SIMPLIFIED MARGINAL LINEARIZATION METHOD IN AUTONOMOUS LIENARD SYSTEMS

Weijing Zhao[1]

Faculty of Electronic Information and Electrical Engineering
Dalian University of Technology
Dalian, Liaoning, 116024
P.R. China
and
College of Air Traffic Management
Civil Aviation University of China
Tianjin, 300300
P.R. China

Hongxing Li

Faculty of Electronic Information and Electrical Engineering
Dalian University of Technology
Dalian, Liaoning, 116024
P.R. China

Yuming Feng

School of Mathematics and Statistics
Chongqing Three Gorges University
Chongqing, 404100
P.R. China

Abstract. In this paper, a simplified marginal linearization method in autonomous Lienard systems is proposed. The new method simplified coefficients of each equation, leads to little calculation, and the time and space complexity are reduced. At last, the simulation results show that the simplified marginal linearization method in autonomous Lienard systems is of high approximation precision.

Keywords: autonomous Lienard systems; marginal linearization; fuzzy systems; rectangle wave.

AMS Subject Classification: 03B52; 65L05.

1. Introduction

In the fields of science and technology, many theoretical issues in physics have been summarized into a large number of ordinary differential equations, most of them

[1]Corresponding author. Tel: +86 411 84706694. Email: 693509394@qq.com (Weijing Zhao).

are nonlinear differential equations. In the history of radio and vacuum tube technology, Lienard systems (equations) were intensely studied as they can be used to model oscillating circuits. H. Cartan and E. Cartan [1] first studied the existence of periodic solutions for differential equation $L\ddot{i}(t) + (r - \psi(t))\dot{i}(t) + \frac{i}{C} = 0$ in telecommunications technology issues, where L, r, C are positive constant, representing inductance, resistance and capacitance respectively. Van der Pol [2] first proposed the famous Van der Pol equation $\ddot{y}(t)+\mu(y^2-1)\dot{y}(t)+y = 0$ ($\mu > 0$) when he studied the equal-amplitude oscillation of triode. In 1928, a French engineer Alfred-Marie Lienard [3] generalized an extensive one: $\ddot{y}(t) + f(y)\dot{y}(t) + g(y) = 0$, i.e., so-called Lienard systems. Lienard systems are widely applied to atmospheric dynamics, physics, biology and other fields. Unfortunately, it is difficult to use it for the simple reason that analytical solutions can't be presented for majority of them. Meanwhile many experts have made great efforts to solve this nonlinear systems problem from different aspects, such as stability of the solution [4], [5], boundedness of solution [6], [7], limit cycle [8], [9], etc.

Since Zadeh [10] first presented the concept of fuzzy sets in 1965, a variety of applications of fuzzy logic have been implemented in various fields ranging from industrial control to financial management. For example, there is a considerable amount of work on hyperoperations defined through fuzzy sets. This study was initiated by Corsini in [14] and then continuated by him together with Leoreanu in [15], [16]. Most notably, fuzzy systems have been successfully applied to control vague, incomplete, and ill-defined systems. Li [11] revealed interpolation mechanism of fuzzy control, i.e., the fuzzy control algorithms used commonly at present are all regarded as some interpolation functions. Li [12] first proposed modelling method based on fuzzy inference (MMFI) for fuzzy control systems, i.e., fuzzy inference is used on a controlled object, and fuzzy inference rule base is transferred into HX equations. It has shown that the mathematical model of a system formed by MMFI can approximate the real mathematical model of the system that is formed by mechanism modelling method. In order to solve the problem that each HX equation is a nonlinear equation, marginal linearization method in modeling on fuzzy control systems is proposed in [13]. This method turned HX equations into some kind of linear differential equations of linear differential equations with constant coefficients. So it provides a way to get approximately analytical solution of nonlinear equations initial value problem.

In this paper, we introduce a simplified marginal linearization method in autonomous Lienard systems. Simplified coefficients of the equations are given, and it needs to solve $p-1$ equations in each segment instead of solving $(p-1)(q-1)$ equations in each piece. So the problem of autonomous Lienard systems is partially resolved from marginal linearization aspect.

The rest of the paper is organized as follows. Some useful concepts and notations are briefly reviewed in Section 2. In Section 3, the proposed simplified marginal linearization method in autonomous Lienard systems is discussed in detail. The simulation experiments of the new method is described in Section 4. Finally, conclusions are drawn in Section 5.

2. Preliminaries

In this section, some useful concepts and notations are introduced.

Definition 2.1. ([6], [9]) Let f and g be two continuous functions on R, with g satisfies Lipschitz condition in any finite interval then the second order ordinary differential equation of the form $\ddot{y}(t)+f(y)\dot{y}(t)+g(y)=0$ is called the autonomous Lienard systems(equations).

Note 2.1. In the previous definition, the hypothesis of f and g are two continuous functions on R, with g satisfies Lipschitz condition in any finite interval guarantees the existence and uniqueness of autonomous Lienard equations.

Lemma 2.1. ([11], [12], [13]) *Let* $Y=[a_1,b_1]$, $\dot{Y}=[a_2,b_2]$ *and* $\ddot{Y}=[a_3,b_3]$ *respectively be the universe of* $y(t)$, $\dot{y}(t)$ *and* $\ddot{y}(t)$, *and* $\mathscr{A}=\{A_i\}_{(1\le i\le p)}$, $\mathscr{B}=\{B_j\}_{(1\le i\le q)}$, $\mathscr{C}=\{C_{ij}\}_{(1\le i\le p\ ,1\le j\le q)}$ *respectively be the fuzzy partition(a group of base elements) of corresponding universe, where* $A_i\in\mathscr{F}(Y)$, $B_j\in\mathscr{F}(\dot{Y})$ *and* $C_{ij}\in\mathscr{F}(\ddot{Y})$, *which are called base element and* y_i, $\dot{y}_j$, $\ddot{y}_{ij}$ *are respectively the peakpoints of* A_i, B_j, C_{ij}, *and with the condition:* $a_1\le y_1<y_2<\cdots<y_p\le b_1$, $a_2\le\dot{y}_1<\dot{y}_2<\cdots<\dot{y}_q\le b_2$, $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ *are regarded as linguistic variables so that a group of fuzzy inference rules is formed as follows:*

$$\text{If } y(t) \text{ is } A_i \text{ and } \dot{y}(t) \text{ is } B_j, \text{ then } \ddot{y}(t) \text{ is } C_{ij}, \tag{1}$$

where $i=1,2,...,p$, $j=1,2,...,q$.

Theorem 2.1. ([13]) *The fuzzy logic system based on* (1) *can be represented as a binary piecewise interpolation function* $F(\cdot,\cdot)$

$$\ddot{y}(t)=F(y(t),\dot{y}(t))\triangleq\sum_{i=1}^{p}\sum_{j=1}^{q}A_i(y(t))B_j(\dot{y}(t))\,y_{ij}. \tag{2}$$

Here, A_i *are taken as "rectangle wave" membership functions,* B_j *are taken as "triangle wave" membership functions (see Fig 1 and Fig 2).*

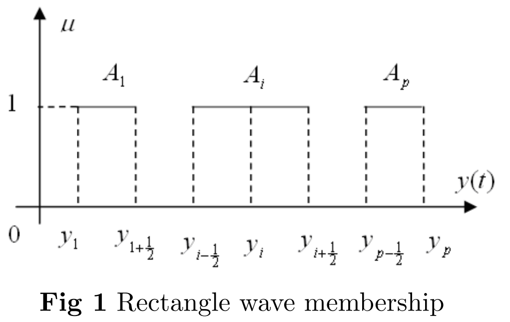

Fig 1 Rectangle wave membership functions of A_i

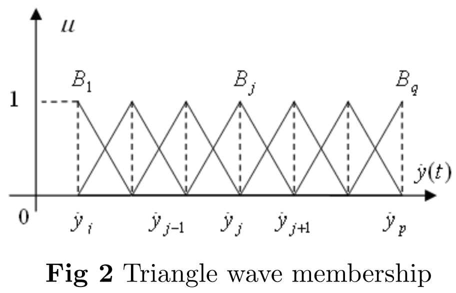

Fig 2 Triangle wave membership functions of B_j

(3) $$A_i(y(t)) = \begin{cases} 1\,, & y_{i-\frac{1}{2}} \leq y(t) < y_{i+\frac{1}{2}}, \\ 0\,, & otherwise. \end{cases}$$

(4) $$B_j(y(t)) = \begin{cases} \dfrac{\dot{y}(t) - \dot{y}_{j-1}}{\dot{y}_j - \dot{y}_{j-1}}, & \dot{y}_{j-1} \leq \dot{y}(t) \leq \dot{y}_j, \\ \dfrac{\dot{y}(t) - \dot{y}_{j+1}}{\dot{y}_j - \dot{y}_{j+1}}, & \dot{y}_j \leq \dot{y}(t) \leq \dot{y}_{j+1}, \\ 0, & otherwise. \end{cases}$$

where $i = 1, 2, ..., p$. *We stipulate that* $y_{1-\frac{1}{2}} = y_1$, $y_{p+\frac{1}{2}} = y_p$; *and also stipulate* $\dot{y}_0 = \dot{y}_1$, $\dot{y}_{q+1} = \dot{y}_q$.

Theorem 2.2. ([13]) **(marginal linearization method)** *Under the previous assumptions and conditions of Theorem* 2.1, *the input-output model of the second order system based on eq.* (1) *can be represented as a second order differential equation with variable coefficients:*

(5) $$\ddot{y}(t) + P_1(y(t)\,, \dot{y}(t))\dot{y}(t) = Q_1(y(t)\,, \dot{y}(t)),$$

where

(6) $$P_1(y(t)\,, \dot{y}(t)) = \sum_{i=1}^{p} \sum_{j=1}^{q-1} P_1^{(i,j)},$$

(7) $$Q_1(y(t)\,, \dot{y}(t)) = \sum_{i=1}^{p} \sum_{j=1}^{q-1} Q_1^{(i,j)},$$

and $P_1^{(i,j)}$, $Q_1^{(i,j)}$ *are defined as local coefficients on the* (i, j)*-th piece as follows:*

(8) $$P_1^{(i,j)} = \begin{cases} \dfrac{\ddot{y}_{ij+1} - \ddot{y}_{ij}}{\dot{y}_j - \dot{y}_{j+1}}, & (y(t), \dot{y}(t)) \in [y_{i-\frac{1}{2}}, y_{i+\frac{1}{2}}] \times [\dot{y}_j, \dot{y}_{j+1}], \\ 0, & otherwise. \end{cases}$$

(9) $$Q_1^{(i,j)} = \begin{cases} \dfrac{\dot{y}_j\ddot{y}_{ij+1} - \dot{y}_{j+1}\ddot{y}_{ij}}{\dot{y}_j - \dot{y}_{j+1}}, & (y(t), \dot{y}(t)) \in [y_{i-\frac{1}{2}}, y_{i+\frac{1}{2}}] \times [\dot{y}_j, \dot{y}_{j+1}], \\ 0, & otherwise. \end{cases}$$

Theorem 2.3. ([13]) *Under condition of Theorem* 2.2, *when* $(y(t), \dot{y}(t)) \in [y_{i-\frac{1}{2}}, y_{i+\frac{1}{2}}] \times [\dot{y}_j, \dot{y}_{j+1}]$, *i.e., when on the* (i, j)*-th piece, equation* (5) *degenerates into a local equation*

(10) $$\ddot{y}(t) + P_1^{(i,j)}\dot{y}(t) = Q_1^{(i,j)}.$$

3. Simplified marginal linearization method in autonomous Lienard systems

In order to deal with the nonlinear system with variable coefficients, marginal linearization method in modeling on fuzzy control systems is proposed in [13]. This method can turn a nonlinear system with variable coefficients into a linear model with variable coefficients in the way that the membership functions of the fuzzy sets in fuzzy partitions of the universes are changed from triangle waves into rectangle waves.

Observe the coefficients in Theorem 2.2, the expression of coefficients in the equations are too complex. And it need to solve $(p-1)(q-1)$ local equations. Therefore, it brings a lot of inconvenience. In this section, we have made an attempt to simplify the calculation process of the autonomous Lienard systems.

Theorem 3.1. (Simplified marginal linearization method in autonomous Lienard system) *Under the condition of eq.* (10)*, autonomous Lienard system* $\ddot{y}(t)+f(y)\dot{y}(t)+g(y)=0$ *can be simplified represented as a second order differential equation with variable coefficients:*

$$\ddot{y}(t)+P_1(y(t))\dot{y}(t)=Q_1(y(t)), \tag{11}$$

where

$$P_1(y(t))=\sum_{i=1}^{p}P_1^{(i)}, \tag{12}$$

$$Q_1(y(t))=\sum_{i=1}^{p}Q_1^{(i)}, \tag{13}$$

and P_1^i, Q_1^i *are defined as local coefficients in the* i*-th segment as follows:*

$$P_1^{(i)}=\begin{cases} f(y_i), & t\in[y_i,y_{i+1}],\\ 0, & otherwise.\end{cases} \tag{14}$$

$$Q_1^{(i)}=\begin{cases} -g(y_i), & t\in[y_i,y_{i+1}],\\ 0, & otherwise.\end{cases} \tag{15}$$

Proof. Here we only proof the expression of Q_1^i, P_1^i can be proved similarly. According to definition 2.1, we have that $\ddot{y}(t)+f(y)\dot{y}(t)+g(y)=0$, therefore,

$$\ddot{y}(t)=-f(y)\dot{y}(t)-g(y).$$

Then, by Lemma 2.1

$$\ddot{y}_{ij}=-f(y_i)\dot{y}_j-g(y_i). \tag{16}$$

According to Definition 2.2, when $(y(t),\dot{y}(t)) \in [y_{i-\frac{1}{2}}, y_{i+\frac{1}{2}}] \times [\dot{y}_j, \dot{y}_{j+1}]$,

$$Q_1^{(i,j)} = \frac{\dot{y}_j \ddot{y}_{ij+1} - \dot{y}_{j+1}\ddot{y}_{ij}}{\dot{y}_j - \dot{y}_{j+1}}. \tag{17}$$

From (16) and (17), we obtain that

$$Q_1^{(i,j)} = \frac{\dot{y}_j \ddot{y}_{ij+1} - \dot{y}_{j+1}\ddot{y}_{ij}}{\dot{y}_j - \dot{y}_{j+1}} = \frac{\dot{y}_j[-f(y_i)\dot{y}_{j+1} - g(y_i)] + \dot{y}_{j+1}[f(y_i)\dot{y}_j + g(y_i)]}{\dot{y}_j - \dot{y}_{j+1}}$$
$$= -\frac{g(y_i)(\dot{y}_i - \dot{y}_{i+1})}{\dot{y}_i - \dot{y}_{i+1}} = -g(y_i).$$

Since $\forall (y(t),\dot{y}(t)) \in [y_{i-\frac{1}{2}}, y_{i+\frac{1}{2}}] \times [\dot{y}_1, \dot{y}_2]$, $[y_{i-\frac{1}{2}}, y_{i+\frac{1}{2}}] \times [\dot{y}_2, \dot{y}_3], \cdots, [y_{i-\frac{1}{2}}, y_{i+\frac{1}{2}}] \times [\dot{y}_{q-1}, \dot{y}_q]$, $Q_1^{(i,j)} = -g(y_i)$ always holds, i.e., when $(y(t),\dot{y}(t)) \in [y_{i-\frac{1}{2}}, y_{i+\frac{1}{2}}] \times ([\dot{y}_1, \dot{y}_2] \cup [\dot{y}_2, \dot{y}_3] \cup \cdots \cup [\dot{y}_{q-1}, \dot{y}_q]) = [y_{i-\frac{1}{2}}, y_{i+\frac{1}{2}}] \times [a_2, b_2]$,

$$Q_1^{(i,j)} = -g(y_i);$$

when $(y(t), \dot{y}(t)) \notin [y_{i-\frac{1}{2}}, y_{i+\frac{1}{2}}] \times [a_2, b_2]$, $Q_1^{(i,j)} = 0$.
Since $\dot{y}(t) \in [a_2, b_2]$ holds inevitably, we only consider the value of $y(t)$. Therefore,

$$Q_1^{(i)} = \begin{cases} -g(y_i), & t \in [y_i, y_{i+1}], \\ 0, & \text{otherwise}. \end{cases}$$

Corollary 3.1. *As to autonomous Lienard systems $\ddot{y}(t) + f(y)\dot{y}(t) + g(y) = 0$, when $y(t) \in [y_i, y_{i+1}]$, a local equation can be simplified as follows:*

$$\ddot{y}(t) + P_1^{(i)}\dot{y}(t) = Q_1^{(i)}. \tag{18}$$

Its coefficients have nothing to do with $\dot{y}_j$.

Proof. The assertions are trivial consequences of Theorems 2.2 and 3.1.

Corollary 3.2. *As to autonomous Lienard systems $\ddot{y}(t) + f(y)\dot{y}(t) + g(y) = 0$, Let $Y = [a_1, b_1]$, $\dot{Y} = [a_2, b_2]$ and $\ddot{Y} = [a_3, b_3]$ respectively be the universe of $y(t)$, $\dot{y}(t)$ and $\ddot{y}(t)$, and $\mathscr{A} = \{A_i\}_{(1\le i\le p)}$, $\mathscr{C} = \{C_i\}_{(1\le i\le p\ )}$ respectively be the fuzzy partition(a group of base elements) of corresponding universe, where $A_i \in \mathscr{F}(Y)$, $C_i \in \mathscr{F}(\ddot{Y})$, which are called base element and y_i, $\ddot{y}_i$ are respectively the peakpoints of A_i, C_i, and with the condition: $a_1 \le y_1 < y_2 < \cdots < y_p \le b_1$, $\mathscr{A}$ and $\mathscr{C}$ are regarded as linguistic variables so that a group of fuzzy inference rules is formed as follows:*

$$\textit{If } y(t) \textit{ is } A_i, \textit{ then } \ddot{y}(t) \textit{ is } C_i, \tag{19}$$

where $i = 1, 2, ..., p$.

Proof. According to Theorem 3.1, in each segments, local coefficients have nothing to do with $\dot{y}_j$, and $\dot{y}(t) \in [a_2, b_2]$ holds inevitably. As a result, it don't need to fuzzy inference on $\dot{y}(t)$.

Corollary 3.3. *As to autonomous Lienard systems $\ddot{y}(t) + f(y)\dot{y}(t) + g(y) = 0$, when applying the simplified marginal linearization method, it doesn't need fuzzy inference on $\dot{y}(t)$, so the partition of rectangular region $[a_1, b_1]\times[a_2, b_2]$ in $(y(t), \dot{y}(t))$-plane can be simplified as the partition of closed interval $[a_1, b_1]$ in a real line. Then, it needs to solve $p-1$ linear time-invariant equations in each segments instead of solving $(p-1)(q-1)p$ equations in each piece.*

Proof. It is obvious from Theorem 3.1 and Corollary 3.2.

Note 3.1. Solving autonomous Lienard systems $\ddot{y}(t) + f(y)\dot{y}(t) + g(y) = 0$ by using the method in [13], we have $(p-1)(q-1)$ linear time-invariant equations to solve. With p and q increase, the number of the equations grows rapidly. So it is difficult to solve these equations (e.g., when $p = 7, q = 8$, it needs to solve 42 equations). According to Corollary 3.3, as to autonomous Lienard systems $\ddot{y}(t) + f(y)\dot{y}(t) + g(y) = 0$, it only needs to solve p equations (e.g., when $p = 7$, $q = 8$ still holds, it only needs to solve 6 equations). The number of equations to be solved are reduced significantly and makes the program simple, so the time complexity is reduced.

Note 3.2. As to autonomous Lienard systems $\ddot{y}(t) + f(y)\dot{y}(t) + g(y) = 0$, it doesn't need to calculate the value of $\ddot{y}_{ij}$ when applying the simplified marginal linearization method. So the space complexity is reduced for it doesn't need to open memory space for $\ddot{y}_{ij}$.

4. Simulation experiments

The novel method proposed in Section 3 has showed that the simplified marginal linearization method in autonomous Lienard systems lead to little calculation, and the time and space complexity are reduced. This section presents whether the novel method is right and effective.

Given a system, for example, we still regard Van der pol equation in [12] as the real model of the system.

$$\ddot{y}(t) + \mu(y^2(t) - 1)\dot{y}(t) + y(t) = 0, \tag{20}$$

where $\mu = 1$. It's a special autonomous Lienard system.

The program design of the simulation follows the following steps:

Step 1. Determine the universes Y and $\dot{Y}$. By using solution (20), find respectively the maximum and the minimum of $y(t)$ and $\dot{y}(t)$: $y_{\max} = \max\{y(t)\}$, $y_{\min} = \min\{y(t)\}$, $\dot{y}_{\max} = \max\{\dot{y}(t)\}$, $\dot{y}_{\min} = \min\{\dot{y}(t)\}$. In order to allow an acceptable range of error, these maximum and minimum values should be extended to such an extent that we can get the universes: $Y = [a_1, b_1]$, $\dot{Y} = [a_2, b_2]$, where

$$a_1 = y_{\min} - 0.1|y_{\min}|,\ b_1 = y_{\max} + 0.1|y_{\max}|,$$
$$a_2 = \dot{y}_{\min} - 0.1|\dot{y}_{\min}|,\ b_2 = \dot{y}_{\max} + 0.1|\dot{y}_{\max}|.$$

Step 2. Calculate the peakpoints. Given a natural number $p > 1$, take $h_1 = (b_1 - a_1)/(p-1)$. And the isometry partition nodal points y_i are computed by the following equation: $y_i = a_1 + (i-1)h_1,\ i = 1, 2, ..., p$.

Step 3. According to (11)-(15), calculate the coefficients in each segment, $P_1^{(i)}$, $Q_1^{(i)}$, $i = 1, 2, ..., p$.

Step 4. Given initial values $y(0) = y_0,\ \dot{y}(0) = \dot{y}_0$, solve the local equation (18) segment by segment, and p local equations should be solved. For this purpose, let $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$, so the local equation (18) becomes a system of local first order differential equations:

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -P_1^{(i)} x_2(t) + Q_1^{(i)}. \end{cases} \tag{21}$$

By using Matlab 6.5, we can easily find the solution to the whole and draw the plots for the curves of $x_1(t)$, $x_2(t)$ and of the phase plane $(x_1(t), x_2(t))$. At the same time, draw the plots for the solution to the real model (20) and the curves of its phase plane, and compare these plots.

Example 1. Take the initial values $x_1(0) = 2$, $x_2(0) = 0$, and let $T = 20$, $p = 7$; Here the state curves $x_1(t)$ and $x_2(t)$, phase plan curves $(x_1(t), x_2(t))$ and the comparison with corresponding curves on real model are shown in Fig 3–Fig 5.

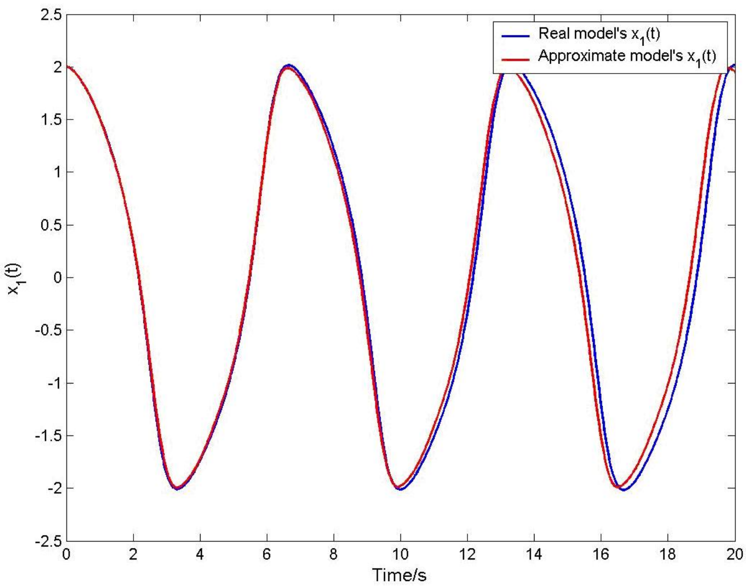

Fig 3 Simulation curves of state $x_1(t)$ under $p = 7$

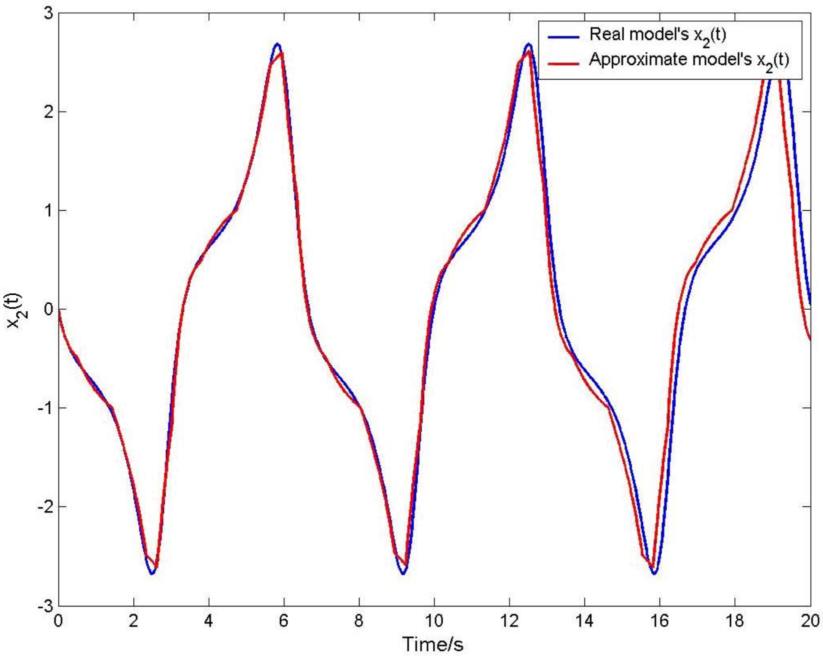

Fig 4 Simulation curves of state $x_2(t)$ under $p = 7$

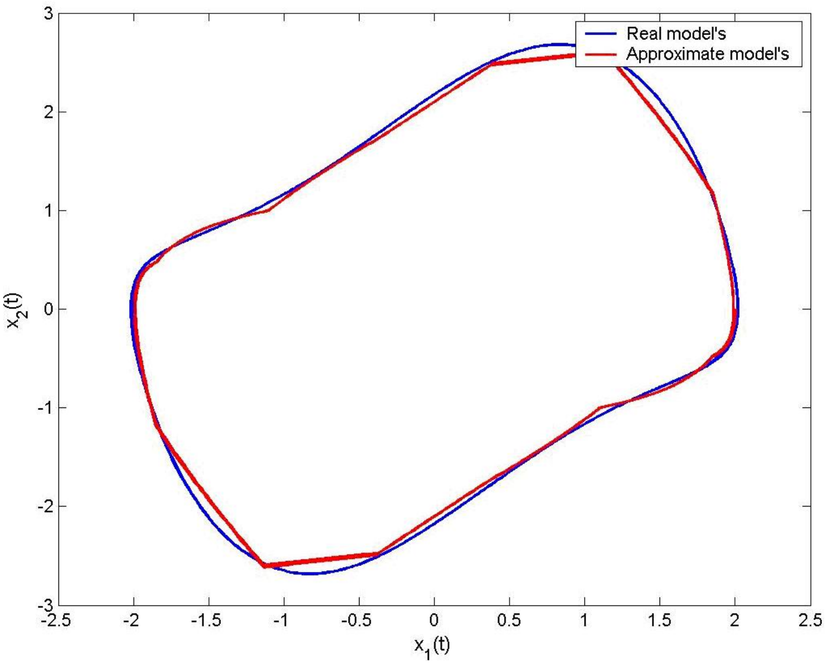

Fig 5 Simulation curves of phase plane $(x_1(t), x_2(t))$ under $p = 7$

Example 2. Let $p = 14$ and other parameters be the same as ones in the former example. The simulation results are shown in Fig 6-Fig 8. It has shown that more fuzzy inference rules are used in simulation, the error will be smaller. It's obvious that the curves of approximate model almost coincide with the ones of real model, which means that the simplified marginal linearization method in autonomous Lienard systems is of high approximation precision. In other words, it is reliable algorithm.

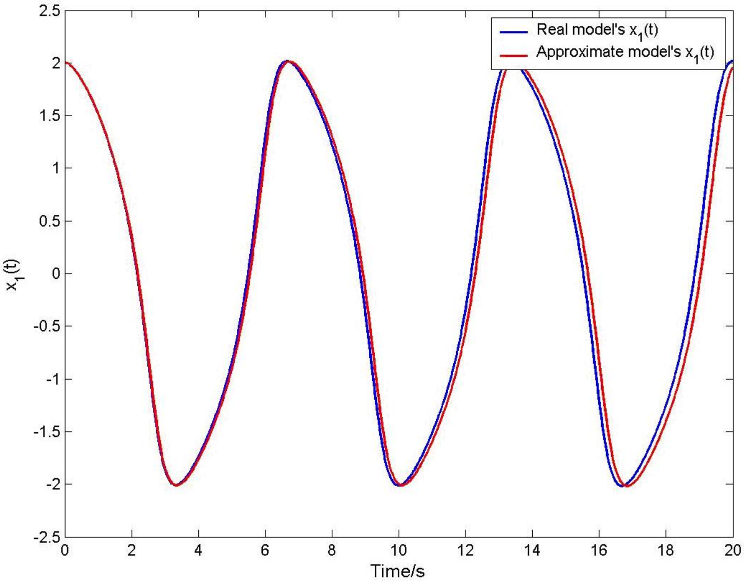

Fig 6 Simulation curves of state $x_1(t)$ under $p = 14$

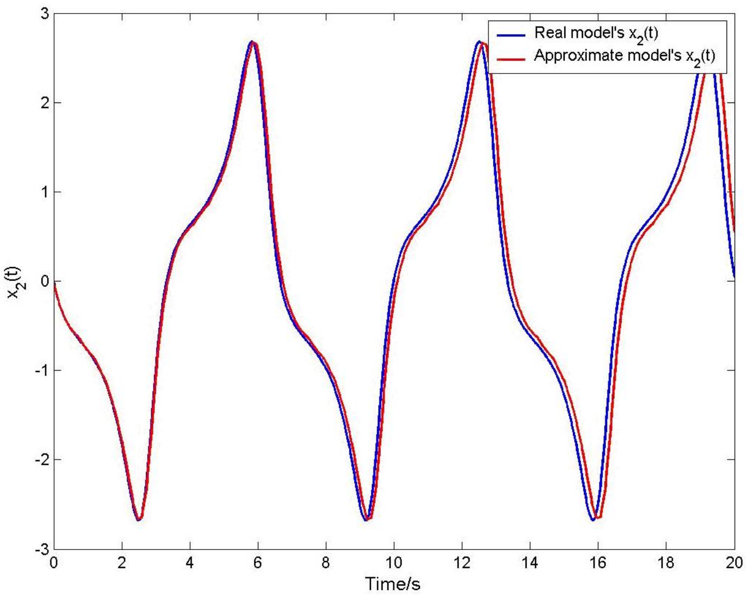

Fig 7 Simulation curves of state $x_2(t)$ under $p = 14$

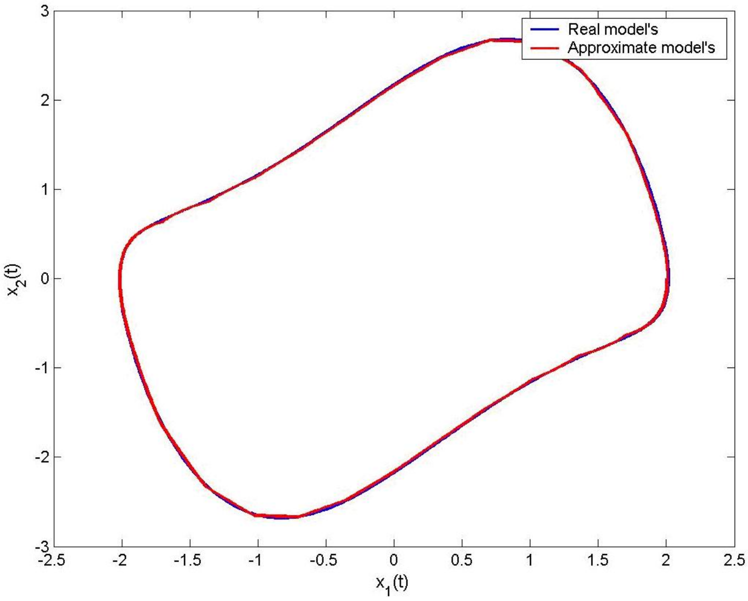

Fig 8 Simulation curves of phase plane $(x_1(t)\,,x_2(t))$ under $p = 14$

5. Conclusions

In this paper, the simplified marginal linearization method in autonomous Lienard systems is proposed. The novel method simplified coefficients of the each equations, leads to little calculation, and the time and space complexity is reduced. From the perspective of fuzzy inference, two-dimensional fuzzy inference of Y and $\dot{Y}$ is simplified as fuzzy inference of Y only. The simulation results show that simplified marginal linearization method in autonomous Lienard system is high approximation precision. Consequently, the theory of marginal linearization is riched.

Acknowledgments. This work is supported by the National Natural Science Foundation of China (Nos. 61074044, 61104038) and Specialized Research Fund for the Doctoral Program of Higher Education (No. 20090041110003).

References

[1] Cartan, H., Cartan, E., *Note sur la generation des oscillations entretenues*, Ann. P.T.T., 14 (1925), 1196-1207.

[2] Van der Pol, B., *Sur les oscillations de relaxation*, The Philos Magazine, 7 (1926), 978-992.

[3] Sugie, J., Amano, Y., *Global asymptotic stability of nonautonomous systems of Lienard type*, J. Math. Anal. Appl., 289 (2) (2004), 673-690.

[4] Yang, Q.G., *Suffcient and necessary conditions for global stability of a system of Lienard type,* Acta Mathematica Sinica 43 (4) (2000), 719-726.

[5] Liu, Z.R., *The Conditions for the global stability of the Lienard equation*, Acta Mathematica Sinica, 38 (5) (1995), 614-620.

[6] Chen, Y.S., Tang, Y., *Modern analytic methods of nonlinear kinetics*, Beijing: Science Press, 2000.

[7] Chen, Y.S., *Nonlinear Vibrations*, Beijing: Higher Education Press, 2003.

[8] Zhang, Z.F., Ding, T.R., et al., *Qualitative theory of differential equations*, Beijing: Science Press, 2000.

[9] Ye, Y.Q., *Theory of limit cycle*, Shanghai: Shanghai Science and Technology Publishing House, 1984.

[10] Zadeh, L.A., *Fuzzy sets*, Information and Control, 8 (1965), 338-353.

[11] Li, H.X., *Interpolation mechanism of fuzzy control*, Science in China (Ser.E), 41 (3) (1998), 312-320.

[12] Li, H.X., Wang, J.Y., et al., *Modeling on fuzzy control systems,* Science in China (Ser. A), 45 (12) (2002) 1506-1517.

[13] Li, H.X., Wang, J.Y., et al., *Marginal linearization method in modeling on fuzzy control systems*, Progress in Natural Science, 13 (7) (2003), 489-496.

[14] Corsini, P., *Join Spaces, Power Sets, Fuzzy Sets,* Proceedings of the Fifth International Congress of Algebraic Hyperstructures and Appl., 1993, Iasi, Romania, Hadronic Press, 1994, 293-303.

[15] Corsini, P., Leoreanu, V., *Fuzzy sets and join spaces associated with rough sets*, Rendiconti Del Circolo Matematico Di Palermo, 51 (2002), 527-536.

[16] Corsini, P., Leoreanu, V., *Join spaces associated with fuzzy sets,* Journal of Combinatorics, Information and System Sciences, 20 (1995), 293-303.

[17] Wang, L.X., *Fuzzy systems are universal approximators,* Proceedings of the IEEE International Conference on Fuzzy Systems, 1992, San Diego, 1163-1169.

Accepted: 07.03.2012