APPROXIMATE SOLUTION OF CONVECTION-DIFFUSION EQUATIONS USING A HAAR WAVELET METHOD

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Abstract. We present here a Haar wavelet method for numerical solution of convection-diffusion equations. Haar wavelet is a powerful mathematical tool used to solve various type of partial differential equations. The solutions obtained by Haar wavelet are more accurate and efficient.

Keywords: convection-diffusion equation, Haar wavelet method, function approximation.

AMS Subject Classification: 65M.

1. Introduction

It has been observed from the literature that many researchers are developing fast, accurate and efficient numerical schemes to handle the different problems arising in several areas of engineering and sciences. In the last few decades wavelet approaches are becoming more popular in the field of numerical approximations. Wavelet, being a powerful mathematical tool, has been widely used in image digital processing, quantum field theory and numerical analysis. Beginning from 1980s, wavelet has been used to find the numerical solution of partial differential equations. Wavelet methods attracts the interest of researchers to find the numerical solution of partial differential equations. Haar wavelet which are Daubechies wavelets of order 1 consists of piecewise constant functions. A drawback of Haar wavelet is their discontinuity. Since the derivatives do not exist in the breaking points, so it is impossible to apply the Haar wavelet for the solution of partial differential equation directly. Several definitions, concepts and modifications of Haar functions and various generalizations have been published and used. Galerkin and collocation techniques were applied to solve partial differential equations using Haar wavelet. The Haar wavelet method for solving partial differential equations has many advantages features such as: it has very high accuracy, small computational costs, sprase representation, fast transformation, simplicity and possibility of implementation of fast algorithms especially if matrix representation is used. Also, it is very convenient for solving boundary value problems. Convection-diffusion equation is a special partial differential equation occur in numerous engineering problems. Convection-diffusion equation is a second order parabolic partial differential equation.

Consider the partial differential equation of the form

$$\frac{\partial u}{\partial t}+\varepsilon\frac{\partial u}{\partial x}=\gamma\frac{\partial^2 u}{\partial x^2},\quad 0<x<1,\ t>0, \tag{1}$$

subject to the initial conditions $u(x,0)=f(x)$, $0\le x\le 1$ and boundary conditions $u(0,t)=0$, $u(1,t)=0$, $t\ge 0$, where γ represents the viscosity coefficient and ε is the phase speed and both are assumed to be positive. The given function f is sufficiently smooth.

Wavelet analysis ia a new branch of mathematics and widely applied in differential equations. Several methods have been proposed to find the numerical solution of different linear and nonlinear partial differential equations. A survey of numerical solutions of differential equation is presented in Hariharan [7]. In 1910, Hungarian mathematicians Alfred Haar [6] introduced a function, known as Haar function which presents a rectangular pulse pair. There are many wavelet families such as Haar wavelet [6], Daubechies wavelet [5], Hermite-type trigonometric wavelet and many more. Among all these wavelet families, Haar is the simplest orthonormal wavelet with compact support. The pioneer work was done by Chen and Hsiao [3] in system analysis with Haar wavelet, who first derive a Haar operational matrix for the integral of the Haar function vector and also presented wavelet approach for optimising dynamic systems in [4]. Haar wavelet has many attractive applications in image coding, edge extraction and binary logic design. Hariharan et al. [10] presented the numerical solution of Fisher's equation using Haar wavelet. Also, Hariharan and Kannan [9] presented the numerical solution of Fitzhugh-Nagumo equation using Haar wavelet. Berwal et al. [1] presented the numerical solution of Telegraph equation. Celik [2] presented the numerical solution of generalized Burger-Huxley equation with Haar wavelet method. Lepik [12, 13] presented the numerical solution of differential and integral equation with Haar wavelet method. Kheiri and Ghafouri [11] presented Haar and Legendre wavelets collocation methods for the numerical solution of Schrondinger and wave equation. Haar wavelet method for solving some nonlinear parabolic equation is presented in [8].

In Section 2, we briefly describe the Haar wavelet method. In Section 3, we have given function approximation. Section 4, a method for solving convection-diffusion equation has been presented, and in Section 5, numerical examples have been solved using the Haar wavelet method to illustrate the efficiency and accuracy of the present method.

2. Haar wavelet method

The Haar functions are an orthogonal family of switched rectangular waveforms where amplitudes can differ from one function to another. They are defined in the interval $[0, 1]$.

$$h_i(x) = \begin{cases} 1, & \alpha \leq x < \beta, \\ -1, & \beta \leq x < \gamma, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

where $\alpha = \frac{k}{m}$, $\beta = \frac{k+0.5}{m}$ and $\gamma = \frac{k+1}{m}$. Integer $m = 2^j$, $(j = 0, 1, 2, 3, 4, ..., J)$ indicates the level of the wavelet, and $k = 0, 1, 2, 3, ..., m-1$ is the translation parameter. Maximal level of resolution is J. The index i is calculated according the formula $i = m + k + 1$. In the case of minimal values, $m = 1$, $k = 0$ we have $i = 2$. The maximal value of i is $i = 2M$, where $M = 2^J$. It is assumed that the value $i = 1$, corresponding to the scaling function in $[0, 1]$

$$h_1(x) = \begin{cases} 1, & 0 \leq x \leq 1, \\ 0, & \text{otherwise}. \end{cases} \tag{3}$$

Let us define the collocation points $x_l = \frac{(l-0.5)}{2M}$, where $l = 1, 2, 3, ..., 2M$ and discretize the Haar function $h_i(x)$. Using the following four notations of Haar functions:

$$h_1(x) = [1, 1, 1, 1],\ h_2(x) = [1, 1, -1, -1],\ h_3 = [1, -1, 0, 0],\ h_4 = [0, 0, 1, -1],$$

we introduce the following notation:

$$H_4(x) = [h_1(x), h_2(x), h_3(x), h_4(x)]^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \tag{4}$$

Here $H_4(x)$ is called Haar coefficient matrix. It is a square matrix of order 4, and is defined as $H(i, l) = (h_i(x_l))$, which has dimension $2M \times 2M$.

Let us integrate equation (2). We get

$$P_{1,i}(x) = \begin{cases} x - \alpha, & x \epsilon [\alpha, \beta), \\ \gamma - x, & x \epsilon [\beta, \gamma), \\ 0, & \text{elsewhere}. \end{cases} \tag{5}$$

$$P_{2,i}(x) = \begin{cases} \frac{1}{2}(x-\alpha)^2, & x \epsilon [\alpha, \beta), \\ \frac{1}{4m^2} - \frac{1}{2}(\gamma - x)^2, & x \epsilon [\beta, \gamma), \\ \frac{1}{4m^2}, & x \epsilon [\gamma, 1), \\ 0, & \text{elsewhere}. \end{cases} \tag{6}$$

3. Function approximation

Any square integrable function $y(x)$ in the interval $[0,1]$ can be expanded by a Haar series of infinite terms:

$$y(x) = \sum_{i=1}^{\infty} c_i h_i(x), \tag{7}$$

where the Haar coefficients c_i are determined as:

$$c_0 = \int_0^1 y(x) h_0(x) dx, \tag{8}$$

$$c_i = 2^j \int_0^1 y(x) h_i(x) dx, \tag{9}$$

where $i = 2^j + k$, $j \geq 0$ and $0 \leq k < 2^j$, $x \in [0,1]$ such that the following integral square error ε is minimized:

$$\varepsilon = \int_0^1 \left[y(x) - \sum_{i=1}^{m} c_i h_i(x) \right]^2 dx, \tag{10}$$

where $m = 2^j$ and $j = 0, 1, 2, 3, \ldots$. Usually, the series expansion of (7) contains infinite terms. If $y(x)$ is piecewise constant by itself or may be approximated as piecewise constant during each subinterval, then $y(x)$ will be terminated at finite m terms. This means

$$y(x) \cong \sum_{i=1}^{m} c_i h_i(x) = {c_m}^T h_m(x), \tag{11}$$

where the coefficients ${c_m}^T$ and the Haar function vectors $h_m(x)$ are defined as: ${c_m}^T = [c_1, c_2, \ldots, c_m]$ and $h_m(x) = [h_1(x), h_2(x), \ldots, h_m(x)]^T$,

4. Method for solving the convection-diffusion equation

Consider the convection-diffusion equation (1) with initial conditions $u(x,0) = f(x)$ and boundary conditions are $u(0,t) = g_0(t)$, $u(1,t) = g_1(t)$. Assume that $\dot{u}''(x,t)$ can be expanded in term of Haar wavelets as follows:

$$\dot{u}''(x,t) = \sum_{i=1}^{2M} a_i h_i(x), \quad t \epsilon (t_s, t_{s+1}]. \tag{12}$$

Integrating the above equation with respect to t from t_s to t, and twice with respect to x, from 0 to x, we get

$$u''(x,t) = (t-t_s)\sum_{i=1}^{2M} a_i h_i(x) + u''(x,t_s), \tag{13}$$

$$u'(x,t) = (t-t_s)\sum_{i=1}^{2M} a_i P_{1,i}(x) + u'(x,t_s) - u'(0,t_s) + u'(0,t), \tag{14}$$

$$\begin{aligned} u(x,t) &= (t-t_s)\sum_{i=1}^{2M} a_i P_{2,i}(x) + u(x,t_s) - u(0,t_s) \\ &\qquad -x[u'(0,t_s) - u'(0,t)] + u(0,t), \end{aligned} \tag{15}$$

$$\dot{u}(x,t) = \sum_{i=1}^{2M} a_i P_{2,i}(x) + x\dot{u}'(0,t) + \dot{u}(0,t). \tag{16}$$

From the initial and boundary conditions, we have the following equations as:

$$u(x,0) = f(x), \quad u(0,t) = g_0(t), \quad u(1,t) = g_1(t), \quad u(0,t_s) = g_0(t_s),$$
$$u(1,t_s) = g_1(t_s), \quad \dot{u}(0,t) = \dot{g}_0(t), \quad \dot{u}(1,t) = \dot{g}_1(t).$$

At $x = 1$ in the formulae (15) and (16) and by using conditions, we have

$$\begin{aligned} u'(0,t) - u'(0,t_s) &= -(t-t_s)\sum_{i=1}^{2M} a_i P_{2,i}(1) + g_1(t) \\ &\qquad -g_1(t_s) + g_0(t_s) - g_0(t), \end{aligned} \tag{17}$$

$$\dot{u}'(0,t) = -\sum_{i=1}^{2M} a_i P_{2,i}(1) - \dot{g}_0(t) + \dot{g}_1(t). \tag{18}$$

If the equations (17) and (18) are substituted into equations (13) – (16), and the results are discretized by assuming $x \to x_l$ and $t \to t_{s+1}$, we obtain

$$u''(x_l,t_{s+1}) = (t_{s+1}-t_s)\sum_{i=1}^{2M} a_i h_i(x_l) + u''(x_l,t_s), \tag{19}$$

$$\begin{aligned} u'(x_l,t_{s+1}) &= (t_{s+1}-t_s)\sum_{i=1}^{2M} a_i P_{1,i}(x_l) + g_1(t_{s+1}) - g_1(t_s) + g_0(t_s) \\ &\qquad -g_0(t_{s+1}) - (t_{s+1}-t_s)\sum_{i=1}^{2M} a_i P_{2,i}(1) + u'(x_l,t_s), \end{aligned} \tag{20}$$

$$\begin{aligned} u(x_l,t_{s+1}) &= (t_{s+1}-t_s)\sum_{i=1}^{2M} a_i P_{2,i}(x_l) + u(x_l,t_s) + g_0(t_{s+1}) - g_0(t_s) \\ &\qquad +x_l\Big[-(t_{s+1}-t_s)\sum_{i=1}^{2M} a_i P_{2,i}(1) + g_1(t_{s+1}) \\ &\qquad -g_0(t_{s+1}) - g_1(t_s) + g_0(t_s)\Big], \end{aligned} \tag{21}$$

$$\begin{aligned}\dot{u}(x_l,t_{s+1}) &= \sum_{i=1}^{2M} a_i P_{2,i}(x_l) \\ &\quad + x_l\Big[-\sum_{i=1}^{2M} a_i P_{2,i}(1) - \dot{g}_0(t_{s+1}) + \dot{g}_1(t_{s+1})\Big] + \dot{g}_0(t_{s+1}).\end{aligned} \tag{22}$$

But, we know that

$$P_{2,i}(1) = \begin{cases} \frac{1}{2}, & i = 1, \\ \\ \frac{1}{4m^2}, & i > 1. \end{cases} \tag{23}$$

In the given scheme

$$\dot{u}(x_l,t_{s+1}) + \varepsilon u'(x_l,t_{s+1}) = \gamma u''(x_l,t_{s+1}), \tag{24}$$

which leads us from the time layer t_s to t_{s+1} is used. From here, wavelet coefficients are calculated and solution of wave equation is obtained.

5. Numerical examples

We present here, some numerical examples of Convection-diffusion equation, which shows the accuracy and efficiency of Haar wavelet method.

Example 1: Consider the convection-diffusion equation of the form

$$\frac{\partial u}{\partial t} + 0.1\frac{\partial u}{\partial x} = 0.01\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \qquad t > 0, \tag{25}$$

subject to the initial conditions $u(x,0) = e^{5x} sin\pi x, \quad 0 \leq x \leq 1$ and boundary conditions $u(0,t) = 0$, $u(1,t) = 0$, $t \geq 0$. The exact solution of this problem is:

$$u(x,t) = e^{5x-(0.25-0.01\pi^2)t} sin\pi x. \tag{26}$$

The process is started with $u(x_l,0) = f(x_l)$, $u'(x_l,0) = f'(x_l)$, $u''(x_l,0) = f''(x_l)$. Numerical results are presented graphically in Figure 1 for $t = 0.1, 0.2, 0.3$ and 5.0. The value of $\Delta t = 0.00001$. Absolute errors for $t = 0.1$, $t = 0.2$ and $t = 0.3$ are presented in Table 1.

xL/64	t= 0.1	t=0.2	t=0.3
1	1.0E-006	3.2E-006	5.9E-006
3	2.2E-006	7.7E-006	1.5E-005
5	2.7E-006	1.0E-005	2.1E-005
7	3.0E-006	1.1E-005	2.4E-005
9	3.3E-006	1.2E-005	2.7E-005
11	3.4E-006	1.3E-005	2.9E-005
13	3.6E-006	1.3E-005	3.0E-005
15	3.6E-006	1.3E-005	3.0E-005
17	3.5E-006	1.3E-005	2.9E-005
29	3.0E-006	1.2E-005	2.8E-005
31	5.8E-006	2.4E-005	5.3E-005
33	9.5E-006	3.8E-005	8.6E-005
59	2.1E-004	8.1E-004	1.7E-003
61	2.3E-004	8.1E-004	1.6E-003
63	1.4E-004	4.4E-004	8.5E-004

Table 1: Comparison of maximum absolute errors for Example 1.

Example 2: Consider the convection-diffusion equation of the form

$$\frac{\partial u}{\partial t} + 0.22\frac{\partial u}{\partial x} = 0.5\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \quad t > 0, \tag{27}$$

subject to the initial conditions $u(x, 0) = e^{0.22x} sin\pi x, \quad 0 \leq x \leq 1$ and boundary conditions $u(0, t) = 0$, $u(1, t) = 0$, $t \geq 0$. The exact solution of this problem is:

$$u(x, t) = e^{0.22x-(0.0242-0.5\pi^2)t} sin\pi x. \tag{28}$$

The process is started with $u(x_l, 0) = f(x_l)$, $u'(x_l, 0) = f'(x_l)$, $u''(x_l, 0) = f''(x_l)$. Numerical results are presented graphically in Figure 2 for $t = 0.1, 0.2, 0.3$. The value of $\Delta t = 0.00001$. Absolute errors for $t = 0.1$, $t = 0.2$ and $t = 0.3$ are presented in Table 2.

Example 3: Consider the convection-diffusion equation of the form

$$\frac{\partial u}{\partial t} + 0.1\frac{\partial u}{\partial x} = 0.2\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \quad t > 0, \tag{29}$$

subject to the initial conditions $u(x, 0) = e^{0.25x} sin\pi x, \quad 0 \leq x \leq 1$ and boundary conditions $u(0, t) = 0$, $u(1, t) = 0$, $t \geq 0$. The exact solution of this problem is:

$$u(x, t) = e^{0.25x-(0.0125-0.2\pi^2)t} sin\pi x. \tag{30}$$

The process is started with $u(x_l, 0) = f(x_l)$, $u'(x_l, 0) = f'(x_l)$, $u''(x_l, 0) = f''(x_l)$. Numerical results are presented graphically in Figure 3 for $t = 0.1, 0.2, 0.3$. The value of $\Delta t = 0.00001$. Absolute errors for $t = 0.1$, $t = 0.2$ and $t = 0.3$ are presented in Table 3.

xL/64	t= 0.1	t=0.2	t=0.3
1	7.0E-007	2.6E-006	5.0E-006
3	2.3E-006	8.4E-006	1.5E-005
5	4.5E-006	1.5E-005	2.7E-005
7	7.2E-006	2.2E-005	4.0E-005
9	1.0E-005	3.0E-005	5.3E-005
11	1.3E-005	3.9E-005	6.7E-005
13	1.6E-005	4.8E-005	8.2E-005
15	2.0E-005	5.7E-005	9.6E-005
17	2.3E-005	6.6E-005	1.1E-004
29	4.2E-005	1.1E-004	1.8E-004
31	4.4E-005	1.1E-004	1.8E-004
33	4.6E-005	1.2E-004	1.9E-004
59	1.8E-005	4.5E-005	6.9E-005
61	1.1E-005	2.8E-005	4.3E-005
63	4.1E-006	9.7E-006	1.4E-005

Table 2: Comparison of maximum absolute errors for Example 2.

xL/64	t= 0.1	t=0.2	t=0.3
1	6.0E-008	9.4E-008	5.2E-007
3	5.0E-008	5.4E-007	1.9E-006
5	1.9E-007	1.4E-006	4.0E-006
7	6.1E-007	2.7E-006	6.7E-006
9	1.1E-006	4.4E-006	9.9E-006
11	1.8E-006	6.3E-006	1.3E-005
13	2.5E-006	8.4E-006	1.7E-005
15	3.3E-006	1.0E-005	2.1E-005
17	4.1E-006	1.3E-005	2.5E-005
29	8.3E-006	2.6E-005	4.9E-005
31	8.8E-006	2.7E-005	5.2E-005
33	9.2E-006	2.9E-005	5.5E-005
59	4.6E-006	1.3E-005	2.4E-005
61	3.0E-006	8.6E-006	1.5E-005
63	1.0E-006	3.0E-006	5.3E-006

Table 3: Comparison of maximum absolute errors for Example 3.

Conclusion

It is conclude that Haar wavelet method is more accurate, simple, fast and computationally attractive than other known methods to solve convection-diffusion equation. The above examples demonstrates the simplicity of the Haar wavelet solution. For getting the necessary accuracy the number of collocation points and the value of M in Equation (12) may be increased.

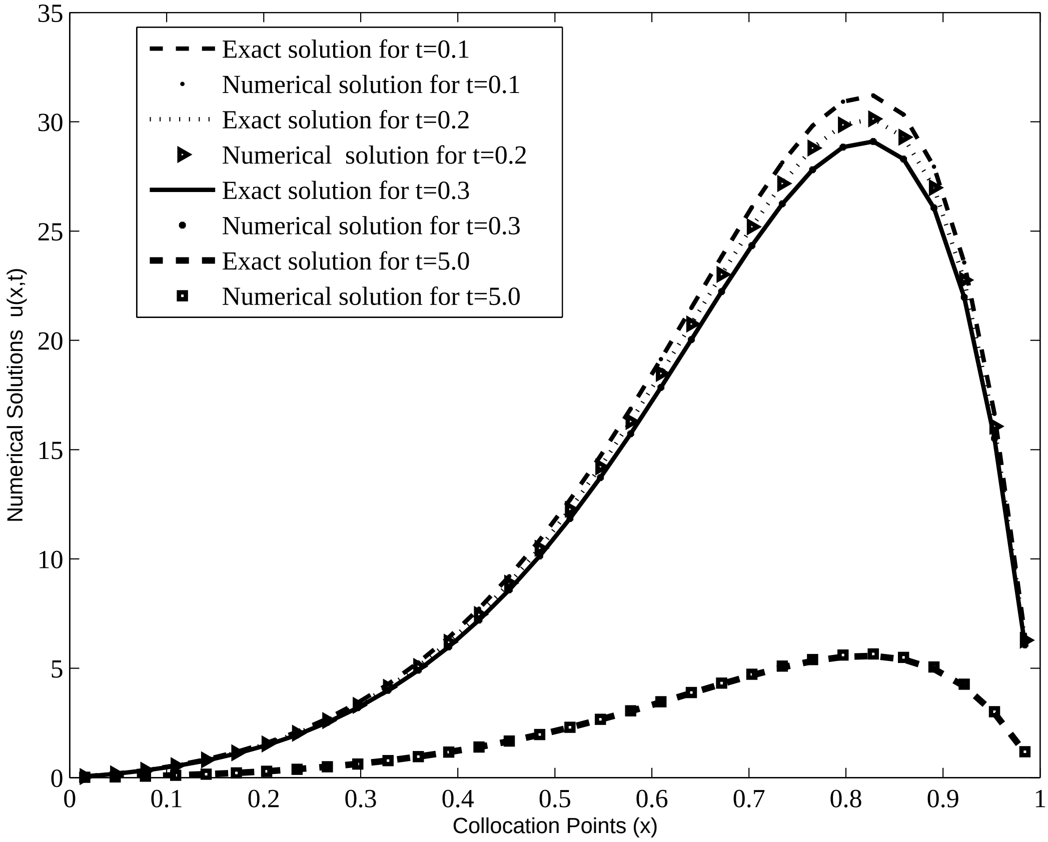

Figure 1: Comparison of numerical and exact solution for Example 1 at $t = 0.1, 0.2, 0.3, 5.0$.

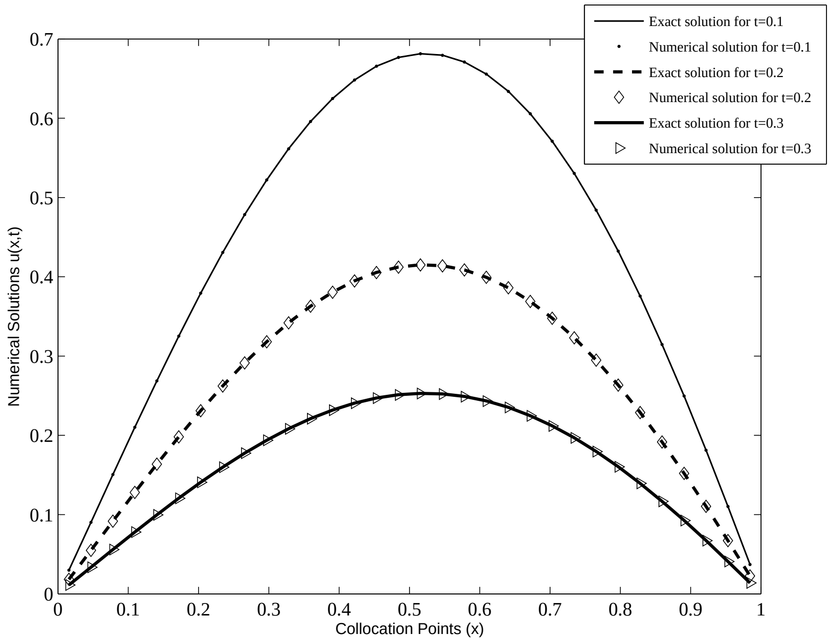

Figure 2: Comparison of numerical and exact solution for Example 2 at $t = 0.1, 0.2, 0.3$.

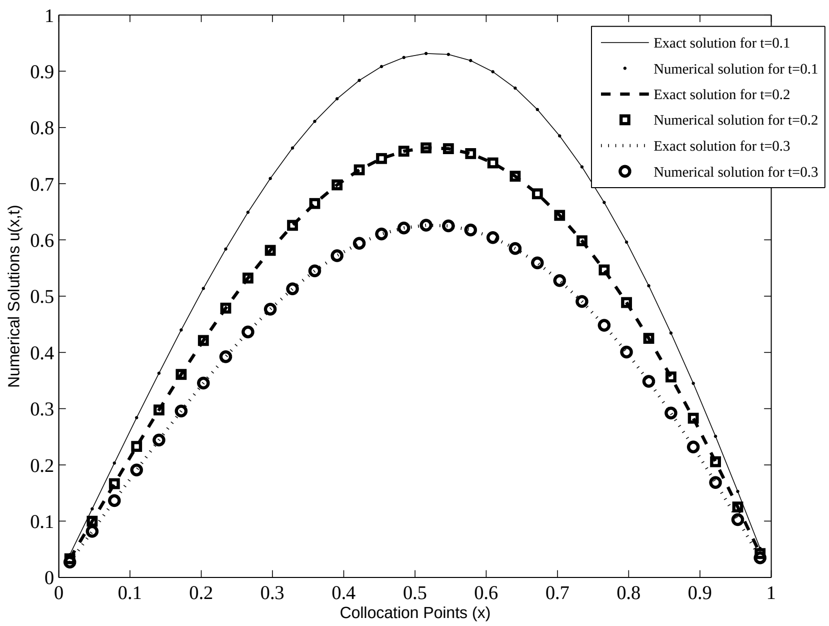

Figure 3: Comparison of numerical and exact solution for Example 3 at $t = 0.1, 0.2, 0.3$.

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