ON ALGEBRAIC AND ANALYTIC CORE II

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Abstract. In this paper, we continue the study of the algebraic core spectrum and the analytic core spectrum of an operator T on the complex Banach space X: $\sigma_{alc}(T) = \{\lambda \in \mathbb{C} : C(\lambda I - T) = \{0\}\}$ and $\sigma_{ac}(T) = \{\lambda \in \mathbb{C} : K(\lambda I - T) = \{0\}\}$ where $C(T)$ and $K(T)$ are respectively the algebraic core and the analytic core for T. We shall be concerned with the relations between $\sigma_{ac}(\cdot)$ $(\sigma_{alc}(\cdot))$ and different classical parts of spectrum: the point spectrum, the approximate point spectrum, the surjectivity spectrum and the Kato spectrum. Moreover, some applications are given.

Keywords: local spectral theory, algebraic core spectrum, analytic core spectrum, Kato resolvent set, quasi-similar operators.

1. Introduction

Throughout, X denotes a complex Banach space and $\mathcal{B}(X)$ denotes the Banach algebra of all bounded linear operators on X, let I be the identity operator, and for $T \in \mathcal{B}(X)$ we denote by T^*, $R(T)$, $R^\infty(T) = \bigcap_{n\geq 0} R(T^n)$, $\rho(T)$, $\sigma(T)$,

$\sigma_p(T)$, $\sigma_{ap}(T)$ and $\sigma_{su}(T)$ respectively the adjoint, the range, the hyper-range, the resolvent set, the spectrum, the point spectrum, the approximate point spectrum and the surjectivity spectrum of T.

Recall that for $T \in \mathcal{B}(X)$ and $x \in X$ the local resolvent of T at x defined as the union of all open subset U of $\mathbb{C}$ for which there is an analytic function $f : U \to X$ such that the equation $(T - \mu I)f(\mu) = x$ holds for all $\mu \in U$. The local spectrum $\sigma_T(x)$ of T at x is defined as $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$. Evidently $\rho(T) \subseteq \rho_T(x)$, $\rho_T(x)$ is open and $\sigma_T(x)$ is closed.

Also, an important concept in local spectral theory is the local spectral subspace for an operator $T \in \mathcal{B}(X)$. For subset Ω of $\mathbb{C}$ the local spectral subspace of T associated with Ω is the set $X_T(\Omega) = \{x \in X : \sigma_T(x) \subseteq \Omega\}$, evidently $X_T(\Omega)$ is a hyperinvariant subspace of T not always closed, if $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{C}$ then $X_T(\Omega_1) \subseteq X_T(\Omega_2)$. We refer the reader to [1], [3], [4], [6] for the properties of the local spectrum and local spectral subspaces .

Next, let $T \in \mathcal{B}(X)$, T is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (SVEP) if for every open neighbourhood $U \subseteq \mathbb{C}$ of λ_0, the only analytic function $f : U \longrightarrow X$ which satisfies the equation $(T - zI)f(z) = 0$ for all $z \in U$ is the function $f \equiv 0$. T is said to have the SVEP if T has the SVEP for every $\lambda \in \mathbb{C}$. Denote by $A(T) = \{\lambda \in \mathbb{C} : T \text{ has the SVEP at } \lambda\}$, by [3, Proposition 1.2.16] $A(T) = \mathbb{C}$ if and only if $X_T(\emptyset) = \{0\}$, if and only if $X_T(\emptyset)$ is closed.

Recall that $T \in \mathcal{B}(X)$ is said to be Kato operator or semi-regular [3], [7] if $R(T)$ is closed and $N(T - \lambda) \subseteq R^\infty(T - \lambda)\}$. Denote by $\rho_K(T)$: $\rho_K(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is Kato }\}$ the Kato resolvent and $\sigma_K(T) = \mathbb{C} \backslash \rho_K(T)$ the Kato spectrum of T. It is well known that $\rho_K(T)$ is on open subset of $\mathbb{C}$ and may be decomposed in connected disjoint open nonempty components [1], $\sigma_K(T)$ play an important role in local spectral theory; in particular, we have

$$\partial\sigma_T(x) \subseteq \sigma_K(T) \subseteq \sigma_{su}(T) \cap \sigma_{ap}(T) \subseteq \sigma(T) \text{ for all } x \in X.$$

According to [1, Definition 1.40], we say that $T \in \mathcal{B}(X)$ admits a generalized Kato decomposition, abbreviated GKD if there exists a pair of T-invariant closed subspaces (M, N) such that $X = M \oplus N$, the restriction $T_{|M}$ is semi-regular, and $T_{|N}$ is quasinilpotent. Obviously, every Kato operator admits a GKD because in this case $M = X$ and $N = \{0\}$, again the quasi-nilpotent operator admits a GKD: Take $M = \{0\}$ and $N = X$. If we suppose that $T_{|N}$ is nilpotent of order $d \in \mathbb{N}$, then T is said to be of Kato type of operator of order d. Finally, T is said essentially semi-regular if it admits a GKD (M, N) such that N is finite-dimensional. Evidently, every essentially semi-regular operator is of Kato type. The Kato type spectrum of T is defined by

$$\sigma_{Kt}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not of Kato type}\},$$

evidently $\sigma_{Kt}(T) \subseteq \sigma_K(T)$. We refer to [1] for more information about the Kato type spectrum.

Let $T \in \mathcal{B}(X)$. The ascent of T is defined by

$$a(T) = \min\{p : N(T^p) = N(T^{p+1})\}.$$

If such p does not exist, we let $a(T) = \infty$. Analogously, the descent of T is $d(T) = \min\{q : R(T^q) = R(T^{q+1})\}$; if such q does not exist, we let $d(T) = \infty$ [4]. It is well known that, if both $a(T)$ and $d(T)$ are finite, then $a(T) = d(T)$ and we have the decomposition $X = R(T^p) \oplus N(T^p)$, where $p = a(T) = d(T)$.

Recall that, for $T \in \mathcal{B}(X)$, the algebraic core $C(T)$ for T is the greatest subspace M of X for which $T(M) = M$. Equivalently,

$$C(T) = \{x \in X : \ \exists (x_n)_{n\geq 0} \subset X, \ \text{ such that } x_0 = x, Tx_n = x_{n-1} \text{ for all } n \geq 1\}$$

Moreover, the analytical core for T is a linear subspace of X defined by:

$$K(T) = \{x \in X : \ \exists (x_n)_{n\geq 0} \subset X \text{ and } \delta > 0 \text{ such that } x_0 = x, \ Tx_n = x_{n-1}, \\ \forall n \geq 1 \text{ and } \|x_n\| \leq \delta^n \|x\|\}$$

There are some relations between the algebraic core and the analytical core, see [1], [3], [5], [9], [11]:

- $T(K(T)) = K(T)$, $T(C(T)) = C(T)$ and $K(T) \subseteq C(T)$.
- If $C(T)$ is closed, then $C(T) = K(T)$.
- $K(T) = X_T(\mathbb{C}\backslash\{0\}) = \{x \in X : 0 \in \rho_T(x)\}$.
- $X_T(\emptyset) \subset K(T) \subseteq C(T) \subset R^\infty(T) \subset R(T)$.
- $N(T - \lambda I) \subseteq K(T - \mu I)$ for all $\lambda \neq \mu$.
- The function : $\lambda \to K(T - \lambda I)$ is constant on component of $\rho_K(T)$.
- If $\lambda \in \rho_K(T)$, then:

$$K(T - \lambda I) = C(T - \lambda I) = X_T(\mathbb{C}\backslash\{\lambda\}) = R^\infty(T - \lambda I).$$

Now, denote

$$\mathcal{R}_{ac}(X) = \{T \in \mathcal{B}(X) : K(T) \neq \{0\}\}$$
$$\mathcal{R}_{alc}(X) = \{T \in \mathcal{B}(X) : C(T) \neq \{0\}\}$$

In [8], we have investigated the study of sets $\mathcal{R}_{ac}(X)$ and $\mathcal{R}_{alc}(X)$, we have showed that these parts of $\mathcal{B}(X)$ are regularities in Kordulla-Müller's sense; consequently

$$\sigma_{alc}(T) \ = \ \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{R}_{alc}(X)\} = \{\lambda \in \mathbb{C} : C(\lambda I - T) = \{0\}\} \text{ and}$$
$$\sigma_{ac}(T) \ = \ \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{R}_{ac}(X)\} = \{\lambda \in \mathbb{C} : K(\lambda I - T) = \{0\}\}$$

respectively the algebraic core spectrum and the analytic core spectrum satisfie the mapping spectral theorem. We refer the reader to [2], [6], [7] for properties of the regularity theory.

Write $\rho_{alc}(T) = \mathbb{C}\backslash\sigma_{alc}(T)$ and $\rho_{ac}(T) = \mathbb{C}\backslash\sigma_{ac}(T)$ respectively the algebraic core resolvent and the analytic core resolvent of T.

In the following section we continue the study of relations between $\sigma_{alc}(.)$, $\sigma_{ac}(\cdot)$ or $\rho_{alc}(\cdot)$, $\rho_{ac}(\cdot)$ and the classical parts of spectrum: $\sigma_p(\cdot)$, $\sigma_{ap}(\cdot)$, $\sigma_{su}(\cdot)$, $\sigma_K(\cdot)$ respectively the point spectrum, the approximate point spectrum, the surjectivity spectrum and the Kato spectrum. On other hand some results and applications are given.

2. Main results

We begin by the following proposition.

Proposition 2.1 *Let $T \in \mathcal{B}(X)$. Then*

$$\sigma_{alc}(T) \subseteq \sigma_{ac}(T) \subseteq \sigma_{su}(T).$$

Proof. Let $\lambda \in \mathbb{C} \setminus \sigma_{su}(T)$, no loss of generality we can assume that $\lambda = 0$, we have $T(X) = X = K(T)$; hence $K(T) \neq \{0\}$ and consequently $0 \in \mathbb{C} \setminus \sigma_{ca}(T)$. ∎

Remarks.

1. We showed already in [11] that $\sigma_{alc}(T) \subseteq \sigma_{ac}(T) \subseteq \sigma_T(x)$ for all $x \in X \backslash \{0\}$; on the other hand, we know that $\sigma_{su}(T) = \bigcup_{x \in X} \sigma_T(x)$. Then, we obtain Proposition 2.1.
2. If, for all $\lambda \in \rho_K(T)$, we have $R^\infty(T - \lambda I) \neq \{0\}$, then

$$\sigma_{alc}(T) \subseteq \sigma_{ac}(T) \subseteq \sigma_K(T) \subseteq \sigma_{su}(T)$$

Proposition 2.2 *Let $T \in \mathcal{B}(X)$. Then $\sigma_{ac}(T) \subseteq A(T)$.*

Proof. Let $\lambda \in \sigma_{ac}(T)$ then $K(T - \lambda I) = \{0\}$. Since $X_{T-\lambda I}(\emptyset) \subseteq K(T - \lambda I)$, then $X_{T-\lambda I}(\emptyset) = \{0\}$; therefore, T satisfies the (SVEP) in λ. ∎

Remarks.

1. If $0 \in \sigma_{ac}(T)$, then $\sigma_p(T) \subseteq \{0\}$. Indeed, we have $N(T-\lambda I) \subseteq K(T) = \{0\}$, for all $\lambda \neq 0$, then $\sigma_p(T) \subseteq \{0\}$.
2. Let $T \in \mathcal{B}(X)$, then $\sigma_{alc}(T) \subseteq \sigma_{ac}(T) \subseteq \sigma(T_{|F})$, for all closed subspace $F \neq \{0\}$ of X. Indeed, let $\lambda \in \sigma_{ca}(T)$, then $K(T - \lambda I) = \{0\}$. If $(T - \lambda I)_{|F}$ is invertible, then $(T - \lambda I)(F) = F$, therefore $F \subseteq K(T - \lambda I) = \{0\}$, contradiction.
3. Let $T \in \mathcal{B}(X)$, assume then T is of Kato type, then $K(T) = R^\infty(T)$. If we suppose that, for all $\lambda \in \rho_{Kt}(T)$, we have $R^\infty(T - \lambda I) \neq \{0\}$, then

$$\sigma_{alc}(T) \subseteq \sigma_{ac}(T) \subseteq \sigma_{Kt}(T)$$

Example 1. Let $T \in \mathcal{B}(X)$ a quasi-nilpotent operator, then $\sigma(T) = \{0\}$, by [1, Corollary 2.28] we have $K(T) = \{0\}$, therefore $\sigma_{ac}(T) = \sigma(T) = \{0\}$.

Example 2. Let $T \in \mathcal{B}(X)$ an injective compact operator. It is well known that $\sigma(T)$ contains at most countable set of point, and each nonzero point of $\sigma(T)$ is an isolated eigenvalue, i.e., $\sigma(T) = \{0\} \bigcup \sigma_p(T)$. Moreover, for each $\lambda \in \sigma(T)\backslash\{0\}$, we know that $T - \lambda I$ is Fredholm operator. By [1, Corollary 3.21], we have $K(T-\lambda I) = R^{\infty}(T-\lambda I) = (T-\lambda I)^p(X)$ where $p = d(T-\lambda I) = a(T-\lambda I)$. Then, we obtain $\sigma_{ac}(T) \subseteq \{0\}$, but if $\sigma_{ac}(T) = \{0\}$ then, by the last remark (1), it follows that $\sigma_p(T) \subseteq \{0\}$, a contradiction. Consequently, $\sigma_{ac}(T) = \emptyset$.

Lemma 2.1 *Let $T \in \mathcal{B}(X)$. Then*

$$\rho_K(T) \cap \sigma_{ap}(T) \subseteq \rho_{ac}(T).$$

Proof. Let $\lambda \in \rho_K(T) \cap \sigma_{ap}(T)$, then $N(T-\lambda I) \neq \{0\}$ and $N(T-\lambda I) \subseteq R^{\infty}(T-\lambda I)$, hence there exists $x \in N(T-\lambda I) \subseteq R^{\infty}(T-\lambda I) = K(T-\lambda I)$, this implies that $K(T-\lambda I) \neq \{0\}$ and consequently $\lambda \in \rho_{ac}(T)$. ∎

Lemma 2.2 *Let $T \in \mathcal{B}(X)$. Then*

$$\rho_K(T) \cap \sigma_{su}(T) \subseteq \rho_{ac}(T^*).$$

Proof. We know that $\sigma_{su}(T) = \sigma_{ap}(T^*)$ and $\rho_K(T) = \rho_K(T^*)$. Therefore, $\rho_K(T) \cap \sigma_{su}(T) = \rho_K(T^*) \cap \sigma_{ap}(T^*) \subseteq \rho_{ac}(T^*)$. ∎

Proposition 2.3 *Let $T \in \mathcal{B}(X)$. Then*

$$\rho_K(T) \cap \sigma(T) \subseteq \rho_{ca}(T^*) \cup \rho_{ac}(T).$$

Proof. It is well known that $\sigma(T) = \sigma_{ap}(T) \cup \sigma_{su}(T)$ and $\sigma_p(T) \subseteq \sigma_{ap}(T)$. Apply now Lemmas 2.1 and 2.2. ∎

Proposition 2.4 *Let $T \in \mathcal{B}(X)$. Then*

$$[\sigma_{ap}(T) \cap \sigma_{su}(T)]\backslash[\rho_{ca}(T) \cap \rho_{ca}(T^*)] \subseteq \sigma_K(T).$$

Proof. By Lemmas 2.1 and 2.2, we have

$$\rho_K(T) \cap \sigma_{ap}(T) \cap \sigma_{su}(T) \subseteq \rho_{ac}(T) \cap \rho_{alc}(T^*).$$

Consequently, $[\sigma_{ap}(T) \cap \sigma_{su}(T)]\backslash[\rho_{ac}(T) \cap \rho_{ac}(T^*)] \subseteq \sigma_K(T)$. ∎

Proposition 2.5 *Let $T \in \mathcal{B}(X)$. Then*

1. $\sigma(T)\backslash\sigma_{ap}(T) \subseteq \rho_K(T) \cap \sigma_{su}(T) \subseteq \rho_{ac}(T^*)$
2. $\sigma(T)\backslash\sigma_{su}(T) \subseteq \rho_K(T) \cap \sigma_{ap}(T) \subseteq \rho_{ac}(T)$

Proof. 1. Let $\lambda \in \sigma(T) \backslash \sigma_{ap}(T)$ then $T - \lambda I$ is not invertible, since $N(T - \lambda I) = \{0\}$ and $R(T - \lambda I)$ is closed, then $\lambda \in \sigma_{su}(T)$ and $\lambda \in \rho_K(T)$. It follows that $\sigma(T) \backslash \sigma_{ap}(T) \subseteq \rho_K(T) \cap \sigma_{su}(T)$, by Lemma 2.2 we conclude 1.

2. is immediate by duality $\sigma_{su}(T) = \sigma_{ap}(T^*)$ and $\sigma_{su}(T^*) = \sigma_{ap}(T)$. ∎

Proposition 2.6 *Let $T \in \mathcal{B}(X)$. Then*

1. $\sigma_K(T) \cup \rho_{ca}(T) = \sigma_{ap}(T) \cup \rho_{ac}(T)$;
2. $\sigma_K(T) \cup \rho_{ca}(T^*) = \sigma_{su}(T) \cup \rho_{ac}(T^*)$.

Proof. 1. Since $\sigma_K(T) \subseteq \sigma_{ap}(T)$, then $\sigma_K(T) \cup \rho_{ac}(T) \subseteq \sigma_{ap}(T) \cup \rho_{ac}(T)$. Suppose that $\lambda \notin \sigma_K(T) \cup \rho_{ac}(T)$, then $R(T - \lambda I)$ is closed and $N(T - \lambda I) \subseteq R^\infty(T - \lambda I) = K(T - \lambda I) = \{0\}$, therefore $\lambda \notin \sigma_{ap}(T)$ and $\lambda \notin \rho_{ac}(T)$.

2. is clair by duality. ∎

Theorem 2.1 *Let $T \in \mathcal{B}(X)$. Then, for all subsets $\Omega \subseteq$ of $\mathbb{C}$, we have*

$$X_T(\Omega) \neq \{0\} \Rightarrow \sigma_{ac}(T) \subseteq \Omega.$$

Proof. Let $\lambda \notin \Omega$ then

$$\{0\} \neq X_T(\Omega) = X_T(\Omega \backslash \{\lambda\}) \subseteq X_T(\mathbb{C} \backslash \{\lambda\}) = K(T - \lambda I).$$

Therefore, $\lambda \notin \sigma_{ac}(T)$. ∎

Proposition 2.7 *Let $T \in \mathcal{B}(X)$, then*

$$\lambda \in \sigma_p(T) \Longrightarrow \sigma_{ac}(T) \subseteq \{\lambda\}.$$

Proof. Let $\lambda \in \sigma_p(T)$. This implies $\{0\} \neq N(T - \lambda I) \subseteq K(T - \mu)$ for all $\lambda \neq \mu$. Consequently, $K(T - \mu I) \neq \{0\}$ for all $\mu \neq \lambda$, hence $\sigma_{ac}(T) \subseteq \{\lambda\}$. ∎

Theorem 2.2 *Let $T \in \mathcal{B}(X)$, if Ω is a connected component of $\rho_K(T)$ we have*

$$\Omega \subseteq \rho_{ac}(T) \Longleftrightarrow \bigcap_{\lambda \in \Omega} R^\infty(T - \lambda I) \neq \{0\}.$$

Proof. Suppose that $\Omega \subseteq \rho_{ac}(T)$. Then we have $K(T - \lambda I) \neq \{0\}$ for all $\lambda \in \Omega$. Since $\Omega \subseteq \rho_{ac}(T)$, then the application $\lambda \to K(T - \lambda I)$ is constant in Ω, and

$$\{0\} \neq K(T - \lambda I) = \bigcap_{\lambda \in \Omega} K(T - \lambda I) = \bigcap_{\lambda \in \Omega} R^\infty(T - \lambda I).$$

So, it follows that

$$\bigcap_{\lambda \in \Omega} R^\infty(T - \lambda I) \neq \{0\}.$$

Conversely, since $\{0\} \neq \bigcap_{\lambda \in \Omega} R^\infty(T - \lambda I) = R^\infty(T - \lambda I) = K(T - \lambda I)$, therefore $K(T - \lambda I) \neq \{0\}$ for all $\lambda \in \Omega$. ∎

Corollary 2.1 *Let $T \in \mathcal{B}(X)$, if Ω is a connected component of $\rho_K(T)$ we have*

$$\rho_{ac}(T) \cap \Omega \neq \emptyset \Longrightarrow \Omega \subseteq \rho_{ac}(T)$$

Proof. Let $\lambda_0 \in \Omega \cap \rho_{ac}(T)$, then for all $\lambda \in \Omega$ we obtain

$$R^{\infty}(T - \lambda I) = K(T - \lambda I) = K(T - \lambda_0 I) \neq \{0\}$$

because $\lambda \to K(T - \lambda I)$ is constant, hence $K(T - \lambda I) \neq \{0\}$ for all $\lambda \in \Omega$ and, therefore, $\Omega \subseteq \rho_{ac}(T)$. ∎

Remark. Know that $\sigma_{ca}(T)$ is closed; then immediately, by Corollary 2.1,

$$\sigma_{ca}(T) \cap \Omega \neq \emptyset \Longrightarrow \overline{\Omega} \subseteq \sigma_{ca}(T).$$

Theorem 2.3 *Let $T \in \mathcal{B}(X)$ and Ω be connected components of $\rho_K(T)$, such that $G \cap \sigma_{ac}(T) \neq \emptyset$. Then*

1. *$\sigma_p(T)$ is empty;*
2. *$\sigma(T)$ and $\sigma_T(x)$ are connected $\forall x \in X$.*

Proof.

1. Suppose that $\lambda \in \sigma_p(T)$, then $\sigma_{ac}(T) \subseteq \{\lambda\}$, this is a contradiction because $\sigma_{ac}(T) \cap \Omega \neq \emptyset \Longrightarrow \Omega \subseteq \sigma_{ac}(T)$.
2. Suppose that there exists $x_0 \in X$ such that $\sigma_T(x_0)$ is non-connected. Then, there is two non-empty closed subsets σ_1 and σ_2 of $\mathbb{C}$ such that $\sigma_T(x_0) = \sigma_1 \cup \sigma_2$ and $\sigma_1 \bigcap \sigma_2 = \emptyset$. By [1, Theorem 2.17] there exists $x_1, x_2 \in X$ such that $\sigma_T(x_1) \subseteq \sigma_1$ and $\sigma_T(x_2) \subseteq \sigma_2$. Therefore,

$$G \subseteq \sigma_{ac}(T) \subseteq \sigma_T(x_1) \cap \sigma_T(x_2) \subseteq \sigma_1 \cap \sigma_2 = \emptyset,$$

a contradiction. Now, since $\sigma_p(T) = \emptyset$ by 1), then T has the SVEP, hence

$$\sigma(T) = \sigma_{su}(T) = \bigcup_{x \in X} \sigma_T(x).$$

Consequently, $\sigma(T)$ is connected. ∎

Example 3. Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $(e_n)_{n\geq 0}$, and let $\omega := (\omega_n)_{n\geq 0}$ be a bounded sequence of strictly positive real numbers. Consider the unilateral weighted right shift operator defined by [3], [10]:

$$Se_n = \omega_n e_{n+1}$$

- The spectrum of S is given by

$$\sigma(S) = \{z \in \mathbb{C} : |z| \leq r(S)\}$$

- The approximate point spectrum of S is the annulus

$$\sigma_{ap}(S) = \{z \in \mathbb{C} : r_1(S) \leq |z| \leq r(S)\}$$

Suppose that $r_1(S) > 0$ and let $\Omega := \{z \in \mathbb{C} : |z| < r_1(S)\}$, then

$$\Omega \subseteq \mathbb{C} \backslash \sigma_{ap}(S) \subseteq \rho_K(S)$$

and G is a connected component of $\rho_K(T)$. We have

$$\bigcap_{n \geq 0} R(S^n) = \{0\},$$

hence $K(T) = \{0\}$ and $0 \in \sigma_{ac}(S)$.

Therefore, $\sigma_{ac}(S) \cap \Omega \neq \emptyset$. By Theorem 2.3, it follows that:

1. $\sigma_p(S)$ is empty;
2. $\sigma_S(x)$ is connected for all $x \in \mathcal{H}$;
3. $\sigma(S)$ is connected.

Let $T, S \in \mathcal{B}(X)$, T and S are said quasi-similarly if there is $R, L \in \mathcal{B}(X)$ injective and has dense range such that $RT = SR$ and $TL = LS$. We said that T and S are similar if there exists $R \in \mathcal{B}(X)$ invertible such that $TR = RS$.

Recall that tow similar operators are some spectral properties (spectrum, approximate point spectrum, essential spectrum...)

In the following result, we show that the algebraic core spectrum and analytic core spectrum are invariant by similarity.

Theorem 2.4 *Let $T, S \in \mathcal{B}(X)$ such that Let T and S are quasi-similar, then*

$$\sigma_{alc}(T) = \sigma_{alc}(S) \quad and \quad \sigma_{ac}(T) = \sigma_{ac}(S).$$

Proof. Since T and S are quasi-similar, then there exists $R, L \in \mathcal{B}(X)$ such that $RT = SR$ and $TL = LS$. Therefore $T - \lambda I$ and $S - \lambda I$ are quasi-similar for all $\lambda \in \mathbb{C}$.

We show that $R(K(T - \lambda I)) \subseteq K(S - \lambda I)$. Indeed, with no loss of the generality we can suppose that $\lambda = 0$. Let $y \in R(K(T))$, then $y = Rx$ such that $x \in K(T)$ or equivalently there exists a sequence $(x_n)_{n\geq 0} \subseteq X$ and $\delta > 0$ satisfying $Tx_n = x_{n-1}$, $x = x_0$ and $||x_n|| < \delta^n ||x||$.

Consider the sequence $(y_n)_{n\geq 0}$, where $y_n = Rx_n$, we have $y_0 = Rx$, $Sy_n = SRx_n = RTx_n = Rx_{n-1} = y_{n-1}$ and $||y_n|| < ||R|| |\delta^n|| |x||$, which implies that $R(K(T)) \subseteq K(S)$. And, by similarity, we prove that $L(K(S - \lambda I)) \subseteq K(T - \lambda I)$.

Now, if $K(S - \lambda I) = \{0\}$ then by injectivity of R we have $K(T - \lambda I) = \{0\}$. Let $\lambda \in \sigma_{ac}(S)$, then $K(S - \lambda I) = \{0\}$ and it follows that $K(T - \lambda I) = \{0\}$ and $\lambda \in \sigma_{ac}(T)$, consequently $\sigma_{ac}(S) \subseteq \sigma_{ac}(T)$.

Similarly, we have $\sigma_{ac}(T) \subseteq \sigma_{ac}(S)$. ∎

Theorem 2.5 *For two injective operators $T, S \in \mathcal{B}(X)$, the following statements hold:*

1. $K(ST-\lambda I) \neq \{0\} \Leftrightarrow K(TS-\lambda I) \neq \{0\}$, *for all* $\lambda \in \mathbb{C}$;
2. $C(ST-\lambda I) \neq \{0\} \Leftrightarrow C(TS-\lambda I) \neq \{0\}$, *for all* $\lambda \in \mathbb{C}$;
3. $\sigma_{alc}(TS) = \sigma_{alc}(ST)$ *and* $\sigma_{ac}(TS) = \sigma_{ac}(ST)$.

Proof. 1. We begin by the implication $K(ST-\lambda I) \neq \{0\} \Longrightarrow K(TS-\lambda I) \neq \{0\}$ $\forall \lambda \in \mathbb{C}$. Of course, if $K(ST-\lambda I) \neq \{0\}$, then there exists a sequence $(x_n)_{n\geq 0} \subseteq X$ and $\delta > 0$ such that $x := x_0 \neq 0$, $(ST-\lambda I)x_n = x_{n-1}$ and $||x_n|| < \delta^n ||x||$.

Let $z_n := Tx_n$. We have $(TS-\lambda I)z_n = (TS-\lambda I)Tx_n = T(ST-\lambda)x_n = Tx_{n-1} = z_{n-1}$. Since T is injective, then $z := z_0 = Tx \neq 0$. On the other hand, $||z_n|| < \delta^n ||z||$. Hence $z \in K(TS-\lambda I)$ and consequently $K(TS-\lambda I) \neq \{0\}$.

Conversely, $K(TS-\lambda I) \neq \{0\}$ implies that there is $(x_n)_{n\geq 0} \subseteq X$ and $\delta > 0$ which $x := x_0 \neq 0$, $(TS-\lambda I)x_n = x_{n-1}$ and $||x_n|| < \delta^n ||x||$.

Let $z_n := Sx_n$, then $(ST-\lambda I)z_n = (ST-\lambda I)Sx_n = S(TS-\lambda I)x_n = Sx_{n-1} = z_{n-1}$. But S is injective then $z := z_0 = Sx \neq 0$, and $||z_n|| < \delta^n ||z||$. Therefore, $z \in K(ST-\lambda I)$ and $K(ST-\lambda I) \neq \{0\}$.

2. Similar to 1.

3. Apply 1, 2 and the definition of $\sigma_{alc}(TS)$ and $\sigma_{ac}(TS)$. ∎

Theorem 2.6 *Let $T, S, R \in \mathcal{B}(X)$ such that T is injective and $TST = TRT$. Let $\lambda \in \mathbb{C}$. Then*

1. $K(ST-\lambda I) \neq \{0\} \Longrightarrow K(TR-\lambda I) \neq \{0\}$;
2. $C(ST-\lambda I) \neq \{0\} \Longrightarrow C(TR-\lambda I) \neq \{0\}$.

Either, if $ST^2 = T^2S$, then

$$K(ST-\lambda I) \neq \{0\} \Longleftrightarrow K(TR-\lambda I) \neq \{0\};$$
$$C(ST-\lambda I) \neq \{0\} \Longleftrightarrow C(TR-\lambda I) \neq \{0\}.$$

Proof. 1. Suppose $K(ST-\lambda I) \neq \{0\}$, then there is a sequence $(x_n)_{n\geq 0} \subseteq X$ and $\delta > 0$ such $x := x_0 \neq 0$, $(ST-\lambda I)x_n = x_{n-1}$, $||x_n|| < \delta^n ||x||$.

Let $z_n := Tx_n$, then $(TR-\lambda I)z_n = (TR-\lambda)Tx_n = T(ST-\lambda I)x_n = Tx_{n-1} = z_{n-1}$. Since T is injective, we have $z := z_0 = Tx \neq 0$ and $||z_n|| < \delta^n ||z||$. Hence $z \in K(TR-\lambda I)$ and, therefore, $K(TR-\lambda I) \neq \{0\}$.

If $ST^2 = T^2R$ we shall prove the converse. Indeed, suppose that $K(TR-\lambda I) \neq \{0\}$, then there is $(x_n)_{n\geq 0} \subseteq X$ and $\delta > 0$ which $x := x_0 \neq 0$, $(TR-\lambda I)x_n = x_{n-1}$ and $||x_n|| < \delta^n ||x||$.

Consider $z_n := Tx_n$, then $(ST-\lambda I)z_n = (ST-\lambda I)Tx_n = T(TR-\lambda I)x_n = Tx_{n-1} = z_{n-1}$. But T is injective then $z := z_0 = Tx \neq 0$, we have $||z_n|| < \delta^n ||z||$. Consequently $z \in K(ST-\lambda I)$, this implies $K(ST-\lambda I) \neq \{0\}$.

2. This is a consequence of 1. ∎

Under the conditions of Theorem 2.6, we have the following results.

Corollary 2.2 *Let $T, S, R \in \mathcal{B}(X)$ such that T is injective and $TST = TRT$. Then*

$$\sigma_c(TR) \subseteq \sigma_c(ST) \text{ and } \sigma_{ca}(TR) \subseteq \sigma_{ca}(ST)$$

Either, if $ST^2 = T^2R$:

$$\sigma_{alc}(TR) = \sigma_{alc}(ST) \quad \text{and} \quad \sigma_{ac}(TR) = \sigma_{ca}(ST).$$

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