ON WEAKLY PRIME L-IDEALS

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Abstract. Let L be a complete lattice. We introduce and study the notion of weakly prime L-ideals of a commutative ring with identity and investigate their properties. In particular we characterize weakly prime L-ideals of R, where R is an integral domain and L is a chain.

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1. Introduction

Throughout this paper, R is a commutative ring with identity and L is a complete lattice. A proper ideal P of R is prime if $ab \in P$ implies that $a \in P$ or $b \in P$, or equivalently, if for ideals A and B of R, $AB \subseteq R$ implies that $A \subseteq P$ or $B \subseteq R$. Weakly prime ideals in a commutative ring with non-zero identity have been introduced and studied by D.D Anderson and E. Smith [3] and weakly primary ideals have been introduced by Sh. Ebrahimi Atani and F. Farzalipour [5]. In fact, a proper ideal P of R is said to be weakly prime if $0 \neq ab \in P$ implies that either $a \in P$ or $b \in P$ and is called weakly primary ideal if $0 \neq ab \in P$ implies that $a \in P$ or $b \in Rad(P)$(the radical of P). So a weakly prime ideal is weakly primary, but since 0 is always weakly primary, a weakly primary ideal does not need to be primary.

In the last few years a considerable amount of work has been done on fuzzy ideals in general and prime fuzzy ideals in particular, and some interesting topological properties of the spectrum of fuzzy prime ideals of a ring are obtained (see [7], [8], [9], [10], [11], [13]).

Zadeh [16] introduced the notion of a fuzzy subset of a non-empty set X as a function from X to the unit interval $I = [0, 1]$. Goguen [6], replaced $I = [0, 1]$ by a complete lattice L in the definition of fuzzy subset and introduced the notion of L-fuzzy subsets and Rosenfeld introduced introduced the notion of fuzzy groups [15].

In this paper we introduce the notion of weakly prime L-ideal of a commutative ring with non-zero identity where L is a complete lattice. In fact our definition is a generalization of the notion of ordinary weakly prime ideal. We investigate some basic properties of weakly prime L-ideals and characterize the weakly prime L-ideals of R where R is an integral domain and L is a chain.

2. Preliminaries

By an *L-subset* μ of a non-empty set X, we mean a function μ from X to L. If $L = [0,1]$, then μ is called a *fuzzy* subset of X. L^X denotes the set of all L-subsets of X. Let $a \in L$ and $y \in M$. For all $x \in M$ define $a_y \in L^M$ as follows:

$$a_y(x) = \begin{cases} a & \text{if } \ x = y \\ 0 & \text{otherwise} \end{cases}$$

a_y is called an *L-point*. Sometimes, we use y_a instead of a_y.

For $\mu, \nu \in L^R$, we say that μ is contained in ν and we write $\mu \subseteq \nu$ if $\mu(x) \leq \nu(x)$, for all $x \in R$. For $\mu, \nu \in L^R$, the intersection and union, $\mu \cup \nu$, $\mu \cap \nu \in L^R$ are defined by

$$(\mu \cup \nu)(x) = \mu(x) \vee \nu(x) \ \ and \ \ (\mu \cap \nu)(x) = \mu(x) \wedge \nu(x), \ \ for \ \ all \ \ x \in R$$

Also for $\mu \in L^R$, $a \in L$ we define μ_a as follows:

$$\mu_a = \{x \in R, \mu(x) \geq a\},$$

μ_a is called *$a-$cut* (or *$a-$level subset*) of μ. In particular, we denote $\mu_{\mu(0)}$ by μ_*, of course

$$\mu_* = \{x \in R | \mu(x) = \mu(0)\}.$$

Let f be a mapping from R into R' and let $\mu \in L^R$, $\nu \in L^{R'}$. Then $f(\mu) \in L^{R'}$ and $f^{-1}(\nu) \in L^R$ are defined as follows

$$f(\mu)(y) = \begin{cases} \vee\{\mu(x) \mid x \in f^{-1}(y)\} & f^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

and

$$f^{-1}(\nu)(x) = \nu(f(x)) \quad \forall x \in R.$$

Let R, R' be rings and $f : R \to N$ be a ring homomorphism. $\mu \in L^R$ is called *f-invariant* if $f(x) = f(y)$ implies that $\mu(x) = \mu(y)$ for all $x, y \in R$.

By the above hypothesis, if $\mu \in LI(R)$ has the sup property, then $f(\mu_*) = (f(\mu))_*$ and if $\nu \in LI(R')$, then $f^{-1}(\nu_*) = (f^{-1}(\nu))_*$.

We recall some definitions and theorems from the book [12].

Definition 2.1. Let $\mu \in L^R$. Then μ is called an *L-ideal* of R if, for every $x, y \in R$, the following conditions are satisfies:

1) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$;

2) $\mu(xy) \geq \mu(x) \vee \mu(y)$.

The set of all L-ideals of R is denoted by $LI(R)$.

Remark 2.2. By previous Definition, it is easily seen that if μ is an L-ideal of R, then $\mu(1) < \mu(x) < \mu(0)$ for all $x \in R$.

Definition 2.3. Let $\mu \in L^R$. Let

$$\langle \mu \rangle = \bigcap \{\nu \in LI(R);\ \mu \subseteq \nu\}.$$

Then $\langle \mu \rangle$ is called the L-ideal of R generated by μ.

In [12] it has been proved that if $\xi = \langle \mu \rangle$, then

$$\xi(x) = \bigvee \{\wedge_{y \in A} \mu(y) \mid A \subseteq R,\ 1 \leq |A| < \infty,\ x \in \langle A \rangle\} \text{ for all } x \in R.$$

Also, $\langle a_x \rangle = a_{\langle x \rangle}$.

Definition 2.4. Let $\mu, \nu \in LI(R)$, We define $\mu \circ \nu \in LI(R)$ as follows:

$$\mu \circ \nu(x) = \bigvee \{\mu(y) \wedge \nu(y) \mid y, z \in R, x = yz\} \text{ for all } x \in R.$$

By the previous definition, it is seen that $\langle a_x \rangle \circ \langle a_y \rangle = \langle (xy)_{a \wedge b} \rangle$.

Definition 2.5. Let R be a ring and $\zeta \in LI(R)$. Then ζ is called *prime L-ideal* of R if ζ is non-constant and for every $\mu, \nu \in LI(R)$, $\mu \circ \nu \subseteq \zeta$ implies that $\mu \subseteq \zeta$ or $\nu \subseteq \zeta$.

Definition 2.6. Let $c \in L \backslash \{1\}$, c is called a *prime element* of L if $a \wedge b \leq c$, then $a \leq c$ or $b \leq c$, for all $a, b \in L$, and c is called a *maximal element* if there does not exist $a \in L \backslash \{1\}$ such that $c < a < 1$.

Theorem 2.7. *Let $\zeta \in L^R$. Then ζ is a prime L-ideal of R if and only if $\zeta(0) = 1$ and $\zeta = 1_{\zeta_*} \cup c_R$ such that ζ_* is a prime ideal of R and c is a prime element of L.*

3. Weakly prime L-ideals

In this section, we introduce the notion of weakly prime L-ideals and investigate some basic properties of them.

Definition 3.1. An L-ideal μ f R is called weakly prime if for every $\alpha, \beta \in LI(R)$ such that $0_R \neq \alpha \circ \beta \subseteq \mu$, then either $\alpha \subseteq \mu$ or $\beta \subseteq \mu$.

Corollary 3.2. *Let μ be a weakly prime L-ideal of R. Then $0_R \neq x_a \circ y_b \subseteq \mu$ implies that $x_a \subseteq \mu$ or $y_b \subseteq \mu$, for all $x, y \in R$ and $a, b \in L$.*

Proof. Let $0_R \neq x_a \circ y_b \subseteq \mu$ for $x, y \in R$ and $a, b \in L$. Then $a \wedge b \neq 0$ and so $\langle x_a \rangle \circ \langle y_b \rangle \neq 0_R$, (since $\langle x_a \rangle \circ \langle y_b \rangle = \langle (xy)_{a \wedge b} \rangle = (a \wedge b)_{\langle xy \rangle}$ and $((a \wedge b)_{\langle xy \rangle})(xy) = a \wedge b \neq 0$). Therefore, $0_R \neq \langle x_a \rangle \circ \langle y_b \rangle \subseteq \mu$. Thus $\langle x_a \rangle \subseteq \mu$ or $\langle y_a \rangle \subseteq \mu$. Hence $x_a \subseteq \mu$ or $y_b \subseteq \mu$. ■

Remark 3.3. It is clear that every prime L-ideal of R is weakly prime. But the converse is not true, for example 0_R is weakly prime ideal which is not prime, because if we consider a, b in L such that $a \wedge b = 0$, then $0_a \circ 0_b \subseteq 0_R$, but neither $0_a \subseteq 0_R$ nor $0_b \subseteq 0_R$. Also, in [12] it has proved that if P is a prime ideal of R and L is regular, then 1_P is a prime L-ideal of R. But it is not true for weakly prime ideals in general. For example $\langle 0 \rangle$ is weakly prime ideal of R, but $1_{\langle 0 \rangle}$ is not weakly prime L-ideal of R.

In the next three propositions, we give some interesting properties of weakly prime L-ideals.

Proposition 3.4. *Let μ be a nonconstant weakly prime L-ideal of R such that $\mu \neq 0_R$. Then $\mu(0) = 1$.*

Proof. Let $\mu(0) < 1$. Since μ is non-constant, there exists $a \in R$ such that $\mu(a) < \mu(0)$. We define two L-subsets on R as follows:

$$\alpha(x) = \begin{cases} 1 & x \in \mu_* \\ 0 & \text{otherwise} \end{cases}$$

and

$$\beta(x) = \mu(0) \quad \forall x \in R$$

Then for $0 \neq x \in R$,

$$(\alpha \circ \beta)(x) = \bigvee \{\alpha(y) \wedge \beta(z) \mid x = yz\} = \bigvee \{\alpha(y) \wedge \mu(0) \mid x = yz\} = \mu(0) \text{ or } 0$$

and again since μ is non-constant

$$(\alpha \circ \beta)(0) = \mu(0) \neq 0.$$

Also $\alpha(0) = 1 > \mu(0)$ and $\beta(a) = \mu(0) > \mu(a)$. Thus $\alpha \nsubseteq \mu$ and $\beta \nsubseteq \mu$ which is a contradiction. Hence $\mu(0) = 1$. ■

By the previous proposition, we get if μ is a weakly prime L-ideal of R, then

$$\mu_* = \{x \in R | \mu(x) = 1\}.$$

Proposition 3.5. *Let μ be a non-constant weakly prime L-ideal of R. Then μ_* is a weakly prime ideal of R.*

Proof. Suppose that μ is a weakly prime L-ideal of R and $0 \neq ab \in \mu_*$ for $a, b \in R$. We put $\alpha = 1_{<a>}$ and $\beta = 1_{<b>}$. Then

$$\alpha \circ \beta = 1_{<a>} \circ 1_{<b>} = 1_{<ab>} \subseteq 1_{\mu_*} \subseteq \mu$$

Also $(\alpha \circ \beta)(ab) = 1 \neq 0$. Since μ is weakly prime, then $1_{<a>} \subseteq \mu$ or $1_{<b>} \subseteq \mu$. So $\mu(a) = 1$ or $\mu(b) = 1$ and thus $a \in \mu_*$ or $b \in \mu_*$. Hence μ_* is weakly prime ideal of R. ■

Definition 3.6. An element $t \neq 1$ in L is called a weakly prime element if $0 \neq a \wedge b \leq t$ implies that either $a \leq t$ or $b \leq t$, for all $a, b \in L$.

Theorem 3.7. *Let μ be a non-constant L-ideal of R. If μ is weakly prime, then $\mu = 1_{\mu_*} \cup c_R$ such that μ_* is a weakly prime ideal of R and c s a weakly prime element of L.*

Proof. Let μ be a weakly prime L-ideal of R. By Proposition 3.5 μ_* is a weakly prime ideal of R and $\mu(0) = 1$. We prove that $|\mu(R) = 2|$ and if $1 \neq c \in \mu(R)$, then c is a weakly prime element. μ is non-constant and $1 \in \mu(R)$, so $|\mu(R)| \geq 2$. Now let $x, y \in R \setminus \mu_*$ and $\mu(x) = c$. There are two cases:

1) $c = 0$. Then $\mu(x) \leq \mu(y)$.

2) $c \neq 0$. Then $1_{\langle x \rangle}$ and c_R are L-ideals, $c_{\langle x \rangle} = \langle x_c \rangle \subseteq \mu$ and
$$0_R \neq 1_{\langle x \rangle} \circ c_R \subseteq 1_{\langle x \rangle} \cap c_R = c_{<x>} \subseteq \mu$$
Since $1_{\langle x \rangle} \nsubseteq \mu$ and μ is weakly prime, then $c_R \subseteq \mu$. Thus, $c \leq \mu(y)$ and so $\mu(x) \leq \mu(y)$. Similarly, we can get $\mu(y) \leq \mu(x)$. Hence $\mu(x) = \mu(y)$ and then $|\mu(R)| = 2$.

Now, suppose that $0 \neq a \wedge b \leq c$ for $a, b \in L$, but $a \nleq c$ and $b \nleq c$. Then, a_R and b_R are L-ideals of R, $0_R \neq a_R \circ b_R = (a \wedge b)_R \subseteq \mu$, but $a_R \nsubseteq \mu$ and $b_R \nsubseteq \mu$ which is a contradiction. Therefore, $\mu = 1_{\mu_*} \cup c_R$ such that μ_* is a weakly prime ideal of R and c is a weakly prime element in L. ∎

Note that if R is an integral domain and L is a chain, then the converse of the previous theorem is true. In fact:

Theorem 3.8. *Let R be an integral domain, L be a chain and μ is an L-ideal of R. If $\mu = 1_{\mu_*} \cup c_R$ such that μ_* is a weakly prime ideal of R and c is an element of L, then μ is prime and so is weakly prime.*

Proof. Every element of L is prime, since L is chain. We prove that μ_* is a prime ideal of R. For this let $ab \in \mu_*$ for $a, b \in R$. If $ab \neq 0$, then $a \in \mu_*$ or $b \in \mu_*$. If $ab = 0$, then $a = 0$ or $b = 0$. So $a \in \mu_*$ or $b \in \mu_*$. Therefore μ_* is prime. Hence, μ is prime and so is weakly prime. ∎

In the following examples, we show that whether R is not integral domain or L is not chain, the converse of Theorem 3.7 is not in general hold.

Example 3.9. Let R be a ring which is not an integral domain. Then $\langle 0 \rangle$ is weakly prime ideal of R which is not prime. Consider $a, b \neq 0$ such that $ab = 0$ and $c \in L$. Put $\mu = 1_{<0>} \cup c_R$, $\alpha = 1_{\langle a \rangle}$ and $\beta = 1_{\langle b \rangle}$. Then $\alpha \circ \beta = 1_{\langle ab \rangle} = 1_{<0>} \neq 0_R$ and $\alpha \circ \beta \subseteq 1_{\langle 0 \rangle} \cup c_R$, but $1_{\langle a \rangle} \nsubseteq \mu$ and $1_{\langle b \rangle} \nsubseteq \mu$. It means that μ is not weakly prime.

Example 3.10. Let $R = \mathbb{Z}$ (the ring of integers) and $L = \{0, a, b, 1\}$ such that $a \wedge b = 0$. Then 0 is a weakly prime element in L. Consider I any prime ideal in $\mathbb{Z}$. Then I is weakly prime. Put $\mu = 1_I \cup 0_R$, $\alpha = 1_I \cup a_R$ and $\beta = 1_I \cup b_R$. So, $0_R \neq \alpha \circ \beta \subseteq \mu$ $((\alpha \circ \beta)(0) = 1 \neq 0$), but $\alpha \nsubseteq \mu$ and $\beta \nsubseteq \mu$.

Corollary 3.11. *Let μ be a non-constant weakly prime L-ideal of R. Then for every $x, y \in R$ such that $xy \neq 0$, either $\mu(xy) = \mu(x)$ or $\mu(xy) = \mu(y)$*

Proposition 3.12. *Let R and S be rings, f be an epimorphism of R to S and μ be an f-invariant L-ideal of R such that $\mu(R)$ is finite. If μ_* is a weakly prime ideal of R, then $f(\mu_*)$ is a weakly prime ideal of S.*

Proof. Since $\mu(R)$ is finite, it has the sup-property. So $f(\mu_*) = f(\mu)_*$. Now, suppose that $0 \neq uv \in f(\mu_*) = f(\mu)_*$. Then there exist $x, y \in R$ such that $u = f(x)$ and $v = f(y)$. So $xy \neq 0$ and $f(xy) = uv \in f(\mu)_*$. Hence

$$f(\mu)(0) = f(\mu)(f(xy)) = \bigvee\{\mu(z) \mid f(z) = f(xy)\} = \mu(0)$$

Therefore, there exists $w \in R$ such that $\mu(w) = \mu(0)$ and $f(w) = f(xy)$. Since μ is f-invariant, $\mu(w) = \mu(xy) = \mu(0)$. Thus $0 \neq xy \in \mu_*$. So $x \in \mu_*$ or $y \in \mu_*$ and then $u \in f(\mu_*)$ or $v \in f(\mu_*)$. It means that $f(\mu_*)$ is weakly prime ideal of R. ∎

Lemma 3.13. *Let f be an epimorphism of R into S and μ be an f-invariant L-ideal of R. If μ is a non-constant weakly prime L-ideal of R,then $f(\mu) = 1_{f(\mu)_*} \cup c_R$ such that $f(\mu)_*$ is a weakly prime ideal of R and c is a weakly prime element in L.*

Proof. By Theorem 3.7, $\mu = 1_{\mu_*} \cup c_R$ such that μ_* is a weakly prime ideal of R and c is a weakly prime element in L. By Proposition 3.12 , it is seen that $f(\mu)_*$ is a weakly prime ideal of S and $f(\mu)(0) = \mu(0) = 1$. Now, for $y \in S$, there exists $x \in R$ such that $f(x) = y$. If $\mu(x) = 1$, we have

$$f(\mu)(y) = f(\mu)(f(x)) = \bigvee\{\mu(z) \mid f(z) = f(x)\} = 1$$

since μ is f-invariant.

Similarly, if $\mu(x) = c$, then $f(\mu)(y) = c$. Hence $f(\mu) = 1_{f(\mu)_*} \cup c_R$ such that $f(\mu)_*$ is a weakly prime ideal of R and c is a weakly prime element in L. ∎

Corollary 3.14. *Let f be an epimorphism from R into S and μ be a non-constant weakly prime f-invariant L-ideal of R. If L is a chain and S is an integral domain, then $f(\mu)$ is a weakly prime L-ideal of S.*

Proof. By Theorem 3.8 and Theorem 3.13. ∎

In the previous corollary, if S is not an integral domain, then the image of a weakly prime L-ideal need not be weakly prime. For an example, if L is a chain, R is an integral domain and S is any ring, then $\mu = 1_{\langle 0\rangle} \cup c_R$ is weakly prime L-ideal of R, but $f(\mu) = 1_{\langle 0\rangle} \cup c_R$ is not a weakly prime L-ideal of S. (By Example 3.9)

Proposition 3.15. *Let S be an integral domain, f be a homomorphism of R into S and $\nu \in LI(S)$. If ν_* is a weakly prime ideal of S, then $f^{-1}(\nu)_*$ is a prime (and so weakly prime) ideal of R.*

Proof. We have $f^{-1}(\nu_*) = f^{-1}(\nu)_*$. So it is enough to prove that $f^{-1}(\nu_*)$ is prime. For this let $xy \in f^{-1}(\nu_*)$. Then $f(x)f(y) = f(xy) \in \nu_*$. If $f(x)f(y) = 0$, then $f(x) = 0$ or $f(y) = 0$. So $x \in f^{-1}(\nu_*)$ or $y \in f^{-1}(\nu_*)$.

If $f(x)f(y) \neq 0$, then $f(x) \in \nu_*$ or $f(y) \in \nu_*$, since ν_* is weakly prime. Therefore, $x \in f^{-1}(\nu_*)$ or $y \in f^{-1}(\nu_*)$. Hence $f^{-1}(\nu_*)$ is prime ideal of R. ▪

Corollary 3.16. *Let S be an integral domain and L be a chain. Suppose that f is a homomorphism of R into S. If ν is a non-constant weakly prime L-ideal of S, then $f^{-1}(\nu)$ is a weakly prime L-ideal of R.*

Proof. First $f^{-1}(\nu)(0) = \nu(f(0)) = 1$ and $f^{-1}(\nu)_*$ is a prime ideal of R, by Proposition 3.15. Now, $\nu = 1_{\nu_*} \cup c_R$ such that ν_* is a weakly prime ideal of S. For $x \in R$, if $\nu(f(x)) = 1$, then $f^{-1}(\nu)(x) = 1$. If $\nu(f(x)) = c$, then $f^{-1}(\nu)(x) = c$. It shows that $f^{-1}(\nu) = 1_{f^{-1}(\nu)_*} \cup c_R$. Thus $f^{-1}(\nu)$ is prime and so is weakly prime L-ideal of R. ▪

References

[1] Ameri, R., Mahjoob, R., *Specrtum of prime fuzzy hyperideals*, Iranian Journal od Fuzzy Systems, 6 (4) (2009), 61-72.

[2] Ameri, R., Mahjoob, R., Mootamani, M., *The zariski topology on the spectrum of prime L-submodules*, Soft Computing, 12 (2008), 901-908.

[3] Anderson, D.D., Smith, E., *Weakly prime ideals*, Houston J. of Math., 29 (4)(2003), 831-840.

[4] Dixit, V.N., Kummar, R., Ajmal, N., *Fuzzy Ideals and Fuzzy Prime Ideals of a Ring*, Fuzzy Sets and Systems, 44 (1991), 127-138.

[5] Ebrahimi Atani, Sh., Farzalipour, F., *On weakly primary ideals*, Georgian Mathematics Journal, 12 (3) (2005), 423-429.

[6] Goguen, J.A., *L-fuzzy Sets*, Journal Math. Appl, 8 (1967), 145-174.

[7] Hadji-Abadi, H., Zahedi, M.M., *Some Results on Fuzzy Prime Spectrum of a Ring*, Fuzzy Sets and Systems, 77 (1996), 235-240.

[8] Kumar, R., *Fuzzy Prime Spectrum of a Ring*, Fuzzy Sets and Systems, 46 (1992), 147-154.

[9] Kumar, R., Kohli, J.K.,, *Fuzzy Prime Spectrum of a Ring II*, Fuzzy Sets and Systems, 59 (1993), 223-230.

[10] Kumbhojkar, H.V., *Some Comments on Spectrum of Prime Fuzzy Ideals of a Ring*, Fuzzy Sets and Systems, 88 (1997), 109-114.

[11] Kumbhojkar, H.V., *Spectrum of Prime Fuzzy Ideals*, Fuzzy Sets and Systems, 62 (1994), 101-109.

[12] Mordeson, J.N., Malik, D.S., *Fuzzy Commutative Algebra*, (1998).

[13] Mukherjee, T.K., Sen, M.K., *On Fuzzy Ideals of a Ring I*, Fuzzy Sets and systems, 21 (1987), 99-104.

[14] Negoita, C.V., Ralescu, D.A., *Application of Fuzzy Systems Analysis*, Birkhauser, Basel, 1975.

[15] Rosenfeld R., *Fuzzy groups*, J. Math. Anal. Appl., 35 (1971), 512-517.

[16] Zadeh, L.A., *Fuzzy Sets*, Inform. and Control, 8 (1965), 338-353.

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