HYPERGROUPS AND FUZZY SETS ASSOCIATED MODULO A SUBGROUP

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Abstract. This paper is the results of the ideas suggested by P. Corsini in his paper [10]. Our investigations take in account the paper [13] of I. Cristea. We study the hypergroup generated by the cosets modulo a subgroup (normal subgroup). We prove that $(G/H, \circ_4)$ is a complete hypergroup. We take many particular examples to illustrate some known results on hypergroups.

Keyword: fuzzy sets, hypergroups, complete hypergroup, hyperoperation.

1. Introduction

Since the introduction of the concept of fuzzy subgroup of a given group, by A. Rosenfeld [22]; the notions of different fuzzy algebraic structures have been introduced and intensively studied worldwide. Many results on group theory have be extended in a natural way to fuzzy groups. Later, B. Davvaz and his collaborators generalized the notion of fuzzy subgroup: They defined the concept of fuzzy subhypergroup of a hypergroup. In the same way many Other relations establishing the relationship between hyperstructures and fuzzy sets were stated by P. Corsini. He associated a join space to a fuzzy set and a fuzzy set to a join space. These connections lead to a sequence of fuzzy sets and join spaces. The sequence ends if two consecutive join spaces are isomorphic. This argument has been studied in depth by I. Cristea in her PhD thesis. Moreover in [28], M. Stefanescu together with I. Cristea studied the above sequence in the general general and particularly for a complete hypergroup. P. Corsini and I. Cristea [5] determined the fuzzy grade of a particular non-complete 1-hypergroup. In the same way I. Cristea investigated [14] the sequences of join spaces associated the case of a hypergroupoid using fuzzy sets endowed with two membership functions.

More connections between algebraic hyperstructures and fuzzy sets can be found in many recent publications [1], [21], [22], [24].

Our contribution to the development of such structure is to study as suggested by P. Corsini, the properties of a hypergroup generated by the cosets modulo a subgroup (normal subgroup). We prove that $(G/H, \circ_4)$ is a complete hypergroup. We illustrate our results by many examples.

2. Preliminaries

First of all, we shall recall some basic definitions of hypergroup theory. Let H be a nonempty set and denote by $\mathcal{P}^*(H)$ the set of all nonempty subsets of H.

Definition 1. A hyperoperation on H is a mapping $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$.

A nonempty set H endowed with a hyperoperation $\circ$ is said to be a hypergroupoid.

The image of the pair $(a, b) \in H \times H$ is usually denoted by $a \circ b$ and called the hyperproduct of a and b.

If A and B are nonempty subsets of H, then we define

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b$$

and call it the hyperproduct of A and B.

Also, for any $a, b \in H$, we define

$$a/b = \{x \in H | a \in x \circ b\}.$$

Definition 2.

1. The hypergroupoid $(H, \circ)$ is called a semihypergroup if the hyperoperation $\circ$ is associative.

2. If $(H, \circ)$ satisfies the reproducibility law $a \circ H = H \circ a = H$, $\forall a \in H$, then we say that it is a quasihypergroup.

3. A hypergroup is a hypergroupoid which is both a semihypergroup and a quasihypergroup.

4. A hypergroup H is said to have a unity or to be unitary if there exists a unique element $1 \in H$ such that $1 \circ a = a \circ 1 = \{a\}$, $\forall a \in H$.

Definition 3. A subset K of hypergroup $(H, \circ)$ is a subhypergroup if and only if the following conditions are satisfied:

1. $a \circ b \subset K$, for all $a, b \in K$,

2. $a \circ K = K \circ a = K$, for all $a \in K$.

Clearly, H is a subhypergroup of itself. Any other subhypergroup of H will be called a proper subhypergroup. In the sequel we recall the following theorems needed to study the fuzzy subset as a function modulo a subgroup H.

Let us introduce some definitions which will be useful.

Definition 4. Let $(G, \circ)$ be a hypergroup and μ be a fuzzy subset of G. μ is called a fuzzy subhypergroup if the following conditions are satisfied.

1. For any $a, b \in G$ one has
$$\min\{\mu(a), \mu(b)\} \leq \min\{\mu(c),\ c \in a \circ b\}.$$
2. For any $a, b \in G$ there exists $c \in G$ such that $a \in b \circ c$ and
$$\min\{\mu(a), \mu(b)\} \leq \mu(c).$$
3. For any $a, b \in G$ there exists $c \in G$ such that $a \in c \circ b$ and
$$\min\{\mu(a), \mu(b)\} \leq \mu(c).$$

If the hyperoperation is commutative, μ is a subhypergroup if the conditions 1 and 2 are satisfied.

Theorem 5 (Cauchy's theorem). *If a prime integer p divides the order of a group G, the group G has at least one element of order p.*

Theorem 6 (Sylow's theorems). *If a group G is of order $p^n m$, p prime and $gcd(p, m) = 1$, then:*

(1) *The group G has at least one subgroup of order p^n called a $p-$Sylow of G.*

(2) *The $p-$Sylow are conjugate.*

(3) *There exists at least one subgroup of order p^α, $\alpha \in \{1, 2, ..., n\}$.*

(4) *The number n_p of $p-$Sylow is such that:*

   (a) $n_p \equiv 1[p]$ *and*
   (b) n_p *divides the integer m.*

3. The main results

Let H be a subgroup of a group $(G, \cdot)$. We know that the relation, $\mathcal{R}$ defined on G by
$$x\mathcal{R}y \iff x \cdot y^{-1} \in H,$$
is an equivalence relation and for any $x \in G$ the equivalence class of x is defined by $\overline{x} = H \cdot x$.

In the sequel, we will be concerned with the relation modulo a subgroup H defined above on a group G.

Proposition 1. *If on set G we define $\circ$ by $a \circ b = \overline{a} \cup \overline{b}$, then $(G, \circ)$ is a commutative hypergroup.*

Proof. 1. The commutativity is clear.

2. Since for any $x \in G$, $\overline{x} \neq \emptyset$, we then have $x \circ y \neq \emptyset$.

3. It can easily proved that $\forall x, y, z \in G$, $(x \circ y) \circ z = \overline{x} \cup \overline{y} \cup \overline{z}$ (the union of sets is associative).

4. For any $a, x \in G$, we have $x \in a \circ x$. So $G \subset a \circ G$ and then $G = a \circ G$ and by the commutativity we also have $G = G \circ a$ for any $a \in G$. ∎

Now, for any $a \in G$, we set

$$\begin{aligned} Q(a) &= \{(x, y) \in G^2 \mid a \in x \circ y\}, \\ q(a) &= |Q(a)|, \\ A(a) &= \sum_{(x,y) \in Q(a)} \frac{1}{|x \circ y|}, \\ \mu_A(a) &= \frac{A(a)}{q(a)} \cdot \end{aligned}$$

Proposition 2. *For any $a \in G$, we set*

$$Q(a) = \{(x, y) \in G^2 \mid a \in x \circ y\} = Ha \times G \cup G \times Ha.$$

Proof. $(x, y) \in Q(a) \iff a \in x \circ y = \overline{x} \cup \overline{y} \iff x \in \overline{a}$ or $y \in \overline{a} \iff (x, y) \in \overline{a} \times G$ or $(x, y) \in G \times \overline{a} \iff (x, y) \in (\overline{a} \times G) \cup (G \times \overline{a})$. ∎

Proposition 3. *If H is a subgroup of order n and G is of order nm, then*

$$\forall a \in G, \ q(a) = |Q(a)| = n^2(2m - 1).$$

Proof. By the above proposition, $|Q(a)| = |Ha \times G| + |G \times Ha| - |Ha \times G \cap G \times Ha| = |H| \times |G| + |G| \times |H| - |Ha \times G \cap G \times Ha|$. On the other hand, $Ha \times G \cap G \times Ha = Ha \times Ha$ so $|Ha \times G \cap G \times Ha| = |H|^2 = n^2$. Finally, $q(a) = |Q(a)| = 2nnm - n^2 = n^2(2m - 1)$. ∎

Proposition 4. *If H is a subgroup of order n and G is of order nm, then*

$$\forall a \in G, \ |A(a)| = nm = |G|.$$

Proof. Regarding the nature of elements of $Q(a)$ for an element $a \in G$, we can divide the elements in three kinds, elements of $(ha, y), y \notin Ha$, the elements of type $(x, ha), x \notin Ha$ and the elements of the type $(ha, h'a)$ where $h, h' \in H$.

1. We have exactly $|H|(|G| - |H|) = n^2(m - 1)$ elements of the first type and we have $ha \circ y = Ha \cup Hy$ and $Ha \cap Hy = \emptyset$ so $|ha \circ y| = 2|H| = 2n$.

2. Similarly, we have exactly $|H|(|G| - |H|) = n^2(m-1)$ elements of the second kind and $x \circ ha = Hx \cup Ha$ and $Hx \cap Ha = \emptyset$ so $|x \circ ha| = 2|H| = 2n$.

3. For the last type, we have n^2 elements and $x \circ y = Ha \cup Ha = Ha$ so $|x \circ y| = n$.

Finally,

$$A(a) = \sum_{(x,y)\in Q(a)} \frac{1}{|x \circ y|} = \sum_{(x,y) \text{ of first type}} \frac{1}{|x \circ y|} + \sum_{(x,y) \text{ of second type}} \frac{1}{|x \circ y|}$$

$$+ \sum_{(x,y) \text{ of third type}} \frac{1}{|x \circ y|} = n^2(m-1) \cdot \frac{1}{2n} + n^2(m-1) \cdot \frac{1}{2n} + n^2 \cdot \frac{1}{n} = nm. \quad \blacksquare$$

Proposition 5. *If H is a subgroup of order n and G is of order nm then the fuzzy function introduced by Corsini is constant on all elements of the group G.*

Proof. For any element $a \in G$, we have the function

$$\mu(a) = \frac{A(a)}{q(a)} = \frac{mn}{n^2(2m-1)} = \frac{m}{n(2m-1)}. \quad \blacksquare$$

From Proposition 8, we can see that the function μ depends only on the cardinalities of both the group G and the subgroup H and not on a specific elements of G, so we denote it by $\mu(nm, n)$. It is easy to prove the following corollaries.

Corollary 6. *If the order n of the group G is fixed, as a function of the order m of the subgroup H, $\mu(nm, n)$ is a decreasing function (the derivative with respect to m is negative).*

Corollary 7. *If the order m of the subgroup H is fixed, as a function of the order n of the group G, $\mu(mn, m)$ is a decreasing function (the derivative with respect to n is negative).*

The above hyperoperation will be denoted by $\circ_1$. We can also introduce a new hyperoperation on a group G denoted by $\circ_2$.

Definition 8. Let H be a subgroup of a group G. For $a, b \in G$, let we define
$a \circ_2 b = \{c \in G \mid abc \in H\}$.
$a \circ_2 b \neq \emptyset$ since $b^{-1}a^{-1} \in a \circ_2 b$.
When it is important to specify the subgroup H, $\circ_2$ will be denoted $\circ_2^H$.

Definition 9. Let H be a subgroup of a group G. For $a \in G$, let we define

$$\mu(a) = \frac{1}{\min\{n \in \mathbb{N}^* \mid a^n \in H\}}.$$

If there is no integer $n \in \mathbb{N}^*$ such that $a^n \in H$, we set $\mu(a) = 0$.

Remark 1.

1. In the case where $H = \{e\}$ the above function is such that $\mu(a) = \frac{1}{ord(a)}$ and $\mu(a) = 0$ if $ord(a)$ is infinite.

2. $h \in H$ if and only if $\mu(h) = 1$.

3. Any element a of finite order of is such $\mu(a) > 0$, since at least

$$\mu(a) \geq \frac{1}{ord(a)}, \ a^{ord(a)} = e \in H.$$

Let us study the case of a cyclic group G of finite order n generated by an element a. In this case the subgroup $H = \langle a^k \rangle$ with $gcd(k,n) > 1$ otherwise H becomes the whole group G.

Proposition 10. *Let G be a group (finite or infinite) generated by a and let H be the subgroup generated by a^k. Then*

1. $gcd(m,k) = 1 \Longrightarrow \mu(a^m) = \frac{1}{k}$.

2. $\mu(a^m) = \frac{1}{k_1}$, *where* $k = dk_1, \ gcd(m,k) = d$.

Proof.
1. We have

(a) $(a^m)^k = (a^k)^m \in H$.

(b) Suppose now that there exists $k_1 < k$ such that $(a^m)^{k_1} \in H$ and then there exists $r \in \mathbb{N}$ such that $(a^m)^{k_1} = (a^k)^r$ and therefore $mk_1 = kr$. The last equality implies that k divides mk_1 but $gcd(m,k) = 1$ then by Gauss's theorem k divides k_1 which implies in particular that $k_1 \geq k$ which is a contradiction. So $k = min\{r \in \mathbb{N} \mid (a^m)^r \in H\}$.

2. Now, let $d = gcd(m,k)$. Then $\exists k_1, m_1 \in \mathbb{N}$ such that $k = dk_1$, $m = dm_1$, $gcd(k_1, m_1) = 1$.

(a) $(a^m)^{k_1} = a^{mk_1} = a^{m_1 dk_1} = (a^{dk_1})^{m_1} = (a^k)^{m_1} \in H$.

(b) Suppose that there exists $k_2 < k_1$ such that $(a^m)^{k_2} \in H$. Then, we have $(a^m)^{k_2} \in H \iff (a^m)^{k_2} = (a^k)^r$ so $mk_2 = kr$, and then $k = dk_1$ divides $mk_2 = dm_1k_2$, and then k_1 divides m_1k_2 but $gcd(k_1, m_1) = 1$, then k_1 divides k_2. We get, in particular, $k_2 \geq k_1$, contradiction. ∎

Definition 11. Let G be a group (finite or infinite) generated by a and let H be the subgroup generated by a^k. We define on G a hyperoperation:

$$a^{m_1} \circ a^{m_2} = \{x \in G \mid min\{\mu(a^{m_1}), \mu(a^{m_2})\} \leq \mu(x) \leq \max\{\mu(a^{m_1}), \mu(a^{m_2})\}\}.$$

Proposition 12. *Let G, H as in Proposition 15 and μ as in the above definition. Then*

1. *If* $gcd(m_1,k){=}gcd(m_2,k) = 1$, *then* $a^{m_1} \circ a^{m_2} {=} \{x \in G \mid x{=}a^m, \ gcd(m,k){=}1\}$.

2. *If* $gcd(m_1, k) = 1$ *and* $gcd(m_2, k) = d > 1$, *then there exists* k_1 *such that* $k = dk_1$. *In this case,* $\mu(a^{m_1}) = \frac{1}{k}$ *and* $\mu(a^{m_2}) = \frac{1}{k_1}$, *so* $(a^{m_1}) \circ (a^{m_2}) = \{a^r \mid \frac{1}{k} \leq \mu(a^r) \leq \frac{1}{k_1}\} = \{a^r \mid gcd(r, k) = 1\} \cup \{a^r \mid gcd(r, k) = d' \leq d\}$.

3. *If* $gcd(m_1, k) = d_1 > 1$, $gcd(m_2, k) = d_2 > 1$, *let* $d_1k_1 = k$ *and* $d_2k_2 = k$. *We can suppose that* $\frac{1}{k_1} \leq \frac{1}{k_2}$. *In this case* $(a^{m_1}) \circ (a^{m_2}) = \{a^r \mid \frac{1}{k_1} \leq \mu(a^r) \leq \frac{1}{k_2}\}$ $= \{a^r \mid d_1 \leq d \leq d_2,\ d = gcd(r, k)\}$.

The next proposition has been proved in [16] (Proposition 2), but we want to prove it in the present study.

Proposition 13 (16, Prop.2)**.** *Let* G, H *as in Proposition* 15 *and* μ *as in the above definition. Then* μ *is a fuzzy subhypergroup of* $(G, \circ)$.

Proof.

1. (a) If $gcd(m_1, k) = gcd(m_2, k) = 1$ then for any $a^m \in a^{m_1} \circ a^{m_2}$ we have $gcd(m, k) = 1$ so

$$min\{\mu(a^{m_1}), \mu(a^{m_2})\} = min\{\mu(c),\ c \in a^{m_1} \circ a^{m_2}\} = \frac{1}{k}.$$

(b) If $gcd(m_1, k) = 1$ and $gcd(m_2, k) = d > 1$ and if $k = dk_1$ by the proposition 17, any element a^m of $a^{m_1} \circ a^{m_2}$ is such that

$$\frac{1}{k} \leq \mu(a^m) \leq \frac{1}{k_1}$$

so

$$min\{\mu(a^{m_1}), \mu(a^{m_2})\} = \frac{1}{k} \leq min\{\mu(c),\ c \in a^{m_1} \circ a^{m_2}\}.$$

(c) If $gcd(m_1, k) = d_1 > 1, gcd(m_2, k) = d_2 > 2$, to fix the ideas, we can suppose that $d_1 \geq d_2$, so there exists two natural integers $k_1 \leq k_2$ such that $k = d_1k_1 = d_2k_2$. In this case, the proposition 17 gives $\frac{1}{k_2} \leq \mu(a^m) \leq \frac{1}{k_1}$ for any $a^m \in a^{m_1} \circ a^{m_2}$ so the following inequality holds.

$$min\{\mu(a^{m_1}), \mu(a^{m_2})\} = \frac{1}{k_2} \leq min\{\mu(c),\ c \in a^{m_1} \circ a^{m_2}\}.$$

2. (a) If $gcd(m_1, k) = gcd(m_2, k) = 1$ and $c = a^{m_1}$ then $a^{m_1} \in a^{m_1} \circ a^{m_2}$ and $min\{\mu(a^{m_1}), \mu(a^{m_2})\} = \frac{1}{k} = \mu(c)$.

(b) If $gcd(m_1, k) = 1$ and $gcd(m_2, k) = d > 1$, it suffices to take in the definition, $c = a^{m_1}$, so $a^{m_1} \in a^{m_1} \circ a^{m_2}$ and then

$$min\{\mu(a^{m_1}), \mu(a^{m_2})\} = \frac{1}{k} \leq \mu(c) = \frac{1}{k}.$$

(c) If $gcd(m_1,k)=d_1>1, gcd(m_2,k)=d_2>2$, d_1 and d_2 are such 1) (c). As in the two precedent cases It suffices to take $c=a^{m_1}$ so $\mu(c)=\frac{1}{k_1}$, $a^{m_1}\in a^{m_2}\circ a^{m_1}$ and

$$min\{\mu(a^{m_1}),\mu(a^{m_2})\}=\frac{1}{k_1}=\mu(c). \qquad \blacksquare$$

Now, we study an example of a complete hypergroup.

Proposition 14. *Let H be a normal subgroup of a group G if $\circ_4$ is defined by $\forall a,b\in G,\ a\circ_4 b=abH$, $(G,\circ_4)$ is a hypergroup.*

Proof.

1. $(G,\circ_4)$ is a hypergroup.
   (a) It is clear that for any $a,b\in G,\emptyset\neq abH\in\mathcal{P}^*(G)$.
   (b) Let $x\in(a\circ_4 b)\circ_4 c$ then there exist $h_1,h_2\in H$ such that $x=abh_1ch_2$. But $x=abh_1ch_2=a(bch_1)h_2\in a\circ_4(bch_1)$. On the other hand, $bch_1\in b\circ_4 c$ so $x\in a\circ_4(b\circ_4 c)$ and this implies $(a\circ_4 b)\circ_4 c\subset a\circ_4(b\circ_4 c)$. In the same way, we can easily get the other inclusion.
   (c) For any element $a\in G$, we have $a\circ_4 G\subset G$. Let $a,x\in G$ there exists an element $y=a^{-1}xh,\ h\in H$ then $a\circ_4 y=ayH=a(a^{-1}xh)H=xH$ and $x\in xH$ so there exists $y\in G$ such that $x\in a\circ_4 y$ and $G\subset a\circ_4 G$.
2. Now, for any x in the quotient group G/H, let $A_x=x$. We then have:
   (a) $G=\bigcup_{x\in G/H}A_x$.
   (b) If $x\neq y$ then $A_x\cap A_y=x\cap y=\emptyset$
   (c) If $(a,b)\in A_x\times A_y$ then $a\circ_4 b=abH=\overline{ab}=\overline{a}\overline{b}=xy=A_{xy}$ since xy is an equivalence class (H is normal). $\blacksquare$

Remark 2. As mentioned in [12] Proposition 3.1, If $|G|=|G/H|$ which case corresponds to $H=\{e\}$, our classes A_x are singletons and the hyperoperation $\circ_4$ is the group binary operation.

Remark 3. If the subgroup H is of index 2, the group G is commutative, the order of H is great or equal than 2. So there exists $a\in G\setminus H$ such that by the above construction we have,

$$b\circ_4 c=\begin{cases} H & \text{if } \exists h\in H \text{ such that } b=ahc^{-1}\\ aH & \text{if } \forall h\in H,\ b\neq ahc^{-1}.\end{cases}$$

Definition 15. Let H be a set endowed with two hyperoperations $*$ and $\star$. $*$ is said to be finer than $\star$ if $\forall a,b\in H, a*b\subset a\star b$.

It is easy to prove the following lemma.

Lemma 16. *Let $H_1 \subset H_2 \subset ... \subset H_n$ be a finite sequence of normal subgroups of a group G, then $xH_i \subset xH_{i+1}$.*

Proposition 17. *Let $H_1 \subset H_2 \subset ... \subset H_n$ be a finite sequence of normal subgroups of a group G. If we denote by $\circ_i$ the hyperoperation associated to G/H_i as in Proposition 20, then $\circ_i$ is finer than $\circ_{i+1}$.*

Proof. Trivial. ∎

Let $H_1 \subset H_2 \subset ... \subset H_n$ be a finite sequence of normal subgroups of a group G. For $a \in G$, let we denote by $Q_i(a) = \{(x, y) \in G \mid a \in x \circ_i y\}$. We then have.

Proposition 18. $q_i(a) = |Q_i(a)| = |G|.|H_i|$, $A_i(a) = |G|$ *and* $\mu_i(a) = \frac{1}{|H_i|}$.

Proof. $Q_i(a)=\{(x, y) \in G\times G \mid a \in x \circ_i y\}=\{(x, x^{-1}ah^{-1}) \mid h \in H_i\}=G\times x^{-1}aH_i$ and then $|Q_i(a)|=|G| \cdot |H_i|$. Now, let $(x, x^{-1}ah) \in Q_i(a)$. Then $x \circ_i x^{-1}ah=aH$ and $|x \circ_i x^{-1}ah| = |aH| = |H|$, and since there exist exactly $|G| \cdot |H_i|$ elements of this type, we deduce $A_i(a) = \frac{|G|\cdot|H_i|}{|G|} = |G|$. The value of $\mu_i(a)$ is then given by $\mu_i(a) = \frac{|G|}{|G|\cdot|H_i|} = \frac{1}{|H_i|}$. ∎

Proposition 19. *The group G endowed with $\circ_2$ is a hypergroupoid and verifies the reproduction property.*

Proof. As seen in the definition 13, it is trivial that $\circ_2$ is an hyperoperation on G.

Let $a \in G$. For any $x \in G$, one has $x \in a \circ_2 z$ for any $z = a^{-1}hx^{-1}$, $h \in H$ and then $x \in \bigcup_{z\in G} a \circ_2 z = a \circ_2 G$. ∎

Remark 4. The hyperoperation $\circ_2$ is not in general associative. To see this, it suffices to take in the symmetric group S_3 the subgroup generated by the transposition $\tau_{1,2}$, that is, $H = \{id, \tau_{1,2}\}$. In this case, it is easy to prove that:

$[(\tau_{1,2} \circ_2 \tau_{2,3}] \circ_2 s_4 = \{id, \tau_{1,2}, \tau_{1,3}, s_4\}$ and $\tau_{1,2} \circ_2 [\tau_{2,3} \circ_2 s_4] = \{\tau_{1,2}, \tau_{1,3}, s_4, s_5\}$, where the permutations s_4 and s_5 are given by

$$s_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \ s_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Definition 20. Let G be a group, for $a, b \in G$ we define $a \circ_3 b = \langle a, b\rangle$, the subgroup generated by a and b.

The following result is well known.

Proposition 21. $(G, \circ_3)$ *is a hypergroup.*

Example 22. If we take in the proposition $G = S_3$, we have: Let

$$S_3 = \left\{id = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \tau_{1,2} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \tau_{1,3} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \right.$$
$$\left. \tau_{2,3} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, s_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, s_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\right\}$$

$\overset{\rightarrow}{\circ}_3$	id	$\tau_{1,2}$	$\tau_{1,3}$	$\tau_{2,3}$	s_1	s_2
id	$\{id\}$	$\{id, \tau_{1,2}\}$	$\{id, \tau_{1,3}\}$	$\{id, \tau_{2,3}\}$	$\{id, s_1, s_2\}$	$\{id, s_1, s_2\}$
$\tau_{1,2}$	$\{id, \tau_{1,2}\}$	$\{id, \tau_{1,2}\}$	S_3	S_3	S_3	S_3
$\tau_{1,3}$	$\{id, \tau_{1,3}\}$	S_3	$\{id, \tau_{1,3}\}$	S_3	S_3	S_3
$\tau_{2,3}$	$\{id, \tau_{2,3}\}$	S_3	S_3	$\{id, \tau_{2,3}\}$	S_3	S_3
s_1	$\{id, s_1, s_2\}$	S_3	S_3	S_3	$\{id, s_1, s_2\}$	$\{id, s_1, s_2\}$
s_2	$\{id, s_1, s_2\}$	S_3	S_3	S_3	$\{id, s_1, s_2\}$	$\{id, s_1, s_2\}$

$Q(id) \ = S_3 \times S_3.$

$Q(\tau_{1,2}) = \{\tau_{1,2}\} \times S_3 \cup S_3 \times \{\tau_{1,2}\} \cup \{\tau_{1,3}\} \times \{\tau_{2,3}, s_1, s_2\} \cup \{\tau_{2,3}, s_1, s_2\} \times \{\tau_{1,3}\}.$

$Q(\tau_{1,3}) = \{\tau_{1,3}\} \times S_3 \cup S_3 \times \{\tau_{1,3}\} \cup \{\tau_{2,3}\} \times \{\tau_{1,2}, s_1, s_2\} \cup \{\tau_{2,3}, s_1, s_2\} \times \{\tau_{2,3}\}.$

$Q(\tau_{2,3}) = \{\tau_{2,3}\} \times S_3 \cup S_3 \times \{\tau_{2,3}\} \cup \{\tau_{1,2}\} \times \{\tau_{2,3}, s_1, s_2\} \cup \{\tau_{2,3}, s_1, s_2\} \times \{\tau_{1,2}\}.$

$Q(s_1) \ = S_3 \times S_3.$

$Q(s_2) \ = S_3 \times S_3.$

So $q(id) = q(s_1) = q(s_2) = 36$ and $q(\tau_{1,3}) = q(\tau_{2,3}) = q(\tau_{1,2}) = 18$.

It is easy to prove that:

1. $A(id) = \frac{67}{6}$,
2. $A(\tau_{1,2}) = A(\tau_{1,3}) = A(\tau_{2,3}) = \frac{23}{6}$,
3. $A(s_1) = A(s_2) = \frac{17}{3}$.

We then deduce that the function μ is such that:

1. $\mu(id) = \frac{67}{216}$,
2. $\mu(\tau_{1,2}) = \mu(\tau_{1,3}) = \mu(\tau_{2,3}) = \frac{23}{108}$,
3. $\mu(s_1) = \mu(s_2) = \frac{17}{108}$.

Remark 5. *We have* $\mu(id) > \mu(\tau_{1,2}) = \mu(\tau_{1,3}) = \mu(\tau_{2,3}) > \mu(s_1) = \mu(s_2)$.

Acknowledgements. The author is highly grateful to Professor P. Corsini for his particular interest to the present paper and to the referee for his precious remarks that make the paper more clear than its initial version.

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Accepted: 04.04.2016