ORTHOGONAL-BASED HYBRID BLOCK METHOD FOR SOLVING GENERAL SECOND ORDER INITIAL VALUE PROBLEMS

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Abstract. The direct integration of general second order initial value problems is considered in this paper. We employ a new class of orthogonal polynomials constructed as basis function to develop One Step Hybrid Block Method (OSHBM) adopting collocation technique. We present the recursive formula of the class of polynomials constructed and give analysis of the basic properties of OSHBM as findings show that the method is accurate and convergent.

Keywords: orthogonal polynomials, algorithm, block method, collocation, interpolation, zero-stable.
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1. Introduction

In many area of physical problems such as in science, engineering and management, second order differential equations of the form

$$(1) \qquad y''(x) = f(x, y(x), y'(x)) \quad y(x_0) = y_0, \;\; y'(x_0) = y_0^{'}, p \leq x \leq q$$

arise frequently. The difficulties encounter in solving such problems has led to development of numerical methods. The solution of (1) has been discussed by various researchers among them are Lambert (1973,1991), Norsett (1989), Sirisena (1999), Gear (1971). However, experience has shown in Lie and Norsett (1989), Fatunla (1988, 1991, 1994), Hairer and Wanner (1993), Lambert (1973, 1991), Brugnano and Trigiante (1998), Onumanyi et al. (1999, 2008) and Jator (2007) that to derive these methods, polynomials play a vital role. Notable among the well-known polynomials are the orthogonal polynomials. Orthogonal polynomial sequence is a family of polynomials such that any two different polynomials in the sequence are orthogonal to each other under some inner product. The first orthogonal polynomials were the Legendre polynomials. Then came the Chebyshev polynomials, the general Jacobi polynomials, the Hermite and the Laguerre polynomials. All these classical orthogonal polynomials play an important role in many applied problems.

Asymptotic formulae for orthogonal polynomials were first discovered by G. Szego, Szego (1975). Lanczos, C. (1938) introduced Chebyshev polynomials as trial function. Several researchers have employed these polynomials as trial functions to formulate algorithms (see Shampine and Watts (1969), Tanner (1979), Dahlquist (1979), Jator (2007), Awoyemi (1991)).
In this work, we shall employ a non-negative weight function to construct a new class of orthogonal polynomials which will serve as trial functions to formulate numerical algorithms for the solution of initial value problems.

2. Construction of orthogonal basis function

We define the orthogonal polynomial of the first kind of degree n over the interval [-1, 1] with respect to weight function $w(x) = (x^2 - 1)^2$ as

$$q_r(x) = \sum_{r=0}^{n} C_r^{(n)} x^r \tag{2a}$$

The following requirements are considered:

$$< q_m(x),\ q_n(x) >= 0,\ \ m = 0, 1, 2, ..., n-1 \tag{2b}$$

For the purpose of constructing the basis function, we adopt the approach discussed extensively in Adeyefa and Adeniyi (2015) and use additional property (the normalization)

$$q_n(1) = 1 \tag{2c}$$

Using (2b) and (2c), equation (2a) yields

$$\begin{cases} q_0(x) = 1 \\ q_1(x) = x \\ q_2(x) = \frac{1}{6}(7x^2 - 1) \\ q_3(x) = \frac{1}{2}(3x^3 - x) \\ q_4(x) = \frac{1}{16}(33x^4 - 18x^2 + 1) \\ q_5(x) = \frac{1}{48}(143x^5 - 110x^3 + 15x) \\ q_6(x) = \frac{1}{32}(143x^6 - 143x^4 + 33x^2 - 1) \\ q_7(x) = \frac{1}{32}(221x^7 - 273x^5 + 91x^3 - 7x) \\ q_8(x) = \frac{1}{384}(4199x^8 - 6188x^6 + 2730x^4 - 364x^2 + 7) \\ q_9(x) = \frac{1}{128}(2261x^9 - 3876x^7 + 2142x^5 - 420x^3 + 21x) \\ q_{10}(x) = \frac{1}{256}(7429x^{10} - 14535x^8 + 9690x^6 - 2550x^4 + 225x^2 - 3) \end{cases} \tag{3}$$

In the spirit of Golub and Fischer (1992), equation (3) must satisfy three-term recurrence relation

$$c_j p(t) = (t - a_j)p_{j-1}(t) - b_j p_{j-2}(t), j = 1, 2, ..., p_{-1}(t) = 0,\ p_0(t) \equiv p_0$$

where
b_j, $c_j > 0$ for $j \geq 1$ (b_1 is arbitrary).
$c_j p(t) = (n+5)P_{n+1}(x),\ (t-a_j)p_{j-1}(t) = (2n+5)xP_n(x),\ b_j p_{j-2}(t) = nP_{n-1}(x),$
$n = 1, 2, ...$
The recursive formula for these orthogonal polynomials is therefore given as

$$P_{n+1}(x) = \frac{1}{n+5}\left[(2n+5)xP_n(x) - nP_{n-1}(x)\right],\ n \geq 1,\ P_0(x) = 1,\ P_1(x) = x$$

This relation, along with the two polynomials P_0(x) and P_1(x), allows the new set of polynomials to be generated recursively.
In what immediately follows, we shall develop an algorithm to integrate second order differential equations where polynomials q_n(x) shall be employed as basis function. Thereafter, the analysis of the method for convergence and implementation of the method through some test problems shall be presented. Finally, conclusion shall be made.

3. Materials and methods

3.1. Development of the method

In this section, our aim is to derive a continuous hybrid scheme which shall serve as direct integrator to second order initial value problems (IVPs) of the form (1). To make this happen, we shall seek an approximant

$$y(x) = \sum_{r=0}^{s+k-1} a_r q_r(x) \tag{4}$$

to obtain the solution of second order initial value problems in ordinary differential equations. Transforming q_n(x) to interval [0, 1], we have $x = \frac{2X-2x_n-ph}{ph}$, where p varies as the method to be developed. In this case, p = 1, s and k in (4) are points of interpolation and collocation respectively. The procedure involves interpolating (4) at points s = $\frac{1}{4}, \frac{1}{2}$ and collocating the second derivative of (4) at points k = 0, $\frac{1}{4}, \frac{1}{3}, \frac{1}{2}$, 1. The a_r, r = 0(1)6 from the resulting system of equations are obtained as

$$\begin{cases} a_0 = y_{n+\frac{1}{2}} - \frac{h^2}{1386}f_n + \frac{h^2}{77}f_{n+\frac{1}{4}} - \frac{81h^2}{4928}f_{n+\frac{1}{3}} + \frac{241h^2}{11088}f_{n+\frac{1}{2}} + \frac{13h^2}{44352}f_{n+1} \\ a_1 = -2y_{n+\frac{1}{4}} + 2y_{n+\frac{1}{2}} - \frac{1703h^2}{506880}f_n + \frac{119h^2}{2376}f_{n+\frac{1}{4}} - \frac{8037h^2}{112640}f_{n+\frac{1}{3}} \\ \qquad\qquad + \frac{10867h^2}{126720}f_{n+\frac{1}{2}}\frac{379h^2}{276480}f_{n+1} \\ a_2 = -\frac{51h^2}{8008}f_n + \frac{120h^2}{1001}f_{n+\frac{1}{4}} - \frac{1215h^2}{8008}f_{n+\frac{1}{3}} + \frac{285h^2}{2002}f_{n+\frac{1}{2}} + \frac{3h^2}{1001}f_{n+1} \\ a_3 = -\frac{h^2}{117}f_n + \frac{40h^2}{351}f_{n+\frac{1}{4}} - \frac{45h^2}{208}f_{n+\frac{1}{3}} + \frac{25h^2}{234}f_{n+\frac{1}{2}} + \frac{23h^2}{5616}f_{n+1} \\ a_4 = -\frac{2h^2}{495}f_n + \frac{16h^2}{165}f_{n+\frac{1}{4}} - \frac{27h^2}{220}f_{n+\frac{1}{3}} + \frac{13h^2}{495}f_{n+\frac{1}{2}} + \frac{7h^2}{1980}f_{n+1} \\ a_5 = -\frac{3h^2}{2860}f_n - \frac{16h^2}{715}f_{n+\frac{1}{4}} + \frac{243h^2}{5720}f_{n+\frac{1}{3}} - \frac{3h^2}{143}f_{n+\frac{1}{2}} + \frac{1h^2}{520}f_{n+1} \\ a_6 = \frac{2h^2}{715}f_n - \frac{64h^2}{2145}f_{n+\frac{1}{4}} + \frac{27h^2}{715}f_{n+\frac{1}{3}} - \frac{8h^2}{715}f_{n+\frac{1}{2}} + \frac{h^2}{2145}f_{n+1} \end{cases} \tag{5}$$

Substituting (5) into (4) yields the continuous implicit method

$$sy(x) = \alpha_{\frac{1}{4}}(x)y_{n+\frac{1}{4}} + \alpha_{\frac{1}{2}}(x)y_{n+\frac{1}{2}} + h^2(\beta_k(x)f_{n+k}),\ \ k = 0, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{2}, 1 \tag{6}$$

Evaluating equation (6) at x = x_{n+m}, m=0, $\frac{1}{3}$, 1 yields the discrete equations

$$\begin{aligned}
&\text{(7a)}\quad y_{n+1} = -2y_{n+\frac{1}{4}} + 3y_{n+\frac{1}{2}} - \tfrac{109h^2}{5120}f_n + \tfrac{41h^2}{120}f_{n+\frac{1}{4}} - \tfrac{5103h^2}{10240}f_{n+\frac{1}{3}} + \tfrac{449h^2}{1280}f_{n+\frac{1}{2}} \\
&\qquad\qquad + \tfrac{451h^2}{30720}f_{n+1} \\
&\text{(7b)}\quad y_{n+\frac{1}{3}} = \tfrac{2}{3}y_{n+\frac{1}{4}} + \tfrac{1}{3}y_{n+\frac{1}{2}} - \tfrac{137h^2}{3732480}f_n + \tfrac{53h^2}{87480}f_{n+\frac{1}{4}} - \tfrac{1817h^2}{276480}f_{n+\frac{1}{3}} - \tfrac{883h^2}{933120}f_{n+\frac{1}{2}} \\
&\qquad\qquad + \tfrac{103h^2}{22394880}f_{n+1} \\
&\text{(7c)}\quad y_n = 2y_{n+\frac{1}{4}} - y_{n+\frac{1}{2}} + \tfrac{71h^2}{15360}f_n + \tfrac{7h^2}{120}f_{n+\frac{1}{4}} - \tfrac{81h^2}{10240}f_{n+\frac{1}{3}} + \tfrac{29h^2}{3840}f_{n+\frac{1}{2}} - \tfrac{h^2}{10240}f_{n+1}
\end{aligned}$$

To develop the block method from the continuous scheme, we adopt the general block formula proposed in Shampine and Watts (1969) in the normalized form given as

$$A^{(0)}Y_m = ey_m + h^{\mu-\lambda}df(y_m) + h^{\mu-\lambda}bF(y_m) \tag{8}$$

Evaluating the first derivative of (6) at x = x_{n+j}, j = 0, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, 1 to obtain the first derivative equations (FDE). Substituting the resulting equations FDE and equation (7) into (8) and solving simultaneously gives a block formulae represented as

$$\text{(9)}\quad \begin{cases}
y_{n+\frac{1}{4}} = \frac{147}{10240}h^2f_n + \frac{7}{144}h^2f_{n+\frac{1}{4}} - \frac{783}{20480}h^2f_{n+\frac{1}{3}} + \frac{17}{2560}h^2f_{n+\frac{1}{2}} \\
\qquad - \frac{23}{184320}h^2f_{n+1} + \frac{1}{4}hy' + y_n \\
y_{n+\frac{1}{3}} = \frac{301}{14580}h^2f_n + \frac{928}{10935}h^2f_{n+\frac{1}{4}} - \frac{13}{216}h^2f_{n+\frac{1}{3}} + \frac{38}{3645}h^2f_{n+\frac{1}{2}} \\
\qquad - \frac{17}{14580}h^2f_{n+1} + \frac{1}{3}hy' + y_n \\
y_{n+\frac{1}{2}} = \frac{1}{30}h^2f_n + \frac{7}{45}h^2f_{n+\frac{1}{4}} - \frac{27}{320}h^2f_{n+\frac{1}{3}} + \frac{1}{48}h^2f_{n+\frac{1}{2}} + \frac{1}{2880}h^2f_{n+1} \\
\qquad + \frac{1}{2}hy' + y_n \\
y_{n+1} = \frac{1}{20}h^2f_n + \frac{32}{45}h^2f_{n+\frac{1}{4}} - \frac{27}{40}h^2f_{n+\frac{1}{3}} + \frac{2}{5}h^2f_{n+\frac{1}{2}} + \frac{1}{72}h^2f_{n+1} + hy' + y_n \\
y'_{n+\frac{1}{4}} = \frac{581}{7680}hf_n + \frac{49}{120}hf_{n+\frac{1}{4}} - \frac{1431}{5120}hf_{n+\frac{1}{3}} + \frac{89}{1920}hf_{n+\frac{1}{2}} - \frac{13}{15360}hf_{n+1} + y'_n \\
y'_{n+\frac{1}{3}} = \frac{61}{810}hf_n + \frac{544}{1215}hf_{n+\frac{1}{4}} - \frac{7}{30}hf_{n+\frac{1}{3}} + \frac{2}{45}hf_{n+\frac{1}{2}} - \frac{1}{1215}hf_{n+1} + y'_n \\
y'_{n+\frac{1}{2}} = \frac{37}{480}hf_n + \frac{2}{5}hf_{n+\frac{1}{4}} - \frac{27}{320}hf_{n+\frac{1}{3}} + \frac{13}{120}hf_{n+\frac{1}{2}} - \frac{1}{960}hf_{n+1} + y'_n \\
y'_{n+1} = -\frac{1}{30}hf_n + \frac{32}{15}hf_{n+\frac{1}{4}} - \frac{27}{10}hf_{n+\frac{1}{3}} + \frac{22}{15}hf_{n+\frac{1}{2}} + \frac{2}{15}hf_{n+1} + y'_n
\end{cases}$$

Equation (9) is our desired block method of which its basic properties shall be discussed in the next section.

3.2. Implementation of the method

Equation (7) and FDE were solved simultaneously to obtain y_{n+j} and y'_{n+j} using the block (8) through maple code. Block equation (9) is applied directly to (1) without requiring any predictor to obtain numerical values for y_{n+j}, y'_{n+j} and f_{n+j}. The necessary starting value is obtained from the last values y_{n+1} and y'_{n+1} of the previous block whose loss of accuracy do not affect subsequent points, thus the order of the algorithm is preserved.

4. Analysis of the method

4.1. Order and error constant

Following Henrici (1962), the approach adopted in Fatunla (1991, 1994) and Lambert (1973), we define the local truncation error associated with equation (9) by the difference operator

$$L[y(x):h]=\sum_{j=0}^{k}\left[\alpha_j y(x_n+jh)-h^2\beta_j f(x_n+jh)\right] \tag{10}$$

where $y(x)$ is an arbitrary function, continuously differentiable on [a, b].

Expanding (10) in Taylor series about the point x, we obtain the expression

$$L[y(x);h]=C_0y(x)+C_1hy'(x)+C_2h^2y''(x)+\ldots+C_{p+3}h^{p+3}y^{p+3}(x)$$

where the $C_0, C_1, C_2, C_p, \ldots$ are obtained as

$$C_0=\sum_{j=0}^{k}\alpha_j,\quad C_1=\sum_{j=1}^{k}j\alpha_j,\quad C_2=\frac{1}{2!}\sum_{j=1}^{k}j^2\alpha_j,$$
$$C_q=\frac{1}{q!}\left[\sum_{j=1}^{k}j^q\alpha_j-q(q-1)(q-2)\sum_{j=1}^{k}\beta_j j^{q-3}\right]$$

According to Lambert (1973), equations (7) and (9) are of order p if

$$C_0=C_1=C_2=\ldots C_p=C_{p+1}=0 \text{ and } C_{p+2}\neq 0$$

The $C_{p+2}\neq 0$ is called the error constant and $C_{p+2}h^{p+2}y^{p+2}(x_n)$ is the principal local truncation error at the pointx_n.

Thus, equations (7) and (9) are all of order 5 with the error constants

$$C_{p+2}=\begin{bmatrix}\frac{-1}{196608} & \frac{329}{6449725440} & \frac{1}{5898240}\end{bmatrix}^T$$

and

$$C_{p+2}=\begin{bmatrix}-\frac{1}{241920} & \frac{1}{1548288} & \frac{13}{35271936} & \frac{59}{247726080} & \frac{7}{4423680} & \frac{13}{8398080} & \frac{1}{552960} & -\frac{1}{34560}\end{bmatrix}^T$$

respectively.

4.2. Zero stability of the method

According to Lambert (1973), a linear multistep method is said to be zero-stable if no root of the first characteristic polynomial $\rho(R)$ has modulus greater than one and if every root of modulus one has multiplicity not greater than the order of the differential equation.

To analyze the zero-stability of the method, we present (9) in vector notation form of column vectors $e = (e_1 \ldots e_r)^T$, $d = (d_1 \ldots d_r)^T$, $y_m = (y_{n+1} \ldots y_{n+r})^T$, $F(y_m) = (f_{n+1} \ldots f_{n+r})^T$ and matrices $A = (a_{ij})$, $B = (b_{ij})$.

Thus, equation (9) forms the block formula

$$A^0 y_m = hBF(y_m) + A^1 y_n + hdf_n \tag{11a}$$

where h is a fixed mesh size within a block.

The first characteristic polynomial of the hybrid block method (9a) is given by

$$\rho(R) = \det(RA^0 - A^1) \tag{11b}$$

where

$$A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{3}} \\ y_{n+\frac{1}{2}} \\ y_{n+1} \\ y'_{n+\frac{1}{4}} \\ y'_{n+\frac{1}{3}} \\ y'_{n+\frac{1}{2}} \\ y'_{n+1} \end{pmatrix},$$

$$A^1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n-\frac{1}{4}} \\ y_{n-\frac{1}{3}} \\ y_{n-\frac{1}{2}} \\ y_{n} \\ y'_{n-\frac{1}{4}} \\ y'_{n-\frac{1}{3}} \\ y'_{n-\frac{1}{2}} \\ y'_{n} \end{pmatrix},$$

$$B = \begin{pmatrix} \frac{7}{144} & \frac{-783}{20480} & \frac{17}{2560} & \frac{-23}{184320} & 0 & 0 & 0 & 0 \\ \frac{928}{10935} & \frac{-13}{216} & \frac{38}{3645} & \frac{-17}{14580} & 0 & 0 & 0 & 0 \\ \frac{7}{45} & \frac{-27}{320} & \frac{1}{48} & \frac{1}{2880} & 0 & 0 & 0 & 0 \\ \frac{32}{45} & \frac{-27}{40} & \frac{2}{5} & \frac{1}{72} & 0 & 0 & 0 & 0 \\ \frac{49}{120} & \frac{-1431}{5120} & \frac{89}{1920} & \frac{-13}{15360} & 0 & 0 & 0 & 0 \\ \frac{544}{1215} & \frac{-7}{30} & \frac{2}{45} & \frac{-1}{1215} & 0 & 0 & 0 & 0 \\ \frac{2}{5} & \frac{-27}{320} & \frac{13}{120} & \frac{-1}{960} & 0 & 0 & 0 & 0 \\ \frac{32}{15} & \frac{-27}{10} & \frac{22}{15} & \frac{2}{15} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n-\frac{1}{4}} \\ f_{n-\frac{1}{3}} \\ f_{n-\frac{1}{2}} \\ f_{n-1} \end{pmatrix},$$

$$d = \begin{pmatrix} 0 & 0 & 0 & \frac{147}{10240} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{301}{14580} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{30} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{20} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{581}{7680} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{61}{810} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{37}{480} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{30} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_{n-\frac{1}{4}} \\ f_{n-\frac{1}{3}} \\ f_{n-\frac{1}{2}} \\ f_n \\ f_{n-1} \\ f_{n-2} \\ f_{n-3} \\ f_{n-4} \end{pmatrix}$$

Substituting A^0 and A^1 in (11b), we obtain $\rho(R) = R^6(R-1)^2$ which implies that $R_1 = ... = R_6 = 0$, $R_7 = R_8 = 1$.

According to Fatunla (1988 , 1991), the our block method equation are zero-stable since from $\rho(R) = 0$ satisfies $|R_j| \leq 1$, $j = 1$ and for those roots with $|R_j| = 1$, the multiplicity does not exceed two.

4.3. Region of absolute stability of the main methods

For the region of absolute stability, the following definitions are considered.

Given the stability polynomial

$$\pi\left(z, \overline{h}\right) = \rho(z) - \overline{h}\sigma(z) = 0 \tag{12}$$

where $\overline{h} = h^2\lambda^2$ and $\lambda = \frac{df}{dy}$ are assumed constants.

The scheme (7) is said to be absolutely stable if for a given $\overline{h}$ all the roots z_s of (12) satisfy $|z_s| < 1$, $s = 1, 2, ...n$, where $\overline{h} = \lambda h$.

Definition 1.1. The region $\Re$ of the complex $\overline{h}$-plane such that the roots of $\pi\left(z, \overline{h}\right) = 0$ lies within the unit circle whenever $\overline{h}$ lies in the interior of the region is called the region of absolute stability.

Remark. Let $\Re$ be the boundary of the region$\Re$. Since the roots of the stability polynomial are continuous functions of $\overline{h}$, $\overline{h}$ will lie on $\Re$ when one of the roots of the $\pi\left(z, \overline{h}\right) = 0$ lies on the boundary of the unit circle. Thus we define (12) in terms of Euler's number, $\exp i\theta$, as follows;

$$\pi(\exp(i\theta), h) = \rho(\exp(i\theta) - \overline{h}\sigma(\exp(i\theta)) = 0 \tag{13}$$

So that, the locus of the boundary $\Re$ is given by

$$\overline{h}(\theta) = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})} \tag{14}$$

From (7a), the boundary of the region of absolute stability is

$$\overline{h}(\theta) = \frac{2\cos\frac{1}{4}\theta - 3\cos\frac{1}{2}\theta - 3i\sin\frac{1}{2}\theta + 2i\sin\frac{1}{4}\theta + \cos\theta + i\sin\theta}{\left(\begin{array}{l} \frac{41}{120}i\sin\frac{1}{4}\theta - \frac{5103}{10240}i\sin\frac{1}{3}\theta + \frac{449}{1280}i\sin\frac{1}{2}\theta + \frac{451}{30720}i\sin\theta - \frac{109}{5120} \\ +\frac{41}{120}\cos\frac{1}{4}\theta - \frac{5103}{10240}\cos\frac{1}{3}\theta + \frac{449}{1280}\cos\frac{1}{2}\theta + \frac{451}{30720}\cos\theta \end{array}\right)}$$

Definition 1.2. According to Widlund (1967), a numerical method is said to be A(α)-stable, $\alpha \in (0, \frac{\pi}{2})$, if its region of absolute stability contains the infinite wedge $W_\alpha = \{h\lambda \,|\, - \ \alpha < \pi - \arg h\lambda \ < \ \alpha\}$.

The A(α)-stability property is shown in Figure 1.

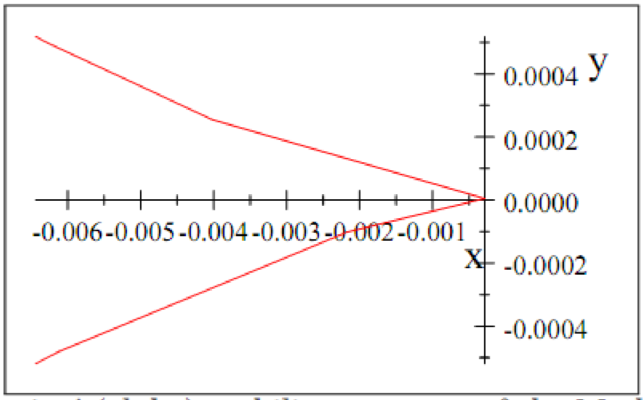

Fig. 1: A(alpha)-stability property of the Method

4.4. Consistency of the method

According to Lambert (1973), a linear multistep method is said to be consistent if it has order at least one. Owing to this definition, equations (7) and (9) are consistent.

4.5. Convergency of the Method

According to the theorem of Dahlquist, the necessary and sufficient condition for a LMM to be convergent is to be consistent and zero stable. Since the method satisfies the two conditions hence its convergence.

4.6. Numerical experiment

Problem 1. We consider here the highly stiff initial value problem

$$y'' = -1001y' - 1000y, y(0) = 1, \ y'(0) = -1, \ h = 0.05$$

Exact Solution: $y(x) = \exp(-x)$.

Problem 2. Here we implement OSHBM using linear initial value problem

$$y" = -\frac{6}{x}y' - \frac{4}{x^2}y, y(1) = 1, \ y'(1) = 1, \ h = \frac{0.1}{32}$$

Exact Solution: $y(x) = \frac{5}{3x} - \frac{2}{3x^4}$.

Problem 3. The Vanderpol's Oscillator Problem

$$y'' = 2\cos x - \cos^3 x - y' - y - y^2 y', y(0) = 0, \ y'(0) = 1, \ h = 0.1$$

whose exact solution is y(x) = sinx is considered as our third test problem.

Problem 4. We consider the second order system equations

$$y_1'' = -4t^2 y_1 - \frac{2y_2}{\sqrt{y_1^2+y_2^2}}, y_1\left(\sqrt{\tfrac{\pi}{2}}\right) = 0, y_1'\left(\sqrt{\tfrac{\pi}{2}}\right) = -2\sqrt{\tfrac{\pi}{2}}$$
$$y_2'' = -4t^2 y_2 + \frac{2y_1}{\sqrt{y_1^2+y_2^2}}, y_2\left(\sqrt{\tfrac{\pi}{2}}\right) = 1, y_2'\left(\sqrt{\tfrac{\pi}{2}}\right) = 0, \quad \sqrt{\tfrac{\pi}{2}} \le t \le 10$$

with exact solution given by $y_1(t) = \cos(t^2), y_2(t) = \sin(t^2)$, see Sommeijer (1993).

Table 1. Numerical Results of Problem 1

X	Error in OSHBM h=0.05, p=5	Error in OSHBM h=0.1, p=5	Error in [2], h=0.1, p=4
0.1	3.23959e-15	3.9996926e-13	2.05e-11
0.2	1.794055e-14	2.75748143e-12	4.39e-11
0.3	6.909555e-14	1.37032147e-11	6.55e-11
0.4	2.3715412e-13	6.272985728e-11	8.38e-11
0.5	7.8100585e-13	2.8095960466e-10	9.86e-11
0.6	2.5337002e-12	1.25130028034e-09	1.10e-10
0.7	8.17574399e-12	5.5650421949e-09	1.19e-10
0.8	2.6332157e-11	2.474161011314e-08	1.24e-10
0.9	8.475545968e-11	1.0998987818295e-07	1.28e-10
1.0	2.7274422458e-10	4.8895562799261e-07	1.30e-10

Table 2. Numerical Results of Problem 2

X	Error in OSHBM h=0.1/32, p=5	Error in OSHBM h=0.1, p=5	Error in [5], h=0.1/32, p=6
0.003125	4.6e-18	1.001124239044e-07	3.8354 E-05
0.00625	4.16e-17	6.413822486422e-07	7.5004E-05
0.009375	1.09e-16	1.136405e-06	1.0592 E-04
0.0125	2.053e-16	1.49966e-06	1.35476 E-04
0.015625	3.288e-16	1.7427e-06	1.55567E-04
0.01875	4.782e-16	1.894407e-06	1.86372E-04
0.021875	6.521e-16	1.980796e-06	1.96055E-04
0.025	8.491e-16	2.021807e-06	2.21045E-04
0.028125	1.0679e-15	2.03174e-06	2.05628E-04
0.03125	1.3073e-15	2.0206e-06	2.77908E-04

Table 3. Numerical Results of Problem 3

X	Exact	OSHBM	Error
0.1	0.09983341664682815231	0.09983341664641143268	4.16719627e-13
0.2	0.19866933079506121546	0.19866933079151260797	3.54860749e-12
0.3	0.29552020666133957511	0.29552020665229235391	9.0472212e-12
0.4	0.38941834230865049167	0.38941834229214808125	1.650241042e-11
0.5	0.47942553860420300027	0.47942553857875939095	2.544360932e-11
0.6	0.56464247339503535720	0.56464247335967945648	3.535590072e-11
0.7	0.64421768723769105367	0.64421768719198266396	4.570838971e-11
0.8	0.71735609089952276163	0.71735609084353295021	5.598981142e-11
0.9	0.78332690962748338846	0.78332690956173874562	6.574464284e-11
1.0	0.84147098480789650665	0.84147098473329359276	7.460291389e-11

Table 4. Numerical Results of Problem 4

Method	p	M=400	M = 800	M = 1600	M = 3200	6400
N4	4	0.6	1.8	3.0	4.2	5.4
H8	8	0.3	2.6	5.2	7.6	10.0
BG8	8	0.9	3.1	5.6	8.0	10.4
OSHBM	5	5.2	6..7	8.2	9.7	11.0

5. Discussion of results

Problem 1 is a highly stiff problem and has been solved by Adeniran and Ogundare (2015) with a method of order 4. The numerical solutions are shown in Table 1. The OSHBM compared favourably well with the Adeniran and Ogundare method though the OSHBM is of order p=5. In Table 2, the solutions of problem 2 is presented as comparison of our order 5 OSHBM is made with order 6 method of Badmus and Yahaya. The superiority of the method has been established numerically. Table 3 shows the desirability of the method as we compare the solution of Problem 3 with the analytical method. Table 4 presents the solutions of Problem 4 which has been integrated in the interval $[\sqrt{\frac{\pi}{2}}, 10]$. This problem has also been solved by Sommeijer (1993) using N4 method of order four, the eighth-order, eight-stage RKN (H8) method constructed by Hairer (1977) and the eight-order, nine-stage method of order 8 (BG8) constructed by Beentjes and Gerritsen (1976). The results of these methods are compared with the OSHBM of order 5 in Table 4. The superiority of OSHBM has been established numerically as it performs better than those in Sommeijer (1993) in terms of accuracy (larger CD values) and efficiency (smaller NFEs).

6. Conclusion

Formulation of initial value problem solver has been developed using a new class orthogonal polynomials with recursive formula. Four test problems have been considered to show the efficiency and accuracy of the method. Tables 1, 2, 3 and 4 display the accuracy and comparison of the numerical results of the OSHBM with existing methods. The method is desirability as its superiority has been established by the numerical results. With little extension, the approach adopted in this paper is viable for the solution of higher order initial value problems of ordinary differential equations.

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