ON BINOMIAL OPERATOR REPRESENTATIONS OF SOME POLYNOMIALS

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Abstract. Based on the technique used by M.A. Khan and A.K. Shukla [2] here finite series representations of bionomial partial differential operators have been used to establish operator representations of various polynomials not considered in the earlier mentioned paper. The results obtained are believed to be new.

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1. Introduction

Recently, in 2009, M.A. Khan and A.K. Shukla [2] evolved a new technique to give operator representations of certain polynomials. They give binomial and trinomial operator representation of certain polynomials. The aim of the present paper is to strengthen the technique evolved by obtaining binomial operator representations of some more polynomials not considered in the above mentioned paper.

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2. The definitions, notations, and results used

In deriving the operational representations of various polynomials use has been made of the fact that

$$D^{\mu}x^{\lambda} = \frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda-\mu)}x^{\lambda-\mu}, \qquad D \equiv \frac{d}{dx} \tag{2.1}$$

where λ and μ, $\lambda \geq \mu$ are arbitrary real numbers.

In particular, use has been made of the following results:

$$D^r\ e^{-x} = \ (-1)^r\ e^{-x} \tag{2.2}$$

$$D^r\ x^{-\alpha} = (\alpha)_r\ (-1)^r\ x^{-\alpha-r},\ \alpha \text{ is not an integer} \tag{2.3}$$

$$D^r\ x^{-\alpha-n} = (\alpha+n)_r\ (-1)^r\ x^{-\alpha-n-r} \tag{2.4}$$

$$D^{n-r}\ x^{\alpha-1+n} = \frac{(\alpha)_n}{(\alpha)_r}\ x^{\alpha-1+r} \tag{2.5}$$

$$D^{n-r}\ x^{-\alpha} = \frac{(\alpha)_n(-1)^n}{(1-\alpha-n)_r}\ \ x^{-\alpha-n+r},\ \alpha \text{ is not an integer,} \tag{2.6}$$

where n and r are denoting the positive integers and

$$(a)_n = a\ (a+1)\cdots\cdots(a+n-1); \qquad (a)_0 = 1.$$

We also need the definitions of the following polynomials in terms of hypergeometric function and also their notations (see [1], [3], [4], [5]).

Cesaro polynomials

It is denoted by the symbol $g_n^{(s)}(x)$ and is defined as

$$g_n^{(s)}(x) = \left(\begin{array}{c} s+n \\ n \end{array}\right)\ {}_2F_1\left[\begin{array}{c} -n,\ \ 1\ ; \\ -s-n\ ; \end{array}\ x\right] \tag{2.7}$$

Miexner polynomials

It is denoted by the symbol $M_n(x;\beta,c)$ and is defined as

$$\begin{aligned} M_n(x;\beta,c) \ &= \ {}_2F_1\left[\begin{array}{cc} -n, & -x\ ; \\ & \beta\ ; \end{array}\ 1-c^{-1}\right], \\ &\beta{>}0,\ o<c<1,\ x=0,1,2,...,N \end{aligned} \tag{2.8}$$

Krawchouk polynomials

It is denoted by the symbol $K_n(x;P,N)$ and is defined as

$$K_n(x;P,N) \;=\; {}_2F_1\left[\begin{matrix}-n, & -x\;; \\ & -N; \end{matrix}\quad P^{-1}\right], \tag{2.9}$$

$$o < P < 1,\;\; x = 0,1,2,...,N$$

Hahn polynomials

It is denoted by the symbol $Q_n(x;\alpha,\beta,N)$ and is defined as

$$Q_n(x;\alpha,\beta,N) \;=\; {}_3F_2\left[\begin{matrix}-n, & -x\;,\alpha+\beta+n+1; \\ & -N,\alpha+1; \end{matrix}\quad 1\right] \tag{2.10}$$

$$\alpha,\beta > -1,\;\; n,x = 0,1,2,...,N$$

Sylvester polynomials

It is denoted by the symbol $\varphi_n(x)$ and is defined as

$$\varphi_n(x) \;=\; \frac{x^n}{n!}\;{}_2F_0\left[\begin{matrix}-n, & x; \\ & -; \end{matrix}\quad x^{-1}\right] \tag{2.11}$$

Gottlieb polynomials

It is denoted by the symbol $l_n(x;\lambda)$ and is defined as

$$l_n(x;\lambda) \;=\; e^{-n\lambda}\;\;{}_2F_1\left[\begin{matrix}-n, & -x\;; \\ & 1; \end{matrix}\quad 1-e^{\lambda}\right] \tag{2.12}$$

Charlier polynomials

It is denoted by the symbol $C_n^a(x)$ and is defined as

$$C_n^a(x) = (-a)^n\;{}_2F_0\left[\begin{matrix}-n,-x; \\ -; \end{matrix}\;\frac{1}{a}\right]$$

Mittag-Leffler polynomials

It is denoted by the symbol $g_n(z,\gamma)$ and is defined as

$$g_n(z,\gamma) = \frac{(-\gamma)_n}{n!}\quad{}_2F_1\left[\begin{matrix}-n, & z\;; \\ & -\gamma; \end{matrix}\quad 2\right] \tag{2.14}$$

Shively's pseudo Laguerre polynomials

It is denoted by the symbol $R_n(a,x)$ and is defined as

$$R_n(a,x) = \frac{(a)_{2n}}{n!(a)_n}\,{}_1F_1\left[\begin{array}{ll} -n\,; & x \\ a+n\,; & \end{array}\right] \tag{2.15}$$

3. Operational representations

If $D_x \equiv \dfrac{\partial}{\partial x}$ *and* $D_y \equiv \dfrac{\partial}{\partial\, y}$, M.A. Khan and A.K. Shukla [2] wrote the binomial expansion for $(D_x + D_y)^n$ as

$$(D_x + D_y)^n \equiv \sum_{r=0}^{n} {}^nC_r D_x^{n-r} D_y^r \tag{3.1}$$

where ${}^nC_r = \dfrac{n!}{r!(n-r)!}$.

By writing the finite series on the right of (3.1) M.A. Khan and A.K. Shukla [2] wrote (3.1) also as

$$(D_x + D_y)^n \equiv \sum_{r=0}^{n} {}^nC_r D_x^r D_y^{n-r} \tag{3.2}$$

If F(x,y) is a function of x and y, they obtained the following from (3.1) and (3.2)

$$(D_x + D_y)^n F(x,y) \equiv \sum_{r=0}^{n} \frac{(-n)_r(-1)^r}{r!} D_x^{n-r} D_y^r\ F(x,y) \tag{3.3}$$

$$(D_x + D_y)^n F(x,y) \equiv \sum_{r=0}^{n} \frac{(-n)_r(-1)^r}{r!} D_x^r D_y^{n-r}\ F(x,y) \tag{3.4}$$

In particular, if $F(x,y) = f(x)g(y)$ then (3.3) and (3.4) in the form

$$(D_x + D_y)^n f(x)g(y) \equiv \sum_{r=0}^{n} \frac{(-n)_r(-1)^r}{r!} D_x^{n-r}\ f(x) D_y^r\ g(y) \tag{3.5}$$

$$(D_x + D_y)^n f(x)g(y) \equiv \sum_{r=0}^{n} \frac{(-n)_r(-1)^r}{r!} D_x^r f(x) D_y^{n-r} g(y) \tag{3.6}$$

Now, by taking special values of $f(x)$ and $g(y)$ in (3.5), we obtain the following partial differential operator representations of the polynomials given above:

(3.7) $$x^{\nu+1}y(D_x+D_y)^n\left\{x^{\nu-1}y^{-1}\right\} = (-1)^n\ n!\ g_n^{(\nu)}\left(\frac{x}{y}\right)$$

(3.8) $$(D_w+D_yD_z)^n\left\{w^{\beta-1+n}y^x e^{(1-c^{-1})z}\right\} = (\beta)_n\ w^{\beta-1}y^x e^{(1-c^{-1})z}M_n(x;\beta,c(w/y))$$

(3.9) $$(D_w+D_yD_z)^n\left\{w^{-N-1+n}y^x e^{p^{-1}z}\right\} = (-N)_n\ w^{-N-1}y^x e^{-p^{-1}z}K_n(x;P,N),\ \text{if } w/y=1$$

(3.10) $$(D_vD_w-D_yD_z)^n\left\{v^{-N-1+n}w^{\alpha+n}y^x z^{-1-\alpha-\beta-n}\right\} = (-N)_n\ (1+\alpha)_n\ v^{-N-1}w^{\alpha}y^x z^{-1-\alpha-\beta-n}Q_n\left(x;\alpha,\beta,N\frac{vw}{yz}\right)$$

(3.11) $$xye^{z/x}(1+D_yD_z)^n\left\{y^{-x}e^{\frac{-1}{x}z}\right\} = n!\varphi_n(y/x)$$

(3.12) $$(D_w+D_yD_z)^n\left\{w^n y^x e^{(1-e^{\lambda})z}\right\} = n!y^x e^{(1-\lambda)z}l_n(x;\lambda),\ \text{when } w/y=1$$

(3.13) $$(1+D_yD_z)^n\left\{y^x e^{\frac{1}{a}z}\right\} = \frac{e^{\frac{1}{a}z}}{(-a)^n}C_n^{(a)}(x)\ \text{for } y=1$$

(3.14) $$(1-D_y)^n\left\{y^{a+2n-1}\right\} = n!y^{a+2n-1}R_n(a,y)$$

To validate the above mentioned equations, here we illustrate some proofs:

Proof of (3.7).

$$\begin{aligned}
&(D_x+D_y)^n\left\{x^{-s-1}y^{-1}\right\}\\
&= \sum_{r=0}^{n}\frac{(-n)_r(-1)^r}{r!}\frac{(1+s)_n(-1)^n}{(-s-n)_r}x^{-s-1-n+r}\ (1)_r(-1)^r y^{-1-r}\\
&= x^{\nu+1}y(-1)^n\ n!\ g_n^{(\nu)}\left(\frac{x}{y}\right)
\end{aligned}$$

Proof of (3.8).

$$\begin{aligned}
&(D_w+D_yD_z)^n\left\{w^{\beta-1+n}y^x e^{(1-c^{-1})z}\right\}\\
&= \sum_{r=0}^{n}\frac{(-n)_r(-1)^r}{r!}D_w^{n-r}w^{\beta-1+n}D_y^r y^x D_z^r e^{(1-c^{-1})z}\\
&= (\beta)_n w^{\beta-1}y^x e^{(1-c)^{-1})z}\sum_{r=0}^{n}\frac{(-n)_r(-x)_r(1-c^{-1})^r w^r}{r!\ (\beta)_r y^r}\\
&= (\beta)_n\ w^{\beta-1}y^x e^{(1-c^{-1})z}\ {}_2F_1\left[\begin{matrix}-n, & -x\ ; \\ & \beta\ ;\end{matrix}\ (1-c)^{-1}(w/y)\right].\\
&= (\beta)_n\ w^{\beta-1}y^x e^{(1-c)^{-1})z}M_n(x;\beta,c(w/y))
\end{aligned}$$

Proof of (3.13).

$$
\begin{aligned}
(D_w + D_y D_z)^n \left\{w^{-\gamma-1+n} y^{-z} e^{2z}\right\} &= \sum_{r=0}^{n} \frac{(-n)_r (-1)^r}{r!} D_w^{n-r}\ w^{-\gamma-1+n} D_y^r y^{-z} D_z^r e^{2z} \\
&= (-\gamma)_n w^{-\gamma-1} y^{-z} e^{2z}\ {}_2F_1 \left[\begin{array}{c} -n, \quad z\ ; \\ -\gamma\ ; \end{array} 2(w/y) \right]
\end{aligned}
$$

Proof of (3.14).

$$
\begin{aligned}
(1 - D_y)^n \left\{y^{a-1+2n}\right\} &= \sum_{r=0}^{n} \frac{(-n)_r (-1)^{n-r}}{r!} \frac{(a+n)_n}{(a+n)_r}\, y^{a+n-1+r} \qquad \text{by (3.6)} \\
&= n!\ \tfrac{(a)_{2n}}{(a)_n}\ x^{a-1+n}\ {}_1F_1 \left[\begin{array}{c} -n\ \ ; \\ a+n; \end{array} y \right]. \\
&= n! y^{a-1+n} R_n(a, y)
\end{aligned}
$$

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