SOME GENERALIZATION OF SUBPULLBACK FLAT

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Abstract. Bulman-Fleming et al. in [*Pullbacks and flatness properties of acts II*, Comm. Algebra, 29(2) (2001), 851-878] initiated the study of three flatness properties of right S-act A_S that can be described by means of when the functor $A_S \otimes -$ preserves certain types of pullbacks. The present paper is extended these results to S-posets and is presented equivalences of them. Moreover, we show that flatness properties of S-posets that mentioned in this paper and in [A. Golchin, L. Nouri, *Subpullbacks and Po-flatness Properties of S-posets*, J. Sci. Islam. Repub. Iran, 25(4) (2014), 369-377], can be transferred to their coproducts, and vice versa.

Keywords: S-poset, subpullback diagram, surjectivity, (weak) po-surjectivity, order embedding.

1. Introduction

Let S be a partially ordered monoid, or briefly, pomonoid with identity element 1. A nonempty poset A is called a right S-poset, usually denoted A_S, if S acts on A from the right, that is there exists a mapping $A \times S \to A$, $(a, s) \mapsto as$, which satisfies the conditions: (i) the action is monotonic in each variable, (ii) $(as)t = a(st)$ and $a1 = a$, for all $a \in A$ and all $s, t \in S$. Left S-poset $_SB$ is defined analogously, and $\Theta_S = \{\theta\}$ is the one-element right S-poset. All left (resp., right) S-posets form a category, denoted **S-POS** (resp., **POS-S**), in which the

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morphisms are the functions that preserve both the action and the order (see [3]). The partially ordered sets form a category, denoted **POS**, in which the morphisms are the functions that preserve the order. In these categories the monomorphisms and epimorphisms are the injective and surjective morphisms, respectively, and a morphism $g : A \to B$ is called *order embedding* if $g(a) \leq g(a')$ implies $a \leq a'$, for all $a, a' \in A$. A surjective order embedding is called an *order isomorphism*.

A nonempty subset I of a pomonoid S, is called an *ordered right ideal* of S if (i) $IS \subseteq I$ and (ii) $a \leq b \in L$ implies $a \in I$, for all $a, b \in S$. An S-subposet B_S of a right S-poset A_S is called *strongly convex* if $a \leq b$ implies $a \in B_S$, for any $a \in A_S$ and $b \in B_S$. Clearly, if I is an ordered right ideal of a pomonoid S, then I is a strongly convex Ssubposet of the S-poset I.

In the 1980s, preliminary work on flatness properties of S-posets was done by Fakhruddin (see [5, 6]). Recently in this area some new articles appeared, such as [3, 1, 4, 13].

In [9] the notation $P(M, N, f, g, Q)$ was introduced to denote the pullback diagram of homomorphisms $f : {}_SM \to {}_SQ$ and $g : {}_SN \to {}_SQ$ in the category of left S-acts, where S is a monoid. Tensoring such a diagram by A_S produces a diagram (in the category of sets) that may or may not be a pullback diagram, depending on whether or not the mapping φ, obtained via the universal property of pullbacks in the category of sets, is bijective. It was shown that, if we require either bijectivity or surjectivity of φ for pullback diagrams of certain types, we not only recover most of the well-known forms of flatness, but also obtain Conditions (WP) and (PWP) as well. In [8] Golchin and Rezaei extended the results from [9] to S-posets and two new Conditions $(WP)_w$ and $(PWP)_w$ were introduced. In [7] Golchin and Nouri described po-flatness properties of S-posets over pomonoids by po-surjectivity of φ corresponding to certain subpullback diagrams. Then they introduced three new Conditions (P_{sw}), $(WP)_{sw}$ and $(PWP)_{sw}$ and describe these properties by po-surjectivity of φ corresponding to certain subpullback diagrams and described Conditions (P_w), $(WP)_w$ and $(PWP)_w$ by weak po-surjectivity of φ corresponding to certain subpullback diagrams.

For basic definitions and terminology relating to S-posets, we refer the reader to [4].

We recall from [3] that in the categories of S-posets and posets the order relation on morphism sets is defined pointwise (i.e. $f \leq g$ for $f, g : A \to B$ if and only if $f(a) \leq g(a)$ for every $a \in A$). In such categories, a diagram

$$\begin{tikzcd} P \arrow[r, "p_1"] \arrow[d, "p_2"'] & M \arrow[d, "f"] \\ N \arrow[r, "g"'] & Q \end{tikzcd}$$

(P_1)

is subcommutative if $fp_1 \leq gp_2$. If $fp_1 \leq gp_2$ and for every left S-poset P' and all homomorphisms $p_1' : P' \to M$ and $p_2' : P' \to N$ such that $fp_1' \leq gp_2'$, there exists a unique homomorphism $h : P' \to P$ such that $p_1 h = p_1'$ and $p_2 h = p_2'$, then diagram (P_1) will be called *subpullback diagram* for f and g. In the category of S-posets or the category of posets, P may in fact be realized as

$$P = \{(m, n) \in \ M \times \ N \mid f(m) \leq g(n)\}$$

where p_1 and p_2 are the first and second coordinate projections (Notice that P is possibly empty). The subpullback diagram (P_1) will be denoted by $P(M, N, f, g, Q)$. Tensoring the subpullback diagram $P(M, N, f, g, Q)$ by any right S-poset A one gets the subcommutative diagram

$$\begin{tikzcd} A \otimes P \arrow[r, "id_A \otimes p_1"] \arrow[d, "id_A \otimes p_2"'] & A \otimes M \arrow[d, "id_A \otimes f"] \\ A \otimes N \arrow[r, "id_A \otimes g"'] & A \otimes Q \end{tikzcd}$$

in the category of posets. For the subpullback of mappings $id_A \otimes f$ and $id_A \otimes g$ in the category of posets we may take

$$P' = \{(a \otimes m, a' \otimes n) \in (A \otimes M) \times (A \otimes N) \mid a \otimes f(m) \leq a' \otimes g(n)\}$$

with p_1' and p_2' the restrictions of the projections. It follows from the definition of subpullbacks that there exists a unique monotone mapping $\varphi : A \otimes_S P \to P'$

such that, in the diagram

$$\begin{tikzcd}
A \otimes P \arrow[dr, "\varphi"] \arrow[drr, "id_A \otimes p_1"] \arrow[ddr, "id_A \otimes p_2"'] & & \\
& P' \arrow[r, "p'_1"'] \arrow[d, "p'_2"] & A \otimes M \arrow[d, "id_A \otimes f"] \\
& A \otimes N \arrow[r, "id_A \otimes g"'] & A \otimes Q
\end{tikzcd}$$

(P_2)

$p'_i\varphi = id_A \otimes p_i$, for $i = 1, 2$. We shall call this mapping *the φ corresponding to the subpullback diagram* $P(M, N, f, g, Q)$. It can be seen that the mapping φ in diagram (P_2) is given by

$$\varphi(a \otimes (m, n)) = (a \otimes m, a \otimes n)$$

for all $a \in A_S$ and $(m, n) \in {}_SP$.

Notice that surjectivity of φ means

$$(\forall a, a' \in A_S)(\forall m \in {}_SM)(\forall n \in {}_SN)[a \otimes f(m) \leq a' \otimes g(n) \Rightarrow$$
$$(\exists a'' \in A_S)(\exists m' \in {}_SM)(n' \in {}_SN)$$
$$(f(m') \leq g(n') \wedge a \otimes m = a'' \otimes m' \wedge a' \otimes n = a'' \otimes n')],$$

po-surjectivity of φ means

$$(\forall a, a' \in A_S)(\forall m \in {}_SM)(\forall n \in {}_SN)[a \otimes f(m) \leq a' \otimes g(n) \Rightarrow$$
$$(\exists a'' \in A_S)(\exists m' \in {}_SM)(n' \in {}_SN)$$
$$(f(m') \leq g(n') \wedge a \otimes m = a'' \otimes m' \wedge a'' \otimes n' \leq a' \otimes n)],$$

and weak po-surjectivity of φ means

$$(\forall a, a' \in A_S)(\forall m \in {}_SM)(\forall n \in {}_SN)[a \otimes f(m) \leq a' \otimes g(n) \Rightarrow$$
$$(\exists a'' \in A_S)(\exists m' \in {}_SM)(n' \in {}_SN)$$
$$(f(m') \leq g(n') \wedge a \otimes m \leq a'' \otimes m' \wedge a'' \otimes n' \leq a' \otimes n)].$$

Obviously, surjectivity $\Rightarrow$ po-surjectivity $\Rightarrow$ weak po-surjectivity.

Also injectivity of φ means that

$$(\forall a, a' \in A_S)(\forall m, m' \in {}_S M)(\forall n, n' \in {}_S N)$$
$$[f(m) \leq g(n) \wedge f(m') \leq g(n') \wedge a \otimes m = a' \otimes m' \wedge a \otimes n = a' \otimes n' \Rightarrow$$
$$a \otimes (m, n) = a' \otimes (m', n')],$$

and φ is order embedding if

$$(\forall a, a' \in A_S)(\forall m, m' \in {}_S M)(\forall n, n' \in {}_S N)$$
$$[f(m) \leq g(n) \wedge f(m') \leq g(n') \wedge a \otimes m \leq a' \otimes m' \wedge a \otimes n \leq a' \otimes n' \Rightarrow$$
$$a \otimes (m, n) \leq a' \otimes (m', n')].$$

Similar to S-acts, coproduct of S-posets are disjoint unions, with S-action and order defined componentwise. If $A = \dot\cup_{i \in I} A_i$, where A_i, $i \in I$ is a strongly convex right S-poset, then by the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$, for A_i, $i \in I$, we mean the unique mapping φ_i which makes the diagram

$$\begin{tikzcd}
A_i \otimes P \arrow[dr, "\varphi_i"] \arrow[drr, "id_{A_i} \otimes p_1"] \arrow[ddr, "id_{A_i} \otimes p_2"'] & & \\
& P'_i \arrow[r, "p'_{1i}"'] \arrow[d, "p'_{2i}"] & A_i \otimes M \arrow[d, "id_{A_i} \otimes f"] \\
& A_i \otimes N \arrow[r, "id_{A_i} \otimes g"'] & A_i \otimes Q
\end{tikzcd}$$

(P_{2i})

commutative, where

$$P'_i = \{(a \otimes m, a' \otimes n) \in (A_i \otimes M) \times (A_i \otimes N) \mid a \otimes f(m) \leq a' \otimes g(n)\},$$

p'_{1i} and p'_{2i} are restrictions of projections to P'_i.

A po-surjective order embedding is called a *po-order isomorphism* and a weak po-surjective order embedding is called a *weak po-order isomorphism*. This paper continues the investigation of the class of right S-posets A_S over S, for which the functor $A_S \otimes -$ has certain subpullback preservation properties. The variations of the types of subpullbacks considered in [7, 10] and in this paper are of the following types.

Here the abbreviations stand for the following properties:
OI= order isomorphism,
POI=po-order isomorphism,
WPOI=weak po-order isomorphism,
WPS=weak po-surjective.

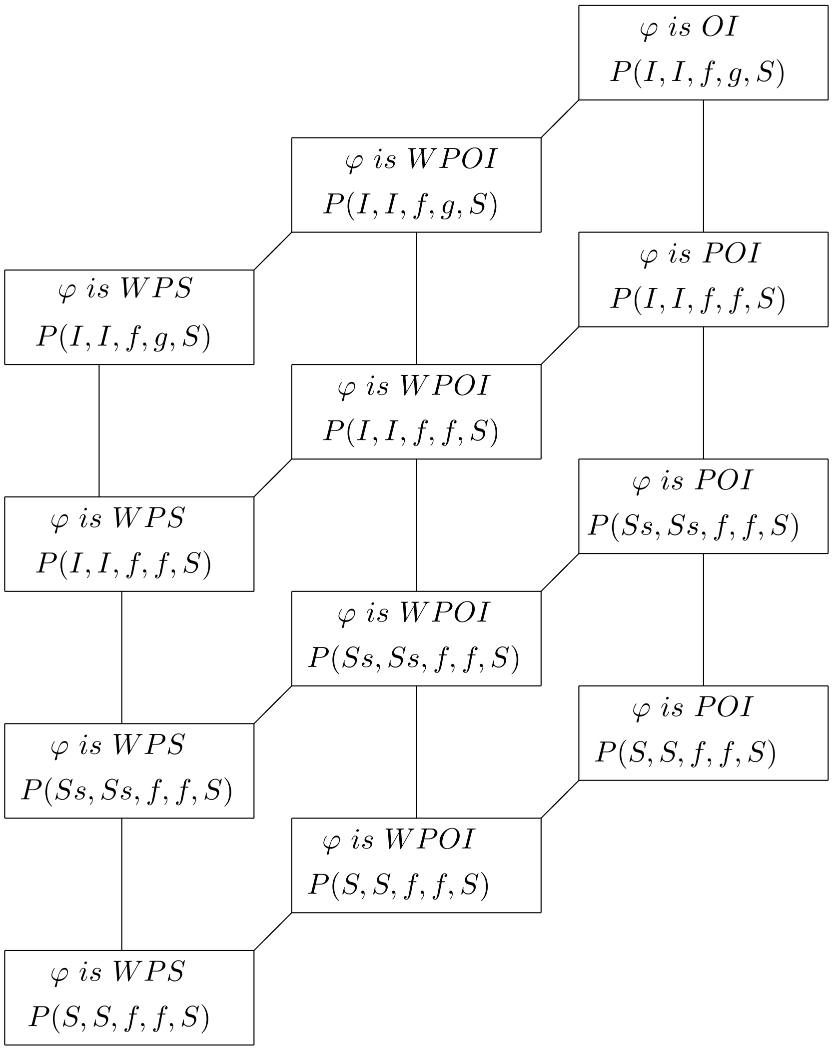

Figure 1.

2. Basic results

In this section, we discuss the classes of right S-poset A_S corresponding to the two and three columns of Figure 1. Also we give equivalences for them. Finally we will show that flatness properties of S-posets that mentioned in this paper and [7], can be transferred to their coproducts, and vice versa.

First we consider the following lemma that formulated in [13].

Lemma 2.1. *Let A_S be a right S-posets and ${}_SB$ be a left S-posets. $a \otimes b \leq a' \otimes b'$ in $A_S \otimes {}_SB$, for $a, a' \in A_S$ and $b, b' \in {}_SB$ if and only if there exist $a_1, a_2, \ldots, a_n \in A_S$, $b_2, \ldots, b_n \in {}_SB$, $s_1, t_1, \ldots, s_n, t_n \in S$, such that*

$$\begin{array}{cc} a \leq a_1 s_1 & \\ a_1 t_1 \leq a_2 s_2 & s_1 b \leq t_1 b_2 \\ a_2 t_2 \leq a_3 s_3 & s_2 b_2 \leq t_2 b_3 \\ \cdots & \cdots \\ a_n t_n \leq a' & s_n b_n \leq t_n b'. \end{array}$$

Definition 2.2. *A right S-poset A_S is called:*

(1) *weakly subpullback po-flat, if the corresponding φ is order isomorphism, for every subpullback diagram $P(I, I, f, g, S)$.*

(2) *weakly subpullback flat, if the corresponding φ is weak po-order isomorphism, for every subpullback diagram $P(I, I, f, g, S)$.*

(3) *$(WKPF)_{sw}$, if the corresponding φ is po-order isomorphism, for every subpullback diagram $P(I, I, f, f, S)$.*

(4) *$(PWKPF)_{sw}$, if the corresponding φ is po-order isomorphism, for every subpullback diagram $P(Ss, Ss, f, f, S)$.*

(5) *$(TKPF)_{sw}$, if the corresponding φ is po-order isomorphism, for every subpullback diagram $P(S, S, f, f, S)$.*

(6) *$(WKPF)_w$, if the corresponding φ is weak po-order isomorphism, for every subpullback diagram $P(I, I, f, f, S)$.*

(7) *$(PWKPF)_w$, if the corresponding φ is weak po-order isomorphism, for every subpullback diagram $P(Ss, Ss, f, f, S)$.*

(8) *$(TKPF)_w$, if the corresponding φ is weak po-order isomorphism, for every subpullback diagram $P(S, S, f, f, S)$.*

An S-poset A_S is called *subpullback flat* if the functor $A_S \otimes -$ takes subpullbacks in **S-POS** to subpullbacks in **POS**. Clearly A_S is subpullback flat if and only if the corresponding φ is order isomorphic, for every subpullback diagram $P(M, N, f, g, Q)$.

Recall from [3] that an S-poset A_S satisfies *Condition* (P), if for all $a, a' \in A_S$ and $s, s' \in S$, $as \leq a's'$ implies $a = a''u$, $a' = a''v$, for some $a'' \in A_S$ and $u, v \in S$, such that $us \leq vs'$. The S-poset A_S satisfies *Condition* (E), if for all $a \in A_S$ and $s, s' \in S$, $as \leq as'$ implies $a = a''u$, for some $a'' \in A_S$ and $u \in S$, such that $us \leq us'$.

Also recall from [12] that an S-poset A_S satisfies *Condition* (PF), if for all $a, a' \in A_S$ and $s, s', t, t' \in S$, $as \leq a's'$ and $at \leq at'$ imply $a = a''u$, $a' = a''v$, for some $a'' \in A_S$ and $u, v \in S$, such that $us \leq vs'$ and $ut \leq vt'$.

Now we give equivalences for subpullback flat.

Theorem 2.3. *For any S-poset A_S, the following statements are equivalent:*

(1) *A_S is subpullback flat;*

(2) *A_S satisfies Condition (PF)*

(3) *A_S satisfies Conditions (P) and (E);*

Proof. (2) $\Leftrightarrow$ (3). It is follows from [12, Proposition 2.3].

(1) $\Rightarrow$ (2). It is obvious that A_S satisfies Condition (P), by [8, Theorem 3.2]. Let $as \leq a's'$ and $at \leq a't'$, for $a, a' \in A_S$ and $s, s', t, t' \in S$. Now it is clear that

$$\begin{tikzcd} {}_SS\prod {}_SS \arrow[r, "p_1"] \arrow[d, "p_2"'] & {}_SS \arrow[d, "c_\theta"] \\ {}_SS \arrow[r, "c_\theta"'] & {}_S\Theta \end{tikzcd}$$

is the subpullback diagram in **S-Act**, where p_1 and p_2 are the projections. Since A_S is subpullback flat, the diagram

$$\begin{tikzcd} A_S \otimes ({}_SS\prod {}_SS) \arrow[r, "id_A \otimes p_1"] \arrow[d, "id_A \otimes p_2"'] & A_S \otimes {}_SS \arrow[d, "id_A \otimes c_\theta"] \\ A_S \otimes {}_SS \arrow[r, "id_A \otimes c_\theta"'] & A_S \otimes {}_S\Theta \end{tikzcd}$$

is a subpullback diagram in **Set**. Applying the canonical $A_S \otimes {}_S S \cong A_S$, (see [12, Lemma 3.2]) we have the diagram

$$\begin{tikzcd} P \arrow[r, "p'_1"] \arrow[d, "p'_2"'] & A_S \arrow[d, "id_A \otimes c_\theta"] \\ A_S \arrow[r, "id_A \otimes c_\theta"'] & A_S \otimes {}_S\Theta \end{tikzcd}$$

is the subpullback diagram in **S-Act**. By assumption the mapping $\varphi : A_S \otimes ({}_S S \prod {}_S S) \to P$, with $\varphi(\bar{a} \otimes (x, y) = (\bar{a}x, \bar{a}y)$, for any $\bar{a} \in A_S$ and $x, y \in S$ is order isomorphic. From $as \leq a's'$ and $at \leq a't'$ follow that $\varphi(a \otimes (s, t)) \leq \varphi(a' \otimes (s', t'))$. Since φ is order embedding, thus $a \otimes (s, t) \leq a' \otimes (s', t')$. Now since A_S satisfies Condition (P), thus by [8, Lemma 3.1], there exist $a'' \in A_S$ and $u, v \in S$, such that $a = a''u$, $a' = a''v$ and $u(s, t) \leq v(s', t')$. Therefore $us \leq vs'$ and $ut \leq vt'$, and so A_S satisfies Condition (PF), as required.

(1) $\Rightarrow$ (2). Since Condition (PF) implies Condition (P), thus the corresponding φ is surjective, for every subpullback diagram $P(M, N, f, g, Q)$, by [8, Theorem 3.2]. Now we show that φ is order embedding. Let

$$f(m) \leq g(n) \wedge f(m') \leq g(n') \wedge a \otimes m \leq a' \otimes m' \wedge a \otimes n \leq a' \otimes n',$$

for $a, a' \in A_S$, $m, m' \in {}_S M$ and $n, n' \in {}_S N$. Then Condition (P) implies that there exist $a_1, a_2 \in A_S$, $u_1, u_2, v_1, v_2 \in S$, such that $a = a_1u_1 = a_2u_2$, $a' = a_1v_1 = a_2v_2$, $u_1m \leq v_1m'$ and $u_2n \leq v_2n'$. Now applying Condition (BF) to the equalities $a_1u_1 = a_2u_2$ and $a_1v_1 = a_2v_2$ implies that there exist $a'' \in A_S$ and $u, v \in S$ such that $a_1 = a''u$, $a_2 = a''v$, $uu_1 \leq vu_2$ and $uv_1 \leq vv_2$. Thus

$$\begin{aligned}(m, n) &= a_1u_1 \otimes (m, n) = a''uu_1 \otimes (m, n) = a'' \otimes uu_1(m, n) \\ &= a'' \otimes (uu_1m, uu_1n) \leq a'' \otimes (uv_1m', uu_1n) \leq a'' \otimes (uv_1m', vu_2n) \\ &= a'' \otimes (vv_2m', vu_2n) \leq a'' \otimes (vv_2m', vv_2n') = a'' \otimes vv_2(m', n') \\ &= a''vv_2 \otimes (m', n') = a' \otimes (m', n'),\end{aligned}$$

and so φ is order embedding, as required. □

Recall from [11] that a pomonoid S is called *left collapsible*, if for any $s, s' \in S$, there exists $u \in S$, such that $us = us'$. A pomonoid S is called *weakly left collapsible*, if for any $s, s', z \in S$, $sz = s'z$ implies that there exists $u \in S$, such that $us = us'$.

Definition 2.4. *A pomonoid S is called strongly weakly left collapsible, if for any $s, s', z, k \in S$, $sz \leq k$ and $s'z \leq k$ implies that there exists $u \in S$, such that $us = us'$.*

It is obvious that every strongly weakly left collapsible pomonoid is weakly left collapsible, but in general, the converse is not true. See the following example.

Example 2.5. Let $S = \mathbb{N}$ be natural numbers, with natural order $\leq$. Obviously S is weakly left collapsible, but S is not strongly weakly left collapsible, since $2.5 < 20$, $3.5 < 20$ and for every $u \in S$, $u.2 \neq u.3$

Recall from [11] that an S-poset A_S satisfies *Condition* (E'), if for all $a \in A_S$, $s, s', z \in S$, $as \leq as'$ and $sz = s'z$ imply $a = a'u$, for some $a' \in A_S$ and $u \in S$ with $us \leq us'$. An S-poset A_S is called *weakly subpullback flat* if it satisfies Conditions (P) and (E'). Now we give the following definition.

Definition 2.6. *An S-poset A_S satisfies Condition $(E')_s$, if for all $a \in A_S$, $s, s', z, k \in S$, $as \leq as'$, $sz \leq k$ and $s'z \leq k$ imply $a = a'u$, for some $a' \in A_S$ and $u \in S$ with $us \leq us'$.*

The proof of the following theorem is similar to that of [1, Thorem 1(3)].

Theorem 2.7. *For any pomonoid S, Θ_S Satisfies Condition $(E')_s$, if and only if S is strongly weakly left collapsible.*

Obviously Condition $(E')_s$ implies Condition (E'), but Example 2.5 shows that the converse is not true.

Definition 2.8. *An S-poset A_S satisfies Condition (PF'), if for all $a, a' \in A_S$ and $s, s', t, t', z, w \in S$, $as \leq a's'$, $at \leq a't'$, $sz \leq tw$ and $s'z \leq t'w$ imply $a = a''u$, $a' = a''v$, for some $a'' \in A_S$ and $u, v \in S$, such that $us \leq vs'$ and $ut \leq vt'$.*

Now we give equivalences for weakly subpullback po-flat.

Theorem 2.9. *For any S-poset A_S, the following statements are equivalent:*

(1) *A_S is weakly subpullback po-flat;*

(2) *the corresponding φ is order isomorphic, for every subpullback diagram $P(Ss, Ss, f, g, S)$;*

(3) *the corresponding φ is order isomorphic, for every subpullback diagram $P(S, S, f, g, S)$;*

(4) *A_S satisfies Condition (PF');*

(5) *A_S satisfies Conditions (P) and $(E')_s$.*

Proof. Implications $(1) \Rightarrow (2) \Rightarrow (3)$ are obvious.

$(3) \Rightarrow (4)$. It is obvious that A_S satisfies Condition (P), by [8, Theorem 3.2]. Now let $as \leq a's'$, $at \leq a't'$, $sz \leq tw$ and $s'z \leq t'w$, for $a, a' \in A_S$ and

$s, s', t, t', z, w \in S$. Define $f, g : {}_SS \to {}_SS$, such that $f(u) = uz$ and $g(u) = uw$, for every $u \in S$. Then $f(s) \leq g(t)$, $f(s') \leq g(t')$, $a \otimes s \leq a' \otimes s'$ and $a \otimes t \leq a' \otimes t'$, and so $\varphi(a \otimes (s,t)) \leq \varphi(a' \otimes (s',t'))$. By assumption, since φ is order embedding, thus $a \otimes (s,t) \leq a' \otimes (s',t')$. Now by [8, Lemma 3.12], there exist $a'' \in A_S$ and $u, v \in S$, such that $a = a''u$, $a' = a''v$ and $u(s,t) \leq v(s',t')$. The last inequality implies that $us \leq vs'$ and $ut \leq vt'$, and so A_S satisfies Condition (PF'), as required.

(4) $\Rightarrow$ (5). Let $as \leq a's'$, for $a, a' \in A_S$ and $s, s' \in$. By putting $t = s$, $t' = s'$ and $z = w = 1$ in the definition of Condition (PF'), at once Condition (P) is followed. Now let $as \leq as'$, $sz \leq k$ and $s'z \leq k$, for $a, a' \in A_S$ and $s, s', k \in S$. By putting $t = t' = 1$ and $w = k$, Condition (PF') implies that there exist $a'' \in A_S$ and $u, v \in S$ such that $a = a''u = a''v$ and $us \leq vs'$, and so A_S satisfies Condition $(E')_s$.

(5) $\Rightarrow$ (1). Since A_S satisfies Condition (P), thus the corresponding φ is surjective, for every subpullback diagram $P(I, I, f, g, S)$. Now let $\varphi(a \otimes (s,t)) \leq \varphi(a' \otimes (s',t'))$, for $a, a' \in A_S$ and $s, s', t, t' \in I$. Then $f(s) \leq g(t)$, $f(s') \leq g(t')$, $a \otimes s \leq a' \otimes s'$ and $a \otimes t \leq a' \otimes t'$. Now $a \otimes s \leq a' \otimes s'$ and $a \otimes t \leq a' \otimes t'$ imply that $as \leq a's'$ and $at \leq a't'$, respectively. By utilization Condition (P) for inequality $at \leq a't'$, there exist $a_1 \in A_S$ and $u_1, v_1 \in S$ such that $a = a_1u_1$, $a' = a_1v_1$ and $u_1t \leq v_1t'$. Then $a_1(u_1s) \leq a_1(v_1s')$,

$$u_1sf(1) = u_1f(s) \leq u_1g(t) = utg(1) \leq v_1t'g(1)$$

and

$$v_1s'f(1) = v_1f(s') \leq v_1g(t') = v_1t'g(1).$$

Condition $(E')_s$ implies that there exist $a'' \in A_S$ and $v \in S$ such that $a_1 = a''v$ and $vu_1s \leq vv_1s'$. Now

$$\begin{aligned} a \otimes (s,t) &= a_1u_1 \otimes (s,t) = a''uu_1 \otimes (s,t) = a'' \otimes uu_1(s,t) \\ &= a'' \otimes (uu_1ms, uu_1t) \leq a'' \otimes (uv_1s', uv_1t') \leq a'' \otimes uv_1(s',t') \\ &= a''uv_1 \otimes (s',t') = a' \otimes (s',t'), \end{aligned}$$

and so φ is order embedding, as required. □

Definition 2.10. *An S-poset A_S satisfies Condition $(PF')_{sw}$, if for all $a, a' \in A_S$ and $s, s', t, t', z, w \in S$, $as \leq a's'$, $at \leq a't'$, $sz \leq tw$ and $s'z \leq t'w$ imply $a = a''u$, $a''v \leq a'$, for some $a'' \in A_S$ and $u, v \in S$, such that $us \leq vs'$ and $ut \leq vt'$.*

Definition 2.11. *An S-poset A_S satisfies Condition $(PF')_w$, if for all $a, a' \in A_S$ and $s, s', t, t', z, w \in S$, $as \leq a's'$, $at \leq a't'$, $sz \leq tw$ and $s'z \leq t'w$ imply $a =\leq a''u$, $a''v \leq a'$, for some $a'' \in A_S$ and $u, v \in S$, such that $us \leq vs'$ and $ut \leq vt'$.*

Recall from [12] that an S-poset A_S satisfies *Condition* (P_w), if for all $a, a' \in A_S$ and $s, s' \in S$, $as \leq a's'$ implies $a \leq a''u$, $a''v \leq a'$, for some $a'' \in A_S$ and $u, v \in S$, such that $us \leq vs'$. Now by a similar argument as in the proof of Theorem 2.9, we can show the following Theorem.

Theorem 2.12. *For any S-poset A_S, the following statements are equivalent:*

(1) *A_S is weakly subpullback flat;*

(2) *the corresponding φ is weak po-order isomorphism, for every subpullback diagram $P(Ss, Ss, f, g, S)$;*

(3) *the corresponding φ is weak po-order isomorphism, for every subpullback diagram $P(S, S, f, g, S)$;*

(4) *A_S satisfies Condition $(PF')_w$;*

(5) *A_S satisfies Conditions (P_w) and $(E')_s$.*

Now we show that flatness properties of S-posets that mentioned in this paper and [7], can be transferred to their coproducts.

Lemma 2.13 ([10])**.** *Let $A = \dot{\bigcup}_{i\in I} A_i$, where A_i, $i \in I$ is a right S-poset. Let ${}_SB$ be a left S-poset. If $a \otimes b \leq a' \otimes b'$ in $A_i \otimes B$, $i \in I$, then $a \otimes b \leq a' \otimes b'$ in $A \otimes B$.*

Lemma 2.14 ([10])**.** *Let $A = \dot{\bigcup}_{i\in I} A_i$, where A_i, $i \in I$ is a right strongly convex S-poset. Let ${}_SB$ be a left S-poset and suppose that $a \otimes b \leq a' \otimes b'$ in $A \otimes B$. Then $a \in A_i$, for some $i \in I$, if and only if $a' \in A_i$.*

Corollary 2.15 ([10])**.** *Let $A = \dot{\bigcup}_{i\in I} A_i$, where A_i, $i \in I$ is a right strongly convex S-poset. Let ${}_SB$ be a left S-poset. If $a \in A_i$, then $a \otimes b \leq a' \otimes b'$ in $A \otimes B$, if and only if $a \otimes b \leq a' \otimes b'$ in $A_i \otimes B$.*

Lemma 2.16 ([10])**.** *Let $A = \dot{\bigcup}_{i\in I} A_i$, where A_i, $i \in I$ is a right strongly convex S-poset. Let ${}_SB$ be a left S-poset and $a \otimes b \in A \otimes B$, for $a \in A$ and $b \in B$. If $a \in A_i$, then $a \otimes b \in A_i \otimes B$.*

Corollary 2.17 ([10])**.** *Let $A = \dot{\bigcup}_{i\in I} A_i$, where A_i, $i \in I$ is a right strongly convex S-poset. Let $\varphi : A \otimes P \to P'$ be the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$, for A. If $\varphi_i = \varphi|_{A_i \otimes P}$, then $\varphi_i : A_i \otimes P \to P'_i$.*

Lemma 2.18 ([10])**.** *Let $A = \dot{\bigcup}_{i\in I} A_i$, where A_i, $i \in I$ is a right strongly convex S-poset. Let $\varphi : A \otimes P \to P'$ be the mapping and suppose that $\varphi_i = \varphi|_{A_i \otimes P}$. Then φ is the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$, for A if and only if φ_i is the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$, for A_i, $i \in I$.*

Theorem 2.19 ([10]). *Let φ be the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$ for A_S, and let φ_i, $i \in I$, be as in Lemma 2.18. Then φ is surjective if and only if φ_i is surjective, for every $i \in I$.*

By a similar argument as in the proof of above theorem, we can show the following Theorems.

Theorem 2.20. *Let φ be the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$ for A_S, and let φ_i, $i \in I$, be as in Lemma 2.18. Then φ is po-surjective if and only if φ_i is po-surjective, for every $i \in I$.*

Theorem 2.21. *Let φ be the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$ for A_S, and let φ_i, $i \in I$, be as in Lemma 2.18. Then φ is weak po-surjective if and only if φ_i is weak po-surjective, for every $i \in I$.*

Theorem 2.22 ([10]). *Let φ be the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$ for A_S, and let φ_i, $i \in I$, be as in Lemma 2.18. Then φ is order embedding if and only if φ_i is order embedding, for every $i \in I$.*

In [7], some flatness properties of S-posets, in the following theorem, investigated by property of φ, for subpullback diagrams of certain types.

Theorem 2.23. *Let S be a pomonoid and $A = \dot{\bigcup}_{i \in I} A_i$, where A_i, $i \in I$ is a right strongly convex S-poset. Then A is weakly subpullback po-flat, weakly subpullback flat, $(WKPF)_{sw}$, $(PWKPF)_{sw}$, $(TKPF)_{sw}$, $(WKPF)_w$, $(PWKPF)_w$, $(TKPF)_w$, po-flat, weakly po-flat, principally weakly po-flat, po-torsion free, and satisfies Conditions (P_{sw}), (P_w), $(WP)_{sw}$, $(WP)_w$, $(PWP)_{sw}$, $(PWP)_w$ if and only if A_i has these properties, foe every $i \in I$.*

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