A RECURSIVE FORMULA FOR POWER MOMENTS OF 2-DIMENSIONAL KLOOSTERMAN SUMS ASSOCIATED WITH GENERAL LINEAR GROUPS[1]

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Abstract. In this paper, we construct a binary linear code connected with the Kloosterman sum for $GL(2,q)$. Here q is a power of two. Then we obtain a recursive formula generating the power moments 2-dimensional Kloosterman sum, equivalently that generating the even power moments of Kloosterman sum in terms of the frequencies of weights in the code. This is done via Pless power moment identity and by utilizing the explicit expression of the Kloosterman sum for $GL(2,q)$.

Keywords: recursive formula, power moment, Kloosterman sum, 2-dimensional Kloosterman sum, general linear group, Pless power moment identity, weight distribution.

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1. Introduction

Let ψ be a nontrivial additive character of the finite field $\mathbb{F}_q$ with $q = p^r$ elements (p a prime), and let m be a positive integer. Then the m-dimensional Kloosterman sum $K_m(\psi; a)$([10]) is defined by

$$K_m(\psi; a) = \sum_{\alpha_1,\ldots,\alpha_m \in \mathbb{F}_q^*} \psi(\alpha_1 + \cdots + \alpha_m + a\alpha_1^{-1} \cdots \alpha_m^{-1}) \quad (a \in \mathbb{F}_q^*).$$

In particular, if $m = 1$, then $K_1(\psi; a)$ is simply denoted by $K(\psi; a)$, and is called the Kloosterman sum. For this, we have the Weil bound(cf. [10])

$$|K(\lambda; a)| \leq 2\sqrt{q}. \tag{1.1}$$

The Kloosterman sum was introduced in 1926([8]) to give an estimate for the Fourier coefficients of modular forms.

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For each nonnegative integer h, we denote by $MK_m(\psi)^h$ the h-th moment of the m-dimensional Kloosterman sum $K_m(\psi; a)$, i.e.,

$$MK_m(\psi)^h = \sum_{a \in \mathbb{F}_q^*} K_m(\psi; a)^h.$$

If $\psi = \lambda$ is the canonical additive character of $\mathbb{F}_q$, then $MK_m(\lambda)^h$ will be simply denoted by MK_m^h. If further $m = 1$, for brevity, MK_1^h will be indicated by MK^h.

Explicit computations on power moments of Kloosterman sums were initiated in the paper [13] of Salié in 1931, where it is shown that for any odd prime q,

$$MK^h = q^2 M_{h-1} - (q-1)^{h-1} + 2(-1)^{h-1} \ (h \geq 1).$$

Here $M_0 = 0$, and for $h \in \mathbb{Z}_{>0}$,

$$M_h = \left| \left\{ (\alpha_1, ..., \alpha_h) \in (\mathbb{F}_q^*)^h | \sum_{j=1}^{h} \alpha_j = 1 = \sum_{j=1}^{h} \alpha_j^{-1} \right\} \right|.$$

For $q = p$ odd prime, Salié obtained MK^1, MK^2, MK^3, MK^4 in that same paper by determining M_1, M_2, M_3.

From now on, let us assume that $q = 2^r$. Carlitz [1] evaluated MK^h for $h \leq 4$. Moisio was able to find explicit expressions of MK^h, for $h \leq 10$ (cf. [12]). This was done, via Pless power moment identity, by connecting moments of Kloosterman sums and the frequencies of weights in the binary Zetterberg code of length $q+1$, which were known by the work of Schoof and Vlugt in [14].

In [5], for both n, q powers of two, a binary linear code $C(SL(n,q))$ associated with the finite special linear group $SL(n,q)$ was constructed in order to produce a recursive formula for the power moments of multi-dimensional Kloosterman sums in terms of the frequencies of weights in that code. On the other hand, in [6], for q a power of three, two infinite families of ternary linear codes associated with double cosets in the symplectic group $Sp(2n,q)$ were constructed in order to generate infinite families of recursive formulas for the power moments of Kloosterman sums with square arguments and for the even power moments of those in terms of the frequencies of weights in those codes.

In this paper, we will utilize one simple identity connecting the Kloosterman sum for $GL(2,q)$ and the ordinary Kloosterman sum (cf. (2.3)). Then we will be able to produce a recursive formula generating the power moments of 2-dimensional Kloosterman sums, equivalently that generating the even power moments of Kloosterman sums. To do that, we construct a binary linear code connected with the Kloosterman sum for $GL(2,q)$.

Theorem 1.1 of the following (cf. (1.2)-(1.4)) is the main result of this paper. Henceforth, we agree that the binomial coefficient $\binom{b}{a} = 0$, if $a > b$ or $a < 0$.

Theorem 1.1 *Let $q = 2^r$. Then we have the following:*

(a) For $r \geq 2$, and $h = 1, 2, ...,$

$$\begin{aligned} MK_2^h =& \sum_{l=0}^{h-1} (-1)^{h+l+1} \binom{h}{l} (q^3 - 2q^2 - q + 1)^{h-l} MK_2^l \\ &+ q^{1-h} \sum_{j=0}^{min\{N,h\}} (-1)^{h+j} C_j \sum_{t=j}^{h} t! S(h,t) 2^{h-t} \binom{N-j}{N-t}, \end{aligned} \tag{1.2}$$

(b) For $r \geq 2$, and $h = 1, 2, ...,$

$$\begin{aligned} MK^{2h} =& \sum_{l=0}^{h-1} (-1)^{h+l+1} \binom{h}{l} (q^3 - 2q^2 + 1)^{h-l} MK^{2l} \\ &+ q^{1-h} \sum_{j=0}^{min\{N,h\}} (-1)^{h+j} C_j \sum_{t=j}^{h} t! S(h,t) 2^{h-t} \binom{N-j}{N-t}, \end{aligned} \tag{1.3}$$

where $N = |GL(2,q)| = q(q-1)(q^2-1)$, and $\{C_j\}_{j=0}^N$ is the weight distribution of $C(GL(2,q))$ given by

$$C_j = \sum \binom{m_0}{\nu_0} \prod_{|t|<2\sqrt{q},\ t \equiv -1(4)} \ \prod_{K(\lambda;\beta^{-1})=t} \binom{m_t}{\nu_\beta} \ (j = 0, \cdots, N), \tag{1.4}$$

with the sum running over all the sets of nonnegative integers $\{\nu_\beta\}_{\beta \in \mathbb{F}_q}$ satisfying

$$\sum_{\beta \in \mathbb{F}_q} \nu_\beta = j \quad and \quad \sum_{\beta \in \mathbb{F}_q} \nu_\beta \beta = 0,$$

$$m_0 = q(2q^2 - 2q - 1) \quad and \quad m_t = q(q^2 - 2q - 1 + t),$$

for all integers t satisfying $|t| < 2\sqrt{q}$ and $t \equiv -1(mod\ 4)$.

In addition, $S(h,t)$ is the Stirling number of the second kind given by

$$S(h,t) = \frac{1}{t!} \sum_{j=0}^{t} (-1)^{t-j} \binom{t}{j} j^h. \tag{1.5}$$

2. Preliminaries

Throughout this paper, the following notations will be used:

$$\begin{aligned} q &= 2^r \ (r \in \mathbb{Z}_{>0}), \\ \mathbb{F}_q &= \text{the finite field with } q \text{ elements}, \\ tr(x) &= x + x^2 + \cdots + x^{2^{r-1}} \text{ the trace function } \mathbb{F}_q \to \mathbb{F}_2, \\ \lambda(x) &= (-1)^{tr(x)} \text{ the canonical additive character of } \mathbb{F}_q. \end{aligned}$$

Then any nontrivial additive character ψ of $\mathbb{F}_q$ is given by $\psi(x) = \lambda(ax)$, for a unique $a \in \mathbb{F}_q^*$.

For any nontrivial additive character ψ of $\mathbb{F}_q$ and $a \in \mathbb{F}_q^*$, the Kloosterman sum $K_{GL(t,q)}(\psi; a)$ for $GL(t,q)$ is defined as

$$K_{GL(t,q)}(\psi; a) = \sum_{g \in GL(t,q)} \psi(Trg + aTrg^{-1}).$$

Observe that, for $t = 1$,$K_{GL(1,q)}(\psi; a)$ denotes the Kloosterman sum $K(\psi; a)$.

In [4], it is shown that $K_{GL(t,q)}(\psi; a)$ satisfies the following recursive relation: for integers $t \geq 2$, $a \in \mathbb{F}_q^*$,

$$K_{GL(t,q)}(\psi; a) = q^{t-1} K_{GL(t-1,q)}(\psi; a) K(\psi; a) + q^{2t-2}(q^{t-1} - 1) K_{GL(t-2,q)}(\psi; a), \tag{2.1}$$

where we understand that $K_{GL(0,q)}(\psi; a) = 1$.

Theorem 2.1 ([2]) *For the canonical additive character λ of $\mathbb{F}_q$, and $a \in \mathbb{F}_q^*$,*

$$K_2(\lambda; a) = K(\lambda; a)^2 - q. \tag{2.2}$$

Our paper will be based on the $t = 2$ case of the identity in (2.1).

Proposition 2.2 *For the canonical additive character λ of $\mathbb{F}_q$, we have:*

$$K_{GL(2,q)}(\lambda; a) = qK(\lambda; a)^2 + q^2(q - 1) = qK_2(\lambda; a) + q^3. \tag{2.3}$$

Proposition 2.3 ([7]) *For $n = 2^s$ ($s \in \mathbb{Z}_{\geq 0}$), λ the canonical additive character of $\mathbb{F}_q$, and $a \in \mathbb{F}_q^*$,*

$$K(\lambda; a^n) = K(\lambda; a). \tag{2.4}$$

Remark 2.4 In fact, (2.4) holds more generally for multi-dimensional Kloosterman sums. For $n = 2^s$ ($s \in \mathbb{Z}_{\geq 0}$), λ the canonical additive character of $\mathbb{F}_q$, $a \in \mathbb{F}_q^*$, and any positive integer m,

$$K_m(\lambda; a^n) = K_m(\lambda; a). \tag{2.5}$$

The order of the general linear group $GL(n,q)$ is given by

$$g_n = \prod_{j=0}^{n-1}(q^n - q^j) = q^{\binom{n}{2}} \prod_{j=1}^{n}(q^j - 1). \tag{2.6}$$

3. Construction of codes

Let

$$N = |GL(2,q)| = q(q - 1)(q^2 - 1). \tag{3.1}$$

Here we will construct a binary linear code C of length N connected with the Kloosterman sum for $GL(2,q)$.

Let $g_1, ..., g_N$ be a fixed ordering of the elements in $GL(2,q)$, and let $v = (Trg_1 + Trg_1{}^{-1}, ..., Trg_N + Trg_N{}^{-1}) \in \mathbb{F}_q^N$. The binary linear code $C = C(GL(2,q))$ is defined as

$$C = \{u \in \mathbb{F}_2^N \mid u \cdot v = 0\}. \tag{3.2}$$

The following Delsarte's theorem is well-known.

Theorem 3.1 ([11]) *Let B be a linear code over $\mathbb{F}_q$. Then $(B|_{\mathbb{F}_2})^{\perp} = tr(B^{\perp})$.*

In view of this theorem, the dual $C^{\perp}$ of C is given by

$$C^{\perp} = \{c(a) = (tr(a(Trg_1 + Trg_1{}^{-1})), ..., tr(a(Trg_N + Trg_N{}^{-1}))) \mid a \in \mathbb{F}_q\}. \tag{3.3}$$

The following estimate is very coarse but will serve for our purpose.

Lemma 3.2 *For any $a \in \mathbb{F}_q^*$, and ψ any nontrivial additive character of $\mathbb{F}_q$,*

$$\begin{aligned} &|K_{GL(n,q)}(\psi; a)| < |GL(n,q)|, \quad \text{for } n \geq 2 \text{ and } q \geq 4, \text{ and} \\ &|K_{GL(1,q)}(\psi; a)| < |GL(1,q)|, \quad \text{for } q \geq 8. \end{aligned} \tag{3.4}$$

Proof. For n=1, this is trivial, since $2\sqrt{q} < q-1$, for $q \geq 8$. For n=2, from (2.1)

$$K_{GL(2,q)}(\psi; a) = qK(\psi; a)^2 + q^2(q-1), \tag{3.5}$$

and hence from (1.1) and (3.5), for $q \geq 4$,

$$|K_{GL(2,q)}(\psi; a)| \leq q^3 + 3q^2 < q(q-1)(q^2-1) = |GL(2,q)|. \tag{3.6}$$

For $n = 3$, from (2.1),

$$K_{GL(3,q)}(\psi; a) = q^2 K_{GL(2,q)}(\psi; a)K(\psi; a) + q^4(q^2-1)K(\psi; a), \tag{3.7}$$

and hence from (1.1), (3.6), and (4.6), for $q \geq 4$,

$$|K_{GL(3,q)}(\psi; a)| < 2q^{\frac{7}{2}}(q^2-1)(2q-1) < q^3(q-1)(q^2-1)(q^3-1) = |GL(3,q)|.$$

Assume now that $n \geq 4$ and that (3.4) holds for all integers less than n and greater than and equal to 2, for $q \geq 4$. Then, from (1.1), (2.1), and (2.6), and for $q \geq 4$,

$$|K_{GL(n,q)}(\psi; a)| < q^{\binom{n}{2}}(q + 2\sqrt{q}) \prod_{j=1}^{n-1}(q^j - 1) < q^{\binom{n}{2}} \prod_{j=1}^{n}(q^j - 1) < |GL(n,q)|. \quad \blacksquare$$

Remark 3.3 It was shown in [3, *Theorem* 2] that, for any nontrivial additive character ψ of $\mathbb{F}_q$ and $a \in \mathbb{F}_q^*$,

$$K_{GL(n,q)}(\psi; a^2) = \sum_{g \in GL(n,q)} \psi(a(Trg + Trg^{-1})) = (-1)^n q^{\binom{n}{2}} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \omega^j \bar{\omega}^{n-j},$$

where ω, $\bar{\omega}$ are complex numbers, depending on ψ and a, with $|\omega|$=$|\bar{\omega}|$=$\sqrt{q}$. Thus

$$|K_{GL(n,q)}(\psi; a^2)| \leq q^{\frac{1}{2}n^2} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q ,$$

and, in particular, we get

$$|K_{GL(2,q)}(\psi; a^2)| \leq q^2 \sum_{j=0}^{2} \begin{bmatrix} 2 \\ j \end{bmatrix}_q = q^2(q+3).$$

Proposition 3.4 *The map $\mathbb{F}_q \to C^{\perp}$ $(a \mapsto c(a))$ is an $\mathbb{F}_2$-linear isomorphism for $q \geq 4$.*

Proof. The map is clearly $\mathbb{F}_2$-linear and surjective. Let a be in the kernel of the map. Then $tr(a(Trg + Trg^{-1})) = 0$, for all $g \in GL(2,q)$. Suppose that $a \neq 0$. Then, on the one hand,

$$(3.8) \quad \begin{aligned} |GL(2,q)| &= \sum_{g \in GL(2,q)} (-1)^{tr(a(Trg+Trg^{-1}))} = \sum_{g \in GL(2,q)} \lambda(a(Trg + Trg^{-1})) \\ &= \sum_{g \in GL(2,q)} \lambda(Trg + a^2 Trg^{-1}) \ (g \to a^{-1}g) = K_{GL(2,q)}(\lambda; a^2). \end{aligned}$$

As $q \geq 4$, (3.8) is on the other hand strictly less than $|GL(2,q)|$ by Lemma 3.2. This is a contradiction. So we must have $a = 0$. ∎

Remark 3.5 (a) If $q = 2$, one checks easily that the kernel of the map $\mathbb{F}_2 \to C^{\perp}$ is $\mathbb{F}_2$.

(b) The fact that the map in Proposition 3.4 is injective follows also from (1.1) and (3.11), since they imply that $n(\beta) > 0$, for all β, provided that $q \geq 4$.

Proposition 3.6 *([7]) Let λ be the canonical additive character of $\mathbb{F}_q$, $m \in \mathbb{Z}_{>0}$, $\beta \in \mathbb{F}_q$. Then*

$$(3.9) \quad \sum_{a \in \mathbb{F}_q^*} \lambda(-a\beta) K_m(\lambda; a) = \begin{cases} qK_{m-1}(\lambda; \beta^{-1}) + (-1)^{m+1}, & \text{if } \beta \neq 0, \\ (-1)^{m+1}, & \text{if } \beta = 0. \end{cases}$$

with the convention $K_0(\lambda; \beta^{-1}) = \lambda(\beta^{-1})$.

Let

$$(3.10) \quad n(\beta) = |\{g \in GL(2,q) | Trg + Trg^{-1} = \beta\}|.$$

Then, with N as in (3.1),

$$\begin{aligned} qn(\beta) &= N + \sum_{\alpha \in \mathbb{F}_q^*} \lambda(-\alpha\beta) \textstyle\sum_{g \in GL(2,q)} \lambda(\alpha(Trg + Trg^{-1})) \\ &= N + \sum_{\alpha \in \mathbb{F}_q^*} \lambda(-\alpha\beta) K_{GL(2,q)}(\lambda; \alpha^2) \\ &= N + \sum_{\alpha \in \mathbb{F}_q^*} \lambda(-\alpha\beta)(qK_2(\lambda; \alpha^2) + q^3)(cf.(2.3)) \\ &= N + q \sum_{\alpha \in \mathbb{F}_q^*} \lambda(-\alpha\beta) K_2(\lambda; \alpha^2) + q^3 \sum_{\alpha \in \mathbb{F}_q^*} \lambda(-\alpha\beta) \\ &= N + q \sum_{\alpha \in \mathbb{F}_q^*} \lambda(-\alpha\beta) K_2(\lambda; \alpha) + q^3 \sum_{\alpha \in \mathbb{F}_q^*} \lambda(-\alpha\beta) \ (cf.\ (2.5)). \end{aligned}$$

Now, from Proposition 3.6, we obtain the following.

Proposition 3.7 *Let $n(\beta)$ be as in* (3.10). *Then we have*

$$n(\beta) = \begin{cases} q(q^2 - 2q - 1 + K(\lambda; \beta^{-1})), & \text{if } \beta \neq 0, \\ q(2q^2 - 2q - 1), & \text{if } \beta = 0. \end{cases} \tag{3.11}$$

4. Power moments of 2-dimensional Kloosterman sums

In this section, we will be able to find, via Pless power moment identity, a recursive formula for the power moments of 2-dimensional Kloosterman sums or equivalently for the even power moments of Kloosterman sums in terms of the frequencies of weights in $C = C(GL(2,q))$.

Theorem 4.1 (Pless power moment identity): Let B be a q-ary $[n,k]$ code, and let B_i (resp. $B_i^{\perp}$) denote the number of codewords of weight i in B (resp. in $B^{\perp}$). Then, for $h = 0, 1, 2, ...,$

$$\sum_{j=0}^{n} j^h B_j = \sum_{j=0}^{min\{n,h\}} (-1)^j B_j^{\perp} \sum_{t=j}^{h} t! S(h,t) q^{k-t} (q-1)^{t-j} \binom{n-j}{n-t}, \tag{4.1}$$

where $S(h,t)$ is the Stirling number of the second kind defined in (1.5).

From now on, we will assume that $q \geq 4$ (i.e., $r \geq 2$), so that every codeword in $C(GL(2,q))^{\perp}$ can be written as $c(a)$, for a unique $a \in \mathbb{F}_q$ (cf. Proposition 3.4). This also allows one to use Theorem 4.5.

Lemma 4.2 *Let* $c(a) = (tr(a(Trg_1 + Trg_1{}^{-1})), ..., tr(a(Trg_N + Trg_N{}^{-1}))) \in C(GL(2,q))^{\perp}$, *for* $a \in \mathbb{F}_q^*$. *Then the Hamming weight $w(c(a))$ can be expressed as follows:*

$$w(c(a)) = \frac{1}{2} q(q^3 - 2q^2 + 1 - K(\lambda; a)^2) \tag{4.2}$$

$$= \frac{1}{2} q(q^3 - 2q^2 - q + 1 - K_2(\lambda; a)). \tag{4.3}$$

Proof.

$$\begin{aligned} w(c(a)) &= \frac{1}{2} \sum_{i=1}^{N} (1 - (-1)^{tr(a(Trg_i + Trg_i^{-1}))}) \\ &= \frac{1}{2} (N - \sum_{g \in GL(2,q)} \lambda(a(Trg + Trg^{-1}))) \\ &= \frac{1}{2} (N - \sum_{g \in GL(2,q)} \lambda(Trg + a^2 Trg^{-1})) \\ &= \frac{1}{2} (N - K_{GL(2,q)}(\lambda; a^2)) \\ &= \frac{1}{2} (N - qK(\lambda; a)^2 - q^2(q-1)) \ (cf. (2.3),\ (2.4)) \\ &= \frac{1}{2} q(q^3 - 2q^2 + 1 - K(\lambda; a)^2) \ (cf. (3.1)) \\ &= \frac{1}{2} q(q^3 - 2q^2 - q + 1 - K_2(\lambda; a)) \ (cf. (2.2)). \end{aligned}$$

■

Let $u = (u_1, ..., u_N) \in \mathbb{F}_2^N$, with ν_β 1's in the coordinate places where $Trg_j + Trg_j{}^{-1} = \beta$, for each $\beta \in \mathbb{F}_q$. Then we see from the definition of the code $C(GL(2,q))$ (cf. (3.2)) that u is a codeword with weight j if and only if $\sum_{\beta \in \mathbb{F}_q} \nu_\beta = j$ and $\sum_{\beta \in \mathbb{F}_q} \nu_\beta \beta = 0$ (an identity in $\mathbb{F}_q$). As there are $\prod_{\beta \in \mathbb{F}_q} \binom{n(\beta)}{\nu_\beta}$ many such codewords with weight j, we obtain the following result.

Proposition 4.3 *Let $\{C_j\}_{j=0}^N$ be the weight distribution of $C(GL(2,q))$, where C_j denotes the frequency of the codewords with weight j in C. Then*

$$C_j = \sum \prod_{\beta \in \mathbb{F}_q} \binom{n(\beta)}{\nu\beta}, \tag{4.4}$$

where the sum runs over all the sets of nonnegative integers $\{\nu_\beta\}_{\beta \in \mathbb{F}_q}$ $(0 \le \nu_\beta \le n(\beta))$, satisfying

$$\sum_{\beta \in \mathbb{F}_q} \nu_\beta = j \text{ and } \sum_{\beta \in \mathbb{F}_q} \nu_\beta \beta = 0. \tag{4.5}$$

Corollary 4.4 *Let $\{C_j\}_{j=0}^N$ be the weight distribution of $C(GL(2,q))$. Then we have:*

$$C_j = C_{N-j},$$

for all j, with $0 \le j \le N$.

Proof. Under the replacements $\nu_\beta \to n(\beta) - \nu_\beta$, for each $\beta \in \mathbb{F}_q$, the first equation in (4.5) is changed to $N - j$, while the second one in (4.5) and the summands in (4.4) are left unchanged. Here the second sum in (4.5) is left unchanged, since $\sum_{\beta \in \mathbb{F}_q} n(\beta)\beta = 0$, as one can see by using the explicit expression of $n(\beta)$ in (3.11).■

Theorem 4.5 ([9]) *Let $q = 2^r$, with $r \ge 2$. Then the range R of $K(\lambda; a)$, as a varies over $\mathbb{F}_q^*$, is given by*

$$R = \{t \in \mathbb{Z} \mid |t| < 2\sqrt{q},\ t \equiv -1 (mod\ 4)\}.$$

In addition, each value $t \in R$ is attained exactly $H(t^2 - q)$ times, where $H(d)$ is the Kronecker class number of d.

Now, we get the following formula in (4.6), by applying the formula in (4.4) to $C(GL(2,q))$, using the explicit values of $n(\beta)$ in (3.11) and taking Theorem 4.5 into consideration.

Theorem 4.6 *Let $\{C_j\}_{j=0}^N$ be the weight distribution of $C(GL(2,q))$. Then*

$$C_j = \sum \binom{m_0}{\nu_0} \prod_{|t|<2\sqrt{q},\ t \equiv -1(4)} \prod_{K(\lambda;\beta^{-1})=t} \binom{m_t}{\nu_\beta} \ (j = 0, ..., N), \tag{4.6}$$

where the sum is over all the sets of nonnegative integers $\{\nu_\beta\}_{\beta \in \mathbb{F}_q}$ *satisfying*

$$\sum_{\beta \in \mathbb{F}_q} \nu_\beta = j \text{ and } \sum_{\beta \in \mathbb{F}_q} \nu_\beta \beta = 0,$$

$$m_0 = q(2q^2 - 2q - 1), \quad \text{and} \quad m_t = q(q^2 - 2q - 1 + t),$$

for all integers t *satisfying* $|t| < 2\sqrt{q}$ *and* $t \equiv -1 (mod\ 4)$.

We now apply the Pless power moment identity in (4.1) to $C(GL(2,q))^\perp$, in order to obtain the results in Theorem 1.1 (cf. (1.2)-(1.4)) about recursive formulas.

Then the left hand side of that identity in (4.1) is equal to

$$\sum_{a \in \mathbb{F}_q^*} w(c(a))^h, \tag{4.7}$$

with the $w(c(a))$ given either by (4.2) or by (4.3).

Using the expression of $w(c(a))$ in (4.3), (4.7) is

$$\begin{aligned} &\left(\frac{q}{2}\right)^h \sum_{a \in \mathbb{F}_q^*} (q^3 - 2q^2 - q + 1 - K_2(\lambda; a))^h \\ &= \left(\frac{q}{2}\right)^h \sum_{a \in \mathbb{F}_q^*} \sum_{l=0}^{h} (-1)^l \binom{h}{l} (q^3 - 2q^2 - q + 1)^{h-l} K_2(\lambda; a)^l \\ &= \left(\frac{q}{2}\right)^h \sum_{l=0}^{h} (-1)^l \binom{h}{l} (q^3 - 2q^2 - q + 1)^{h-l} M K_2^l. \end{aligned} \tag{4.8}$$

Equivalently, using the expression of $w(c(a))$ in (4.2), (4.7) is

$$\left(\frac{q}{2}\right)^h \sum_{l=0}^{h} (-1)^l \binom{h}{l} (q^3 - 2q^2 + 1)^{h-l} M K^{2l}. \tag{4.9}$$

On the other hand, the right hand side of the identity in (4.1) is

$$q \sum_{j=0}^{min\{N,h\}} (-1)^j C_j \sum_{t=j}^{h} t! S(h,t) 2^{-t} \binom{N-j}{N-t}. \tag{4.10}$$

Our main results in Theorem 1.1 (cf. (1.2)-(1.4)) now follow by equating (4.8) and (4.10), and (4.9) and (4.10). Also, one has to separate the term corresponding to $l = h$ in (4.8) and (4.9), and note $dim_{\mathbb{F}_2} C(GL(2,q)) = r$.

Note here that, in view of (2.2), obtaining power moments of 2-dimensional Kloosterman sums is equivalent to getting even power moments of Kloosterman sums.

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