ALGEBRAIC HYPERSTRUCTURES OF SOFT SETS ASSOCIATED WITH TERNARY SEMIHYPERGROUPS

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Abstract. Molodtsov introduced the concept of soft set, which can be seen and used as a new mathematical tool for dealing with uncertainty. In this paper we introduce and initiate the study of soft ternary semihypergroups by using soft set theory. The notions of soft ternary semihypergroups, soft ternary subsemihypergroups, soft left (right, lateral) hyperideals, soft hyperideals, soft quasi-hyperideals and soft bi-hyperideals are introduced, and several related properties are investigated.

Keywords: ternary semihypergroup, soft set, soft ternary semihypergroup, soft left (lateral, right) hyperideal, soft hyperideal, soft quasi (bi)-hyperideal.

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1. Introduction and preliminaries

The Hyperstructure theory was introduced in 1934, at the eighth Congress of Scandinavian Mathematicians, when F. Marty [1] defined hypergroups based on the notion of hyperoperation, began to analyze their properties and applied them to groups. In the following decades and nowadays, a number of different hyperstructures are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics, such as in fuzzy sets and rough set theory, optimization theory, theory of discrete event dynamical systems, cryptography, codes, analysis of computer programs, automata, formal language theory, combinatorics, artificial intelligence, probability, graphs and hypergraphs, geometry, lattices and binary relations (see [2], [3], [4], [5]).

Ternary algebraic operations were considered in the 19th century by several mathematicians. Cayley [6] introduced the notion of "cubic matrix" which was generalized by Kapranov, et al. in 1990 [7]. Ternary structures and their generalization, the so-called n-ary structures, raise certain hopes in view of their possible applications in physics and other sciences. Some significant physical applications in Nambu mechanics, although still hypothetical, in the fractional quantum Hall effect, the non-standard statistics (the anyons), supersymmetric theories, Yang-Baxter equation etc. can be seen in [22], [25], [8], [27], [28], [29]). The notion of an n-ary group was introduced in 1928 by W. Dörnte [15] (under inspiration of Emmy Noether). The idea of investigations of n-ary algebras, i.e., sets with one n-ary operation, seems to be going back to Kasner's lecture [16] at the 53rd annual meeting of the American Association of the Advancement of Science in 1904. Sets with one n-ary operation having different properties were investigated by many authors. Such systems have many applications in different branches. For example, in the theory of automata [24] are used n-ary systems satisfying some associative laws, some others n-ary systems are applied in the theory of quantum groups [14] and combinatorics [30]. Different applications of ternary structures in physics are described by R. Kerner in [8]. In physics there are also used such structures as n-ary Filippov algebras (see [9]) and n-Lie algebras (see [22]). Some n-ary structures induced by hypercubes have applications in error-correcting and error-detecting coding theory, cryptology, as well as in the theory of (t, m, s)-nets (see for example [26]). Ternary semigroups are universal algebras with one associative operation. The theory of ternary algebraic systems was introduced by D. H. Lehmer [17] in 1932. He investigated certain algebraic systems called triplexes which turn out to be commutative ternary groups. The notion of ternary semigroups was introduced by S. Banach (cf. [23]). By an example he showed that a ternary semigroup does not necessary reduce to an ordinary semigroup.

n-ary generalizations of algebraic structures is the most natural way for further development and deeper understanding of their fundamental properties. In [10], Davvaz and Vougiouklis introduced the concept of n-ary hypergroups as a generalization of hypergroups in the sense of Marty. Also, n-ary hypergroups can be seen as an interesting generalization of n-ary groups. Davvaz and et. al. in [11] considered a class of algebraic hypersystems which represent a generalization of semigroups, hypersemigroups and n-ary semigroups. Ternary semihypergroups are algebraic structures with one associative hyperoperation and they are a particular case of an n-ary semihypergroup (n-semihypergroup) for $n = 3$ (cf. [11], [12], [10, 13]). Recently, Hila et. al. [20], [21] introduced and studied some classes of hyperideals in ternary semihypergroups.

Several difficult problems in economics, engineering, environment, social sciences, medicine and many other fields involve uncertain data. We usually come across the data which is blurred and contains many types of uncertainties. There are several well-known theories developed to overcome difficulties which arise due to uncertainty. For instance, probability theory, fuzzy sets theory [31], rough sets theory [32] and other mathematical tools. But all these theories have their inherited difficulties as pointed out by Molodtsov [33]. In probability theory, we have to make a great deal of experiments in order to check samples. In social sciences and economics, it is not always possible to make so many experiments.

Pawlak [32] introduced the theory of rough sets in 1982. It was a significant approach for modeling vagueness. In this theory, equivalence classes are used to approximate crisp subsets by their upper and lower approximations. This theory has been applied to many problems successfully, yet has its own limitations. It is not always possible to have an appropriate equivalence relation among the elements of a given set, so we cannot have equivalence classes to get upper and lower approximations of a subset. Later, some authors [34] tried to have approximations not with the help of equivalence relation, but by using a relation in general.

Fuzzy set theory was developed by Zadeh [31]. It is the most appropriate approach to deal with uncertainties. However, some authors [33] think that the difficulties in fuzzy set theory are due to the inadequacy of parametrization tools of this theory.

In 1999, Molodtsov introduced the concept of soft sets as a new mathematical tool for dealing with uncertainties that is free from the difficulties affecting existing methods. Soft set theory has rich potential for applications in several directions, few of which had been demonstrated by Molodtsov in his pioneer work [33]. Molodtsov also showed how Soft Set Theory (SST) is free from parametrization inadequacy syndrome of Fuzzy Set Theory (FST), Rough Set Theory (RST), Probability Theory, and Game Theory. SST is a very general framework. Many of the established paradigms appear as special cases of SST. Applications of Soft Set Theory in other disciplines and real life problems are now catching momentum. At present, research in the theory of soft sets is in progress. Theoretical and application aspects of soft set theory are discussed in several papers, such as [35], [36], [37], [38]. Maji et al. [35] applied soft sets to a decision making problem and studied several operations on the theory of soft sets. Pei et al. [36] discussed the relationship between soft sets and information systems. Roy et al. [37] applied fuzzy soft set theory to a decision making problem. Maji et al. [38] studied several operations on the theory of soft sets. Ali et al. [39] also studied some new notions such as the restricted intersection, the restricted union, the restricted difference, and the extended intersection of two soft sets. The algebraic structure of soft sets has been studied by several authors. For example, Aktas and Cagman [40] introduced the basic concepts of soft set theory and compared soft sets to the related concepts of fuzzy sets and rough sets. They also discussed the notion of soft groups and drove their basic properties using Molodtsov's definition of soft sets. Other applications of soft set theory in different algebraic structure can be found in [41], [42], [43], [44], [45], [46], [47], [48], [49] etc. Recently, Leoreanu-Fotea and Corsini [19] and then Davvaz et. al. [50], [51], [52] introduced and analyzed several types of soft hyperstructures: soft hypergroupoids, soft semihypergroups, soft hypergroups and soft polygroups.

In this paper, we introduce and initiate the study of soft ternary semihypergroups by using soft set theory. The notions of soft ternary semihypergroups, soft ternary subsemihypergroups, soft left (lateral, right) hyperideals, soft hyperideals, soft quasi-hyperideals and soft bi-hyperideals are introduced, and several related properties are investigated.

Recall first the basic terms and definitions from hyperstructure theory and soft set theory.

2. Ternary semihypergoups

Let H be a nonempty set and let $\mathcal{P}^*(H) = \mathcal{P}(H)\backslash\{\emptyset\}$ denote the set of all nonempty subsets of H. A map $\circ : H \times H \to \mathcal{P}^*(H)$ is called a *hyperoperation* or *join operation* on H. A *hypergroupoid* is a pair $(H, \circ)$, where $\circ$ is a hyperoperation on H. A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all $x, y, z \in H$, $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

If $x \in H$ and A, B are non-empty subsets of H then

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, A \circ x = A \circ \{x\} \text{ and } x \circ B = \{x\} \circ B.$$

A nonempty subset B of a semihypergroup H is called a *sub-semihypergroup* of H if $B \circ B \subseteq B$. A semihypergroup $(H, \circ)$ is a *hypergroup* if it satisfies the reproduction axiom: for all $a \in H$, $a \circ H = H \circ a = H$.

A map $f : H \times H \times H \to \mathcal{P}^*(H)$ is called a *ternary hyperoperation* on H. A *ternary hypergroupoid* is a pair (H, f) where f is a ternary hyperoperation on H.

If A, B, C are non-empty subsets of H, then we define

$$f(A, B, C) = \bigcup_{a \in A, b \in B, c \in C} f(a, b, c).$$

A ternary hypergroupoid (H, f) is called a *ternary semihypergroup* if for all $a_1, a_2, ..., a_5 \in H$, we have

$$(*) \qquad f(f(a_1, a_2, a_3), a_4, a_5) = f(a_1, f(a_2, a_3, a_4), a_5) = f(a_1, a_2, f(a_3, a_4, a_5)).$$

Since we can identify the set $\{x\}$ with the element x, any ternary semigroup [17] is a ternary semihypergroup.

Due to the associative law in a ternary semihypergroup (H, f), for any elements $x_1, x_2, ..., x_{2n+1} \in H$ and positive integers m, n with $m \leq n$, one may write

$$\begin{aligned} f(x_1, x_2, ..., x_{2n+1}) &= f(x_1, ..., x_m, x_{m+1}, x_{m+2}, ..., x_{2n+1}) \\ &= f(x_1, ..., f(f(x_m, x_{m+1}, x_{m+2}), x_{m+3}, x_{m+4}), ..., x_{2n+1}). \end{aligned}$$

A ternary hypergroupoid (H, f) is *commutative* if for all $a_1, a_2, a_3 \in H$ and for all $\sigma \in \mathbb{S}_3$, $f(a_1, a_2, a_3) = f(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$.

Let (H, f) be a ternary semihypergroup. A nonempty subset T of H is called a *ternary subsemihypergroup* if $f(T, T, T) \subseteq T$.

An element e of a ternary semihypergroup (H, f) is called a *left identity* element of H if for all $a \in H$, $f(e, a, a) = \{a\}$. An element $e \in H$ is called an *identity* element of H if for all $a \in H$, $f(a, a, e) = f(e, a, a) = f(a, e, a) = \{a\}$. Hence $f(e, e, a) = f(e, a, e) = f(a, e, e) = \{a\}$.

A nonempty subset I of a ternary semihypergroup H is called a *left* (*right, lateral*) *hyperideal* of H if

$$f(H, H, I) \subseteq I \quad (f(I, H, H) \subseteq I,\ f(H, I, H) \subseteq I).$$

A nonempty subset I of a ternary semihypergroup H is called a *hyperideal* of H if it is a left, right and lateral hyperideal of H. A nonemtpy subset I of a ternary semihypergroup H is called *two-sided hyperideal* of H if it is a left and right hyperideal of H.

Different examples can be found in [20], [21].

3. Preliminaries of soft set theory

Let U be an initial universe set and E be a set of parameters. The power set of U is denoted by $\mathcal{P}(U)$ and A is a subset of E.

Definition 3.1. A pair (F, A) is called a *soft set* over U, where $F : A \to \mathcal{P}(U)$ is a mapping.

In the other words, a soft set over U is a parameterized family of subsets of the universe U. For $a \in A, F(a)$ may be considered as the set of a-approximate elements of the soft set (F, A). Notice that a soft set is not a set, as Molodtsov pointed out in several examples in [33].

Let us consider the following example. We quote it directly from [38].

Example 3.2. [38] Let us consider a soft set (F, E), which describes the "attractiveness of houses", considered for purchase. Suppose that there are six houses in the universe U, given by $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ and $E = \{e_1, e_2, e_3, e_4, e_5\}$ is a set of decision parameters, where $e_i (i = 1, 2, 3, 4, 5)$ stand for the parameters "expensive", "beautiful", "wooden", "cheap" and "in green surroundings", respectively. Consider a mapping $F : E \to \mathcal{P}(U)$. For instance, suppose that $F(e_1) = \{h_2, h_4\}$, $F(e_2) = \{h_1, h_3\}$, $F(e_3) = \{h_3, h_4, h_5\}$, $F(e_4) = \{h_1, h_3, h_5\}$, $F(e_5) = \{h_1\}$. The soft set (F, E) is a parameterized family $\{F(e_i), i = 1, ..., 5\}$ of subsets of the set U, and can be viewed as a *collection* of approximations: (F, E)=\{expensive houses = $\{h_2, h_4\}$, beautiful houses =$\{h_1, h_3\}$, wooden houses =$\{h_3, h_4, h_5\}$, cheap houses =$\{h_1, h_3, h_5\}$, in green surroundings houses =$\{h_1\}$\}. Each approximation has two parts: a *predicate* and an *approximate value set.*

In [47], for a soft set (F, A), the set $Supp(F, A) = \{x \in A | F(x) \neq \emptyset\}$ is called the *support* of the soft set (F, A). If $Supp(F, A) \neq \emptyset$, then (F, A) is called non-null.

Definition 3.3. Let (F, A) and (G, B) be two soft sets over a common universe U. The *extended intersection* of (F, A) and (G, B), denoted by $(F, A) \bigcap_E (G, B)$ is the soft set (K, C), satisfying the following conditions: (i) $C = A \cup B$; (ii) for all $e \in C$,

$$K(e) = \begin{cases} F(e) & \text{if } e \in A \backslash B, \\ G(e) & \text{if } e \in B \backslash A, \\ F(e) \cup G(e) & \text{if } e \in A \cap B. \end{cases}$$

Definition 3.4. Let (F, A) and (G, B) be two soft sets over a common universe U. The *restricted intersection* of (F, A) and (G, B), denoted by $(F, A) \bigcap_R (G, B)$ is the soft set (K, C), satisfying the following conditions: (i) $C = A \cap B$; (ii) for all $e \in C$, $K(e) = F(e) \cap G(e)$. We write $(F, A) \widetilde{\cap} (G, B) = (K, C)$.

Definition 3.5. Let (F, A) and (G, B) be two soft sets over a common universe U. The *bi-intersection* of (F, A) and (G, B) is the soft set (K, C), where $C = A \cap B$ and $K : C \to \mathcal{P}(U)$ is a mapping given by $K(x) = F(x) \cap G(x)$ for all $x \in G$. We write $(F, A) \widetilde{\sqcap} (G, B) = (K, C)$.

Definition 3.6. Let (F, A) and (G, B) be two soft sets over a common universe U. The *extended union* of (F, A) and (G, B), denoted by $(F, A) \bigcup_E (G, B)$ is the soft set (K, C), satisfying the following conditions: (i) $C = A \cup B$; (ii) for all $e \in C$,

$$K(e) = \begin{cases} F(e) & \text{if } e \in A \backslash B, \\ G(e) & \text{if } e \in B \backslash A, \\ F(e) \cup G(e) & \text{if } e \in A \cap B. \end{cases}$$

Definition 3.7. Let (F, A) and (G, B) be two soft sets over a common universe U. The *restricted union* of (F, A) and (G, B), denoted by $(F, A) \bigcup_R (G, B)$, is the soft set (K, C) satisfying the following conditions: (i) $C = A \cap B$; (ii) for all $e \in C, K(e) = F(e) \cup G(e)$. We write $(F, A) \widetilde{\cup} (G, B) = (K, C)$.

Definition 3.8. Let $(F_i, A_i)_{i \in I}$ be a nonempty family of soft sets over a common universe U. The union of these soft sets is the soft set (G, B) such that $B = \bigcup_{i \in I} A_i$ and for all $x \in B$, $G(x) = \bigcup_{i \in I(x)} F_i(x)$, where $I(x) = \{i \in I | x \in A_i\}$. We write $\widetilde{\bigcup_{i \in I}} (F_i, A_i) = (G, B)$.

Definition 3.9. Let (F, A) and (G, B) be two soft sets over a common universe U. Then (F, A) AND (G, B) denoted by $(F, A) \widetilde{\wedge} (G, B)$ is defined by $(F, A) \widetilde{\wedge} (G, B) = (K, A \times B)$, where $K(x, y) = F(x) \cap G(y)$ for all $(x, y) \in A \times B$.

Definition 3.10. Let (F, A) and (G, B) be two soft sets over a common universe U. Then (F, A) OR (G, B) denoted by $(F, A) \widetilde{\vee} (G, B)$ is defined by $(F, A) \widetilde{\vee} (G, B) = (K, A \times B)$, where $K(x, y) = F(x) \cup G(y)$ for all $(x, y) \in A \times B$.

Definition 3.11. Let (F, A) and (G, B) be two soft sets over a common universe U. We say that (F, A) is a *soft subset* of (G, B), denoted by $(F, A)\widetilde{\subseteq}(G, B)$, if it satisfies: (i) $A \subseteq B$; (ii) $F(a) \subseteq G(a)$ for all $a \in A$.

We say that (F, A) and (G, B) are *soft equal*, denoted by $(F, A) = (G, B)$ if $(F, A) \subseteq (G, B)$ and $(G, B) \subseteq (F, A)$.

4. Soft ternary semihypergroups

Hereafter, we shall consider soft sets over a ternary semihypergroup (H, f).

Definition 4.1. The *restricted hyperproduct* of soft sets $(F_1, B_1), (F_2, B_2), (F_3, B_3)$ over a ternary semihypergroup H, denoted by $\widehat{f}((F_1, B_1), (F_2, B_2), (F_3, B_3))$, is defined as a soft set

$$(K, D) = \widehat{f}((F_1, B_1), (F_2, B_2), (F_3, B_3)),$$

where $D = \bigcap\{B_i | i = 1, 2, 3\} \neq \emptyset$ and $K : D \to \mathcal{P}(H)$ defined by

$$K(d) = f(F_1(d), F_2(d), F_3(d)) \text{ for all } d \in D.$$

Definition 4.2. Let (F, A) be a non-null soft set over a ternary semihypergroup (H, f). Then, (F, A) is called a *soft ternary semihypergroup* over H if $F(x)$ is a ternary subsemihypergroup of H for all $x \in Supp(F, A)$, i.e.

$$\widehat{f}((F, A), (F, A), (F, A)) \subseteq (F, A).$$

A soft set (H, E) over H is said to be an *absolute soft set* over H, if for all $e \in E$, $H(e) = H$.

Proposition 4.3. *A soft set (F, A) over a ternary semihypergroup H is a soft ternary semihypergroup over H if and only if for all $a \in A, F(a) \neq \emptyset$ is a ternary subsemihypergroup of H.*

Proof. Let (F, A) be a soft ternary hypergroup over H and $a \in A$ be such that $F(a) \neq \emptyset$. By definition

$$\widehat{f}((F, A), (F, A), (F, A)) = (K, A \cap A \cap A) = (K, A)$$

where K is defined by

$$K(a) = f(F(a), F(a), F(a)) \text{ for all } a \in A.$$

Since $\widehat{f}((F, A), (F, A), (F, A)) \subseteq (F, A)$, it follows that $K(a) \subseteq F(a)$ for all $a \in A$, and so $f(F(a), F(a), F(a)) \subseteq F(a)$. This means that $F(a)$ is a ternary subsemihypergroup of H.

Conversely, let $F(a) \neq \emptyset$ be a ternary semihypergroup of H for all $a \in A$. We have $f(F(a), F(a), F(a)) \subseteq F(a)$, that is $K(a) \subseteq F(a)$. Thus, $(K, A) \subseteq (F, A)$, which implies that

$$\widehat{f}((F,A),(F,A),(F,A)) \subseteq (F,A)$$

Hence (F,A) is a ternary semihypergroup over H. ∎

Example 4.4. Let $H = \{a,b,c,d,e,g\}$ and $f(x,y,z) = (x*y)*z$ for all $x,y,z \in H$, where $*$ is defined by the table:

$*$	a	b	c	d	e	g
a	a	$\{a,b\}$	c	$\{c,d\}$	e	$\{e,g\}$
b	b	b	d	d	g	g
c	c	$\{c,d\}$	c	$\{c,d\}$	c	$\{c,d\}$
d	d	d	d	d	d	d
e	e	$\{e,g\}$	c	$\{c,d\}$	e	$\{e,g\}$
g	g	g	d	d	g	g

Then (H,f) is a ternary semihypergroup. We define $F : H \to \mathcal{P}(H)$ by $F(a) = \{a,b\}, F(b) = \{b,d,g\}, F(c) = \{c,d\}, F(d) = \{d\}, F(e) = \{a,b,e,g\}, F(g) = \{d,g\}$. It is clear that for all $x \in H$, $F(x)$ is a ternary subsemihypergroup of H. Hence, (F,H) is a soft ternary semihypergroup over H. Notice that not every soft set over a ternary semihypergroup H, is a soft ternary semihypergroup over H. Let $G : H \to \mathcal{P}(H)$ defined by $G(a) = \{a,b,c\}, G(b) = \{b,d,g\}, G(c) = \{c,d\}, G(d) = \{d\}, G(e) = \{a,b,e,g\}, G(g) = \{d,e,g\}$. Then (G,H) is a soft set over H, but it is not a soft ternary semihypergroup over H, because $G(a)$ and $G(g)$ are not ternary subsemihypergroups of H.

Proposition 4.5. *Let (F,A) and (G,B) be two soft ternary semihypergroups over H such that $A \cap B = \emptyset$. Then $(F,A)\bigcup_E(G,B)$ is a soft ternary semihypergroup over H.*

Proof. By definition, $(K,C) = (F,A)\bigcup_E(G,B)$, where $C = A\cup B$ and $A\cap B = \emptyset$. Then for all $c \in C$, either $c \in A\backslash B$ or $c \in B\backslash A$. If $c \in A\backslash B$, then $K(c) = F(c)$ and if $c \in B\backslash A$, then $K(c) = G(c)$. So in both cases $K(c)$ is a ternary subsemihypergroup of H. Therefore, (K,C) is a soft ternary semihypergroup over H. ∎

Proposition 4.6. *Let (F,A) and (G,B) be two soft ternary semihypergroups over H such that $A \cap B \neq \emptyset$. Then $(F,A)\bigcup_E(G,B)$ is a soft ternary semihypergroup over H.*

Proof. Let us assume that $A \cap B \neq \emptyset$. Let $(K,C) = (F,A)\bigcup_E(G,B)$ where

$$K(c) = \begin{cases} F(c) & \text{if } c \in A\backslash B, \\ G(c) & \text{if } c \in B\backslash A, \\ F(c)\cup G(c) & \text{if } c \in A\cap B. \end{cases}$$

To show that (K,C) is a soft ternary semihypergroup over H, we have to show that

$$\widehat{f}((K,C),(K,C),(K,C)) \subseteq (K,C).$$

We have

$$\widehat{f}((K,C),(K,C),(K,C)) = (P,C).$$

and

$$P(c) = f(K(c),K(c),K(c)) \text{ for all } c \in C.$$

For $c \in C$, we have

$$\begin{array}{rcl} P(c) & = & f(K(c),K(c),K(c)) \\ & = & f((F,A) \cup_E (G,B),(F,A) \cup_E (G,B),(F,A) \cup_E (G,B)) \\ & \subseteq & \begin{cases} F(c) & \text{if } c \in A \backslash B, \\ G(c) & \text{if } c \in B \backslash A, \\ F(c) \cap G(c) & \text{if } c \in A \cap B. \end{cases} \end{array}$$

Then $P(c) \subseteq K(c)$ for all $c \in C$. Therefore, $(F,A) \cup_E (G,B)$ is a soft ternary semihypergroup over H. ∎

Proposition 4.7. *Let (F,A) and (G,B) be two soft ternary semihypergroups over H such that $A \cap B \neq \emptyset$. Then $(F,A) \cap_R (G,B)$ is a soft ternary semihypergroup over H.*

Proof. Let us assume that $A \cap B \neq \emptyset$. Let $(K,C) = (F,A) \cap_R (G,B)$ where $C = A \cap B \neq \emptyset$ and $K(c) = F(c) \cap G(c)$ for all $c \in C$. To show that (K,C) is a soft ternary semihypergroup over H, we have to show that

$$\widehat{f}((K,C),(K,C),(K,C)) \subseteq (K,C).$$

We have

$$\widehat{f}((K,C),(K,C),(K,C)) = (P,C).$$

and

$$P(c) = f(K(c),K(c),K(c)) \text{ for all } c \in C.$$

For $c \in C$, we have

$$\begin{array}{rcl} P(c) & = & f(K(c),K(c),K(c)) \\ & = & f(F(c) \cap G(c), F(c) \cap G(c), F(c) \cap G(c)) \\ & \subseteq & F(c) \cap G(c) \\ & = & K(c). \end{array}$$

Then $P \subseteq K$. Therefore, $(F,A) \cap_R (G,B)$ is a soft ternary semihypergroup over H. ∎

Proposition 4.8. *Let (F,A) and (G,B) be two soft ternary semihypergroups over H. Then $(F,A) \wedge (G,B)$ is a soft ternary semihypergroup over H.*

Proof. Let $(F, A) \wedge (G, B) = (K, A \times B) = (K, C)$ where $K(a, b) = F(a) \cap G(b)$ for all $(a, b) \in A \times B$. We have to show that $(K, A \times B)$ is a soft ternary semihypergroup over H. We have

$$(P, C) = \widehat{f}((K, A \times B), (K, A \times B), (K, A \times B))$$

where $C = A \times B$ and $P(a, b) = f(K(a, b), K(a, b), K(a, b))$ for all $(a, b) \in C$. For $(a, b) \in C$, we have

$$\begin{aligned} P(a, b) &= f(K(a, b), K(a, b), K(a, b)) \\ &= f(F(a) \cap G(b), F(a) \cap G(b), F(a) \cap G(b)) \\ &\subseteq F(a) \cap G(b) \\ &= K(a, b). \end{aligned}$$

Then $P \subseteq K$. Therefore, $(F, A) \wedge (G, B)$ is a soft ternary semihypergroup over H. ∎

Definition 4.9. Let $(F_1, A_1), (F_2, A_2), (F_3, A_3)$ be three soft sets over a ternary semihypergroup (H, f). Define

$$f^*((F_1, A_1), (F_2, A_2), (F_3, A_3)) = (K, A_1 \times A_2 \times A_3)$$

be a soft set where $K(a_1, a_2, a_3) = f((F_1(a_1), F_2(a_2), F_3(a_3))$.

Proposition 4.10. *Let (H, f) be a commutative ternary semihypergroup. If (F_1, A_1), (F_2, A_2), (F_3, A_3) are soft ternary semihypergroup over H, then $f^*((F_1, A_1), (F_2, A_2), (F_3, A_3))$ is a soft ternary semihypergroup over H.*

Proof. Let $f^*((F_1, A_1), (F_2, A_2), (F_3, A_3)) = (K, A_1 \times A_2 \times A_3)$ where

$$K(a_1, a_2, a_3) = f(F_1(a_1), F_2(a_2), F_3(a_3))$$

for all $(a_1, a_2, a_3) \in A_1 \times A_2 \times A_3$. We have

$$(P, C) = \widehat{f}((K, A_1 \times A_2 \times A_3), ..., (K, A_1 \times A_2 \times A_3))$$

where $C = A_1 \times A_2 \times A_3$ and

$$P(a_1, a_2, a_3) = f(K(a_1, a_2, a_3), K(a_1, a_2, a_3), K(a_1, a_2, a_3))$$

for all $(a_1, a_2, a_3) \in C$. For $(a_1, a_2, a_3) \in C$, since H is commutative, we have

$$\begin{aligned} P(a_1, a_2, a_3) &= f(K(a_1, a_2, a_3), K(a_1, a_2, a_3), K(a_1, a_2, a_3)) \\ &= f(f(F_1(a_1), F_2(a_2), F_3(a_3)), f(F_1(a_1), F_2(a_2), F_3(a_3))) \\ &\subseteq f(F_1(a_1), F_2(a_2), F_3(a_3)) \\ &= K(a_1, a_2, a_3). \end{aligned}$$

Then $P \subseteq K$. This completes the proof. ∎

5. Soft ternary subsemihypergroups and soft hyperideals

Definition 5.1. Let (F, A) and (G, B) be two soft sets over H such that (F, A) is a ternary semihypergroup and $(G, B) \subseteq (F, A)$. Then (G, B) is called a *soft ternary subsemihypergroup (hyperideal)* of (F, A) if $G(b)$ is a ternary subsemihypergroup (hyperideal) of $F(b)$ for all $b \in B$.

Proposition 5.2. *Let (F, A) be a soft ternary semihypergroup over H and let $\{(K_i, A_i)|i \in I\}$ be a non-empty family of soft ternary subsemihypergoups of (F, A). Then the following hold:*

1. $\cap_{R_{i\in I}}(K_i, A_i)$ *is a soft ternary subsemihypergroup of* (F, A).
2. $\wedge_{i\in I}(K_i, A_i)$ *is a soft ternary subsemihypergroup of* $\wedge_{i\in I}(F, A)$.
3. $\cup_{E_{i\in I}}(K_i, A_i)$ *is a soft ternary subsemihypergroup of* (F, A) *if the set* $\{A_i|i \in I\}$ *is pairwise disjoint.*

Proof. Straightforward. ∎

Definition 5.3. A soft set (F, A) over a ternary semihypergroup H is called a *soft left (right, lateral) hyperideal* over H, if $\widehat{f}((H, E), (H, E), (F, A)) \subseteq (F, A)$. A soft set (F, A) over H is called a *soft hyperideal* over H, if it is a soft left, soft right and a soft lateral hyperideal over H.

Proposition 5.4. *A soft set (F, A) over H is a soft left (right, lateral) hyperideal over H if and only if for all $a \in A, F(a) \neq \emptyset$ is a left (right, lateral) hyperideal of H.*

Proof. Let us suppose that (F, A) is a soft left hyperideal over H. We show that $F(a) \neq \emptyset$ is a left hyperideal of H. By the definition we have

$$\widehat{f}((H, E), (H, E), (F, A)) = (K, E \cap E \cap A) = (K, A).$$

where $f(H(a), H(a), F(a)) = K(a)$, for all $a \in A$. That is, $f(H, H, F(a)) = K(a)$. Since $\widehat{f}((H, E), (H, E), (F, A)) \subseteq (F, A)$, where $(K, A) \subseteq (F, A)$. Thus we have, $(K(a) \subseteq F(a)$ for all $a \in A$. Therefore, $f(H, H, F(a)) \subseteq F(a)$. This shows that $F(a)$ is a left hyperideal of H.

Conversely, let us assume that $F(a) \neq \emptyset$ is a left hyperideal of H. We will show that (F, A) is a soft left hyperideal over H. By the definition

$$\widehat{f}((H, E), (H, E), (F, A)) = (K, E \cap E \cap A) = (K, A)$$

where $f(H(a), H(a), F(a)) = K(a)$, for all $a \in A$. That is, $f(H, H, F(a)) = K(a)$. But $f(H, H, F(a)) \subseteq F(a)$. This implies $K(a) \subseteq F(a)$ and so $(K, A) \subseteq (F, A)$. Thus $\widehat{f}((H, E), (H, E), (F, A)) \subseteq (F, A)$. Hence, (F, A) is a soft left hyperideal over H. ∎

The soft hyperideal over H defined above is different from the soft hyperideal of a soft ternary semihypergroup. The following example shows this.

Example 5.5. Let $H = \{a, b, c, d, e, f\}$ and $f(x, y, z) = (x \circ y) \circ z$ for all $x, y, z \in H$ with the hyperoperation $\circ$ given by the following table:

$\circ$	0	a	b	c	d	e	f
0	0	0	0	0	0	0	0
a	0	a	$\{a, b\}$	c	$\{c, d\}$	e	$\{e, f\}$
b	0	b	b	d	d	f	f
c	0	c	$\{c, d\}$	c	$\{c, d\}$	c	$\{c, d\}$
d	0	d	d	d	d	d	d
e	0	e	$\{e, f\}$	c	$\{c, d\}$	e	$\{e, f\}$
f	0	f	f	d	d	f	f

Then (H, f) is a ternary semihypergroup. Let we consider the soft set (F, H), where $F : H \to \mathcal{P}(H)$ is defined as $F(0) = \{0\}, F(a) = \{0, d\}, F(b) = \{0, b, d, f\}$, $F(c) = \{0, c, d\}, F(d) = \{0, d, f\}, F(e) = \{0, e, f\}, F(f) = \{0, a, b\}$. It is clear that (F, H) is a soft ternary semihypergroup over H. Let we consider now the soft set $(G, \{b\})$ in which $G : \{b\} \to \mathcal{P}(H)$ is defined as $G(b) = \{0, d\}$. As $\{b\} \subseteq H$ and $G(b)$ is a hyperideal of $F(b)$, therefore $(G, \{b\})$ is a soft hyperideal of (F, H). But $G(b) = \{0, d\}$ is not a hyperideal over H, so $(G, \{b\})$ is not a soft hyperideal over H.

Proposition 5.6. *Let (F, A) and (G, B) be any two soft hyperideals over a ternary semihypergroup H, with $A \cap B \neq \emptyset$. Then $(F, A) \cap_R (G, B)$ is a soft hyperideal over H contained in both (F, A) and (G, B).*

Proof. Straightforward. ∎

Proposition 5.7. *Let (F, A) and (G, B) be any two soft hyperideals over a ternary semihypergroup H. Then $(F, A) \cup_E (G, B)$ is a soft hyperideal over H containing both (F, A) and (G, B).*

Proof. By the definition, $(K, C) = (F, A) \cup_E (G, B)$, where $C = A \cup B$ for all $c \in C = A \cup B$, either $c \in A \backslash B$ or $c \in B \backslash A$ or $c \in A \cap B$. If $c \in A \backslash B$, then $K(c) = F(c)$ if $c \in B \backslash A$, then $K(c) = G(c)$, and if $c \in A \cap B$, then $K(c) = F(c) \cup G(c)$, in all the cases $K(c)$ is a hyperideal of H. Hence, (K, C) is soft hyperideal over H. Since $A \subseteq A \cup B, B \subseteq A \cup B$ and $F(c) \subseteq K(c), G(c) \subseteq K(c)$ for all $c \in C$. Therefore, by the definition of soft subsets $(F, A) \subseteq (K, C)$ and $(G, B) \subseteq (K, C)$. ∎

Proposition 5.8. *Let (F, A) and (G, B) be any two soft hyperideals over a ternary semihypergroup H. Then $(F, A) \wedge (G, B)$ is a soft hyperideal over H.*

Proof. Straightforward. ∎

Proposition 5.9. *Let (F, A) and (G, B) be any two soft hyperideals over a ternary semihypergroup H. Then $(F, A) \vee (G, B)$ is a soft hyperideal over H.*

Proof. Straightforward. ∎

Proposition 5.10. *Let (F, A) be a soft ternary semihypergroup over H and let $\{(K_i, A_i)|i \in I\}$ be a non-empty family of soft hyperideals of (F, A). Then the following hold:*

1. $\cap_{R_{i\in I}}(K_i, A_i)$ *is a soft hyperideal of* (F, A).
2. $\wedge_{i\in I}(K_i, A_i)$ *is a soft hyperideal of* $\wedge_{i\in I}(F, A)$.
3. $\cup_{R_{i\in I}}(K_i, A_i)$ *is a soft hyperideal of* (F, A).
4. $\vee_{i\in I}(K_i, A_i)$ *is a soft hyperideal of* $\vee_{i\in I}(F, A)$.

Proof. Straightforward. ∎

6. Soft quasi-hyperideals over ternary semihypergroups

Definition 6.1. A soft set (F, A) over a ternary semihypergroup H is called a *soft quasi-hyperideal* over H if

1. $\widehat{f}((F, A), (H, E), (H, E)) \cap_R \widehat{f}((H, E), (F, A), (H, E))$
   $\cap_R \widehat{f}((H, E), (H, E), (F, A)) \subseteq (F, A)$.
2. $\widehat{f}((F, A), (H, E), (H, E)) \cap_R \widehat{f}((H, E), (H, E), (F, A), (H, E), (H, E))$
   $\cap_R \widehat{f}((H, E), (H, E), (F, A)) \subseteq (F, A)$.

where (H, E) is the absolute soft set over H.

Proposition 6.2. *A soft set (F, A) over a ternary semihypergroup H is a soft quasi-hyperideal over H if and only if for all $a \in A, F(a) \neq \emptyset$ is a quasi-hyperideal of H.*

Proof. Let us suppose that a soft set (F, A) over H is a soft quasi-hyperideal over H. We show that $F(a)$ is a quasi-hyperideal of H. By the definition of restricted product,

(1) $\quad \widehat{f}((F, A), (H, E), (H, E)) = (G, A \cap E \cap E) = (G, A)$

(2) $\quad \widehat{f}((H, E), (F, A), (H, E)) = (S, E \cap A \cap E) = (S, A)$

(3) $\quad \widehat{f}((H, E), (H, E), (F, A)) = (I, E \cap E \cap A) = (I, A)$

(4) $\quad \widehat{f}((H, E), (H, E), (F, A), (H, E), (H, E)) = (J, E \cap E \cap A \cap E \cap E) = (J, A)$

Equations (1), (2), (3) imply that

(5) $\quad \widehat{f}((F, A), (H, E), (H, E)) \cap_R \widehat{f}((H, E), (F, A), (H, E))$
$\cap_R \widehat{f}((H, E), (H, E), (F, A)) = (G, A) \cap_R (S, A) \cap_R (I, A)$

(6) $\quad \widehat{f}((F, A), (H, E), (H, E)) \cap_R \widehat{f}((H, E), (F, A), (H, E))$
$\cap_R \widehat{f}((H, E), (H, E), (F, A)) = (K, A)$

But (F,A) is a soft quasi-hyperideal over H. Thus $\widehat{f}((F,A),(H,E),(H,E)) \cap_R \widehat{f}((H,E),(F,A),(H,E)) \cap_R \widehat{f}((H,E),(H,E),(F,A)) \subseteq (F,A)$. From equation (6), $(K,A) \subseteq (F,A)$, that is, $K(a) \subseteq F(a)$ for all $a \in A$. Again, equation (6) implies that, $\forall a \in A$,

$$f(F(a),H(a),H(a)) \cap f(H(a),F(a),H(a)) \cap f(H(a),H(a),F(a)) = K(a).$$

This implies that, $\forall a \in A$,

$$f(F(a),H(a),H(a)) \cap f(H(a),F(a),H(a)) \cap f(H(a),H(a),F(a)) \subseteq F(a). \tag{7}$$

Similarly, from equations (1), (3), (4), we have

$$\begin{aligned}&\widehat{f}((F,A),(H,E),(H,E)) \cap_R \widehat{f}((H,E),(H,E),(F,A),(H,E),(H,E))\\ &\cap_R \widehat{f}((H,E),(H,E),(F,A)) = (P,A).\end{aligned} \tag{8}$$

Since (F,A) is a soft quasi-hyperideal over H, we have

$$\begin{aligned}&\widehat{f}((F,A),(H,E),(H,E)) \cap_R \widehat{f}((H,E),(H,E),(F,A),(H,E),(H,E))\\ &\cap_R \widehat{f}((H,E),(H,E),(F,A)) \subseteq (F,A).\end{aligned}$$

Therefore, $(P,A) \subseteq (F,A)$, that is, for all $a \in A$, $P(a) \subseteq F(a)$. From equation (8), we have for all $a \in A$,

$$f(F(a),H(a),H(a)) \cap f(H(a),H(a),F(a),H(a),H(a)) \cap f(H(a),H(a),F(a)) = P(a).$$

This implies that

$$\begin{aligned}&f(F(a),H(a),H(a)) \cap f(H(a),H(a),F(a),H(a),H(a)) \cap f(H(a),H(a),F(a))\\ &\subseteq F(a).\end{aligned} \tag{9}$$

From equations (7) and (9), it is clear that $F(a)$ is a quasi-hyperideal of H.

Conversely, let $F(a) \neq \emptyset$ be a quasi-hyperideal of H for all $a \in A$. We will show that (F,A) is a soft quasi-hyperideal over H. From equations (1), (2), (3) we have

$$\begin{aligned}&\widehat{f}((F,A),(H,E),(H,E)) \cap_R \widehat{f}((H,E),(F,A),(H,E))\\ &\cap_R \widehat{f}((H,E),(H,E),(F,A)) = (G,A) \cap_R (S,A) \cap_R (I,A) = (K,A).\end{aligned}$$

By the definition, $\forall a \in A$, we have

$$f(F(a),H(a),H(a)) \cap f(H(a),F(a),H(a)) \cap f(H(a),H(a),F(a)) = K(a).$$

But $F(a)$ is a quasi-hyperideal of H. Therefore,

$$K(a) = f(F(a),H(a),H(a)) \cap f(H(a),F(a),H(a)) \cap f(H(a),H(a),F(a)) \subseteq F(a),$$

and so, $(K,A) \subseteq (F,A)$. Thus

$$\begin{aligned}&\widehat{f}((F,A),(H,E),(H,E)) \cap_R \widehat{f}((H,E),(F,A),(H,E))\\ &\cap_R \widehat{f}((H,E),(H,E),(F,A)) \subseteq (F,A).\end{aligned} \tag{10}$$

From equations (1), (3), and (4) we have for, all $a \in A$,

$$\widehat{f}((F,A),(H,E),(H,E)) \cap_R \widehat{f}((H,E),(H,E),(F,A),(H,E),(H,E))$$
$$\cap_R \widehat{f}((H,E),(H,E),(F,A)) = (G,A) \cap_R (I,A) \cap_R (J,A) = (P,A)$$

and also

$$f(F(a),H(a),H(a)) \cap f(H(a),H(a),F(a),H(a),H(a))$$
$$\cap f(H(a),H(a),F(a)) = P(a).$$

Since $F(a)$ is a quasi-hyperideal of H, so we have

$$P(a) = f(F(a),H(a),H(a)) \cap f(H(a),H(a),F(a),H(a),H(a))$$
$$\cap f(H(a),H(a),F(a)) \subseteq F(a)$$

and thus $(P,A) \subseteq (F,A)$. Hence we have

$$\widehat{f}((F,A),(H,E),(H,E)) \cap_R \widehat{f}((H,E),(H,E),(F,A),(H,E),(H,E))$$
$$\cap_R \widehat{f}((H,E),(H,E),(F,A)) \subseteq (F,A). \tag{11}$$

From equations (10) and (11), (F,A) is a soft quasi-hyperideal over H. ∎

Proposition 6.3. *Let (R,A), (L,B) and (M,C) be soft right, soft left and soft lateral hyperideals over H, respectively. Then $(R,A) \cap_R (M,C) \cap_R (L,B)$ is a soft quasi-hyperideal over H.*

Proof. It is straightforward. ∎

Proposition 6.4. *Let (R,A), (L,B) and (M,C) be soft right, soft left and soft lateral hyperideals over H, respectively, such that $A \cap B \cap C = \emptyset$. Then $(R,A) \cap_E (M,C) \cap_E (L,B)$ is a soft quasi-hyperideal over H.*

Proof. By definition, $(S,D) = (R,A) \cap_E (M,C) \cap_E (L,B)$, where $D = A \cup B \cup C$, $A \cap B \cap C = \emptyset$, and

$$S(d) = \begin{cases} R(d) & \text{if } d \in A \backslash B \cap C \\ M(d) & \text{if } d \in C \backslash A \cap B, \\ L(d) & \text{if } d \in B \backslash A \cap C. \end{cases}$$

for any $d \in D$. In each case, $S(d)$ is a quasi-hyperideal of H. Since every left, right and lateral hyperideal of a ternary semihypergroup H is a quasi-hyperideal of H, thus, by definition, $(S,D) = R,A) \cap_E (M,C) \cap_E (L,B)$ is a soft quasi-hyperideal over H. ∎

Proposition 6.5. *Every soft left (right, lateral) hyperideal over a ternary semihypergroup H is a soft quasi-hyperideal over H.*

Proof. Let (L,A) be a soft left hyperideal over H. Then $L(a)$ is a left hyperideal of H. Since each left hyperideal of H is a quasi-hyperideal of H, therefore $L(a)$ is a quasi-hyperideal of H. Hence (L,A) is a soft quasi-hyperideal over H. ∎

The converse of the above proposition is not true in general as the following example shows.

Example 6.6. Let we consider the ternary semihypergroup (H, f) of the Example 5.5. Let we consider $A = \{b\}$ and $G : \{b\} \to \mathcal{P}(H)$ defined as $G(b) = \{0, b, d, f\}$. Then it can be easily verified that (G, B) is a soft quasi-hyperideal over H. But it is not a soft left and a soft lateral hyperideal over H.

Proposition 6.7. *Every soft left (right, lateral) hyperideal over a ternary semihypergroup H is a soft ternary semihypergroup over H.*

Proof. It is straightforward. ∎

Proposition 6.8. *Every soft quasi-hyperideal is a soft ternary semihypergroup over H.*

Proof. It is straightforward. ∎

Proposition 6.9. *Let (R, A), (L, B) and (M, C) be soft right, soft left and soft lateral hyperideals over H, respectively. Then $(R, A) \wedge (M, C) \wedge (L, B)$ is a soft quasi-hyperideal over H.*

Proof. By definition, $(S, D) = (R, A) \wedge (M, C) \wedge (L, B)$, where $D = A \times C \times B$, and for any $(a, c, b) \in A \times C \times B, S(a, c, b) = R(a) \cap M(c) \cap L(b)$ is a quasi-hyperideal of H. Since the intersection of a left, right and a lateral hyperideal is a quasi-hyperideal of H, then $(R, A) \wedge (M, C) \wedge (L, B)$ is a soft quasi-hyperideal over H. ∎

Proposition 6.10. *Let (F, A) and (G, B) be two soft quasi-hyperideals over a ternary semihypergroup H. Then the following hold:*

1. *$(F, A) \cap_R (G, B)$ is a soft quasi-hyperideal over H.*
2. *$(F, A) \cap_E (G, B)$ is a soft quasi-hyperideal over H.*
3. *$(F, A) \wedge (G, B)$ is a soft quasi-hyperideal over H.*
4. *$(F, A) \cup_E (G, B)$ is a soft quasi-hyperideal over H, whenever $A \cap B = \emptyset$.*

Proof. It is straightforward. ∎

Proposition 6.11. *Let (F, A) be a soft quasi-hyperideal and (G, B) a soft ternary semihypergroup H. Then $(F, A) \cap_R (G, B)$ is a soft quasi-hyperideal of (G, B).*

Proof. By definition, $(S, C) = (F, A) \cap_R (G, B)$, where $C = A \cap B \neq \emptyset$ and $S(c) = F(c) \cap G(c)$ for all $c \in C$, since $S(c) \subseteq F(c)$ and $S(c) \subseteq G(c)$. We show that $S(c)$ is a quasi-hyperideal of $G(c)$. Since $S(c) \subseteq G(c)$,

$$\begin{aligned}
&f(S(c), G(c), G(c)) \cap f(G(c), S(c), G(c), G(c)) \cap f(G(c), G(c), S(c)) \\
&\subseteq f(G(c), G(c), G(c)) \cap f(G(c), G(c), G(c), G(c)) \cap f(G(c), G(c), G(c)) \\
&\subseteq f(G(c), G(c), G(c)) \subseteq G(c)
\end{aligned}$$

because $G(c)$ is a ternary subsemihypergroup of H. This implies that

(1) $$f(S(c),G(c),G(c))\cap f(G(c),S(c),G(c),G(c))\cap$$
$$f(G(c),G(c),S(c))\subseteq G(c).$$

Also $S(c)\subseteq F(c)$. So

$$\begin{aligned}&f(S(c),G(c),G(c))\cap f(G(c),S(c),G(c),G(c))\cap f(G(c),G(c),S(c))\\&\subseteq f(F(c),G(c),G(c))\cap f(G(c),F(c),G(c),G(c))\cap f(G(c),G(c),F(c))\\&\subseteq f(F(c),H(c),H(c))\\&\cap f(H(c),F(c),H(c),H(c))\cap f(H(c),H(c),F(c))\subseteq F(c)\end{aligned}$$

because $F(c)$ is a quasi-hyperideal of H. Thus

(2) $$f(S(c),G(c),G(c))\cap f(G(c),S(c),G(c),G(c))\cap$$
$$f(G(c),G(c),S(c))\subseteq F(c).$$

From equations (1) and (2), we have

(3) $$f(S(c),G(c),G(c))\cap f(G(c),S(c),G(c),G(c))\cap f(G(c),G(c),S(c))$$
$$\subseteq F(c)\cap G(c)=S(c).$$

Similarly, we can show that

(4) $$f(S(c),G(c),G(c))\cap f(G(c),G(c),S(c),G(c),G(c))$$
$$\cap f(G(c),G(c),S(c))\subseteq S(c).$$

From equation (3) and (4), $S(c)$ is a quasi-hyperideal of $G(c)$. Thus $(F,A)\cap_R (G,B)$ is a soft quasi-hyperideal of (G,B). ∎

7. Soft bi-hyperideals over ternary semihypergroups

Definition 7.1. A soft set (F,A) over a ternary semihypergroup H is called a *soft bi-hyperidea* over H if

1. (F,A) is a soft ternary semihypergroup over H.
2. $\widehat{f}((F,A),(H,E),(F,A),(H,E),(F,A))\subseteq(F,A)$ where (H,E) is the absolute soft set over H.

Proposition 7.2. *A soft set (F,A) over a ternary semihypergroup H is a soft bi-hyperideal over H if and only if for all $a\in A$, $F(a)\neq\emptyset$ is a bi-hyperideal of H.*

Proof. Let (F,A) be a soft bi-hyperideal over a ternary semihypergroup H. Then by definition, (F,A) is a soft ternary semihypergroup over H. By Proposition 4.3, for any $a\in A, F(a)\neq\emptyset$ is a ternary subsemihypergroup of H. Moreover, since (F,A) is a soft bi-hyperideal over H, we have
$\widehat{f}((F,A),(H,E),(F,A),(H,E),(F,A))\subseteq(F,A)$ where (H,E) is the absolute soft set over H. It follows that $\widehat{f}(F(a),H,F(a),H,F(a))\subseteq F(a)$, which shows that $F(a)$ is a bi-hyperideal of H.

Conversely, let us suppose that (F,A) is a soft set over H such that for all $a\in A, F(a)$ is a bi-hyperideal of H, whenever $F(a)\neq\emptyset$. Then it is clear that each $F(a) \neq \emptyset$ is a ternary subsemihypergroup of H. Hence, by Proposition 4.3, (F, A) is a soft ternary semihypergroup over H. Furthermore, since $F(a) \neq \emptyset$ is a bi-hyperideal of H, then for all $a \in A$, $\widehat{f}(F(a), H, F(a), H, F(a)) \subseteq F(a)$. Hence, we conclude that $\widehat{f}((F, A), (H, E), (F, A), (H, E), (F, A)) \subseteq (F, A)$. This shows that (F, A) is a soft bi-hyperideal over H. ∎

Proposition 7.3. *Every soft quasi-hyperideal over a ternary semihypergroup H is a soft bi-hyperideal over H.*

Proof. It is straightforward. ∎

Proposition 7.4. *Let (F, A) be a soft bi-hyperideal over H and (G, B) a soft ternary semihypergroup over H. Then $(F, A) \cap_R (G, B)$ is a soft bi-hyperideal of (G, B).*

Proof. By the definition, $(S, C) = (F, A) \cap_R (G, B)$ where $C = A \cap B \neq \emptyset$, and S is defined by $S(c) = F(c) \cap G(c)$ for all $c \in C$. We will show that (S, C) is a soft bi-hyperideal of (G, B). We have

$$\begin{aligned}
&\widehat{f}((S, C), (S, C), (S, C)) \\
&= \widehat{f}(((F, A) \cap_R (G, B)), ((F, A) \cap_R (G, B)), ((F, A) \cap_R (G, B))) \\
&\subseteq \widehat{f}((F, A), (F, A), (F, A)) \subseteq (F, A),
\end{aligned}$$

because (F, A) is a soft bi-hyperideal over H. This implies that

$$\widehat{f}((S, C), (S, C), (S, C)) \subseteq (F, A) \tag{1}$$

Also

$$\begin{aligned}
&\widehat{f}((S, C), (S, C), (S, c)) \\
&= \widehat{f}(((F, A) \cap_R (G, B)), ((F, A) \cap_R (G, B)), ((F, A) \cap_R (G, B))) \\
&\subseteq \widehat{f}((G, B), (G, B), (G, B)) \subseteq (G, B),
\end{aligned}$$

because (G, B) is a soft ternary semihypergroup over H. This implies that

$$\widehat{f}((S, C), (S, C), (S, C)) \subseteq (G, B). \tag{2}$$

From equations (1) and (2), we have

$$\widehat{f}((S, C), (S, C), (S, C)) \subseteq (F, A) \cap_R (G, B) = (S, C).$$

This implies that (S, C) is a soft ternary semihypergroup over H. Also

$$\begin{aligned}
&\widehat{f}((S, C), (G, B), (S, C), (S, C)) \\
&= \widehat{f}(((F, A) \cap_R (G, B)), (G, B), ((F, A) \cap_R (G, B)), (G, B), ((F, A) \cap_R (G, B))) \\
&\subseteq \widehat{f}((G, B), (G, B), (G, B)) \\
&\subseteq \widehat{f}((G, B), (G, B), (G, B)) \subseteq (G, B),
\end{aligned}$$

and so

$$\widehat{f}((S, C), (G, B), (S, C), (S, C)) \subseteq (G, B). \tag{3}$$

Again

$$\begin{aligned}
&\widehat{f}((S,C),(G,B),(S,C),(S,C))\\
&=\widehat{f}(((F,A)\cap_R (G,B)),(G,B),((F,A)\\
&\cap_R(G,B)),(G,B),((F,A)\cap_R (G,B)))\\
&\subseteq \widehat{f}((F,A),(H,E),(F,A))\\
&\subseteq (F,A),
\end{aligned}$$

because (F,A) is a soft bi-hyperideal over H. This implies that

$$\widehat{f}((S,C),(G,B),(S,C),(S,C)) \subseteq (F,A). \tag{4}$$

From equations (3) and (4), we have

$$\widehat{f}((S,C),(G,B),(S,C),(S,C)) \subseteq (F,A)\cap_R (G,B)=(S,C).$$

Hence (S,C) is a soft bi-hyperideal of (G,B). ∎

Definition 7.5. A soft hyperideal (F,A) over a ternary semihypergroup H is soft idempotent if $\widehat{f}((F,A),(F,A),(F,A))=(F,A)$.

Theorem 7.6. *Let (F,A) be a soft bi-hyperideal over H and (G,B) a soft bi-hyperideal of (F,A) such that $\widehat{f}((G,B),(G,B),(G,B))=(G,B)$. Then (G,B) is a soft bi-hyperideal over H.*

Proof. By the condition, we have $\widehat{f}((G,B),(G,B),(G,B))\subseteq(G,B)$. This implies that (G,B) is a soft ternary semihypergroup over H. Since $(G,B)\subseteq(F,A)$ and $\widehat{f}((G,B),(G,B),(G,B))=(G,B)$, we have

$$\begin{aligned}
&\widehat{f}((G,B),(H,E),(G,B),(H,E),(G,B))\\
&=\widehat{f}(\widehat{f}((G,B),(G,B),(G,B)),(H,E),(G,B),(H,E),\widehat{f}((G,B),(G,B),...,(G,B)))\\
&=\widehat{f}((G,B),(G,B),\widehat{f}((G,B)),(H,E),(G,B),(H,E),(G,B)),(G,B),(G,B))\\
&\subseteq \widehat{f}((G,B),(G,B),\widehat{f}((F,A)),(H,E),(F,A),(H,E),(F,A)),(G,B),(G,B))\\
&\subseteq \widehat{f}((G,B),(G,B),(F,A),(G,B),(G,B)) \text{ since } (F,A) \text{ is a soft bi-hyperideal of } H\\
&=\widehat{f}((G,B),(G,B),...,(F,A),(G,B),\widehat{f}((G,B),(G,B),(G,B)))\\
&=\widehat{f}((G,B),\widehat{f}((G,B),(F,A),(G,B),(G,B),(G,B),(G,B)),(G,B))\\
&\subseteq \widehat{f}((G,B),\widehat{f}((G,B),(F,A),(G,B),(G,B)),(G,B))\\
&\subseteq \widehat{f}((G,B),(G,B),(G,B))=(G,B)
\end{aligned}$$

because (G,B) is a soft bi-hyperideal of (F,A). This implies that

$$\widehat{f}((G,B),(H,E),(G,B),(H,E),(G,B))\subseteq(G,B).$$

Hence, (G,B) is a soft bi-hyperideal over H. ∎

Proposition 7.7. *Let (F, A) be a non-empty soft subset over a ternary semihypergroup H. If $(G_i, B_i), i = 1, 2, 3$ are soft left, lateral and right hyperideals respectively, over H such that*

$$\widehat{f}((G_1, B_1), (G_2, B_2), (G_3, B_3)) \subseteq (F, A) \subseteq (G_1, B_1) \cap_R (G_2, B_2) \cap_R (G_3, B_3),$$

then (F, A) is a soft bi-hyperideal over H.

Proof. By definition,

$\widehat{f}((F, A), (F, A), (F, A)) \subseteq \widehat{f}((G_1, B_1) \cap_R (G_2, B_2) \cap_R (G_3, B_3), (G_1, B_1)$
$\cap_R (G_2, B_2) \cap_R (G_3, B_3), (G_1, B_1) \cap_R (G_2, B_2) \cap_R (G_3, B_3))$
$\subseteq \widehat{f}((G_1, B_1), (G_2, B_2), (G_3, B_3)) \subseteq (F, A)$.

This implies that $\widehat{f}((F, A), (F, A), (F, A)) \subseteq (F, A)$. Thus (F, A) is a soft ternary semihypergroup over H. Let us consider again

$\widehat{f}((F, A), (H, E), (F, A), (H, E), (F, A)) \subseteq \widehat{f}((G_1, B_1) \cap_R (G_2, B_2) \cap_R (G_3, B_3),$
$(H, E), (G_1, B_1) \cap_R (G_2, B_2) \cap_R (G_3, B_3), (H, E), (G_1, B_1) \cap_R (G_2, B_2) \cap_R (G_3, B_3))$
$\subseteq \widehat{f}((G_1, B_1), (H, E), (G_2, B_2), (H, E), (G_3, B_3))$
$\subseteq \widehat{f}((G_1, B_1), (G_2, B_2), (G_3, B_3)) \subseteq (F, A)$

because (G_2, B_2) is a soft lateral hyperideal over H. This implies that

$$\widehat{f}((F, A), (H, E), (F, A), (H, E), (F, A)) \subseteq (F, A).$$

Hence, (F, A) is a soft bi-hyperideal over H. ∎

Proposition 7.8. *Let (F, A) and (G, B) be two non-empty soft sets over a ternary semihypergroup H. Then the soft set $(S, C) = \widehat{f}((F, A), (H, E), (G, B))$ is a soft bi-hyperideal over H.*

Proof. By definition, we have

$\widehat{f}((S, C), (S, C), (S, C))$
$= \widehat{f}(\widehat{f}((F, A), (H, E), (G, B)), \widehat{f}((F, A), (H, E), (G, B)), \widehat{f}((F, A), (H, E), (G, B)))$
$= \widehat{f}((F, A), \widehat{f}((H, E), (G, B), (F, A)), \widehat{f}((H, E), (G, B), (F, A)), (H, E), (G, B))$
$\subseteq \widehat{f}((F, A), \widehat{f}((H, E)), (H, E), (H, E)), \widehat{f}((H, E)), (H, E), (H, E)), (H, E), (G, B))$
$\subseteq \widehat{f}((F, A), \widehat{f}((H, E)), (H, E), (H, E)), (G, B)) \subseteq \widehat{f}((F, A)), (H, E), (G, B))=(S, C)$.

This implies that $\widehat{f}((S, C)), (S, C), (S, C)) \subseteq (S, C)$. Thus, (S, C) is a soft ternary semihypergroup over H. Also, we have

$\widehat{f}((S, C), (H, E), (S, C), (H, E), (S, C))$
$= \widehat{f}(\widehat{f}((F, A), (H, E), (G, B)), (H, E), \widehat{f}((F, A), (H, E), (G, B)), (H, E),$
$\widehat{f}((F, A), (H, E), (G, B)))$
$= \widehat{f}((F, A), \widehat{f}((H, E), (G, B), (H, E)), \widehat{f}((F, A), (H, E), (G, B)),$
$\widehat{f}((H, E), (F, A), (H, E)), (G, B))$

$$\subseteq \widehat{f}((F,A), \widehat{f}((H,E)), (H,E), (H,E)), \widehat{f}((H,E)), (H,E), (H,E)),$$
$$\widehat{f}((H,E)), (H,E), (H,E)), (G,B))$$
$$\subseteq \widehat{f}((F,A), \widehat{f}((H,E)), (H,E), (H,E)), (G,B))$$
$$\subseteq \widehat{f}((F,A)), (H,E), (G,B)) = (S,C).$$

This implies that $\widehat{f}((S,C)), (H,E), (S,C), (H,E), (S,C)) \subseteq (S,C)$. Hence (S,C) is a soft bi-hyperideal over H. ∎

8. Conclusion

Molodtsov introduced the concept of soft set theory, which can be seen and used as a new mathematical tool for dealing with uncertainly. In the present paper, we introduce and initiate the study of soft ternary semihypergroups by using soft sets. The notions of soft ternary semihypergroups, soft ternary subsemihypergroups, soft left (lateral, right) hyperideals, soft hyperideals, soft quasi-hyperideals and soft bi-hyperideals are introduced here, and several related properties are investigated. Based on these results, we could apply soft sets to other types of hyperideals in ternary semihypergroups and do some further work on the properties of soft ternary semihypergroups. Moreover, one may characterize several classes of ternary semihypergroups such as regular ternary semihypergroups by the properties of soft hyperideals (quasi and bi)-hyperideals. Also, by using soft set theory, one may consider other soft algebraic hyperstructures such as soft ternary semihyperrings etc.

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