THE SOLUTIONS OF PAULI EQUATION IN DE SITTER SPACE BACKGROUND AND HOMOGENEOUS MANIFOLD $SU(2)/U(1)$

F. Safari
H. Jafari[1]

Department of Mathematics
University of Mazandaran
Babolsar
Iran

J. Sadeghi

Department of Physics
Faculty of Basic Sciences
University of Mazandaran
Babolsar
Iran

Abstract. In this paper, we consider a particle with spin $\frac{1}{2}$ in de Sitter space-time. Procedure for transition to the Pauli approximation is conducted in the equation in the variable (t, r), obtained after separating the angular dependence of (θ, φ) from the wave function. We make the suitable second order equation corresponding to de Sitter space time for particle spine $\frac{1}{2}$ we then compare this equation to Jacobi polynomial and obtain the wave function and eigenvalues (energy spectrum) which is important for the corresponding system.
Also, by taking the advantage from weight and main function in Jacobi polynomial and obtain the corresponding algebra.

Keywords: factorization method; Dirac equation; Jacobi polynomial; raising and lowering operators.

1. Introduction

De Sitter and Anti de Sitter models have taken attraction in the context of development of the quantum theory in curved space-time for a long time. The most attention about such curved space-time was to solve the wave function for field with different spin in relativistic [1]– [5].

Now, we want to make the equation corresponding to spine $\frac{1}{2}$ particle nonrelativistic in de Sitter space-time. This equation lead us to compare it with Jacobi polynomial [6], [7]. By comparing two equation order by order, one can obtain the exact solution

[1]Corresponding author. E-mail: jafari@umz.ac.ir

for the wave function and eigenvalues (energy spectra). Finally, we take advantage from weight and main function from Jacobi polynomial [9], [10], and obtain the generator of algebra for the corresponding system. This form of algebra gives the symmetry of the system which is important in physics and mathematics point of view. For this reason, first we have to make the Pauli equation in de Sitter space-time.

2. The Pauli equation in de Sitter space

The de Sitter space-time Dirac equation in orthogonal coordinates is,

$$\left[i\gamma^k \left(e^{\alpha}_{(k)}\partial_\alpha + A_k\right) - m\right]\Psi = 0, \tag{1}$$

where

$$A_k(x) = \frac{1}{2}e^{\alpha}_{(k);\alpha}(x) = \frac{1}{2}\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}\sqrt{-g}e^{\alpha}_{(k)}, \tag{2}$$

In order to arrange the $A_k(x)$, we have to introduce the de Sitter metric background,

$$dS^2 = dt^2 - \cosh^2 t dr^2 - \cosh^2 t \sin^2 r d\theta^2 - \cosh^2 t \sin^2 r \sin^2 \theta d\phi^2, \tag{3}$$

where

$$\begin{aligned} g_{00} &= 1, & g_{11} &= -\cosh^2 t, \\ g_{22} &= -\cosh^2 t \sin^2 r, & g_{33} &= -\cosh^2 t \sin^2 r \sin^2 \theta, \end{aligned} \tag{4}$$

and

$$g = det(g_{i,j}), \qquad i, j = 0, .., 3,$$

Also, the corresponding tetrad,

$$\begin{aligned} e^{\alpha}_{(0)} &= (1,0,0,0), & e^{\alpha}_{(1)} &= \left(0,0,\frac{1}{\cosh t \sin r},0\right), \\ e^{\alpha}_{(2)} &= \left(0,0,0,\frac{1}{\cosh t \sin r \sin\theta}\right), & e^{\alpha}_{(3)} &= \left(0,\frac{1}{\cosh t},0,0\right), \end{aligned} \tag{5}$$

So, by using (2), (3), (4) and (5) we can rewrite equation (1) as,

$$\left[i\,\gamma^0 \cosh t\left(\partial_t+\frac{3}{2}\tanh t\right)+\frac{1}{\sin r}\Sigma_{\theta,\phi}+i\,\gamma^3\left(\partial_r+\frac{1}{\tan r}\right)-m\cosh t\right]\Psi=0, \tag{6}$$

where $\Sigma_{\theta,\phi} = i\,\gamma^1\partial_\theta + \gamma^2\frac{i\,\partial_\phi+i\,\sigma^{12}\cos\theta}{\sin\theta}$.

Now, we choose the change of variable for the wave function which is given by

$$\Psi(x) = \frac{1}{\sin r \cosh^{\frac{3}{2}} t}\varphi(x), \tag{7}$$

we obtain a simpler equation

$$\left[i\,\gamma^0\cosh t\frac{\partial}{\partial t}+i\,\gamma^3\frac{\partial}{\partial r}+\frac{1}{\sin r}\Sigma_{\theta\phi}-m\cosh t\right]\varphi=0, \tag{8}$$

Here, we need to describe the φ in spin system, as we know the φ have a three indexes as j, m and δ. We use the Wigners function j which define spin as $\nu = j+\frac{1}{2}$ and m angular momentum and δ is parity. So, we consider the following expression

$$\varphi_{jm\delta}=\begin{vmatrix} f_1(t,r)D_{-\frac{1}{2}} \\ f_2(t,r)D_{+\frac{1}{2}} \\ f_3(t,r)D_{-\frac{1}{2}} \\ f_4(t,r)D_{+\frac{1}{2}} \end{vmatrix}=\begin{vmatrix} f_1(t,r)D_{-\frac{1}{2}} \\ f_2(t,r)D_{+\frac{1}{2}} \\ \delta f_2(t,r)D_{-\frac{1}{2}} \\ \delta f_1(t,r)D_{+\frac{1}{2}} \end{vmatrix} \tag{9}$$
$$\delta=\pm 1,\qquad f_4=\delta f_1,\qquad f_3=\delta f_2,$$

Here we take $\delta = 1$, so the corresponding equation will he as,

$$\begin{gathered}\left(\frac{\partial}{\partial r}+\frac{\nu}{\sin r}\right)f+\left(i\,\cosh t\frac{\partial}{\partial t}+\delta m\cosh t\right)g=0,\\ \left(\frac{\partial}{\partial r}-\frac{\nu}{\sin r}\right)g-\left(i\,\cosh t\frac{\partial}{\partial t}-\delta m\cosh t\right)f=0,\end{gathered} \tag{10}$$

Insted of $f_1(t,r)$ and $f_2(t,r)$, we use the linear combination which are given by,

$$f(t,r)=\frac{f_1+f_2}{\sqrt{2}},\qquad\qquad g(t,r)=\frac{f_1-f_2}{i\,\sqrt{2}}.$$

First, we separate the rest energy by the formal change,

$$i\,\frac{\partial}{\partial t}\qquad\Rightarrow\qquad M+i\,\frac{\partial}{\partial t}.$$

consider just $\delta = 1$,

$$\begin{gathered}\frac{1}{\cosh t}\left(\frac{\partial}{\partial r}+\frac{\nu}{\sin r}\right)f+\left(M+i\,\frac{\partial}{\partial t}+M\right)g=0,\\ \frac{1}{\cosh t}\left(\frac{\partial}{\partial r}-\frac{\nu}{\sin r}\right)g-\left(M+i\,\frac{\partial}{\partial t}-M\right)f=0,\end{gathered} \tag{11}$$

We solve the up system, therefore, non-relativistic functions $f(t,r)$ and $g(t,r)$ will obtain of equation following [8],

$$i\frac{\partial f}{\partial t}=-\frac{1}{2M}\frac{1}{\cosh^2 t}\left(\frac{\partial^2}{\partial r^2}-\frac{\nu^2+\nu\cos r}{\sin^2 r}\right)f, \tag{12}$$

and

$$g=-\frac{1}{2M}\frac{1}{\cosh t}\left(\frac{\partial}{\partial r}+\frac{\nu}{\sin r}\right)f, \tag{13}$$

so, we solve equation (12) with the following separation of variable [8],

$$f(t,r) = e^{-iE\tanh t} f(r), \tag{14}$$

where

$$\left(\frac{d^2}{dr^2} - \frac{\nu^2 + \nu\cos r}{\sin^2 r} + 2ME\right) f(r) = 0. \tag{15}$$

Now, we use a new variable, $z = -\cos r, z \in (-1, +1)$ in equation (15), so we have

$$(1-z^2)\frac{d^2 f}{dz^2} - z\frac{df}{dz} + \left(\nu\frac{z}{1-z^2} - \frac{\nu^2}{1-z^2} + 2ME\right) f = 0. \tag{16}$$

In order to solve equation (16), we use of factorization method and following change of variable

$$f(r) = U(z)P(z) \tag{17}$$

Substituting (17) in (16) leads to

$$\begin{aligned}&(1-z^2)P''(z) + \left(2\frac{U'}{U}(1-z^2) - z\right) P'(z)\\ &+\left((1-z^2)\frac{U''}{U} - z\frac{U'}{U} + \frac{\nu z}{1-z^2} - \frac{\nu^2}{1-z^2} + 2ME\right) P(z) = 0.\end{aligned} \tag{18}$$

In order to obtain function U, we compared (18) with the following associated Jacobi differential equation

$$\begin{aligned}&(1-z^2)P''^{(\alpha,\beta)}_{n,m}(z) - [\alpha - \beta + (\alpha+\beta+2)z]P'^{(\alpha,\beta)}_{n,m}(z)\\ &+\big[n(\alpha+\beta+n+1) - \frac{m(\alpha+\beta+m+(\alpha-\beta)z)}{1-z^2}\big]P^{(\alpha,\beta)}_{n,m}(z) = 0.\end{aligned} \tag{19}$$

So we obtain $U(z)$ as,

$$U(z) = N_0(1+z)^{\frac{2\beta+1}{4}}(1-z)^{\frac{2\alpha+1}{4}}, \tag{20}$$

Therefore, the corresponding eigenfunction will be as,

$$f(z) = N_0(1+z)^{\frac{2\beta+1}{4}}(1-z)^{\frac{2\alpha+1}{4}} P^{(\alpha,\beta)}_{n,m}(z).$$

By comparing multiples $P^{(\alpha,\beta)}_{n,m}$ and P in equations (18) and (19), we can obtain α, β and E as

$$\alpha = \frac{m-j}{2m-1}, \qquad \beta = -1-\alpha, \qquad 2ME = n^2. \tag{21}$$

Finally, we have

$$f(r) = N_0\left(\cot\frac{r}{2}\right)^{\frac{2\alpha+1}{2}} \psi^{(\alpha)}_{n,m}(r),$$

where

$$\psi^{(\alpha)}_{n,m}(r) = P^{(\alpha,-1-\alpha)}_{n,m}(-\cos\theta) = a_n(-1)^m \tag{22}$$
$$\left[(1-x)^{-\frac{2m+2\alpha-1}{4}}(1+x)^{-\frac{2m-2\alpha-3}{4}}\left(\frac{d}{dx}\right)^{n-m}(1-x)^{n+\alpha}(1+x)^{n-1-\alpha}\right]_{x=-\cos\theta}.$$

Also, in view of normalization condition in quantum mechanics, we obtain the normalization coefficient N_0 as,

$$\int_{-\infty}^{\infty} f(r)f^*(r)dz = 1, \qquad N_0^2\int_{-\infty}^{\infty}(\psi^{(\alpha)}{}_{n,m}(r))^2U^2(r)d(r) = 1,$$

Thus, solutions of the Pauli equation in non-static de Sitter metrics have been constructed as follow

$$\Psi^{Pauli}_{Ejm}(t,r,\theta,\phi) = e^{iE\tanh t}f(r)\begin{vmatrix} D^j_{-m,-\frac{1}{2}}(\phi,\theta,0) \\ D^j_{-m,+\frac{1}{2}}(\phi,\theta,0)\end{vmatrix}$$

Spectral parameter E (it does not represent the energy of stationary states) is quantized according to the rule (the same for state with different parities) of quantum mechanic

$$2ME = n^2. \tag{23}$$

Here, we can obtain the raising and lowering operators corresponding to equation (15) with corresponding α and β [6]. So, we factorized equation (21) as

$$B(m)A(m)\psi^{(\alpha)}_{n,m}(r) = E(n,m)\psi^{(\alpha)}_{n,m}(r),\;\; A(m)B(m)\psi^{(\alpha)}_{n,m-1}(r) = E(n,m)\psi^{(\alpha)}_{n,m-1}(r)$$

where

$$B(m) = \frac{d}{dr} + W_m(r),\; A(m) = -\frac{d}{dr} + W_m(r),\; E(n,m) = n^2 - (m-1)^2, \tag{24}$$

where the super potential $W_m(r)$ will be as $W_m(r) = (1-m)\cot r - \frac{1}{2}(1+2\alpha)\csc r$. By using (24), we obtain operators $J_\pm(m)$ that describe the shape invariance symmetry on the homogeneous manifold $SU(2)/U(1)$ parametrized by $\{\theta,\varphi\}$ which are ([11])

$$J_\pm(m) \;=\; \pm\frac{\partial}{\partial\theta} - \frac{\imath}{\sin\theta}\frac{\partial}{\partial\varphi} - \frac{m-1}{2\tan\theta}. \tag{25}$$

So, the shape invariance equations on the homogeneous manifold $SU(2)/U(1)$ are obtained by following equation,

$$\begin{aligned} J_+(m)J_-(m)\phi_{n,m}(\theta,\varphi) &= E(n,m)\phi_{n,m}(\theta,\varphi), \\ J_-(m)J_+(m)\phi_{n,m-1}(\theta,\varphi) &= E(n,m)\phi_{n,m-1}(\theta,\varphi), \end{aligned} \tag{26}$$

where spectrum $E(n,m)$ have already been introduced in equations (24) and the bases of representation will be as,

$$\phi_{n,m}(\theta,\varphi) = e^{\frac{\imath\varphi(1+2\alpha)}{2}}P^{(\alpha,-1-\alpha)}_{n,m}(-\cos\theta). \tag{27}$$

3. Conclusion

In this paper, we studied the Dirac system with spin $\frac{1}{2}$ equation. On the other hand we introduced the Jacobi equation and obtained the first order operators for such equation. We then compered the Dirac equation with Jacobi second order equation and obtained the exact solution for the wave function and energy spectrum. Finally we take advantage from Jacobi equation and factorized the corresponding Dirac system with first order equations. These first order equation in physics point of view will be form of raising and lowering operators. In here we had two cases, the first one we factorized second order Dirac equation in terms of indices n and m . In second case just was factorized in terms of indices m. In future, two operators can be applied for different system for the coherent states. Also, such operators help us to discuss the generators of supersymmetry as super charges. Also, these operators for the partner potential and partner Hamiltonian may be play important role.

References

[1] DIRAC, P.A.M., *The electron wave equation in the de Sitter space*, Ann. Math., 36 (1935), 657-669.

[2] GOTO, K., *Wave equations in de Sitter space*, Progr. Theor. Phys., 6 (1951), 1013-1014.

[3] NACHTMANN, O., *Quantum theory in de-Sitter space*, Commun. Math. Phys., 6 (1967), 1-16.

[4] CHERNIKOV, N.A., TAGIROV, E.A., *Quantum theory of scalar field in de Sitter space-time*, Ann. Inst. H. Poincaré. A. 9 (1968), 109-141.

[5] MISHIMA, T., NAKAYAMA, A., *Particle production in de Sitter spacetime*, Progr. Theor. Phys., 77 (1987), 218-222.

[6] SADEGHI, J., *Raising and lowering of generalized Hulthèn potential from supersymmetry approaches*, Int. J. Theor. Phys., 46 (2007), 492-502.

[7] FAKHRI, H., SADEGHI, J., *Supersymmetry approaches to the bound states of the generalized Woods-Saxon potential*, Mod. Phys. Lett. A 19 (2004), 615.

[8] OVSIYUK, E.M., KAZMERCHUK, K.V., *On solutions of the Pauli equation in non-static de Sitter metrics*, 2015.

[9] JAFARIZADEH, M.A., FAKHRI, H., *Parasupersymmetry and shape invariance in differential equations of mathematical physics and quantum mechanics,* Ann. Phys. (New York) 262 (1998), 260-276.

[10] JAFARIZADEH, M.A., FAKHRI, H., *Supersymmetry and shape invariance in differential equations of mathematical physics*, Phys. Lett. A 230 (1997), 164-170.

[11] FAKHRI, H., *Solution of the Dirac equation on the homogeneous manifold $SL(2,c)/GL(1,c)$ in the presence of a magnetic monopole field*, Journal of Physics A: Mathematical and General 33 (2000), 293-305.

Accepted: 07.07.2016