ROUGH FUZZY GROUPS AND ROUGH SOFT GROUPS

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Abstract. By means of Dubois and Prade's idea, we introduce the concept of rough fuzzy normal subgroups and investigate some related properties. Combining rough sets and soft sets, rough soft normal subgroups with respect to a normal subgroup of groups in Pawlak approximation spaces are introduced. Moreover, we investigate the relationships between lower and upper rough soft normal subgroups with respect to normal subgroups. Finally, decision making methods in rough soft sets are put forward and some related algebraic and applied examples are given.

Keywords: rough set; soft set; rough fuzzy normal subgroup; rough soft normal subgroup.

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1. Introduction

As far as known that rough set theory was firstly proposed by Pawlak [15] in 1982. In general, the Pawlak approximation operators are defined by an equivalence relation. However, the equivalence relations in Pawlak rough sets are too restrictive in applications. Based on the reason, several researcher put forward some general models (see e.g. [11], [19]). In particular, Zhang [22] researched the union and intersection operations of rough sets based on various approximations spaces. Later, by combining rough sets and fuzzy sets, the concepts of fuzzy rough sets and rough fuzzy sets were introduced by Dubois and Prade [6] in 1990. Nowadays, this theory has been applied successfully to many areas, such as machine learning, information sciences, and knowledge discovery, and so on.

At the same time, some researchers applied this theory to algebraic structures. In 1994, Biswas et al. [3] applied rough set theory to groups. Moreover, Davvaz [5] gave the notion of rough ideals with respect to an ideal of rings in 2004. After that, rough set theory in different algebraic structures was investigated by many

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authors [10, 18]. In particular, by means of the idea of Dubois and Prade [6], Zhan [20] studied rough fuzzy (fuzzy rough) strong h-ideals of hemirings and researched some important properties of them.

In 1999, Molodtsov [14] firstly introduced the concept of soft set as a new mathematical tool for dealing with uncertainties. Based on this novel idea, the research on the soft set theory is developing rapidly. The new operations on soft sets were discussed and the applications of soft set theory in decision making problems were researched by Maji et al. [12], [13]. Later, Feng [9] researched the generalized uni-int decision making schemes based on choice value soft set and Alcantud [2] put forward a novel algorithm for fuzzy soft set based decision making from multiobserver input parameter data set. Aktaş et al. [1], [17] gave the definitions of soft groups and normalistic soft groups and some related properties were derived. Moreover, soft semirings and some related notions were put forward in [7]. In particular, by combining Pawlak rough sets and soft sets, the rough soft sets were proposed by Feng et al. [8]. According to Feng's idea in [8], rough soft set theory was firstly applied to hemirings and some vital results and properties were investigated by Zhan [21].

This paper is organized as follows: In section 2, some concepts and results on groups, rough sets and soft sets were recalled. Based on the ideal of Zhan [20], we introduce the concept of rough fuzzy normal subgroups and obtain some related properties in section 3. In section 4, we give the definition of rough soft normal subgroups and investigate the relationship between lower and upper approximation operations. In section 5, we give the decision making method of rough soft groups.

2. Preliminaries

Throughout this paper, G is a group. A fuzzy subset μ of G is called a fuzzy subgroup of G if $\mu(xy) \geq \min\{\mu(x)$ and $\mu(y)\}$ and $\mu(x^{-1}) \geq \mu(x)$ for all $x, y \in G$. A fuzzy subgroup μ of G is said to be a fuzzy normal subgroup of G if $\mu(xyx^{-1}) \geq \mu(y)$ for all $x, y \in G$ [4, 16].

Let μ be a fuzzy set of G and $t \in [0,1]$. Then the sets $\mu_t = \{x \in G | \mu(x) \geq t\}$ and $\mu_t^s = \{x \in G | \mu(x) > t\}$ are called t-level subset and strong t-level subset of μ, respectively.

Proposition 2.1 [4] *A fuzzy subset μ of a group G is a fuzzy normal subgroup of G if and only if for all $t \in [0,1]$, μ_t (μ_t^s) is a normal subgroup of G.*

Molodtsov [14] defined the soft set in the following way: A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$ for all $A \subset E$, where U is an initial universe, E is a set of parameters and $P(U)$ is the power set of U. Let (F, A) and (H, B) be two soft sets over U. The inclusion symbol "$\subseteq$" of (F, A) and (H, B), denoted by $(F, A) \subseteq (H, B)$, is defined as $F(x) \subseteq H(x)$ for all $x \in A \subseteq B$.

Definition 2.2 [1, 17] (1) Let (F, A) be a soft set over group G. Then (F, A) is said to be a soft group over G if and only if $F(x) < G$ for all $x \in A$.

(2) Let G be a group and (F, A) be a non-null soft set over G. Then (F, A) is called a normalistic soft group over G if $F(x)$ is a normal subgroup of G for all $x \in \mathrm{Supp}(F, A)$.

Definition 2.3 (1) Let $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (H, B)$ be two soft sets over a common group G, then the "multiplication" of $\mathfrak{S}$ and $\mathfrak{T}$, denoted by $\mathfrak{S} \cdot \mathfrak{T}$ is defined by $\mathfrak{S} \cdot \mathfrak{T} = (F, A) \cdot (H, B) = (J, A * B)$, where $J(x, y) = F(x) \cdot H(y)$ for all $(x, y) \in A * B$.

(2) Let M be a subset of G and $\mathfrak{S} = (F, A)$ a soft set over G. Then the "multiplication" of M and $\mathfrak{S}$, denoted by $M \cdot \mathfrak{S}$ is defined by $M \cdot \mathfrak{S} = M \cdot F(x)$ for all $x \in A$.

Definition 2.4 [6] Let R be an equivalence relation on the universe U and μ a fuzzy set of U. Then we define the two fuzzy sets $\underline{R}(\mu)$ and $\overline{R}(\mu)$ as follows:

$$\underline{R}(\mu)(x) = \wedge\{\mu(y) | y \in [x]_R\}$$

and

$$\overline{R}(\mu)(x) = \vee\{\mu(y) | y \in [x]_R\},$$

for all$x \in U$.

The fuzzy sets $\underline{R}(\mu)$ and $\overline{R}(\mu)$ are called, resp., the lower and upper approximations of μ. Moreover, $R(\mu) = (\underline{R}(\mu), \overline{R}(\mu))$ is called a rough fuzzy set of U if $\underline{R}(\mu) \neq \overline{R}(\mu)$.

Definition 2.5 [8] Let R be an equivalence relation on the universe U, (U, R) a Pawlak approximation space and $\mathfrak{S} = (F, A)$ a soft set over U. The lower and upper rough approximations of $\mathfrak{S}$ w.r.t. (U, R) are defined by:

$\underline{R}(\mathfrak{S}) = (\underline{F}, A)$ and $\overline{R}(\mathfrak{S}) = (\overline{F}, A)$, which are soft sets over U with $\underline{F}(x) = \underline{R}(F(x)) = \{y \in U | [y]_R \subseteq F(x)\}$ and $\overline{F}(x) = \overline{R}(F(x)) = \{y \in U | [y]_R \cap F(x) \neq \emptyset\}$, for all $x \in A$.

The soft sets $\underline{R}(\mathfrak{S})$ $(\overline{R}(\mathfrak{S}))$ is called a lower (upper) approximation. Moreover, $\mathfrak{S}$ is called a rough soft set of U, if $\underline{R}(\mathfrak{S}) \neq \overline{R}(\mathfrak{S})$, for all $x \in A$.

3. Rough fuzzy normal subgroups

In this section, we introduce the concept of rough fuzzy normal subgroups of groups and investigate some related properties.

Let N be a normal subgroup of G. Then we define a relation $\equiv_N$ by $x \equiv_N y \Leftrightarrow xy^{-1} \in N$. It is clear that the relation $\equiv_N$ is a congruence relation on G. Let $[x]_N$ denote the equivalence class of x w.r.t. N. So it is easy to obtain the following lemma.

Lemma 3.1 *Let N be a normal subgroup of G. If $x, y \in G$, then*

(1) $x \in [y]_N$ *if and only if* $x \in yN$,

(2) $[x]_N \cdot [y]_N = [xy]_N$.

Proof. (1) If $x \in [y]_N$, then there exists $a \in N$ such that $xy^{-1} = a$. So $x = ay \in Ny = yN$. Conversely, if $x \in yN$, then there exists $a \in N$ such that $x = ya$, that is $xy^{-1} = a \in N$, hence $x \in [y]_N$.

(2) If $z \in [x]_N \cdot [y]_N$, then there exists $a \in [x]_N$ and $b \in [y]_N$ such that $z = ab$. Thus we have $z = ab \in (xN)(yN) = (xy)N$, hence $z \in [xy]_N$. Conversely, if $z \in [xy]_N$, then $z \in (xy)N = (xN)(yN)$. Thus there exists $a \in xN$ and $b \in yN$, that is, $a \in [x]_N$ and $b \in [y]_N$ such that $z = ab$, and so $z = ab \in [x]_N \cdot [y]_N$. Consequently, $[x]_N \cdot [y]_N = [xy]_N$. ∎

By means of the idea of Lemma 3.1, we introduce the concepts of rough fuzzy normal subgroups of groups.

Definition 3.2 Let N be a normal subgroup of G and μ a fuzzy set of G. Then we define two fuzzy sets $\underline{Apr}_N(\mu)$ and $\overline{Apr}_N(\mu)$ as follows:

$\underline{Apr}_N(\mu)(x) = \bigwedge_{y \in [x]_N} \mu(y)$ and $\overline{Apr}_N(\mu)(x) = \bigvee_{y \in [x]_N} \mu(y)$, for all $x \in G$.

The fuzzy sets $\underline{Apr}_N(\mu)$ and $\overline{Apr}_N(\mu)$ are called, resp., the lower and upper approximations of μ w.r.t. N of G. Moreover, $Apr_N(\mu) = (\underline{Apr}_N(\mu), \overline{Apr}_N(\mu))$ is called a rough fuzzy set w.r.t. N of G if $\underline{Apr}_N(\mu) \neq \overline{Apr}_N(\mu)$.

Definition 3.3 Let N be a normal subgroup of G and μ a fuzzy set of G. Then μ is called a lower (upper) rough fuzzy normal subgroup w.r.t. N of G if $\underline{Apr}_N(\mu)$ ($\overline{Apr}_N(\mu)$) is a fuzzy normal subgroup of G. Moreover, $Apr_N(\mu) = (\underline{Apr}_N(\mu), \overline{Apr}_N(\mu))$ is called a rough fuzzy normal subgroup w.r.t. N of G if both $\underline{Apr}_N(\mu)$ and $\overline{Apr}_N(\mu)$ are fuzzy fuzzy normal subgroups of G.

Example 3.4 Consider $G = \{\pm 1, \pm i\}$ be the group where $i^2 = -1$. Let $N = \{1, -1\}$, it is easy to find that N is a normal subgroup of G. Moreover, $[1]_N = N$, $[i]_N = \{i, -i\}$.

Define a fuzzy set μ of G by $\mu(1) = 0.7$ and $\mu(-1) = \mu(i) = \mu(-i) = 0.5$.

By calculations, we have

$$\underline{Apr}_N(\mu) = \frac{0.5}{1} + \frac{0.5}{-1} + \frac{0.5}{i} + \frac{0.5}{-i}$$

and

$$\overline{Apr}_N(\mu) = \frac{0.7}{1} + \frac{0.7}{-1} + \frac{0.5}{i} + \frac{0.5}{-i}.$$

This implies that $Apr_N(\mu)$ is a rough fuzzy normal subgroup w.r.t. N of G.

Proposition 3.5 *Let N and K be any two normal subgroups of G and μ a fuzzy set of G. If $N \subseteq K$, then*

(1) $\underline{Apr}_N(\mu) \supseteq \underline{Apr}_K(\mu)$,

(2) $\overline{Apr}_N(\mu) \subseteq \overline{Apr}_K(\mu)$.

Proof. For any $y \in [x]_N$, since $N \subseteq K$, then we have $y \in [x]_K$, by the Definition 3.2, hence $\underline{Apr}_N(\mu) \supseteq \underline{Apr}_K(\mu)$. Similarly, we can get $\overline{Apr}_N(\mu) \subseteq \overline{Apr}_K(\mu)$. ∎

We can get the following corollary directly from Proposition 3.5

Corollary 3.6 *Let N and K be any two normal subgroups of G and μ a fuzzy set of G. If $N \subseteq K$, then*

(1) $\underline{Apr}_{N\cap K}(\mu) \supseteq \underline{Apr}_N(\mu) \supseteq \underline{Apr}_{N\cdot K}(\mu)$,

(2) $\overline{Apr}_{N\cap K}(\mu) \subseteq \overline{Apr}_N(\mu) \subseteq \overline{Apr}_{N\cdot K}(\mu)$.

Proof. Since N and K are any two normal subgroups of G, so are $N \cap K$ and NK, and it is easy to obtain the inclusion relation: $N \cap K \subseteq N \subseteq N \cdot K$. ∎

Proposition 3.7 *Let N be a normal subgroup of G and μ, ν any two fuzzy sets of G. If $\mu \subseteq \nu$, then*

(1) $\underline{Apr}_N(\mu) \subseteq \underline{Apr}_N(\nu)$,

(2) $\overline{Apr}_N(\mu) \subseteq \overline{Apr}_N(\nu)$.

Proof. For any $x \in G$, by hypothesis, we have $\mu(x) \leq \nu(x)$. So $\underline{Apr}_N(\mu)(x) = \bigwedge_{y\in[x]_N} \mu(y) \leq \bigwedge_{y\in[x]_N} \nu(y) = \underline{Apr}_N(\nu)(x)$. Similarly, we can get $\overline{Apr}_N(\mu) \subseteq \overline{Apr}_N(\nu)$. This completes the proof. ∎

Now, we give the (strong) level subset of lower and upper rough approximations of a fuzzy set μ w.r.t. N of groups.

Theorem 3.8 *Let N be a normal subgroup of G. If μ is a fuzzy set of G and $t \in [0,1]$. Then*

(1) $(\underline{Apr}_N(\mu))_t = \underline{Apr}_N(\mu_t)$,

(2) $(\overline{Apr}_N(\mu))_t^s = \overline{Apr}_N(\mu_t^s)$.

Proof. (1) For any $x \in G$,we have

$$\begin{aligned} x \in (\underline{Apr}_N(\mu))_t &\Leftrightarrow \underline{Apr}_N(\mu)(x) \geq t \\ &\Leftrightarrow \bigwedge_{y\in[x]_N} \mu(y) \geq t \\ &\Leftrightarrow \forall y \in [x]_N, \mu(y) \geq t \\ &\Leftrightarrow [x]_N \subseteq \mu_t \\ &\Leftrightarrow x \in \underline{Apr}_N(\mu_t). \end{aligned}$$

(2)

$$\begin{aligned} x \in (\overline{Apr}_N(\mu))_t^s &\Leftrightarrow \overline{Apr}_N(\mu)(x) > t \\ &\Leftrightarrow \bigvee_{y\in[x]_N} \mu(y) > t \\ &\Leftrightarrow \exists y \in [x]_N, \mu(y) > t \\ &\Leftrightarrow [x]_N \cap \mu_t^s \neq \emptyset \\ &\Leftrightarrow x \in \overline{Apr}_N(\mu_t^s). \end{aligned}$$

This completes the proof. ∎

Lemma 3.9 *Let N and A be two normal subgroups of G. Then*

(1) $\overline{Apr}_N(A) = N \cdot A$ *and* $\overline{Apr}_N(A)$ *is a normal subgroup of* G*;*

(2) *if* $\underline{Apr}_N(A) \neq \emptyset$*, then* $\underline{Apr}_N(A) = A$ *and* $\underline{Apr}_N(A)$ *is a normal subgroup of* G*.*

Proof. (1) Let $\forall x \in \overline{Apr}_N(A)$. Then $xN \cap A \neq \emptyset$, so there exists $a \in G$ such that $a \in xN \cap A$, that is, $x \in aN$ and $a \in A$, hence $x \in A \cdot N = N \cdot A$.

Conversely, if $x \in N \cdot A$. Then there exists $a \in N$ and $b \in A$ such that $x = ab$, hence $b \in xN$ and so $b \in xN \cap A$, that is, $x \in \overline{Apr}_N(A)$. Therefore, $\overline{Apr}_N(A) = N \cdot A$. Since N and A are two normal subgroups of G, so $\overline{Apr}_N(A)$ is a normal subgroup of G.

(2) By hypothesis, if $\underline{Apr}_N(A) \neq \emptyset$, so there exists $s \in \underline{Apr}_N(A) \subseteq A$, that is $aN \subseteq A$. Thus $\forall y \in N$, $\exists b \in A$ such that $b = ay$. Since A is a normal subgroup of G, we have $y \in A$ and so $N \subseteq A$. And then, for all $b \in A$, we have $bN \subseteq A$, this implies that $b \in \underline{Apr}_N(A)$, so $A \subseteq \underline{Apr}_N(A)$, hence $\underline{Apr}_N(A) = A$ and so $\underline{Apr}_N(A)$ is a normal subgroup of G. This completes the proof. ■

Theorem 3.10 Let N be a normal subgroup of G. If μ is a fuzzy normal subgroup of G, then

(1) $\overline{Apr}_N(\mu)$ is a fuzzy normal subgroup of G,

(2) if $\underline{Apr}_N(\mu) \neq \emptyset$, then $\underline{Apr}_N(\mu)$ is a fuzzy normal subgroup of G.

Proof. (1) Let μ be a fuzzy normal subgroup of G. For any $t \in [0, 1]$, by Theorem 3.8 (2), $(\overline{Apr}_N(\mu))_t^s = \overline{Apr}_N(\mu_t^s)$. By Proposition 2.1, we know that μ_t^s is a normal subgroup of G. Hence, by Lemma 3.9 (1), $\overline{Apr}_N(\mu_t^s)$ is a normal subgroup of G. Then by Proposition 2.1, $\overline{Apr}_N(\mu)$ is a fuzzy normal subgroup of G.

(2) Since $\underline{Apr}_N(\mu) \neq \emptyset$, by Theorem 3.8 (1), there exists $t \in [0, 1]$ such that $(\underline{Apr}_N(\mu))_t = \underline{Apr}_N(\mu_t)$. Let t be any value that fulfills the above property. Then it is clear that $\mu_t \neq \emptyset$, and we know from Proposition 2.1 that μ_t is a normal subgroup of G. Hence, by Lemma 3.9 (2), $(\underline{Apr}_N(\mu))_t$ is a normal subgroup of G. Then, by Proposition 2.1, $\underline{Apr}_N(\mu)$ is a fuzzy normal subgroup of G. This completes the proof. ■

By Theorem 3.10, we can obtain the following corollary.

Corollary 3.11 *Let N be a normal subgroup of G. If μ is a fuzzy normal subgroup of G and $\underline{Apr}_N(\mu) \neq \emptyset$, then $Apr_N(\mu)$ is a rough fuzzy normal subgroup of G.*

4. Rough soft normal subgroups

Definition 4.1 Let N be a normal subgroup of G, (G, N) a Pawlak approximation space and $\mathfrak{S} = (F, A)$ a soft set over G. The lower and upper rough approximations of $\mathfrak{S} = (F, A)$ w.r.t. (G, N) are denoted by:

$$\underline{Apr}_N(\mathfrak{S}) = (\underline{F}_N, A) \quad and \quad \overline{Apr}_N(\mathfrak{S}) = (\overline{F}_N, A),$$

which are soft sets over G with $\underline{F}_N(x) = \underline{Apr}_N(F(x)) = \{y \in G | yN \subseteq F(x)\}$ and $\overline{F}_N(x) = \overline{Apr}_N(F(x)) = \{y \in G | yN \cap F(x) \neq \emptyset\}$, for all $x \in A$.

(1) $\underline{Apr}_N(\mathfrak{S}) = \overline{Apr}_N(\mathfrak{S})$, the soft set $\mathfrak{S}$ is said to be definable;

(2) $\underline{Apr}_N(\mathfrak{S}) \neq \overline{Apr}_N(\mathfrak{S})$, $\underline{Apr}_N(\mathfrak{S})$ $(\overline{Apr}_N(\mathfrak{S}))$ is called a lower (upper) rough soft normal subgroup w.r.t. N over G, if $\underline{F}_N(x)$ $(\overline{F}_N(x))$ is a normal subgroup of G, for all $x \in \mathrm{Supp(F, A)}$. Moreover, $\mathfrak{S}$ is called a rough soft normal subgroup w.r.t. N over G, if $\underline{F}_N(x)$ and $\overline{F}_N(x)$ are normal subgroups of G, for all $x \in \mathrm{Supp(F, A)}$.

Example 4.2 Let $G = \{1, 3, 5, 7\}$ be a group of non-negative integers module 8 with usual multiplication. Let $N = \{1, 3\}$, then N is a subgroup of G. Define a soft set $\mathfrak{S} = (F, A)$ over G, $A = \{a, b\}$, where $F(a) = G$, $F(b) = \{1, 3, 5\}$. By calculations, we have $\underline{F}_N(a) = G$, $\underline{F}_N(b) = \{1, 3\}$, $\overline{F}_N(a) = G$, $\overline{F}_N(b) = G$. This shows that $\underline{F}_N(x)$ and $\overline{F}_N(x)$ are normal subgroups of G for all $x \in \mathrm{Supp(F, A)}$. So $\mathfrak{S}$ is a rough soft normal subgroup w.r.t. N over G.

Lemma 4.3 *Let N be a normal subgroup of G and $\mathfrak{S} = (F, A)$ a soft set over G. Then $\overline{Apr}_N(\mathfrak{S}) = N \cdot \mathfrak{S}$.*

Proof. Let y be any element of $\overline{F}_N(x)$. Then $yN \cap F(x) \neq \emptyset$, $\forall x \in A$, and so there exists $a \in G$ such that $a \in yN \cap F(x)$, that is, $y \in aN$ and $b \in F(x)$, hence $y \in N \cdot F(x)$.

Conversely, if $y \in N \cdot F(x)$, then there exists $a \in N$ and $b \in F(x)$ such that $y = ab$, hence $b \in yN$ and so $b \in yN \cap F(x)$, that is, $y \in \overline{F}_N(x)$. Therefore, $\overline{F}_N(x) = N \cdot F(x)$, $\forall x \in A$, that is, $\overline{Apr}_N(\mathfrak{S}) = N \cdot \mathfrak{S}$. ∎

Theorem 4.4 *Let N be a normal subgroup of G and $\mathfrak{S} = (F, A)$ a normalistic soft group over G. Then*

(1) $\mathfrak{S}$ *is an upper rough soft normal subgroup w.r.t. N over G,*

(2) $\underline{Apr}_N(\mathfrak{S}) \neq \emptyset \Leftrightarrow \underline{Apr}_N(\mathfrak{S}) = \mathfrak{S}$ *and $\mathfrak{S}$ is a lower rough soft normal subgroup w.r.t. N over G.*

Proof. (1) By Lemma 4.3, we can obtain easily.

(2) By the assumption, if $\underline{Apr}_N(\mathfrak{S}) \neq \emptyset$, in other words, for all $x \in \mathrm{Supp(F, A)}$, there exists $a \in \overline{F}_N(x) \subseteq F(x)$, that is $aN \subseteq F(x)$. Thus for any $y \in N$, $\exists b \in F(x)$ such that $ay = b$. Since $\mathfrak{S}$ is a normalistic soft group over G, we have $y \in F(x)$ and so $N \subseteq F(x)$. And then, for $\forall c \in F(x)$, we have $cN \subseteq F(x)$, this implies that $c \in \overline{F}_N(x)$, so $F(x) \subseteq \overline{F}_N(x)$, hence $F(x) = \overline{F}_N(x)$, $\forall x \in \mathrm{Supp(F, A)}$, that is $\underline{Apr}_N(\mathfrak{S}) = \mathfrak{S}$.

Conversely, if $\underline{Apr}_N(\mathfrak{S}) = \mathfrak{S}$, it is clear that $\underline{Apr}_N(\mathfrak{S}) \neq \emptyset$. This implies that $\underline{Apr}_N(\mathfrak{S}) \neq \emptyset \Rightarrow \underline{Apr}_N(\mathfrak{S}) = \mathfrak{S} \Rightarrow \underline{Apr}_N(\mathfrak{S}) \neq \emptyset$.

Since $\mathfrak{S} = (F, A)$ is a normalistic soft group over G, so $\mathfrak{S}$ is a lower rough soft normal subgroup w.r.t. N over G. This completes the proof. ∎

By Theorem 4.4, we have the following corollary.

Corollary 4.5 *Let N be a normal subgroup of G and $\mathfrak{S} = (F, A)$ a normalistic soft group over G such that $\underline{Apr}_N(\mathfrak{S}) \neq \emptyset$. then $\mathfrak{S}$ is a rough soft normal subgroup w.r.t. N over G.*

The following example shows that $\mathfrak{S}$ does not need to be a normalistic soft group over G if $\mathfrak{S}$ is a rough soft normal subgroup over G.

Example 4.6 In Example 4.2, we can find that $\underline{F}_N(x)$ and $\overline{F}_N(x)$ are normal subgroups of G for all $x \in \mathrm{Supp(F, A)}$, so $\mathfrak{S}$ is a rough soft normal subgroup over G, but $\mathfrak{S}$ is not a normalistic soft group over G.

Theorem 4.7 *Let N be a normal subgroup of G. If $\mathfrak{S}$ is a lower (resp. upper) rough soft normal subgroup over G, then $\underline{Apr}_N(\mathfrak{S})$ (resp. $\overline{Apr}_N(\mathfrak{S})$) is a rough soft normal subgroup over G.*

Proof. Assume that $\mathfrak{S}$ is a lower rough soft normal subgroup over G. Summing up the discussions of Theorem 4.4, $\overline{Apr}_N(\mathfrak{S})$ is a normalistic soft group over G and $\underline{Apr}_N(\mathfrak{S})$ is an upper soft normal subgroup over G. On the other hand, it is clear that $\underline{Apr}_N(\underline{Apr}_N(\mathfrak{S})) \subseteq \underline{Apr}_N(\mathfrak{S})$. Let $a \in \underline{F}_N(x)$, where $x \in \mathrm{Supp(F, A)}$. Then $aN \subseteq F(x)$, and so, $aN \cdot N \subseteq F(x)$, in other words, $a \in \underline{Apr}_N(\underline{Apr}_N(\mathfrak{S}))$. Hence $\underline{Apr}_N(\underline{Apr}_N(\mathfrak{S})) = \underline{Apr}_N(\mathfrak{S})$ and thus $\underline{Apr}_N(\mathfrak{S})$ is a lower rough soft normal subgroup over G. Therefore, $\underline{Apr}_N(\mathfrak{S})$ is a rough soft normal subgroup over G. Similarly, we can get $\overline{Apr}_N(\mathfrak{S})$ is a rough soft normal subgroup over G. This completes the proof. ■

Proposition 4.8 *Let N be a normal subgroup of G, $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (H, B)$ any two non-null soft sets over G. Then*

(1) $\overline{Apr}_N(\mathfrak{S}) \cdot \overline{Apr}_N(\mathfrak{T}) = \overline{Apr}_N(\mathfrak{S} \cdot \mathfrak{T})$;

(2) $\underline{Apr}_N(\mathfrak{S}) \cdot \underline{Apr}_N(\mathfrak{T}) \subseteq \underline{Apr}_N(\mathfrak{S} \cdot \mathfrak{T})$.

Proof. (1) According to Lemma 4.3, $\forall x \in \mathrm{Supp(F, A)}$, $y \in \mathrm{Supp(H, B)}$, we have $\overline{F}_N(x) \cdot \overline{H}_N(y) = (N \cdot F(x))(N \cdot H(y)) = N \cdot N \cdot F(x) \cdot H(y) = N \cdot F(x) \cdot H(y) = \overline{Apr}_N(F(x) \cdot H(y))$. This completes the proof.

(2) $\forall x \in \mathrm{Supp(F, A)}$, $y \in \mathrm{Supp(H, B)}$, let $\forall c \in \underline{F}_N(x) \cdot \underline{H}_N(y)$. Then $c = ab$ for some $a \in \underline{F}_N(x)$ and $b \in \underline{H}_N(y)$. Therefore $aN \subseteq F(x)$ and $bN \subseteq H(y)$. Since $aN \cdot bN = abN = cN \subseteq F(x) \cdot H(y)$. Hence $c \in \underline{Apr}_N(F(x) \cdot H(y))$. This completes the proof. ■

Proposition 4.9 *Let N be a normal subgroup of G, $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (H, B)$ any two normalistic soft groups over G. Then*

$$\underline{Apr}_N(\mathfrak{S}) \cdot \underline{Apr}_N(\mathfrak{T}) = \underline{Apr}_N(\mathfrak{S} \cdot \mathfrak{T}).$$

Proof. By Theorem 4.4 (2), it is obvious. ■

Combining Proposition 4.8 and 4.9, we have the following proposition.

Proposition 4.10 *Let N be a normal subgroup of G, $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (H, B)$ any two normalistic soft groups over G, then*

(1) *$\mathfrak{S} \cdot \mathfrak{T}$ is an upper rough soft normal subgroup w.r.t. N over G,*

(2) *if $\underline{Apr}_N(\mathfrak{S}) \neq \emptyset$ and $\underline{Apr}_N(\mathfrak{T}) \neq \emptyset$, $\mathfrak{S} \cdot \mathfrak{T}$ is a lower rough soft normal subgroup w.r.t. N over G.*

Corollary 4.11 *Let N be a normal subgroup of G, $\mathfrak{S} = (F, A)$ and $\mathfrak{T} = (H, B)$ any two normalistic soft groups over G such that $\underline{Apr}_N(\mathfrak{S}) \neq \emptyset$ and $\underline{Apr}_N(\mathfrak{T}) \neq \emptyset$, then $\mathfrak{S} \cdot \mathfrak{T}$ is a rough soft normal subgroup w.r.t. N over G.*

Proposition 4.12 *Let N and K be two normal subgroups of G such that $N \subseteq K$, $\mathfrak{S} = (F, A)$ a non-null soft set over G. Then*

(1) *$\underline{Apr}_N(\mathfrak{S}) \supseteq \underline{Apr}_K(\mathfrak{S})$;*

(2) *$\overline{Apr}_N(\mathfrak{S}) \subseteq \overline{Apr}_K(\mathfrak{S})$.*

Proof. (1) For all $x \in A$, let $a \in \underline{Apr}_K(F(x)) = \underline{F}_K(x)$, then $aK \subseteq F(x)$, we have $aN \subseteq aK \subseteq F(x)$, hence $a \in \underline{Apr}_N(F(x))$, that is $\underline{Apr}_K(F(x)) \subseteq \underline{Apr}_N(F(x))$, $\forall x \in A$. This completes the proof.

(2) By Lemma 4.3, $\overline{Apr}_N(\mathfrak{S}) = N \cdot \mathfrak{S}$, $\overline{Apr}_K(\mathfrak{S}) = K \cdot \mathfrak{S}$. By hypothesis, we have $N \cdot \mathfrak{S} \subseteq K \cdot \mathfrak{S}$. Thus, $\overline{Apr}_N(\mathfrak{S}) \subseteq \overline{Apr}_K(\mathfrak{S})$. ∎

5. Applications of rough soft groups in decision making

In this section, we give a new decision making method for rough soft sets.

Decision making method. We will put forward the new method to find which is the best parameter e of a given soft set $\mathfrak{S} = (F, A)$. In other words, $F(e)$ is the nearest accurate subgroup on $\mathfrak{S}$ w.r.t. a normal subgroup of a group.

Let G be a group and E a set of related parameters. Let $A=\{e_1, e_2, ..., e_m\}\subseteq E$ and $\mathfrak{S} = (F, A)$ be an original description soft set over G. Let N be a normal subgroup of G and (G, N) a Pawlak approximation space. Then we present the decision algorithm for rough soft groups as follows:

Step 1. Input the original description group G, soft set $\mathfrak{S}$ and Pawlak approximation space (G, N), where N is a normal subgroup of G.

Step 2. Compute the lower and upper rough soft approximation operations $\underline{Apr}_N(\mathfrak{S})$ and $\overline{Apr}_N(\mathfrak{S})$ on $\mathfrak{S}$, respectively.

Step 3. Compute the different values of $\|F(e_i)\|$, where $\|F(e_i)\| = \frac{|\overline{F}_N(e_i)|-|\underline{F}_N(e_i)|}{|F(e_i)|}$.

Step 4. Find the minimum value $\|F(e_k)\|$ of $\|F(e_i)\|$, where $\|F(e_k)\|=\min\limits_i \|F(e_i)\|$.

Step 5. The decision is $F(e_k)$.

Example 5.1 Assume that we want to find the nearest accurate group on a soft set $\mathfrak{S}$. Let $G = S_3 = \{(1), (12), (13), (23), (123), (132)\}$ be a group, then we can check that $N = \{(1), (12)\}$ is a normal subgroup of G, and define a soft set $\mathfrak{S} = (F, A)$ over S_3, the set of parameters is $A = \{e_1, e_2, e_3, e_4\}$. The tabular representation of the soft set $\mathfrak{S}$ is given in Table 1.

Table 1. Table for soft set $\mathfrak{S}$

	(1)	(12)	(13)	(23)	(123)	(132)
e_1	1	1	1	1	0	0
e_2	1	1	1	0	0	0
e_3	1	0	0	0	0	0
e_4	1	0	1	0	1	1

By Definition 4.1, we have two soft sets $\underline{Apr}_N(\mathfrak{S})$ and $\overline{Apr}_N(\mathfrak{S})$ over G. The tabular representations of the soft sets are given by Table 2 and 3,respectively.

Table 2: Table for soft set $\underline{Apr}_N(\mathfrak{S})$

	(1)	(12)	(13)	(23)	(123)	(132)
e_1	1	1	0	0	0	0
e_2	1	1	0	0	0	0
e_3	0	0	0	0	0	0
e_4	0	0	1	0	0	1

Table 3: Table for soft set $\overline{Apr}_N(\mathfrak{S})$

	(1)	(12)	(13)	(23)	(123)	(132)
e_1	1	1	1	1	1	1
e_2	1	1	1	0	0	1
e_3	1	1	0	0	0	0
e_4	1	0	1	1	1	1

By calculations, we find $\|F(e_1)\|$=1, $\|F(e_2)\|$=0.67, $\|F(e_3)\|$=2, $\|F(e_4)\|$=0.75. Thus the minimum value for $\|F(e_i)\|$ is $\|F(e_2)\|$=0.67. That is $F(e_2)$ is the nearest accurate subgroup on $\mathfrak{S}$.

Remark 5.2 Generally speaking, we always tend to get the optimal choices in the actual decision problems. In the above method, we want to obtain the best parameter e of the given soft set $\mathfrak{S} = (F, A)$. By using the decision algorithm in the step 4, we can find that the smaller the values of $\|F(e_i)\|$ are, the more our options are optimal. By the formula of $\|F(e_i)\|$, we obtain that the values of $\|F(e_i)\|$ are smaller while the difference between $|\overline{F}_N(e_i)|$ and $|\underline{F}_N(e_i)|$ are smaller, that is, the parameter e is the optimal choice and $F(e_i)$ is the nearest accurate subgroup structure.

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