GENERALIZED FUZZY ALGEBRAIC HYPERSYSTEMS

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Abstract. The concept of quasi-coincidence of a fuzzy interval value with an interval valued fuzzy set, which is a generalization of quasi-coincidence of a fuzzy point with a fuzzy set, is introduced. Using this new idea, the notion of interval valued (α, β)-fuzzy subalgebraic hypersystems in an algebraic hypersystem, which is a generalization of a fuzzy subalgebraic system, is defined, and related properties are investigated. We also discuss entropy of interval valued (α, β)-fuzzy subalgebraic hypersystems. In particular, the study of interval valued $(\in, \in \vee q)$-fuzzy subalgebraic hypersystems of an algebraic hypersystem is dealt with. Finally, we consider the concept of implication-based interval valued fuzzy subalgebraic hypersystems.

Keywords: Algebraic hypersystem;interval valued (α, β)-fuzzy subalgebraic hypersystem; interval valued $(\in, \in \vee q)$-fuzzy subalgebraic hypersystem; entropy; fuzzy logic; implication operator.

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1. Introduction

The study of algebraic hyperstructures (or hypersystems) is a well established branch of classical algebraic theory. Hyperstructure theory was born in 1934 when Marty [19] defined hypergroups, began to analyse their properties and applied them to groups, rational functions and algebraic functions. Later on, people

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have observed that hyperstructures have many applications to several branches of both pure and applied sciences, for example, semihypergroups are the simplest algebraic hyperstructures which possess the properties of closure and associativity. A comprehensive review of the theory of hyperstructures appears in [5], [6], [23].

After the introduction of fuzzy sets by Zadeh [26], reconsideration of the concept of classical mathematics began. On the other hand, because of the importance of group theory in mathematics, as well as its many areas of application, the notion of fuzzy subgroups was defined by Rosenfeld [22] and its structure was investigated. Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces and so on. This provides sufficient motivations to researchers to review various concepts and results from the realm of abstract algebra in the broader framework of fuzzy setting, see [20]. In 1975, Zadeh [27] introduced the concept of interval valued fuzzy subset, where the values of the membership functions are intervals of numbers instead of the numbers. In [4], Biswas defined interval valued fuzzy subgroups of the same nature of Rosenfeld's fuzzy subgroups. A new type of fuzzy subgroup (viz, $(\in, \in \vee q)$-fuzzy subgroup) was introduced in an earlier paper of Bhakat and Das [3] by using the combined notions of "belongingness" and "quasicoincidence" of fuzzy points and fuzzy sets, which was introduced by Pu and Liu [21]. In fact, $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroups. Recently, Davvaz ([13]) introduced the concept of $(\in, \in \vee q)$-fuzzy subnear-rings(ideals) and investigated some interesting results.

Fuzzy sets and hyperstructures introduced by Zadeh and Marty, respectively, are now used in the world both on the theoretical point of view and for their applications. The relations between fuzzy sets and algebraic hyperstructures (structures) have been already considered by Corsini, Davvaz, Leoreanu, Ameri, Vougiouklis, Zahedi, Zhan and others, for instance, see [1], [5]-[18], [23]-[25], [28]-[33]. In Section 2, we recall some basic defnitions and results about algebraic hypersystems. In Section 3, we discuss entropy of interval valued (α, β)-fuzzy subalgebraic hypersystems in an algebraic hypersystem. Since the concept of interval valued $(\in, \in \vee q)$-fuzzy subalgebraic hypersystems is an important and useful generalization of ordinary fuzzy algebraic hypersystem in an algebraic hypersystem, some fundamental aspects of interval valued $(\in, \in \vee q)$-fuzzy subalgebraic hypersystems in an algebraic hypersystem have been discussed in section 4. Finally, in section 5, we consider the concept of implication-based interval valued fuzzy subalgebraic hypersystems.

2. Preliminaries

General algebraic hypersystems and some of their related concepts are introduced in this section. Examples of some familiar algebraic hypersystems are given.

Let H be a non-empty set and f a mapping $f : H \times H \to \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ is the set of all the non-empty subsets of H. Then f is called a binary hyperoperation on H. In general, a mapping $f : H^n \to \mathcal{P}^*(H)$ is called an n-ary hyperoperation and n is called the order of hyperoperation.

A non-empty set and one or more n-ary hyperoperations on the set will be called an algebraic hypersystem. We shall denote an algebraic hypersystem by $< H, \Gamma >$, where H is a non-empty set and $\Gamma = \{f_1, f_2, ...\}$ is a set of hyperoperations on H.

The algebraic hypersystem $(H, \{f_i\}_{i\geq 1})$ induces a universal algebra

$$(\mathcal{P}^*(H), \{F_i\}_{i\geq 1})$$

with the operations

$$F_i(A_1, ..., A_n) = \cup\{f_i(a_1, ..., a_n) \mid a_k \in A_k,\ 1 \leq i \leq n\}$$

for $A_1, ..., A_n \in \mathcal{P}^*(H)$.

If Γ is a singleton $\Gamma = \{f\}$ and f is a 2-ary hyperoperation, the algebraic hypersystem is called hypergroupoid, the hyperoperation is denoted by $\circ$ and the image of the pair (x, y) is denoted by $x \circ y$. Hypergroups, polygroups and hyperrings are algebraic hypersystems.

Let (H, Γ) be an algebraic hypersystem. A subset S of H is said to be closed under the n-ary hyperoperation f if $(x_1, ..., x_n) \in S \times S \times ... \times S($ where S occurs n times) implies $f(x_1, ..., x_n) \in \mathcal{P}^*(S)$. S is called a subalgebraic hypersystem of H, if S is closed under any hyperoperation in Γ.

Now, we give the following concept:

Definition 2.1. Let (H, Γ) be an algebraic hypersystem and F a fuzzy subset of H. Then F is called a fuzzy subalgebraic hypersystem of H, if for any n-ary hyperoperation $f \in \Gamma$ and for all $x_i \in H(i = 1, 2, ..., n)$,

(HF1) $\inf\limits_{y\in f(x_1,x_2,...,x_n)} F(y) \geq \min\{F(x_1), F(x_2), ..., F(x_n)\}.$

Example 2.2.

(i) Consider $H = \{e, a, b\}$ and define $\circ$ on H with the following table:

$\circ$	e	a	b
e	$\{e\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{a, b\}$	$\{e, a\}$
b	$\{b\}$	$\{e, a\}$	$\{e\}$

Define a fuzzy set $F : H \to [0, 1]$ by $F(a) \leq F(b) \leq F(e)$. Then F is a fuzzy subalgebraic hypersystem of H.

(ii) Let $(H = \{1, -1, i, -i\}, \circ)$, where $\circ$ defined on H with the following table:

$\circ$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

Note that every algebraic system is an algebraic hypersystem. Define $F : H \longrightarrow [0,1]$ by F(1)=1, $F(-1) = 0.5$ and $F(i) = F(-i) = 0$. Clearly F is a fuzzy subalgebraic system of H.

(iii) Let $(H = \{x, y, z, t\}, \{f_1, f_2\})$ be an algebraic hypersystem, where

f_1	x	y	z	t
x	$\{x\}$	$\{y\}$	$\{z\}$	$\{t\}$
y	$\{y\}$	$\{x, y\}$	$\{z\}$	$\{t\}$
z	$\{z\}$	$\{z\}$	$\{x, y, t\}$	$\{z, t\}$
t	$\{t\}$	$\{t\}$	$\{z, t\}$	$\{x, y, z\}$

f_2	x	y	z	t
x	x	y	z	t
y	x	y	z	t
z	x	y	z	t
t	x	y	z	t

We define a fuzzy set $F : H \longrightarrow [0,1]$ by $F(x) = 0.7$, $F(y) = 0.5$ and $F(z) = F(t) = 0.3$. Routine calculations give that F is a fuzzy subalgebraic hypersystem of H.

Let F be a fuzzy set. For every $t \in [0,1]$, the set $U(F;t) = \{x \in H | F(x) \geq t\}$ is called the level subset of F.

Theorem 2.3. *Let H be an algebraic hypersystem and F a fuzzy set of H. Then F is a fuzzy subalgebraic hypersystem of H if and only if for any $t \in [0,1]$, $U(F;t)(\neq \emptyset)$ is a subalgebraic hypersystem of H.*

By an interval number $\tilde{a}$ we mean (cf. [27]) an interval $[a^-, a^+]$, where $0 \leq a^- \leq a^+ \leq 1$. The set of all interval numbers is denoted by $D[0,1]$. The interval $[a, a]$ is identified with the number $a \in [0,1]$.

For interval numbers $\widetilde{a}_i = [a_i^-, a_i^+], \widetilde{b}_i = [b_i^-, b_i^+] \in D[0,1]$, $i \in I$, we define

$$\text{rmax}\{\widetilde{a}_i, \widetilde{b}_i\} = [\max(a_i^-, b_i^-), \max(a_i^+, b_i^+)],$$

$$\text{rmin}\{\widetilde{a}_i, \widetilde{b}_i\} = [\min(a_i^-, b_i^-), \min(a_i^+, b_i^+)],$$

$$\text{rinf}\widetilde{a}_i = \left[\bigwedge_{i\in I} a_i^-, \bigwedge_{i\in I} a_i^+\right], \qquad \text{rsup}\widetilde{a}_i = \left[\bigvee_{i\in I} a_i^-, \bigvee_{i\in I} a_i^+\right]$$

and put

(1) $\widetilde{a}_1 \leq \widetilde{a}_2 \Longleftrightarrow a_1^- \leq a_2^-$ and $a_1^+ \leq a_2^+$,

(2) $\widetilde{a}_1 = \widetilde{a}_2 \Longleftrightarrow a_1^- = a_2^-$ and $a_1^+ = a_2^+$,

(3) $\widetilde{a}_1 < \widetilde{a}_2 \Longleftrightarrow \widetilde{a}_1 \leq \widetilde{a}_2$ and $\widetilde{a}_1 \neq \widetilde{a}_2$,

(4) $k\widetilde{a} = [ka^-, ka^+]$, whenever $0 \leq k \leq 1$.

It is clear that $(D[0,1], \leq, \vee, \wedge)$ is a complete lattice with $0 = [0,0]$ as the least element and $1 = [1,1]$ as the greatest element.

Interval valued fuzzy sets provide a more adequate description of uncertainty than traditional fuzzy sets; it is therefore important to use interval valued fuzzy sets in applications. One of main applications of fuzzy sets is fuzzy control, and one of the most computationally intensive part of fuzzy control is defuzzification. Since a transition to interval valued fuzzy sets usually increase the amount of computations, it is vitally important to design faster algorithms for the corresponding defuzzification.

By an interval valued fuzzy set F on X we mean the set

$$F = \{(x, [\mu_F^-(x), \mu_F^+(x)]) \mid x \in X\},$$

where μ_F^- and μ_F^+ are two fuzzy subsets of X such that $\mu_F^-(x) \leq \mu_F^+(x)$ for all $x \in X$. Putting $\widetilde{\mu_F}(x) = [\mu_F^-(x), \mu_F^+(x)]$, we see that $F = \{(x, \widetilde{\mu_F}(x)) \mid x \in X\}$, where $\widetilde{\mu_F} : X \to D[0,1]$.

If A, B are two interval valued fuzzy sets of X, then we define

$A \subseteq B$ if and only if for all $x \in X, \mu_A^-(x) \leq \mu_B^-(x)$ and $\mu_A^+(x) \leq \mu_B^+(x)$,

$A = B$ if and only if for all $x \in X, \mu_A^-(x) = \mu_B^-(x)$ and $\mu_A^+(x) = \mu_B^+(x)$.

Also, the union, intersection and complement are defined as follows: let A, B be two interval valued fuzzy sets of X, then

$A \cup B = \{(x, [\max\{\mu_A^-(x), \mu_B^-(x)\}, \max\{\mu_A^+(x), \mu_B^+(x)\}] | x \in X\}$,

$A \cap B = \{(x, [\min\{\mu_A^-(x), \mu_B^-(x)\}, \min\{\mu_A^+(x), \mu_B^+(x)\}] | x \in X\}$,

$A^c = \{(x, [1 - \mu_A^+(x), 1 - \mu_A^-(x)]) | x \in X\}$,

where the operation "c" is the complement of interval valued fuzzy set in X.

3. Entropy of interval valued (α, β)-fuzzy subalgebraic hypersystems

Based on [2], [3], we can extend the concept of quasi-coincidence of fuzzy point with a fuzzy set to the concept of quasi-coincidence of a fuzzy interval value with an interval valued fuzzy set as follows.

An interval valued fuzzy set $F = \{(x, \widetilde{\mu_F}(x)) \mid x \in H\}$ of an algebraic hypersystem H of the form

$$\widetilde{\mu_F}(y) = \begin{cases} \widetilde{t}(\neq[0,0]) & \text{if} \quad y = x \\ [0,0] & \text{if} \quad y \neq x \end{cases}$$

is said to be fuzzy interval value with support x and interval value $\widetilde{t}$ and is denoted by $U(x; \widetilde{t})$. A fuzzy interval value $U(x; \widetilde{t})$ is said to belong to (resp. be quasi-coincident with) an interval valued fuzzy set F, written as $U(x; \widetilde{t}) \in F$ (resp. $U(x; \widetilde{t}) qF$) if $\widetilde{\mu_F}(x) \geq \widetilde{t}$ (resp. $\widetilde{\mu_F}(x) + \widetilde{t} > [1,1]$). If $U(x; \widetilde{t}) \in F$ or (resp. and) $U(x; \widetilde{t}) qF$, then we write $U(x; \widetilde{t}) \in \vee q$ (resp. $\in \wedge q$) F. The symbol $\overline{\in \vee q}$ means $\in \vee q$ does not hold.

In what follows, let (H, Γ) be an algebraic hypersystem, $f \in \Gamma$ any n-ary hyperoperation on H, and α and β will denote any one of $\in, q, \in \vee q$ or $\in \wedge q$ unless otherwise specified. Also we emphasis $\widetilde{\mu_F}(x) = [\mu_F^-(x), \mu_F^+(x)]$ must satisfy the following properties:

$$[\mu_F^-(x), \mu_F^+(x)] < [0.5, 0.5] \text{ or } [0.5, 0.5] \leq [\mu_F^-(x), \mu_F^+(x)], \text{ for all } x \in H.$$

Definition 3.1. An interval valued fuzzy set F of H is called an interval valued (α, β)-fuzzy subalgebraic hypersystem of H with $\alpha \neq \in \wedge q$, if it satisfies for all $t_i \in (0,1]$ and $x_i \in H (i = 1, 2, ..., n)$,

(HF2) $U(x_i; \widetilde{t_i})\alpha F$ implies $U(y; \text{rmin}\{\widetilde{t_1}, \widetilde{t_2}, ..., \widetilde{t_n}\})\beta F$, for all $y \in f(x_1, x_2, ..., x_n)$.

Let F be an interval valued fuzzy set of H such that $\widetilde{\mu_F}(x) \leq [0.5, 0.5]$ for all $x \in H$. Let $x \in H$ and $t \in (0,1]$ be such that $U(x; \widetilde{t}) \in \wedge qF$, then $\widetilde{\mu_F}(x) \geq \widetilde{t}$ and $\widetilde{\mu_F}(x) + \widetilde{t} > [1,1]$. It follows that $[1,1] < \widetilde{\mu_F}(x) + \widetilde{t} \leq \widetilde{\mu_F}(x) + \widetilde{\mu_F}(x) = 2\widetilde{\mu_F}(x)$, which implies $\widetilde{\mu_F}(x) > [0.5, 0.5]$. This means that $\{U(x; \widetilde{t}) | U(x; \widetilde{t}) \in \wedge qF\} = \emptyset$. Therefore, the case $\alpha = \in \wedge q$ in Definition 3.1 will be omitted.

Proposition 3.2.

(i) *Every interval valued* $(\in \vee q, \in \vee q)$*-fuzzy subalgebraic hypersystem of* H *is an interval valued* $(\in, \in \vee q)$*-fuzzy subalgebraic hypersystem of* H*;*

(ii) *Every interval valued* $(\in, \in)$*-fuzzy subalgebraic hypersystem of* H *is an interval valued* $(\in, \in \vee q)$*-fuzzy subalgebraic hypersystem of* H*.*

The converse of Propositions 3.2 is not true. Consider Klein's 4-group $H = \{e, x, y, z\}$. Defined an interval valued fuzzy set F of H by

$$\widetilde{\mu_F}(e) = [0.6, 0.7], \quad \widetilde{\mu_F}(x) = [0.7, 0.8] \quad \text{and} \quad \widetilde{\mu_F}(y) = \widetilde{\mu_F}(z) = [0.4, 0.5].$$

Then, F is an interval valued $(\in, \in \vee q)$-fuzzy subalgebraic hypersystem of H. We note that F is not an interval valued (α, β)-fuzzy algebraic hypersystem of H, where $(\alpha, \beta) \neq (\in, \in),\ (q, \in \vee q),\ (\in \vee q, \in \vee q)$.

Lemma 3.3.

(i) *If* A *is a subalgebraic hypersystem of* H*, then the characteristic function* χ_A *of* A *is an interval valued* $(\in, \in)$*-fuzzy subalgebraic hypersystem of* H*;*

(ii) *For any subset* A *of* H*,* χ_A *is an interval valued* $(\in, \in \vee q)$*-fuzzy subalgebraic hypersystem of* H *if and only if* A *is a subalggebraic hypersystem of* H*.*

Now, we give the main result on a general interval valued (α, β)-fuzzy subalgebraic hypersystem of algebraic hypersystems.

Theorem 3.4. *Let* F *be a non-zero interval valued* (α, β)*-fuzzy subalgebraic hypersystem of* H*. Then the set* $U(F; [0,0]) = \{x \in H | \widetilde{\mu_F}(x) > [0,0]\}$ *is a subalgebraic hypersystem of* H*.*

Proof. Let $x_i \in U(F; [0,0])$, for all $i = 1, 2, ..., n$, then $\widetilde{\mu_F}(x_i) > [0,0]$, $i = 1, 2, ..., n$. Assume that $\widetilde{\mu_F}(y) = [0,0]$, for some $y \in f(x_1, x_2, ..., x_n)$. If $\alpha \in \{\in, \in \vee q\}$, then $U(x_i; \widetilde{\mu_F}(x_i))\alpha F$, for some $i = 1, 2, ..., n$, but $U(y; \text{rmin}\{\widetilde{\mu_F}(x_1), ..., \widetilde{\mu_F}(x_n)\})\overline{\beta} F$, for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Note that $U(x_i; [1,1])qF$ $(i = 1, 2, ..., n)$, but, for all $y \in f(x_1, x_2, ...x_n), U(y, \text{rmin}\{[1,1], [1,1], ...[1,1]\}) = U(y; [1,1])\overline{\beta} F$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, this is a contradiction. Hence, for all $y \in f(x_1, x_2, ..., x_n), \widetilde{\mu_F}(y) > [0,0]$, that is, $y \in U(F;[0,0])$, and so $f(x_1, x_2, ..., x_n) \subseteq U(F;[0,0])$. This completes the proof. ∎

Entropy and similarity measure of fuzzy sets are two important topics in fuzzy set theory. Entropy of a fuzzy set describes the fuzziness degree of fuzzy set. Many scholars have studied it from different points of view. We denote IVFSs for the sets of all interval valued (α, β)-fuzzy subalgebraic hypersystems of H.

A real function $E : IVFSs \to [0,1]$ is called an entropy on IVFSs, if E satisfies the following properties:

(1) $E(A) = 0$ if and only if A is a crisp subalgebraic hypersystem,

(2) $E(A) = 1$ if and only if $\mu_A^-(x) + \mu_A^+(x) = 1$,

(3) $E(A) \leq E(B)$ if A is less fuzzy than B,

(4) $E(A) = E(A^c)$.

For $M = \{x_1, ..., x_n\}$, we can give the following formulas to calculate the entropy of interval valued (α, β)-fuzzy subalgebraic hypersystem A:

$$E_1(A) = 1 - \frac{1}{n}\sum_{i=1}^{n} |\mu_A^-(x_i) + \mu_A^+(x_i) - 1|,$$

$$E_2(A) = 1 - \sqrt{\frac{1}{n}\sum_{i=1}^{n} (\mu_A^-(x_i) + \mu_A^+(x_i) - 1)^2}.$$

A real function $\varphi; IVFSs \times IVFSs \to [0,1]$ is called similarity measure of interval valued (α, β)-fuzzy subalgebraic hypersystems, if φ satisfies the following properties:

(1) $\varphi(A, A^c) = 0$ if A is a crisp subalgebraic hypersystem,

(2) $\varphi(A, B) = 1 \Leftrightarrow A = B$,

(3) $\varphi(A, B) = \varphi(B, A)$,

(4) if $A \subseteq B \subseteq C$, then $\varphi(A, C) \leq \varphi(A, B)$ and $\varphi(A, C) \leq \varphi(B, C)$.

Based on the point of view, then we have the following statement:

Theorem 3.5.

(i) $\varphi(A^-, (A^+)^c)$ *is entropy of interval valued* (α, β)*-fuzzy subalgebraic hypersystem* A*;*

(ii) *Suppose* φ *be similarity measure of interval valued* (α, β)*-fuzzy subalgebraic hypersystems and* $A \in IVFSs$*, then* $\varphi(A, A^c)$ *is entropy of interval valued* (α, β)*-fuzzy subalgebraic hypersystem* A*.*

4. Interval valued $(\in, \in \vee q)$-fuzzy subalgebraic hypersystems

In this section, we mainly discuss some fundamental aspects of interval valued $(\in, \in \vee q)$-fuzzy subalgebraic hypersystems of an algebraic hypersystem H. First, we can extend the concept of fuzzy subalgebraic hypersystems to the concept of interval valued fuzzy subalgebraic hypersystems in an algebraic hypersystem as follows:

Definition 4.1. An interval valued fuzzy set F of H is said to be an interval valued fuzzy subalgebraic hypersystem of H, if for all $x_i \in H$ $(i = 1, 2, ..., n)$,

(HF3) $$\operatorname*{rinf}_{y \in f(x_1, x_2, ..., x_n)} \widetilde{\mu_F}(y) \geq \mathrm{rmin}\{\widetilde{\mu_F}(x_1), \widetilde{\mu_F}(x_2), ..., \widetilde{\mu_F}(x_n)\}.$$

Example 4.2. Consider Example 2.2 (ii). Define an interval valued fuzzy set F by $\widetilde{\mu_F}(1) = [1, 1], \widetilde{\mu_F}(-1) = [0.5, 0.6]$ and $\widetilde{\mu_F}(i) = \widetilde{\mu_F}(-i) = [0, 0]$. Then F is an interval valued fuzzy subalgebraic hypersystem of H.

Let F be an interval valued fuzzy set. For every $t \in [0, 1]$, the set $U(F; \widetilde{t}) = \{x \in H | \widetilde{\mu_F}(x) \geq \widetilde{t}\}$ is called the interval valued level subset of F. Now, we characterize interval valued fuzzy subalgebraic hypersystems by their level subalgebraic hypersystems.

Theorem 4.3. *An interval valued fuzzy set F of H is an interval valued fuzzy subalgebraic hypersystem of H if and only if for any $[0, 0] < \widetilde{t} \leq [1, 1]$, $U(F; \widetilde{t})(\neq \emptyset)$ is a subalgebraic hypersystem of H.*

Proof. For every $x_1, x_2, ..., x_n \in U(F; \widetilde{t})$, we have $\mathrm{rmin}\{\widetilde{\mu_F}(x_1), ..., \widetilde{\mu_F}(x_n)\} \geq \widetilde{t}$, and so $\mathrm{rinf}\{\widetilde{\mu_F}(y) | y \in f(x_1, x_2, ..., x_n)\} \geq \widetilde{t}$. Therefore, for every $y \in f(x_1, x_2, ...x_n)$, we have $y \in U(F; \widetilde{t})$, so $f(x_1, x_2, ..., x_n) \subseteq U(F; \widetilde{t})$.

Conversely, assume that for every $[0, 0] < \widetilde{t} \leq [1, 1]$, $U(F; \widetilde{t})(\neq \emptyset)$ is a subalgebraic hypersystem of H. For every $x_1, x_2, ..., x_n \in H$, we have $\widetilde{\mu_F}(x_i) \geq \mathrm{rmin}\{\widetilde{\mu_F}(x_1), ..., \widetilde{\mu_F}(x_n)\} = \widetilde{t_0}$. Then, $x_i \in U(F; \widetilde{t_0})$, for $i = 1, 2, ..., n$. So, $f(x_1, x_2, ..., x_n) \subseteq U(F; \widetilde{t_0})$. Therefore, for every $y \in f(x_1, x_2, ..., x_n)$, we have $\widetilde{\mu_F}(y) \geq \widetilde{t_0}$ implying

$$\mathrm{rinf}\{\widetilde{\mu_F}(y) | y \in f(x_1, x_2, ..., x_n)\} \geq \mathrm{rmin}\{F(x_1), ..., F(x_n)\} \geq \widetilde{t} \qquad \blacksquare$$

Further, we define the following concept:

Definition 4.4. An interval valued fuzzy set F of H is said to be an interval valued $(\in, \in \vee q)$-fuzzy subalgebraic hypersystem of H if for all $t_i \in (0, 1]$ and $x_i \in H (i = 1, 2, ..., n)$,

(HF4) $U(x_i; \widetilde{t_i}) \in F$ implies $U(y; \mathrm{rmin}\{\widetilde{t_1}, \widetilde{t_2}, ..., \widetilde{t_n}\}) \in \vee q\, F$, for all $y \in f(x_1, x_2, ...x_n)$.

Note that if F is an interval valued fuzzy subalgebraic hypersystem of H according to Definition 4.1, then F is an interval valued $(\in, \in \vee q)$-fuzzy subalgebraic hypersystem of H according to Definition 4.4. But the converse is not true shown by the following example:

Example 4.5.

(i) Let $H = \{e, a, b\}$ be an algebraic hypersystem defined by the following table:

$\circ$	e	a	b
e	$\{e\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{e, b\}$	$\{a, b\}$
b	$\{b\}$	$\{a, b\}$	$\{e, a\}$

Define an interval valued fuzzy set F by $\widetilde{\mu_F}(e) = [0.8, 0.9], \widetilde{\mu_F}(a) = [0.7, 0.8]$ and $\widetilde{\mu_F}(b) = [0.6, 0.7]$. Then it is easy to see that F is an interval valued $(\in, \in \vee q)$-fuzzy subalgebraic hypersystem of H, but is not an interval valued fuzzy subalgebraic hypersystem of H.

(ii) Consider Example 2.2 (ii), and define

$$[0.4, 0.5] \leq \widetilde{\mu_F}(i) = \widetilde{\mu_F}(-i) \leq \widetilde{\mu_F}(-1) = \widetilde{\mu_F}(1).$$

Then, F not only is an interval valued $(\in, \in \vee q)$-fuzzy subalgebraic system of H, but also is an interval valued fuzzy subalgebraic system of H.

(iii) Consider Example 2.2 (iii), define an interval valued fuzzy set F by $\widetilde{\mu_F}(x) = [0.7, 0.8]$, $\widetilde{\mu_F}(y) = [0.5, 0.6]$ and $\widetilde{\mu_F}(z) = \widetilde{\mu_F}(t) = [0.3, 0.4]$, then F is an interval valued $(\in, \in \vee q)$-fuzzy subalgebraic system of H.

Theorem 4.6. *An interval valued fuzzy set F of H is an interval valued $(\in, \in \vee q)$-fuzzy subalgebraic hypersystem of H if and only if for all $x_i \in H (i = 1, 2, ..., n)$,*

$$\text{(HF5)} \qquad \text{rmin}\{\widetilde{\mu_F}(x_1), \widetilde{\mu_F}(x_2), ..., \widetilde{\mu_F}(x_n), [0.5, 0.5]\} \leq \underset{y \in f(x_1, x_2, ..., x_n)}{\text{rinf}} \widetilde{\mu_F}(y).$$

Proof. Assume that F is an interval valued $(\in, \in \vee q)$-fuzzy subalgebraic hypersystem of H. Let $x_i \in H (i = 1, 2, ..., n)$, we consider the following cases:

(i) $\text{rmin}\{\widetilde{\mu_F}(x_1), \widetilde{\mu_F}(x_2), ..., \widetilde{\mu_F}(x_n)\} < [0.5, 0.5]$,

(ii) $\text{rmin}\{\widetilde{\mu_F}(x_1), \widetilde{\mu_F}(x_2), ..., \widetilde{\mu_F}(x_n)\} \geq [0.5, 0.5]$.

Case (i): Assume that there exists $y \in f(x_1, x_2, ..., x_n)$ such that

$$\widetilde{\mu_F}(y) < \text{rmin}\{\widetilde{\mu_F}(x_1), \widetilde{\mu_F}(x_2), ..., \widetilde{\mu_F}(x_n), [0.5, 0.5]\},$$

which implies $\widetilde{\mu_F}(y) < \text{rmin}\{\widetilde{\mu_F}(x_1), ..., \widetilde{\mu_F}(x_n)\}$. Choose t such that

$$\widetilde{\mu_F}(y) < \widetilde{t} < \text{rmin}\{\widetilde{\mu_F}(x_1), \widetilde{\mu_F}(x_2), ..., \widetilde{\mu_F}(x_n)\}.$$

Then, $U(x_i; \widetilde{t}) \in F$, but $U(y; \widetilde{t}) \overline{\in \vee q} F$, which contradicts (HF4).

Case (ii): Assume that $\widetilde{\mu_F}(y) < [0.5, 0.5]$ for some $y \in f(x_1, x_2, ..., x_n)$. Then, $U(x_i; [0.5, 0.5]) \in F, i = 1, 2, ..., n$. But $U(y; [0.5, 0.5])\overline{\in \vee q}F$, a contradiction. Hence (HF5) holds.

Conversely, let $U(x_i; \widetilde{t_i}) \in F(i = 1, 2, ..., n)$, then $\widetilde{\mu_F}(x_i) \geq \widetilde{t_i}$ $(i = 1, 2, ..., n)$. For any $y \in f(x_1, x_2, ..., x_n)$, we have

$$\begin{aligned} \widetilde{\mu_F}(y) &\geq \operatorname{rmin}\{\widetilde{\mu_F}(x_1), \widetilde{\mu_F}(x_2), ..., \widetilde{\mu_F}(x_n), [0.5, 0.5]\} \\ &\geq \operatorname{rmin}\{\widetilde{t_1}, \widetilde{t_2}, ..., \widetilde{t_n}, [0.5, 0.5]\}. \end{aligned}$$

If $\operatorname{rmin}\{\widetilde{t_1}, \widetilde{t_2}, ..., \widetilde{t_n}\} > [0.5, 0.5]$, then $\widetilde{\mu_F}(y) \geq [0.5, 0.5]$, which implies $\widetilde{\mu_F}(y) + \operatorname{rmin}\{\widetilde{t_1}, \widetilde{t_2}, ..., \widetilde{t_n}\} > [1, 1]$.

If $\operatorname{rmin}\{\widetilde{t_1}, \widetilde{t_2}, ..., \widetilde{t_n}\} \leq [0.5, 0.5]$, then $\widetilde{\mu_F}(y) \geq \operatorname{rmin}\{\widetilde{t_1}, \widetilde{t_2}, ..., \widetilde{t_n}\}$. Therefore, $U(y; \operatorname{rmin}\{\widetilde{t_1}, \widetilde{t_2}, ..., \widetilde{t_n}\}) \in \vee q\ F$ for all $y \in f(x_1, x_2, ..., x_n)$. This completes the proof. ∎

Theorem 4.7. *Let F be an interval valued $(\in, \in \vee q)$-fuzzy subalgebraic hypersystem of H. Then for all $[0, 0] < \widetilde{t} \leq [0.5, 0.5]$, $U(F; \widetilde{t})$ is an empty set or a subalgebraic hypersystem of H. Conversely, if F is an interval valued fuzzy set of H such that $U(F; \widetilde{t})(\neq \emptyset)$ is a subalgebraic hypersystem of H for all $[0, 0] < \widetilde{t} \leq [0.5, 0.5]$, then F is an interval valued $(\in, \in \vee q)$-fuzzy subalgebraic hypersystem of H.*

Proof. Let F be an interval valued $(\in, \in \vee q)$-fuzzy subalgebraic hypersystem of H and $[0, 0] < \widetilde{t} \leq [0.5, 0.5]$. Let $x_i \in U(F; \widetilde{t}), i = 1, 2, ..., n$, then $\widetilde{\mu_F}(x_i) \geq \widetilde{t}$. Now,

$$\begin{aligned} \operatorname{rinf}\{\widetilde{\mu_F}(y) | y \in f(x_1, x_2, ..., x_n)\} &\geq \operatorname{rmin}\{\widetilde{\mu_F}(x_1), \widetilde{\mu_F}(x_2), ...\widetilde{\mu_F}(x_n), [0.5, 0.5]\} \\ &\geq \operatorname{rmin}\{\widetilde{t}, [0.5, 0.5]\} = \widetilde{t}. \end{aligned}$$

Therefore, for every $y \in f(x_1, x_2, ..., x_n)$, we have $\widetilde{\mu_F}(y) \geq \widetilde{t}$, and so $y \in U(F; \widetilde{t})$, which implies, $f(x_1, x_2, ..., x_n) \subseteq U(F; \widetilde{t})$. Therefore $U(F; \widetilde{t})$ is a subalgebraic hypersystem of H.

Conversely, let F be an interval valued fuzzy set of H such that $U(F; \widetilde{t})$ $(\neq \emptyset)$ is a subalgebraic hypersystem of H for all $[0, 0] < \widetilde{t} \leq [0.5, 0.5]$. For every $x_i \in H(i = 1, 2, ..., n)$, we can write

$$\widetilde{\mu_F}(x_i) \geq \operatorname{rmin}\{\widetilde{\mu_F}(x_1), \widetilde{\mu_F}(x_2), ...\widetilde{\mu_F}(x_n), [0.5, 0.5]\} = \widetilde{t_0},\ i = 1, 2, ..., n,$$

then $x_i \in U(F; \widetilde{t_0})$, and so $f(x_1, x_2, ..., x_n) \supseteq U(F; \widetilde{t_0})$. Therefore, for every $y \in f(x_1, x_2, ...x_n)$, we have $\widetilde{\mu_F}(y) \geq \widetilde{t_0}$, which implies $\widetilde{\mu_F}(y) \geq \widetilde{t_0}$ for all $y \in f(x_1, x_2, ..., x_n)$. Therefore, F is an interval valued $(\in, \in \vee q)$-fuzzy subalgebraic hypersystem of H. ∎

Naturally, a corresponding result should be considered when $U(F; \widetilde{t})$ is a subalgebraic hypersystem of H for all $[0.5, 0.5] < \widetilde{t} \leq [1, 1]$.

Theorem 4.8. *Let F be an interval valued fuzzy set of H. Then $U(F; \widetilde{t})(\neq \emptyset)$ is a subalgebraic hypersystem of H for all $[0.5, 0.5] < \widetilde{t} \leq [1, 1]$ if and only if*

(HF6) $\mathrm{rmin}\{\widetilde{\mu_F}(x_1), \widetilde{\mu_F}(x_2), ..., \widetilde{\mu_F}(x_n)\}$
$\leq \mathrm{rinf}\{\mathrm{rmax}\{\widetilde{\mu_F}(y), [0.5, 0.5]\} | y \in f(x_1, x_2, ..., x_n)\}$,
for all $x_i \in H,\ i = 1, 2, ..., n$.

Proof. Assume that $U(F; \widetilde{t})$ is a subalgebraic hypersystem of H. If there exist $x_i, y \in H (i = 1, 2, ..., n)$ with $y \in f(x_1, x_2, ..., x_n)$ such that

$$\mathrm{rmax}\{\widetilde{\mu_F}(y), [0.5, 0.5]\} < \mathrm{rmin}\{\widetilde{\mu_F}(x_1), \widetilde{\mu_F}(x_2), ..., \widetilde{\mu_F}(x_n)\} = \widetilde{t},$$

then $[0.5, 0.5] < \widetilde{t} \leq [1, 1], \widetilde{\mu_F}(y) < \widetilde{t}, x_i \in U(F; \widetilde{t})$ $(i = 1, 2, ..., n)$. Since $x_i \in U(F; \widetilde{t})$ and $U(F; \widetilde{t})$ is a subalgebraic hypersystem, so $f(x_1, x_2, ...x_n) \subseteq U(F; \widetilde{t})$ and $\widetilde{\mu_F}(y) \geq \widetilde{t}$ for all $y \in f(x_1, x_2, ..., x_n)$, which is a contradiction with $\widetilde{\mu_F}(y) < \widetilde{t}$. Therefore,

$$\mathrm{rmin}\{\widetilde{\mu_F}(x_1), \widetilde{\mu_F}(x_2), ..., \widetilde{\mu_F}(x_n)\} \leq \mathrm{rmax}\{\widetilde{\mu_F}(y), [0.5, 0.5]\},$$

for all $x_i, y \in H$ $(i = 1, 2, ..., n)$ with $y \in f(x_1, x_2, ..., x_n)$, which implies,

$$\mathrm{rmin}\{\widetilde{\mu_F}(x_1), \widetilde{\mu_F}(x_2), ..., \widetilde{\mu_F}(x_n)\} \leq \mathrm{rinf}(\mathrm{rmax}\{\widetilde{\mu_F}(y), [0.5, 0.5]\} | y \in f(x_1, x_2, ..., x_n)),$$

for all $x_i \in H$. Hence (HF6) holds.

Conversely, assume that $([0.5, 0.5], [1, 1]]$ and $x_i \in U(F; \widetilde{t})$ $(i = 1, 2, ..., n)$. Then,

$$\begin{aligned} [0.5, 0.5] < \widetilde{t} &\leq \mathrm{rmin}\{\widetilde{\mu_F}(x_1), \widetilde{\mu_F}(x_2), ..., \widetilde{\mu_F}(x_n)\} \\ &\leq \mathrm{rinf}\{\mathrm{rmax}\{\widetilde{\mu_F}(y), [0.5, 0.5]\} | y \in f(x_1, x_2, ..., x_n)\}. \end{aligned}$$

It follows that, for every $y \in f(x_1, x_2, ..., x_n)$,

$$[0.5, 0.5] < \widetilde{t} \leq \mathrm{rmax}\{\widetilde{\mu_F}(y), [0.5, 0.5]\},$$

and so $\widetilde{t} \leq \widetilde{\mu_F}(y)$, which implies $y \in U(F; \widetilde{t})$. Hence $f(x_1, x_2, ..., x_n) \subseteq U(F; \widetilde{t})$, that is, $U(F; \widetilde{t})$ is a subalgebraic hypersystem of H. ∎

Let F be an interval valued fuzzy set of an algebraic hypersystem H and $J = \{\alpha | \alpha \in (0, 1]$ and $U(F; [0.5, 0.5])$ is an empty set or a subalgebraic hypersystem of $H\}$. In particular, if $J = (0, 1]$, then F is an ordinary interval valued fuzzy subalgebraic hypersystem of H (Theorem 4.3); if $J = (0, 0.5)$, F is an interval valued $(\in, \in \vee q)$-fuzzy subalgebraic hypersystem of H (Theorem 4.7).

In [25], Yuan et al. gave the definition of a fuzzy subgroup with thresholds which is a generalization of Rosenfeld's fuzzy subgroup, and Bhkat and Das's fuzzy subgroup. Based on [25], we can extend the concept of a fuzzy subgroup with thresholds to the concept of fuzzy sublagebraic hypersystems with thresholds in the following way:

Definition 4.9. Let $s, t \in [0, 1]$ and $\widetilde{s} < \widetilde{t}$, then an interval valued fuzzy set F of H is called an interval valued fuzzy subalgebraic hypersystem with thresholds $(\widetilde{s}, \widetilde{t})$ of H if it satisfies:

(HF7) $\text{rmin}\{\widetilde{\mu_F}(x_1), \widetilde{\mu_F}(x_2), ..., \widetilde{\mu_F}(x_n), \widetilde{t}\}$
$\leq \inf\{\text{rmax}\{\widetilde{\mu_F}(y), \widetilde{s}\} | y \in f(x_1, x_2, ..., x_n)\}$,

for all $x_i \in H,\ i = 1, 2, ..., n$.

Remark. If F is an interval valued fuzzy subalgebraic hypersystem with thresholds of H, then we can conclude that F is an ordinary interval valued fuzzy subalgebraic hypersystem when $\widetilde{s} = [0, 0], \widetilde{t} = [1, 1]$; and F is an interval valued $(\in, \in \vee q)$-fuzzy subalgebraic hypersystem when $\widetilde{s} = [0, 0], \widetilde{t} = [0.5, 0.5]$.

Now, we characterize interval valued fuzzy subalgebraic hypersystems with thresholds by their level subalgebraic hypersystems.

Theorem 4.10. *An interval valued fuzzy set F of H is an interval valued fuzzy subalgebraic hypersystem with thresholds $(\widetilde{s}, \widetilde{t})$ of H if and only if $U(F; \widetilde{\alpha})$ $(\neq \emptyset)$ is ,a subalgebraic hypersystem of H for all $\widetilde{s} \leq \widetilde{\alpha} \leq \widetilde{t}$.*

Proof. Let F be an interval valued fuzzy subalgebraic hypersystem with thresholds $(\widetilde{s}, \widetilde{t})$ of H and $\widetilde{s} \leq \widetilde{\alpha} \leq \widetilde{t}$. Let $x_i \in U(F; \widetilde{\alpha})$, then $\widetilde{\mu_F}(x_i) \geq \widetilde{\alpha}, i = 1, 2, ..., n$. Now

$$\begin{aligned}
&\text{rmin}\{\widetilde{\mu_F}(x_1), \widetilde{\mu_F}(x_2), ..., \widetilde{\mu_F}(x_n), \widetilde{t}\} \\
&\quad \leq \text{rinf}\{\text{rmax}\{\widetilde{\mu_F}(y), \widetilde{s}\} | y \in f(x_1, x_2, ..., x_n)\} \\
&\quad \geq \text{rmin}\{\widetilde{\alpha}, \widetilde{t}\} \geq \widetilde{\alpha} > \widetilde{s}.
\end{aligned}$$

So, for every $\alpha \in f(x_1, x_2, ..., x_n)$, we have $\text{rmax}\{\widetilde{\mu_F}(y), \widetilde{s}\} > \widetilde{\alpha} > \widetilde{s}$, which implies $\widetilde{\mu_F}(y) > \widetilde{\alpha}$, and so $y \in U(F; \widetilde{\alpha})$. Hence $f(x_1, x_2, ..., x_n) \subseteq U(F; \widetilde{\alpha})$. Therefore, $U(F; \widetilde{\alpha})$ is a subalgebraic hypersystem of H for all $\widetilde{\alpha} \in (\widetilde{s}, \widetilde{t}]$.

Conversely, let F be an interval valued fuzzy set of H such that $U(F; \widetilde{\alpha})$ $(\neq \emptyset)$ is a subalgebraic hypersystem of H for all $\widetilde{s} \leq \widetilde{\alpha} \leq \widetilde{t}$. If there exist x_i $(i = 1, 2, ..., n)$, $y \in H$ with $y \in f(x_1, x_2, ..., x_n)$ such that $\text{rmax}\{\widetilde{\mu_F}(y), \widetilde{s}\} < \text{rmin}\{\widetilde{\mu_F}(x_1), ..., \widetilde{\mu_F}(x_n), \widetilde{t}\} = \widetilde{\alpha}$, then $\widetilde{\alpha} \in (\widetilde{s}, \widetilde{t}], \widetilde{\mu_F}(y) < \widetilde{\alpha}, x_i \in U(F; \widetilde{\alpha})$, $i = 1, 2, ..., n$. Since $U(F; \widetilde{\alpha})$ is a subalgebraic hypersystem of H and $x_i \in U(F; \widetilde{\alpha})$ for all $y \in f(x_1, x_2, ..., x_n)$, so $f(x_1, x_2, ..., x_n) \subseteq U(F; \widetilde{\alpha})$. Hence $\widetilde{\mu_F}(y) \geq \widetilde{\alpha}$, for all $y \in f(x_1, x_2, ..., x_n)$. This is a contradiction with $\widetilde{\mu_F}(y) < \widetilde{\alpha}$. Therefore $\text{rmin}\{\widetilde{\mu_F}(x_1), ..., \widetilde{\mu_F}(x_n), \widetilde{t}\} \leq \text{rmax}\{\widetilde{\mu_F}(y), \widetilde{s}\}$, for all $x_i, y \in H$ with $y \in f(x_1, x_2, ..., x_n)$. This proves that F is an interval valued fuzzy subalgebraic hypersystem with thresholds $(\widetilde{s}, \widetilde{t})$ of H. ■

5. Implication-Based interval valued fuzzy subalgebraic hypersystems

Fuzzy logic is an extension of set theoretic variables (or terms of the linguistic variable truth). Some operators, like $\wedge, \vee, \neg, \rightarrow$ in fuzzy logic are also defined by using truth tables, the extension principle can be applied to derive definitions of the operators.

In the fuzzy logic, truth value of fuzzy proposition P is denoted by $[P]$. In the following, we display the fuzzy logical and corresponding set-theoretical notions:

$[x \in A] = A(x)$;
$[x \notin A] = 1 - A(x)$;
$[P \wedge Q] = \min\{[P], [Q]\}$;
$[P \vee Q] = \max\{[P], [Q]\}$;
$[P \to Q] = \min\{1, 1 - [P] + [Q]\}$;
$[\forall x P(x)] = \inf[P(x)]$;
$\models P$ if and only if $[P] = 1$ for all valuations.

Of course, various implication operators have been defined. We only show a selection of them in the next able, α denotes the degree of truth (or degree of membership) of the premise, β the respective values for the consequence, and I the resulting degree of truth for the implication:

Name	Definition of Implication Operators
Early Zadeh	$I_m(\alpha, \beta) = \max\{1 - \alpha, \min\{\alpha, \beta\}\}$
Lukasiewicz	$I_a(\alpha, \beta) = \min\{1, 1 - \alpha + \beta\}$
Standard Star(Godel)	$I_g(\alpha, \beta) = \begin{cases} 1 & \text{if} \alpha \leq \beta \\ \beta & \text{if } \alpha > \beta \end{cases}$
Contraposition of Godel	$I_{cg}(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha \leq \beta \\ 1 - \alpha & \text{if } \alpha > \beta \end{cases}$
Gaines-Rescher	$I_{gr}(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha \leq \beta \\ 0 & \text{if } \alpha > \beta \end{cases}$
Kleene-Dienes	$I_b(\alpha, \beta) = \max\{1 - \alpha, \beta\}$

The "quality" of these implication operators could be evaluated either empirically or axiomatically.

In the following definition, we considered the implication operators in the Lukasiewicz system of continuous-valued logic.

Definition 5.1. An interval valued fuzzy set F of H is called an interval valued fuzzifying subalgebraic hypersystem of H if it satisfies, for all $x_i \in H$, $i = 1, 2, ..., n$,

$$\models [\text{rinf}\{x_i \in F\} \to [\forall y \in f(x_1, x_2, ..., x_n),\ y \in F]].$$

Clearly, Definition 5.1 is equivalent to Definition 4.1. Therefore, an interval valued fuzzifying subalgebraic hypersystem is an ordinary interval valued fuzzy subalgebraic hypersystem.

Now, we introduce the concept of interval valued t-tautology, i.e.,

$$\models_{\widetilde{t}} P \text{ if and only if } [P] \geq \widetilde{t} \text{ for all valuations.}$$

Based on [25], we can extend the concept of implication-based fuzzy subalgebraic hypersystems in the following way:

Definition 5.2. Let F be an interval valued fuzzy set of H and $t \in (0,1]$ is a fixed number. Then F is called a t-implication-based interval valued fuzzy subalgebraic hypersystem of H, if for all $x_i \in H,\ i = 1,2,...,n$,

$$\models_{\widetilde{t}} [\mathrm{rinf}\{x_i \in F\} \to [\forall y \in f(x_1,x_2,...,x_n),\ y \in F]].$$

Now, let I be an implication operator, then we have

Corollary 5.3. *An interval valued fuzzy set F of H is a t-implication-based interval valued fuzzy subalgebraic hypersystem of H if and only if, for all $x_i \in H$, $i = 1,2,...,n$,*

$$I(\mathrm{rinf}\{\widetilde{\mu_F}(x_i)\}, \mathrm{rinf}\{\widetilde{\mu_F}(y) | y \in f(x_1,x_2,...,x_n)\}) \geq \widetilde{t}.$$

Let F be an interval valued fuzzy set of H, then we have the following results:

Theorem 5.4.

(i) *Let $I = I_{gr}$, then F is an 0.5-implication-based fuzzy interval valued subalgebraic hypersystem of H if and only if F is an interval valued fuzzy subalgebraic hypersystem with thresholds $(\widetilde{r} = [0,0], \widetilde{s} = [1,1])$ of H;*

(ii) *Let $I = I_g$, then F is an 0.5-implication-based interval valued fuzzy subalgebraic hypersystem of H if and only if F is an interval valued fuzzy subalgebraic hypersystem with thresholds $(\widetilde{r} = [0,0], \widetilde{s} = [0.5,0.5])$ of H;*

(iii) *Let $I = I_{cg}$, then F is an 0.5-implication-based interval valued fuzzy subalgebraic hypersystem of H if and only if F is an interval valued fuzzy subalgebraic hypersystem with thresholds $(\widetilde{r} = [0.5,0.5], \widetilde{s} = [1,1])$ of H.*

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References

[1] Ameri, R., Zahedi, M.M., *T-fuzzy hyperalgebraic systems*, Lecture Notes in Computer Science, Springer-Verlag Heidelberg, 2275 (2002).

[2] Bhakat, S.K., $(\in, \in \vee q)$*-fuzzy normal, quasinormal and maximal subgroups,* Fuzzy Sets and Systems, 112 (2000), 299-312.

[3] Bhakat, S.K., Das, P., $(\in, \in \vee q)$*-fuzzy subgroups*, Fuzzy Sets and Systems, 80 (1996), 359-368.

[4] Biswas, R., *Rosenfeld's fuzzy subgroups with interval valued membership functions*, Fuzzy Sets and Systems, 63 (1994), 87-90.

[5] Corsini, P., *Prolegomena of Hypergroup Theory*, Second edition, Aviani Editore, 1993.

[6] Corsini, P., Leoreanu, V., *Applications of Hyperstructure Theory*, Advances in Mathematics (Dordrecht), Kluwer Academic Publishers, Dordrecht, 2003.

[7] Davvaz, B., *Fuzzy H_v-subgroups*, Fuzzy Sets and Systems, 101 (1999), 191-195.

[8] Davvaz, B., *Fuzzy H_v-submodules,* Fuzzy Sets and Systems, 117 (2001), 477-484.

[9] Davvaz, B., *Interval-valued fuzzy subhypergroups,* Korean J. Comput. Appl. Math., 6 (1999), 197-202.

[10] Davvaz, B., *Approximations in H_v-modules,* Taiwanese J. Math., 6 (4) (2002), 499-506.

[11] Davvaz, B., *Characterizations of sub-semihypergroups by various triangular norms,* Czechoslovak Math. J., 55 (130) (2005), 923–932.

[12] Davvaz, B., *On connection between uncertainty algebraic hyperstructures and probability theory,* International Journal of Uncertainty, Fuzziness and Knowledge Based Systems, 13 (3) (2005), 337-345.

[13] Davvaz, B., $(\in, \in \vee q)$*-fuzzy subnear-rings and ideals,* Soft Computing, 10 (2006), 206-211.

[14] Davvaz, B., Corsini, P., *Redefined fuzzy H_v-subhypermodules and many valued implications,* Inform. Sci., 177 (2007), 865-875.

[15] Davvaz, B., Dudek,W.A., . Jun, Y.B., *Intuitionistic fuzzy H_v-submodules*, Inform. Sci., 176 (2006), 285-300.

[16] Deschrijver, G., *Arithmetric operators in interval-valued fuzzy theory,* Inform. Sci., 177 (2007), 2906-2924.

[17] Dudek, W.A., *Fuzzication of n-ary groupoids,* Quasigroups and Related Systems, 7 (2000), 45-66.

[18] Dudek, W.A., Davvaz, B., Jun, Y.B., *On intuitionistic fuzzy subhypergroups of hypergroups*, Inform. Sci., 170 (2005), 251-262.

[19] Marty, F., *Sur une generalization de la notation de groupe,* 8th Congress Math. Scandianaves, Stockholm, 1934, 45-49.

[20] Mordeson, J.N., Malik, M.S., *Fuzzy Commutative Algebra,* World Publishing, 1998.

[21] Pu, P.M., Liu, Y.M., *Fuzzy topology.* I: *Neighourhood struture of a fuzzy point and Moore-Smith convergence,* J. Math. Anal. Appl., 76 (1980), 571-599.

[22] Rosenfeld, A., *Fuzzy groups,* J. Math. Anal. Appl., 35 (1971), 512-517.

[23] Vougiouklis, T., *Hyperstructures and their Representations*, Hadronic Press Inc., Palm Harbor, USA, 1994.

[24] Yuan, X.H., Lee, E.S., *A fuzzy algebraic system based on the theory falling shadows*, J. Math. Anal. Appl., 208 (1997), 243-251.

[25] Yuan, X.H., Zhang, C., Ren, Y.H., *Generalized fuzzy groups and many valued applications,* Fuzzy Sets and Systems, 138 (2003), 205-211.

[26] Zadeh, L.A., *Fuzzy sets*, Inform. and Control, 8 (1965), 338-353.

[27] Zadeh, L.A., *The concept of a lingistic variable and its application to approximate reason*, Inform. Control, 18 (1975), 199-249.

[28] Zeng, W., Li, H., *Relationship between similarity measure and intropy of interval valued fuzzy set,* Fuzzy Sets and System,s 157 (2006), 1477-1484.

[29] Zhan, J., *On properties of fuzzy hyperideals in hypernear-rings with t-norms*, J. Appl. Math. Computing, 20 (2006), 255-277.

[30] Zhan, J., Davvaz, B., Shum, K.P., *On fuzzy isomorphism theorems of hypermodules,* Soft Computing, 11 (2007), 1053-1057.

[31] Zhan, J., Dudek, W.A., *Interval valued intuitionistic (S,T)-fuzzy H_v-submodules,* Acta. Math. Sinica., English Series, 22 (2006), 963-970.

[32] Zhan, J., Dudek, W.A., *Fuzzy h-ideals of hemirings,* Inform. Sci., 177 (2007), 876-886.

[33] Zhan, J., Ma, X., *Approximations in hypernear-rings,* Soochow J. Math., 32 (2006), 421-431.

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