MORE ON UPPER AND LOWER FAINTLY ω-CONTINUOUS MULTIFUNCTIONS

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Abstract. The aim of this paper is to find under what minimal conditions the class of faintly ω-continuous multifunctions and the class of ω-continuous multifunctions agree. Similarly, we investigate when the class of faintly ω-continuous multifunctions and the class of faintly continuous multifunctions are the same.

Keywords: ω-open sets, ω-closed sets, faintly ω-continuous multifunctions.

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1. Introduction

It is well known that the class of faintly ω-continuous multifunctions contains properly the class of ω-continuous multifunctions and faintly continuous multifunctions. Recently, Zorlutuna et al. [12] introduced and studied the concept of upper (lower) faintly continuous multifunctions and obtain some characterizations. Also, some relationships between the concept of faint continuity and known concepts of continuity and weak continuity are given. Noiri et al. [8] introduced a new class of functions called a faintly m-continuous function, which is a generalization of both faintly continuous functions and m-continuous functions. Carpintero et al. [3] introduced and studied upper (lower) faintly ω-continuous multifunctions and obtain some characterizations and properties. Carpintero et al. [2] introduced and studied a new generalization of ω-continuous multifunctions called weakly ω-continuous multifunctions and obtain some characterizations. Also, some relationships between the concept of faint ω-continuity and known concepts of ω-continuity and weak ω-continuity are given. In this paper, we find some minimal conditions in order to obtain when the class of faintly ω-continuous multifunctions and the class of ω-continuous multifunctions agree, in the same form the class of faintly ω-continuous multifunctions and the class of faintly continuous multifunctions are the same.

2. Preliminaries

Throughout this paper, (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces in which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. For a subset A of (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure of A with respect to τ and the interior of A with respect to τ, respectively. A subset A is said to be regular closed [10] if $A = \mathrm{Cl}(\mathrm{Int}(A)$. Recently, as generalization of closed sets, the notion of ω-closed sets were introduced and studied by Hdeib [6]. A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A subset A is said to be ω-closed [6] if it contains all of its condensation points. The complement of an ω-closed set is said to be an ω-open set. It is well known that a subset W of a space (X, τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U \backslash W$ is countable. The family of all ω-open subsets of a topological space (X, τ) denoted by $\omega O(X, \tau)$, forms a topology on X which is finer than τ. The ω-closure and the ω-interior, that can be defined in the same way as $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively, will be denoted by $\omega\,\mathrm{Cl}(A)$ and $\omega\,\mathrm{Int}(A)$, respectively. We set $\omega O(X, x) = \{A : A \in \omega O(X, \tau) \text{ and } x \in A\}$. A subset N of a topological space (X, τ) is said to be ω-neighborhood of a point $x \in X$, if there exists an ω-open set V such that $x \in V \subset N$. A point $x \in X$ is called a θ-cluster point of A [11] if $\mathrm{Cl}(V) \cap A \neq \varnothing$ for every open set V of X containing x. The set of all θ-cluster points of A is called the θ-closure of A and is denoted by $\mathrm{Cl}_\theta(A)$. If $A = \mathrm{Cl}_\theta(A)$, then A is said to be θ-closed [11]. The complement of a θ-closed set is said to be a θ-open set [11]. The union of all θ-open sets contained in $A \subset X$ is called the θ-interior of A and is denoted by $\mathrm{Int}_\theta(A)$. It follows from [11] that the collection of θ-open sets in a topological space (X, τ) forms a topology τ_θ on X. By a multifunction $F : (X, \tau) \to (Y, \sigma)$, we mean a point-to-set correspondence from X into Y, also we always assume that $F(x) \neq \varnothing$ for all $x \in X$. For a multifunction

$F : (X, \tau) \to (Y, \sigma)$, the upper and lower inverse of any subset A of Y are denoted by $F^+(A)$ and $F^-(A)$, respectively, that is $F^+(A) = \{x \in X : F(x) \subseteq A\}$ and $F^-(A) = \{x \in X : F(x) \cap A \neq \varnothing\}$. In particular, $F^-(y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be surjective if $F(X) = Y$. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be lower ω-continuous [13] (resp. upper ω-continuous) multifunction if $F^-(V) \in \omega O(X, \tau)$ (resp. $F^+(V) \in \omega O(X, \tau)$) for every $V \in \sigma$. A multifunction F is said to be upper(lower) faintly continuous [12] if for each $x \in X$ and for each open set V of Y such that $x \in F^+(V)$ ($x \in F^-(V)$), there exists an open set U of X containing x such that $U \subset F^+(\mathrm{Int}(\mathrm{Cl}(V)))$ ($U \subset F^-(\mathrm{Int}(\mathrm{Cl}(V)))$). A multifunction F is faintly continuous [12] if it is both upper faintly continuous and lower faintly continuous. More generally F is said to be upper(lower) almost ω-continuous [5] if for each $x \in X$ and for each open set V of Y such that $x \in F^+(V)$ ($x \in F^-(V)$), there exists a $U \in \omega O(X, x)$ such that $U \subset F^+(\mathrm{Int}(\mathrm{Cl}(V)))$ ($U \subset F^-(\mathrm{Int}(\mathrm{Cl}(V)))$). A multifunction F is almost ω-continuous [5] if it is both upper almost ω-continuous and lower almost ω-continuous.

3. Faintly ω-continuous multifunctions

Definition 1 [3] A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be:

1. upper faintly ω-continuous at $x \in X$ if for each θ-open set V of Y containing $F(x)$, there exists $U \in \omega O(X)$ containing x such that $F(U) \subset V$;
2. lower faintly ω-continuous at $x \in X$ if for each θ-open set V of Y such that $F(x) \cap V \neq \varnothing$, there exists $U \in \omega O(X)$ containing x such that $F(u) \cap V \neq \varnothing$ for every $u \in U$;
3. upper (lower) faintly ω-continuous if it has this property at each point of X.
4. faintly ω-continuous if it is both upper faintly ω-continuous and lower faintly ω-continuous.

Remark 3.1 It is clear that the class of faintly ω-continuous multifunctions contains properly the classes of ω-continuous multifunctions and faintly continuous multifunctions. As we can seen in the following examples.

Example 3.2 Let $X = \mathbb{R}$ with the topology $\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - \mathbb{Q}\}$. Define a multifunction $F : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau)$ as follows:

$$F(x) = \begin{cases} \mathbb{Q} & \text{if } x \in \mathbb{R} - \mathbb{Q} \\ \mathbb{R} - \mathbb{Q} & \text{if } x \in \mathbb{Q}. \end{cases}$$

Then F is upper faintly ω-continuous but is not upper ω-continuous.

In a similar form, we can find a multifunction G that is lower faintly ω-continuous but is not lower ω-continuous.

Example 3.3 Let $X = \mathbb{R}$ with the following topologies: $\tau_1 = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$ and $\tau_2 = \{\emptyset, \mathbb{R}, \mathbb{R} - \mathbb{Q}, \mathbb{Q}\}$. Define a multifunction $F : (\mathbb{R}, \tau_1) \to (\mathbb{R}, \tau_2)$ as follows:

$$F(x) = \begin{cases} \mathbb{Q} & \text{if } x \in \mathbb{Q} \\ \mathbb{R} - \mathbb{Q} & \text{if } x \in \mathbb{R} - \mathbb{Q}. \end{cases}$$

Then F is upper faintly ω-continuous but is not upper faintly-continuous.

In a similar form, we can find a multifunction G that is lower faintly ω-continuous but is not lower faintly-continuous.

Remark 3.4 There exist any relations between the classes of ω-continuous multifunctions and faintly continuous multifunctions? In general there are no relations between them, as we can see in the following examples.

Example 3.5 Let $X = \mathbb{R}$ with the usual topology τ and $Y = \{a, b, c\}$ with topology $\tau_Y = \{\emptyset, Y, \{a\}\}$. Define a multifunction $F : (\mathbb{R}, \tau) \to (Y, \tau_Y)$ as follows:

$$F(x) = \begin{cases} \{a\} & \text{if } x < 0 \\ \{a, b\} & \text{if } x = 0 \\ \{c\} & \text{if } x > 0. \end{cases}$$

Then F is lower faintly continuous but is not lower ω-continuous.

In a similar form, we can find a multifunction G that is upper faintly continuous but is not upper ω-continuous.

Example 3.6 Let $X = \mathbb{R}$ with the following topologies: $\tau_1 = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$ and $\tau_2 = \{\emptyset, \mathbb{R}, \mathbb{R} - \mathbb{Q}, \mathbb{Q}\}$. Define a multifunction $F : (\mathbb{R}, \tau_1) \to (\mathbb{R}, \tau_2)$ as follows:

$$F(x) = \begin{cases} \mathbb{Q} & \text{if } x \in \mathbb{Q} \\ \mathbb{R} - \mathbb{Q} & \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

Then F is upper ω-continuous but is not upper faintly continuous.

In a similar form, we can find a multifunction G that is lower ω-continuous but is not lower faintly continuous.

It is possible to give some additional condition to a multifunction $F : (X, \tau) \to (Y, \sigma)$ that is upper (lower) faintly ω-continuous in order to obtain that F is upper (lower) faintly continuous? In the same form to a multifunction $F : (X, \tau) \to (Y, \sigma)$ that is upper (lower) faintly ω-continuous in order to obtain that F is upper (lower) ω-continuous? Under what condition are equivalents upper (lower) faintly continuous and upper (lower) ω-continuous multifunctions?

Definition 2 [2] A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be:

1. lower weakly ω-continuous, if for each $x \in X$ and each open set V of Y such that $x \in F^-(V)$, there exists $U \in \omega O(X, x)$ such that $U \subset F^-(\mathrm{Cl}(V))$,

2. upper weakly ω-continuous, if for each $x \in X$ and each open set V of Y such that $x \in F^+(V)$, there exists $U \in \omega O(X, x)$ such that $U \subset F^+(\mathrm{Cl}(V))$,

3. weakly ω-continuous, if it is both upper weakly ω-continuous and lower weakly ω-continuous.

Remark 3.7 It is clear that the class of weakly ω-continuous multifunctions contains properly the classes of almost ω-continuous multifunctions.

Theorem 3.8 *If $F : (X, \tau) \to (Y, \sigma)$ is lower (upper) weakly ω-continuous then F is lower (upper) faintly ω-continuous. But the converse is not true in general.*

Example 3.9 Let $X = \mathbb{R}$ with topology $\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - \mathbb{Q}, \mathbb{Q}\}$ and $Y = \{a, b, c\}$ with topology $\sigma = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}$. Take $A \subset \mathbb{Q}$ and define $F : (\mathbb{R}, \tau) \to (Y, \sigma)$ as follows:

$$F(x) = \begin{cases} \{a\}, & \text{if } x \in \mathbb{Q} - A \\ \{b\}, & \text{if } x \in \mathbb{R} - \mathbb{Q} \\ \{c\}, & \text{if } x \in A. \end{cases}$$

Then F is upper faintly ω-continuous but is not upper weakly ω-continuous.

In a similar form, we can find a multifunction G that is lower faintly-ω-continuous but is not lower-ω-continuous.

Theorem 3.10 *Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is an open set of Y for each $x \in X$. Then the following assertions are equivalent:*

(1) *F is lower ω-continuous,*

(2) *F is lower weakly ω-continuous.*

Proof. (1)⇒(2): Obvious.

(2)⇒(1): Let $x \in X$ and V be an open set of Y such that $F(x) \cap V \neq \emptyset$. Then there exists an ω-open set U containing x such that $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ for each $u \in U$. Since $F(u)$ is open, $F(u) \cap V \neq \emptyset$ for each $u \in U$ and hence F is lower ω-continuous. ∎

Theorem 3.11 *Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is closed in Y for each $x \in X$ and Y is normal. Then the following assertions are equivalent:*

(1) *F is upper weakly ω-continuous,*

(2) *F is upper ω-continuous.*

Proof. (2)⇒(1): Obvious.

(1)⇒(2): Let $x \in X$ and G be an open set containing $F(x)$. Since $F(x)$ is closed in Y, by the normality of Y, there exists an open set V such that $F(x) \subset V \subset \mathrm{Cl}(V) \subset G$. Since F is upper weakly ω-continuous, there exists an ω-open set U containing x such that $F(U) \subset \mathrm{Cl}(V) \subset G$. This shows that F is upper ω-continuous. ∎

Definition 3 A topological space X is said to be almost regular if for each regular closed set F of X and each point $x \notin F$, there exist disjoint open sets U and V of X such that $x \in U$ and $F \subset V$.

The following characterization of almost regular spaces obtained in [7] will be useful in the sequel.

Theorem 3.12 *A topological space X is almost regular if and only if, for every open set U in X,* $\mathrm{Int}_\theta(\mathrm{Cl}(U))$ *is a θ-open set.*

Theorem 3.13 *Let $F : (X, \tau) \to (Y, \sigma)$ be an upper (lower) faintly ω-continuous multifunction and (Y, σ) be an almost regular space, then F is an upper (lower) weakly ω-continuous multifunction.*

Proof. We proof the theorem for upper (lower) weakly ω-continuous multifunctions. Let $x \in X$ and V any open set in Y with $x \in F^+(V)$ $(x \in F^-(V))$. There exists $y \in F(x) \subset V$ $(y \in F(x) \cap V)$, since $y \in V$ and $V \subset \mathrm{Cl}(V)$, $y \in \mathrm{Int}_\theta(\mathrm{Cl}(V))$. Using the fact that (Y, σ) is almost regular, $\mathrm{Int}_\theta(\mathrm{Cl}(V))$ is a θ-open set in Y and $x \in F^+(\mathrm{Int}_\theta(\mathrm{Cl}(V)))$ $(x \in F^-(\mathrm{Int}_\theta(\mathrm{Cl}(V))))$. Since F is upper (lower) faintly ω-continuous, there exists an ω-open set U of X containing x, such that $F(U) \subset \mathrm{Int}_\theta(\mathrm{Cl}(V))$ $(z \in U$ implies $F(z) \cap \mathrm{Int}_\theta(\mathrm{Cl}(V)) \neq \emptyset)$. Hence $F(U) \subset \mathrm{Cl}(V)$ $(F(U) \cap \mathrm{Int}_\theta(\mathrm{Cl}(V)) \subseteq F(U) \cap \mathrm{Cl}(V) \neq \emptyset)$. Therefore, F is upper (lower) weakly ω-continuous. ∎

The following definition is a particular case of Definition 3.4 of [9].

Definition 4 [4] A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be:

1. upper slightly ω-continuous at $x \in X$ if for each clopen set V of Y containing $F(x)$, there exists $U \in \omega O(X)$ containing x such that $F(U) \subset V$;
2. lower slightly ω-continuous at $x \in X$ if for each clopen set V of Y such that $F(x) \cap V \neq \varnothing$, there exists $U \in \omega O(X)$ containing x such that $F(u) \cap V \neq \varnothing$ for every $u \in U$;
3. upper (lower) slightly ω-continuous if it has this property at each point of X.

Remark 3.14 It is clear that every upper ω-continuous multifunction is upper slightly ω-continuous. But the converse is not true in general, as the following example shows.

Example 3.15 Let $X = \mathbb{R}$ with the topology $\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - \mathbb{Q}\}$. Define a multifunction $F : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau)$ as follows:

$$F(x) = \begin{cases} \mathbb{Q} & \text{if } x \in \mathbb{R} - \mathbb{Q} \\ \mathbb{R} - \mathbb{Q} & \text{if } x \in \mathbb{Q}. \end{cases}$$

Then F is upper slightly ω-continuous but is not upper ω-continuous.

Theorem 3.16 *Every upper (lower) weakly ω-continuous multifunction is upper (lower) slightly ω-continuous. But the converse is not true in general.*

Example 3.17 The multifunction defined in Example 3.9 is upper (lower) slightly ω-continuous but not upper (lower) weakly ω-continuous multifunction.

Theorem 3.18 *Every upper (lower) faintly ω-continuous multifunction is upper (lower) slightly ω-continuous. But the converse is not true in general.*

Proof. Since every clopen set is a θ-open set, the proof follows. ∎

The following example shows that the converse of Theorem 3.18 is not true in general.

Example 3.19 Let $X = \mathbb{R}$ with topology $\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - \mathbb{Q}\}$ and $Y = \mathbb{R}$ with topology σ as the usual topology. Define $F : (\mathbb{R}, \tau) \to (\mathbb{R}, \sigma)$ as $F(x) = \{x\}$ for all $x \in \mathbb{R}$. It is easy to see that F is upper (lower) slightly ω-continuous but not upper (lower) faintly ω-continuous.

Definition 5 A topological space (X, τ) is said to be an ω-space [1] if every ω-open set of X is open.

Theorem 3.20 *Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction and (X, τ) be an ω-space. Then F is faintly ω-continuous if and only if F is faintly continuous.*

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