SCOTT CLOSED INJECTIVITY AND RETRACTNESS OF DIRECTED COMPLETE POSET ACTS

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Abstract. Domain theory, which studies directed complete partially ordered sets, was introduced by Scott in the 1970s as a foundation for programming semantics and provides an abstract model of computation, and has grown into a respected field on the borderline between mathematics and computer science.

In this paper, we consider actions of a semigroup (monoid or group) on directed complete posets and study the algebraic notions of injectivity and retractness with respect to Scott closed embeddings in the categories so obtained.

Keywords: Dcpo, S-Dcpo, Scott closed embedding, injective object, retract.

2010 Mathematical Subject Classification: 06F05, 08B30, 68Q70, 20M30, 20M50.

1. Introduction and preliminaries

Injectivity and retractness are crucial notions in many branches of mathematics. Many mathematicians studied these notions in different categories with respect to different classes of monomorphisms and investigated their relations, see for example [5], [8], [6], [11], [9]. In this paper we study these notions with respect to the class of Scott closed embeddings in the category of actions of a monoid on directed complete posets.

First we recall some preliminaries needed in the sequel. The reader can find more details in [2], [4], [10], [13]. Let **Pos** denote the category of all partially ordered sets (posets) with order-preserving (monotone) maps between them. A non-empty subset D of a partially ordered set is called *directed*, denoted by $D \subseteq^d P$, if for every $a, b \in D$ there exists $c \in D$ such that $a, b \leq c$; and P is called *directed complete*, or briefly a *dcpo*, if for every $D \subseteq^d P$, the directed join $\bigvee^d D$ exists in P.

A *dcpo map* or a *continuous map* $f : P \to Q$ between dcpo's is a map with the property that for every $D \subseteq^d P$, $f(D)$ is a directed subset of Q and $f(\bigvee^d D) = \bigvee^d f(D)$. Thus we have the category **Dcpo** of all dcpo's with continuous maps between them.

A *po-monoid* (*po-semigroup, po-group*) S is a monoid (semigroup, group) with a partial order $\leq$ which is compatible with its binary operation (that is, for $s, t, s', t' \in S$, $s \leq t$ and $s' \leq t'$ imply $ss' \leq tt'$). Similarly, a *dcpo-monoid* (*group*) is a monoid (group) which is also a dcpo whose binary operation is a continuous map.

Recall that a (*right*) *S-act* or *S-set* for a monoid S is a set A equipped with an *action* $A \times S \to A$, $(a, s) \rightsquigarrow as$, such that $ae = a$ (e is the identity element of S) and $a(st) = (as)t$, for all $a \in A$ and $s, t \in S$. Let **Act**-S denote the category of all S-acts with action preserving maps ($f : A \to B$ with $f(as) = f(a)s$, for all $a \in A, s \in S$). Let A be an S-act. An element $a \in A$ is called a *zero, fixed,* or *a trap element* if $as = a$, for all $s \in S$.

For a po-monoid S, a (*right*) *S-poset* is a poset A which is also an S-act whose action $\lambda : A \times S \to A$ is order-preserving, where $A \times S$ is considered as a poset with componentwise order. The category of all S-posets with action preserving monotone maps between them is denoted by **Pos**-S.

Also, for a dcpo-monoid S, a (*right*) *S-dcpo* is a dcpo A which is also an S-act whose action $\lambda : A \times S \to A$ is a continuous map.

A non-empty subset B of an S-dcpo A is called a *sub S-dcpo* of A if B is a sub dcpo (a subset which is closed under directed joins) and subact of A. In this case, A is said to be an *extension* of B.

By an *S-dcpo map* between S-dcpo's, we mean a map $f : A \to B$ which is both continuous and action preserving. We denote the category of all S-dcpo's and S-dcpo maps between them by **Dcpo**-S.

A morphism $f : A \to B$ in the category of all S-dcpo's is called *order-embedding*, or briefly *embedding*, if for all $x, y \in A$, $f(x) \leq f(y)$ if and only if $x \leq y$.

Because of the fact that sub S-dcpo's are exactly subsets for which the inclusion map is order-embedding, we consider an arbitrary order-embedding as an inclusion from a sub S-dcpo.

Notice that order-embeddings are monomorphisms in the category of all S-dcpo's. But the converse is not necessarily true. For example, take S to be the one element dcpo-monoid, $A = \{\perp, a, a'\}$ with the order $\perp \leq a, a'$, $a \parallel a'$, and B the three element chain $\mathbf{3} = \{0, 1, 2\}$. Let $g : A \to B$ be defined as $g(\perp) = 0$, $g(a) = 1$, $g(a') = 2$. Then, g is one-one and hence a monomorphism in **Dcpo**-S, but it is not an embedding.

Finally, recall that for a class $\mathcal{M}$ of monomorphisms in a category $\mathcal{C}$, an object $A \in \mathcal{C}$ is called $\mathcal{M}$-*injective* if for each $\mathcal{M}$-*morphism* $f : B \to C$ and morphism $g : B \to A$ there exists a morphism $h : C \to A$ such that $hf = g$.

Also, an object A of a category $\mathcal{C}$ is called $\mathcal{M}$-*absolute retract* if it is a retract of each of its $\mathcal{M}$-extensions; that is, for each $\mathcal{M}$-morphism $f : A \to C$ there exists a morphism $h : C \to A$ such that $hf = id_A$, in which case h is said to be a *retraction*.

We say that $\mathcal{C}$ has enough $\mathcal{M}$-injective objects if for each $A \in \mathcal{C}$ there exists an $\mathcal{M}$-injective $\mathcal{M}$-extension of A.

In this paper, we consider $\mathcal{M}$ to be the class *sc* of all Scott closed embeddings (will be introduced in the following) in the above mentioned category of dcpo-monoid actions. Then, we investigate injectivity and retractness with respect to Scott closed embeddings in the category **Dcpo**-S and some of its subcategories.

2. Scott closed injectivity in Dcpo-S

In this section, we investigate injectivity with respect to Scott closed embeddings in the category **Dcpo**-S and give a non-trivial *sc*-injective object. Then, we consider the behaviour of *sc*-injective objects with products and coproducts.

Definition 2.1. A possibly empty subset B of a dcpo C is said to be *Scott closed* in C if B is a downward closed subset of C and for every $D \subseteq^d B$, $\bigvee^d D \in B$. An embedding $f : B \to C$ is said to be *Scott closed* or *sc-embedding* if B is Scott closed in C.

Definition 2.2. An S-dcpo A is said to be *Scott closed injective* or *sc-injective* if it is injective with respect to *sc*-embeddings.

Lemma 2.3. *If A is an sc-injective object in* **Dcpo**-S*, then it has a zero top element.*

Proof. Let A be an *sc*-injective in **Dcpo**-S. Consider the S-dcpo $C = A \oplus \{\theta\}$ where θ is taken to be a zero element. Since A is *sc*-injective, there exists a retraction $h : C \to A$. Then, the zero element $h(\theta)$ is the top element of A. ∎

Recall from [13] that, for a dcpo A and dcpo-monoid S, the cofree S-dcpo on A is the set $A^{(S)}$, of all dcpo maps from S to A, with pointwise order and the action given by $(fs)(t) = f(st)$, for $s, t \in S$ and $f \in A^{(S)}$.

Theorem 2.4. *Let A be a dcpo which has a top element. Then $A^{(S)}$ is an sc-injective S-dcpo.*

Proof. Let A be a dcpo with the top element T_A, $i : B \hookrightarrow C$ be an *sc*-embedding in **Dcpo**-S and $g : B \to A^{(S)}$ be an S-dcpo map. We must find an S-dcpo map $h : C \to A^{(S)}$ which extends g. Define $h : C \to A^{(S)}$ where

$$h(x)(s) = \begin{cases} g(xs)(e) & \text{if } \ xs \in B, \\ \top_A & \text{otherwise.} \end{cases}$$

Now, we show that h is an S-dcpo map. First, we prove that h is well-defined and $h(x)$ is a dcpo-map. We know $h(x)$ preserves the order. To see this, let $s, s' \in S$

where $s \leq s'$. So $xs \leq xs'$. If $xs' \in B$ then, since B is Scott closed in C, we get $xs \in B$. Therefore, since g preserves the order, we have $g(xs) \leq g(xs')$ and so

$$h(x)(s) = g(xs)(e) \leq g(xs')(e) = h(x)(s').$$

Otherwise, if $xs' \notin B$ then, $h(x)(s') = \top_A$ and it is clear that $h(x)(s) \leq h(x)(s') = \top_A$. Now, we show that $h(x)$ is continuous. To see this, let $T \subseteq^d S$. Since $h(x)$ preserves the order, we get $h(x)(T) \subseteq^d A$. Now, if $x \bigvee^d T \in B$ then, since B is Scott closed in C and $xt \leq x \bigvee^d T \in B$ for every $t \in T$, we get $xt \in B$, for every $t \in T$. Since g is continuous and B is an S-dcpo, we get

$$h(x)\left(\bigvee^d T\right) = g\left(x \bigvee^d T\right)(e) = g\left(\bigvee^d_{t\in T} xt\right)(e) = \bigvee^d_{t\in T} g(xt)(e) = \bigvee^d_{t\in T} h(x)(t).$$

Now, if $x \bigvee^d T \notin B$, then there exists $t_0 \in T$ such that $xt_0 \notin B$. The latter is because if, on the contrary, for every $t \in T$, $xt \in B$ then, since B is Scott closed in C, we get $x \bigvee^d T \in B$, which is a contradiction. So

$$\top_A = h(x)(t_0) \leq \bigvee^d_{t\in T} h(x)(t) \quad \text{and} \quad h(x)\left(\bigvee^d T\right) = \top_A = \bigvee^d_{t\in T} h(x)(t).$$

Hence $h(x) \in A^{(S)}$ and h is well-defined. Now, we prove that h is an S-dcpo map. First, we show that h preserves the order. To see this, let $x, x' \in C$ where $x \leq x'$. So for every $s \in S$, we get $xs \leq x's$. If $x's \in B$, then, $xs \in B$. Since g preserves the order, $g(xs) \leq g(x's)$ and so $h(x)(s) = g(xs)(e) \leq g(x's)(e) = h(x')(s)$. Otherwise, if $x's \notin B$ then, two cases may occur.

Case (i): If $xs \in B$, then $h(x)(s) = g(xs)(e) \leq \top_A = h(x')(s)$.

Case (ii): If $xs \notin B$, then $h(x)(s) = \top_A = h(x')(s)$.

Therefore, $h(x) \leq h(x')$ and h preserves the order. Now, to show that h is continuous, let $D \subseteq^d C$. Then, since h preserves the order, we have $h(D) \subseteq^d A^{(S)}$. Now, two cases may occur.

Case (i): $D \subseteq B$. In this case, $\bigvee^d D \in B$. This is because B is Scott closed in C. So, for every $s \in S$, $(\bigvee^d D)s \in B$. On the other hand,

$$ds \leq \bigvee^d_{d\in D} ds = \left(\bigvee^d D\right) s \in B,$$

for every $d \in D$. Since B is Scott closed in C, we get $ds \in B$, for every $d \in D$. Thus, for every $s \in S$, we have:

$$h\left(\bigvee^d D\right)(s) = g\left(\left(\bigvee^d D\right) s\right)(e) = g\left(\bigvee^d_{d\in D} ds\right)(e) = \bigvee^d_{d\in D} g(ds)(e) = \bigvee^d_{d\in D} h(d)(s).$$

So

$$h\left(\bigvee^d D\right) = \bigvee^d_{d\in D} h(d).$$

Case (ii): $D \nsubseteq B$. In this case, $\bigvee^d D \notin B$. This is because, on the contrary, if $\bigvee^d D \in B$, then, since B is Scott closed in C and for every $d \in D$, $d \leq \bigvee^d D \in B$ we get $d \in B$, for every $d \in D$. It contradicts $D \nsubseteq B$. Now, two cases may occur. If $(\bigvee^d D)s \in B$ then, for every $d \in D$ we get $ds \in B$. This is because, for every $d \in D$, $ds \leq (\bigvee^d D)s \in B$ and B is Scott closed in C. So

$$h\left(\bigvee^d D\right)(s) = g\left(\left(\bigvee^d D\right)s\right)(e) = g\left(\bigvee^d_{d\in D} ds\right)(e) = \bigvee^d_{d\in D} g(ds)(e) = \bigvee^d_{d\in D} h(d)(s).$$

Otherwise, if $(\bigvee^d D)s \notin B$ then, there exists $d_0 \in D$ such that $d_0 s \notin B$. The latter is because if, on the contrary, for every $d \in D$, $ds \in B$ then, since B is Scott closed in C, we have

$$\bigvee^d_{d\in D} ds = \left(\bigvee^d D\right)s \in B,$$

which is a contradiction. So

$$\top_A = h(d_0)(s) \leq \bigvee^d_{d\in D} h(d)(s) \quad \text{and} \quad h\left(\bigvee^d D\right)(s) = \top_A = \bigvee^d_{d\in D} h(d)(s).$$

Now, we show that h preserves the action. To see this, let $x \in C$ and $t \in S$. We prove that for every $s \in S$, $h(xt)(s) = (h(x)t)(s)$. Let $s \in S$. Then, two cases may occur. If $(xt)s = x(ts) \in B$ then, since g preserves the action, we get $h(xt)(s) = g((xt)s)(e) = g(x(ts))(e) = h(x)(ts) = (h(x)t)(s)$. Otherwise, if $(xt)s = x(ts) \notin B$, then $h(xt)(s) = \top_A$. On the other hand, $(h(x)t)(s) = h(x)(ts) = \top_A$. Hence $h(xt)(s) = (h(x)t)(s)$, for every $s \in S$ and h preserves the action.

Finally, $h|_B = g$. To prove this, let $x \in B$. Then, for every $s \in S$ we have $h(x)(s) = g(xs)(e) = (g(x)s)(e) = g(x)(s)$, as required. ∎

In the following, we consider the behaviour of *sc*-injective S-dcpo's with products and coproducts. First, we recall the following remark from [12].

Remark 2.5. Let $\{A_i\}_{i\in I}$ be a family of S-dcpo's. Then:

(i) The product of A_i's in the category **Dcpo**-S, is their cartesian product $\prod_{i\in I} A_i$ with componentwise action and order.

(ii) The coproduct of A_i's in the category **Dcpo**-S, is their disjoint union $\dot{\bigcup}_{i\in I} A_i$ with the order given by $x \leq y$ in $\dot{\bigcup}_{i\in I} A_i$ if and only if $x, y \in A_i$ and $x \leq y$ in A_i, for some $i \in I$; and with the action as A_i, for $a \in A_i$ and $s \in S$.

Theorem 2.6. *Let $\{A_i : i \in I\}$ be a family of S-dcpo's. Then, the product $\prod_{i\in I} A_i$ is sc-injective if and only if each A_i is sc-injective.*

Proof. If each A_i is sc-injective then, by the universal property of products, it is clear that $\prod_{i\in I} A_i$ is sc-injective. For the converse, let $\prod_{i\in I} A_i$ be sc-injective, and $j \in I$. Then A_j is sc-injective. To see this, consider the diagram where f is an sc-embedding and g is an S-dcpo map. Define $\bar{g} : B \to \prod_{i\in I} A_i$ where

$$\bar{g}(b)(i) = \begin{cases} g(b) & \text{if} \quad i = j, \\ \theta_i & \text{if} \quad i \neq j, \end{cases}$$

where for $i \in I$, θ_i is a zero element of A_i which exists since $\prod_{i\in I} A_i$ has a zero top element by Remark 2.3, and the i-th component of that zero element is a zero element of A_i, $i \in I$. We show that $\bar{g}$ is an S-dcpo map. Since g preserves the order, it is clear that $\bar{g}$ preserves the order. Now, we prove that $\bar{g}$ is continuous. To see this, let $D \subseteq^d B$. Then we have $\bar{g}(D) \subseteq^d \prod_{i\in I} A_i$, since $\bar{g}$ preserves the order. Thus we get:

$$\bar{g}\left(\bigvee^d D\right)(i) = \begin{cases} g\left(\bigvee^d D\right) = \bigvee^d_{d\in D} g(d) & \text{if} \quad i = j, \\ \theta_i & \text{if} \quad i \neq j, \end{cases}$$

and

$$\bar{g}(d)(i) = \begin{cases} g(d) & \text{if} \quad i = j, \\ \theta_i & \text{if} \quad i \neq j, \end{cases}$$

for every $d \in D$. So, for all $i \in I$,

$$\bar{g}\left(\bigvee^d D\right)(i) = \bigvee^d_{d\in D} (\bar{g}(d)(i)) = \left(\bigvee^d_{d\in D} \bar{g}(d)\right)(i).$$

Therefore,

$$\bar{g}\left(\bigvee^d D\right) = \bigvee^d_{d\in D} \bar{g}(d).$$

Finally, $\bar{g}$ preserves the action. This is because, for every $s \in S$ we have:

$$\bar{g}(bs)(i) = \begin{cases} g(bs) = g(b)s & \text{if} \quad i = j, \\ \theta_i & \text{if} \quad i \neq j. \end{cases}$$

Also

$$(\bar{g}(b)(i))s = \begin{cases} g(b)s & \text{if} \quad i = j, \\ \theta_i s = \theta_i & \text{if} \quad i \neq j, \end{cases}$$

and hence, for all $i \in I$,

$$\bar{g}(bs)(i) = (\bar{g}(b)(i))s = (\bar{g}(b)s)(i) \quad \text{and} \quad \bar{g}(bs) = \bar{g}(b)s.$$

Now, since $\prod_{i\in I} A_i$ is *sc*-injective, there exists an S-dcpo map $\bar{h} : C \to \prod_{i\in I} A_i$, where $\bar{h}f = \bar{g}$. It is clear that $\pi_j\bar{h} : C \to A_j$ extends g, where $\pi_j : \prod_{i\in I} A_i \to A_j$ is the j-th projection map. So A_j is *sc*-injective.

∎

Theorem 2.7. *Let $\{A_i : i \in I, |I| > 1\}$ be an arbitrary family of S-dcpo's. Then, $\coprod_{i\in I} A_i$ is not sc-injective.*

Proof. By Remark 2.5, $\coprod_{i\in I} A_i$ is not bounded and so by Lemma 2.3, it is not *sc*-injective. ∎

3. Scott closed injectivity versus absolute retract in Dcpo-S

In this section, we investigate the relation between *sc*-injectivity and *sc*-absolute retractness in the category **Dcpo**-S and its full subcategories **R**-**Dcpo**-S and **SR**-**Dcpo**-S of reversible and strongly reversible S-dcpo's, respectively.

First, we mention that, similar to Lemma 2.3, we have:

Lemma 3.1. *If A is an sc-absolute retract in* **Dcpo**-S*, then it has a zero top element.*

Theorem 3.2. *Let S be a dcpo-monoid with any one of the following properties:*

(1) $\forall s \in S$, $\exists t \in S$, $e \leq st$.

(2) $\forall s \in S$, $e \leq s^2$.

(3) *S is a dcpo-group.*

(4) $\perp_S = e$.

(5) *S is a right zero semigroup with an adjoined identity and has a top element.*

Then, for object A in **Dcpo**-S*, the following statements are equivalent:*

(i) *A has a zero top element.*

(ii) *A is sc-injective.*

(iii) *A is sc-absolute retract.*

Proof. (i)$\Rightarrow$(ii) (1) Let S be a dcpo-monoid with the property that for every $s \in S$, there exists $t \in S$ such that $e \leq st$ and A be an S-dcpo with zero top element $\top$. Then, we prove that A is an sc-injective S-dcpo. To see this, let $f : B \to C$ be an sc-embedding in **Dcpo**-S and $g : B \to A$ be an S-dcpo map. Then, define $h : C \to A$ as:

$$h(x) = \begin{cases} g(x) & \text{if } x \in B, \\ \top & \text{otherwise.} \end{cases}$$

We prove that h is an S-dcpo map. First, we show that h preserves the order. To see this, take $x, x' \in C$ where $x \leq x'$. Two cases may occur. If $x' \in B$ then, since B is Scott closed in C, we get $x \in B$ and so $h(x) = g(x) \leq g(x') = h(x')$. Otherwise, if $x' \notin B$, then we have $h(x) \leq h(x') = \top$. Now, we show that h is continuous. Let $D \subseteq^d B$. Then, since h preserves the order, we get $h(D) \subseteq^d A$. Two cases may occur:

Case (i): $D \subseteq B$.
In this case, since B is Scott closed in C, we have $\bigvee^d D \in B$. So,

$$h\left(\bigvee^d D\right) = g\left(\bigvee^d D\right) = \bigvee_{d \in D}^d g(d) = \bigvee_{d \in D}^d h(d).$$

Case (ii): $D \nsubseteq B$.
In this case $\bigvee^d D \notin B$. This is because on the contrary, if $\bigvee^d D \in B$ then, since B is Scott closed in C we get $d \in B$, for all $d \in D$. It contradicts $D \nsubseteq B$. By the assumption, there exists $d_0 \in D$ where $d_0 \notin B$. So,

$$\top = h(d_0) \leq \bigvee_{d \in D}^d h(d) \quad \text{and} \quad h\left(\bigvee^d D\right) = \top = \bigvee_{d \in D}^d h(d).$$

Finally, h preserves the action. To prove this, let $x \in C$ and $s \in S$. Then, two cases may occur. If $xs \in B$ then, since for $s \in S$, there exists $t \in S$ such that $x \leq xst \in B$ and B is Scott closed in C, we get $x \in B$. So, $h(xs) = g(xs) = g(x)s = h(x)s$. Otherwise, if $xs \notin B$ then, $x \notin B$. The latter is because, on the contrary, if $x \in B$ then, since B is an S-dcpo, we have $xs \in B$ which is a contradiction. Therefore, $h(xs) = \top = \top s = h(x)s$.

Notice that if S is a dcpo-monoid with one of the properties (2)-(3) or (4), then, S is a dcpo-monoid with the property (1). So we get the result.

(5) Let S be a right zero semigroup with an adjoined identity and has a top element. Also, let A be an S-dcpo with zero top element $\top$, $f : B \to C$ be an sc-embedding in **Dcpo**-S and $g : B \to A$ be an S-dcpo map. Then, h is defined similar to the proof of case (1). The only part which needs to be changed, is showing that h preserves the action. Let $x \in C$ and $s \in S$. Then, if $xs \in B$ we get $x\top_S = (xs)\top_S \in B$. On the other hand, $x = xe \leq x\top_S$. Therefore,

$x \in B$, since B is Scott closed in C. Hence, $h(xs) = g(xs) = g(x)s = h(x)s$. Otherwise, if $xs \notin B$ then, $x \notin B$. The latter is because, on the contrary, if $x \in B$ then, since B is an S-dcpo, we have $xs \in B$, which is a contradiction. Therefore, $h(xs) = \top = \top s = h(x)s$.

(ii)⇒(iii) It is clear.

(iii)⇒(i) By Lemma 3.1, we get the result. ∎

Corollary 3.3. *If S is a dcpo-monoid with any one of the properties mentioned in Theorem* 3.2*, then* **Dcpo**-*S has enough injective objects with respect to sc-embeddings.*

Proof. Let S be a dcpo-monoid with any one of the properties mentioned in Theorem 3.2 and A be an S-dcpo. Then, consider the *sc*-embedding $i : A \hookrightarrow A \oplus \top$ in **Dcpo**-S, where $\top$ is a zero element. Now by Theorem 3.2, we get the result. ∎

Definition 3.4. An S-dcpo A is called *reversible* if for every $a \in A$ and $s \in S$, there exists $t \in S$ such that $ast = a$.

So, we have the category **R**-**Dcpo**-S of all reversible S-dcpo's and S-dcpo maps between them.

Theorem 3.5. *For object A in* **R**-**Dcpo**-*S, the following conditions are equivalent:*

(i) *A has a zero top element.*

(ii) *A is sc-injective.*

(iii) *A is sc-absolute retract.*

Proof. (i)⇒ (ii) Suppose that B and C are reversible S-dcpo's, A is a reversible S-dcpo with zero top element $\top$, $f : B \to C$ is an *sc*-embedding in **R**-**Dcpo**-S and $g : B \to A$ is an S-dcpo map. Define $h : C \to A$ where:

$$h(x) = \begin{cases} g(x) & \text{if } x \in B, \\ \top & \text{otherwise.} \end{cases}$$

We show that h is an S-dcpo map. The proof of the fact that h is continuous, is similar to the proof of Theorem 3.2. The only part which needs to be changed, is that h preserves the action. To show this, let $x \in C$ and $s \in S$. Then, two cases may occur. If $xs \in B$ then, since B is a reversible S-dcpo, there exists $t \in S$ such that $x = xst \in B$. Therefore, $h(xs) = g(xs) = g(x)s = h(x)s$. Otherwise, if $xs \notin B$ then, $x \notin B$. The latter is because, on the contrary, if $x \in B$ then, since B is an S-dcpo, we have $xs \in B$ which is a contradiction. Therefore, $h(xs) = \top = \top s = h(x)s$.

(ii)⇒(iii) is clear, and the proof of (iii)⇒(i) is similar to the proof of Lemma 3.1. ∎

Now, similar to Corollary 3.3, we have:

Corollary 3.6. **R-Dcpo**-S *has enough injective objects with respect to sc-embeddings.*

Definition 3.7. An S-dcpo A is called *strongly reversible* if for every $a \in A$ and $s \in S$, we have $as^2 = a$.

So, we have the category **SR-Dcpo-**S of all strongly reversible S-dcpo's and S-dcpo maps between them.

Theorem 3.8. *For object A in* **SR-Dcpo**-S, *the following conditions are equivalent:*

(i) *A has a zero top element.*

(ii) *A is sc-injective.*

(iii) *A is sc-absolute retract.*

Proof. (i)⇒(ii) Suppose that B and C are strongly reversible S-dcpo's, A is a strongly reversible S-dcpo with zero top element $\top$, $f : B \to C$ is an sc-embedding in **SR-Dcpo**-S and $g : B \to A$ is an S-dcpo map. Define $h : C \to A$ as:

$$h(x) = \begin{cases} g(x) & \text{if } x \in B, \\ \top & \text{otherwise.} \end{cases}$$

We show that h is an S-dcpo map. The proof of the fact that h is continuous, is similar to the proof of Theorem 3.2. The only part which needs to be changed, is to show that h preserves the action. To see this, let $x \in C$ and $s \in S$. Then, two cases may occur. If $xs \in B$ then, since B is a strongly reversible S-dcpo, $x = xs^2 \in B$. Therefore, $h(xs) = g(xs) = g(x)s = h(x)s$. Otherwise, if $xs \notin B$ then, $x \notin B$. The latter is because, on the contrary, if $x \in B$ then, since B is an S-dcpo, we have $xs \in B$, which is a contradiction. Therefore, $h(xs) = \top = \top s = h(x)s$.

(ii)⇒(iii) is clear, and the proof of (iii)⇒(i) is similar to the proof of Lemma 3.1. ∎

Corollary 3.9. **SR-Dcpo**-S *has enough injective objects with respect to sc-embeddings.*

Let $\mathcal{C}'$ be the full subcategory of **Dcpo**-S whose objects are S-dcpo's A with the property that for every $a \in A$ and $s \in S$, $a \leq as$. In the following we see a similar result to the categories of reversible and strongly reversible S-dcpo's for $\mathcal{C}'$.

Theorem 3.10. *For any object A in $\mathcal{C}'$, the following conditions are equivalent:*

(i) *A has a zero top element.*

(ii) *A is sc-injective.*

(iii) *A is sc-absolute retract.*

Proof. (i)$\Rightarrow$(ii) Suppose that B and C are objects of $\mathcal{C}'$, A is an object of $\mathcal{C}'$ with the zero top element $\top$, $f : B \to C$ is an *sc*-embedding in $\mathcal{C}'$ and $g : B \to A$ is an S-dcpo map. Define $h : C \to A$ as:

$$h(x) = \begin{cases} g(x) & \text{if } x \in B, \\ \top & \text{otherwise.} \end{cases}$$

We show that h is an S-dcpo map. The proof of the fact that h is continuous, is similar to the proof of Theorem 3.2. The only part which needs to be changed, is showing that h preserves the action. To see this, let $x \in C$ and $s \in S$. Then, two cases may occur. If $xs \in B$ then, since for every $s \in S$, $x \leq xs$ and B is Scott closed in C, we get $x \in B$ and $h(xs) = g(xs) = g(x)s = h(x)s$. Otherwise, if $xs \notin B$ then, $x \notin B$. The latter is because, on the contrary, if $x \in B$ then, since B is an S-dcpo, we have $xs \in B$, which is a contradiction. Therefore, $h(xs) = \top = \top s = h(x)s$.

(ii)$\Rightarrow$(iii) is clear, and the proof of (iii)$\Rightarrow$(i) is similar to the proof of Lemma 3.1. ∎

Acknowledgment. The authors gratefully acknowledge Professor M. Mehdi Ebrahimi's comments and conversations during this work.

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Accepted: 08.02.2015