PARTITIONED FRAMES IN DISCRETE BAK SNEPPEN MODELS

Livio Clemente Piccinini
Maria Antonietta Lepellere
Ting Fa Margherita Chang
Luca Iseppi

Dipartimento di Ingegneria Civile e Architettura
Via delle Scienze 208
Universit di Udine
33100 Udine
Italy
e-mails: piccinini@uniud.it
lepellere@uniud.it
chang@uniud.it
iseppi@uniud.it

Abstract. In this paper, we wish to present some simplified cases of discrete Bak-Sneppen models in which explicit computations via Markov chains are possible, hence reaching a better understanding of some rather hidden phenomena of the general case: in particular "avalanches" can be read in terms of mean waiting times and in terms of transitions between structures. The simple models allow us to introduce new frames that do not seem to have been considered in the previous literature, namely the case of partitioned Bak-Sneppen frames, that appear more realistic from the point of view of speed of evolution and do not present a unique criticality level, but a staircase tending towards a final equilibrium level, cadenced by an increasing sequence of footholds. The introduction summarizes Bak-Sneppen models, starting from the central model due to Bak and Sneppen, and recalls their use in applied sciences. The first section gives the general frame of models where locality and globality coexist, the second section shows the simplest case of a matching between locality and globality, that will become exemplar in the most complex frames of Bak-Sneppen processes. The main quantitative theorems are stated and proved in the third section and finally the fourth section presents examples that illustrate the more sophisticated points of our paper and the use (and limits) of experimental results, while the fifth section considers real world situations where Bak Sneppen partitioned schemes can be tailored to represent the core of their evolution.

Keywords: Self organizing criticalities, Markov Chains, Bak Sneppen processes, Econophysics, Socio-economic evolution staircase.

AMS Mathematics Subject Classification: 34C28, 37H99, 60J10, 60J22, 91B80.

Introduction

Bak-Sneppen ([1], [2]) original model (BS) can be defined as follows. There are n species arranged on a circle, each of which has been assigned a random fitness. The fitness values are independent and uniformly distributed on $(0; 1)$. At each discrete time step the system is updated by locating the lowest fitness and replacing this fitness, and those of its two neighbors, by independent and uniform $(0; 1)$ random variables. This model is the result of a powerful synthesis of non equilibrium systems displaying self organized criticality, a concept introduced by P. Bak, C. Tang and K. Wiesenfeld [3]. One of the most fundamental characteristics of a system in a self-organized critical state is to exhibit a stationary state with a long-range power law decay of both spatial and temporal correlations.

Usually, the self organized state is attained only after a very long period of transient, a minor change in the system can cause colossal instabilities called avalanches. Intermittent burst of activity separating long periods of quiescence is called punctuated equilibrium.

An f_0-avalanche is defined as the event when all fitness initially above a threshold f_0 are perturbed such that for a certain time there are some below f_0. The event ends as soon as all fitness are again above f_0. For a certain value of f_0, namely $f_0 = f_c$ one obtains scale free avalanches, i.e. their distribution in size and duration follows a power law. The exponent of this power law is not easy to measure and still debated ([12], [23]). The distribution above f_c seems to be uniform, and asymptotically one expects a step function for the distribution of fitness. The value of f_c is given in ([33], [23]) as $f_c = 0,66702$ (a simulation value).

BS models can be defined on a wide range of graphs using the same update rule as above. What the BS model illustrates is that even random processes can result in self-organization to a critical state, see [31] for a discussion. We cannot describe the many studies that have arisen in physics, probability, econophysic following the first paper; we just recall some developments that are connected with the present paper. There is anyhow a strong believing in the power of these methods for constructing economic models that should suggest what actually happens in many-agent phaenomena. Though the provisional validity is small, these models can be richer than many econometric sophisticated simulations.

The authors in particular think that a sound basis for applying BS model of contact with neighbors is given by Duesenberry demonstration effect. Its first presentation can be found in Duesenberry's [17], while many application in consumer's economy and sociology can be found for example in Cavalli [6]. We may also recall Cuniberti et al. [11] and Rotundo–Scozzari [38], Rotundo–Ausloos [37].

It has been debated if changing the microscopic dynamical rules in BS model does or does not change the self organized universality class of the system [34], [15]. It is therefore interesting to study the robustness of BS-type models when the interaction rules are changed. A number of variants of Bak and Sneppen original model have been introduced which evolve according to different criteria. One variant is the anisotropic Bak Sneppen model [26], [30], [22], in which, in addition to the least fit species, only its right-hand nearest neighbor is replaced.

This model also gives rise to a threshold value $f_c = 0.724$ [22]. Another variant on the BS model which eliminates topology is the mean-field version analysed in [21], [13], [29], in which one replaces the smallest fitness and a fixed number of randomly chosen other ones.

In [22] the authors perform a detailed investigation of the effect of symmetry on the scaling behavior of the BS model. In their generalized model, the site with the minimum fitness plus a neighbors on the left side and b neighbors on the right are replaced with independent random numbers. If $a = b = 1$ it is recovered the original model; if $a = 0$ and $b = 1$ the anisotropic BS model; if $a \neq b$ is obtained the modified BS models with asymmetric dynamics. They conjectured that all dynamics which preserve the reflection symmetry of the original BS model possess the same critical exponents as the original model, while asymmetric dynamics lead to the exponents of the anisotropic BS model, reinforcing the evidence for two symmetry-based universality classes [26]. We suppose that, apart segmentation, that obviously perturbs also the symmetric class, the asymmetric models do not belong to a unique class, but rather to a certain number of slightly differentiated classes as it is shown in the examples of 19 and 24 nodes (see Example 3).

Motivated by the difficulty of analyzing rigorously even the one-dimensional version of the BS model, J. Barbay and C. Kenyon [5] proposed a still simpler model with discrete fitness values.[1] In their model, each species has fitness 0 or 1, and each new fitness is drawn from the Bernoulli distribution with parameter p. Since there are typically several least fit species, the process then repeatedly chooses a species at random for mutation among the least fit species. They proved bounds on the average numbers of ones in the stationary distribution and presented experimental results. Parameter p can substitute up to a certain level a plurality of values, but it cannot explain the staircase phenomenon of Example 1 in Section 5. Hence binary structure, though simple and appealing, is not sufficient for a thorough description of what may happen.[2]

In the first section we give a frame for studying local and global evolution.

Section 2 discusses a basic model of teleological local-global evolution, that will be used to clarify the content of Sections 3 and 4 (the first example, where overtaking and footholds arise).

Partitioned BS frames are the object of Section 3, while Section 4 shows how this model clearly explains the overtaking of competitors with respect to species that seem to be well assessed and recalls some of our experimental data.

Finally, Section 5 considers real world situations where Bak Sneppen partitioned schemes can be tailored to represent the core of their evolution.

[1] Also this case is by no means trivial, as it was shown by Meester and Znamenski (see [32]).

[2] A further case is dealt by C. Bandt in [4], who shows that the discrete BS model behaves exactly like the contact process, on an arbitrary graph, thus all results which have been shown for Contact process will immediately extend to discrete BS model.

1. A frame for studying local and global evolution

The idea of BS model is to glue globality with locality. For this reason in this section it is pointed out the way how locality and globality coexist in the same system and how they interact. For sake of simplicity the present paper will deal with finite graphs and with discrete time evolution systems.

Let us consider a finite set of nodes

$$X = \{K_1, K_2, ..., K_n\}$$

To each node it is appended a unique cell that contains the node itself and in general some other nodes. This will be called locality cell of the node:

$$L_k = \{K_k, K_{(k1)}, ..., K_{(kh)}\}$$

The node that labels the locality cell is called the kernel. Remark that unlike in a partition the cells usually overlap, so that a node could belong to different locality cells, but in particular it belongs to its own cell, in which it is the kernel. The two trivial cases are the atomistic locality system, where $h = 0$ for all k, so that each cell is built only by its kernel and the cells form a partition of the set X, and the totally global system where all cells coincide with the whole set X. To each node K_k, a vector of features is associated. The dimension of the vector does not need to be the same at all nodes, and it can reduce to 1. In many cases, the features are expressed by values taken from some ordered set, or even from numeric sets, discrete or continuous. In order to apply the scheme to numerical calculus, it is useful to associate a duplicate of the vector that registers the next step of the evolution.

The meaning of the cell is that all evolutions in the cell depend only on the data contained in the cell, and affect only elements of the cell. Generally the evolution affects only the features of the kernel of the cell, but sometimes it can affect also other elements, as it happens in BS processes. Usually more than one type of evolution can affect a cell, or a parametric evolution, so when the cell is chosen for evolution it must receive also the information of the action that it must perform.

Globality is controlled by the Global Controller (GC). He has at least some counters and can access all data stored in the kernels and using these data he can perform global operations, if it is required. According to the counters and the result of operations he will define the next cell that will evolve, the choice of the type of evolution and its parameters, if any. For example the approximate solution of a differential equation (using for example fourth order Runge Kutta method) has at least one more parameter, namely the length of the step. More sophisticated controls arise when there is a guess of the error by dividing the step and comparing the solutions. The choice of the step may thus become adaptive.

A case in which globality performs a fundamental task is when the choice of the evolving cell is no longer sequential, but is connected to some global evaluation of all the cells. This happens for example in DNA analysis, where longest

matching is chosen for improvement. The simplest case is the random maximization, that is performed on atomistic local system: a cell is chosen randomly and then to its kernel is assigned a random number. After a transitory phase in which some memory of the original values still survives, it is obtained a purely random distribution of values, that coincides in probability with a casual extraction with replacement. The process becomes much more interesting if some rule for the choice of the evolution cell is added. Till now, no local relation is at hand. The idea of Bak and Sneppen is to join each kernel with two adjacent nodes, getting three-node local cells. The cell that evolves is the one whose kernel is minimum, but all the three nodes, not only the kernel, will receive a new value. In this case, very interesting connection structures tend to become more likely than purely casual distributions, usually long chains of high values, alternated with shorter chains of lower values is obtained.

An interesting process can arise also without ordered structures of values. It is enough to choose a cell where the value is not modal, and replace it at random. The structure will converge anyhow to a mode involving all kernels but one; in particular if, at the starting time, more than half of the kernels have the same value, this will also be the final mode, otherwise we have some function of membership depending on the original distribution of features. A somewhat opposite process arises when it is chosen one of the modal cells; in this case the evolution tends to a division into groups that have all the same mode, or a mode differing by one. Both schemes arise in political analysis (band-wagon in one case, segmentation in the other case, see [36])

2. Binary Bak Sneppen linear models

Let us introduce a modified discrete BS process denoted by $(l_1, l_2)-$BS. Let $X = \{0, 1, ..., n-1\}$ be the set of nodes in the global system. The nodes will be arranged on a circle (the operation are *mod* n), each of which has been assigned a random fitness. The fitness values are independent and uniformly distributed on the set $\{0, 1, ..., s-1\}$. Sometimes it will be useful to think the set $\{0, 1, ..., s-1\}$ as a partition of the unitary interval $[0, 1]$ so that the element $i \in \{0, 1, ..., s-1\}$ will be identified with the central value $\frac{i}{s} + \frac{1}{2s}$. Let l_1 and l_2 be natural numbers $(l_1 < l_2)$.

> *$(l_1, l_2)-$BS is a process such that at each discrete time step, the node $i \in X$ with the lowest fitness (in the case of more then one element the choice will be made randomly between all the candidates) is chosen. Then the fitness of i, $i + l_1$ and $i + l_2$, (so the locality cell of i is $L_i = \{i, i + l_1, i + l_2\}$) will be replaced by independent and uniform $\{0, 1, ..., s-1\}$ random variables.*

Since the states may be described by a number of n digits in base s, the total number of states is s^n and the evolution of the system beginning from an initial

state chosen randomly to the successive one, according to the law just described, can be modeled by an irreducible Markov chain with transition matrix, say M (to simplify the reading we will consider the transposed matrix of the usual one). Let us recall, for the convenience of the reader, some important definitions about Markov chains.

Definition 1 Two states x and y communicate each other, and we write $x \leftrightarrow y$, if they are equal or if it is possible (with positive probability) to get from either one to the other in a finite amount of time. This is an equivalence relation and the equivalence classes are called communication classes.

There are two types of communication classes: recurrent and transient.

Definition 2 A communication class Z is called transient if, starting from any $x \in Z$, it is possible to return to x only a finite number of times with probability one, otherwise it is called recurrent or persistent.

It is also useful to remaind the following result.

Theorem 1 *Given an irriducible Markov chain with transition matrix M, there is a unique probability distribution π on the state space such that*

$$M\pi = \pi \tag{1}$$

π is called the stationary distribution of the Markov chain.

It is easy to see that every stochastic matrix has the all-1 vector as a left eigenvector corresponding to the eigenvalue 1. The above theorem says that the corresponding right eigenvector is also non-negative, and that there is only one eigenvector corresponding to eigenvalue 1 if the matrix corresponds to an irreducible Markov chain.

The mean waiting time from state x to state y is the expected number of iterations for reaching state y for the first time starting from state x.

Definition 3 Let Z be a communication class, a state $e \in Z$ will be called an exchange state of Z if it is the unique state such that $Z \backslash \{e\}$ is not a communication class.

From a theoretical point of view, it is possible to construct the transition matrix M of the process for any finite dimension, but dimension grows as s^n. Hence the most critical factor is the number of sections, and this explains the importance of two section frame, where critical information are left to the probability of the digits as in [5]. The authors have developed a computer program that draws the transition matrix. From this matrix many information can be derived: the most important parameter associated to the process is its average:

$$A_s(n; l_1, l_2) = \sum_{i=0}^{s^n-1} \pi(i) w(i) \tag{2}$$

where $w(i)$ is the value of the state, in particular, in the standard case of sections on the unitary interval, denoting by $C_s(i,j)$ the number of j-digits present in the representation of i in base s, it amounts to

$$w(i) = \frac{1}{s}\sum_{j=0}^{s-1} C_s(i,j)j + \frac{1}{2s} \tag{3}$$

When the number of states is greater than 2 an important value is also the frequency of the sections (associated to the digit j ranging from 0 to $s-1$):

$$F_s(j;n;l_1,l_2) = \frac{1}{n}\sum_{i=0}^{s^n-1} \pi(i)C_s(i,j) \tag{4}$$

2.1. Example

As an example of non trivial discrete $(l_1,l_2)-$BS model we choose what seems to be the simplest meaningful case: 6 nodes and 2 values. In this case, when we identify the states apart the rotation, the 64 states can be reduced to the following 14:

0=000000	1=000001	3=000011	5=000101	7=000111	9=001001
11=001011	13=001101	15=001111	21=010101	23=010111	27=011011
31=011111	63=111111				

The transition matrix depends on the probability of each digit and on the system of locality cells. Since many minima may arise, we suppose that the locality cell is chosen at random among the candidates (different rules give rise to different transition matrix and may even be not consistent with a representation of states reduced by rotation, these evolutions will be discussed in another paper).

For sake of simplicity, we consider only the case of constant probability. For the couples (l_1,l_2) there are the following cases:

- $(1,5)$ (that is equal to the classical BS model)
- $(1,2)$ (that a posteriori coincides with the results of (1,3) and of (2,3))
- $(1,4)$ (that a posteriori coincides with the results of (2,5))
- $(2,4)$ (that as shown later allows also a more suitable state representation)

Next table will summarize the results, the first part shows the frequencies of each state, the second one shows the averages. We have added also a non BS case, namely the random one, where the locality cell coincides with the whole space for each kernel, and the two neighbors are chosen at random (compare [21], [13], [29]).

Configuration	Structure (1,5)	Structures (1,2),(1,3),(2,3)	Structures (1,4),(2,5)	Structure (2,4)	Structure Random
0	0,002643	0,0033320	0,004112	0,000000	0,003662
1	0,024136	0,0304099	0,035908	0,000000	0,032801
3	0,049081	0,0446415	0,050715	0,000000	0,050041
5	0,035378	0,0446415	0,050715	0,000000	0,050041
7	0,108473	0,0626871	0,069095	0,000000	0,079790
9	0,016208	0,0280814	0,028844	0,000000	0,025020
11	0,060323	0,1002903	0,076068	0,000000	0,079790
13	0,060323	0,0742085	0,098396	0,000000	0,079790
15	0,167864	0,1221499	0,120348	0,000000	0,128153
21	0,016527	0,0208957	0,023032	0,125000	0,026597
23	0,122677	0,1221499	0,120348	0,375000	0,128153
27	0,044115	0,0798766	0,074824	0,000000	0,064076
31	0,230218	0,2115085	0,197502	0,375000	0,201231
63	0,062034	0,0551272	0,050093	0,125000	0,050854

l_1	l_2	$A_2(6; l_1, l_2)$
1	5	0,568695
1	2	0,560366
1	4	0,553865
2	4	0,625
random	random	0,556145

Table 1. *Binary BS model on a frame of* 6 *nodes: frequencies and averages*

The most striking value is the case of (2,4) and will be discussed in detail below. Usually it is expected that in BS processes (self organizing criticality) the average is higher than in a random process, since BS lets arise chains of adjoining high values and chains are in some sense stable. Table 1 shows that, in a structured process, it is possible to get an average lower than in a random process, namely the strongly asymmetric case (1,4) in which the two operating cells are just opposite, hence tend to break long chains of maximums wherever they could be located.

In the case (2,4), the set of nodes is split in two subsets: even $E = \{0, 2, 4\}$ and odd $O = \{1, 3, 5\}$. For each even kernel its locality cell is given by E, while for odd kernels their locality cell is given by O. There is no overlapping between the two sets of locality cells. Hence the system operates as if it was built by two subsystems of three nodes each. Until both in E and in O there exist 0's, the change may happen in any of the two sets, and the probability is given according to the frequency of 0's (in a set of three elements BS is trivial since it coincides with random process). When the state 111 is reached in one of the subsets, only the other one may be changed, until the exchange state 111, 111 is reached. From

this state the process will go on in one of the two subsets, but the other one will maintain the values of 111, so that 0's can appear at most in one of two subsets. This means that the original set of 14 states reduces to a persistent class formed only by states 21, 23, 31, 63, that is one set with three 0's, one set with two 0's, one set with one 0 and the exchange state of all 1's. This reduced system does not give a complete information since it identifies the two subsets O and E, that up to a rotation are coincident.

An equivalent (but more expressive) transition matrix could be derived from the 14-element matrix taking into account only the couples of numbers of 1's in the two sets. We get the following 10 states

00 10 11 20 21 22 30 31 32 33

where ij means i ones in one set and j ones in the other. The correspondences are obtained by direct check: 00={0}; 10={1}; 11={3, 9}; 20={5}; 21={7,11,13}; 22={15,27}; 30={21}; 31={23}; 32={31}; 33={63}.

Gluing together the rows and columns in the 14 nodes transition matrix actually we get the reduced transition matrix M, given by:

	00	10	11	20	21	22	30	31	32	33
00	1/8	2/40	0	1/32	0	0	0	0	0	0
10	3/8	9/40	1/8	3/32	1/24	0	0	0	0	0
11	0	9/40	3/8	0	3/24	0	0	0	0	0
20	3/8	6/40	0	6/32	2/24	1/8	0	0	0	0
21	0	9/40	3/8	9/32	9/24	3/8	0	0	0	0
22	0	0	0	9/32	6/24	3/8	0	0	0	0
30	1/8	2/40	0	1/32	0	0	1/8	1/8	1/8	1/8
31	0	3/40	1/8	0	1/24	0	3/8	3/8	3/8	3/8
32	0	0	0	3/32	2/24	1/8	3/8	3/8	3/8	3/8
33	0	0	0	0	0	0	1/8	1/8	1/8	1/8

Table 2. *Transition matrix of the reduced representation for the* $(2,4)$*-BS process of* 6 *nodes.*

Let us remark that from all the six non persistent states $(00, 10, 11, 20, 21, 22)$ the total probability of reaching a persistent state $(30, 31, 32, 33)$ is $1/8$, but it is not possible to reach directly the exchange state 33, that can be attained only starting from a persistent state.

In the next section we shall review this example in the context of a general theory of partitioned structures. This section ends with some remarks about equivalence of different (l_1, l_2)-BS like processes. In fact a condition of equivalence is the possibility of rearrangement of rows and columns of the transition matrix. A trivial condition of equivalence is symmetry: in a non symmetric BS process such as (1,2), this is equivalent to $(n-1, n-2)$. A more sophisticate condition can be achieved when a Hamiltonian permutation of the nodes is available. In order to avoid complication due to partitioned frames, consider a prime number of

nodes n. Then, for any given couple (l_1, l_2), all the couples ([$(k\ l_1)$ mod n],[$(k\ l_2)$ mod n]) are obtained by a Hamiltonian permutation that allows to rearrange the transition matrix so that it coincides with the original matrix associated to (l_1, l_2). Let us remark that this result does not require the actual construction of the transition matrix, and it holds for any dimension of nodes and sections. Consider, for example, the case of 5 nodes: the couples are (1,2); (1,3); (1,4) (equivalent to standard BS); (2,3); (2,4); (3,4). The couple (3,4) is the symmetric of (1,2) and (2,4) is the symmetric of (1,3). We get $(1, 2) \equiv (2, 4)$, hence $(1, 2) \equiv (1, 3)$. Finally, $(1, 4) \equiv (2, 3)$; the latter equivalence is usually read as $(1, -1) \equiv (2, -2)$ (that is a formal extension of symmetric BS, compare [22]).

A somewhat more complex model is obtained for a higher prime number of nodes. For example, in the case of 19 nodes there are at most 9 possible equivalent structures. We list them according to decreasing (statistical) average. There is statistical evidence that the two couples (1,2), (1,17) and (1,3), (1,16) are actually different even if it is not proved that the two components of the couple are different. On the contrary, there is no statistical evidence that the last four cases have really different averages.

For reader's information, next table gets the statistical average values for 19 nodes. Even if this section is dealing with binary processes, it is added also information about the division in 4 sections. The frequencies in the case of 2 sections are immediately recovered from the average subtracting 0.25 and multiplying by 2. The first five structures are kept distinct, while the remaining 4 are merged into the class "other".

Name	Av. 2 sect.	Av. 4 sect.	$F_s(0)$	$F_s(1)$	$F_s(2)$	$F_s(3)$
(1,18)	0.67826	0.70015	0.045065	0.090301	0.383608	0.481026
(1,2)	0.653533	0.66260	0.049383	0.156482	0.387682	0.406453
(1,17)	0.640368	0.661908	0.050324	0.157052	0.388094	0.40453
(1,3)	0.62702	0.64451	0.052896	0.195466	0.373361	0.378277
(1,16)	0.62455	0.64583	0.053179	0.191047	0.37481	0.380964
other	0.621289	0.64137	0.05363	0.20193	0.36977	0.37468

Table 3. *Statistical results for a frame of* 19 *nodes,* 2 (*respectively* 4) *sections.*

The reader can make many remarks. The most obvious is that passing from 2 sections to 4 sections the average is increased; this fact depends from the central value that is assigned to each section. Recall that the limit distribution in Bak and Sneppen infinite dimensional model is a step function of 0 value up to the critical value, and then it is constant. In general, it is a monotone function that becomes asymptotically constant. Giving the central value to each interval it is obtained a good estimate for the asymptotical section of the distribution, but we underestimate the lower sections, the more the less is the number of sections.

3. Partitioned frames

In this section we will analyze a much more interesting phenomenon, that may arise only in the case of a non prime number of nodes (hence the reason for choosing 6 nodes for the simplest case). In Table 1 the average of the last case (2,4) is much greater than all the remaining averages. In the transition matrix of the associated Markov chain there exists a persistent class that is smaller than the whole set of 14 structures. Namely it contains only 4 structures (apart rotation): 01 01 01, 01 01 11, 01 11 11, 11 11 11, while the remaining 10 form a transient class: this means that there exist sets of transformations that allow to pass from any of them to any other, but there is the possibility of falling outside into the persistent class without the possibility of returning back. In this simple case the analysis is straightforward: nodes 2 and 4, together with the minimum conventionally placed at 0, form a subset that has no interference with nodes 1-3-5, that, on the contrary, are activated when the minimum is attained in one of those nodes. Whenever an operation is performed, one 3-element subset is left unchanged, while the other one is totally changed at random. The trick is that in this case the two subsets do not change between different operations, what on the contrary happens in all the remaining cases. The process cannot anyhow be divided into two independent subprocesses, since the minimum must be looked for among the elements of both subsets (for example a subset containing 011 enters in this search, while a subset 111 enters in the search only if also the other one is a 111 subset, in which case there are 6 minima of value 1). During the transient phase a further interaction is given by the number of minima; for example in the case 000 011 the first subset has probability 3/4 of being chosen, while the other one only 1/4. When one set reaches the configuration 111 (probability 1/8) the transient phase is terminated and the process becomes stable, in the sense that it can be changed only if the global minimum is 1, that is the configuration 111 111. This is the exchange state. In this case at least one of the two subsets will save the configuration 111, that can no longer be destroyed. The four structures above in fact can be read, using the reduced state description of the last section, as 03= 000 111, 13 = 001 111, 23 = 011 111, 33 = 111 111. In the particular case of 6 nodes the transition matrix becomes trivial since there is no longer a BS structure.

In particular the mean waiting time for reaching the persistent class starting from any configuration of the transient class is 8, and does not depend on the initial structure. The mean waiting time for reaching the top (and exchange) configuration 111 111 starting from 000 000 (or any other transient) is 16. Remark that in the standard BS process this last mean waiting time is 23.07572. The increase in time is due to the impossibility of protecting the structure 111 from decay once it is achieved for the first time.

A richer, but similar situation, arises in the binary case of 8 nodes. The subsets are formed by four elements and the different structures are 0000, 0001, 0011, 0101, 0111, 1111; the persistent class is thus formed by a subset 1111 coupled with any of these six structures. The estimations become less trivial because the structure with four elements is already BS non trivial, inasmuch one element

is not changed at random and preserves memory of the past. The average is therefore somewhat increased, up to 0.63785. Using the transition matrix, one can compute the mean waiting time. In particular, in this case it depends on the starting configuration, in particular from 0000 0000 it is 14.79873, while it attains a minimum from 0011 0011 or 0111 0111 where anyhow the 1 is saved. In this case the mean waiting time is 10.98887.

We wish now to state formally the theorems that describe the main factors of partitioned frames. In the first theorem there is no need to define explicitly what a partitioned frame is, since it is simply built by the distinct cosets of the cyclic group of n elements when n is not prime. In the general case of the second theorem some definition will be needed, and also more information about the substructures will be required.

Theorem 2 *Let n be a non prime number of nodes. Let $m = MCD(n, l_1, l_2)$* [3] *be the greatest common divisor, and denote by $Max = (s - 1/2)/s$ the central value of the upmost section, then the average is*

$$A_s(n; l_1, l_2) = \frac{m-1}{m} Max + \frac{1}{m} A_s\left(\frac{n}{m}; \frac{l_1}{m}, \frac{l_2}{m}\right) \tag{5}$$

and the frequencies of the single sections are

$$F_s(j; n; l_1, l_2) = \frac{1}{m} F_s\left(j; \frac{n}{m}; \frac{l_1}{m}, \frac{l_2}{m}\right) \tag{6}$$

for $j < s - 1$, and

$$F_s(s-1; n; l_1, l_2) = \frac{m-1}{m} + \frac{1}{m} F_s\left(s-1; \frac{n}{m}; \frac{l_1}{m}, \frac{l_2}{m}\right) \tag{7}$$

Proof. The theorem is a particular case of Theorem 2. Symmetry considerations would allow a straightforward proof, but we prefer to use the proof of the much more powerful Theorem 2. As for the numerical aspect see the comment after the proof of theorem 2. ∎

We give now a formal definition of partitioned scheme.

Definition 4 Let X be the space of nodes on which a locality cell system L_i is given; X is said to be partitioned into a subsystem $(X_1, X_2, ..., X_k)$ if for any cell L_h it holds

$$\#\{i : X_i \cap L_h \neq \} = 1$$ [4]

[3] Of course, if $m = 1$, the formulas still hold, but are not meaningful.

[4] That is, each locality cell is contained exactly in one subset of the partition.

With n_i we will denote the cardinality of set X_i and so $n = \sum_{i=1}^{k} n_i$.[5]

In a partitioned scheme, the configurations may be rearranged in order to be consistent with the partition in subsets. To a set X_k we associate a set S_k formed by the configurations that involve the nodes of the component X_k. A set of configurations suitable for the construction of the transition matrix is thus represented by the Cartesian product

$$S = S_1 S_2 ... S_k \ ^{6}$$

The use of a reduced transition space is allowed by the fact that changes can happen only inside one subspace at each time. When the maximum is the same for the whole system (as it usually happens) the exchange configuration is $E = m_1 m_2 ... m_h$, where m_i denotes the state in which all nodes in the subset X_i attain the maximum value. If in the punctuated k-dimensional structure we associate the state m_i with the plane $x_i = 0$, then the persistent class is formed by the coordinate axes and the exchange configuration is the origin. In the partitioned BS scheme the transition matrix thus assumes the form given in the following table.

	E	C_1	C_2	...	C_h	Z
E	$*$	$*$	$*$	$*$	$*$	0
C_1	$*$	$*$	0	0	0	$*$
C_2	$*$	0	$*$	0	0	$*$
...	$*$	0	0	$*$	0	$*$
C_h	$*$	0	0	0	$*$	$*$
Z	0	0	0	0	0	T

Table 4. *Transition matrix in a partitioned BS scheme.*

Let us remark that C_i denotes the set of states derived from the exchange state E changing only the i-th component of the Cartesian product, and leaving all the remaining components at the value m_s.[7] For passing from one state to another it is compulsory to pass through the exchange state E. Here $*$ denotes the elements that can be non zero, while 0 indicates the components that are necessarily 0. T is the transition matrix of the transient class Z in itself, and also can be built up by a recursive matrix of partitioned frames. This happens in particular whenever the sections are more than 2, as it will be shown in the examples of the next section.

[5]Very often it may be convenient to look for the finest partition available, but our results do not require minimality conditions.

[6]An example was given for the case of 6 nodes when we passed from the 14 state full representation to the 10 state reduced representation.

[7]Hence it must not be confounded with S_i since it is a proper subset of the whole Cartesian product.

The restrictions of the process to each subset X_i can be described using a transition matrix M_i defined on the system of configurations S_i. Thus we get the usual information such as frequencies of each configuration y_i that we shall denote by F_i (y_i), in particular we shall be particularly interested to F_i (E_i), where E_i denotes the projection of E on the space X_i. Another important information is of course the average value A_i. When we pass from the partitioned matrixes M_i to the global matrix M, we require one more information, namely the probability p_i of going from the exchange state E to each state of the subsystems C_i. Let us remark that in BS processes this is given by the ratio between the number of maximum elements in each set (i.e n_i/n). In particular if all the sets X_i have the same number of nodes, p_i is constant. We describe the persistent matrix, dropping the transitions described by the last column and the last row. All data outside the first column of the transition matrix of Table 4 are the same as in the single transition matrixes M_h, and a formal change is the addition of the full description of the other subsets that remain unchanged. The only change happens in the first column, where the terms outside the first row are the corresponding terms of the first rows of matrixes M_i multiplied by p_i, and $M(E, E)$ is the complement to 1 of the column. Different schemes of course could be foreseen, provided the exchange rule is maintained.

The most interesting results allow to calculate global frequencies and averages from the frequencies and averages of the single subsystems, taking obviously into account the cardinality of the subsets, the value of the exchange state (usually the maximum) and the probability of different exits from the exchange state. For sake of generality we denote by V_i the value of each component of the exchange state, hence its value is given by the sum of these items.

Theorem 3 *Let M be the transition matrix of a partitioned frame generated by matrixes M_i. Let E be the exchange state, p_i the probability of different exits from the exchange state, F_i the family of frequencies for each subsystem. Then the average A is a weighted sum of the single averages according to the formula*

$$A = \frac{1}{N} \sum_{i=1}^{k} \frac{p_i}{F_i(E_i))} \left(\sum_{j \neq i} \frac{n_j}{n} V_j + \frac{n_i}{n} A_i \right) \tag{8}$$

where the normalization factor satisfies the relation

$$N = \sum_{i=1}^{k} \frac{p_i}{F_i(E_i)} \tag{9}$$

The frequency of the global configurations is 0 for the transient configurations. As for the persistent configurations, we denote by B_i the configuration

$$(E_1, ..., E_{i-1}, b_i, E_{i+1}, ..., E_k)$$

and we get

$$F(B_i) = \frac{1}{N}\frac{p_i}{F(E_i)}F_i(b_i) \tag{10}$$

for $B_i \neq E$, *and*

$$F(E) = \frac{1}{N}\cdot \tag{11}$$

Proof. For sake of simplicity, we can consider the case of two subsets, since the general case may be proved by induction. We represent the transition matrixes M_1 and M_2 by the following notations

M_a	E_a	a_1	...	a_r
E_a	e_{a0}	a_{01}	...	a_{0r}
a_1	e_{a1}	a_{11}	...	a_{1r}
...	...	...	...	...
a_r	e_{ar}	a_{r1}	...	a_{rr}

M_b	E_b	b_1	...	b_s
E_b	e_{b0}	b_{01}	...	b_{0s}
b_1	e_{b1}	b_{11}	...	b_{1s}
...	...	...	...	...
b_s	e_{bs}	b_{s1}	...	b_{ss}

Let $v_0 = F(E_a)$, $v_1 = F(a_1)$,..., $v_r = F(a_r)$, and $w_0 = F(E_b)$, $w_1 = F(b_1)$, ..., $w_s = F(b_s)$, and denote by A the submatrix of M_a without the first row and the first column, and similarly by B the submatrix of M_b without the first row and the first column, by e_a the column vector $(e_{a1}, ..., e_{ar})^*$, by e_b the column vector $(e_{b1}, ..., e_{bs})^*$, an eigenvector associated to eigenvalue 1 is given by $(x_0, x_1, ..., x_r)$ where $x_0 = 1$ and $x_1, ..., x_n$ satisfy the system

$$(A - I)\mathbf{x} = -e_a \tag{12}$$

Normalizing, the frequency vector is given by $\mathbf{v} \equiv (v_0, x_1v_0, ..., x_rv_0)$ or $\mathbf{x} = \mathbf{v}/F(E_a)$. Respectively, let $y_1, y_2, ..., y_s$ satisfy the system

$$(B - I)\mathbf{y} = -e_b \tag{13}$$

Normalizing, we get $\mathbf{w} \equiv (w_0, y_1w_0, ...y_sw_0)$, or $\mathbf{y} = \mathbf{w}/F(E_b)$

In the complete system, we are interested only to the part that corresponds to the persistent class. The matrix is thus

M	E	A_1	...	A_r	B_1	...	B_s
E	$p_ae_{a0} + p_be_{b0}$	a_{01}	...	a_{0r}	b_{01}	...	b_{0s}
A_1	p_ae_{a1}	a_{11}	...	a_{1r}	0	...	0
...	...	...	...	...	...	...	
A_r	p_ae_{ar}	a_{r1}	...	a_{rr}	0	...	0
B_1	p_be_{b1}	0	...	0	b_{11}	...	b_{1s}
...	...	...	...	...	...	...	
B_s	p_be_{bs}	0	...	0	b_{s1}	...	b_{ss}

Table 5: *Joint transition matrix of a partitioned process*

In order to obtain the eigenvectors we solve thus the system

$$\begin{cases} (A-I)\mathbf{x}' + 0\mathbf{y}' = -p_a e_a \\ 0\mathbf{x}' + (B-I)\mathbf{y}' = -p_b e_b \end{cases}$$

obtaining

$$\mathbf{x}' = p_a\mathbf{x} = p_a/F(E_a)\mathbf{v},\ \mathbf{y}' = p_b\mathbf{y} = p_b/F(E_b)\mathbf{w}.$$

The normalization of the vector leads to the normalizing factor N given in equation (9) that appears in the statement of the theorem.

$$\begin{aligned} 1+\sum_{i=1}^{r} x_i' + \sum_{j=1}^{s} y_i' &= 1 + \frac{p_a}{F_a(E_a)}\sum_{i=1}^{r} v_i + \frac{p_b}{F_b(E_b)}\sum_{i=1}^{s} w_i \\ &= 1 + \frac{p_a}{F_a(E_a)}(1-F_a(E_a)) + \frac{p_b}{F_b(E_b)}(1-F_b(E_b)) \\ &= 1 + \frac{p_a}{F_a(E_a)} + \frac{p_b}{F_b(E_b)} - p_a - p_b \\ &= \frac{p_a}{F_a(E_a)} + \frac{p_b}{F_b(E_b)} = N \end{aligned} \tag{14}$$

Once formulas (10) and (11) are proved, the main formula (8) is easily derived keeping in mind that the average value of the configuration is the weighted sum of the averages of the single cartesian components, and the weight is just given by the proportion of configurations that belong to each S_i. ∎

We wish to remark that there is an essential difference between the two theorems: in fact in Theorem 1 in order to get the average it is not required to know the frequencies $F_i(E_i)$. From a numerical point of view these frequencies, very near to 0, are difficult to be experimentally determinated, while on the contrary the averages in non partitioned schemes can be easily experimentally estimated. Let us remark that this is not true for the case of partitioned schemes, where transitory phase may be very long. A computational trick is thus to start directly from the exchange configuration or anyhow from the interior of the persistent class, so that transitory period is skipped away.

4. Examples and staircase of critical configurations with overtaking

In this section, we give some examples for the theorems of the previous section, but before we highlight the main features of the partitions that do not allow to study these processes simply using binary representations joint to a change in the probability system.

4.1. Example 1

We come to a more general case of partition of the BS process. The simplest case that shows the main features requires 2 subsets of 3 nodes each, and we consider a ternary set of values, say 0,1,2 [8]. This corresponds to 6 nodes and displacements of 2-4. A simplified analysis is the following: we attach label 0 to any subset in which the minimum is 0 (not regarding their number), label 1 to any subset in which the minimum is 1 (not regarding their number), and label 2 to the set $\{2, 2, 2\}$.

Next table shows the transition matrix.

	00	01	10	11	02	20	12	21	22
00	38/54								
01	7/54	38/54		19/54					
10	7/54		38/54	19/54					
11		14/54	14/54	14/54					
02	1/54				38/54		38/54		19/54
20	1/54					38/54		38/54	19/54
12			2/54	1/54	14/54		14/54		7/54
21		2/54		1/54		14/54		14/54	7/54
22					2/54	2/54	2/54	2/54	2/54

Tab le 6. *Transitions in a* 6 *node bipartite Bak Sneppen set with* 3 *values.*

There is one transient class (01, 11, 10) with exchange state 11 and one persistent class (20, 02, 22, 21, 12) with exchange state 22. The asymptotic average, normalized to the scale $[0, 1]$ is thus 0.667, since one subset has the form $\{2, 2, 2\}$, that corresponds to 0.8333 and the second one is random on the three values 012 and corresponds to 0.5. A more complex frame with a ternary partition, but in reduced form, was introduced in [35] and [36]. From states 00, 01, 10, 11 the mean waiting time to the persistent class is 27. Let us remark that the topological structure of our representation is the same as what would be achieved in section 3 when dealing with random optimization for the case of three values on three nodes, and becomes exactly the same if we set $p(0) = 19/27$, $p(1) = 7/27$, $p(2) = 1/27$.

Unlike the ternary partition there exists a path that touches all states even if mean waiting time between non persistent states still remains infinity. This remark a peculiar "overtaking law", that namely concerns the transition from 10 to 12 and from 01 to 21. The subset that is changed into the optimal label 2 is not one labeled with 1, but one that is labeled with 0, since its minimum must be lower than that of the best subset. For example, in a situation $A = \{1, 2, 2\}$, $B = \{0, 1, 1\}$, it is impossible that A is transformed into $\{2, 2, 2\}$, while this is possible, even if unlikely, for B.

[8]The example was first presented during the AMASES meeting of 2011 [35], see also [36]

4.2. Example 2

We go back now to the examples that illustrate theorems 1 and 2. We give an example of a non symmetric partitioned frame in which BS processes happen. We suppose that the total number of nodes is $n = 8$, and that we have a subset X_a of $n_a = 5$ nodes and a subset X_b of $n_b = 3$ nodes. The number of sections is 2. On both subsets the locality cells are whose of the standard BS process, namely each cell contains the kernel and the right and left neighbor; obviously in the case of three nodes (random case) the locality cell is the same for all the kernels and coincides with the subset.

We write down separately the two evolution matrixes and we make the usual computations of frequency and of average. The representation is minimal, hence S_a contains 8 items instead of 32 and S_b contains 4 items instead of 8. In the case of S_a we get the transition matrix of Table 7, while the trivial transition matrix of S_b in presented in Table 8.

	31	0	1	3	5	7	11	15	F	$\sharp 1's$
31	0	0	0	0.0416667	0	0.125	0	0.125	0.071267	5
0	0	0.125	0.0625	0	0.0416667	0	0	0	0.0098982	0
1	0.125	0.375	0.25	0.0833333	0.2083333	0	0.125	0	0.0735294	1
3	0	0.25	0.1875	0.125	0.1666667	0.125	0.125	0.125	0.1348982	2
5	0.25	0.125	0.1875	0.1666667	0.2083333	0	0.25	0	0.0975679	2
7	0.125	0.125	0.125	0.1666667	0.125	0.25	0.125	0.25	0.196267	3
11	0.375	0	0.125	0.2083333	0.1666667	0.125	0.25	0.125	0.1589367	3
15	0.125	0	0.0625	0.2083333	0.0833333	0.375	0.125	0.375	0.2576357	4
								Average	0.549095	

Table 7. *Transition matrix for Bak Sneppen binary process on five nodes.*

	7	0	1	3	F	$\sharp 1's$
7	0.125	0.125	0.125	0.125	0.125	3
0	0.125	0.125	0.125	0.125	0.125	0
1	0.375	0.375	0.375	0.375	0.375	1
3	0.375	0.375	0.375	0.375	0.375	2
				Average	0.5	

Table 8. *Trivial transition matrix of binary BS process on three nodes.*

The exchange state is $(31, 7)$, that corresponds to section 1 in all the 8 nodes. The probability of changing state S_a (resp. S_b) is proportional to the number of nodes, so we have $p_a = 0.625$, $p_b = 0.325$, $n_a = 5$, $n_b = 3$. The remaining coefficients are already known from the elaboration of the transition matrixes, we have namely $F(E_a) = 0.071267$, $F(E_b) = 0,125$, $m_a = 0.5491$ and $m_b = 0.5$ (obviously, since it is a random process). We get thus the coefficients $p_a/F(E_a) = 8.76984$, $p_b/F(E_b) = 3$, $N = 11.7698$. We recall that, in view of our convention about the middle of the sections, $V_a = V_b = 0.75$. Finally we get from theorem

2, formula (8), the final value $A = 0.632544$. The case is simple enough that it can be handled directly by constructing the transition matrix. We obtain the following table:

	31-7	0-7	1-7	3-7	5-7	7-7	11-7	15-7	31-0	31-1	31-3	F	$1's$
31-7	0.125	0	0	0.041	0	0.125	0	0.125	0.125	0.125	0.125	0.084	8
0-7	0	0.125	0.062	0	0.041	0	0	0	0	0	0	0.007	3
1-7	0	0.375	0.25	0.083	0.208	0	0.125	0	0	0	0	0.055	4
3-7	0.078	0.25	0.187	0.125	0.166	0.125	0.125	0.125	0	0	0	0.100	5
5-7	0	0.125	0.187	0.166	0.208	0	0.25	0	0	0	0	0.072	5
7-7	0.156	0.125	0.125	0.166	0.125	0.25	0.125	0.25	0	0	0	0.146	6
11-7	0.078	0	0.125	0.208	0.166	0.125	0.25	0.125	0	0	0	0.118	6
15-7	0.234	0	0.062	0.208	0.083	0.375	0.125	0.375	0	0	0	0.191	7
31-0	0.046	0	0	0	0	0	0	0	0.125	0.125	0.125	0.031	5
31-1	0.140	0	0	0	0	0	0	0	0.125	0.125	0.125	0.095	6
31-3	0.140	0	0	0	0	0	0	0	0.375	0.375	0.375	0.095	7
										Average	0.632		

Table 9. *The transition matrix of an asymmetrically partitioned Bak Sneppen frame*

A further comparison of the frequencies is consistent with formulas (10) and (11).

We end this example comparing the result with the symmetrical partition of the 8 nodes system as given by formula (5) in Theorem 1. We need to know $A_2(4; 1, 3)$.

The transition matrix is given below and the average is 0.525735.

	15	0	1	3	5	7	F	$\sharp 1's$
15	0.125	0	0.041667	0.125	0	0.125	0.088235	4
0	0	0.125	0.083333	0	0.125	0	0.036765	0
1	0.125	0.375	0.291667	0.125	0.375	0.125	0.198529	1
3	0.25	0.25	0.25	0.25	0.25	0.25	0.25	2
5	0.125	0.125	0.125	0.125	0.125	0.125	0.125	2
7	0.375	0.125	0.208333	0.375	0.125	0.375	0.301471	3
						average	0.525735	

Table 10. *Transition matrix for BS binary process on four nodes*

By this computation we also know $F(15)$, but we do not require it [9]. We remark that, as usual, $V(E_a) = V(E_b) = 0.75$. Hence we get

$$A_4(8; 2, 6) = [0.75 + A_2(4; 1, 3)]/2 = 0.637868.$$

The symmetrical partition gives an average slightly greater than the unbalanced partition, according to the general conjecture of [26] and to some experimental results we recall now in the end of this section.

[9]The advantage of not requiring the knowledge of $F(E)$ will become evident in the next example.

4.3. Example 3

This example shows a partitioned frame in a case particularly rich, namely 24 nodes. Here also some of the partitions may still be decomposed. The results of Theorem 2 do not depend on the number of sections, but the numerical simulation can become heavily unstable, since it can take a very long transient period before the exchange state is reached for the first time. If the numerical simulations are needed for analyzing the persistent state, the best strategy is to start the simulation from the exchange state or anyhow from the inside of the persistent class. Since in Theorem 1 all the partitions are equal, there is no problem if it takes a great number of iterations to rejoin the exchange state in order to pass to another partition.

The problem would arise in the case of Theorem 3, where the single partitions must necessarily be simulated separately, and the further estimate of the frequency of exchange state for each partition would be needed.

We describe now the structure of the 24 node system. The experimental data (*) are referred to 4 sections, where in particular $M = 0.875$. In the next table we report only the reduced classical BS of each partition.
8 subsets of 3 elements, 6 subsets of 4 elements, 4 subsets of 6 elements, 3 subsets of 8 elements, 2 subsets of 12 elements:

$$\begin{array}{ll} A_s(24,8,16) = 7/8M + 1/8A_s(3,1,2) & A_4(24,8,16)^* = 0.828166 \\ A_s(24,6,18) = 5/6M + 1/6A_s(4,1,3) & A_4(24,6,18)^* = 0.817772 \\ A_s(24,4,20) = 3/4M + 1/4A_s(6,1,5) & A_4(24,4,20)^* = 0.802184 \\ A_s(24,3,21) = 2/3M + 1/3A_s(8,1,7) & A_4(24,3,21)^* = 0.789492 \\ A_s(24,2,22) = 1/2M + 1/2A_s(12,1,11) & A_4(24,2,22)^* = 0.768288 \\ A_4(24,1,23) & = 0.715446 \end{array}$$

For reader's convenience we add also

$$A_4(24,1,2)^* = A_4(24,5,10)^* = A_4(24,7,14)^* = A_4(24,11,22)^* = 0.676339$$

The remaining non reducible structures have averages ranging between 0.649575 and 0.663925.

The 19 (prime number) case is non decomposable and its range should be compared only with non decomposable cases of 24; but the whole range reaches, as shown before, 0.8281, hence much greater than the case of 19 nodes. These results are to be compared with the case of a prime number of nodes, where the dependence on the structure has much smaller effects. In the case of 19 nodes the total range is between 0.640 and 0.700, while here the average ranges between 0.660 and 0.828. While the non decomposable frames show only small differences due to the increase in the number of nodes, the main differences arise as the dimension of the subsets diminishes down to the elementary case of 3 elements.

5. Examples from the socio-economic world

As mentioned in the introduction, the BS model is suitable to analyze multi-agent economic and social phenomena. Many variations are obtained by structuring the network of connections differently, giving different laws of transition, establishing different criteria for choosing the item that has to undergo the change. In the case of partitioned schemes the basic element is the segmentation of the system of nodes, which may possibly accompany the other structural changes. The situation presented in the theoretical sections and in the section of mathematical examples is clearly an extreme situation, but it summarizes trends that can be found in the external reality: the links that join subsets can become very weak at the level of connection between different sets, while can remain strong within the subsets. An analysis of the fundamental components then leads to an approximation of the model that ends up consisting of separate components as it happens in Hsu technique of generalized cell-mapping ([27]), used by the authors in territorial analysis ([9]). Example 3 of the previous section shows that there are large differences in mean values as the division into subsets becomes finer, while in the absence of splitting the average values are much lower. Actually the process of approximation is more articulated, because the transition takes place progressively changing the laws of proximity between nodes. In the examples there was a pure dichotomy, with the node connected or not connected without gradients, while it could be supposed that the transition to the partitioned cases occurs by a progressive change of the probability of proximity. So the model provides a useful partitioned limit-schema for understanding (but not for quantitative prediction).

An aspect of particular interest was highlighted in Example 1 where the phenomenon of overtaking is presented. It is typically characterized by a low probability, but becomes gradually higher as the set shrinks. The phenomenon of overtaking can occur along a scale of discrete values more complex than the ternary system (and as it is obvious it cannot be recognized in the binary system), and as the steps go up it becomes more and more unlikely, but instead acquires great stability. A first discussion of the phenomenon was given in [36].

The territorial systems are those which because of their nature are more subjected to segmentation, both for the influence of metric distance, and for the effect of border barriers, that may be physical but also regulatory, economic, social. In urbanism the various districts are subject to town plans that generally do not interface with their neighbors (even if it seems absurd), and thus to a BS evolution model that is subject to segmentation. This encourages overtaking, which often is found in the restoration of old urban centers or deteriorated areas of the city, as has often occurred in the case of the old port areas (Liverpool, Valencia, the East End of London.) The drag effects fall in the broader study of spontaneous synchronization phenomena that characterize complex systems. A regional system of central places (as proposed by Christaller and perfected by many geographers) can enter in the pattern of BS models provided the local districts are normalized to make them balanced in dimensions (combinations of secondary sites or subdivisions of the central place) or in connections (asymmetrical links). However, when the system is substantially changed, becoming a lattice, it is necessary to

pass to a partitioned model, where the restricted set of nodes in the network is separate from the system of central places (which is rated lower). All the same, if in this system subsystems of small size are created, it is possible that in turn some of them can perform overtaking. That is the case for many small countries that were not originally nodes in the network system, such as Luxembourg, San Marino, Monaco, United Arab Emirates, where the ability to get out of law restrictions allows the creation of nuclei inserted in a strong economic system both because of the connections with the rich system and for the possibility of tax havens. Singapore has not been added to this list, as it, like other strategic ports, has always represented an exceptional situation, so it would be improper to speak of overtaking, while it is more correct to see a continuity in its position of privilege.

Partitioned models typically represent abnormal situations, as the evolution of the individual subsets may differ significantly. At the global level between the US and Europe there are sensitive differences in terms of the law even where there are international rules. An interesting case concerning different rules for the production of compost was discussed in [20], particularly about the rights on the green waste, where a standard obedience was expected. Even at the local level BS models of evolution bring to unexpected gradients as soon as region borders are overcome ([28]) and it seems that these gradients may not be statistically included in a simple random fluctuation.

Economic interdependence is a field that suggests various applications. There is a horizontal network of relationships between companies operating in the same industry and a network of vertical relations of sale and purchase between the different branches. This is assessed at national level through various types of input-output matrices based on Leontiev Model. The input-output matrices correspond to an oriented network connecting the various nodes (branches) of the economy and allow to study how internal relations influence economic evolution. There are several ways to reorder the matrix and then to analyze the interrelationships between the subsystems of the network. This leads to the possibility of comparisons between different countries, as was done in section 3 of [10]. You may recall that in theory the French model emphasizes the Hamiltonian circuit, so aims to an economy in which the structure should recall the basic circular model of BS. However, the partitioned models can lead to higher average returns even if the imbalances between single subsets may be higher.

The segmentation is due to different skills and different technologies but also to the entry barriers that individual subsystems seek to set up for their own defense. On the other side they evaluate the possibility of overcoming thus becoming aggressive and the effects of this unstable equilibrium can be evaluated by means of dedicated indicators (see for example [7]). At the horizontal level the slight initial segmentation due to natural aggregations disappears along the trend towards oligopoly, which creates a smaller number of subsets, moreover strongly segmented between them. The vertical expansion of these oligopolistic subsets is not very strong but it is an important phenomenon in terms of the BS model, as it enhances concentration and promotes the tendency to a network system of

oligopolists that crosses the widespread system of small agents. The problem has been studied in particular in two fields where the concentrated system coexists with the distributed system, namely in the field of hotels and restaurants by [16] and in the agrifood chain by [8].

In a socio-cultural context, interesting cases occur in the evolution of science when overtaking happens in those areas where large investments of capital are not required. Mathematics is one of these areas. Countries with good basic skills but no former tradition of innovative research may suddenly succeed since there is the possibility to bypass the plethora of detailed information by directly accessing the nodal points of the evolution of the search. According to some scholars, in particular Guerraggio (see [25]), that is what happened in the golden age of Italian mathematics in the years 1880 to 1910, when the new state saw active almost simultaneously U. Dini, V. Volterra, G. Peano, F. Enriques, Severi F.G. Castelnuovo, just to remember some of the famous names who were honored by the greatest international appraisal. The importance of Italy was confirmed by the allocation of the third International Congress (Rome 1908) and by the astonishing development of Mathematical Circle of Palermo. In its acts (e.g. [24]) the formidable team of world-famous associates at the beginning of the century can still be read: among them Poincaré considered Palermo Circle as the most important mathematical organization in the world. It is worth to mention the sudden overtaking in regularity theorems performed by De Giorgi in the 50's of last century (De Giorgi-Nash theorem). The very De Giorgi in a private conversation said he had profited of his restricted knowledge of the relevant literature, so that he did not follow the paths already beaten. In fact he pointed strongly on the knowledge of Caccioppoli inequality and on the isoperimetric refinement of Sobolev inequality, adding the construction and the solution of an ingenious system of finite difference equations that allowed him to close the chain of inequalities.

In the world economic development, the examples of sudden overtaking are very common. Starting two centuries ago from Germany industry, when England compelled German production to advertise that it was "made in Germany", and not in England, passing through Japan and, again, Germany after the Second World War, and arriving at the emerging economic and industrial powers of China, India, Brazil, South Africa. Self defense of the leading countries creates the segmentation that can lead to the overtaking, that will be discovered only when it is actually too late. A counterpart can be found in protectionism, that encourages the first phases of development, defending from the risks of global, and mostly unfair, competition, but according to Example 1 bears the risk of stopping a long time on one of the lower steps of the development staircase, without the incentive to new (and perhaps risky) steps in connection with global evolution. Bak and Sneppen partitioned schemes are sensitive to these real world situations. In these models, as sometimes in the real life, the principle is *quieta non movere* (Let quiet things stay). The idea is that movement requires material or intellectual dissatisfaction, while further movements are caused by some form of nested neighborhood, with unsatisfied people that can in turn generate new dissatisfaction. To this purpose we can remind the *happiness paradox* of Easterlin ([18], [19]).

6. Conclusions

In this paper, the authors have shown that discrete models of local-global processes may give relevant information on systems where self organizing criticalities arise. They have shown that a relevant feature is the existence of sequences of footholds that allow also long time persistency, but are subjected to phenomena of overtaking (what has also a very important economical and behavioural meaning). From a technical point of view the authors have shown that among the many generalizations of BS processes, there exist a lot of cases where some form of decomposition is possible, allowing optimization at higher levels. Random optimization, that seems to represent the simplest form of local-global process, allows a comparison with BS processes on partitioned frames, even if it does not allow sharp estimates on the average time required to ascend all the scale of increasing footholds. The definition of partitioned frames is somewhat more general than the pure definition that arises from BS models. The main theorems give a sharp information about the average and the frequencies of the states of the global processes when the single processes that are glued together are known, thus allowing in particular a reduction of the numerical instabilities that conflict with a good knowledge of self organizing criticalities.

This scheme could be used in very general partitioned processes; in some cases two processes could coexist independently from each other, if the transition matrix of the system S_k does not depend from the remaining S_i's, but this case would have little interest. In general the evolution depends also on what is the actual global state of the system even if it perturbs only one subset at a time. The single transition matrix might even not depend explicitly on the rest of the system, but it is enough that some selection rules are given from the GC that allow or forbid some subsystems to evolve, according to a suitable law of choice, deterministic or probabilistic. Actually when dealing with BS processes, much harder bounds are put on partitioned schemes. In fact a subset can be changed only if it has at least one node that attains the minimum. When in a configuration all the nodes of a subset attain the maximum, they cannot be changed until some lower term exists in the system. This means that in this terminal (persistent) class the only admissible configurations are those in which all the subsets but one get the maximum value. In particular should it happen that some subset can attain values greater than the rest of the system, then when it attains this "exceptional" maximum it will never change any longer. The socioeconomic examples of Section 5 suggest the use of these generalizations.

In our numerical simulations we experimented mainly the case of four values since for lower number of values the standard BS distribution is anyhow too concentrated on the top value, hence the different steps are confounded with the casual fluctuation. As soon as the dimension of the subsets is increased it becomes more and more difficult to reach the stable steps; for example already subsets of 18 nodes very often require more than 100,000 iterations in order to reach the persistent class. Let us remark that the transition, when it happens, is very similar to an "avalanche" and average suddenly increase. In some cases the

process is not complete, hence it can be reverted, but finally it happens that the threshold is reached. Increasing the number of values, multiple thresholds arise; the lower levels can be easily overcome, while the top levels can prove to be almost unreachable. This is for example the case of ten values and forty nodes, in which the top level has never been reached in ten simulations of 1,000,000 iterations.

A further remark is that in complex cases (more than two subsets, more than two levels) the evolution staircase can change between simulations, since each step is not reversible, but different steps have different compatibility, so that different development paths can arise starting from similar original situations.

The experienced reader can remark that the Global Controller is not a new concept, since such a figure was introduced in full detail already in the Middle Ages by Dante's poetry, and was called Fortuna (Fortune):

> *"He made earth's splendors by a like decree*
> *and posted as their minister this high Dame,*
> *The Lady of Permutations. All earth's gear*
> *she changes from nation to nation, from house to house,*
> *in changeless change through every turning year."*

Dante, Inferno, 7, 77-81 translation J. Ciardi

References

[1] Bak, P., *How Nature Works: The Science of Self-Organized Criticality*, Springer, 1996.

[2] Bak, P., Sneppen, K., *Punctuated equilibrium and criticality in a simple model of evolution*, Phys. Rev. Lett., 71 (24) (1993), 4083-4086.

[3] Bak, P., Tang, K., Wiesenfeld, K., *Self-Organized Criticality*, Phys. Rev., A 38 (1988), 364-374.

[4] Bandt, C., *The discrete evolution model of Bak and Sneppen is conjugate to classical contact process*, J. Stat. Phys., 120 (3-4) (2005), 685-693.

[5] Barbay, J., Kenyon, C., *On the discrete Bak-Sneppen model of self-organized criticality*, Proc. 12th Annual ACM-SIAM Symposium on discrete Algorithms, Washington DC, 2001.

[6] Cavalli, A., *Sociologia* , Enciclopedia delle Scienze Sociali VIII Roma, 1998.

[7] Chang, T.F.M., Iseppi L., *Specialization versus Diversification in EU Economies: a Challenge for Agro-food?*, Transition Studies Review, 18, 1 (2011), 16-37, Springer.

[8] Chang, T.F.M., Iseppi L., *EU Agro-Food Chain and Vertical Integration Potentiality: a Strategy for Diversification?*, Transition Studies Review, 19, 1 (2012), 107-130, Springer.

[9] Chang, T.F.M., Piccinini, L.C., *A new Method for Analyzing Urban-Rural Relations through Commutation Data*, H. Hillebrand, R. Goetgeluk, H. Hetsen (eds.) Plurality and Rurality (The role of the countryside in urbanized Regions), 1 LEI The Hague (2000), 102-122.

[10] Chang, T.F.M., Piccinini, L.C., Iseppi, L., Lepellere, M.A., *The black box of economic interdependence in the process of structural change. EU and EA on the stage*, Italian Journal of Pure and Applied Mathematics, 31 (2013), 285-306.

[11] Cuniberti, G., Valleriani, A., Vega, J.L., *Effects of regulation on a self-organized market*, Quant. Fin., 1 (2001), 332-335.

[12] Datta, K. Christensen, K., Jensen, H.J., *On the physical relevance of extremal Dynamics*, Europhys. Lett., 50 (2) (2000), 162-168.

[13] de Boer, J., Derrida, B., Flyvbjerg, H., Jackson, A.D., Wettig, T., *Simple-model of self-organized biological evolution*, Phys. Rev. Lett, 73 (1994), 906-909.

[14] De Los Rios, P., Marsili, M., Vendruscolo, M., *Self organized criticality in deterministic systems with disorder*, Phys. Rev. Lett., 80 (1998), 6451-6459.

[15] De Los Rios, P., Marsili, M., Vendruscolo, M., *High-dimensional Bak-Sneppen model*, Phys. Rev. Lett., 80 (1998), 5746-5749.

[16] Droli, M., Chang, T.F.M., Iseppi, L., Piccinini, L.C., *Managing trade contacts in HotRest intermediate markets: a resource-based view analysis in EU countries* Tourism Economics, August, 20, 4 (2014), 757-778.

[17] Duesenberry, J.S., *Income, Savings and Theory of Consumer Behaviour*, Cambridge Mass, (1949), 89-125.

[18] Easterlin, R.A., *Does economic Growth improve the human Lot?*, In P.A.David and M.W.Reder eds. Nations and Households in Economic Growth Acad. Press N.Y., 1974.

[19] Easterlin, R.A., *Income and Happiness: Towards a Unified Theory*, The Econ. J., (2001), 465-484.

[20] Eggerth, L.L., Diaz, L.F., Chang, T.F.M., Iseppi, L., *Marketing of Composts*. In Diaz L.F., de Bertoldi M., Bidlingmaier W. Stentiford E., *Compost Science and Technology*, Waste Management Series, vol. 8 (2007), 325-355, Elsevier Science Ltd, Amsterdam, the Netherlands.

[21] Flyvbjerg, H., Sneppen, K., Bak, P., *Mean Field Theory for a Simple Model of Evolution*, Phys. Rev. Lett., 74 (1993), 4087-4090.

[22] Garcia, G.J.M., Dickman, R., *On singular probability densities generated by extremal dynamics*, Phys. A, 332 (2004), 318-336.

[23] Grassberger, P., *The Bak-Sneppen model for punctuated evolution*, Phys. Lett., A 200 (1995), 277-282.

[24] G.B. Guccia Supplemento Rendiconti Circolo Matematico Palermo, 1909, 4-9.

[25] Guerraggio, A., *La bella stagione*. In WMY 2000 Anno mondiale della Matematica, Bollati Boringlieri, 2000, 50-56.

[26] Head, D.A., Rodgers, G.J., *The anisotropic Bak-Sneppen model*, Phys., A 31 (1998), 3977-3988.

[27] Hsu, C.S., *Cell to cell mapping. A method of global analysis for non linear systems*, Springer Vg, NY, 1987.

[28] Iseppi, L., Chang, T.F.M., Droli, M., *Lombardy and Veneto Biocultural Fingerprint: a Driving Force for Tourism and Residential Attraction.* In: Society, Integration, Education Proceedings of the International Scientifical Conference. Sabiedriba, Integracija, Izglitiba, 2 (2013), 353-363.

[29] Labzowsky, G.L., Pismak, Y.M., *Exact analytical results for the Bak-Sneppen model with arbitrary number of randomly interacting species*, Phys. Lett., A 246 (1998), 377-383.

[30] Maslov, S., De Los Rios, P., Marsili, M., Zhang, Y.C., *Critical exponents of the anisotropic Bak-Sneppen model*, Phys. Rev., E 58 (1998), 7141-7145.

[31] Meester, R., Quant, C., *Connections between "classical" and "self-organized" criticality*, Markov Process. Related Fields, 11 (2) (2005), 355-370.

[32] Meester, R., Znamenski, D., *Non-triviality of a discrete Bak-Sneppen evolution model*, J. Stat. Phys., 109 (2002), 987-1004.

[33] Paczuski, M., Maslov, S., Bak, P., *Avalanche dynamics in evolution, growth, and depinning models*, Phys. Rev., E 53 (1996), 414-443.

[34] Paczuski, M., Maslov, S., Bak, P., *Field-theory for a model of self-organized criticality*, Europhys. Lett., 27 (1994), 97-102.

[35] Piccinini, L.C., Lepellere, M.A., Chang, M., *Partitioned Frames in Bak Sneppen Models*, Atti del XXX convegno AMASES, 2011.

[36] Piccinini, L.C., Lepellere, M.A., Chang, M., *Utopias of Perfection and their Dystopias*, Society, Integration, Education, Proc. of Intern. Conference: Sabiedriba, Integracija, Izglitiba, 3 (2013), 189-200.

[37] Rotundo, G., Ausloos, M., *Microeconomic co-evolution model for financial technical analysis signals* Phys. A, 373 (2007), 569-585.

[38] Rotundo, G., Scozzari, A., *Coevolutive Models for Firms Dynamics*, Lecture Notes in Econom. Math. Systems, 613 (2008), 143-158.

Accepted: 21.09.2014