APPROXIMATE SOLUTIONS TO THE GENERALIZED TIME-FRACTIONAL ITO SYSTEM

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Abstract. In this work, we consider the time-fractional version of the well-known integrable Ito system to study the effect of its fractional index "memory index". A modified approach of a relatively new method called residual power series (RPS) is applied to construct an analytical solution for the fractional system. For purpose of comparison, we derive one of the classical Ito system solitary wave solution using Tanh method.

Keywords: time-fractional Ito model, Tanh method, residual power series method.

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1. Introduction

The literature has recently been enriched by significant studies on the theory of fractional calculus and its applications. Systematic development on the mathematical structures and physical interpretations of different types was the start point in this field [1]-[7]. A progress on better understanding of the realistic of fractional calculus urged researchers to seek solutions of fractional differential equations (FDEs) by conducting well-posed numerical and analytical schemes. For example, the adomian decomposition method (ADM) [8], [9], the modified (G'/G)-expansion method [10], the differential transform method [11], [12], the homotopy perturbation method [13], the variational iteration method [14], [15], the finite difference method [16], the finite element method [17], [18], the simplified bilinear method [19]-[22] and other methods [23]-[26]. Recently, new trends in the field of fractional nonlinear equations has been approached by scholars by considering fractional versions of well-known nonlinear evolution systems connecting their findings with "Memory-index" feature may such models possess [38], [39]. In this work, we proceed with this trend and we will explore the time-fractional integrable Ito system [27] that reads

$$\begin{aligned} u_t &= v_x \\ v_t &= -2v_{xxx} - 6(uv)_x. \end{aligned} \tag{1.1}$$

Tam, Hu and Wang [28] implemented the Hirota bilinear method and obtained soliton solutions of (1.1). Also, they studied the integrability of this system in the sense of existence of Lax pairs, infinitely many conservations laws and N-soliton solutions.

The study on searching for exact or approximate solutions to partial differential equations enable us to understand physical models better. Therefore, the purpose of this paper is twofold. First, Finding soliton solutions of system (1.1) by using the tanh method [29]-[34]. Second, we use a modified implementation of the RPS method [35]-[43] to find approximate solution to a fractional version of system (1.1) which is

$$\begin{aligned} D_t^{\alpha} u &= v_x \\ D_t^{\alpha} v &= -2v_{xxx} - 6(uv)_x, \end{aligned} \tag{1.2}$$

where α represent the fractional derivative in Caputo sense and also is regarded as the memory index of this model.

2. Survey of the Tanh method

One of the most efficient solitary wave methods is the Tanh method. This technique is based on the assumption that the traveling wave solutions can be expressed in terms of the tanh function [29]-[34]. We therefore introduce a new independent variable

$$Y = \tanh(\mu\zeta). \tag{2.1}$$

Then, the solution can be introduced as a finite power series in Y in the form:

$$u(\mu\zeta) = S(Y) = \sum_{i=0}^{M} a_i Y^i. \tag{2.2}$$

The index M is a positive integer, in most cases, that will be determined by using a balance procedure, where by comparing the behavior of Y^i in the highest derivative against its counterpart within the nonlinear terms. Once M is determined, the coefficients a_i can be determined by setting the coefficients of powers of Y in the resulting equation to zero.

3. Solitary wave solutions to the Ito system

The wave variable $\zeta = x - ct$ carries (1.1) into the ordinary differential system

$$\begin{aligned} -cu' &= v' \\ -cv' &= -2v''' - 6(uv)'. \end{aligned} \tag{3.1}$$

Integrating the above differential equations (DEs) with respect to ζ and considering the constant of integration to be zero, we reach to the following single DE

$$-\frac{c}{2}u + u'' + 3(u^2) = 0. \tag{3.2}$$

Now, by means of Tanh method, u can be assumed as

$$u(\zeta) = \sum_{i=0}^{M} a_i \tanh^i(\mu\zeta). \tag{3.3}$$

Balancing u'' with u^2 in (3.3) gives $M + 2 = 2M$. Thus, $M = 2$.

Now, substituting (3.3) in (3.2) and collecting the coefficients of $\tanh^i$, we obtain a system of algebraic equations for $a_0,\ a_1,\ \ldots\ a_M$ and $\mu,\ \zeta$. By solving this system, one of the obtained solutions is read as:

$$\begin{aligned} u(x,t) &= \frac{2\mu^2}{3} - 2\mu^2 \tanh^2(\mu(x + 8\mu^2 t)), \\ v(x,t) &= 8\mu^2 \left(\frac{2\mu^2}{3} - 2\mu^2 \tanh^2(\mu(x + 8\mu^2 t)) \right). \end{aligned} \tag{3.4}$$

4. Analytical solution of the fractional Ito system

Consider the time-fractional Ito system

$$\begin{aligned} D_t^{\alpha} u &= v_x \\ D_t^{\alpha} v &= -2v_{xxx} - 6(uv)_x. \end{aligned} \tag{4.1}$$

subject to the initial conditions:

$$\begin{aligned} u(x,0) &= f(x), \\ v(x,0) &= g(x). \end{aligned} \tag{4.2}$$

Now we write the solution of Eqs. (4.1-4.2) as a fractional power series about $t = 0$

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} a_n(x) \, \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \\ v(x,t) &= \sum_{n=0}^{\infty} b_n(x) \, \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}. \end{aligned} \tag{4.3}$$

Provided that $0 < \alpha \leq 1$. For the RPS purposes, we let $u_j(x,t),\ v_j(x,t)$ to denote the j-th truncated series of $u(x,t),\ v(x,t)$, respectively, i.e.

$$\begin{aligned} u_j(x,t) &= \sum_{n=0}^{j} a_n(x) \, \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \\ v_j(x,t) &= \sum_{n=0}^{j} b_n(x) \, \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}. \end{aligned} \tag{4.4}$$

Applying the conditions given in (4.2), the 0-th RPS approximate solutions of $u(x,t),\ v(x,t)$ are

$$\begin{aligned} u_0(x,t) &= a_0(x) = u(x,0) = f(x) \\ v_0(x,t) &= b_0(x) = v(x,0) = g(x). \end{aligned} \tag{4.5}$$

Therefore, Eqs. (4.4) can be written as

$$\begin{aligned} u_j(x,t) &= f(x) + \sum_{n=1}^{j} a_n(x) \, \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \\ v_j(x,t) &= g(x) + \sum_{n=1}^{j} b_n(x) \, \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \end{aligned} \tag{4.6}$$

where $j = 1, 2, 3, ...$

Now, we define the residual functions, $Res_u,\ Res_v$, for equations (4.1)

$$\begin{aligned} Res_u(x,t) &= D_t^{\alpha} u - v_x \\ Res_v(x,t) &= D_t^{\alpha} v + 2v_{xxx} + 6(uv)_x, \end{aligned} \tag{4.7}$$

and therefore, the j-th residual functions, $Res_{u,j},\ Res_{v,j}$ are

$$\begin{aligned} Res_{u,j}(x,t) &= D_t^{\alpha} u_j - \frac{\partial v_j}{\partial x} \\ Res_{v,j}(x,t) &= D_t^{\alpha} v_j + 2\frac{\partial^3 v_j}{\partial x^3} + 6\frac{\partial (u_j v_j)}{\partial x}. \end{aligned} \tag{4.8}$$

To determine the coefficients $a_n(x),\ b_n(x):\ n = 1,2,3,...,j$ in Eqs. (4.6), we solve the system

$$\begin{aligned} D_t^{(j-1)\alpha} Res_{u,j}(x,0) &= 0, \\ D_t^{(j-1)\alpha} Res_{v,j}(x,0) &= 0, \end{aligned} \tag{4.9}$$

Now, we are ready to formulate the following steps.

Step 1. To determine $a_1(x),\ b_1(x)$, we consider $(j = 1)$ in (4.8)

$$\begin{aligned} Res_{u,1} &= D_t^{\alpha} u_1 - \frac{\partial v_1}{\partial x} \\ Res_{v,1} &= D_t^{\alpha} v_1 + 2\frac{\partial^3 v_1}{\partial x^3} + 6\frac{\partial (u_1 v_1)}{\partial x}. \end{aligned} \tag{4.10}$$

But, $u_1(x,t) = f(x) + a_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)}$ and $v_1(x,t) = g(x) + b_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)}$. Therefore,

$$\begin{aligned} Res_{u,1}(x,t) &= a_1(x) - \left(g'(x) + b_1'(x)\frac{t^\alpha}{\Gamma(1+\alpha)}\right), \\ Res_{v,1}(x,t) &= b_1(x) + 2\left(g'''(x) + b_1'''(x)\tfrac{t^\alpha}{\Gamma(1+\alpha)}\right) \\ &+ 6\left(f(x) + a_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)}\right)\left(g'(x) + b_1'(x)\frac{t^\alpha}{\Gamma(1+\alpha)}\right) \\ &+ 6\left(f'(x) + a_1'(x)\frac{t^\alpha}{\Gamma(1+\alpha)}\right)\left(g(x) + b_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)}\right). \end{aligned} \tag{4.11}$$

Using equations (4.9), we have $Res_{u,1}(x,0) = 0,\ Res_{v,1}(x,0) = 0$. Therefore,

$$\begin{aligned} a_1(x) &= g'(x), \\ b_1(x) &= -2g'''(x) - 6(f(x)g(x))'. \end{aligned} \tag{4.12}$$

Step 2. To determine $a_2(x),\ b_2(x)$, we consider $(j = 2)$ in (4.8), where $u_2(x,t) = f(x) + a_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)} + a_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$ and $v_1(x,t) = g(x) + b_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)} + b_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$. Then, we solve the corresponding algebraic equations $D_t^\alpha Res_{u,2}(x,0) = 0$ and $D_t^\alpha Res_{v,2}(x,0) = 0$. Equivalently, this step can be done easily by considering the coefficient of the variable t^α in each of the functions $D_t^\alpha Res_{u,2}(x,t)$ and $D_t^\alpha Res_{v,2}(x,t)$. Then we multiply each obtained coefficient by the factor $\Gamma(1+\alpha)$ and set to zero to find the unknown required functions $a_2(x),\ b_2(x)$. The reason of adopting this approach the fact that by Caputo derivative, $D_t^\alpha (t^\alpha) = \Gamma(1+\alpha)$ and $D_t^\alpha (t^b) \downarrow_{t=0} = 0,\ b > \alpha$. Using this argument, leads to the following result formulas.

$$\begin{aligned} a_2(x) &= b_1'(x), \\ b_2(x) &= -2b_1'''(x) - 6\left(f(x)b_1(x)\right)' - 6\left(a_1(x)g(x)\right)'. \end{aligned} \tag{4.13}$$

Step 3. To find the functions $a_3(x),\ b_3(x)$, we find the coefficient of the variable $t^{2\alpha}$ in the resulting expansion of $D_t^{2\alpha} Res_{u,3}(x,t)$, $D_t^{2\alpha} Res_{v,3}(x,t)$ and multiply it by the factor $\Gamma(1+2\alpha)$ and set to zero. Provided that $D_t^{2\alpha}\left(t^{2\alpha}\right)=\Gamma(1+2\alpha)$ and $D_t^{2\alpha}\left(t^{b}\right)\downarrow_{t=0}=0,\ b>2\alpha$. Therefore,

$$\begin{aligned} a_3(x) &= b_2'(x) \\ b_3(x) &= -2b_2'''(x)-6\left(f(x)b_2(x)\right)'-6\left(a_2(x)g(x)\right)' \\ &\quad -6\frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}\left(a_1(x)b_1(x)\right)' \end{aligned} \tag{4.14}$$

Finally, following the above routine, we reach at,

$$\begin{aligned} a_4(x) &= b_3'(x) \\ b_4(x) &= -2b_3'''(x)-6\left(f(x)b_3(x)\right)'-6\left(a_3(x)g(x)\right)' \\ &\quad -\frac{6\ \Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\left[\left(b_1(x)a_2(x)\right)'+\left(b_2(x)a_1(x)\right)'\right]. \end{aligned} \tag{4.15}$$

5. Discussions and concluding remarks

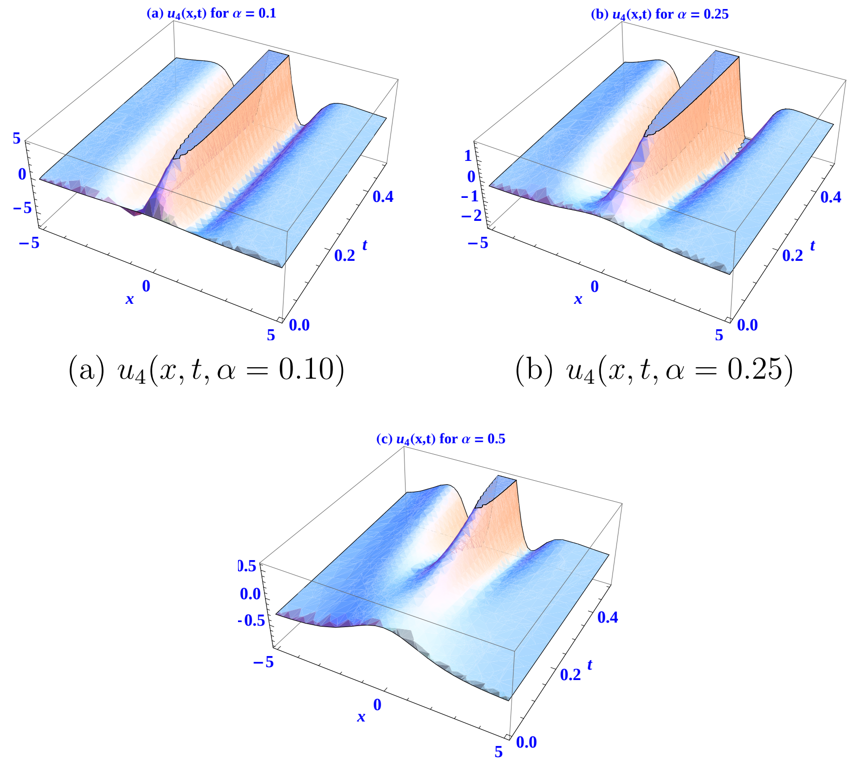
In this section, we study the nature of the solution of the fractional Ito system as the fractional derivative parameter varies from 0 to 1. We solve the system subject to the initial conditions

$$\begin{aligned} u(x,0) &= f(x)=\frac{1}{6}-\frac{1}{2}\tanh^2\left(\frac{1}{2}x\right) \\ v(x,0) &= g(x)=\frac{1}{3}-\tanh^2\left(\frac{1}{2}x\right). \end{aligned} \tag{5.1}$$

It is to be noted that the exact solution of this system when $\alpha=1$ is (see Section 3, when $\mu=\frac{1}{2}$):

$$\begin{aligned} u(x,t) &= \frac{1}{6}-\frac{1}{2}\tanh^2\left(\frac{x}{2}+t\right) \\ v(x,t) &= \frac{1}{3}-\tanh^2\left(\frac{x}{2}+t\right). \end{aligned} \tag{5.2}$$

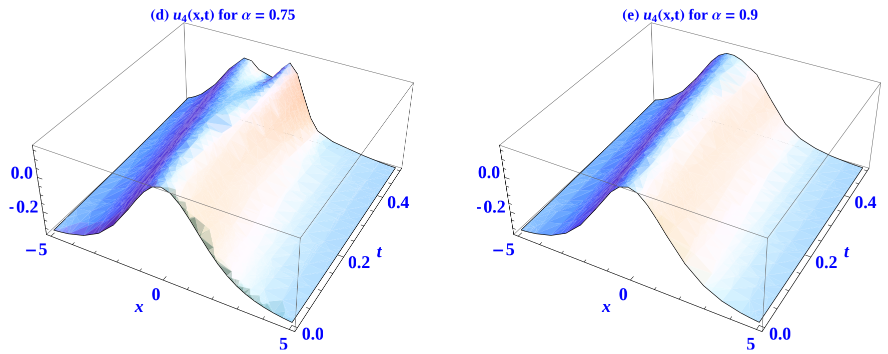
Figures 1 and 2, represent the effect of the fractional derivative parameter α on the chaotic behavior of the obtained solution. It is been observed that as the memory index "The fractional order" getting smaller as to 0, the RPS solutions bifurcate and has a wave-like shape. On the other hand, when the fractional derivative reaches 1, the RPS solution is a complete wave.



(a) $u_4(x, t, \alpha = 0.10)$ (b) $u_4(x, t, \alpha = 0.25)$

(c) $u_4(x, t, \alpha = 0.50)$

Figure 1: The 4-th RPS approximate solution.



(d) $u_4(x, t, \alpha = 0.75)$ (e) $u_4(x, t, \alpha = 0.90)$.

Figure 2: The 4-th RPS approximate solution.

In order to illustrate that the obtained RPS solution is efficient and accurate, we first present the comparison of the numerical solution using the present method "4-th RPS solution" and the exact solution given in (5.2) when $\alpha = 1$, see Figure 3.

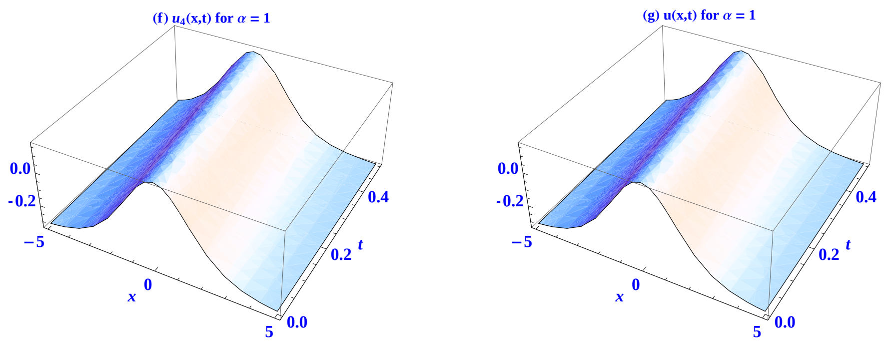


Figure 3: The approximate and exact solutions, respectively, when $-5 < x < 5$ and $0 < t < 0.5$ and $\alpha = 1$.

To validate the accuracy of the numerical scheme, we give explicit values of the variables x, t and compute the absolute error between the exact solution and the 5-th RPS solution when $\alpha = 1$, see Table 1.

$x\|t$	0.1	0.15	0.2	0.25	0.3
-4	2.47×10^{-9}	2.92×10^{-8}	1.71×10^{-7}	6.76×10^{-7}	2.09×10^{-6}
-2	1.14×10^{-8}	1.47×10^{-8}	9.35×10^{-7}	3.97×10^{-6}	1.31×10^{-5}
0	1.88×10^{-7}	2.13×10^{-6}	1.18×10^{-5}	4.46×10^{-5}	1.32×10^{-5}
2	5.93×10^{-9}	5.39×10^{-8}	2.32×10^{-7}	6.27×10^{-7}	1.15×10^{-6}
4	2.12×10^{-9}	2.33×10^{-8}	1.26×10^{-7}	4.65×10^{-7}	1.33×10^{-6}

Table 1: Absolute error of the RPS approximate solution against the exact solution when $\alpha = 1$

In conclusion, the Residual power series method has been successfully applied to find an analytical solution to the fractional Ito system. Comparison of the result obtained by the present method when $\alpha = 1$ with the exact solution obtained by Tanh method evidences that the numerical results are harmonious and can be easily improved by adding new terms of the power series approximations. Applications, concluding remarks and some theory regards the Residual power series method can be found in ([40]-[42]) and ([35]-[39]).

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