$  maps  $a_i$  to a power of  $a_i$  for all  $i \in \{1, 2, \dots, n\}$  (see [5, Definition 1.4]).

**Lemma 2.6** ([5, Lemma 2.2]). *The following statements for a group  $G$  are true:*

- (a) *Suppose that  $G = H\langle x \rangle$ , where  $H$  is a normal nilpotent subgroup of  $G$ . If  $x$  induces a semi-power-automorphism on  $H/\Phi(H)$ , then  $G$  is supersolvable.*
- (b) *If  $N$  is a normal nilpotent subgroup of a supersolvable group  $G$ , then each element  $x$  of  $G$  induces a semi-power-automorphism on  $N/\Phi(N)$ .*

### 3. Proofs of theorems

In this section, we prove Theorems 1.3, 1.4, and 1.5:

**The proof of the Theorem 1.3:** (a)  $\Rightarrow$  (b): Suppose that  $G$  is not nilpotent. Take the primary decomposition of  $|G|$  as

$$|G| = p_1^{m_1} p_2^{m_2} \cdots p_s^{m_s}, m_i \geq 1.$$

where  $p_1 < p_2 < \cdots < p_s$ . We split the proof into the following steps:

- (1)  $G$  is solvable.

Let  $P_i$  be a Sylow  $p_i$ -subgroup of  $G$ , where  $i \in \{1, \dots, s\}$ . Then from our assumption there exists a subgroup  $H$  of  $G$  with order  $|G : P_i|$ . This implies that  $H$  is a Hall  $p_i'$ -subgroup of  $G$ . By a well-known theorem of P. Hall [6, 9.1.8], we can see that  $G$  is solvable.

(2)  $G$  is  $p_1$ -nilpotent and the normal  $p_1$ -complement is nilpotent.

Since  $G$  has a subgroup of order  $p_1$ ,  $G$  has a subgroup  $K$  of index  $p_1$  which is either subnormal or abnormal. By using the permutation representation of  $G$  on  $K$ , we see that  $G/K_G$  is isomorphic to a subgroup of the symmetric group  $S_{p_1}$  of degree  $p_1$ . The minimality of  $p_1$  forces  $K = K_G$ . It follows that  $K$  is normal in  $G$ . Let  $q \in \{p_2, \dots, p_s\}$  and  $H$  be a Hall  $q'$ -subgroup of  $G$ . Then, by assumption, there exists a Sylow  $q$ -subgroup  $S$  of  $G$  which is either abnormal or subnormal in  $G$ . Since  $K$  is normal in  $G$ , Lemma 2.1 (d) implies that  $S$  is subnormal. Since  $S$  is a Sylow  $q$ -subgroup of  $G$ , Lemma 2.1 (e) gives  $S$  is normal. Hence the product of all the Sylow  $q$ -subgroups,  $q \in \{p_2, \dots, p_s\}$  is a normal  $p_1$ -complement and this normal  $p$ -complement is nilpotent.

(3) Let  $P$  be a Sylow  $p_1$ -subgroup of  $G$ , then  $P$  is abnormal in  $G$ .

Let  $B$  be a Hall  $p_1'$ -subgroup of  $G$ . Then from our assumption there exists a subgroup  $H$  of order  $|G : B|$  in  $G$  such that  $H$  is either subnormal or abnormal in  $G$ . If  $H$  is subnormal in  $G$ , then it follows from Lemma 2.1 (e) that  $H$  is normal in  $G$ , whence  $H = P$ . By Statement (2), we have that  $G$  is nilpotent. This contradiction shows that  $H$  is abnormal in  $G$ . By Lemma 2.1, we obtain that  $P$  is abnormal in  $G$ .

(4)  $|G/F(G)| = |P/O_{p_1}(G)| = p_1$ .

If  $m_1 = 1$ , then Statement (4) follows from Statements (2) and (3). Hence we may assume that  $m_1 \geq 2$ . Let  $B_1$  be a subgroup of  $G$  of order  $p_1$  and  $M$  a normal  $p_1$ -complement of  $G$ . Then  $B_1M$  is a subgroup of  $G$ . Set  $B = B_1M$ . By hypothesis, there exists a subgroup  $H$  of  $G$  of order  $|G : B| = p_1^{m_1-1}$  such that  $H$  is either subnormal or abnormal in  $G$ . By Sylow's theorem, we have that  $H < P^x$  for some  $x \in G$ , whence  $H < N_G(H)$ . It follows from Lemma 2.1 (b) that  $H$  is subnormal in  $G$  and so  $H \leq O_{p_1}(G)$ . On the other hand, by Statement (3),  $P$  is abnormal in  $G$ , which means that  $O_{p_1}(G) < P$ . Therefore  $H = O_{p_1}(G)$  by comparing the orders and hence  $P/O_{p_1}(G)$  is a cyclic group of order  $p_1$ . The above proofs and Statement (2) are combined to give that  $G/F(G)$  is also a cyclic group of order  $p_1$ .

(5)  $G/O_{p_1}(G)$  is a Frobenius group with Frobenius complement  $P/O_{p_1}(G)$ .

Set  $\bar{G} = G/O_{p_1}(G)$ . Then  $\bar{G} = \bar{P} \ltimes F(\bar{G})$ . If  $\bar{G}$  is not a Frobenius group, then there is a non-trivial element  $\bar{y}$  in  $\bar{P}$  such that  $C_{F(\bar{G})}(\bar{y}) \neq \bar{1}$ . We can select a non-trivial element  $\bar{x} \in C_{F(\bar{G})}(\bar{y})$ , then  $[\bar{x}, \bar{y}] = \bar{1}$  and so  $[\bar{x}, \bar{P}] = \bar{1}$  as  $|\bar{P}| = p_1$ . On the other hand, by Statement (3) and Lemma 2.1 (c), we can see that  $\bar{P}$  is abnormal in  $\bar{G}$ . This contradiction shows that  $C_{F(\bar{G})}(\bar{y}) = \bar{1}$  and hence  $\bar{P}$  induces a fixed-point-free automorphism on  $F(\bar{G})$ . It follows from Lemma 2.2 that  $\bar{G}$  is a Frobenius group with Frobenius complement  $\bar{P}$ .

(6)  $G$  is supersolvable.

In view of Statement (2), we can see that  $G/M \cong P$ , where  $M$  is the normal  $p_1$ -complement of  $G$ . Hence we only need to show that every chief factor of  $G$  below  $M$  is cyclic in order to conclude that  $G$  is supersolvable. Now, let  $K/L$  be a chief factor of  $G$  such that  $K \leq M$ . Then  $K/L$  is an elementary

abelian  $q$ -group for some prime  $q \in \pi(M)$ . By [4, VI, 5.4 Satz], we can see that  $F(G) \leq C_G(K/L)$ . Then the automorphism group induced by  $G$  on  $K/L$  is of order  $p_1$ . We may assume without loss generality that  $q = p_2$ . By Statement (2),  $M$  is nilpotent and so  $M$  contains a subgroup  $B$  of order  $p_2^{m_2-1} p_3^{m_3} \cdots p_s^{m_s}$ . By hypothesis, there exists a subgroup  $H$  of order  $|G : B| = p_1^{m_1} p_2 = p_1^{m_1} q$  in  $G$  such that  $H$  is either subnormal or abnormal in  $G$ . Then  $H$  contains a Sylow  $p_1$ -subgroup of  $G$ , say  $P$ . Applying Statement (3),  $P$  is abnormal in  $G$  and hence  $H$  is abnormal in  $G$  by Lemma 2.1 (a) too. Let  $Q$  be a Sylow  $q$ -subgroup of  $H$ . Then  $Q$  is of order  $q$  and is contained in  $M$  by Statement (2). It follows that  $Q = H \cap M \trianglelefteq H$ . If  $C_H(Q) = H$ , then  $Q$  centralizes  $P$ , in contradiction to the fact that  $P$  is abnormal in  $G$ . Thus  $C_H(Q) < H$ . Since  $[O_{p_1}(G), Q] = 1$ , we obtain that  $H/C_H(Q)$  is cyclic of order  $p_1$ . It follows that  $H/C_H(Q)$  is isomorphic to a subgroup of  $\text{Aut}(Q)$ , a cyclic group of order dividing  $q-1$ . This means that  $p_1 | (q-1)$ . According to [7, Chapter 1, Lemma 1.3], we conclude that  $K/L$  is cyclic of order  $q$ , as desired.

(b) $\Rightarrow$ (a): If  $G$  is nilpotent, then  $G$  is a  $\mathcal{C}_1$ -group by Lemma 2.4. Hence we may assume that  $G$  is not nilpotent. Let  $B$  be a subgroup of  $G$  of index  $p_1^{mt}$ , where  $(p_1, t) = 1$  and  $m \geq 0$ . We proceed in two cases:

**Case 1:**  $m < m_1$ .

By Statement (b-2), we can see that  $p_1^{mt}$  is a divisor of the order of  $F(G)$ . In view of Statement (b-1) and Lemma 2.5, there exists a subgroup of order  $p_1^{mt}$  in  $F(G)$  such that it is normal in  $G$ , as desired.

**Case 2:**  $m = m_1$ .

It is clear that  $t$  is a divisor of the order of  $F(G)$ . By (b-1) and Lemma 2.5, there exists a subgroup  $H$  of order  $t$  in  $F(G)$  such that  $H$  is normal in  $G$ . Then the subgroup  $PH$  is of order  $p_1^{mt}$ . Set  $\bar{G} = G/O_{p_1}(G)$ , then  $\bar{G} = \bar{P} \rtimes F(\bar{G})$ . If  $\bar{P}$  is not self-normalizing, then there is a non-trivial element  $\bar{x}$  in  $N_{\bar{G}}(\bar{P})$  but not in  $\bar{P}$ . It follows from the theorem of Burnside [4, IV, 2.6 Hauptsatz and 2.7 Satz] that  $\langle \bar{x} \rangle \bar{P}$  is  $p$ -nilpotent and so  $[\bar{x}, \bar{P}] = \bar{1}$ . The Statement (b-3) provides a contradiction and thus  $N_{\bar{G}}(\bar{P}) = \bar{P}$ . We have that  $\bar{P}$  is pronormal in  $\bar{G}$  since  $\bar{P}$  is a Sylow subgroup of  $\bar{G}$ . By Lemma 2.1, we obtain that  $\bar{P}$  is abnormal in  $\bar{G}$ . According to Lemma 2.1 (c), we can see that  $P$  is abnormal in  $G$  and so is  $PH$  by Lemma 2.1 (a), as desired.

The Statement (a) is now completely proved by means of the previous cases.

This completes the proof of the theorem.  $\square$

**The proof of the Theorem 1.4:** This is transparent from Theorem 1.3 and Lemma 2.3.  $\square$

**The proof of the Theorem 1.5:** Let  $G = P \rtimes M$ . We assume that  $G$  is a  $\mathcal{C}_1$ -group. We can assume that  $G$  is not nilpotent and  $G$  satisfies Statements (b-1), (b-2) and (b-3) of Theorem 1.3. By (b-2), we have  $M \leq F(G)$  and so  $M$  is nilpotent.

Now, Statement (b-1) and Lemma 2.6 imply Statements (1) and (3). Statement (2) follows from Statement (b-3).

Conversely, by the theorem of Thompson [6, Theorem 10.5.4], we can see that  $M$  is nilpotent. Let  $K = \langle x \in P \mid a^x = a, a \in M \rangle$ . Then  $P/K$  is of order  $p$  by hypothesis and so  $K = O_p(G)$ . Thus  $F(G) = KM$  and  $G/F(G)$  is cyclic of order  $p$ . It follows that  $G/O_p(G)$  is a Frobenius group, whose Frobenius complement  $P/O_p(G)$  is cyclic of order  $p$ . Applying Lemma 2.6, we can see that  $G$  is supersolvable. Therefore  $G$  is a  $\mathcal{C}_1$ -group by Theorem 1.3.  $\square$

### Acknowledgements

The work is supported by the National Natural Science Foundation of China (No. 11671324 and 11471266), China Postdoctoral Science Foundation (No. 2016M592617), Chongqing Research Program of Basic Research and Frontier Technology (No. cstc2016jcyjA0786).

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Accepted: 14.12.2017