ROUGH FUZZY n-ABSORBING (n-ABSORBING PRIMARY) IDEAL IN COMMUTATIVE RINGS

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Abstract. In this paper, we introduce the notions of rough n-absorbing (n-absorbing primary) ideals and rough fuzzy n-absorbing (n-absorbing primary) ideals in a ring, and give some properties of such ideals. Also, we discuss the relations between the upper and lower rough n-absorbing (n-absorbing primary) ideals and the upper and lower approximations of their homomorphism images.

Keywords: fuzzy set, rough set, lower approximation; upper approximation, ring, ideal, prime ideal, primary ideal, n-absorbing ideal, n-absorbing primary ideal, fuzzy ideal.

1. Introduction

The notion of rough sets was introduced by Z. Pawlak in the year 1982 [13] as a formal tool for modelling and processing incomplete information in information systems. Some authors have studied the algebraic properties of rough sets. Biswas and Nanda [5] introduced the notion of rough subgroups. Kuroki in [10], introduced the notion of a rough ideal in a semigroup. Kuroki and Wang [11] gave some properties of the lower and upper approximations with respect to the normal subgroups. In [6], Davvaz concerned a relationship between rough sets and ring theory and considered a ring as a universal set and introduced the notion of rough ideals and rough subrings with respect to an ideal of a ring. In [9], Kazanci and Davvaz introduced the notions of rough prime (primary) ideals and rough fuzzy prime (primary) ideals in a ring. Now, this paper is a continuation of ideas presented by Kazanci and Davvaz [9]. We introduce the notions of rough n-absorbing (n-absorbing primary) ideals and rough fuzzy n-absorbing (n-absorbing primary) ideals in a ring, and give some properties of such ideals.

2. Preliminaries

For the sake of completeness, we would like to recall some definitions and results proposed in this field earlier. Throughout this paper R is a ring and n be a positive integer. Recall that an equivalence relation θ on R is a reflexive, symmetric, and transitive binary relation on R. If θ is an equivalence relation on R then the equivalence class of $x \in R$ is the set $\{y \in R \mid (x, y) \in \theta\}$. We write it as $[x]_\theta$.

Definition 2.1. [9] Let θ be an equivalence relation on R, then θ is called a full congruence relation if $(a, b) \in \theta$ implies $(a + x, b + x), (ax, bx)$ and $(xa, xb) \in \theta$ for all $x \in R$.

A full congruence relation θ on R is called complete if it satisfies

$$[ab]_\theta = \{xy \mid x \in [a]_\theta, y \in [b]_\theta\}, \text{ for all } a, b \in R.$$

Definition 2.2. [9] Let θ be a full congruence relation on R and A a subset of R. Then, the sets

$$\theta_-(A) = \{x \in R \mid [x]_\theta \subseteq A\} \text{ and } \theta^-(A) = \{x \in R \mid [x]_\theta \cap A \neq \emptyset\}$$

are called, respectively, the θ-lower and θ-upper approximations of the set A.

$\theta(A) = (\theta_-(A), \theta^-(A))$ is called a rough set with respect to θ if

$$\theta_-(A) \neq \theta^-(A).$$

Definition 2.3. [9] Let θ be a full congruence relation on R and A a subset of R. A non-empty subset A of a ring R is called an upper (resp. lower) rough ideal of R if $\theta^-(A)$ (resp. $\theta_-(A)$)is an ideal of R.

Theorem 2.4. [9] *Let θ be a full congruence relation on R. If A is an ideal of R, then $\theta^-(A)$ is an ideal of R.*

Theorem 2.5. [9] *Let θ be a full congruence relation on R and A is an ideal of R. If $\theta_-(A)$ is a nonempty set, then it is equal to A.*

Theorem 2.6. [9] *Let φ be an epimorphism (an onto homomorphism) of a ring R_1 to a ring R_2 and let θ_2 be a full congruence relation on R_2. Then*

(1) $\theta_1 = \{(a, b) \in R_1 \times R_1 \mid (\varphi(a), \varphi(b)) \in \theta_2\}$ *is a full congruence relation on R_1.*

(2) *If θ_2 is complete and φ is one to one, then θ_1 is complete.*

(3) $\varphi(\theta_1^-(A)) = \theta_2^-(\varphi(A))$.

(4) $\varphi(\theta_{1-}(A)) \subseteq \theta_{2-}(\varphi(A))$. *If* φ *is one to one, then* $\varphi(\theta_{1-}(A)) = \theta_{2-}(\varphi(A))$

As it is well known in the fuzzy theory established by Zadeh [14], a fuzzy subset μ of R is defined as a map from R to the unit interval $[0,1]$. Let μ and λ be two fuzzy subsets of R. The inclusion $\lambda \subseteq \mu$ is denoted by $\lambda(x) \leq \mu(x)$ for all $x \in R$, and $\mu \cap \lambda$ is defined by $\mu \cap \lambda(x) = \mu(x) \wedge \lambda(x)$ for all $x \in R$ and the expression $\mu o \lambda$ is defined by

$$(\mu o \lambda)(x) = \bigvee_{yz=x} (\mu(y) \wedge \lambda(z)).$$

Definition 2.7. [9] Let θ be a full congruence relation on R and μ a fuzzy subset of R. Then we define the fuzzy sets $\theta_-(\mu)$ and $\theta^-(\mu)$ as follow:

$$\theta_-(\mu)(x) = \bigwedge_{a \in [x]_\theta} \mu(a) \ \ and \ \ \theta^-(\mu)(x) = \bigvee_{a \in [x]_\theta} \mu(a).$$

The fuzzy sets $\theta_-(\mu)$ and $\theta^-(\mu)$ are called, respectively the θ-lower and θ-upper approximations of the fuzzy set μ.

$\theta(\mu) = (\theta_-(\mu), \theta^-(\mu))$ is called a rough fuzzy set with respect to θ if

$$\theta_-(\mu) \neq \theta^-(\mu).$$

Liu [12] introduced and examined the notion of a fuzzy ideal of a ring.

Definition 2.8. A fuzzy subset μ of a ring R is called a fuzzy ideal of R if it has the following properties:

(i) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in R$

(ii) $\mu(xy) \geq \mu(x) \vee \mu(y)$ for all $x, y \in R$

Example 1. [9] Let I be any ideal of R and let $\alpha \leq \beta \neq 0$ be numbers in $[0,1]$ then the fuzzy subsets μ defined by

$$\mu(x) = \begin{cases} \beta & \text{if } x \in I \\ \alpha & \text{otherwise} \end{cases}$$

is a fuzzy ideal.

Let μ and λ be fuzzy ideals of a ring R. Then, $\mu \cap \lambda$ is also a fuzzy ideal of R. A fuzzy subset μ of a ring R is called an lower rough fuzzy ideal of R if $\theta_-(\mu)$ is a fuzzy ideal of R. Similarly we can define upper rough fuzzy ideal. Recall that if μ, λ are fuzzy ideals of R, then $(\theta_-(\mu \cap \lambda), \theta^-(\mu \cap \lambda))$ is a rough fuzzy ideal of R. Let μ be a fuzzy subset of R. Then the sets

$$\mu_t = \{x \in R \mid \mu(x) \geq t\}, \mu_t^s = \{x \in R \mid \mu(x) > t\},$$

where $t \in [0,1]$ are called respectively, t-level subset and t-strong level subset of μ (see [9]).

Theorem 2.9. [8] *Let μ be a fuzzy subset of R. μ is a fuzzy ideal of R if and only if μ_t and μ_t^s are, if they are nonempty, ideals of R for every $t \in [0, 1]$.*

Theorem 2.10. [9] *Let θ be a full congruence relation on R. If μ is a fuzzy subset of R and $t \in [0, 1]$, then*

(1) $(\theta_-(\mu))_t = \theta_-(\mu_t)$

(2) $(\theta^-(\mu))_t^s = \theta^-(\mu_t^s)$.

3. Rough n-absorbing (n-absorbing primary) ideals in commutative rings

In the rest of this paper, R is a commutative ring with identity.

Let R be a ring. Recall that a nonzero proper ideal I of R is called an n-absorbing ideal of R if whenever $x_1, ..., x_{n+1} \in R$ and $x_1 \cdots x_{n+1} \in I$, then there are n of the x_i's whose product is in I (there exist $i \in \{1, ..., n + 1\}$ such that $x_1 \cdots x_{i-1}x_{i+1} \cdots x_{n+1} \in I$ (see [1], [2]).

In $\mathbb{Z}$, the ideal $< p_1 \cdots p_n >= \{p_1 \cdots p_n k | k \in \mathbb{Z}\}$, where p_i is a prime number of $\mathbb{Z}$, is n-absorbing ideal of $\mathbb{Z}$.

Every prime ideal is n-absorbing but the converse is not true. For example, let $n = 3$. Then the ideals $I_1 = 27\mathbb{Z}$, $I_2 = 18\mathbb{Z}$, and $I_3 = 30\mathbb{Z}$ are 3-absorbing, but not prime ideals of $\mathbb{Z}$.

Let θ be a congruence relation on a ring R. Then a subset A of R is called a lower rough n-absorbing ideal of R if $\theta_-(A)$ is a n-absorbing ideal of R. Similarly we can define upper rough n-absorbing ideal.

Theorem 3.1. *Let θ be a complete congruence relation on R and P an n-absorbing ideal of R such that $\theta^-(P) \neq R$, then P is an upper rough n-absorbing ideal of R.*

Proof. Since P is an ideal of R, by Theorem 2.4, we know that $\theta^-(P)$ is an ideal of R. Now, let $x_1 \cdots x_{n+1} \in \theta^-(P)$, then $[x_1 \cdots x_{n+1}]_\theta \cap P \neq \emptyset$. Since θ is complete, so there exist $a_1 \in [x_1]_\theta, ..., a_{n+1} \in [x_{n+1}]_\theta$ such that $a_1 \cdots a_{n+1} \in P$. Since P is an n-absorbing ideal, there exist $i \in \{1, ..., n+1\}$ such that $a_1 \cdots a_{i-1}a_{i+1} \cdots a_{n+1} \in P$. Since $a_1 \cdots a_{i-1}a_{i+1} \cdots a_{n+1} \in [x_1 \cdots x_{i-1}x_{i+1} \cdots x_{n+1}]_\theta$, it follows that

$$[x_1 \cdots x_{i-1}x_{i+1} \cdots x_{n+1}]_\theta \cap P \neq \emptyset.$$

Hence, we have $x_1 \cdots x_{i-1}x_{i+1} \cdots x_{n+1} \in \theta^-(P)$. Therefore, $\theta^-(P)$ is a n-absorbing ideal of R. ∎

Theorem 3.2. *Let θ be a full congruence relation on R and P be an n-absorbing ideal of R. If $\theta_-(P)$ is a nonempty set, then P is a lower rough n-absorbing ideal of R.*

Proof. The proof is straightforward by Theorem 2.5. ∎

Definition 3.3. A proper ideal I of R is n-absorbing primary if whenever $x_1, ..., x_{n+1} \in R$, $x_1 \cdots x_{n+1} \in I$ and $x_1 \cdots x_n \notin I$, there exist $i \in \{1, ..., n\}$ such that
$$(x_{n+1}x_1 \cdots x_{i-1}x_{i+1} \cdots x_n)^k \in I,$$
for some positive integer k.

Let θ be a congruence relation on a ring R. Then a subset A of R is called a lower rough n-absorbing primary ideal of R if $\theta_-(A)$ is an n-absorbing primary ideal of R. Similarly we can define upper rough n-absorbing primary ideal.

We call Q a rough n-absorbing (n-absorbing primary) ideal of R if it is both a lower and an upper rough n-absorbing (n-absorbing primary) ideal of R.

Theorem 3.4. *Let θ be a complete congruence relation on R and Q an n-absorbing primary ideal of R such that $\theta^-(Q) \neq R$. Then, Q is an upper rough n-absorbing primary ideal of R.*

Proof. Let $x_1 \cdots x_{n+1} \in \theta^-(Q)$ and $x_1 \cdots x_n \notin \theta^-(Q)$. Then $[x_1 \cdots x_{n+1}]_\theta \cap Q \neq \emptyset$. Since θ is complete, so there exist $a_1 \in [x_1]_\theta, ..., a_n \in [x_n]_\theta$ and $a_{n+1} \in [x_{n+1}]_\theta$, $a_1 \cdots a_{n+1} \in Q$. Since Q is an n-absorbing primary ideal, it follows that there exist $i \in \{1, ..., n\}$ such that $(a_{n+1}a_1 \cdots a_{i-1}a_{i+1} \cdots a_n)^k \in Q$ for some positive integer k. Since θ is complete, it follows that
$$a_{n+1}a_1 \cdots a_{i-1}a_{i+1} \cdots a_n \in [x_{n+1}x_1 \cdots x_{i-1}x_{i+1} \cdots x_n]_\theta.$$
We get
$$(a_{n+1}a_1 \cdots a_{i-1}a_{i+1} \cdots a_n)^k \in [(x_{n+1}x_1 \cdots x_{i-1}x_{i+1} \cdots x_n)^k]_\theta.$$
Hence,
$$[(x_{n+1}x_1 \cdots x_{i-1}x_{i+1} \cdots x_n)^k]_\theta \cap Q \neq \emptyset,$$
which implies that $(x_{n+1}x_1 \cdots x_{i-1}x_{i+1} \cdots x_n)^k \in \theta^-(Q)$. Therefore, $\theta^-(Q)$ is an n-absorbing primary ideal of R. ∎

Theorem 3.5. *Let θ be a full congruence relation on R and Q be an n-absorbing primary ideal of R. If $\theta_-(Q)$ is a nonempty set, then Q is a lower rough n-absorbing primary ideal of R.*

Proof. The proof is straightforward by Theorem 2.5. ∎

Theorem 3.6. *Let φ be an epimorphism of a ring R_1 to a ring R_2 and let θ_2 be a full congruence relation on R_2. Let A be a subset of R_1.*

If $\theta_1 = \{(a, b) \in R_1 \times R_1 \mid (\varphi(a), \varphi(b)) \in \theta_2\}$, then

(i) *If θ_2 is complete, $\theta_1^-(A)$ is an n-absorbing ideal of R_1 if and only if $\theta_2^-(\varphi(A))$ is an n-absorbing ideal of R_2.*

(ii) *If θ_2 is complete, $\theta_1^-(A)$ is an n-absorbing primary ideal of R_1 if and only if $\theta_2^-(\varphi(A))$ is an n-absorbing primary ideal of R_2.*

Proof. (i) Assume that $\theta_1^-(A)$ is an n-absorbing ideal of R_1. Suppose that $x_1, ..., x_{n+1} \in R_2$ such that $x_1 \cdots x_{n+1} \in \theta_2^-(\varphi(A))$. Then there exist $a_1, ..., a_{n+1} \in R_1$ such that $\varphi(a_1) = x_1, ..., \varphi(a_{n+1}) = x_{n+1}$. Thus $[\varphi(a_1) \cdots \varphi(a_{n+1})]_{\theta_2} \cap \varphi(A) \neq 0$. Since θ_2 is complete, there exists elements $\varphi(u_1) \in [\varphi(a_1)]_{\theta_2}, ..., \varphi(u_{n+1}) \in [\varphi(a_{n+1})]_{\theta_2}$ such that $\varphi(u_1) \cdots \varphi(u_{n+1}) = \varphi(u_1 \cdots u_{n+1}) \in \varphi(A)$. Then we have $u_1 \in [a_1]_{\theta_1}, ..., u_{n+1} \in [a_{n+1}]_{\theta_1}$ and there exists $c \in A$ such that $\varphi(c)=\varphi(u_1 \cdots u_{n+1})$. Hence $u_1 \cdots u_{n+1} \in [a_1 \cdots a_{n+1}]_{\theta_1}$ and $c \in [u_1 \cdots u_{n+1}]_{\theta_1}$. Thus $[a_1 \cdots a_{n+1}]_{\theta_1} \cap A \neq \emptyset$ which implies that $a_1 \cdots a_{n+1} \in \theta_1^-(A)$. Since $\theta_1^-(A)$ is an n-absorbing ideal of R_1, there exist $i \in \{1, ..., n+1\}$ such that $a_1 \cdots a_{i-1} a_{i+1} \cdots a_{n+1} \in \theta_1^-(A)$. From this and [9, Theorem 2.9], we obtain that $x_1 \cdots x_{i-1} x_{i+1} \cdots x_{n+1} = \varphi(a_1 \cdots a_{i-1} a_{i+1} \cdots a_{n+1}) \in \varphi(\theta_1^-(A)) = \theta_2^-(\varphi(A))$. This means that $\theta_2^-(\varphi(A))$ is an n-absorbing ideal of R_2.

Conversely, assume that $\theta_2^-(\varphi(A))$ is an n-absorbing ideal of R_2. Suppose that $x_1 \cdots x_{n+1} \in \theta_1^-(A)$ for some $x_1, ..., x_{n+1} \in R_1$, then by [9, Theorem 2.9], we obtain that $\varphi(x_1 \cdots x_{n+1}) = \varphi(x_1) \cdots \varphi(x_{n+1}) \in \varphi(\theta_1^-(A)) = \theta_2^-(\varphi(A))$. Since $\theta_2^-(\varphi(A))$ is an n-absorbing ideal of R_2, we have n of the $\varphi(x_i)$'s whose product is in $\in \varphi(\theta_1^-(A))$, then there exists $a \in \theta_1^-(A)$ such that $\varphi(x_1 \cdots x_{i-1} x_{i+1} \cdots x_{n+1}) = \varphi(a)$. Thus $[a]_{\theta_1} \cap A \neq \emptyset$ and $x_1 \cdots x_{i-1} x_{i+1} \cdots x_{n+1} \in [a]_{\theta_1}$. So $[x_1 \cdots x_{i-1} x_{i+1} \cdots x_{n+1}]_{\theta_1} \cap A \neq \emptyset$, which implies $x_1 \cdots x_{i-1} x_{i+1} \cdots x_{n+1} \in \theta_1^-(A)$. This means that $\theta_1^-(A)$ is an n-absorbing ideal of R_1.

(ii) The proof is similar to the proof of (i). ∎

Theorem 3.7. *Let φ be an isomorphism of a ring R_1 to a ring R_2 and let θ_2 be a complete congruence relation on R_2. Let A be a subset of R_1.*
If $\theta_1 = \{(a, b) \in R_1 \times R_1 \mid (\varphi(a), \varphi(b)) \in \theta_2\}$, then

(i) *$\theta_{1-}(A)$ is an n-absorbing ideal of R_1 if and only if $\theta_{2-}(\varphi(A))$ is an n-absorbing ideal of R_2,*

(ii) *$\theta_{1-}(A)$ is an n-absorbing primary ideal of R_1 if and only if $\theta_{2-}(\varphi(A))$ is an n-absorbing primary ideal of R_2.*

Proof. By [9, Theorem 2.9], we have $\varphi(\theta_{1-}(A)) = \theta_{2-}(\varphi(A))$. Now, the proof is similar to the proof of Theorem 3.6. ∎

4. Rough fuzzy n-absorbing(n-absorbing primary) ideals

Definition 4.1.

(i) A fuzzy ideal ρ is called n-absorbing if for all $x_1, ..., x_{n+1} \in R$, there exist $i \in \{1, ..., n+1\}$ such that $\rho(x_1 \cdots x_{n+1}) = \rho(x_1 \cdots x_{i-1} x_{i+1} \cdots x_{n+1})$.

(ii) A fuzzy ideal ρ is called n-absorbing primary if for all $x_1, ..., x_{n+1} \in R$, there exist $i \in \{1, ..., n+1\}$ such that $\rho(x_1 \cdots x_{n+1}) = \rho((x_1 \cdots x_{i-1} x_{i+1} \cdots x_{n+1})^k)$ for some integer k.

Example 2. Let I be any n-absorbing ideal of R and let $\alpha \leq \beta \neq 0$ be numbers in $[0, 1]$ then the fuzzy subsets μ defined by

$$\mu(x) = \begin{cases} \beta & \text{if } x \in I \\ \alpha & \text{otherwise} \end{cases}$$

is a fuzzy n-absorbing ideal.

Theorem 4.2. *Let μ be a fuzzy ideal of a commutative ring with identity R. Then*

(i) *If μ is an n-absorbing fuzzy ideal then the set*

$$\{\alpha_i = \mu(x_1 \cdots x_{i-1}x_{i+1} \cdots x_{n+1})/x_i \in R,\ i \in \{1, ..., n+1\}\}$$

is totally ordered set with the least element $\mu(1)$ and the greatest element $\mu(0)$.

(ii) *μ is n-absorbing if and only if μ_t is n-absorbing for all $t > \mu(1)$. For $t = \mu(1)$, $\mu_t = R$.*

Proof. (i) If $x \in R$, then $\mu(0) = \mu(x - x) \geq \mu(x) = \mu(x.1) \geq \mu(1)$. Further, if $\alpha_i \in [0, 1]$ with $\mu(x_1 \cdots x_{i-1}x_{i+1} \cdots x_{n+1}) = \alpha_i$, then $\mu(x_1 \cdots x_{n+1}) = \alpha_i$ for some $i \in \{1, ..., n+1\}$, then $\alpha_i = \mu(x_1 \cdots x_{n+1}) \geq \mu(x_1 \cdots x_{j-1}x_{j+1} \cdots x_{n+1}) = \alpha_j$ for all $j \in \{1, ..., n+1\}$.

(ii) Let μ_t be n-absorbing for all $t > \mu(1)$ and $x_1, ..., x_{n+1} \in R$. If $\mu(x_1 \cdots x_{n+1}) = \mu(1)$, then as $\mu(1)$ is the smallest element of $\mu(R)$, $\mu(x_1 \cdots x_{i-1}x_{i+1} \cdots x_{n+1}) = \mu(x_1 \cdots x_{n+1})$ for all $i \in \{1, ..., n+1\}\}$. If $\mu(x_1 \cdots x_{n+1}) = \alpha > \mu(1)$, then μ_t being n-absorbing, there is $i \in \{1, ..., n+1\}\}$ such that $x_1 \cdots x_{i-1}x_{i+1} \cdots x_{n+1} \in \mu_t$. Accordingly, $\mu(x_1 \cdots x_{i-1}x_{i+1} \cdots x_{n+1}) = \mu(x_1 \cdots x_{n+1})$ for some $i \in \{1, ..., n+1\}\}$. The converse is obvious. ■

The following proposition is straightforward.

Theorem 4.3. *Let Q be an ideal of R.*

(i) *The characteristic function χ_Q is an n-absorbing primary fuzzy ideal if and only if Q is an n-absorbing primary ideal.*

(ii) *If ϱ is an n-absorbing primary fuzzy ideal, then $\varrho_t (\neq \emptyset)$ is an n-absorbing primary ideal for all $0 \leq t \leq 1$.*

(iii) *Every n-absorbing fuzzy ideal is n-absorbing primary.*

Theorem 4.4. *Let φ be an epimorphism of a ring R_1 to a ring R_2 and let θ_2 be a complete full congruence relation on R_2. Let μ be a fuzzy subset of R_1.*

If $\theta_1 = \{(a, b) \in R_1 \times R_1 \mid (\varphi(a), \varphi(b)) \in \theta_2\}$, then

(i) *$\theta_1^-(\mu)$ is a fuzzy ideal of R_1 if and only if $\theta_2^-(\varphi(\mu))$ is a fuzzy ideal of R_2.*

(ii) *$\theta_1^-(\mu)$ is a fuzzy n-absorbing ideal of R_1 if and only if $\theta_2^-(\varphi(\mu))$ is a fuzzy n-absorbing ideal of R_2.*

(iii) *$\theta_1^-(\mu)$ is a fuzzy n-absorbing primary ideal of R_1 if and only if $\theta_2^-(\varphi(\mu))$ is a fuzzy n-absorbing primary ideal of R_2.*

Moreover, if φ is one-to-one, then we have

(iv) *$\theta_{1-}(\mu)$ is a fuzzy ideal of R_1 if and only if $\theta_2^-(\varphi(\mu))$ is a fuzzy ideal of R_2.*

(v) *$\theta_{1-}(\mu)$ is a fuzzy n-absorbing ideal of R_1 if and only if $\theta_2^-(\varphi(\mu))$ is a fuzzy n-absorbing ideal of R_2.*

(vi) *$\theta_{1-}(\mu)$ is a fuzzy n-absorbing primary ideal of R_1 if and only if $\theta_2^-(\varphi(\mu))$ is a fuzzy n-absorbing primary ideal of R_2.*

Proof. (i) By [9, Theorem 3.6], we obtain that $\theta_1^-(\mu)$ is a fuzzy ideal of R_1 if and only if $(\theta_1^-(\mu))_t^s$ is, if it is nonempty, an ideal of R_1 for every $t \in [0,1]$. Using [9, Theorem 3.7], we have $(\theta_1^-(\mu))_t^s = (\theta_1^-(\mu_t^s))$. By Theorem 4.2, we obtain that $\theta_1^-(\mu_t^s)$ is an ideal of R_1 if and only if $\theta_2^-(\varphi(\mu_t^s))$ is an ideal of R_2. It is clear that $\varphi(\mu_t^s) = (\varphi(\mu))_t^s$. From this and [9, Theorem 3.6], we have

$$\theta_2^-(\varphi(\mu_t^s)) = \theta_2^-((\varphi(\mu))_t^s) = (\theta_2^-(\varphi(\mu)))_t^s$$

By [9, Theorem 3.6], we obtain $(\theta_2^-(\varphi(\mu)))_t^s$ is an ideal of R_2 for every $t \in [0,1]$ if and only if $\theta_2^-(\varphi(\mu))$ is a fuzzy ideal of R_2.

The proof of other parts is similar to the proof of (i). ∎

Theorem 4.5. *Let μ be a fuzzy n-absorbing ideal of R.*

(i) *If θ is a complete congruence relation on R and $\theta_-(\mu) \neq \emptyset$, then μ is a lower rough fuzzy n-absorbing ideal of R.*

(ii) *If θ is a complete congruence relation on R, then μ is an upper rough fuzzy n-absorbing ideal of R.*

Proof. (i) Since μ is a fuzzy n-absorbing ideal, by Theorem 4.2, we know that $\mu_t(t > \mu(1))$ is, if it is non-empty, an n-absorbing ideal of R. Then by Theorem 3.2, we obtain that $\theta_-(\mu_t)$, if it is non-empty, is an n-absorbing ideal of R. From this and [9, Theorem 3.7], we know that $(\theta_-(\mu))_t$ is an n-absorbing ideal of R. Now, by Theorem 4.2, we obtain that $\theta_-(\mu)$ is a fuzzy n-absorbing ideal of R.

(ii) It can be seen in a similar way. ∎

Theorem 4.6. *Let θ be a full congruence relation on R. Then μ is a lower [an upper] rough fuzzy n-absorbing ideal if and only if μ_t, μ_t^s are, if they are nonempty, lower [upper] rough n-absorbing ideals of R for every $t \in [0,1]$.*

Proof. It is straightforward. ∎

5. Conclusions

The rough sets theory is regarded as a generalization of the classical sets theory. A key notion in rough set is an equivalence relation. An equivalence is sometime difficult to be obtained in rearward problems due to vagueness and incompleteness of human knowledge. The combination of fuzzy set and rough set theory lead to various models. The relations between fuzzy sets, rough sets and ring theory have been already considered by the second author in [6], [7], [9]. In the present paper, we introduced the notions of rough n-absorbing (n-absorbing primary) ideals and rough fuzzy n-absorbing (n-absorbing primary) ideals in a commutative ring. We discussed the relations between upper (lower) rough n-absorbing and n-absorbing primary ideals and upper (lower) approximations of their homomorphism images. Also, we proved that a fuzzy set is a rough fuzzy n-absorbing ideal if and only if its level set is a rough n-absorbing ideal.

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References

[1] Anderson, D.F., Badawi, A., *On n-absorbing ideals of commutative rings*, Comm. Algebra, 39 (2011), 1646-1672.

[2] Anderson, D.D., Bataineh, M., *Generalizations of prime ideals*, Comm. Algebra, 36 (1008), 686-696.

[3] Badawi, A., *On 2-absorbing ideals of commutative rings*, Bull. Austral. Math. Soc., 75 (2007), 417-429.

[4] Badawi, A., Tekir, U., Yetkin, E., *On 2-absorbing primary ideals in commutative rings*, Bull. Korean Math. Soc., 51 (4) (2014), 1163-1173.

[5] Biswas, R., Nanda, S., *Rough groups and rough subgroups*, Bull. Polish Acad. Sci. Math., 42 (1994), 251-254.

[6] Davvaz, B., *Roughness in rings*, Inform. Sci., 164 (2004), 147-163.

[7] Davvaz, B., *Roughness based on fuzzy ideals*, Inform. Sci., 176 (2006), 2417-2437.

[8] Jun, Y.B., *Roughness of ideals in BCK-algebras*, Sci. Math. Japonica, 57 (1) (2003), 165-169.

[9] Kazanci, O., Davvaz, B., *On the structure of rough prime (primary) ideals and rough fuzzy prime (primary) ideals in commutative rings*, Information Sciences, 178 (2008), 1343-1354.

[10] Kuroki, N., *Rough ideals in semigroups*, Inform. Sci., 100 (1997), 139-163.

[11] Kuroki, N., Wang, P.P., *The lower and upper approximations in a fuzzy group*, Inform. Sci., 90 (1996), 203-220.

[12] Liu, W.J., *Fuzzy invariant subgroups and fuzzy ideals*, Fuzzy Sets Syst., 8 (1982), 133-139.

[13] Pawlak, Z., *Rough sets*, International Journal of Computer and Information Sciences, 11 (1982), 341-356.

[14] Zadeh, L.A., *Fuzzy sets*, Inform. Control, 8 (1965), 338-353.

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