ON UPPER AND LOWER ALMOST CONTRA-ω-CONTINUOUS MULTIFUNCTIONS

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Abstract. In this paper, we introduce and study the almost contra-ω-continuous multifunctions between topological spaces and obtain some characterizations and properties of such multifunctions.

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1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, separation axioms, etc., by utilizing generalized open sets. One of the most well known notions and also an inspiration source is the notion of ω-open [10] sets introduced by H.Z. Hdeib in 1982 and used by Al-Zoubi and Al-Nashef [1] in 2003. Various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good many of them have been extended to the setting of multifunction [2], [6],[7], [9], [17], [18]. In this paper, we introduce and study almost contra-ω-continuous multifunctions between topological spaces and obtain some characterizations of such multifunctions.

2 Preliminaries

Throughout this paper, (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces in which no separation axioms are assumed unless explicitly stated. For a subset A of (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure of A with respect to τ and the interior of A with respect to τ, respectively. Recently, as generalization of closed sets, the notion of ω-closed sets were introduced and studied by Hdeib [10]. A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A is said to be ω-closed [10] if it contains all its condensation points. The complement of an ω-closed set is said to be ω-open. It is well known that a subset W of a space (X, τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U \backslash W$ is countable. The family of all ω-open subsets of a topological space (X, τ) denoted by $\omega O(X, \tau)$, forms a topology on X finer than τ and the family of all ω-closed subsets of a topological space (X, τ) is denoted by $\omega C(X, \tau)$. The ω-closure and the ω-interior, that can be defined in the same way as $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively, will be denoted by $\omega\,\mathrm{Cl}(A)$ and $\omega\,\mathrm{Int}(A)$, respectively. We set $\omega O(X, x) = \{A : A \in \omega O(X) \text{ and } x \in A\}$ and $\omega C(X, x) = \{A : A \in \omega C(X) \text{ and } x \in A\}$. A subset A is said to be regular open [20] (resp. semiopen [13], preopen [14], α-open [15], semi-preopen [3]) if $A = \mathrm{Int}(\mathrm{Cl}(A))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$). The complement of a regular open (resp. semiopen, preopen, semi-preopen) set is called a regular closed (resp. semiclosed, preclosed, semi-preclosed). The intersection of all semiclosed (resp. preclosed, α-closed, semi-preclosed) subsets of (X, τ) containing $A \subset X$ is called the semiclosure (resp. preclosure, α-closure, semi-preclosure) of A and is denoted by $s\,\mathrm{Cl}(A)$ (resp. $p\,\mathrm{Cl}(A)$, $\alpha\,\mathrm{Cl}(A)$, $sp\,\mathrm{Cl}(A)$). The θ-semiclosure [12] of A, denoted by $s\,\mathrm{Cl}_\theta(A)$, is defined to be the set of all $x \in X$ such that $A \cap \mathrm{Cl}(U) \neq \varnothing$ for every semiopen set U containing x. A subset A is called θ-semiclosed [12] if and only if $A = s\,\mathrm{Cl}_\theta(A)$. The complement of a θ-semiclosed set is called a θ-semiopen set [12]. The family of all regular open (resp. regular closed, semiopen, semiclosed, α-open, semi-preopen, semi-preclosed) sets of (X, τ) is denoted by $RO(X)$ (resp. $RC(X)$, $SO(X)$, $SC(X)$, $\alpha O(X)$, $SPO(X)$, $SPC(X)$). By a multifunction $F : (X, \tau) \to (Y, \sigma)$, following [4], we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X : F(x) \subset B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. In particular, $F^-(Y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$ and for each $A \subset X$, $F(A) = \cup_{x \in A} F(x)$. Then F is said to be surjection if $F(X) = Y$ and injection if $x \neq y$ implies $F(x) \cap F(y) = \emptyset$.

Definition 2.1 A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be:

1. upper almost ω-continuous [5] if for each point $x \in X$ and each open set V containing $F(x)$, there exists $U \in \omega O(X, x)$ such that $U \subset F^+(\mathrm{Int}(\mathrm{Cl}(V)))$;
2. lower almost ω-continuous [5] if for each point $x \in X$ and each open set V such that $F(x) \cap V \neq \emptyset$, there exists $U \in \omega O(X, x)$ such that $U \subset F^-(\mathrm{Int}(\mathrm{Cl}(V)))$.
3. upper contra-ω-continuous [6] if for each point $x \in X$ and each closed set V containing $F(x)$, there exists $U \in \omega O(X, x)$ such that $U \subset F^+(V)$;
4. lower contra-ω-continuous [6] if for each point $x \in X$ and each closed set V such that $F(x) \cap V \neq \emptyset$, there exists $U \in \omega O(X, x)$ such that $U \subset F^-(V)$;
5. upper weakly ω-continuous [7] if for each $x \in X$ and each open set V of Y such that $x \in F^+(V)$, there exists $U \in \omega O(X, x)$ such that $U \subset F^+(\mathrm{Cl}(V))$;
6. lower weakly ω-continuous [7] if for each $x \in X$ and each open set V of Y such that $x \in F^-(V)$, there exists $U \in \omega O(X, x)$ such that $U \subset F^-(\mathrm{Cl}(V))$.

Definition 2.2 A subset K of a space X is said to be S-closed [21] (resp. ω-compact [2]) relative to X if every cover of K by regular closed (resp. ω-open) sets of X has a finite subcover. A space X is said to be S-closed (resp. ω-compact) if X is S-closed (resp. ω-compact) relative to X.

Lemma 2.3 [1] *Let A and B be subsets of a space (X, τ).*

1. *If $A \in \omega O(X)$ and $B \in \tau$, then $A \cap B \in \omega O(B)$;*
2. *If $A \in \omega O(B)$ and $B \in \omega O(X)$, then $A \in \omega O(X)$.*

Lemma 2.4 [17] *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following holds:*

1. $G_F^+(A \times B) = A \cap F^+(B)$;
2. $G_F^-(A \times B) = A \cap F^-(B)$,

for any subset A of X and B of Y, where $G_F : X \to X \times Y$ is defined as $G_F(x) = \{x\} \times F(x)$ for every $x \in X$.

Definition 2.5 [2] A function $f : (X, \tau) \to (Y, \sigma)$ is said to be almost contra-ω-continuous if $f^{-1}(W) \in \omega O(X)$ for every $W \in RC(Y)$.

3. On upper and lower almost contra-ω-continuous multifunctions

Definition 3.1 A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be:

1. upper almost contra-ω-continuous if for each point $x \in X$ and each regular closed set V with $x \in F^+(V)$, there exists $U \in \omega O(X, x)$ such that $U \subset F^+(V)$;
2. lower almost contra-ω-continuous if for each point $x \in X$ and each regular closed set V with $x \in F^-(V)$, there exists $U \in \omega O(X, x)$ such that $U \subset F^-(V)$.

Theorem 3.2 *If $F : (X, \tau) \to (Y, \sigma)$ is an upper (lower) almost contra-ω-continuous multifunction, then it is upper (lower) weakly ω-continuous.*

Proof. Let $x \in X$ and V be an open subset of Y with $F(x) \subset V$. This implies that $\mathrm{Cl}(V)$ is a regular closed set with $F(x) \subset \mathrm{Cl}(V)$. Since F is upper almost contra-ω-continuous, there exists $U \in \omega O(X, x)$ such that $U \subset F^+(\mathrm{Cl}(V))$. Hence, F is upper weakly ω-continuous. ∎

The following example shows that the converse of the above Theorem 3.2 is not true in general.

Example 3.3 Let $X = \Re$ with the topologies $\tau = \{\emptyset, \Re, \Re - Q\}$ and $\sigma = \{\emptyset, \Re, \Re - Q\}$. Define $F : (\Re, \tau) \to (\Re, \sigma)$ as follows: $F(x) = \{x\}$. Then F is upper weakly-ω-continuous multifunction but is not upper almost contra-ω-continuous multifunction.

Corollary 3.4 *If $F : (X, \tau) \to (Y, \sigma)$ is almost contra-ω-continuous, then it is weakly ω-continuous.*

Theorem 3.5 *If $F : (X, \tau) \to (Y, \sigma)$ is an upper (lower) contra-ω-continuous multifunction, then it is upper (lower) almost contra ω-continuous multifunction.*

Proof. The proof is obvious. ∎

The following example shows that the converse of the above Theorem 3.5 is not true in general.

Example 3.6 Let $X = \Re$ with the topology $\tau = \{\emptyset, \Re, \Re - Q\}$. And $Y = \{a, b, c\}$ with the topology $\sigma = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Take a fixed number $e \in Q$, and define $F : (\Re, \tau) \to (Y, \sigma)$ as follows:

$$F(x) = \begin{cases} \{b\} & \text{if } x \in Q - \{e\} \\ \{c\} & \text{if } x \in (\Re - Q) \cup \{e\}. \end{cases}$$

Then F is upper almost contra-ω-continuous multifunction but is not upper contra-ω-continuous multifunction.

Corollary 3.7 [2] *If $F : (X, \tau) \to (Y, \sigma)$ is contra-ω-continuous, then F is almost contra-ω-continuous.*

Theorem 3.8 *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

1. *F is upper almost contra-ω-continuous;*
2. *$F^+(A) \in \omega O(X)$ for every regular closed A of Y;*
3. *$F^-(U) \in \omega C(X)$ for every regular open subset U of Y;*
4. *$F^-(\mathrm{Int}(\mathrm{Cl}(A))) \in \omega C(X)$ for every open subset A of Y;*
5. *$F^+(\mathrm{Cl}(\mathrm{Int}(A))) \in \omega O(X)$ for every closed subset A of Y;*
6. *for each $x \in X$ and for each $V \in SO(Y)$ with $F(x) \subset V$, there exists $U \in \omega O(X, x)$ such that $F(U) \subset \mathrm{Cl}(V)$;*
7. *$F^+(V) \subset \omega\,\mathrm{Int}(F^+(\mathrm{Cl}(V)))$ for every $V \in SO(Y)$.*

Proof. (1) $\Leftrightarrow$ (2): Let $A \in RC(Y)$ and $x \in F^+(A)$. Since F is upper almost contra-ω-continuous, there exists $U \in \omega O(X, x)$ such that $U \subset F^+(A)$. Thus, $F^+(A) \in \omega O(X)$. The converse is obvious.

(2) $\Leftrightarrow$ (3) and (4) $\Leftrightarrow$ (5): It follows from the fact that $F^+(Y \backslash A) = X \backslash F^-(A)$ for every subset A of Y.

(3) $\Leftrightarrow$ (4): Let A be an open subset of Y. Since $\mathrm{Int}\,\mathrm{Cl}(A))$ is regular open, then $F^-(\mathrm{Int}(\mathrm{Cl}(A)))$ is ω-closed. The converse is obvious.

(5) $\Leftrightarrow$ (2): It is similar to that of (3) $\Leftrightarrow$ (4).

(6) $\Rightarrow$ (7): Let $V \in SO(Y)$ and $x \in F^+(V)$. Then $F(x) \subset V$. By (6), there exists $U \in \omega O(X, x)$ such that $F(U) \subset \mathrm{Cl}(V)$. This implies that $x \in U \subset F^+(\mathrm{Cl}(V))$. Hence, $x \in \omega\,\mathrm{Int}(F^+(\mathrm{Cl}(V)))$ and $F^+(V) \subset \omega\,\mathrm{Int}(F^+(\mathrm{Cl}(V)))$.

(7) $\Rightarrow$ (2): Let $A \in RC(Y)$. Since $A \in SO(Y)$, then $F^+(A) \subset \omega\,\mathrm{Int}(F^+(\mathrm{Cl}(A)))$; hence $F^+(A) \in \omega O(X)$.

(2) $\Rightarrow$ (6): Let $x \in X$ and $V \in SO(Y)$ with $F(x) \subset V$. Since $\mathrm{Cl}(V) \in RC(Y)$, there exists $A \in \omega O(X, x)$ such that $A \subset F^+(\mathrm{Cl}(V))$. Hence $F(A) \subset \mathrm{Cl}(V)$. ■

Theorem 3.9 *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

1. *F is lower almost contra-ω-continuous;*
2. *$F^-(A) \in \omega O(X)$ for every regular closed A of Y;*
3. *$F^+(U) \in \omega C(X)$ for every regular open subset U of Y;*
4. *$F^+(\mathrm{Int}(\mathrm{Cl}(A))) \in \omega C(X)$ for every open subset A of Y;*

5. $F^{-}(\mathrm{Cl}(\mathrm{Int}(A))) \in \omega O(X)$ *for every closed subset* A *of* Y*;*

6. *for each* $x \in X$ *and for each* $V \in SO(Y)$ *with* $F(x) \cap V \neq \emptyset$*, there exists* $U \in \omega O(X, x)$ *such that* $F(u) \cap \mathrm{Cl}(V) \neq \emptyset$ *for each* $u \in U$*;*

7. $F^{-}(V) \subset \omega\,\mathrm{Int}(F^{-}(\mathrm{Cl}(V)))$ *for every* $V \in SO(Y)$*.*

Proof. The proof is similar to that of Theorem 3.8. ∎

Definition 3.10 [8] Let U be a subset of a topological space (X, τ). The set $\cap\{V \in RO(X) : U \subset V\}$ is called the r-kernel of U and is denoted by $r\text{-}Ker(U)$.

Lemma 3.11 [8] *The following properties hold for subsets* U, V *of a space* X*:*

1. $x \in r\text{-}Ker(U)$ *if and only if* $U \cap V \neq \emptyset$ *for any regular closed set* V *containing* x*.*

2. $U \subset r\text{-}Ker(U)$ *and* $U = r\text{-}Ker(U)$ *if* U *is regular open in* X*.*

3. *If* $U \subset V$ *then* $r\text{-}Ker(U) \subset r\text{-}Ker(V)$*.*

Corollary 3.12 [2] *For a function* $f : (X, \tau) \to (Y, \sigma)$*, the following statements are equivalent:*

1. f *is almost contra-*ω*-continuous;*

2. $f^{-1}(F) \in \omega O(X)$ *for every* $F \in RC(Y)$*;*

3. *for each* $x \in X$ *and each* $F \in RC(Y, f(x))$*, there exists* $U \in \omega O(X, x)$ *such that* $f(U) \subset F$*;*

4. *for each* $x \in X$ *and each* $U \in RO(Y, f(x))$*, there exist* $V \in \omega C(X, x)$ *such that* $f(V) \subset U$*;*

5. $f^{-1}(\mathrm{Int}(\mathrm{Cl}(G))) \in \omega C(X)$ *for every open subset* G *of* Y*;*

6. $f^{-1}(\mathrm{Cl}(\mathrm{Int}(F))) \in \omega O(X)$ *for every closed subset* F *of* Y*;*

7. $f(\omega\,\mathrm{Cl}(A)) \subset r\,\mathrm{Ker}(f(A))$ *for every subset* A *of* X*;*

8. $\omega\,\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(r\,\mathrm{Ker}(B))$ *for every subset* B *of* Y*.*

Theorem 3.13 *For a multifunction* $F : (X, \tau) \to (Y, \sigma)$*, the following statements are equivalent:*

1. F *is lower almost contra-*ω*-continuous;*

2. $F^{-}(A) \in \omega O(X)$ *for every* θ*-semiopen* A *of* Y*;*

3. $F^{+}(U) \in \omega C(X)$ *for every* θ*-semiclosed subset* U *of* Y*;*

4. $\omega\,\mathrm{Cl}(F^{+}(\mathrm{Int}(\mathrm{Cl}(B)))) \subset F^{+}(s\,\mathrm{Cl}(B)$ *for every subset* B *of* Y*;*

5. $\omega\,\mathrm{Cl}(F^+(B)) \subset F^+(s\,\mathrm{Cl}_\theta(B))$ *for every subset* B *of* Y*;*

6. $F(\omega\,\mathrm{Cl}(A)) \subset s\,\mathrm{Cl}_\theta(F(A))$ *for every subset* A *of* X*.*

Proof. (1) $\Rightarrow$ (2): Let G be any θ-semiopen set of Y. There exists a family of regular closed sets $\{K_\alpha : \alpha \in \Delta\}$ such that $G = \cup\{K_\alpha : \alpha \in \Delta\}$. It follows from Theorem 3.9 (ii) that $F^-(G) = \cup\{F^-(K_\alpha) : \alpha \in \Delta\}$ is ω-open.

(2) $\Rightarrow$ (3): This is obvious.

(3) $\Rightarrow$ (4): Let B be any subset of Y. Then $\mathrm{Int}(\mathrm{Cl}(B))$ is regular open and it is θ-semiclosed in Y. Therefore, we have that $F^+(\mathrm{Int}(\mathrm{Cl}(B)))$ is ω-closed and $\omega\,\mathrm{Cl}(F^+(\mathrm{Int}(\mathrm{Cl}(B)))) = F^+(\mathrm{Int}(\mathrm{Cl}(B))) \subset F^+(s\,\mathrm{Cl}(B))$.

(4) $\Rightarrow$ (5): Let B be any subset of Y. For any regular open set V with $B \subset V$, we have $\omega\,\mathrm{Cl}(F^+(B)) \subset \mathrm{Cl}(F^+(V)) = \omega\,\mathrm{Cl}(F^+(\mathrm{Int}(\mathrm{Cl}(V)))) \subset F^+(s\,\mathrm{Cl}(V)) = F^+(V)$. Therefore, $\omega\,\mathrm{Cl}(F^+(B)) \subset F^+(\cap\{V \in RO(Y) : B \subset V\}) = F^+(s\,\mathrm{Cl}_\theta(B))$.

(5) $\Rightarrow$ (1): Let V be any semiopen set of Y. Then we have $X\backslash\omega\,\mathrm{Int}(F^-(\mathrm{Cl}(V))) = \omega\,\mathrm{Cl}(F^+(Y\backslash\,\mathrm{Cl}(V))) \subset F^+(s\,\mathrm{Cl}_\theta(Y\backslash\,\mathrm{Cl}(V))) = F^+(Y\backslash\,\mathrm{Cl}(V)) = X\backslash F^-(\mathrm{Cl}(V))$. Therefore, we obtain $F^-(V) \subset F^-(\mathrm{Cl}(V)) \subset \omega\,\mathrm{Int}(F^-(\mathrm{Cl}(V)))$. By Theorem 3.9 (7), F is lower almost contra-ω-continuous.

(5) $\Rightarrow$ (6): Let A be a subset of X and $B = F(A)$. Then $A \subset F^+(B)$ and $\omega\,\mathrm{Cl}(A) \subset \omega\,\mathrm{Cl}(F^+(B)) \subset F^+(s\,\mathrm{Cl}_\theta(B))$. Therefore, we have $F(\omega\,\mathrm{Cl}(A)) \subset F(F^+(s\,\mathrm{Cl}_\theta(B))) \subset s\,\mathrm{Cl}_\theta(B) = s\,\mathrm{Cl}_\theta(F(A))$.

(6) $\Rightarrow$ (5): Let B be any subset of Y. Then we have $F(\omega\,\mathrm{Cl}(F^+(B))) \subset s\,\mathrm{Cl}_\theta(F(F^+(B))) \subset s\,\mathrm{Cl}_\theta(B)$; hence $\omega\,\mathrm{Cl}(F^+(B)) \subset F^+(s\,\mathrm{Cl}_\theta(B))$. ∎

Corollary 3.14 *For a function* $f : (X, \tau) \to (Y, \sigma)$*, the following properties are equivalent:*

1. *f is almost contra-ω-continuous;*

2. $f^{-1}(V) \in \omega O(X)$ *for each θ-semiopen set V of Y;*

3. $f^{-1}(F) \in \omega C(X)$ *for each θ-semiclosed set F of Y.*

4. *for each $x \in X$ and each $U \in SO(Y, f(x))$, there exist $V \in \omega O(X, x)$ such that $f(V) \subset \mathrm{Cl}(U)$;*

5. $f^{-1}(V) \subset \omega\,\mathrm{Int}(f^{-1}(\mathrm{Cl}(V)))$ *for every $V \in SO(Y)$.*

6. $f(\omega\,\mathrm{Cl}(A)) \subset s\,\mathrm{Cl}_\theta(f(A))$ *for every subset A of X;*

7. $\omega\,\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(s\,\mathrm{Cl}_\theta(B))$ *for every subset B of Y.*

8. $\omega\,\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}(s\,\mathrm{Cl}_\theta(V))$ *for every open subset V of Y.*

9. $\omega\,\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}(s\,\mathrm{Cl}(V))$ *for every open subset V of Y.*

10. $\omega\,\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}(\mathrm{Int}(\mathrm{Cl}(V)))$ *for every open subset V of Y.*

Theorem 3.15 *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

1. *F is upper almost contra-ω-continuous;*
2. *$\omega\,\mathrm{Cl}(F^-(\mathrm{Int}(K))) \subset F^{-1}(K)$ for every semiclosed set K of Y;*
3. *$\omega\,\mathrm{Cl}(F^-(\mathrm{Int}(s\,\mathrm{Cl}(B)))) \subset F^-(s\,\mathrm{Cl}(B))$ for every $B \subseteq Y$;*
4. *$F^+(s\,\mathrm{Int}(B)) \subset \omega\,\mathrm{Int}(F^+(\mathrm{Cl}(s\,\mathrm{Int}(B))))$ for every $B \subseteq Y$.*

Proof. (1) $\Rightarrow$ (2): Let K be a semiclosed set of Y. Then $Y\backslash K$ is semiopen. By Theorem 3.8 (7), it follows that $F^+(Y\backslash K) \subset \omega\,\mathrm{Int}(F^+(Y\backslash \mathrm{Int}(K)))$. Hence $X\backslash F^-(K) \subset \omega\,\mathrm{Int}(F^+(Y\backslash \mathrm{Int}(K)))=\omega\,\mathrm{Int}(X\backslash F^-(\mathrm{Int}(K)))=X\backslash\omega\,\mathrm{Cl}(F^-(\mathrm{Int}(K)))$. Hence, $\omega\,\mathrm{Cl}(F^-(\mathrm{Int}(K))) \subset F^{-1}(K)$.

(2) $\Rightarrow$ (3): Let B be any subset of Y. Then $s\,\mathrm{Cl}(B)$ is semiclosed in Y and hence $\omega\,\mathrm{Cl}(F^-(\mathrm{Int}(s\,\mathrm{Cl}(B)))) \subset F^-(s\,\mathrm{Cl}(B))$.

(3) $\Rightarrow$ (4): Let B be any subset of Y. Then we have

$$\begin{aligned} X\backslash F^+(s\,\mathrm{Int}(B)) &= F^-(s\,\mathrm{Cl}(Y\backslash B)) \supset \omega\,\mathrm{Cl}(F^-(\mathrm{Int}(s\,\mathrm{Cl}(Y\backslash B)))) \\ &= \omega\,\mathrm{Cl}(F^-(\mathrm{Int}(Y\backslash s\,\mathrm{Int}(B)))) = \omega\,\mathrm{Cl}(F^-(Y\backslash \mathrm{Cl}(s\,\mathrm{Int}(B)))) \\ &= \omega\,\mathrm{Cl}(X\backslash F^+(\mathrm{Cl}(s\,\mathrm{Int}(B)))) = X\backslash\omega\,\mathrm{Int}(F^+(\mathrm{Cl}(s\,\mathrm{Int}(B)))). \end{aligned}$$

Hence, $F^+(s\,\mathrm{Int}(B)) \subset \omega\,\mathrm{Int}(F^+(\mathrm{Cl}(s\,\mathrm{Int}(B))))$.

(4) $\Rightarrow$ (1): Let V be any semiopen set of Y. Then $V = s\,\mathrm{Int}(V)$ and hence $F^+(V) \subset \omega\,\mathrm{Int}(F^+(\mathrm{Cl}(V)))$. By Theorem 3.8 (7), F is upper almost contra-ω-continuous. ∎

Theorem 3.16 *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

1. *F is lower almost contra-ω-continuous;*
2. *$\omega\,\mathrm{Cl}(F^+(\mathrm{Int}(K))) \subset F^+(K)$ for every semiclosed set K of Y;*
3. *$\omega\,\mathrm{Cl}(F^+(\mathrm{Int}(s\,\mathrm{Cl}(B)))) \subset F^+(s\,\mathrm{Cl}(B))$ for every $B \subseteq Y$;*
4. *$F^-(s\,\mathrm{Int}(B)) \subset \omega\,\mathrm{Int}(F^-(\mathrm{Cl}(s\,\mathrm{Int}(B))))$ for every $B \subseteq Y$.*

Proof. The proof is similar to that of Theorem 3.15. ∎

Recall that a topological space is said to be extremely disconnected if the closure of every open set is open in the space.

Theorem 3.17 *Let (Y, σ) be an extremely disconnected space. Then a multifunction $F : (X, \tau) \to (Y, \sigma)$ is upper almost contra-ω-continuous if and only if it is upper almost ω-continuous.*

Proof. Let $x \in X$ and V be any regular open set of Y containing $F(x)$. Since (Y, σ) is extremely disconnected, V is regular closed and hence semiopen. By Theorem 3.8, there exists $U \in \omega O(X, x)$ such that $F(U) \subset \mathrm{Cl}(V) = V$. Then F is upper almost ω-continuous. Conversely, let K be any regular closed subset of Y. Since (Y, σ) is extremely disconnected, K is also regular open and by Theorem 3.4 of [5], $F^+(K)$ is ω-open. By Theorem 3.8, F is upper almost contra-ω-continuous. ∎

Theorem 3.18 *Let (Y, σ) be an extremely disconnected space. Then a multifunction $F : (X, \tau) \to (Y, \sigma)$ is lower almost contra-ω-continuous if and only if F is lower almost ω-continuous.*

Proof. The proof is similar to that of Theorem 3.17. ∎

Theorem 3.19 *The following statements are equivalent for a multifunction $F : (X, \tau) \to (Y, \sigma)$:*

1. *F is upper (lower) almost contra-ω-continuous;*
2. *$F^+(\mathrm{Cl}(V))(F^-(\mathrm{Cl}(V)))$ is ω-open in X for every $V \in SPO(Y)$;*
3. *$F^+(\mathrm{Cl}(V))(F^-(\mathrm{Cl}(V)))$ is ω-open in X for every $V \in SO(Y)$;*
4. *$F^-(\mathrm{Int}(\mathrm{Cl}(V)))(F^+(\mathrm{Int}(\mathrm{Cl}(V))))$ is ω-closed in X for every $V \in PO(Y)$.*

Proof. (1) $\Rightarrow$ (2): Suppose that V is any semi-preopen set of Y. Since $\mathrm{Cl}(V) \in RC(Y)$, by Theorem 3.8, $F^-(\mathrm{Cl}(V))$ is ω-open in X.

(2) $\Rightarrow$ (3): This is obvious, since $SO(Y) \subset SPO(Y)$.

(3) $\Rightarrow$ (4): Let $V \in PO(Y)$. Then $Y \setminus \mathrm{Int}(\mathrm{Cl}(V))$ is regular closed and hence it is semiopen. Then, we have $X \setminus F^-(\mathrm{Int}(\mathrm{Cl}(V))) = F^+(Y \setminus \mathrm{Int}(\mathrm{Cl}(V))) = F^+(\mathrm{Cl}(Y \setminus \mathrm{Int}(\mathrm{Cl}(V)))) \in \omega O(X)$. Hence $F^-(\mathrm{Int}(\mathrm{Cl}(V))) \in \omega C(X)$.

(4) $\Rightarrow$ (1): If V is any regular open set of Y. Then $V \in PO(Y)$ and hence $F^-(V) = F^-(\mathrm{Int}(\mathrm{Cl}(V)))$ is ω-closed in X. Therefore, F is upper almost contra-ω-continuous.

The proof of the second case is similar. ∎

Lemma 3.20 [16] *For a subset V of a topological space (Y, σ), the following properties hold:*

1. *$\alpha\,\mathrm{Cl}(V) = \mathrm{Cl}(V)$ for every $V \in SPO(Y)$;*
2. *$p\,\mathrm{Cl}(V) = \mathrm{Cl}(V)$ for every $V \in SO(Y)$;*
3. *$s\,\mathrm{Cl}(V) = \mathrm{Int}(\mathrm{Cl}(V))$ for every $V \in PO(Y)$.*

Corollary 3.21 *The following statements are equivalent for a multifunction $F : (X, \tau) \to (Y, \sigma)$:*

1. *F is upper (lower) almost contra-ω-continuous;*
2. $F^+(\alpha \operatorname{Cl}(V))(F^-(\alpha \operatorname{Cl}(V)))$ *is ω-open in X for every $V \in SPO(Y)$;*
3. $F^+(p \operatorname{Cl}(V))(F^-(p \operatorname{Cl}(V)))$ *is ω-open in X for every $V \in SO(Y)$;*
4. $F^-(s \operatorname{Cl}(V))(F^+(s \operatorname{Cl}(V)))$ *is ω-closed in X for every $V \in PO(Y)$.*

Proof. This is an immediate consequence of Theorem 3.19 and Lemma 3.20. ∎

Theorem 3.22 *The following statements are equivalent for a multifunction $F : (X, \tau) \to (Y, \sigma)$:*

1. *F is upper almost contra-ω-continuous;*
2. $\omega \operatorname{Cl}(F^-(V)) \subset F^-(\operatorname{Int}(\operatorname{Cl}(V))$ *for every open subset V of Y;*
3. $\omega \operatorname{Cl}(F^-(V)) \subset F^-(s \operatorname{Cl}(V))$ *for every open subset V of Y.*

Proof. (2) ⇒ (1): Let $V \in RO(Y)$. Then $\omega \operatorname{Cl}(F^-(V)) \subset F^-(\operatorname{Int}(\operatorname{Cl}(V)) = F^-(V)$. This implies that $F^-(A)$ is ω-closed and hence F is upper almost contra-ω-continuous.

(1) ⇒ (2): Let V be an open set. We have $\operatorname{Int}(\operatorname{Cl}(V)) \in RO(Y)$. By (1), $F^-(\operatorname{Int}(\operatorname{Cl}(V))$ is ω-closed. Since $V \subset \operatorname{Int}(\operatorname{Cl}(A))$, then $F^-(A) \subset F^-(\operatorname{Int}(\operatorname{Cl}(A)))$. Thus, $\omega \operatorname{Cl}(F^-(V)) \subset F^-(s \operatorname{Cl}(V))$.

(2) ⇔ (3): It follows from Lemma 3.20. ∎

Theorem 3.23 *The following statements are equivalent for a multifunction $F : (X, \tau) \to (Y, \sigma)$:*

1. *F is lower almost contra-ω-continuous;*
2. $\omega \operatorname{Cl}(F^+(V)) \subset F^+(\operatorname{Int}(\operatorname{Cl}(V))$ *for every open subset V of Y;*
3. $\omega \operatorname{Cl}(F^+(V)) \subset F^+(s \operatorname{Cl}(V))$ *for every open subset V of Y.*

Proof. The proof is similar to that of Theorem 3.22. ∎

Theorem 3.24 *Let $F : (X, \tau) \to (Y, \sigma)$ be any multifunction. If $\omega \operatorname{Cl}(F^-(V)) \subset F^-(r\text{-}Ker(V))$ for every subset V of Y, then F is upper almost contra-ω-continuous.*

Proof. Let $V \in RO(Y)$. By Lemma 3.11, $\omega \operatorname{Cl}(F^-(V)) \subset F^-(r\text{-}Ker(V)) = F^-(V)$. This implies that $\omega \operatorname{Cl}(F^-(V)) = F^-(V)$; hence $F^-(V)$ is ω-closed. By Theorem 3.8, F is upper almost contra-ω-continuous. ∎

Theorem 3.25 *Let $F : (X, \tau) \to (Y, \sigma)$ be any multifunction. If $F(\omega \operatorname{Cl}(V)) \subset r\text{-}Ker(F(V))$ for every subset V of Y, then F is lower almost contra-ω-continuous.*

Proof. Let H be a regular open set of X. Then $F(\omega\,\mathrm{Cl}(F^+(H))) \subset r\text{-}Ker(H)$ and $\omega\,\mathrm{Cl}(F^+(H)) \subset F^+(r\text{-}Ker(H))$. By Lemma 3.11,

$$\omega\,\mathrm{Cl}(F^+(H)) \subset F^+(r\text{-}Ker(H)) = F^+(H).$$

We have $\omega\,\mathrm{Cl}(F^+(H)) = F^+(H)$. This implies that $F^+(H)$ is ω-closed in X. By Theorem 3.9, F is lower almost contra-ω-continuous. ∎

Definition 3.26 A topological space (X, τ) is said to ω-T_2 [2], if for each pair of distinct points x and y in X, there exist disjoint ω-open sets U and V in X such that $x \in U$ and $y \in V$.

Lemma 3.27 [19] *If A and B are disjoint compact subsets of an Urysohn space X, there exist open sets U and V of X such that $A \subset U$, $B \subset V$ and $\mathrm{Cl}(U) \cap \mathrm{Cl}(V) = \emptyset$.*

Theorem 3.28 *If $F : (X, \tau) \to (Y, \sigma)$ is an upper almost contra-ω-continuous injective multifunction into an Urysohn space Y and $F(x)$ is compact for each $x \in X$, then X is ω-T_2.*

Proof. For any distinct points $x_1, x_2 \in X$, we have $F(x_1) \cap F(x_2) = \emptyset$, since F is injective and $F(x_1)$ and $F(x_2)$ are disjoint compact sets, by Lemma 3.27, there exist open sets V_1 and V_2 such that $F(x_1) \subset V_1$, $F(x_2) \subset V_2$ and $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) = \emptyset$. Since $\mathrm{Cl}(V_1)$ and $\mathrm{Cl}(V_2)$ are regular closed sets and F is upper almost contra-ω-continuous, there exist $U_1 \in \omega O(X, x_1)$ and $U_2 \in \omega O(X, x_2)$ such that $F(U_1) \subset \mathrm{Cl}(V_1)$, $F(U_2) \subset \mathrm{Cl}(V_2)$; hence $U_1 \cap U_2 = \emptyset$ and X is ω-T_2. ∎

Definition 3.29 A subset A of a topological space (X, τ) is said to be ω-dense in X if $\omega\,\mathrm{Cl}(A) = X$.

Definition 3.30 A multifunction $F : (X, \tau) \to (Y, \sigma)$ is called upper weakly continuous [19], if for each open set V containing $F(x)$ and for each $x \in X$, there exists an open set U containing x such that $F(U) \subset \mathrm{Cl}(V)$.

Theorem 3.31 *Let X be a topological space and Y an Urysohn space. If the following four conditions are satisfied:*

1. *$F : (X, \tau) \to (Y, \sigma)$ is an upper weakly continuous multifunction,*
2. *$G : X \to Y$ is an upper almost contra-ω-continuous multifunction,*
3. *$F(x)$ and $G(x)$ are compact sets of Y for each $x \in X$,*
4. *$A = \{x \in X : F(x) \cap G(x) \neq \emptyset\}$,*

then A is ω-closed. Moreover if $F(x) \cap G(x) \neq \emptyset$ for each point $x \in X$ in a ω-dense set D, then $F(x) \cap G(x) \neq \emptyset$ for each $x \in X$.

Proof. Suppose that $x \in X \backslash A$. Then we have $F(x) \cap G(x) = \emptyset$. Since $F(x)$ and $G(x)$ are disjoint compact sets of an Urysohn space, by Lemma 3.27, there exist open sets V and W such that $F(x) \subset V$ and $G(x) \subset W$ and $\mathrm{Cl}(V) \cap \mathrm{Cl}(W) = \emptyset$. Since F is upper weakly continuous and $F(x) \subset V$, there exists an open set U_1 containing x such that $F(U_1) \subset \mathrm{Cl}(V)$. Since $\mathrm{Cl}(W)$ is regular closed, $G(x) \subset W$ and G is upper almost contra-ω-continuous, there exists $U_2 \in \omega O(X, x)$ such that $G(U_2) \subset \mathrm{Cl}(W)$. Let $U = U_1 \cap U_2$, then U is ω-open and $U \cap A = \emptyset$. Therefore, $x \in X \backslash \omega \,\mathrm{Cl}(A)$ and hence A is ω-closed. On the other hand, if $F(x) \cap G(x) \neq \emptyset$ on an ω-dense set D of X, then we have $X = \omega \,\mathrm{Cl}(D) \subset \omega \,\mathrm{Cl}(A) = A$. Therefore, we obtain $F(x) \cap G(x) \neq \emptyset$ for each $x \in X$. ∎

Definition 3.32 A subset A of a space (X, τ) is said to be:

1. α-regular [11], if for each $a \in A$ and any open set U containing a, there exists an open set V of X such that $a \in V \subset \mathrm{Cl}(V) \subset U$;
2. α-paracompact [11], if every X-open cover A has an X-open refinement which covers A and is locally finite for each point of X.

Lemma 3.33 [11] *If A is an α-regular and α-paracompact subset of a space X and U is an open neighborhood of A, then there exists an open set V of X such that $A \subset V \subset \mathrm{Cl}(V) \subset U$.*

For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the multifunction $\mathrm{Cl}(F) : X \to Y$ is defined by $\mathrm{Cl}(F)(x) = \mathrm{Cl}(F(x))$ for each point $x \in X$. Similarly, we denote $s\,\mathrm{Cl}(F)$, $p\,\mathrm{Cl}(F)$, $\alpha\,\mathrm{Cl}(F)$, $sp\,\mathrm{Cl}(F)$, $\omega\,\mathrm{Cl}(F)$.

Lemma 3.34 [18] *If $F : (X, \tau) \to (Y, \sigma)$ is a multifunction such that $F(x)$ is α-paracompact α-regular for each $x \in X$, then for each open set V of Y, $G^+(V) = F^+(V)$ and for each closed set K of Y, $G^-(K) = F^-(K)$, where G denotes $\mathrm{Cl}(F)$, $s\,\mathrm{Cl}(F)$, $p\,\mathrm{Cl}(F)$, $\alpha\,\mathrm{Cl}(F)$, $sp\,\mathrm{Cl}(F)$, $\omega\,\mathrm{Cl}(F)$.*

Lemma 3.35 [18] *If $F : (X, \tau) \to (Y, \sigma)$ is a multifunction, then for each open set V of Y, $G^-(V) = F^-(V)$ and for each closed set K of Y, $G^+(K) = F^+(K)$, where G denotes $\mathrm{Cl}(F)$, $s\,\mathrm{Cl}(F)$, $p\,\mathrm{Cl}(F)$, $\alpha\,\mathrm{Cl}(F)$, $sp\,\mathrm{Cl}(F)$, $\omega\,\mathrm{Cl}(F)$.*

Theorem 3.36 *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is upper almost contra-ω-continuous if and only if G is upper almost contra-ω-continuous.*

Proof. Let K be a regular closed set of Y. By Theorem 3.8 and Lemma 3.35, $G^+(K) = F^+(K)$ is a ω-open set of X. Hence, G is upper almost contra-ω-continuous. Conversely, Let K be a regular closed set of Y. By Theorem 3.8 and Lemma 3.35, $F^+(K) = G^+(K)$ is a ω-open set of X. Hence, G is upper almost contra-ω-continuous. ∎

Theorem 3.37 *Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is α-regular α-paracompact for each $x \in X$. Then F is lower almost contra-ω-continuous if and only if G is lower almost contra-ω-continuous.*

Proof. Let K be a regular closed set of Y. By Theorem 3.9 and Lemma 3.34, $G^{-}(K) = F^{-}(K)$ is an ω-open set of X. Hence, G is lower almost contra-ω-continuous. Conversely, Let K be a regular closed set of Y. By Theorem 3.9 and Lemma 3.34, $F^{-}(K) = G^{-}(K)$ is an ω-open set of X. Hence, G is lower almost contra-ω-continuous. ∎

Theorem 3.38 *Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction and U be an open subset of X. If F is a lower (upper) almost contra-ω-continuous, then $F|_U : U \to Y$ is lower (upper) almost contra-ω-continuous.*

Proof. Let V be any regular closed set of Y. Let $x \in U$ and $x \in (F|_U)^{-}(V)$. Since F is a lower almost contra-ω-continuous multifunction, then there exists $G \in \omega O(X, x)$ such that $G \subset F^{-}(V)$. Then by Lemma 2.3, $x \in G \cap U \in \omega O(U)$ and $G \cap U \subset (F|_U)^{-}(V)$. This shows that $F|_U$ is a lower almost contra-ω-continuous multifunction. The proof of the second case is similar. ∎

Theorem 3.39 *Let $\{U_i : i \in \Delta\}$ be an open cover of a space X. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is upper almost contra-ω-continuous if and only if the restriction $F|_{U_i} : U_i \to Y$ is upper almost contra-ω-continuous for each $i \in \Delta$.*

Proof. Suppose that F is an upper almost contra-ω-continuous multifunction. Let $i \in \Delta$, $x \in U_i$ and V be a regular closed set of Y containing $F|_{U_i}(x)$. Since F is an upper almost contra-ω-continuous multifunction and $F(x) = F|_{U_i}(x)$, there exists $G \in \omega O(X, x)$ such that $F(G) \subset V$. Set $U = G \cap U_i$, then $x \in U \in \omega O(U_i, x)$ and $F|_{U_i}(U) = F(U) \subset V$. Therefore, $F|_{U_i}$ is upper almost contra-ω-continuous. Conversely, let $x \in X$ and $V \in RC(Y)$ containing $F(x)$. There exists $i \in \Delta$ such that $x \in U_i$. Since $F|_{U_i}$ is upper almost contra-ω-continuous and $F(x) = F|_{U_i}(x)$, there exists $U \in \omega O(U_i, x)$ such that $F|_{U_i}(U) \subset V$. Then we have $U \in \omega O(X, x)$ and $F(U) \subset V$. Therefore, F is upper almost contra-ω-continuous. ∎

Theorem 3.40 *Let $\{U_i : i \in \Delta\}$ be an open cover of a space X. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is lower almost contra-ω-continuous if and only if the restriction $F|_{U_i} : U_i \to Y$ is lower almost contra-ω-continuous for each $i \in \Delta$.*

Proof. The proof is similar to that of Theorem 3.39 and is thus omitted. ∎

For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the graph multifunction $G_F : X \to Y$ is defined as follows: $G_F(x) = \{x\} \times F(x)$ for every $x \in X$.

Theorem 3.41 *If $G_F : X \to X \times Y$ is an upper almost contra-ω-continuous multifunction, then $F : (X, \tau) \to (Y, \sigma)$ is an upper almost contra-ω-continuous multifunction.*

Proof. Let $x \in X$ and $K \in RC(Y)$ with $F(x) \subset K$. Since $X \times K$ is regular closed in $X \times Y$ and $G_F(x) \subset X \times K$, then there exists $U \in \omega O(X, x)$ such that $G_F(U) \subset X \times K$. By Lemma 2.4, $U \subset G_F^{+}(X \times K) = F^{+}(K)$ and $F(U) \subset K$. Thus, F is an upper almost contra-ω-continuous multifunction. ∎

Theorem 3.42 *If $G_F : X \to X \times Y$ is a lower almost contra-ω-continuous multifunction, then $F : (X, \tau) \to (Y, \sigma)$ is a lower almost contra-ω-continuous multifunction.*

Proof. Let $x \in X$ and $K \in RC(Y)$ with $x \in F^-(K)$. Since $X \times K$ is regular closed in $X \times Y$ and $G_F(x) \cap (X \times K) = (\{x\} \times F(x)) \cap (X \times K) = \{x\} \times (F(x) \cap K) \neq \emptyset$. Since G_F is lower almost contra-ω-continuous, then there exists $U \in \omega O(X, x)$ such that $U \subset G_F^-(X \times K)$. Then $U \subset F^-(K)$. Hence, F is a lower almost contra-ω-continuous. ■

Theorem 3.43 *Let $F : (X, \tau) \to (Y, \sigma)$ be an upper almost contra-ω-continuous surjective multifunction and $F(x)$ is a S-closed relative to Y for each $x \in X$. If A is a ω-compact relative to X, then $F(A)$ is a S-closed relative to Y.*

Proof. Let $\{V_i : i \in \Delta\}$ be any cover of $F(A)$ by regular closed sets of Y. For each $x \in A$, there exists a finite subset $\Delta(x)$ of Δ such that

$$F(x) \subset \cup\{V_i : i \in \Delta(x)\}.$$

Put $V(x) = \cup\{V_i : i \in \Delta(x)\}$. Then $F(x) \subset V(x)$ and there exists $U(x) \in \omega O(X, x)$ such that $F(U(x)) \subset V(x)$. Since $\{U(x) : x \in A\}$ is a cover of A by ω-open sets in X, there exists a finite number of points of A, say, x_1, x_2,....x_n such that $A \subset \cup\{U(x_i) : 1 = 1, 2, ...n\}$. Therefore, we obtain

$$F(A) \subset F\left(\bigcup_{i=1}^{n} U(x_i)\right) \bigcup_{i=1}^{n} F(U(x_i)) \bigcup_{i=1}^{n} V(x_i) \subset \bigcup_{i=1}^{n} \bigcup_{i \in \Delta(x_i)} V_i.$$

This shows that $F(A)$ is a S-closed relative to Y. ■

Theorem 3.44 *Let X and X_i be topological spaces for $i \in I$. If $F : X \to \prod_{i \in I} X_i$ is an upper (lower) almost contra-ω-continuous multifunction, then $P_i \circ F$ is an upper (lower) almost contra-ω-continuous multifunction for each $i \in I$, where $P_i : \prod_{i \in I} X_i \to X_i$ is the projection for each $i \in I$.*

Proof. Let H_i be a regular closed subset of X_j. We have

$$(P_j \circ F)^+(H_j) = F^+(P_j^+(H_j)) = F^+\left(H_j \times \prod_{i \neq j} X_i\right).$$

Since F is an upper almost contra-ω-continuous multifunction, $F^+\left(H_j \times \prod_{i \neq j} X_i\right)$ is ω-open in X. Hence, $P_i \circ F$ is an upper (lower) almost contra-ω-continuous. ■

Theorem 3.45 *Let X_i and Y_i be topological spaces and $F_i : X_i \to Y_i$ be a multifunction for each $i \in I$. If $F : \prod_{i\in I} X_i \to \prod_{i\in I} Y_i$, defined by $F(x_i) = \prod_{i\in I} F_i(x_i)$, is upper (lower) almost contra-ω-continuous multifunction, then F_i is upper (lower) almost contra-ω-continuous multifunction for each $i \in I$.*

Proof. Let $H_i \subset Y_i$ be a regular closed subset. Since F is upper almost contra-ω-continuous multifunction, $F^+(H_i \times \prod_{i\neq j} Y_j) = F_i^+(H_i) \times \prod_{i\neq j} X_j$ is an ω-open set. Thus, $F_i^+(H_i)$ is an ω-open set; hence F is upper almost contra-ω-continuous multifunction. ∎

References

[1] Al-Zoubi, K., Al-Nashef, B., *The topology of ω-open subsets*, Al-Manarah, (9) (2003), 169-179.

[2] Al-Omari, A., Noorani, M.S.M., *Contra-ω-continuous and almost ω-continuous functions*, Int. J. Math. Math. Sci. (9) (2007), 169-179.

[3] Andrijevic, D., *Semi-preopen sets*, Mat. Vesnik, 38 (1986), 24-32.

[4] Banzaru, T., *Multifunctions and M-product spaces*, Bull. Stin. Tech. Inst. Politech. Timisoara, Ser. Mat. Fiz. Mer. Teor. Apl., 17 (31) (1972), 17-23.

[5] Carpintero, C., Rajesh, N., Rosas, E., Saranyasri, S., *Some properties of Upper/lower almost ω-continuous functions*, Sci. Stud. Res. Ser. Math. Inform., por aparecer en 2014.

[6] Carpintero, C., Rajesh, N., Rosas, E., Saranyasri, S., *On upper and lower contra-ω-continuous multifunctions*, to appear in Novi Sad J. Math., (2014).

[7] Carpintero, C., Rajesh, N., Rosas, E., Saranyasri, S., *Upper and lower weakly ω-continuous functions* (submitted).

[8] Ekici, E., *On contra R-continuity and a weak form,* Indian J. Math., 46 (2-3) (2004), 267-281.

[9] Ekici, E., Jafari, S., Popa V., *On almost contra-continuous multifunctions,* Lobachesvskii J. Math., 30 (2) (2009), 124-131.

[10] Hdeib, H.Z., *ω-closed mappings*, Revista Colombiana Mat., 16 (1982), 65-78.

[11] Kovacevic, Subsets and paracompactness, Univ. u. Novom Sadu, Zb. Rad. Prirod. Mat. Fac. Ser. Mat., 14 (1984), 79-87.

[12] Joseph, J.E., Kwack, M.H., *On S-closed spaces*, Proc. Amer. Math. Soc., 80 (1980), 341-348.

[13] Levine, N., *Semiopen sets and semicontinuity in topological spaces*, Amer. Math. Monthly, 70 (1963), 36-41.

[14] Mashhour, A.S., Abd El-Monsef, M.E., El-Deep, S.N., *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt, 53 (1982), 47–53.

[15] Njastad, O., *On some classes of nearly open sets,* Pacific J. Math., 15 (1965), 961-970.

[16] Noiri, T., *On almost continuous functions*, Indian J. Pure Appl. Math., 20 (1989), 571-576.

[17] Noiri, T., *Almost weakly continuous multifunctions*, Demonstratio Math., 26 (1993), 363-380.

[18] Noiri, T., Popa, V., *A unified theory of weak continuity for multifunctions*, Stud. Cerc. St. Ser. Mat. Univ. Bacau, 16 (2006), 167-200.

[19] Simithson, R.E., *Almost and weak continuity for multifunctions,* Bull. Calcutta Math. Soc., 70 (1978), 383-390.

[20] Stone, M., *Applications of the theory of boolean rings to general topology,* Trans. Amer. Math. Soc., 41 (1937), 374-381.

[21] Thompson, T., *S-closed spaces*, Proc. Amer. Math. Soc., 60 (1976), 335-338.

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