ANALYTIC SOLUTION FOR RLC CIRCUIT OF NON-INTEGER ORDER

Jignesh P. Chauhan
Department of Applied Mathematics & Humanities
S.V. National Institute of Technology
Surat – 395007
India
e-mail: impulse.nit07@gmail.com

Pratik V. Shah
Department of Applied Sciences
C.K. Pithawalla College of Engineering and Technology
Surat – 395007
India
e-mail: pratikshah8284@yahoo.co.in

Ranjan K. Jana
Department of Applied Mathematics & Humanities
S.V. National Institute of Technology
Surat – 395007
India
e-mail: rkjana2003@yahoo.com

Ajay K. Shukla
Department of Applied Mathematics & Humanities
S.V. National Institute of Technology
Surat – 395007
India
e-mail: ajayshukla2@rediffmail.com, aks@ashd.svnit.ac.in

Abstract. In this paper, we discuss second order fractional differential equation model for RLC circuit with order $1 < \alpha \leq 2$ and $0 < \beta \leq 1$. Further, we use Laplace transform method including convolution theorem to obtain the solution.
Keywords: Mittag-Leffler function, Laplace transform, mathematical modeling, RLC circuit.
2010 Mathematics Subject Classification: 00A71, 33E12, 34A08, 44A10.

1. Introduction

Fractional calculus goes back to the beginning of the theory of differential calculus. Fractional calculus has become new mathematical method for solution of diverse problems in mathematics, sciences and engineering as it is generalization of classical calculus to derivatives and integrals of fractional non-integer order. The applications of fractional calculus has just emerged in last two decades, due to progress in the area of chaos that revealed subtle relationship with fractional calculus concepts. Apparently, mathematical models involving fractional order derivatives has became a powerful and widely used tool for better modeling and control of processes in many areas of science and engineering as it reflects study of intrinsic dissipative processes that are complicated in nature, particularly for systems where memory or hereditary properties plays a significant role. Large amount of research oriented studies were developed concerning the applications of fractional calculus in the field of physics. Fractional model for electrical circuits such as RL, RC, RLC have already been proposed. Recently, Gómez *et al.* [3] considered Caputo derivatives and Numerical Laplace transform to get the solution of RL and RC circuits. Further, they also analysed RLC circuit in time domain and found solution in terms of Mittag-Leffler function. Shah *et. al* [8] has obtained analytic solution of RL electrical circuit described by a fractional differential equation of the order $0 < \alpha \leq 1$ and used the Laplace transform of the fractional derivative in the Caputo sense. In order to stimulate more interest in subject and to show its utility, this paper is devoted to new and recent application of fractional calculus i.e. RLC electrical circuit considering the second order fractional differential equation with parameters $\alpha \in (1, 2], \beta \in (0, 1]$ and obtained analytic solution in terms of Mittag-Leffler function using integral transform method.

2. Used integral transforms and special functions

Definition. In 1903, the Swedish mathematician Gósta Mittag-Leffler introduced the function

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0. \tag{1}$$

The Mittag-Leffler function in two parameters $E_{\alpha,\beta}(z)$ was introduced by Agarwal [4] and reported by Goenflo *et al.* [6] in their book,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \quad \beta > 0. \tag{2}$$

Its k-th derivative is given by,

$$E^{(k)}_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{(j+k)! z^j}{j! \Gamma(\alpha j + \alpha k + \beta)}, \; k = 0, 1, 2, 3, \ldots \tag{3}$$

The Mittag-Leffler type function [7] is defined by,

$$\mathcal{E}_k(z, y; \alpha, \beta) = z^{k\alpha+\beta-1} E^{(k)}_{\alpha,\beta}(y z^\alpha) \quad k = 0, 1, 2, 3, \ldots \tag{4}$$

Definition [1]. Let $f(t)$ be an arbitrary function defined on the interval $0 \leq t < \infty$. Then,

$$F(s) = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt \tag{5}$$

is the Laplace transform of $f(t)$, provided that the integral exists.

For existence of integral (5), the function $f(t)$ must be piecewise continuous and of exponential order α, original function $f(t)$ can be restored from the Laplace transform $F(s)$ with the help of inverse Laplace transform,

$$f(t) = \mathcal{L}^{-1}\{F(s);t\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)ds, \quad c = Re(s). \tag{6}$$

Definition. The *Caputo's* definition of fractional derivative is given by

$${}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^n(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau, \tag{7}$$

where $\alpha \in R$ is order of fractional derivative, $n-1 < \alpha \leq n$ and $n \in N = \{1,2,3,\dots\}$, $f^n(\tau) = \frac{d^n}{dt^n}f(\tau)$ and $\Gamma(.)$ is Euler Gamma function.

Laplace transform of fractional derivative [1], [7].

The Laplace transform of Caputo fractional derivative has the form

$$\mathcal{L}\left\{{}_0^C D_t^p f(t)\right\} = s^p F(s) - \sum_{k=0}^{n-1} s^{p-k-1}f^{(k)}(0), \qquad (n-1 < p \leq n). \tag{8}$$

where $f^{(k)}$ is k-th order derivative.

Convolution theorem [1]. *If* $\mathcal{L}\{f(t)\} = \bar{f}(s)$ *and* $\mathcal{L}\{g(t)\} = \bar{g}(s)$*, then*

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = \bar{f}(s)\bar{g}(s), \tag{9}$$

or, equivalently,

$$\mathcal{L}^{-1}\left\{\bar{f}(s)\bar{g}(s)\right\} = f(t) * g(t), \tag{10}$$

where $f(t) * g(t)$ *is called the convolution of* $f(t)$ *and* $g(t)$*, and is defined by the integral*

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau. \tag{11}$$

The integral in (11) is often referred to as the convolution integral and is denoted simply by $(f * g)(t)$.

Laplace transform of Mittag-Leffler function [7].

A two parameter function of Mittag-Leffler type is defined by series expansion,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \quad (\alpha > 0,\ \beta > 0). \tag{12}$$

Laplace transform of Mittag-Leffler function is given by [7],

$$\mathcal{L}\left[z^{k\alpha+\beta-1}E^{(k)}_{\alpha,\beta}(\pm a z^{\alpha})\right] = \frac{k!\ s^{\alpha-\beta}}{(s^{\alpha} \mp a)^{k+1}} \qquad k = 0, 1, 2, 3, \ldots . \tag{13}$$

3. Mathematical model for RLC electrical circuit

The integrated process of translating real world problem into mathematical problem is termed as mathematical modeling. It includes mathematical concepts such as function, variables, constants, inequality, etc. taken from different branches of mathematics. Here, we formulate the model for electrical circuit which are widely used in branch of physics and engineering sciences.

The oscillating electrical circuit in which resistor (R), inductor (L) and capacitor (C) are connected with voltage (E) is known as RLC circuit. There are number of ways of connecting these elements across voltage supply, we consider electrical circuit where these elements are connected in series with voltage as shown in figure (1). Here, the capacitance (C), the inductance (L) and the resistor (R) are considered as positive constants.

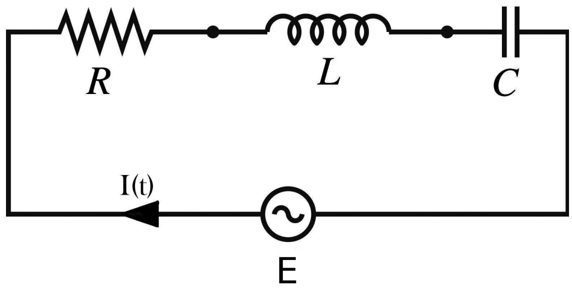

Figure 1: RLC Circuit

Nomenclature

$E(t)$	–	The voltage of the power source (measured in volts = V)
$I(t)$	–	The current in the circuit at time t (measured in amperes = A)
R	–	The resistance of the resistor (measured in ohms = V/A)
L	–	The inductance of the inductor (measured in henry = H)
C	–	The capacitance of the capacitor (measured in farads = F = C/V)

4. Formulation of fractional differential equation model and its solution

Considering the RLC circuit, the constitutive equations associated with three elements, i.e., resistor, inductor and capacitor are:

- The voltage drop across inductor, i.e., $U_L(t) = \frac{d}{dt}I(t)$.
- The voltage drop across resistor, i.e., $U_R(t) = RI(t)$.
- The voltage drop across capacitor, i.e., $U_C(t) = \frac{1}{C}\int_0^t I(\xi)d\xi$.

As per Kirchhoff's voltage law, around any loop in a circuit, the voltage rises must equal to the voltage drops. We have,

$$U_L(t) + U_R(t) + U_C(t) = E(t), \tag{14}$$

or

$$LI' + RI + \frac{1}{C}\int I dt = E\left(t\right). \tag{15}$$

to get rid of integral, we differentiate (15) with respect to t, which yields non-homogeneous second order ordinary differential equation,

$$LI'' + RI' + \frac{1}{C}I = E(t), \tag{16}$$

In this paper, we develop model for RLC circuit in the form of fractional differential equation as,

$$LD^{\alpha}I\left(t\right) + RD^{\beta}I\left(t\right) + \frac{1}{C}I\left(t\right) = E\left(t\right), \tag{17}$$

where $D^{\alpha}I(t) = \frac{d^{\alpha}I}{dt^{\alpha}}$ and $D^{\beta}I(t) = \frac{d^{\beta}I}{dt^{\beta}}$, $1 < \alpha \leq 2$, $0 < \beta \leq 1$, when $\lim\limits_{\alpha\to 2}\frac{d^{\alpha}I}{dt^{\alpha}} = \frac{d^2I}{dt^2}$ and $\lim\limits_{\beta\to 1}\frac{d^{\beta}I}{dt^{\beta}} = \frac{dI}{dt}$.

Taking the Laplace transform on both sides of (17) by considering the initial condition $I(0) = A$ and $I'(0) = B$, and further using (8), we get,

$$\mathcal{L}\left\{L\,D^{\alpha}I(t)\right\} + \mathcal{L}\left\{R\,D^{\beta}I(t)\right\} + \mathcal{L}\left\{\frac{1}{C}I(t)\right\} = \mathcal{L}\left\{E\left(t\right)\right\}.$$

Using (8), we have,

$$\begin{aligned} &L\{s^{\alpha}I(s) - s^{\alpha-1}I(0) - s^{\alpha-2}I'(0)\} + R\{s^{\beta}I(s) - s^{\beta-1}I(0)\} + \frac{1}{C}\{I(s)\} = E(s) \\ &\Rightarrow L\left\{s^{\alpha}I\left(s\right) - s^{\alpha-1}A - s^{\alpha-2}B\right\} + R\left\{s^{\beta}I\left(s\right) - s^{\beta-1}B\right\} + \frac{1}{C}\left\{I\left(s\right)\right\} = E\left(s\right) \\ &\Rightarrow I\left(s\right) = \frac{E\left(s\right)}{\left\{Ls^{\alpha} + Rs^{\beta} + \frac{1}{C}\right\}} + LA\frac{s^{\alpha-1}}{\left\{Ls^{\alpha} + Rs^{\beta} + \frac{1}{C}\right\}} \\ &\qquad + LB\frac{s^{\alpha-2}}{\left\{Ls^{\alpha} + Rs^{\beta} + \frac{1}{C}\right\}} + RB\frac{s^{\beta-1}}{\left\{Ls^{\alpha} + Rs^{\beta} + \frac{1}{C}\right\}} \end{aligned} \tag{18}$$

Solution of (18) is obtained by taking inverse Laplace transform and using (4), (13), we get,

$$
\begin{aligned}
I(t) = {} & C\int_0^t E(t-u)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{1}{(LC)^{k+1}}\mathcal{E}_k\left(u,\frac{-R}{L};\alpha-\beta,\alpha+\beta k\right)du \\
& +\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{ALC}{(LC)^{k+1}}\ \mathcal{E}_k\left(t,-\frac{R}{L};\alpha-\beta,\beta k+1\right) \\
& +\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{BLC}{(LC)^{k+1}}\ \mathcal{E}_k\left(t,-\frac{R}{L};\alpha-\beta,\beta k+2\right) \\
& +\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{RAC}{(LC)^{k+1}}\ \mathcal{E}_k\left(t,-\frac{R}{L};\alpha-\beta,\beta(k-1)+\alpha+1\right),
\end{aligned}
\tag{19}
$$

where $1<\alpha\leq 2$ and $0<\beta\leq 1$ also $\alpha-\beta>0$.

Some cases for different values of $E(t)$ are discussed below.

Case I. When constant electromotive force is applied, i.e., $E(t)=E_0$, then (19) takes the following form,

$$
\begin{aligned}
I(t) = {} & CE_0\int_0^t \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{1}{(LC)^{k+1}}\ \mathcal{E}_k\left(u,-\frac{R}{L};\alpha-\beta,\alpha+\beta k\right)du \\
& +\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{ALC}{(LC)^{k+1}}\ \mathcal{E}_k\left(t,-\frac{R}{L};\alpha-\beta,\beta k+1\right) \\
& +\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{BLC}{(LC)^{k+1}}\ \mathcal{E}_k\left(t,-\frac{R}{L};\alpha-\beta,\beta k+2\right) \\
& +\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{RAC}{(LC)^{k+1}}\ \mathcal{E}_k\left(t,-\frac{R}{L};\alpha-\beta,\beta(k-1)+\alpha+1\right).
\end{aligned}
\tag{20}
$$

Case II. When periodic electromotive force is applied, i.e., $E(t)=E_0\cos\omega t$, where E_0 and ω are constants, then (19) yields,

$$
\begin{aligned}
I(t) = {} & CE_0\int_0^t \cos\omega(t-u)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{1}{(LC)^{k+1}}\,\mathcal{E}_k\left(u,-\frac{R}{L};\alpha-\beta,\alpha+\beta k\right)du \\
& +\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{ALC}{(LC)^{k+1}}\ \mathcal{E}_k\left(t,-\frac{R}{L};\alpha-\beta,\beta k+1\right) \\
& +\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{BLC}{(LC)^{k+1}}\ \mathcal{E}_k\left(t,-\frac{R}{L};\alpha-\beta,\beta k+2\right) \\
& +\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{RAC}{(LC)^{k+1}}\ \mathcal{E}_k\left(t,-\frac{R}{L};\alpha-\beta,\beta(k-1)+\alpha+1\right).
\end{aligned}
\tag{21}
$$

Case III. When periodic electromotive force is applied, i.e. $E(t)=E_0\sin\omega t$, where E_0 and ω are constants, then (19) reduces to

$$
\begin{aligned}
I(t) = {} & CE_0\int_0^t \sin\omega(t-u)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{1}{(LC)^{k+1}}\,\mathcal{E}_k\left(u,-\frac{R}{L};\alpha-\beta,\alpha+\beta k\right)du \\
& +\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{ALC}{(LC)^{k+1}}\;\mathcal{E}_k\left(t,-\frac{R}{L};\alpha-\beta,\beta k+1\right) \\
& +\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{BLC}{(LC)^{k+1}}\;\mathcal{E}_k\left(t,-\frac{R}{L};\alpha-\beta,\beta k+2\right) \\
& +\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{RAC}{(LC)^{k+1}}\;\mathcal{E}_k\left(t,-\frac{R}{L};\alpha-\beta,\beta(k-1)+\alpha+1\right).
\end{aligned}
\tag{22}
$$

Some special cases of Equation (17)**.** When we take $\beta = 1$, (17) reduces to the form,

$$
L\ D^{\alpha}I(t)+R\frac{dI}{dt}+\frac{1}{C}I(t)=E(t)\,,\qquad 1<\alpha\le 2. \tag{23}
$$

On further simplification yields,

$$
\begin{aligned}
I(t) = {} & C\int_0^t E(t-u)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{1}{(LC)^{k+1}}\;\mathcal{E}_k\left(u,-\frac{R}{L};\alpha-1,\alpha+k\right)du \\
& +\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{ALC}{(LC)^{k+1}}\;\mathcal{E}_k\left(t,-\tfrac{R}{L};\alpha-1,k+1\right) \\
& +\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{BLC}{(LC)^{k+1}}\;\mathcal{E}_k\left(t,-\frac{R}{L};\alpha-1,k+2\right) \\
& +\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{RAC}{(LC)^{k+1}}\;\mathcal{E}_k\left(t,-\frac{R}{L};\alpha-1,\alpha+k\right),
\end{aligned}
\tag{24}
$$

where $1<\alpha\le 2 \Rightarrow \alpha-1>0$.

Similarly, on setting $\alpha = 2$ in (17) with $1<\alpha\le 2,\ 0<\beta\le 1$, this reduces to following form,

$$
L\frac{d^2I}{dt^2}+R\ D^{\beta}I(t)+\frac{1}{C}I(t)=E(t)\,. \tag{25}
$$

Solution of this equation can also be obtained by aforesaid method, which gives,

$$
\begin{aligned}
I(t) = {} & C\int_0^t E(t-u)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{1}{(LC)^{k+1}}\,\mathcal{E}_k\left(u,-\frac{R}{L};2-\beta,2+\beta k\right)du \\
& +\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{ALC}{(LC)^{k+1}}\;\mathcal{E}_k\left(t,-\frac{R}{L};2-\beta,\beta k+1\right) \\
& +\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{BLC}{(LC)^{k+1}}\;\mathcal{E}_k\left(t,-\frac{R}{L};2-\beta,2+\beta k\right) \\
& +\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{RAC}{(LC)^{k+1}}\;\mathcal{E}_k\left(t,-\frac{R}{L};2-\beta,\beta(k-1)+3\right).
\end{aligned}
\tag{26}
$$

4. Conclusion

Fractional calculus has been recognized as advantageous mathematical tool in modeling and control of dynamical systems. We have obtained the analytic solution of the second order fractional differential equation associated with a RLC electrical circuit in time domain using Caputo derivative in terms of Mittag-Leffler type function which can be implemented for computational study of behaviour of current.

Acknowledgement. The first author is thankful to Sardar Vallabhbhai National Institute of Technology, Surat, Gujarat, INDIA for awarding Junior Research Fellowship.

References

[1] Debnath, L., Bhatta, D., *Integral Transforms and their Applications,* Chapman & Hall, CRC Press, Boca Raton, 2007.

[2] Gómez, F., Rosales, J., Guĺa, M., *RLC electrical ciruit of non-integer order,* Cent. Eur. J. Phys., 2013, 1361-1365.

[3] Gómez, J.F., Astorga, C.M., Escobar, R.F., Medina, M.A., Guzmán, R., González, A., Baleanu, D., *Overview of simulation of fractional differential equation using numerical Laplace transform*, Cent. Eur. J. Phys., 2012, 1-14.

[4] Humbert, P., Agarwal, R.P., *Sur la fonction de Mittag-Leffler et quelquenes de ses généralisationes,* Bull. Sci. Math. (Ser. II), 1953, 180-185.

[5] Miller, K.S., Ross, B., *An Introduction to the Fractional Calculus and Fractional Differential Equations,* John Wiley & Sons, New York, 1993.

[6] Gorenflo, R., Kilbas, A.A., Mainardi, F., Rogosin, S.V., *Mittag-Leffler functions, related topics and applications*, Springer, 2014.

[7] Podlubny I., *Fractional Differential Equations,* Technical University of Kosice, Academic Press, 1999.

[8] Shah, P.V., Patel, A.D., Salehbhai, I.A., Shukla, A.K., *Analytic Solution for the RL electric circuit model in fractional order,* Abstract and Applied Analysis, 2014, (http://dx.doi.org/10.1155/2014/343814).

Accepted: 06.05.2016